PART IV

PHILOSOPHY OF MATHEMATICS

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Wholes, Parts, and Numbers (1997)

If it's not one thing, it's two.
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I present here a puzzle that arises in the area of overlap among the philosophy of logic, the philosophy of mathematics, and the philosophy of language. The puzzle also concerns a host of issues in metaphysics, insofar as it crucially involves wholes, their parts, and the relation of part to whole. Almost entirely nontechnical, the puzzle is disarmingly simple to state. What little technicality I introduce below is mostly of a purely logical nature, and mostly inessential to the puzzle's central thrust. I discovered the puzzle nearly twenty years ago. (See note 4 below.) It had been my intention since that time to publish the puzzle together with its solution, but finding a solution that I was strongly inclined to accept proved difficult. I have presented the puzzle orally and informally to a number of philosophers, including several of the world's greatest thinkers in the philosophy of logic and the philosophy of mathematics. None offered a solution that strikes me as definitively striking to the heart of the matter. Indeed, I was in no position to make others appreciate the full philosophical significance of the problem. I present here a couple of my own proposals for its solution, an acknowledgement of some shortcomings of those proposals, and a final nod in the direction of the solution I currently think is best.

The Problem: There are several pieces of fruit, including exactly three whole oranges, on top of the table.[1] I cut one of the oranges exactly in half, eat one of the halves, and leave the remaining half on the table. Consider now the following question:

Q Exactly how many oranges are there remaining on the table?

Any schoolboy is able to calculate that the correct answer to (*Q*) is:

A There are exactly 2½ oranges remaining on the table.

But there is a proof that (*A*) is incorrect: Consider the orange-half (as it may be called) that remains on the table, and whose Siamese twin I have eaten. By Excluded Middle, either it is itself an orange on the table, or else it is not itself an orange on the table. If the former, then there are not only 2½ oranges on the table but 3—the two whole oranges together with the orange-half. If the latter, then there are not as many as 2½ oranges on the table but only 2—together with something that is not itself an

[1] Throughout, in saying that there are exactly *n F*'s, I mean that there are at least and at most *n F*'s, no more and no fewer.

orange on the table. Of course, there are many additional non-oranges on the table: four whole apples, two pear slices, and a kiwi fruit. The presence of non-oranges does not alter the fact that (on this horn of our dilemma) the correct answer to (Q) is:

A' There are exactly 2 oranges remaining on the table.

In either case, then, the exact number of oranges on the table is not a number between 2 and 3. We seem forced to the conclusion that the schoolboy's answer (*A*) to question (Q) is incorrect, specifying either too few or too many.

One should note that the first horn of our dilemma, on which there are exactly three oranges on the table, may be eliminated by changing the example slightly. Suppose I go on to eat exactly half of the remaining orange-half, so that the schoolboy's answer to (Q) now becomes that there are exactly 2¼ oranges on the table. Surely an orange-quarter is not itself an orange; it is only a fractional portion of an orange.[2] On this modified version of the problem, we may construct a simpler proof that the correct answer to (Q) is in fact (A').

The problem is that common sense tells us the correct answer to (Q) is not (A'). It is (*A*).

One solution: A solution to the problem lies somewhere in the very meaning, or perhaps what is called the 'logical form', of sentences like (*A*) and (A'). These sentence have, or at least appear to have, the common form:

F There are exactly n objects x such that ϕ_x.

There is a tradition in philosophical logic and the philosophy of mathematics of glossing the phrase 'there are exactly *n*', as it occurs in (F), as a special kind of quantifier: a *numerical quantifier*. The traditional logicist conception of number fits perfectly with the notion that the numerals '1', '2', '3', etc. are quantifiers (hence, not singular terms), thought of now as second-order predicates. As is well known, in the case of whole numbers, the corresponding numerical quantifiers are contextually definable in first-order logic by making use of the traditional quantifiers '∀' and '∃' in combination with '='. For example, the sentence '2*xFx*' (read 'There are exactly 2 objects x such that *Fx*') may be taken as shorthand for:

$$\exists x \exists y[Fx \wedge Fy \wedge x \neq y \wedge \forall z(Fz \supset x = z \vee y = z)].$$

[2] By contrast, a quarter of any (non-negligible sized) portion of an orange is another portion of an orange. One might say of an orange-quarter, with justification, that it is 'an orange,' in order to distinguish it, for example, from a pear (or a pear-portion). But it seems likely that this is a special use of the predicate 'is an orange' by the speaker to mean *orange-portion* (as opposed to a portion of some other kind of fruit). In saying that a orange-quarter is not itself an orange, I am relying on the intuition (which many, including myself, share) that an orange-quarter is not an element of the semantic extension of the count noun 'orange' in English. One does not produce many oranges from a single orange simply by slicing; instead one produces orange-slices. Although I will often assume in what follows that orange-halves are likewise not themselves oranges, I must emphasize here that nothing I say depends crucially on this assumption. Each of the arguments can be made, *mutatis mutandis*, on the opposite assumption that proper portions of oranges are themselves oranges (elements of the English extension of the noun 'orange')—and even (e.g. by a dilemma form of argument) on the assumption that the noun 'orange' is ambiguous, having one English meaning that includes, and another that excludes, proper orange-portions.

And the sentence '$3xFx$' may be taken as shorthand for:

$$\exists x \exists y \exists z[Fx \wedge Fy \wedge Fz \wedge x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall w(Fw \supset x = w \vee y = w \vee z = w)].$$

The sentence '$3xFx$' thus says something particular about the class of F's: that it has exactly three elements. (Or if one prefers, the sentence says the corresponding thing about the Fregean characteristic function λxFx, which assigns truth to F's and falsity to non-F's.) If we follow the Frege-Carnap-Church tradition in distinguishing for expressions of every type between reference/extension on the one hand and content/intension on the other, then the numeral '3' itself may be taken as expressing the property of classes (or alternatively the concept) of having exactly three elements (the content of '3'), and as referring to the class of all such classes (the extension). This directly yields the Frege-Russell conception of number.[3]

The number 2½ is not a whole number, and (*A*) cannot be taken as shorthand in the same way for any first-order sentence whose only nonlogical component is a predicate for being an orange on the table. But not to despair. We may simply introduce a new expression, say '2.5', as a primitive numerical quantifier, giving (*A*) the particular form '$2.5xFx$', and similarly for other rational numbers. The question 'Exactly how many F's are there?' may now be taken as an instruction to provide the particular numerical quantifier Q such that $\ulcorner QxFx \urcorner$ is true.

One immediate problem with this proposal is that, as we have seen, besides '2.5' there is another numerical quantifier Q such that $\ulcorner Qx$ (*x* is an orange on the table)$\urcorner$ is true, namely '2'. Perhaps we must take the question 'Exactly how many F's are there?' instead as an instruction to provide the *greatest* numerical quantifier Q such that $\ulcorner QxFx \urcorner$ is true, in the standard numerical ordering of numerical quantifiers. As any schoolboy knows, $2.5 > 2$.

One remaining problem is that this proposal does not provide any reason to hold that (A') is actually false. On the contrary, on the proposal now before us, both (*A*) and (A') are deemed literally true. Why, then, does the latter not count, along with the former, as an alternative but equally accurate answer to (Q)?

There is a more serious problem. In classical formal semantics, a quantifier Q in a formula $\ulcorner QxFx \urcorner$ may be regarded as a second-order predicate, one that says something quantitative about the class of F's (or about its characteristic function). The standard universal quantifier '$\forall$'—or alternatively the English word 'everything'—expresses the concept of *universality*; '$\forall xFx$' says that the class of F's is universal. The

[3] It is often said that the theory of classes (or sets), if it is consistent, offers a selection of equally legitimate constructions for the sequence of whole numbers (the von Neumann construction, the Zermelo construction, etc.), no one of which may be singled out as the 'right' one, exactly capturing metaphysically the actual, genuine numbers 0, 1, 2, 3, etc. However, only the Frege-Russell construction fits as well as it does with the treatment of whole-number numerals as first-order quantifiers. Philosophically, this gives the traditional logicist construction a stronger *prima facie* claim than its rivals to capturing the authentic numbers, since number terms ('two', '2½', etc.) are like color terms ('blue', etc.) in that the noun form (singular term) seems essentially parasitic on the adjective form (predicate or quantifier), which is fundamental. (In the final section of this paper, however, I shall present a new challenge to the claim that the logicist conception fits well with the treatment of numerals as quantifiers.)

standard existential quantifier '$\exists$'—or 'something'—expresses the concept of being non-empty. The quantifier 'nothing' expresses the complementary concept of being empty. Similarly for the whole-number numerical quantifiers. They specify the cardinality of the class; '$2xFx$' says that the class of F's has exactly two elements, '$3xFx$' that the class has exactly three elements, and so on. This is precisely how the whole-number numerical quantifiers yield the Frege-Russell conception of number. But what exactly does the '2.5' in '$2.5xFx$' say about the class of F's? That it has exactly 2½ elements? What is that supposed to mean? How does an element of a class come to be counted merely as one-half, rather than as one, in determining the class's cardinality?

It does not. The class of oranges on the table has exactly two elements, no more and no less. The orange-half on the table is not an orange, and hence is not in the class of oranges on the table. It therefore cannot affect the cardinality of that class in any way. (Alternatively, if it is an orange, then the class of oranges on the table has exactly three elements, no more or no less. *Cf.* note 2 above.)

This point may be sharpened by considering an alternative first-order contextual definition of '2', *qua* numerical quantifier. We may contextually define the *lower-bound quantifiers* 'there are no fewer than 2' and 'there are no fewer than 3', respectively, as follows:

$$\geq 2xFx =_{\text{def}} \exists x \exists y (Fx \wedge Fy \wedge x \neq y).$$

$$\geq 3xFx =_{\text{def}} \exists x \exists y \exists z (Fx \wedge Fy \wedge Fz \wedge x \neq y \wedge x \neq z \wedge y \neq z).$$

Consider now the conjunction:

$$\geq 2xFx \wedge \sim \geq 3xFx,$$

i.e., there are no fewer than two F's but it is not the case that there are no fewer than three F's. Letting the predicate letter 'F' mean *orange on the table*, this conjunction is in fact true (assuming that the orange-half is not itself an orange). Since $2 \leq 2.5 < 3$, the conjunction may appear to cohere with the truth of '$2.5xFx$'. But the conjunction is in fact provably equivalent (in first-order logic, using no non-logical hypotheses) to '$2xFx$', as contextually defined above. Once again, analysis in terms of numerical quantifiers leads us to (A'), rather than (A), as our answer to (Q).[4]

The lesson is this: Insofar as (A), not (A'), is the correct answer to (Q), the '2½' in (A) does not say, or does not merely say, something quantitative about the class of oranges on the table. If it says anything quantitative at all about that class, it also says something more, something not about the class.

An Alternative Solution: Have we construed our question (Q) excessively literally? Perhaps it asks something not merely about the class of oranges on the table,

[4] I discovered the puzzle, quite by chance, when teaching my logic students at Princeton University how to express the numerical quantifiers in terms of the lower-bound quantifiers, and how to express the latter in terms of '$\forall$', '$\exists$' and '$=$'. One student, who failed to see the various connections clearly, innocently asked how the conjunction 'There are at least 2, but there are not at least 3, F's' (as defined in the text), requires that there be no more than 2 F's, rather than some number between 2 and 3, like 2½. Pondering what it means to say that there are exactly two and one-half F's, I realized that the student's confusion was not only hers, and not unwarranted.

properly so-called, but about the class of (proper and improper) pieces or portions of orange on the table. No; the latter class has exactly three elements, not two and one-half. Unless one counts undetached orange-parts as pieces of orange. And in that case, the class of pieces of orange on the table has some very large cardinality, far greater than 2½. If (*A*) is the correct answer to (*Q*), then (*Q*) does not ask for the number of objects that are pieces of orange remaining on the table.

Perhaps (*Q*) is concerned not with how many, but with *how much*. The question may be this: Exactly how much orange-stuff is there on the table? Certainly this is a legitimate question. It is the sort of question one might ask if one needs to make a specific amount of orange juice. It is not so much the quantity of oranges that matters as the quantity of orange-stuff. One should probably see this how-much question as asking for a measure of mass or weight. But since oranges do not typically vary greatly in size and weight—unlike, say, pumpkins—in some contexts (*A*) may yield a correct answer to the question of how much orange-stuff there is on the table.

Perhaps. But there is another way to construe the question. The *count* (how many) construal is at least as legitimate as the *mass* (how-much) construal, if no more so. (Some questions may even require the count construal, e.g. 'How many oranges make up three pounds of orange-stuff?') If there are exactly two pumpkins in the yard, one of which weighs a few ounces and the other six hundred pounds, it is still correct to answer the question 'Exactly how many pumpkins are there in the yard?' by saying that there are exactly two—even though this does not yield an answer to the question of how much pumpkin there is (which in this case is the equivalent of a substantial number of middle-sized pumpkins). Ever when our question (*Q*) is explicitly put forward as a count question, and not as a mass question, the correct answer still appears to be (*A*) rather than (*A′*). Ask the schoolboy, 'Never mind how much orange-stuff there is on the table, exactly how many oranges are there?' The answer comes back: Two and a half.

A Preferable Solution: Let us write (*A*) out in longhand, replacing all mathematical notation with genuine English:

There are exactly two and one-half oranges remaining on the table.

With a modicum of word-processing magic, and some finesse, this might be rewritten as the following conjunction:

A″: There are exactly two oranges remaining on the table and there is exactly one orange-half remaining on the table.

We are now in a position to grant that there is something right about this alternative analysis of (*A*). There are indeed two objects such that each is an orange on the table, and no more than two. In addition, there is indeed one orange-half on the table, and in *some* sense, no more than one. As we have seen, there are also thousands of undetached orange-halves on the table. When we say that there is exactly one orange-half on the table, we mean that there is exactly one detached orange-half (or at least that there is exactly one mostly detached orange-half, or something similar).

There are serious difficulties with our new proposal, though. The shuffling around of characters that transformed (*A*) into (*A″*) produced a sea change in logical form.

Our new answer (A'') is evidently not a numerical-quantifier generalization of the form '*nxFx'* at all, but a conjunction of distinct generalizations. It is in fact the conjunction of our formerly rejected answer (A') with something else. What else? A new numerical-quantifier generalization of the particular form '1*xGx*', where '*G'* stands in for the phrase 'orange-half on the table' (or perhaps I should say, for the phrase 'mostly detached orange-half on the table'). The patient has undergone massive surgical reconstruction. The numeral '2' occurring in (*A*) has been separated from its accompanying fraction, and now performs as a solo numerical quantifier. The fraction itself has been severely mutilated. The numeral '1', which appears as the fraction's numerator in (*A*), has ascended to the status of an antonymous quantifier, functioning independently both of its former denominator and of the quantifier in the first conjunct. At the same time, the word 'half' appearing in the longhand version of (*A*) has been reassigned, from quantifier position to predicate position. In effect, the mixed-number expression '2½', occurring as a unit in (*A*), has been blown to smithereens, its whole integer now over here, the fraction's numerator now over there, the fraction's denominator someplace else. Even those of us who have survived major earthquakes need some time to adjust to reconfiguration on this scale.[5]

Of particular philosophical interest is the word 'half', which on this proposal, is attached by a hyphen to a count noun like 'orange' (*qua* noun rather than adjective) to form a new count noun with a new extension (and hence, of course, with a new intension as well). One may well doubt that this device can be sensibly attached to each and every count noun. If one cuts a television set down the middle, for example, does one thereby obtain two television-set-halves (or two half-television-sets)? Well, perhaps one does. In either case, the device seems clearly applicable at least to a great many nouns, especially the names of a wide variety of fruits and vegetables, and perhaps to such expressions as 'cup of coffee'. ('Exactly how many cups of coffee are there on the table? Two and a half.') And indeed, one advantage of this account of the English word 'half' is that it may provide a semantically-based explanation (of a sort, anyway) for the uneasiness one feels in such weird constructions as 'There are exactly two and one-half television sets in the storage room'.

On the other hand, the word 'half' occurring in the pure-English version of (*A*) is evidently the English counterpart of the fraction's denominator in the original (*A*). Can it be that one obtains a correct analysis of fractional quantifiers by stripping the denominator in numerical-quantifier position of a numeral's customary status as quantifier, and reclassifying it altogether as a special nonmathematical operator on nouns? If so, the apparent unity of the fraction is a mirage. Fractions emerge as fragmented entities, comprised by both a numerical quantifier (the numerator) and a noun operator (the denominator)—hybrid entities that are part mathematical and part non-mathematical in form and function. What are entities like that doing in a purely deductive discipline like mathematics? The whole things smells fishy.

[5] Living in Santa Barbara, my family was spared the great trauma that my friends at Cal State Northridge and their loved ones have had to endure since January 17, 1994. Although our own experience of the great Northridge Quake amounted to little more than an inconvenience, we are no strangers to massive deconstruction, having endured the full fury of Hurricane Iniki in a demolished condo on the south shore of Kauai only a few years before. And now this.

Even the schoolboy knows that the phrase 'and a half' in the sentence 'There are exactly two and a half oranges on the table' goes with the 'two' and not with the 'orange'.

A related problem: As we have seen, (A'') is the conjunction of (A') with 'There is exactly one orange-half on the table'. But (Q) asks simply for the number of oranges on the table. On the proposal under consideration, (A') correctly specifies that number. The second conjunct '$1x$ (x is an orange-half on the table)' merely provides extraneous information, information that was not explicitly requested. Why, then, do we not simply give (A') in answer to (Q), holding the second conjunct in reserve, in case we are later asked exactly how many orange-halves there are on the table? Instead, we persist in giving (A) as our answer, even though no one has asked separately for the number of orange-halves on the table.

There is a more concrete problem. Our new proposal puts (A'') forward as an analysis of the schoolboy's answer (A) to (Q). This analysis is subject to direct disproof. For (A'') to be a correct answer to (Q), it would have to be true. And this would require its first conjunct, (A'), to be true. Now for any pair of numbers n and m, if there are exactly n F's, no more and no less, and also exactly m F's, no more and no less, then n and m must be exactly equal. But $2\frac{1}{2} \neq 2$. The two alternatives, (A) and (A'), are not teammates but competing rivals. Hence, since it entails (A'), if (A'') is a correct answer to (Q), then (A) is not. In a word, (A'') and (A) are incompatible. Therefore, the former cannot provide a correct analysis of the latter.

Instead of precluding (A)'s rival, (A'), by entailing its negation, (A'') does exactly the opposite, directly asserting (A') itself and then something further. Suppose one were to ask a question for which something analogous to (A'') would be a correct reply—such as, for example, 'Exactly how many *whole* oranges are there remaining on the table, and exactly how many orange-halves?'. Here one might well reply, 'Exactly two of the former and exactly one of the latter.' A response instead of only the first conjunct would be regarded as compatible with the right answer, correct as far as it goes but essentially incomplete.[6] Even if one were to ask a question for which something analogous to (A'') is only part of the correct reply ('What portions of oranges are there remaining on the table, and exactly how many of each?' 'Exactly two whole oranges, exactly one orange-half, and nothing more'), we should still regard the first conjunct as compatible with the right answer. But in giving (A) as our answer to the original question (Q), we also reject (A')—not merely as incomplete, but as flatly incorrect. There are *not* exactly two oranges remaining on the table. On the contrary, there are exactly two and one-half.

In fact, (A) may also clash with the second conjunct of (A''). For (A) does not, or at least need not, pretend to specify the total number of orange-halves on the table. Suppose I cut one of the whole oranges exactly in half, placing the two

[6] I assume here that there are exactly two whole oranges remaining on the table, not exactly two and one-half. The phrase 'two and one-half whole oranges' is taken here to be an oxymoron. This is largely a matter of terminology. Call them what one will, there are exactly two (rather than two and one-half) of *something* orange-like remaining on the table. The orangey things of which there are exactly two on the table are what I call *whole oranges*. (Modifying Ledford, if it's less than one F but more than none, there's some sort G such that it's no G.)

orange-halves back on the table. Now there is only one whole orange together with three orange-halves on the table. Our current proposal would answer (Q) under these circumstances by saying that there are one and three-halves (1³⁄₂) oranges on the table. But one might still answer our original question (Q) with the same old answer (A), adding now that one of the oranges has been cut in half. And indeed, $1\frac{3}{2} = 2\frac{1}{2}$. It is especially tempting to count (A) as still a correct answer since two of the three orange-halves on the table do indeed come from the same orange. While no longer whole, the orange in question might still be deemed to exist as a (slightly scattered) orange on the table. The configuration of the oranges on the table has changed, but not their number. By contrast, (A'') is not in any way a correct description of the new situation. There is not only one orange-half on the table. Rather, there are exactly three orange-halves on the table (together with one whole orange). This further demonstrates that (A) and (A'') are not equivalent.

Return to the original situation, with two whole oranges and a single orange-half on the table. As the schoolboy knows, $2\frac{1}{2} = \frac{5}{2}$. Substituting into ($A$), we obtain, as an alternative answer to (Q), that there are exactly five-halves oranges on the table. Special care must be taken here to distinguish syntactically between 'there are five-halves oranges' and 'there are five half-oranges'. The proposal under consideration regards the distinction as purely syntactic, a distinction without a difference. On that proposal, the claim that there are exactly five-halves oranges on the table amounts to the claim that there are exactly five orange-halves on the table—the sort of thing that would be true if there were one orange-half from each of exactly five different oranges on the table. The proposal cannot suppress the inevitable protest that there is only one orange-half on the table, not five—together, of course, with two whole oranges. Can we, as it were, *grok* two whole oranges alternatively as four orange-halves, without actually cutting into them? The four orange-halves would have to be undetached orange-halves. Well, then, which four undetached orange-halves? There are a great many undetached orange-halves *in* those two whole oranges. Why do we say only four? Is it not at least as accurate to say instead that there is one detached orange-half on the table and in addition hundreds, perhaps thousands, of undetached orange-halves on the table?

Perhaps we mean something like this: Cut them up into orange-halves any way you like, there will be exactly five *non-overlapping* orange-halves on the table. But probably we do not. Certainly the original schoolboy's answer (A) does not literally and explicitly make any dispositional assertion about what *would* result from performing certain hypothetical cutting procedures. The mere substitution of the notation '$\frac{5}{2}$' for '$2\frac{1}{2}$' cannot introduce any counterfactual or dispositional notions that were not there to begin with. It is far more likely that our latest proposal errs in equating the claim that there are exactly five-halves oranges on the table with the (apparently false) claim that there are exactly five orange-halves on the table.[7]

[7] Frank McGuinness points out that since the expression '$2\frac{1}{2}$' is not a singular term on this solution, but a mishmash of numerical quantifiers, a truth-functional connective, and a predicate operator, the solution effectively blocks any straightforward application of Leibniz's Law (Substitutivity of Equality) in the manner proposed. This observation illustrates the extent to which the solution fails to respect the import of standard mathematical notation. Indeed, the solution

What, then, does the former claim mean? If it is correct, it can only mean something mathematically equivalent to (A)—which is, or at least seems, incompatible with anything mathematically equivalent to (A'). The meaning of (A') is reasonably clear. But the exact meaning of (A) still is not.

The Preferred Solution? We considered some fairly definite phenomena that led us to dismiss the numerical-quantifier analysis. That proposal should now be re-evaluated in light of our dissatisfaction with alternative analyses. We have already acknowledged that the alleged mixed-numerical quantifier '2.5' is not contextually definable in first-order logic. Maybe it is *sui generis*. We are not compelled to say that the sentence '$2.5x$ (x is an orange on the table)' says something quantitative about the class of oranges on the table. Surely (A) does not say anything primarily about the *class* of oranges on the table. The number 2½ is not the number of elements of the class of oranges on the table. It is a mixed number, while finite classes have only whole-number cardinalities. Mixed number though it is, it is also exactly how many oranges there are on the table.

Perhaps our numerically quantified sentence '$2.5x$ (x is an orange on the table)' says something quantitative not about the class of oranges on the table, nor anything similar (like the characteristic function of that class), but about . . . well, . . . *the oranges on the table*—the property, if you will, of being such an orange, or better, the *plurality* (group, collective), i.e. the oranges themselves. There are not only two, but two and one-half, of *those things*.

Pluralities are what plural terms like 'the oranges on the table' and 'those things', and conjunctive-enumerative terms like 'Sid and Nancy', refer to. A plurality is essentially not one but many. It is well known that pluralities differ in various ways from the separate individuals, taken individually, and also from their corresponding unity, the class of the individuals. When, in one of my fondest fantasies, C. Anthony Anderson, Anthony Brueckner, and I lift the Philosophy Department's photocopier to throw it out the window, no one of us lifts the machine individually (although it does seem that Brueckner and I put forth more than our fair share of effort). Still less does the class of all three of us—a causally inert abstract entity—lift the machine. It is not a unity, but a threesome, that lifts the machine. Note, however, that the threesome is not a fourth entity, over and above the three of us. It *is* the three of us—or better put, the three of us are not a single entity at all but three, and therefore not a fourth entity. Talk of 'pluralities' may be regarded as a manner of speaking. The crucial idea is that some properties are exemplified or possessed by individuals taken collectively, in concert, rather than taken individually and rather than by the corresponding class. The property of lifting the photocopier is such a property.[8]

makes a complete mystery of equations like '$2½ = 5/2$'. What is this equation supposed to mean, if not something that licenses the substitution of '5/2;' for '2½' in an ordinary extensional context (like 'There are exactly ___ gallons of fuel remaining in the tank', as opposed to 'Anderson believes that there are exactly ___ gallons of fuel remaining in the tank')?

[8] This idea also seems to lie behind our tendency to anthropomorphize groups, as when it is said, for instance, that the public favors one policy over another. A plurality should not be confused with the mereological sum or fusion of individuals. A mereological sum is a unit composed of many, the plurality is/are the many of which the sum is composed. The former is one, the latter essentially more than one. One might cash out the collective exemplification of a property in terms of the

On my proposal, yet another respect in which a 'plurality'—a many rather than a one—may differ from its (more accurately, from their) corresponding class is in regard to number. What numbers number are not classes but pluralities, things taken together, collectively and not individually. The class of objects that are oranges on the table has cardinality 2. Each individual orange on the table has a different number, namely 1. The class itself is also one. But the oranges themselves number some 2½.

How does the plurality of oranges on the table come to have a mixed number rather than a whole number? The orange-half is not itself an orange. Nor, therefore, is it one of the oranges on the table. And yet it is included, by virtue of its quantity of orange-stuff, in the plurality of oranges on the table. When sizing up a plurality, different individuals are given different weight. Some may have fractional shares, counting for less than one but more than none. Though not itself an orange on the table, the orange-half is counted among the oranges on the table. It is not *one* of those things. But it is *of* those things. And among those things, it counts for less than one—for one-half, in fact.[9] To be sure, this is not at all how the cardinality of a class is measured; instead, each element counts equally as one. The quantity of a plurality is measured differently. Among the F's, a part of a whole F counts for part of a whole number, i.e. it counts for a fraction.

Strictly speaking, on this proposal numbers are not merely properties of pluralities *simpliciter*, but relativized properties. They are properties of pluralities *relative to* some sort or counting property. Typically, the sort or counting property to which the number of a plurality is relativized is a sort or property of the individuals so numbered. The oranges on the table are two and one-half in number, but the detached orange-portions, proper and not, on the table are three—even though these are the very same things. The orange-half counts for 1 if one is counting detached orange-portions, but only counts for one-half if one is counting oranges—and only counts for zero if one is counting *whole* oranges, since it is not among the whole oranges.[10] One may also define an absolute notion of the number of a plurality, in terms of the number relative to a counting property, by taking the counting property

holding of a relation among the participants. Lifting a particular photocopier would appear to be a property, not an n-ary relation for any $n > 1$. Yet it may happen that two individuals, or three, or more, co-operate to lift the photocopier in concert. If lifting a photocopier is a relation, the relation must be *multigrade*, allowed to be n-ary for any of a wide range of whole numbers n (uniary, binary, ternary, etc.). Property or multigrade relation, there is a difficulty either way. Given only a predicate (monadic or multiadic) for the attribute of lifting a photocopier, a monadic predicate for the property of being a full professor in the UCSB philosophy department holding a doctorate degree from UCLA, and the full resources of standard first-order logic, it is not possible to write a sentence saying that the UCSB full professors of philosophy with doctorates from UCLA are lifting the photocopier. A mechanism for plural reference is needed.

[9] The fact that it does not count for one may be why it is itself not *an* orange. An orange is one orange. The orange-half is not one orange; it is only *one-half* of an orange. These points are not essential to the solution proposed here, however, whose core ideas are compatible also with the opposing view that the orange-half is an orange, as long as it counts not for one but for one-half. See note 2 above.

[10] See note 6. Husserl held, against Frege, that numbers are properties of 'multiplicities' (*Mannigfaltigkeit*). Peter M. Simons cites Husserl while defending the view that numbers are properties of the referents of plural terms, in his 'Numbers and Manifolds,' in B. Smith, ed., *Parts*

to be fixed as the universal property of *being an object* or *being a thing*. The two and one-half oranges on the table are three things. The number of oranges that are on the table is two and one-half, whereas the number of things on the table that are oranges is two.[11] The number of things that are such-and-such is always a whole number. The sort *orange* includes a provision for fractional shares; the sort *thing* does not. Ledford's variant of Murphy's Law may be generalized as follows: If it's not one thing, it's a plurality of n for some whole number $n > 1$. But if it's not one fruit, it could be less.

If something along the lines of this proposal is right, then there is a serious rift between (A') and the first-order formula that had been given as a definition for its formal counterpart '$2x$ (x is an orange on the table)':

$$\exists x \exists y[x \text{ is an orange on the table} \wedge y \text{ is an orange on the table} \wedge x \neq y \wedge \forall z(z \text{ is an orange on the table} \supset x = z \vee y = z)].$$

This first-order formula is true. There are exactly two objects such that each is an orange on the table. This is, or is at least tantamount to, a statement of the cardinality of the class of oranges on the table.[12] The first-order formula does not attempt to specify the quantity of the plurality of oranges on the table. That is precisely what (A') does, and it does so unsuccessfully. (A') is false; the number of oranges on the table is correctly given by (A).

Given this rift, it is left for us to decide whether the quasi-formal sentence '$2x$ (x is an orange on the table)' is to mean the same as (A') or instead the same as the formula displayed above. Since we already have a way to symbolize the latter (namely, the latter itself), it would be better to let the numerically quantified sentence symbolize the former. On this solution, not only the mixed-number quantifier '2.5' but even whole-number quantifiers like '2', as they occur in numerically quantified sentences like (A'), are strictly not definable using the traditional universal and existential quantifiers together with identity. Any quantified statement of

and Moments: Studies in Logic and Formal Ontology (Munich: Philosophia Verlog, 1982), pp. 160–198. See also Glenn Kessler, 'Frege, Mill, and the Foundations of Arithmetic,' *Journal of Philosophy*, 77, 2 (February 1980), pp. 65–79; and Simons' reply, 'Against the Aggregate Theory of Number,' *Journal of Philosophy*, 79, 3 (March 1982), pp. 163–167. Kessler defends the view that numbers are properties relativized to properties. However, Kessler treats numbers properties of aggregates (rather than pluralities), relativized to 'individuating properties' of parts of those aggregates. Byeong-uk Yi endorses the view that numbers are properties of pluralities, although he does not accept my proposal that they are relativized to counting properties or that pluralities like the oranges on the table have mixed numbers. Instead he believes the correct answer to (Q) is (A'). Simons (p. 160) also restricts his account to whole numbers. One significant advantage of treating numbers as properties of pluralities, however, is precisely that doing so in the way I propose here—with some individuals of a plurality counting for more than none but less than one, relative to a counting property—justifies, and seems to underlie, our giving (A) as the correct answer to (Q) while rejecting (A'). (Thanks to Ronald McIntyre, Kevin Mulligan, and Yi for scholarly references.)

[11] See again note 2 above. One may substitute here the claim that there are exactly three things on the table that are oranges.

[12] Like the property of being a thing, the property of being an element of a set does not include a provision for fractional shares. There are exactly two and one-half oranges that are elements of the class of detached pieces of fruit on the table, but only two elements of that class are oranges (or alternatively, in accordance with the preceding note, all of three elements are).

classical first-order logic concerns classes rather than pluralities. And it would appear that the quantity of a plurality may sometimes diverge, at least by a fraction, from the cardinality of the corresponding class. Another blow to traditional logicism.

One way to represent plural descriptions of the form 'the F's' would be by means of a variable-binding plurality-abstraction operator. We may read '$\mathscr{P}xFx$' as 'the objects x such that Fx'. (Note the plural form 'objects'.) This expression may be regarded as being of a special logico-syntactic type, which may be called a *plural term* (as opposed to a singular term). Conjunctive-enumerative terms and plural indexicals should count equally as plural terms. On the proposal I am making here, it may be said that numerical quantifiers like '2.5' are what some philosophers have called 'plural quantifiers'.[13] Given an appropriate numerical predicate, our numerical quantifier '2.5' might be contextually defined so that '$2.5xFx$' is taken to mean the same as '$\mathscr{P}xFx$ are 2½ in number'. A full treatment should introduce plural variables (corresponding to the English pronoun 'they'). Doing so would allow for the formalization of a plural description like 'the individuals who lifted the photocopier' using the variable-binding definite-description operator attached to a plural variable, in the manner of 'the plurality of individuals who are such that they lifted the photocopier'. (Notice that it would be incorrect to attempt to capture this plural description by means of the plurality-abstraction operator '$\mathscr{P}$' attached to a singular variable, since no one of the threesome who lifted the machine did so individually. See note 8.)

A couple of interestingly odd (though, I think, not unacceptable) consequences, or possible consequences, should be noted. First, if an orange-half from a fourth orange is placed alongside the two and one-half oranges already on the table, and question (Q) is posed anew, it is difficult (although not impossible) to resist the conclusion that the answer becomes that there are exactly three oranges on the table, since $2½ + ½ = 3$, despite the fact that the two detached orange-halves now on the table do not come from a single orange. Insofar as one is inclined to reject this answer, and to claim instead that there are only two oranges on the table (together with two orange-halves which do not comprise a third orange), one might likewise proffer (A') in place of (A) as the correct answer to the original question (Q), as posed before the placement of the second orange-half on the table. My own intuitions balk at (A') as the correct answer to (Q) in the original circumstances much more strongly than they balk at the answer 'exactly three' in the new circumstances—although a solution that avoids both would clearly be preferable. If (A') rather than (A) is the correct answer in the original circumstances, then one wonders whether there can be any true statement of the form 'There are exactly two and one-half F's'. Surely, for example, there can be exactly two and one-half gallons of orange juice in the tank. And if another half-gallon of juice is added to the tank, there will then be exactly three gallons. But then why not exactly three oranges on the table when the new orange-half is placed alongside the two and one-half oranges

[13] For valuable discussion of plural quantification, see George Boolos, 'To Be is To Be the Value of A Variable (or To Be Some Values of Some Variables),' *Journal of Philosophy*, 81 (1984), pp. 430–449; 'Nominalist Platonism,' *The Philosophical Review*, 94, 3 (July 1985), pp. 327–344; and David Lewis, *Parts of Classes* (Oxford: Basil Blackwell, 1991), at pp. 62–71.

already there? Perhaps because comparing gallons of orange juice to oranges is comparing apples and oranges. If one puts two half-gallons of juice together in the same tank, the result is a single gallon. But put two orange-halves from different oranges together on the same table, and the result is . . . what? Two orange-halves put together. Is that a scattered orange?[14]

Second, if we follow this path, numerical quantifiers like '2' may emerge as *nonextensional* operators. The phrase 'there are exactly 2' in (A') does not express a numerical property that is attributed to the semantic extension of the phrase (general term) 'orange on the table', i.e. to the class. Instead (A') assigns that number to the plurality semantically determined by the phrase, relative to the property expressed by the phrase. Since the orange-half counts itself among the plurality but is not itself an element of the phrase's extension, that extension does not determine the plurality. Nor, as we have seen, does the extension determine the property relative to which the elements are two and one-half in number. The non-extensionality of the numerical quantifier '2' manifests itself in the fact that although the phrase 'orange on the table' has exactly the same extension as the phrase 'whole orange on the table', the sentence 'There are exactly two whole oranges on the table' is true of the original example whereas the sentence 'There are exactly two oranges on the table' is false. The truth value of a statement of the form ⌜There are exactly n F's⌝ depends not on some feature (e.g. the cardinality) of the class of F's, but on a feature of the F's themselves, taken collectively *qua* F's—or if one prefers, on a feature of the property of *being an F.* A numerical-quantifier phrase ⌜there are exactly n objects x such that⌝ is thus less like the phrase 'the class of objects x such that' than it is like the phrase 'the property of *being an object x such that*'.[15]

[14] Some, though not all, of the discomfort one feels in answering that there are now exactly three oranges on the table may stem from an inclination to interpret this answer as meaning that there are exactly three *whole* oranges on the table. This answer is presumably false of the expanded example. If it is true there are exactly three oranges on the table, then one of the three is the scattered mereological sum of two orange-halves, and hence not a whole orange. (This issue may be partly terminological. See again note 6. An inclination to interpret the answer 'exactly three oranges' as concerning whole oranges may be a result of the fact that 3 is a whole number.)

Robin Jeshion urged in discussion that one, such as me, who favors the preferred solution in the text should resist the inference that there are exactly three oranges when there are exactly two together with exactly two orange-halves. I am inclined to agree. But I remain troubled by the nagging fact that $2\frac{1}{2} + \frac{1}{2} = 3$. Why does this mathematical equation apply straightforwardly to gallons of orange juice but not to oranges, to yards but not to yardsticks, etc.?

[15] See again note 2. See also note 10. If instead the orange-half is included in the semantic extension of the phrase 'orange on the table', then that class is also the extension of the phrase 'proper or improper detached orange-portion on the table'. The quantity of the plurality determined by the latter phrase, relative to the property expressed, is 3. This is greater by ½ than the quantity of the plurality determined by the former phrase relative to the property expressed (i.e. the number of oranges on the table). The plurality-abstraction operator '$\mathscr{P}$' is therefore likewise a nonextensional operator.

One may want to distinguish here between the singular phrase 'orange on the table' and its pluralization 'oranges on the table'. It may be held, for example, that the latter phrase does not have an extension, as a single unified object, but instead applies to the plurality, i.e. to the oranges themselves, including the orange-half in its second-class status. Pluralization may thus emerge as a nonextensional operation.

Notice that the nonextensionality of numerical quantifiers 'there are exactly n objects x such that' induces failures of substitution of co-extensional expressions only within the numerical quantifier's

Insofar as either, or both, of these alleged consequences is deemed genuine and undesirable, an alternative solution to the original problem is wanted. I have canvassed here all of the promising solutions I can think of, and in each case I have noted consequences that strike me as being at least as undesirable as the two possible consequences just noted. It is possible, of course, that there is some alternative solution that is free of all such difficulties. The reader is hereby invited to discover that solution. I would welcome hearing from you.[16]

operand matrix. More precisely, there are formulas ϕ_α and ψ_α, containing α as a free variable, such that $\ulcorner$there are exactly n objects α such that $\phi_\alpha\urcorner$ and $\ulcorner\forall_\alpha(\phi_\alpha \equiv \psi_\alpha)\urcorner$ are both true whereas $\ulcorner$there are exactly n objects α such that $\psi_\alpha\urcorner$ is false. This does not entail that there are failures of substitution *within* the quantifier phrase. That is, we have no reason to deny that if $\ulcorner$there are exactly n objects α such that $\phi_\alpha\urcorner$ and $\ulcorner n = m\urcorner$ are both true, then so is $\ulcorner$there are exactly m objects α such that $\phi_\alpha\urcorner$. (Contrast with note 7 above.)

[16] A version of the present essay was delivered as the fifth annual *Philosophical Perspectives* Lecture at California State University, Northridge. I thank that institution for its invitation. As indicated above, I have discussed the problem presented here with a number of philosophers. Though I found their proposals unconvincing, I am grateful to them for their reactions. I am especially grateful to Ilhan Inan for discussion, and to Takashi Yagisawa and Byeong-uk Yi for correspondence, concerning the favored solution proposed in the final section, and to my audience at CSUN for their helpful comments.