Disappearing Diamonds

Fitch-Like Results in Bimodal Logic*

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Abstract

Augment the propositional language with two modal operators: $\Box$ and $\blacksquare$. Define $\blacklozenge$ to be the dual of $\blacksquare$, i.e. $\blacklozenge \equiv \neg\blacksquare\neg$. Whenever (X) is of the form $\varphi \to \psi$, let ($X^{\blacklozenge}$) be $\varphi \to \blacklozenge\psi$. ($X^{\blacklozenge}$) can be thought of as the modally qualified counterpart of (X)—for instance, under the metaphysical interpretation of $\blacklozenge$, where (X) says φ implies ψ, ($X^{\blacklozenge}$) says φ implies *possibly* ψ. This paper shows that for various interesting instances of (X), fairly weak assumptions suffice for ($X^{\blacklozenge}$) to imply (X)—so, the modally qualified principle is as strong as its unqualified counterpart. These results have surprising and interesting implications for issues spanning many areas of philosophy.

1 Introduction

Augment the propositional language with two modal operators: $\Box$ and $\blacksquare$.[1] Define $\blacklozenge$ to be the dual of $\blacksquare$, i.e. $\blacklozenge \equiv \neg\blacksquare\neg$. Whenever (X) is of the form $\varphi \to \psi$, let ($X^{\blacklozenge}$) be $\varphi \to \blacklozenge\psi$. ($X^{\blacklozenge}$) can be thought of as the modally qualified counterpart of (X)—for instance, under the metaphysical interpretation of $\blacklozenge$, where (X) says φ implies ψ, ($X^{\blacklozenge}$) says φ implies *possibly* ψ. This paper shows that for various interesting instances of (X), fairly weak assumptions suffice for ($X^{\blacklozenge}$) to imply (X)—so, the modally qualified

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[1]Where no risk of confusion between use and mention arises, we omit quotation marks.

principle is as strong as its unqualified counterpart. These results have surprising and interesting implications for issues spanning many areas of philosophy.

The Church–Fitch Theorem—or as it is more commonly known, 'Fitch's Paradox' or the 'Paradox of Knowability'—is a well-known instance of the kind of result just outlined.[2] The theorem is often described as the result that if every truth is possibly known, then every truth is known. Although the result is often explicated in epistemic-alethic terms, we can abstract away from specific interpretations of the modal operators. From a technical point of view, the result (as formalised in a propositional bimodal logic) simply shows that any logic (satisfying certain weak assumptions) that contains ($\mathrm{S}_{\Box}^{\blacklozenge}$) $p \to \blacklozenge\Box p$ also contains ($\mathrm{S}_{\Box}$) $p \to \Box p$. ($\mathrm{S}_{\Box}^{\blacklozenge}$) differs from ($\mathrm{S}_{\Box}$) only in that it qualifies the consequent of ($\mathrm{S}_{\Box}$) using $\blacklozenge$. The Church–Fitch Theorem shows that such a qualification is redundant—weak assumptions suffice to make the $\blacklozenge$ in ($\mathrm{S}_{\Box}^{\blacklozenge}$) disappear. It is for this reason that Jenkins (2009) describes the air of paradoxicality surrounding the result as one concerning "The Mystery of the Disappearing Diamond". Borrowing from Jenkin's apt description, we shall refer, more generally, to cases of principles of the form ($\mathrm{X}^{\blacklozenge}$) implying their unqualified counterparts (X) as cases of 'disappearing diamonds'.

This paper shows that interesting cases of disappearing diamonds extend beyond the Church–Fitch case and are more widespread than is commonly recognised. For instance, one case concerns ($4_{\Box}$) $\Box p \to \Box\Box p$ and ($5_{\Box}$) $\Diamond p \to \Box\Diamond p$, and their qualified counterparts, ($4_{\Box}^{\blacklozenge}$) $\Box p \to \blacklozenge\Box\Box p$ and ($5_{\Box}^{\blacklozenge}$) $\Diamond p \to \blacklozenge\Box\Diamond p$. In §3, we show that given fairly weak assumptions, ($4_{\Box}^{\blacklozenge}$) and ($5_{\Box}^{\blacklozenge}$) jointly imply ($4_{\Box}$) and ($5_{\Box}$). §4 and §5 note some other cases of disappearing diamonds and finally, §6 concludes by raising some outstanding questions. But first, we begin by introducing some preliminaries.

2 Preliminaries

We assume in the background a propositional bimodal language whose sentences are defined recursively as follows:

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \blacksquare\varphi$$

where p ranges over the members of a countable set At of propositional letters. We take $\neg$ and $\wedge$ to be our only primitive connectives, from which

[2]In Fitch's original 1963, he attributes the result to an anonymous referee, who was later discovered to be Alonzo Church (see Church (2009)).

Name	Principle	Frame Condition
$(\mathrm{K}_\Box)$	$\Box(p \to q) \to (\Box p \to \Box q)$	Trivial
$(\mathrm{D}_\Box)$	$\Box p \to \Diamond p$	Serial
$(\mathrm{T}_\Box)$	$p \to \Diamond p$	Reflexive
$(\mathrm{U}_\Box)$	$\Box(\Box p \to p)$	Shift-Reflexive
$(\mathrm{B}_\Box)$	$p \to \Box\Diamond p$	Symmetric
$(4_\Box)$	$\Box p \to \Box\Box p$	Transitive
$(5_\Box)$	$\Diamond p \to \Box\Diamond p$	Euclidean
$(\mathrm{S}_\Box)$	$p \to \Box p$	Subidentity

Table 1: Some common principles and their frame conditions

the other connectives are defined. $\Diamond$ abbreviates $\neg\Box\neg$ and $\blacklozenge$ abbreviates $\neg\blacksquare\neg$. We call $\Box$ and $\Diamond$ (and similarly, $\blacksquare$ and $\blacklozenge$) each other's *duals*. We use $\varphi, \psi, \ldots$ as metavariables ranging over sentences of the language.

A *modal logic* L is a set of sentences which contains all truth-functional tautologies and is closed under modus ponens (MP) and uniform substitution (US). φ is a *theorem* of L ($\vdash_{\mathsf{L}} \varphi$) iff $\varphi \in \mathsf{L}$. If $\mathsf{L} \subseteq \mathsf{L}'$, we say that L' is an *extension* of L. Table 1 lists some common modal principles and their corresponding frame conditions.[3] (The corresponding principles for $\blacksquare$ can be obtained by substituting each occurrence of $\Box$ and $\Diamond$ with $\blacksquare$ and $\blacklozenge$ respectively.)

L is $\Box$-*congruential* if it is closed under:

$(\mathrm{RC}_\Box)$ If $\vdash \varphi \leftrightarrow \psi$, then $\vdash \Box\varphi \leftrightarrow \Box\psi$.

$\Box$-congruential logics are also closed under:

$(\mathrm{RC}_\Diamond)$ If $\vdash \varphi \leftrightarrow \psi$, then $\vdash \Diamond\varphi \leftrightarrow \Diamond\psi$.

We use M, M_1, M_2, $\ldots$ to denote arbitrary modalities, where a *modality* is a string $O_1 \ldots O_n$, with each O_i $(1 \leq i \leq n)$ being either $\Box$ or $\Diamond$, for any $n \geq 0$ (with the case where $n = 0$ being the empty modality). $\tilde{M}$ denotes the *dual string* of M (e.g. if $M = \Box\Diamond\Diamond\Box$, then $\tilde{M} = \Diamond\Box\Box\Diamond$). More precisely, if $M = O_1 \ldots O_n$, then $\tilde{M} = \tilde{O}_1 \ldots \tilde{O}_n$ (where $\tilde{O}_i$ is the dual of O_i, for $1 \leq i \leq n$). In the case of the empty modality, its dual string is itself.

[3] The frame condition of 'subidentity' for $(\mathrm{S}_\Box)$ is the condition that for each point in the frame, either it accesses no point or it accesses only itself.

We will make use of the following familiar facts about $\Box$-congruential logics (see, for instance, (Chellas, 1980, 233)):

Proposition 1. *Let* L *be closed under* ($\mathrm{RC}_\Box$). *Then,* $\vdash_{\mathsf{L}} Mp \leftrightarrow \neg\tilde{M}\neg p$.

Proposition 2 (Duality)**.** *Let* L *be closed under* ($\mathrm{RC}_\Box$). *If* $\vdash_{\mathsf{L}} M_1 p \to M_2 p$, *then* $\vdash_{\mathsf{L}} \tilde{M}_2 p \to \tilde{M}_1 p$.

In sufficiently strong modal logics, modalities distribute over conjunctions, i.e. $\vdash M(p \wedge q) \to (Mp \wedge Mq)$. Let (DIST) be the modally qualified counterpart of that fact. That is, let it be the following schema (ranging over arbitrary M):[4]

$$\text{(DIST)} \qquad M(p \wedge q) \to \blacklozenge(Mp \wedge Mq).$$

And let the Necessitation Rule for $\blacksquare$ be:

$$(\mathrm{RN}_\blacksquare) \qquad \text{If } \vdash \varphi\text{, then } \vdash \blacksquare\varphi.$$

We can now prove the following key lemmas:

Lemma 3 (Moore Lemma)**.** *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$), *and let it contain* (DIST). *If* $\vdash_{\mathsf{L}} M_1 p \to \tilde{M}_1 M_2 p$, *then* $\vdash_{\mathsf{L}} \neg M_1(p \wedge \neg M_2 p)$.

Proof. Assume $\vdash_{\mathsf{L}} M_1 p \to \tilde{M}_1 M_2 p$.

(1)	$\vdash_{\mathsf{L}} M_1 p \to \tilde{M}_1 M_2 p$	assumption
(2)	$\vdash_{\mathsf{L}} \neg(M_1 p \wedge \neg\tilde{M}_1 M_2 p)$	(1)
(3)	$\vdash_{\mathsf{L}} \neg\tilde{M}_1 M_2 p \leftrightarrow M_1\neg M_2 p$	Prop 1
(4)	$\vdash_{\mathsf{L}} \neg(M_1 p \wedge M_1\neg M_2 p)$	(2), (3)
(5)	$\vdash_{\mathsf{L}} \blacksquare\neg(M_1 p \wedge M_1\neg M_2 p)$	(4), ($\mathrm{RN}_\blacksquare$)
(6)	$\vdash_{\mathsf{L}} M_1(p \wedge \neg M_2 p) \to \blacklozenge(M_1 p \wedge M_1\neg M_2 p)$	(DIST), (US)
(7)	$\vdash_{\mathsf{L}} \blacksquare\neg(M_1 p \wedge M_1\neg M_2 p) \to \neg M_1(p \wedge \neg M_2 p)$	(6), contraposition
(8)	$\vdash_{\mathsf{L}} \neg M_1(p \wedge \neg M_2 p)$	(5), (7)

$\Box$

[4] It will turn out that at all the points where we appeal to (DIST), its unqualified counterpart also suffices. The reason that we focus on (DIST) instead is that on various interpretations, (DIST) is intuitively weaker than its unqualified counterpart. The focus on (DIST) is to emphasise that the results to follow don't depend on the actual distributivity of the modal operators over conjunctions (for discussions on the role of the assumption of distributivity in the derivation of the Church–Fitch Theorem, see, for instance, Williamson (1993) and Chapter 4 of Kvanvig (2006)).

Lemma 4 (Church–Fitch Lemma)**.** *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$)*, and let it contain* (DIST)*. If* $\vdash_\mathsf{L} M_1p \to \tilde{M}_1M_2p$ *and* $\vdash_\mathsf{L} M_3p \to \blacklozenge M_1p$*, then* $\vdash_\mathsf{L} \neg M_3(p \wedge \neg M_2p)$*.*

Proof. Assume $\vdash_\mathsf{L} M_1p \to \tilde{M}_1M_2p$ and $\vdash_\mathsf{L} M_3p \to \blacklozenge M_1p$.

(1)	$\vdash_\mathsf{L} M_1p \to \tilde{M}_1M_2p$	assumption
(2)	$\vdash_\mathsf{L} \neg M_1(p \wedge \neg M_2p)$	(1), Moore Lemma
(3)	$\vdash_\mathsf{L} \blacksquare\neg M_1(p \wedge \neg M_2p)$	(2), ($\mathrm{RN}_\blacksquare$)
(4)	$\vdash_\mathsf{L} M_3p \to \blacklozenge M_1p$	assumption
(5)	$\vdash_\mathsf{L} \blacksquare\neg M_1p \to \neg M_3p$	(4), contraposition
(6)	$\vdash_\mathsf{L} \blacksquare\neg M_1(p \wedge \neg M_2p) \to \neg M_3(p \wedge \neg M_2p)$	(5), (US)
(7)	$\vdash_\mathsf{L} \neg M_3(p \wedge \neg M_2p)$	(3), (6)

$\Box$

A version of the Church–Fitch Theorem easily follows:

Theorem 5 (Church–Fitch Theorem)**.** *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$)*, and let it contain* (DIST)*. If* $\vdash_\mathsf{L} \Box p \to \Diamond\Box p$ *and* $\vdash_\mathsf{L} p \to \blacklozenge\Box p$*, then* $\vdash_\mathsf{L} p \to \Box p$.[5]

Proof. Assume $\vdash_\mathsf{L} \Box p \to \Diamond\Box p$ and $\vdash_\mathsf{L} p \to \blacklozenge\Box p$. Let $M_1 = \Box$, $M_2 = \Box$, and M_3 be the empty string. Then, $\vdash_\mathsf{L} \Box p \to \Diamond\Box p$ is $\vdash_\mathsf{L} M_1p \to \tilde{M}_1M_2p$ and $\vdash_\mathsf{L} p \to \blacklozenge\Box p$ is $\vdash_\mathsf{L} M_3p \to \blacklozenge M_1p$. Thus, by the Church–Fitch Lemma, $\vdash_\mathsf{L} \neg M_3(p \wedge \neg M_2p)$, which is $\vdash_\mathsf{L} \neg(p \wedge \neg\Box p)$—or equivalently, $\vdash_\mathsf{L} p \to \Box p$. $\Box$

The reason that the Church–Fitch Theorem follows so easily from the Church–Fitch Lemma is because the lemma is really just a generalisation of the reasoning that underlies the usual proof of the Church–Fitch Theorem. However, as we are about to see, it isn't an idle generalisation; it will allow us to prove other similar results of philosophical interest.

[5]Standard proofs of the theorem appeal either to the factivity of $\Box$ (i.e. ($\mathrm{T}_\Box$) $p \to \Diamond p$) or the limited factivity of $\Box$ (i.e. $\Box p \to \Diamond\Box p$, as above). In another paper, I prove a generalisation of the Church–Fitch Theorem which shows a more general assumption suffices. In particular, let $|M_1 - M_2| = n$ be the difference in the number of modal operators in M_1 and M_2 (e.g. if $M_1 = \Box$ and $M_2 = \Diamond\Diamond\Box\Diamond$, then $|M_1 - M_2| = 3$). I show that given weak background assumptions, if $\vdash M_1p \to M_2p$ (where $|M_1 - M_2| = n$) and $\vdash_\mathsf{L} p \to \blacklozenge\Box p$, then $\vdash p \leftrightarrow \Box^n p$ (where $\Box^n$ abbreviates n iterations of $\Box$ and $\Box^0$ is the empty string, and similarly for $\Diamond^n$). An upshot is that, contrary to conventional wisdom, the modal collapse identified in the Church–Fitch Theorem does not rely fundamentally on the factivity or limited factivity of $\Box$ or on Moorean sentences of the form '$p \wedge \neg\Box p$'. Rather, the modal collapse has to do more generally with modal level-bridging principles of the form $M_1p \to M_2p$, of which $\Box p \to \Diamond\Box p$ and $p \to \Diamond p$ are mere instances.

3 Disappearing Diamonds 1: Recovering S5

Recall the notation introduced earlier: if (X) is of the form $\varphi \to \psi$, ($X^\blacklozenge$) is $\varphi \to \blacklozenge\psi$. Thus, since ($4_\Box$) is $\Box p \to \Box\Box p$, ($B_\Box$) is $p \to \Box\Diamond p$, and ($5_\Box$) is $\Diamond p \to \Box\Diamond p$, ($4^\blacklozenge_\Box$) is $\Box p \to \blacklozenge\Box\Box p$, ($B^\blacklozenge_\Box$) is $p \to \blacklozenge\Box\Diamond p$, and ($5^\blacklozenge_\Box$) is $\Diamond p \to \blacklozenge\Box\Diamond p$.[6] In this section, we will show that given fairly weak assumptions, we can recover ($4_\Box$) and ($B_\Box$) from their qualified counterparts, ($4^\blacklozenge_\Box$) and ($B^\blacklozenge_\Box$). An immediate consequence of this will be that we can also recover ($4_\Box$) and ($5_\Box$) from their qualified counterparts, ($4^\blacklozenge_\Box$) and ($5^\blacklozenge_\Box$).

We start by sketching a quick informal proof of the result for normal modal logics. L is *normal* if it contains ($K_\Box$) and ($K_\blacksquare$) and is closed under the Necessitation Rules ($RN_\Box$) ($\varphi/\Box\varphi$) and ($RN_\blacksquare$) ($\varphi/\blacksquare\varphi$). We let $\mathsf{KX}^1 \ldots \mathsf{X}^n_\Box \oplus \mathsf{KX}^1 \ldots \mathsf{X}^{n'}_\blacksquare$ denote the smallest normal modal logic containing ($X^1_\Box$), ..., ($X^n_\Box$) and ($X^1_\blacksquare$), ..., ($X^{n'}_\blacksquare$). Thus, $\mathsf{K}_\Box \oplus \mathsf{K}_\blacksquare$ is the smallest normal modal logic and $\mathsf{KT}_\Box \oplus \mathsf{K4}_\blacksquare$, for instance, is the smallest normal modal logic containing ($T_\Box$) and ($4_\blacksquare$). Sometimes, we use more common notation—e.g. $\mathsf{S4}$ for $\mathsf{KT4}$, $\mathsf{S5}$ for $\mathsf{KT5}$, and Triv for KTS.

We want to show:

Theorem 6 (S5-Recovery)**.** *Let* L *be a normal extension of* $\mathsf{KT}_\Box \oplus \mathsf{K}_\blacksquare$. *If* L *contains* ($4^\blacklozenge_\Box$) *and either* ($B^\blacklozenge_\Box$) *or* ($5^\blacklozenge_\Box$)*, then* L *extends* $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$.

Informal proof: Suppose L contains ($4^\blacklozenge_\Box$) and ($B^\blacklozenge_\Box$). First, we will show that $\vdash_\mathsf{L} p \to \Box\Box\Diamond p$. Suppose $p \wedge \neg\Box\Box\Diamond p$ for a contradiction. By ($B^\blacklozenge_\Box$), $\blacklozenge\Box\Diamond(p \wedge \neg\Box\Box\Diamond p)$. Thus, by ($4^\blacklozenge_\Box$), $\blacklozenge\blacklozenge\blacklozenge\Box\Box\Box\Diamond(p \wedge \neg\Box\Box\Diamond p)$. Distributing '$\Box\Box\Box\Diamond$' over the conjunction, $\blacklozenge\blacklozenge\blacklozenge(\Box\Box\Box\Diamond p \wedge \Box\Box\Box\Diamond\neg\Box\Box\Diamond p)$. By the duality of $\Box$ and $\Diamond$, $\blacklozenge\blacklozenge\blacklozenge(\Box\Box\Box\Diamond p \wedge \Box\Box\Box\neg\Box\Box\Box\Diamond p)$. Thus, by ($T_\Box$), $\blacklozenge\blacklozenge\blacklozenge(\Box\Box\Box\Diamond p \wedge \neg\Box\Box\Box\Diamond p)$. Contradiction. Thus, $\vdash_\mathsf{L} p \to \Box\Box\Diamond p$.

[6] A note of caution: Proposition 2 (Duality) shows that axioms of the form $M_1 p \to M_2 p$ have equivalent dual formulations of the form $\tilde{M}_2 p \to \tilde{M}_1 p$ (equivalent in the sense that any $\Box$-congruential logic contains one iff it contains the other). For instance, ($T_\Box$) can be taken to be either $p \to \Diamond p$ or $\Box p \to p$. Often, there is no canonical formulation—which formulation is identified with ($T_\Box$) depends on which interpretation of $\Box$ one has in mind. For instance, under the metaphysical interpretation, perhaps $p \to \Diamond p$ might be preferred to capture the thought that anything actual is possible, whereas under the epistemic interpretation, $\Box p \to p$ is preferred since it captures the idea that knowledge is factive. But, having introduced the ($X^\blacklozenge$) notation, care needs to be exercised since which formulation we take ($T_\Box$) to be will affect whether we take ($T^\blacklozenge_\Box$) to be $p \to \blacklozenge\Diamond p$ or $\Box p \to \blacklozenge p$—which we cannot assume to be equivalent, even though $p \to \Diamond p$ and $\Box p \to p$ are. Thus, given the ($X^\blacklozenge$) notation, we need a canonical formulation of ($T_\Box$). The same point applies to ($4_\Box$), ($B_\Box$) ($5_\Box$), and so on. For instance, one should not confuse ($4^\blacklozenge_\Box$) as defined above with $\Diamond\Diamond p \to \blacklozenge\Diamond p$. With this in mind, we take the canonical formulations of the principles to be as presented in Table 1.

Then, by ($\mathrm{T}_\Box$), it follows that $\vdash_{\mathsf{L}} p \to \Box\Diamond p$. So, L contains ($\mathrm{B}_\Box$). Furthermore, by Duality, it follows from ($\mathrm{B}_\Box$) that $\vdash_{\mathsf{L}} \Diamond\Box p \to p$. So, by the normality of L, $\vdash_{\mathsf{L}} \Box\Box\Diamond\Box p \to \Box\Box p$. Furthermore, from $\vdash_{\mathsf{L}} p \to \Box\Box\Diamond p$, it follows by (US) that $\vdash_{\mathsf{L}} \Box p \to \Box\Box\Diamond\Box p$. Thus, $\vdash_{\mathsf{L}} \Box p \to \Box\Box p$; so, L contains ($4_\Box$). And since $\mathsf{KT4B} = \mathsf{S5}$, L extends $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$. This also holds if we had assumed ($5^\blacklozenge_\Box$) instead of ($\mathrm{B}^\blacklozenge_\Box$), since given ($\mathrm{T}_\Box$), the former entails the latter. That concludes the informal proof of Theorem 6. But as we will see, the assumption of normality is dispensable. A similar result holds in extremely weak non-normal modal logics. And from that result, Theorem 6 immediately follows.

In sufficiently strong modal logics, $\Box^n$ distributes out of conjunctions, i.e. $\vdash (\Box^n p \wedge \Box^n q) \to \Box^n(p \wedge q)$. Let (DIST*) be the modally qualified counterpart of that fact. That is, let (DIST*) be the schema:[7]

$$\text{(DIST}^*\text{)} \qquad (\Box^n p \wedge \Box^n q) \to \blacklozenge\Box^n(p \wedge q).$$

We can show:

Lemma 7. *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$), *and let it contain* (DIST*). *For any* $n \geq 0$, *if* $\vdash_{\mathsf{L}} \neg\Box^n(p \wedge \neg Mp)$, *then* $\vdash_{\mathsf{L}} \Box^n p \to \Diamond^n Mp$.

Proof. Assume $\vdash_{\mathsf{L}} \neg\Box^n(p \wedge \neg Mp)$. Then, by ($\mathrm{RN}_\blacksquare$), $\vdash_{\mathsf{L}} \blacksquare\neg\Box^n(p \wedge \neg Mp)$. Given (DIST*), it follows by (US) that $\vdash_{\mathsf{L}} (\Box^n p \wedge \Box^n \neg Mp) \to \blacklozenge\Box^n(p \wedge \neg Mp)$. So, by contraposition, $\vdash_{\mathsf{L}} \blacksquare\neg\Box^n(p \wedge \neg Mp) \to \neg(\Box^n p \wedge \Box^n\neg Mp)$. Thus, from $\vdash_{\mathsf{L}} \blacksquare\neg\Box^n(p \wedge \neg Mp)$, it follows that $\vdash_{\mathsf{L}} \neg(\Box^n p \wedge \Box^n\neg Mp)$—or equivalently, $\vdash_{\mathsf{L}} \Box^n p \to \neg\Box^n\neg Mp$. By Proposition 1, $\vdash_{\mathsf{L}} \Diamond^n Mp \leftrightarrow \neg\Box^n\neg Mp$. Thus, $\vdash_{\mathsf{L}} \Box^n p \to \Diamond^n Mp$. ∎

Lemma 8. *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$), *and let it contain* (DIST), (DIST*), *and* ($\mathrm{T}_\Box$). *If* L *contains* ($4^\blacklozenge_\Box$), *then* $\vdash_{\mathsf{L}} \Box p \to \Diamond\Box^n p$, *for any* $n \geq 0$.

Proof. Assume L contains ($4^\blacklozenge_\Box$), i.e. $\vdash_{\mathsf{L}} \Box p \to \blacklozenge\Box\Box p$. The proof that $\vdash_{\mathsf{L}} \Box p \to \Diamond\Box^n p$ is by induction on the number of iterations of $\Box$. The base case where $n = 0$ (i.e. $\vdash_{\mathsf{L}} \Box p \to \Diamond p$) follows immediately from the familiar fact that ($\mathrm{T}_\Box$) implies ($\mathrm{D}_\Box$).

[7] As with (DIST), it will turn out that at all the points where we appeal to (DIST*), its unqualified counterpart also suffices. The reason that we focus on (DIST*) instead is that on various interpretations, (DIST*) is intuitively weaker than its unqualified counterpart. The focus on (DIST*) is to emphasise that the results to follow don't depend on $\Box^n$ actually distributing out of conjunctions.

For the inductive step, let the inductive hypothesis (IH) be $\vdash_{\mathsf{L}} \Box p \to \Diamond\Box^n p$. We will show that $\vdash_{\mathsf{L}} \Box p \to \Diamond\Box^{n+1} p$.

(1)	$\vdash_{\mathsf{L}} \Box p \to \Diamond\Box^n p$	(IH)
(2)	$\vdash_{\mathsf{L}} \Box\Box p \to \Diamond\Box^{n+1} p$	(1), (US)
(3)	$\vdash_{\mathsf{L}} p \to \Diamond p$	($\mathrm{T}_\Box$)
(4)	$\vdash_{\mathsf{L}} \Diamond\Box^{n+1} p \to \Diamond\Diamond\Box^{n+1} p$	(3), (US)
(5)	$\vdash_{\mathsf{L}} \Box\Box p \to \Diamond\Diamond\Box^{n+1} p$	(2), (4)

Now, let $M_1 = \Box\Box$, $M_2 = \Box^{n+1}$, and $M_3 = \Box$. So, (5) is $\vdash_{\mathsf{L}} M_1 p \to \tilde{M}_1 M_2 p$. Furthermore, by assumption, $\vdash_{\mathsf{L}} \Box p \to \blacklozenge\Box\Box p$, which is $\vdash_{\mathsf{L}} M_3 p \to \blacklozenge M_1 p$. Thus, by the Church–Fitch Lemma, $\vdash_{\mathsf{L}} \neg M_3(p \wedge \neg M_2 p)$, which is $\vdash_{\mathsf{L}} \neg\Box(p \wedge \neg\Box^{n+1} p)$. Thus, by Lemma 7, $\vdash_{\mathsf{L}} \Box p \to \Diamond\Box^{n+1} p$. □

Lemma 9. *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$)*, and let it contain* (DIST), (DIST*)*, and* ($\mathrm{T}_\Box$)*. If* L *contains* ($4^\blacklozenge_\Box$) *and* ($\mathrm{B}^\blacklozenge_\Box$)*, then* $\vdash_{\mathsf{L}} p \to \Box^n \Diamond p$*, for any* $n \geq 0$*.*

Proof. Assume L contains ($4^\blacklozenge_\Box$) and ($\mathrm{B}^\blacklozenge_\Box$) (i.e. $\vdash_{\mathsf{L}} \Box p \to \blacklozenge\Box\Box p$ and $\vdash_{\mathsf{L}} p \to \blacklozenge\Box\Diamond p$). By Lemma 8, $\vdash_{\mathsf{L}} \Box p \to \Diamond\Box^{n+1} p$, for any $n \geq 0$. So, by (US), $\vdash_{\mathsf{L}} \Box\Diamond p \to \Diamond\Box^{n+1}\Diamond p$. Now, let $M_1 = \Box\Diamond$, $M_2 = \Box^n\Diamond$, and M_3 be the empty string. So, $\vdash_{\mathsf{L}} \Box\Diamond p \to \Diamond\Box^{n+1}\Diamond p$ is $\vdash_{\mathsf{L}} M_1 p \to \tilde{M}_1 M_2 p$ and $\vdash_{\mathsf{L}} p \to \blacklozenge\Box\Diamond p$ is $\vdash_{\mathsf{L}} M_3 p \to \blacklozenge M_1 p$. Thus, by the Church–Fitch Lemma, $\vdash_{\mathsf{L}} \neg M_3(p \wedge \neg M_2 p)$, which is $\vdash_{\mathsf{L}} \neg(p \wedge \neg\Box^n\Diamond p)$—or equivalently, $\vdash_{\mathsf{L}} p \to \Box^n\Diamond p$. □

We can then easily show that ($4^\blacklozenge_\Box$) and ($\mathrm{B}^\blacklozenge_\Box$) imply their unqualified counterparts.

Theorem 10. *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$)*, and let it contain* (DIST), (DIST*)*, and* ($\mathrm{T}_\Box$)*. If* L *contains* ($4^\blacklozenge_\Box$) *and* ($\mathrm{B}^\blacklozenge_\Box$)*, then* L *contains* ($4_\Box$) *and* ($\mathrm{B}_\Box$)*.*

Proof. Assume L contains ($4^\blacklozenge_\Box$) and ($\mathrm{B}^\blacklozenge_\Box$). Then, by Lemma 9, L contains ($\mathrm{B}_\Box$) (simply let $n = 1$). Furthermore, L contains ($4_\Box$):

(1)	$\vdash_{\mathsf{L}} p \to \Box\Diamond p$	Lemma 9
(2)	$\vdash_{\mathsf{L}} \Diamond\Box p \to p$	(1), Prop 2
(3)	$\vdash_{\mathsf{L}} \Diamond\Box\Box\Box p \to \Box\Box p$	(2), (US)
(4)	$\vdash_{\mathsf{L}} \Box p \to \Diamond\Box\Box\Box p$	Lemma 8
(5)	$\vdash_{\mathsf{L}} \Box p \to \Box\Box p$	(3), (4)

□

From this, it follows that $(4^{\blacklozenge}_{\Box})$ and $(5^{\blacklozenge}_{\Box})$ also imply their unqualified counterparts:

Theorem 11. *Let* L *be closed under* $(\mathrm{RC}_{\Box})$ *and* $(\mathrm{RN}_{\blacksquare})$*, and let it contain* (DIST)*,* (DIST*)*, and* $(\mathrm{T}_{\Box})$*. If* L *contains* $(4^{\blacklozenge}_{\Box})$ *and* $(5^{\blacklozenge}_{\Box})$*, then* L *contains* $(4_{\Box})$ *and* $(5_{\Box})$*.*

Proof. Assume L contains $(4^{\blacklozenge}_{\Box})$ and $(5^{\blacklozenge}_{\Box})$. Since $(5^{\blacklozenge}_{\Box})$ implies $(\mathrm{B}^{\blacklozenge}_{\Box})$ given $(\mathrm{T}_{\Box})$, L contains $(\mathrm{B}^{\blacklozenge}_{\Box})$. So, by Theorem 10, L contains $(4_{\Box})$ and $(\mathrm{B}_{\Box})$. All that's left to show is that L contains $(5_{\Box})$. Given $(\mathrm{T}_{\Box})$ and $(4_{\Box})$, $\vdash_{\mathsf{L}} \Box p \leftrightarrow \Box\Box p$. So, by $(\mathrm{RC}_{\Diamond})$, $\vdash_{\mathsf{L}} \Diamond\Box p \leftrightarrow \Diamond\Box\Box p$. And by Duality, $\vdash_{\mathsf{L}} \Box\Diamond p \leftrightarrow \Box\Diamond\Diamond p$. Now, by (US) and $(\mathrm{B}_{\Box})$, $\vdash_{\mathsf{L}} \Diamond p \to \Box\Diamond\Diamond p$. Thus, $\vdash_{\mathsf{L}} \Diamond p \to \Box\Diamond p$. $\Box$

From Theorems 10 and 11, Theorem 6 easily follows.[8] The upshot is that given some weak background assumptions, accepting $(4^{\blacklozenge}_{\Box})$ and $(\mathrm{B}^{\blacklozenge}_{\Box})$ or $(5^{\blacklozenge}_{\Box})$ is tantamount to accepting their unqualified counterparts $(4_{\Box})$, $(\mathrm{B}_{\Box})$, and $(5_{\Box})$—and thus, in the setting of normal modal logics, tantamount to accepting an $\mathsf{S5}$ logic for $\Box$. In the monomodal setting, $\mathsf{S5}$ can be obtained by adding to KT either (4) and (B), or (5). The results above show that, to obtain an $\mathsf{S5}$ logic for $\Box$ in the bimodal setting, it suffices to add to $\mathsf{KT}_{\Box} \oplus \mathsf{K}_{\blacksquare}$ the axioms $(4^{\blacklozenge}_{\Box})$ and either $(\mathrm{B}^{\blacklozenge}_{\Box})$ or $(5^{\blacklozenge}_{\Box})$.

This result might seem surprising. On various interpretations of $\Box$ and $\blacksquare$, $(4^{\blacklozenge}_{\Box})$ and $(5^{\blacklozenge}_{\Box})$ can appear extremely weak. How could it be that they turn out to have the full logical strength of their unqualified counterparts? However, the initial surprise diminishes upon reflection on the Church–Fitch Theorem. The Church–Fitch Theorem shows that the Church–Fitch principle, $p \to \blacklozenge\Box p$, is much stronger than it appears. In particular, for normal modal logics satisfying weak assumptions, the Church–Fitch principle gives rise to an extremely strong Triv logic for $\Box$. In light of this, the results above are not so surprising. After all, conceived as schemas, the $(4^{\blacklozenge}_{\Box})$-schema $(\Box\varphi \to \blacklozenge\Box\Box\varphi)$ and the $(5^{\blacklozenge}_{\Box})$-schema $(\Diamond\varphi \to \blacklozenge\Box\Diamond\varphi)$ are really just restrictions of the Church–Fitch schema $(\varphi \to \blacklozenge\Box\varphi)$ to $\Box\varphi$ and $\Diamond\varphi$ sentences, respectively. Given the dramatic strengthening effects of the Church–Fitch schema, it is not so surprising that the restriction of it to $\Box\varphi$ and $\Diamond\varphi$ sentences should remain fairly strong.

Several things are worth noting. First: We have shown that given $\mathsf{KT}_{\Box} \oplus \mathsf{K}_{\blacksquare}$, $(4^{\blacklozenge}_{\Box})$ and $(5^{\blacklozenge}_{\Box})$ are jointly sufficient to give rise to an $\mathsf{S5}$ logic for $\Box$. But we can also show that neither is individually sufficient. That is, $(4^{\blacklozenge}_{\Box})$, on

[8] While normal extensions of $\mathsf{KT}_{\Box} \oplus \mathsf{K}_{\blacksquare}$ need not contain (DIST) or (DIST*), they contain their unqualified counterparts, which as has already been noted, also suffice for proving everything that we used (DIST) or (DIST*) to prove.

its own, is not strong enough to generate an $\mathsf{S5}$ logic for $\Box$. And neither is $(5^{\blacklozenge}_{\Box})$ (see Appendix A).

Second: The results above have straightforward implications for monomodal logics. Let $\mathscr{L}_{\Box}$ be the fragment of our bimodal language without any occurrences of $\blacksquare$. A monomodal logic (in $\mathscr{L}_{\Box}$) is a set of $\mathscr{L}_{\Box}$-sentences containing all truth-functional tautologies and closed under (MP) and (US). A monomodal logic is normal if it contains $(\mathrm{K}_{\Box})$ and is closed under $(\mathrm{RN}_{\Box})$. $\mathsf{KX}^1 \ldots \mathsf{X}^n{}_{\Box}$ denotes the smallest normal monomodal logic containing $(\mathrm{X}^1_{\Box})$ $, \ldots, (\mathrm{X}^n_{\Box})$. An immediate consequence of our results is:

Corollary 12. *The smallest normal monomodal extension of* $\mathsf{KT}_{\Box}$ *containing* $\Box p \to \Diamond\Box\Box p$ *and* $p \to \Diamond\Box\Diamond p$ *is* $\mathsf{S5}_{\Box}$.[9]

Proof. Same as the proof of Theorem 6, since nothing in the proof depended on $\Box$ and $\blacksquare$ being distinct operators. $\Box$

Thus, instead of adding $(4_{\Box})$ and $(\mathrm{B}_{\Box})$ to $\mathsf{KT}_{\Box}$, $\mathsf{S5}_{\Box}$ can already be axiomatised using the weaker axioms $\Box p \to \Diamond\Box\Box p$ and $p \to \Diamond\Box\Diamond p$.

Finally, it is worth noting briefly that the results above have interesting and important implications for philosophical issues ranging across many different areas. For instance, interpreting $\Box$ in terms of knowledge, $(4_{\Box})$ corresponds to the much-debated KK-principle: if one knows p, then one knows that one knows p. And $(5_{\Box})$ corresponds to the almost universally rejected K¬K-principle: if one doesn't know p, then one knows that one doesn't know p. And interpreting $\blacksquare$ in terms of logical necessity, $(4^{\blacklozenge}_{\Box})$ corresponds to an extremely weak variant of KK: if one knows p, then it's *logically possible* that one knows that one knows p. And $(5^{\blacklozenge}_{\Box})$ corresponds to an extremely weak variant of K¬K: if one doesn't know p, then it's *logically possible* that one knows that one doesn't know p. By our results, given weak assumptions, these extremely weak variants in fact have the full logical strength of KK and K¬K. This has many important implications for foundational epistemological issues to do with the nature and structure of knowledge and knowability, their limits, and so on. I explore these issues in greater detail in another paper.

For another application of the results, consider the interpretation of $\Box$ in terms of metaphysical modality. According to the orthodoxy, $\mathsf{S5}$ is the logic for metaphysical modality. However, some, like Chandler (1976) and Salmon (1989), challenge this orthodoxy, rejecting both $(4_{\Box})$ and $(5_{\Box})$.

[9] In fact, it is easy to see that instead of $\Box p \to \Diamond\Box\Box p$ and $p \to \Diamond\Box\Diamond p$, this also holds more generally given $\Box p \to \Diamond^m\Box\Box p$ and $p \to \Diamond^n\Box\Diamond p$, for any m, n.

Under the metaphysical reading, ($4_\Box$) is the principle that whatever is necessary is necessarily so and ($5_\Box$) the principle that whatever is possible is necessarily so. By Theorem 6, rejecting metaphysical $\mathsf{S5}$ requires rejecting at least one of ($4^{\blacklozenge}_\Box$) and ($5^{\blacklozenge}_\Box$), for any interpretation of $\blacksquare$ for which the assumption of normality is justified. Thus, for instance, interpreting $\blacksquare$ in terms of logical necessity, opponents of metaphysical $\mathsf{S5}$ must deny either that whatever is metaphysically necessary is logically possibly metaphysically necessarily so or that whatever is metaphysically possible is logically possibly metaphysically necessarily so.

4 Disappearing Diamonds 2: Recovering KT

This section shows that given ($\mathrm{U}_\Box$) $\Box(\Box p \to p)$ and some other weak assumptions, ($\mathrm{T}_\Box$) $p \to \Diamond p$ can be recovered from its modally qualified counterpart, ($\mathrm{T}^{\blacklozenge}_\Box$) $p \to \blacklozenge\Diamond p$. Thus, given normality and ($\mathrm{T}^{\blacklozenge}_\Box$), a KT logic for $\Box$ can be recovered from $\mathsf{KU}_\Box \oplus \mathsf{K}_\blacksquare$.

Theorem 13. *Let* L *be closed under* ($\mathrm{RC}_\Box$) *and* ($\mathrm{RN}_\blacksquare$), *and let it contain* ($\mathrm{U}_\Box$). *If* L *contains* ($\mathrm{T}^{\blacklozenge}_\Box$), *then* L *contains* ($\mathrm{T}_\Box$).

Proof. Assume L contains ($\mathrm{T}^{\blacklozenge}_\Box$).

(1)	$\vdash_{\mathsf{L}} \Box(\Box p \to p)$	($\mathrm{U}_\Box$)
(2)	$\vdash_{\mathsf{L}} \neg\Diamond(\Box p \wedge \neg p)$	(1)
(3)	$\vdash_{\mathsf{L}} \blacksquare\neg\Diamond(\Box p \wedge \neg p)$	(2), ($\mathrm{RN}_\blacksquare$)
(4)	$\vdash_{\mathsf{L}} p \to \blacklozenge\Diamond p$	($\mathrm{T}^{\blacklozenge}_\Box$)
(5)	$\vdash_{\mathsf{L}} \blacksquare\neg\Diamond p \to \neg p$	(4), contraposition
(6)	$\vdash_{\mathsf{L}} \blacksquare\neg\Diamond(\Box p \wedge \neg p) \to \neg(\Box p \wedge \neg p)$	(5), (US)
(7)	$\vdash_{\mathsf{L}} \neg(\Box p \wedge \neg p)$	(3), (6)
(8)	$\vdash_{\mathsf{L}} \Box p \to p$	(7)
(9)	$\vdash_{\mathsf{L}} p \to \Diamond p$	(8), Prop 2

$\Box$

It follows that:

Theorem 14 (KT-Recovery)**.** *Let* L *be a normal extension of* $\mathsf{KU}_\Box \oplus \mathsf{K}_\blacksquare$. *If* L *contains* ($\mathrm{T}^{\blacklozenge}_\Box$), *then* L *extends* $\mathsf{KT}_\Box \oplus \mathsf{K}_\blacksquare$.

And:

Corollary 15. *Let* L *be a normal extension of* $\mathsf{K5}_\Box \oplus \mathsf{K}_\blacksquare$. *If* L *contains* ($\mathrm{T}^{\blacklozenge}_\Box$), *then* L *extends* $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$.

Proof. By Theorem 14 and the fact that $\mathsf{K5}_\Box \oplus \mathsf{K}_\blacksquare$ is a normal extension of $\mathsf{KU}_\Box \oplus \mathsf{K}_\blacksquare$. $\Box$

As before, nothing in the proofs depended on $\Box$ and $\blacksquare$ being distinct operators. So:

Corollary 16. *The smallest normal monomodal extension of* $\mathsf{KU}_\Box$ *containing* $p \to \Diamond\Diamond p$ *is* $\mathsf{KT}_\Box$*. And the smallest normal monomodal extension of* $\mathsf{K5}_\Box$ *containing* $p \to \Diamond\Diamond p$ *is* $\mathsf{S5}_\Box$.[10]

The results above also have interesting implications. For instance, doxastic logic is standardly taken to be $\mathsf{KD45}$. So, where $\Box$ is given a doxastic interpretation, $(\mathrm{T}_\Box^\blacklozenge)$ must be rejected for every interpretation of $\blacksquare$ for which the assumption of normality is justified. For otherwise, by Corollary 15, we would end up with an $\mathsf{S5}$ doxastic logic, in which belief is factive. Thus, for instance, the principle that whatever is necessarily believed is true must be rejected.[11]

For another application, consider the interpretation of $\Box$ as 'normally'. The logic for normalcy is also sometimes taken to be at least as strong as $\mathsf{KD45}$ but not $\mathsf{S5}$ (see (Smith, 2007, 114)). If so, then by Theorem 14, $(\mathrm{T}_\Box^\blacklozenge)$ must be rejected for every interpretation of $\blacksquare$ for which the assumption of normality (in the sense of having a normal modal logic) is justified. Thus, for instance, the principle that whatever is necessarily normal is true must be rejected. Similarly, deontic logic is also sometimes taken to be at least as strong as $\mathsf{KD45}$ but not $\mathsf{S5}$. So, $(\mathrm{T}_\Box^\blacklozenge)$ must also be rejected for every interpretation of $\blacksquare$ for which the assumption of normality is justified.

5 Disappearing Diamonds 3: Recovering Triv

For the purposes of this section, let $(\mathrm{Y}_\Box)$ be an arbitrary sentence of the form $M_1 p \to M_2 p$, where M_1 doesn't end in $\Box$ and M_2 does.[12] For instance, one such principle is the McKinsey axiom, $(\mathrm{M}_\Box)$ $\Box\Diamond p \to \Diamond\Box p$. Another is $(\mathrm{S}_\Box)$ $p \to \Box p$. This section shows that given $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$, $(\mathrm{Y}_\Box^\blacklozenge)$ implies $(\mathrm{Y}_\Box)$ and $(\mathrm{Y}_\Box^\blacklozenge)$ gives rise to a Triv logic for $\Box$.

[10] In fact, it is easy to see that instead of $p \to \Diamond\Diamond p$, this holds more generally for $p \to \Diamond^n p$, for any $n \geq 1$.

[11] $(\mathrm{T}_\Box^\blacklozenge)$ and $\blacksquare\Box p \to p$ are equivalent, in the sense that normal modal logics containing one also contain the other.

[12] Where, in general, $M = O_1 \dots O_n$ *ends in* O_n.

It is a well-known fact that in $\mathsf{S5}$, a modality is reducible to its innermost operator:

Proposition 17 (Modal Reduction)**.** *Let* L *be a normal extension of* $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$. *If* M *ends in* $\Box$, *then* $\vdash_\mathsf{L} Mp \leftrightarrow \Box p$. *If* M *ends in* $\Diamond$, *then* $\vdash_\mathsf{L} Mp \leftrightarrow \Diamond p$.

Using this, we can show:

Lemma 18. *Let* L *be a normal extension of* $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$. *If* L *contains* $(\mathrm{Y}^{\blacklozenge}_{\Box})$, *then* L *contains* $(\mathrm{S}^{\blacklozenge}_{\Box})$ $p \to \blacklozenge\Box p$.

Proof. Suppose L contains $(\mathrm{Y}^{\blacklozenge}_{\Box})$, i.e. $\vdash_\mathsf{L} M_1 p \to \blacklozenge M_2 p$, where M_1 doesn't end in $\Box$ and M_2 does. By Proposition 17, $\vdash_\mathsf{L} M_2 p \leftrightarrow \Box p$. And thus, by $(\mathrm{RC}_\blacklozenge)$, $\vdash_\mathsf{L} \blacklozenge M_2 p \leftrightarrow \blacklozenge\Box p$.[13] Thus, it follows from $\vdash_\mathsf{L} M_1 p \to \blacklozenge M_2 p$ that $\vdash_\mathsf{L} M_1 p \to \blacklozenge\Box p$.

And by assumption, M_1 doesn't end in $\Box$ so either it is the empty string or it ends in $\Diamond$. If it is the empty string, then $\vdash_\mathsf{L} M_1 p \leftrightarrow p$. And so, it follows from $\vdash_\mathsf{L} M_1 p \to \blacklozenge\Diamond p$ that $\vdash_\mathsf{L} p \to \blacklozenge\Box p$. If M_1 ends in $\Diamond$, then by Proposition 17, $\vdash_\mathsf{L} M_1 p \leftrightarrow \Diamond p$. Thus, given $M_1 p \to \blacklozenge\Box p$, it follows that $\vdash_\mathsf{L} \Diamond p \to \blacklozenge\Box p$. But by $(\mathrm{T}_\Box)$, $\vdash_\mathsf{L} p \to \Diamond p$ and so, $\vdash_\mathsf{L} p \to \blacklozenge\Box p$. $\Box$

Using this lemma, we can prove:

Theorem 19 (Triv-Recovery)**.** *Let* L *be a normal extension of* $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$. *If* L *contains* $(\mathrm{Y}^{\blacklozenge}_{\Box})$, *then* L *contains* $(\mathrm{Y}_\Box)$ *and* L *extends* $\mathsf{Triv}_\Box \oplus \mathsf{K}_\blacksquare$.

Proof. Suppose L contains $(\mathrm{Y}^{\blacklozenge}_{\Box})$. By Lemma 18, L contains $(\mathrm{S}^{\blacklozenge}_{\Box})$. Thus, by the Church–Fitch Theorem, L contains $(\mathrm{S}_\Box)$ $p \to \Box p$.[14] Thus, L extends $\mathsf{Triv}_\Box \oplus \mathsf{K}_\blacksquare$. And in $\mathsf{Triv}_\Box$, any modality is equivalent to every other modality so $\vdash_\mathsf{L} M_1 p \to M_2 p$. And so, L contains $(\mathrm{Y}_\Box)$. $\Box$

6 Conclusion

An insufficiently appreciated lesson of the Church–Fitch Theorem is that principles concerning how different modalities interact can be treacherous.[15]

[13] Normal modal logics are closed under $(\mathrm{RC}_\blacklozenge)$: if $\vdash \varphi \leftrightarrow \psi$, then $\vdash \blacklozenge\varphi \leftrightarrow \blacklozenge\psi$.

[14] While L might not contain (DIST), it does contain its unqualified counterpart, which as we have already noted, also suffices for proving everything that we used (DIST) to prove.

[15] What is meant by 'modalities' here is not the technical notion introduced earlier, but rather 'modalities' in the sense of alethic modalities, physical modalities, epistemic

Though seemingly weak, principles like 'if something is true, then it is possible to know it' are far from innocuous. The aim of this paper has been to further reinforce this lesson.

In particular, we showed that, given modest background assumptions, various bimodal principles of the form $\varphi \to \blacklozenge\psi$ have the full strength of their unqualified counterparts, $\varphi \to \psi$. Specifically:

1. given $\mathsf{KT}_\Box \oplus \mathsf{K}_\blacksquare$, $(4^\blacklozenge_\Box)$ and $(\mathrm{B}^\blacklozenge_\Box)$ entail $(4_\Box)$ and $(\mathrm{B}_\Box)$, and thus give rise to an $\mathsf{S5}$ logic for $\Box$;
2. given $\mathsf{KT}_\Box \oplus \mathsf{K}_\blacksquare$, $(4^\blacklozenge_\Box)$ and $(5^\blacklozenge_\Box)$ entail $(4_\Box)$ and $(5_\Box)$, and thus give rise to an $\mathsf{S5}$ logic for $\Box$;
3. given $\mathsf{KU}_\Box \oplus \mathsf{K}_\blacksquare$, $(\mathrm{T}^\blacklozenge_\Box)$ entails $(\mathrm{T}_\Box)$ and thus gives rise to a KT logic for $\Box$;
4. given $\mathsf{K5}_\Box \oplus \mathsf{K}_\blacksquare$, $(\mathrm{T}^\blacklozenge_\Box)$ entails $(\mathrm{T}_\Box)$ and thus gives rise to an $\mathsf{S5}$ logic for $\Box$;
5. given $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$, $(\mathrm{Y}^\blacklozenge_\Box)$ $M_1p \to M_2p$, where M_1 doesn't end in $\Box$ and M_2 does, entails $(\mathrm{Y}_\Box)$ and gives rise to a Triv logic for $\Box$.

An overarching theme that emerges is that the logical distance between various modal systems is not as great as commonly thought. It does not take much to close the logical gap between KU and KT, KT and $\mathsf{S5}$, $\mathsf{K5}$ and $\mathsf{S5}$, or $\mathsf{S5}$ and Triv. Consequently, it also does not take much to traverse the distance between some of the weakest normal modal logics, like KU, and some of the strongest, like Triv.

The results raise various further technical questions. One question concerns whether there is a general recipe that determines when a principle (X) of the form $\varphi \to \psi$ falls out of its qualified counterpart $(\mathrm{X}^\blacklozenge)$ $\varphi \to \blacklozenge\psi$. The hope is that the results of this paper can ultimately be subsumed as special instances of some such general result. A converse question concerns when a bimodal principle of the form $\varphi \to \blacklozenge\psi$ *doesn't* give rise to its unqualified counterpart—or even more strongly, when it doesn't strengthen the underlying logic for $\Box$ in any way. These questions are topics for future investigation.

modalities, and so on.

A Appendix

Theorem 6 shows that given $\mathsf{KT}_\Box \oplus \mathsf{K}_\blacksquare$, $(4^{\blacklozenge}_{\Box})$ and $(5^{\blacklozenge}_{\Box})$ together give rise to an $\mathsf{S5}$ logic for $\Box$. We will show that though jointly sufficient, $(4^{\blacklozenge}_{\Box})$ and $(5^{\blacklozenge}_{\Box})$ are not individually sufficient to give rise to an $\mathsf{S5}$ logic for $\Box$.

First, given $\mathsf{KT}_\Box \oplus \mathsf{K}_\blacksquare$, $(4^{\blacklozenge}_{\Box})$ isn't strong enough by itself to generate an $\mathsf{S5}$ logic for $\Box$:

Theorem 20. *The smallest normal extension of* $\mathsf{KT}_\Box \oplus \mathsf{K}_\blacksquare$ *containing* $(4^{\blacklozenge}_{\Box})$ *does not extend* $\mathsf{S5}_\Box \oplus \mathsf{K}_\blacksquare$*.*

Proof. As before, nothing depends on $\Box$ and $\blacksquare$ being distinct operators. So, it suffices to show that the smallest normal monomodal extension of $\mathsf{S4}_\Box$ containing $\Box p \to \Diamond\Box\Box p$ does not extend $\mathsf{S5}_\Box$. And that is obvious, since $\Box p \to \Diamond\Box\Box p$ is already a theorem of $\mathsf{S4}_\Box$. $\Box$

Now, we show, by a semantic argument, that $(5^{\blacklozenge}_{\Box})$ also isn't strong enough by itself to generate an $\mathsf{S5}$ logic for $\Box$. First, some preliminaries: A *Kripke frame* is a structure $\mathfrak{F} =< W, R_\Box, R_\blacksquare >$, where the domain W is a non-empty set, whose elements we shall refer to as 'worlds', and $R_\Box \subseteq (W \times W)$ and $R_\blacksquare \subseteq (W \times W)$ are binary relations on W. A *Kripke model* $\mathfrak{M} =< \mathfrak{F}, V >$ is a frame with a valuation function V which maps each propositional letter to a set of worlds. If $\mathfrak{M} =< \mathfrak{F}, V >$, we say that $\mathfrak{M}$ is based on $\mathfrak{F}$. A *pointed Kripke model* $< \mathfrak{M}, w >$ is a model $\mathfrak{M}$ together with a world w in the domain of $\mathfrak{M}$ (by the domain of $\mathfrak{M}$, we mean the domain of the frame on which $\mathfrak{M}$ is based). Satisfaction in a pointed model is defined:

$\mathfrak{M}, w \Vdash p$	iff	$w \in V(p)$, for every $p \in At$;
$\mathfrak{M}, w \Vdash \neg\varphi$	iff	not $\mathfrak{M}, w \Vdash \varphi$;
$\mathfrak{M}, w \Vdash (\varphi \wedge \psi)$	iff	$\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}, w \Vdash \psi$;
$\mathfrak{M}, w \Vdash \Box\varphi$	iff	for every $v \in W$ such that $< w, v > \in R_\Box$, $\mathfrak{M}, v \Vdash \varphi$.
$\mathfrak{M}, w \Vdash \blacksquare\varphi$	iff	for every $v \in W$ such that $< w, v > \in R_\blacksquare$, $\mathfrak{M}, v \Vdash \varphi$.

$\mathfrak{M} \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \varphi$ for all w in the domain of $\mathfrak{M}$. And $\mathfrak{F} \Vdash \varphi$ iff $\mathfrak{M} \Vdash \varphi$ for every model $\mathfrak{M}$ based on $\mathfrak{F}$.

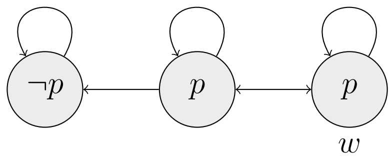
It is easy to show that any frame satisfying the condition in the antecedent of the lemma below validates $(5^{\blacklozenge}_{\Box})$:

Lemma 21. *If* $\mathfrak{F} \vDash \forall wv(wR_{\Box}v \to \exists u(wR_{\blacksquare}u \wedge \forall t(uR_{\Box}t \to tR_{\Box}v)))$*, then* $\mathfrak{F} \Vdash \Diamond p \to \blacklozenge\Box\Diamond p$*.*

Thus:

Theorem 22. *Let* L *be the smallest normal extension of* $\mathsf{KT}_{\Box} \oplus \mathsf{K}_{\blacksquare}$ *containing* $(5^{\blacklozenge}_{\Box})$*. Then,* L *doesn't extend* $\mathsf{S4}_{\Box} \oplus \mathsf{K}_{\blacksquare}$ *(and thus also doesn't extend* $\mathsf{S5}_{\Box} \oplus \mathsf{K}_{\blacksquare}$*).*

Proof Sketch. Consider the model $\mathfrak{M}$ below, where the arrows represent $R_{\Box}$ (let $R_{\blacksquare}$ be the universal accessibility relation):



$R_{\Box}$ is reflexive, so $\mathfrak{M}$ is a $\mathsf{KT}_{\Box} \oplus \mathsf{K}_{\blacksquare}$-model. Furthermore, checking that the condition in the antecedent of the previous lemma is satisfied is a tedious but straightforward exercise. Thus, the model is an L-model. However, $\mathfrak{M}, w \Vdash \Box p \wedge \neg\Box\Box p$, so $\mathfrak{M}$ is a countermodel to $(4_{\Box})$. Thus, L doesn't extend $\mathsf{S4}_{\Box} \oplus \mathsf{K}_{\blacksquare}$ and thus also doesn't extend $\mathsf{S5}_{\Box} \oplus \mathsf{K}_{\blacksquare}$. $\Box$

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