Natural Mathematics

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I.

One of the main difficulties that has beset philosophical attempts to accommodate mathematics within a broadly naturalistic perspective is that mathematical knowledge appears to be responsive to an objective reality of mathematical facts. Mathematical objects, however, are commonly characterized as causally inert and located outside of space and time. Now, if these objects belong to a non-spatio-temporal, acausal realm, how do we know about them? How do creatures located in space and time gather information about entities that are not so located? If the latter are causally impotent, and thus cannot affect us in any way, it is certainly puzzling how our cognitive apparatus could be responsive to them. But then, how could we possibly have any knowledge of mathematics?

The naive position just sketched—an ontological position broadly endorsed by a majority of working mathematicians—is not a standing metaphysical dogma, but something that calls for philosophical treatment. As a matter of fact, it has been vigorously challenged, and mainly so because of the epistemological puzzles it raises. It is beyond the scope of my present paper, however, to discuss the reasons why many philosophers have found this position unpalatable or to assess the strength of their alternative proposals. Instead, I will draw on recent cognitive research in order to account for the actual ways an individual can come to know specific, apparently objective mathematical truths.

Current approaches to mathematical cognition divide into two major camps. Cognitive studies try to render mathematical intuition—the faculty that gives us immediate and authoritative knowledge of mathematics—respectable on scientific grounds. Cultural studies, on the other hand, regard mathematics as a form of cultural achievement, like literature or architecture. Both positions have their own shortcomings. While cognitive approaches are limited in scope and fail to account for complex mathematical developments, cultural approaches are short of detailed answers as to what enables us to participate in a common mathematical practice. This situation evinces a need to balance a cognitive perspective on mathematical culture against a cultural perspective on mathematical cognition. Whereas certain numerical concepts, such as the concept of square root, and of real, imaginary or complex numbers, are only ever accessible to a tiny proportion of educated human adults in a subset of cultures, other numerical abilities are quite widespread—even among nonhuman species.

According to current behavioral and neuropsychological evidence, the complex, uniquely human, culture-specific mathematical skills exhibited by human adults rest on a set of psychological and neural mechanisms that (a) are shared by other animals, and (b) emerge early in human development, continue to function throughout the lifespan, and thus are common to infants, children and adults. It has been proposed that these common and evolutionary ancient mechanisms account for humans' basic "number sense" and form the building blocks for the development of more sophisticated numerical skills. Indeed, infants leave animals far behind in their numerical sophistication. What boosts this developmental difference? How do human beings acquire mathematical concepts such as the concept of natural number? First, I will specify the representations that are the building blocks for the target concepts. Second, I will describe how the target concepts differ from these basic representations. And finally, I will characterize the learning mechanisms that enable the construction of the target concepts out of those prior representations.

I will argue that the power of the resulting conceptual system derives from the combination and integration of previously distinct representational systems, capitalizing on the human capacity for creating and using external symbols: human beings can only develop their distinct conceptual abilities due to their original embeddedness in both the physical world and, most importantly, in a rich milieu of cultural resources. Thus, an important developmental source of number representations, in addition to the preverbal systems mentioned above, is the representation of numbers within natural language.

II.

The world is a dangerous, messy, inscrutable and overwhelmingly complex environment in which animals try to make a living. Animals are designed to tidy up their immediate environments building nests, dens, ambush sites, scent trails, etc. "They do all this to help them keep better track of the things that matter—predators and prey, mates, etc. These are done by 'instinct': automatized routines for improving the environment of action, making a better fit between agent and world" (Dennett 2000). Among these routines, we may include systems for representing number that create simple *numerical expectations* about the immediate environment (in order to keep track of predators, for instance).

In fact, comparative literature of animals and developmental studies of infants provide evidence for a shared "number sense": a set of distinct representational systems that serves as the foundational core of human sophisticated numerical abilities. These "core systems" have several "signature properties". First, they are domain-specific, that is, these systems are tuned to specific types of numerical information. Second, they are task-specific, hence, each system addresses specific questions about the world, like "How many ___ are there?" Third, these representational systems are relatively encapsulated, in other words, they are relatively impervious to explicitly held beliefs and goals—though the triggering of a particular numerical response in a given situation may be goal-directed (see below). And, finally, they are robust across modalities of input (e.g., auditory or visual modalities). These properties can be used to identify the particular systems responsible for human basic *nonverbal* numerical competence: (1) a small precise number system, and (2) a large approximate number system.

The small precise number system accounts for a subject's ability to identify small numbers of individual objects. Psychologists have christened this pre-attentive, unconscious process "subitizing" (Butterworth 1999; Hurford 2001). Subitizing has distinct signature properties brought to light by variations on Wynn's celebrated '1+1=2' task. Subjects represent the number of items in visual arrays and auditory sequences up to a set size limit of 3-4. Animals, as well as infants and adults, fail to represent number of items greater than 3 (for animals and infants, "failure to represent" is interpreted as looking equally long at possible and impossible outcomes). In addition, small number discrimination is affected by variations in continuous variables. Instead of computing discrete number, subjects often respond to continuous variables in terms of "amount" of motion, amount of sound, or amount of "stuff", depending on their "goals" (e.g., in comparative judgements, infants presented with one large cracker versus two crackers totaling half the area of the large one reliably prefer the larger one, while in choices of 1 vs. 2 equal-sized crackers, infants spontaneously prefer the larger quantity).

The large approximate number system yields a noisy representation of approximate number and has the following signature properties: large number discrimination varies relative to the ratio between numerosities, that is, discriminability depends on the set size ratio; and second, contrary to small number discrimination, large number discrimination is robust over variations in continuous variables. Interestingly, numerical discrimination increases in precision over development. 6-month-old infants can discriminate numerosities with a 1:2 but not a 2:3 ratio, whereas 10-month-old toddlers also succeed with the latter. Adults, on the other hand, can discriminate ratios as small as 7:8.

To sum up, infants' processing of large versus small numbers exhibit two dissociations. First, large approximate number discrimination accuracy varies with the ratio between numerosities, whereas small number discrimination varies relative to the absolute number of items, with a set size limit of about 3. Second, large number discrimination is impervious to variations in continuous variables, whereas small number discrimination is often affected by such continuous properties. These dissociations suggest that large and small numerosities are the province of different systems with different functions: large arrays primarily activate a system for representing sets and comparing their approximate numerosities (i.e., estimated cardinal value). Small arrays primarily activate a system for representing and tracking numerically distinct items, which allows for computations of either their continuous quantitative properties or of the number of items in the array.

These basic representational systems are common across many species. When given similar tasks to those presented to human infants and adults, animals show the same signature properties, indicating that such systems depend on mechanisms with a long evolutionary history. Moreover, recent neurophysiological findings show how the functional architecture of our "number sense" (composed by those two dissociated core systems for numerical representation) gets implemented in our human brains (Dehaene and Piazza 2004).

However, there still remain some open questions. For instance, what factors determine which representational system is deployed in a given situation? We may allow the numerical representation of a given situation to be sensitive to various contextual and "top-down" influences (Chalmers, French and Hofstadter 1995). Recall that small number discrimination is affected by fluctuations in continuous variables. What determines whether the computation is performed on discrete number or on continuous variables? In a given situation, the representation of discrete number or of continuous variables may depend on explicitly (and probably also implicitly) held beliefs and goals, that is, be open to contextual and top-down influences. (This explains why core systems of representation are said to be only *relatively* encapsulated.)

III.

What we have to figure out now is how human conceptual knowledge differs from these basic representations. According to Marc Hauser, some animals may have evolved a number category, but not a number concept, which is a distinctly human "acquisition".

> A number category ... is a category by virtue of the fact that it *refers to specific things* on the basis of their properties. In the case of number, the essential property is the *countable item, action, or event, independent of its physical attributes*. ... In contrast, a number concept represents a symbol that has a particular relationship to other symbols within the number domain. ... [N]umber concepts have unique roles by virtue of the arithmetical operations that can be performed on them and with them. (2000, p. 50; emphasis mine)

Hank Davis phrases this distinction in slightly different terms (Butterworth 1999). He refers to a number concept as an "absolute numerosity", which is—according to him—a "distinctly human invention". "No nonhuman animal needs this form of numerical competence in order to lead to a successful, totally normal life" (p. 159). Animals use "relative numerosity" (a number category) to compare different collections of objects: "they do not need to understand that *each time* they see a collection of three things, it has the same numerosity, only that *in the course of comparing it with other collections*, they can tell which has more, i.e. it has more than 2 and less than 4 things" (ibid.; my italics). Thus, a number category is relative to the collection of objects it refers to "in the course of comparing it with other collections" (e.g. three *rivals* vs. three *defenders*), whereas a number concept is "absolute" in so far as it bears a particular relationship to other number concepts independently of its particular instantiations (e.g. *three* equals *three* "come what may").

Notice that we have taken a crucial step: from what might be regarded as a numerical representation "in use" (in the course of "counting" and "comparing"), a representation that exists *for* a particular collection of objects, to a deliberate numerical representation.

How do these basic representations *in* the organism—embedded in procedures for interacting with the environment—become available *to* the organism? In other words, how do these basic representations get re-described (to borrow Karmiloff-Smith's term) into more sophisticated systems of representation up to properly called conceptual systems that are inferentially articulated? Our major challenge is to specify the bootstrapping mechanisms that result in representational systems with more expressive power than the hemi-semi-demi-numerical systems antecedently available.

> The progressivist and continuist ... is ... embarrassed by the fact that one cannot even begin to see how the capacity to count up to infinity could arise little by little in a finite evolutionary time (chimps can, at best, count up to four or five. A 'favorable' mutant could have arisen, mastering numbers up to ten, then another mutant up to twenty... Could we believe that *this* is the story?). Our unique, species specific and unprecedented capacity to deal with numbers is (just like language itself) the epitome of evolutionary gratuity and discontinuity. (Piatelli-Palmarini 1989, p. 35, footnote 12)

We don't want to believe *that* story either! Besides, has anybody ever literally *counted* up to infinity? The closest we've come is to create systems of representation that contain generative principles for yielding (potentially) infinite lists (cf. Boolos 1998). The concept of infinity "is the fruit of a slow process of invention over thousands of years" rather than the "epitome of evolutionary gratuity and discontinuity". For example, the Oksapmin of Papua New Guinea have a counting system using body parts that only goes up to 27 and has no base structure. This system is adequate for the simple numerical tasks of traditional life. However, without a base structure, counting beyond 27 or performing simple computations becomes very messy, and ultimately unfeasible. With the advent of the money economy, the Oksapmin began to transform their "body parts" counting system toward a base system. And this socially and culturally mediated bootstrapping process may, in turn, pave the way for further refinements and extensions of their basic "number sense". (The case is reported in Karmiloff-Smith 1992, as well as the entry 'Numeracy and culture' in Wilson and Keil 1999).

However, as Susan Carey notes, the choice of metaphor may seem puzzling: *it is impossible to pull oneself up by one's own bootstrap*. Have we just been sent off on a wild goose chase?

Well, let's sort things out. We develop concepts, also mathematical concepts, in order to "navigate" (or articulate our experience of) the world. Hence, concepts are individuated on the basis of two kinds of considerations: (a) their reference to the world and (b) their relation to each other in a system of inferential relations. Our ongoing concern is how to best characterize the learning mechanisms that enable the construction of properly called number concepts out of prior basic numerical representations.

IV.

The best way to tackle the question is examining the cognitive abilities that are involved in "counting". When we see a small collection of objects, we often determine their number instantaneously. *This* is not counting. This process, which permits 'rapid numerical quantification', is called *subitization*: a fundamental, unconscious system for recognizing patterns that we share with other animals (monkeys or chimpanzees, for instance). Subitization is limited; if the collection has more than three or four objects, we must count (Hurford 2001). This might explain why rhesus monkeys fail to "understand" 2+2=4 in Wynn's classical '1+1=2' task. We have already met subitization before, so we don't need to dwell on it now.

Following Gallistel and Gelman (1978), we can distinguish five 'core principles' that are entailed in counting. First, there has to be a *one-to-one correspondence* between the objects to be counted and the labels or symbols we apply to them. Second, the principle of *ordinality* requires a stable count sequence: the labels have to be applied in order. Third, there is no restriction on the sorts of things one can count (objects, actions or events): this is the principle of *property indifference*. Fourth, the principle of *order indifference* establishes that we can count the objects in any order. Finally, the last label applied in the count sequence represents the total number of objects, the *cardinality* of the set.

The count list ('one', 'two', 'three', and so on) is a system of representation that has the power to represent the positive integers, so long as it contains a generative system for creating an infinite list. When deployed in counting, it provides a representation of exact integer values based on the successor function. That is, when applied in order, in one-one correspondence with the objects in a set, the ordinal position of the last number word in the count provides a representation of the cardinal value of the set—of how many items it contains.

As suggested above, the symbolic, count list representation of number transcends the representational power of the core systems of numerical representation available to preverbal infants. Now we need to explain how we arrive at this representational summit.

> The sort of learning we human beings can achieve just by contemplating symbolic representations of knowledge depends not on our merely, in some sense, perceiving them, but also understanding them, and my [Dennett's] rather curious suggestion is that in order to arrive at this marvelous summit, we must climb steps in which we *perceive but don't understand* our own representations. (Dennett 1993)

Indeed, Dennett's rather curious suggestion has been confirmed to some extent. Cross-cultural developmental studies (see Carey 2004 for a recent review) show that children go through different stages in generating an ordered list of integers and then working out the meanings of number words in the count list. Two-year-olds learn to recite the count list and even engage in pretend "counting" routines, but need another year and a half to figure out how counting represents number. As the child lays down more associations between the auditory and articulatory processes of reciting a seemingly meaningless serially ordered list of scribble on the one hand (say "eeny, meeny, miney, mo,..." or, for that matter, "one, two, three, four,..."), and other patterns of concurrent activity on the other (for instance, touching objects in a set one by one as they recite the list), number words gradually become more and more familiar, even without being readily understood. It is these "anchors of familiarity" (Dennett's expression) that give a word, a "label", saliency within a representational system and pave the way for genuine understanding. At that point, "the *mere contemplation* of a representation is sufficient to call to mind all the appropriate lessons; we have become *understanders* of the objects we have created” (Dennett 1993).

Now, let me zoom in on the actual developmental steps that children must take in order to work out how counting represents number. First, we may allow the child the prenumerical capacity for representing serial order—to engage in an initially meaningless mouthing of serially ordered lists of gibberish. Eventually, children learn to recite the sequence of number words “one, two, three, etc.” but do not have a clue about their meaning yet. Then, children become “one-knowers” (the term is borrowed from Carey 2004), taking ‘one’ to contrast with all the other words in the list, meaning “more than one” or “some”. Two-year-olds will give you one object if you ask for ‘one’, but grab a bunch (always greater than one) if you ask for ‘two’, ‘three’, ‘four’, or ‘nine’. Oddly enough, they do not create a larger set for ‘two’ than for ‘nine’. Six to nine months later, children learn what “two” means and hence become “two-knowers”. After some months being two-knowers, children become “three-knowers”. And, some months later, they finally induce how counting works. The crucial inductive step has the following form (I_F): if number word x refers to a set with cardinal value n, then the next number word in the list refers to a set with cardinal value n+1.

This whole bootstrapping process draws on both the small precise and the large approximate systems for numerical representation, in addition to a prenumerical system for “voiced” representation of serial order, in order to generate the count list and work out the meanings of the number words in the list. The small precise number system supports this process from “below” (see how nicely “one”, “two” and “three”-knowers match up with its signature limit), the large approximate system sustains it from “above”, while verbal representation of serial order molds the representations of the former systems into a counting list. The original representations yielded by these basic representational systems are re-described into “knowing” representations through a mapping process that connects patterns of concurrent action (e.g. touching and reciting) into a new system of “salient” or “visible” representations. Finally, the inferential structure of the resulting representational system articulates the meanings of the number words according to the general form (I_F): for instance, ‘five’ meaning “one more than ‘four’, which is one more than ‘three’, which is...”

V.

By way of conclusion, I want to anticipate and address two objections that might be raised to the present account. The claim is that children acquire the concept of number “bootstrapping” their way through the count sequence. Again, let me summarize the actual developmental steps that children take in order to work out how counting represents number. First, children have a mapping between the first few number words and their respective quantities, but do not—at least not at this early stage—have any further understanding of the number system nor of its formal (i.e. arithmetical) properties. Second, children realize that the last label in the count sequence denotes the numerical quantity, the total number of items, of the collection being counted. Finally, children notice that each successive count term picks out a numerical quantity that is precisely one more than the term that precedes it and so induce how counting works: (I_F) if number word x refers to a set with cardinal value n, then the next number in the list refers to a set with cardinal value n+1.

Now, one might argue that the second step in my account of children’s gradual mastery of the count sequence is too *strong* and hence potentially misleading. Once children realize that the final word in a count refers to the number of items in a collection, they come to know everything there is to know about counting. Hence, no need for a further inductive step (I_F).

At first glance, the objection seems compelling: figuring out how counting works (third step) amounts to the realization that counting is a way of enumerating a collection of items (second step). However, I believe that each step reflects a different degree of numerical understanding. Children may notice *that* counting works—that the final label in a count sequence corresponds to the number of items in a collection—without thereby having to understand *how* (or even *why*) it works. (I_F), that every word in the count picks out a quantity with a numerical value of precisely one more than its predecessor, is something that children only grasp at a later stage, in which they come to understand subsequent count terms and the inferential relations that hold between the corresponding number concepts that they encode.

On the other hand, one might make quite a different objection and argue that my account is to *weak* to grant children's proper understanding of number. What prevents children, say "four-knowers", from applying the same count term, take "five", to any arbitrarily large collection of more than four objects, thus labeling any set larger than four "five"? This challenge allows for the possibility that children's application of the count terms systematically differs from the norm in cases outside the test range. So how do we know, in general, whether children are interpreting a count term correctly?

Counting is naturally grounded on children's "number sense", the ability to selectively respond to different numerosities, drawing on both the small precise and the large approximate systems for numerical representation. The bootstrapping process—by which children work out the meanings of the number concepts—is supported by the small precise number system from "below", while the large approximate number system sustains it from "above".

Notice, however, that these systems do not constrain children's interpretation of a count term—any count term—so as to rule out the skeptic's possible scenario. For one thing, the large approximate number system only yields a *noisy* representation of approximate number. And, although numerical discrimination increases in precision over development, many instances will fall outside our discriminability threshold.

In order to circumvent such difficulties, I allowed for a prenumerical system for "voiced" (or "written" or, for that matter, any other stable format of your choice) representation of serial order. The idea being that verbal representation of serial order molds the representations yielded by these basic nonverbal representational systems into a counting list. In addition, recall that I distinguished five 'counting principles' to work out the meanings of the number words in the list: the point is that the skeptical scenario violates the so-called "correspondence principle", the one-to-one correspondence between the items to be counted and the labels or symbols we apply to them, which prevents a count term to be applied more than once in a count.

VI.

Bootstrapping processes are not only present during ontogenetic time, archeologists and anthropologists have described processes of cultural construction over thousands of years of historical time (Ascher 1998, 2002). This raises an important question: Is the developmental process that has just been described already pre-specified in infants' innate dispositions, or do the resulting capacities rather depend to an important degree (how important?) on the cultural resources (e.g., pre-designed inferential structures) that the child acquires as they are moved from the child's surrounding culture into its brain? This is another instance of the old nature/nurture debate, where a cognitive perspective on mathematical culture needs to be balanced against a cultural perspective on mathematical cognition. And, as with other chicken and egg problems, it's far from clear how to go about gaging it.

Nevertheless, so much is clear: without some innately specified attention biases and principles, numerical competence cannot develop. On the other hand, without an appropriate cultural environment, number competence cannot develop *either*! Consider the case of the Pirahâ, an isolated Amazonian tribe in Brazil (Karmiloff-Smith 1992; Carey 2004). The Pirahâ people are literally "two-knowers", they only posses a 1/2/many system for representing number, and have no representations of large exact numerical values at all. This indicates that whatever numerical abilities develop subsequently they are not "already there". Rather the capacity to develop such abilities is there. Development transforms this capacity, sometimes in fundamental ways.

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