

Relative arithmetic

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In nonstandard mathematics, the predicate ‘ x is standard’ is fundamental. Recently, ‘relative’ or ‘stratified’ nonstandard theories have been developed in which this predicate is replaced with ‘ x is y -standard’. Thus, objects are not (non)standard in an absolute sense, but (non)standard relative to other objects and there is a whole stratified universe of ‘levels’ or ‘degrees’ of standardness. Here, we study stratified nonstandard arithmetic and the related transfer principle. Using the latter, we obtain the ‘reduction theorem’ which states that arithmetical formulas can be reduced to equivalent bounded formulas. Surprisingly, the reduction theorem is also equivalent to the transfer principle. As applications, we obtain a truth definition for arithmetical sentences and we formalize Nelson’s notion of impredicativity.

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1 Introduction

Nonstandard analysis was developed in the early 1960’s by Abraham Robinson. It was among the first rigorous theories for calculus with infinitesimals. Although the latter had been used centuries before by Archimedes, Euler, Leibniz and others, Robinson was the first to produce a formal framework which was free of the inconsistencies that plagued the earlier ‘intuitive’ infinitesimal calculus (see [16]). There are several different approaches to nonstandard mathematics: from Robinson’s original type theory ([16]), the superstructure method by Robinson and Zakon ([17]), and Chang and Keisler ([3]), Luxemburg’s ultrafilter approach ([11]), to axiomatic theories like Hrbacek’s **HST** ([6, 14]) and Nelson’s **IST** ([12]). All these theories somehow introduce the concept of standard and nonstandard objects and, in particular, of finite and infinite numbers. Theories of nonstandard mathematics which involve this dichotomy of finite versus infinite and standard versus nonstandard will be called ‘classical’ nonstandard theories.

Recently, a new class of theories, called ‘relative’ or ‘stratified’ nonstandard mathematics, has been introduced (see e.g. [5, 7, 15, 18]). Instead of just two levels of objects (standard and nonstandard), the new theories involve several ‘levels’ or ‘degrees’ of standardness. To achieve this, the predicate ‘ x is standard’ is replaced by ‘ x is y -standard’, which stratifies the two levels ‘standard’ and ‘nonstandard’ into infinitely many levels of standardness. In this way, an object x can be nonstandard compared to z , but standard relative to another object y . In particular, the unary number predicate ‘ x is infinite’ is replaced with the binary predicate ‘ x is y -infinite’. It is clear that stratified nonstandard mathematics is a refinement of the classical nonstandard framework, not a departure from it.

There are several convincing arguments why stratified nonstandard mathematics is an improvement over its classical counterpart. First of all, by renaming the predicate ‘ x is y -infinite’ to ‘ x is very large compared to y ’, stratified nonstandard analysis becomes a formal framework for physics which preserves physical intuition, in particular the intuitive calculus of ‘small’ and ‘large’ quantities prevalent in physics (see [8] and [18]). Second,

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calculus (and even analysis) can finally be done in a quantifier-free way in stratified nonstandard analysis ([5]), even in weak theories of arithmetic (see [18]). In contrast to classical nonstandard mathematics, the stratified framework is completely free of the ε - δ -method. Third, formulas can be transferred almost without any limitations, greatly reducing the dependence on logic. Indeed, in classical nonstandard mathematics, transfer is limited to standard formulas and it takes some technical machinery to validate this principle. Hence, there is a greater reliance on logic than is common in most of mathematics and this seems to be an important obstacle for the adoption of the nonstandard framework in mainstream mathematics.

In this paper, we study the classical transfer principle in stratified nonstandard arithmetic. The theories at hand range in strength from $I\Delta_0$ to Peano arithmetic (see theorem 4.2). Using the transfer principle, we can prove the ‘reduction theorem’ (see theorem 3.1) which reduces arithmetical formulas to equivalent Δ_0 -formulas. Thus, it is possible to collapse the arithmetical hierarchy onto Δ_0 . Surprisingly, the reduction theorem is also equivalent to the aforementioned transfer principle (see theorem 5.3). As applications, we define a truth definition for arithmetical sentences and formalize Nelson’s notion of impredicativity (see [13]).

2 Stratified nonstandard arithmetic

In this section, we describe stratified nonstandard arithmetic and its fundamental features. Let L be the language of arithmetic. We introduce a new binary predicate ‘ $x \sqsubseteq y$ ’ which applies to all natural numbers. For better readability we write ‘ x is y -finite’ instead of $x \sqsubseteq y$. This notation is purely symbolic and we may also read $x \sqsubseteq y$ as e.g. ‘ x is not very large compared to y ’. The following axiom set describes the properties of $x \sqsubseteq y$. These axioms are not intended to be minimal.

Axiom 2.1 (NS)

1. The numbers 0, 1 and x are x -finite.
2. If x and y are z -finite, so are $x + y$ and $x \times y$.
3. If x is y -finite and $z \leq x$, then z is y -finite.
4. If x is y -finite and y is z -finite, then x is z -finite.
5. Either x is y -finite or y is x -finite.
6. There is a number y that is not x -finite.

Definition 2.2 A number y is called ‘ x -infinite’ if it is not x -finite. We denote this by ‘ $x \ll y$ ’. A number is also called ‘ x -standard’ if it is x -finite.

By item (6) of the previous schema, the set of natural numbers is ‘stratified’ in different ‘levels’ or ‘degrees’ of magnitude. Intuitively, numbers of the same level are ‘finite’ (or ‘not very large’) relative to each other and ‘infinite’ (or ‘very large’) compared to numbers of lower levels. The numbers 0 and 1 are at the lowest level.

It should be noted that we do *not* expand the set of natural numbers; we only define a new predicate $x \sqsubseteq y$ which can be interpreted in several ways (see also section 7). More technically, we use Nelson’s ‘internal’ view of nonstandard mathematics rather than Robinson’s ‘external’ viewpoint (see [5, 12]). This choice is only motivated by aesthetics.

Definition 2.3 A formula is called ‘internal’ if it does not involve the predicate ‘ x is y -finite’ for any x and y . Non-internal formulas are called ‘external’.

In the following, we assume that the classes Δ_0 , Σ_n and Π_n of the arithmetical hierarchy are limited to internal formulas, i.e. they carry their usual meaning. We also assume that all parameters are shown, unless explicitly stated otherwise.

Notation 2.4 We write ‘ $(\exists^{x-st} y)\varphi(y)$ ’ instead of $(\exists y)(y \text{ is } x\text{-finite} \wedge \varphi(y))$ and we write ‘ $(\forall^{x-st} y)\varphi(y)$ ’ instead of $(\forall y)(y \text{ is } x\text{-finite} \rightarrow \varphi(y))$.

Now consider the following transfer principle.

Axiom schema 2.5 (Σ_n -TRANS) For every formula $\varphi \in \Delta_0$ and x -finite \vec{y} ,

$$(\exists x_1)(\forall x_2) \dots (Qx_n)\varphi(x_1, \dots, x_n, \vec{y}) \quad (1)$$

is equivalent to

$$(\exists^{x-st} x_1)(\forall^{x-st} x_2) \dots (Q^{x-st} x_n)\varphi(x_1, \dots, x_n, \vec{y}). \quad (2)$$

Depending on whether n is odd or even, ' (Qx_n) ' is ' $(\exists x_n)$ ' or ' $(\forall x_n)$ '.

For fixed x and $\varphi \in \Delta_0$, the previous schema is just the usual transfer principle for Σ_n -formulas, relative to the level of magnitude of x . Thus, Σ_n -TRANS expresses Leibniz's principle that the same laws should hold for all numbers, standard or nonstandard alike, relative to the level at which the numbers occur. For brevity, we write 'TRANS' for ' $\cup_{n \in \mathbb{N}} \Sigma_n$ -TRANS'.

By contraposition, the schema Σ_n -TRANS immediately yields the following equivalent transfer principle.

Axiom schema 2.6 (Π_n -TRANS) For every formula $\varphi \in \Delta_0$ and x -finite \vec{y} ,

$$(\forall x_1)(\exists x_2) \dots (Qx_n)\varphi(x_1, \dots, x_n, \vec{y}) \quad (3)$$

is equivalent to

$$(\forall^{x-st} x_1)(\exists^{x-st} x_2) \dots (Q^{x-st} x_n)\varphi(x_1, \dots, x_n, \vec{y}). \quad (4)$$

Depending on whether n is even or odd, ' (Qx_n) ' is ' $(\exists x_n)$ ' or ' $(\forall x_n)$ '.

The following lemma greatly reduces the number of applications of transfer in a proof. We sometimes refer to it as the 'transfer lemma'.

Lemma 2.7 For every formula $\varphi \in \Delta_0$ and x -finite \vec{y} , if Σ_n -TRANS is available,

$$(\exists x_1)(\forall x_2) \dots (Qx_n)\varphi(x_1, \dots, x_n, \vec{y}) \quad (5)$$

is equivalent to

$$(\exists^{x-st} x_1)(\forall x_2) \dots (Qx_n)\varphi(x_1, \dots, x_n, \vec{y}), \quad (6)$$

and, for $y \gg x$, to

$$(\exists^{y-st} x_1)(\forall^{y-st} x_2) \dots (Q^{y-st} x_n)\varphi(x_1, \dots, x_n, \vec{y}). \quad (7)$$

Proof. The equivalence between (5) and (7) follows immediately from Σ_n -TRANS and the implication '(6) \rightarrow (5)' is trivial. For the implication '(5) \rightarrow (6)', by Σ_n -transfer, (5) implies (2). Fix x -finite x'_1 such that $(\forall^{x-st} x_2) \dots (Q^{x-st} x_n)\varphi(x'_1, \dots, x_n, \vec{y})$ and apply Π_{n-1} -transfer. The resulting formula implies (6). \square

3 The reduction theorem

In this section, we describe a procedure which reduces a Σ_n -formula with x -standard parameters to a Δ_0 -formula. The resulting formula is equivalent to the original one, if Σ_n -TRANS is available. Thus, the following theorem is proved in the theory $I\Delta_0 + NS + \Sigma_n$ -TRANS.

Theorem 3.1 For $\varphi \in \Delta_0$ and x -standard \vec{y} , the formula

$$(\exists x_1)(\forall x_2) \dots (Qx_n)\varphi(x_1, \dots, x_n, \vec{y}) \quad (8)$$

is equivalent to

$$(\exists x_1 \leq c_1)(\forall x_2 \leq c_2) \dots (Qx_n \leq c_n)\varphi(x_1, \dots, x_n, \vec{y}), \quad (9)$$

whenever $x \ll c_1 \ll \dots \ll c_n$.

Proof. Let φ , x and \vec{y} be as stated and fix numbers c_i such that $x \ll c_1 \ll \dots \ll c_n$. For better readability, we suppress the x -standard parameters \vec{y} in φ . We first prove the implication ‘(8) \rightarrow (9)’. Assume n is even. The case for odd n is treated below. From

$$(\exists x_1)(\forall x_2) \dots (\forall x_n)\varphi(x_1, \dots, x_n), \quad (10)$$

there follows, by the transfer lemma,

$$(\exists^{x\text{-}st} x_1)(\forall x_2)(\exists x_3) \dots (\forall x_n)\varphi(x_1, x_2, \dots, x_n).$$

As $x \ll c_1$, this implies

$$(\exists x_1 \leq c_1)(\forall x_2 \leq c_2)(\exists x_3)(\forall x_4) \dots (\forall x_n)\varphi(x_1, x_2, x_3, \dots, x_n). \quad (11)$$

Fix suitable $x'_1 \leq c_1$ such that for all $x'_2 \leq c_2$ there holds

$$(\exists x_3)(\forall x_4)(\exists x_5) \dots (\forall x_n)\varphi(x'_1, x'_2, x_3, \dots, x_n).$$

This formula is in Σ_{n-2} and x'_1 and x'_2 are now amongst its parameters. Repeat the steps that produce (11) from (10), with $x = c_2$. This yields

$$(\exists x_3 \leq c_3)(\forall x_4 \leq c_4)(\exists x_5) \dots (\forall x_n)\varphi(x'_1, x'_2, x_3, \dots, x_n),$$

which implies

$$(\exists x_1 \leq c_1)(\forall x_2 \leq c_2)(\exists x_3 \leq c_3)(\forall x_4 \leq c_4)(\exists x_5) \dots (\forall x_n)\varphi(x_1, \dots, x_n).$$

Now keep repeating the above process until we obtain (9).

If n is odd, we apply the same process as in the even case to obtain

$$(\exists x_1 \leq c_1)(\forall x_2 \leq c_2) \dots (\forall x_{n-1} \leq c_{n-1})(\exists x_n)\varphi(x_1, \dots, x_n).$$

Applying Σ_1 -transfer to the innermost existential formula yields

$$(\exists x_1 \leq c_1)(\forall x_2 \leq c_2) \dots (\forall x_{n-1} \leq c_{n-1})(\exists^{c_{n-1}\text{-}st} x_n)\varphi(x_1, \dots, x_n),$$

and since $c_n \gg c_{n-1}$, this implies (9).

For the reverse implication, we treat the case where n is even; the case where n is odd can be treated analogously. In the former case, we have

$$(\exists x_1 \leq c_1)(\forall x_2 \leq c_2) \dots (\exists x_{n-1} \leq c_{n-1})(\forall x_n \leq c_n)\varphi(x_1, \dots, x_n).$$

As $c_n \gg c_{n-1}$, this implies

$$(\exists x_1 \leq c_1) \dots (\exists x_{n-3} \leq c_{n-3})(\forall x_{n-2} \leq c_{n-2})(\exists^{c_{n-1}\text{-}st} x_{n-1})(\forall^{c_{n-1}\text{-}st} x_n)\varphi(x_1, \dots, x_n),$$

and the transfer lemma, applied to the innermost Σ_2 -formula, yields

$$(\exists x_1 \leq c_1) \dots (\exists x_{n-3} \leq c_{n-3})(\forall x_{n-2} \leq c_{n-2})(\exists^{c_{n-2}\text{-}st} x_{n-1})(\forall^{c_{n-2}\text{-}st} x_n)\varphi(x_1, \dots, x_n).$$

As $c_{n-2} \gg c_{n-3}$, this implies

$$(\exists x_1 \leq c_1) \dots (\exists^{c_{n-3}\text{-}st} x_{n-3})(\forall^{c_{n-3}\text{-}st} x_{n-2})(\exists^{c_{n-2}\text{-}st} x_{n-1})(\forall^{c_{n-2}\text{-}st} x_n)\varphi(x_1, \dots, x_n),$$

Again applying the transfer lemma to the innermost Σ_2 -formula yields

$$(\exists x_1 \leq c_1) \dots (\exists^{c_{n-3}\text{-}st} x_{n-3})(\forall^{c_{n-3}\text{-}st} x_{n-2})(\exists^{c_{n-3}\text{-}st} x_{n-1})(\forall^{c_{n-3}\text{-}st} x_n)\varphi(x_1, \dots, x_n).$$

Repeat this process until all n quantifiers are exhausted. Note that at later stages it will be the innermost Σ_4 , Σ_6 , etc. subformulas that have to be transferred. Thus, we obtain

$$(\exists^{c_1\text{-}st} x_1)(\forall^{c_1\text{-}st} x_2) \dots (\exists^{c_1\text{-}st} x_{n-1})(\forall^{c_1\text{-}st} x_n)\varphi(x_1, \dots, x_n),$$

and Σ_n -transfer with $x = c_1$ yields (8). □

Theorem 3.1 states that a Σ_n -statement (with x -finite parameters) about all numbers can be reduced to a Δ_0 -statement about a certain initial segment. Thus, this theorem is called the ‘ Σ_n -reduction theorem’ or just ‘reduction theorem’, if the class of formulas is clear from the context. If we interpret ‘ $y \ll z$ ’ as ‘ z is very large compared to y ’, then the reduction theorem tells us that a Σ_n -statement about numbers of size at most x can be reduced to a bounded statement if we have access to n -many higher levels of ‘largeness’.

The best-known way to remove quantifiers from a formula is by introducing Herbrand or Skolem functions (see [1] or [4]). However, the predicate $x \sqsubseteq y$ makes it possible to remove all quantifiers *simultaneously* while keeping the newly introduced objects simple. Indeed, in contrast to Skolemization or Herbrandization, the reduction theorem only introduces new constants c_i . On the other hand, Skolemization removes quantifiers from all formulas of the skolemized language, while our procedure only works for formulas of the original language (without \sqsubseteq).

To conclude this section, we point out an application of the reduction theorem in Reverse Mathematics (see [19]). In [10], Keisler presents a nonstandard version of each of the ‘Big Five’ theories of Reverse Mathematics. To this end, he formalizes nonstandard arithmetic in second-order arithmetic (see [10, §3 and §4]), using Robinson’s external view. After formalizing the stratified framework in second-order arithmetic in the same way (in particular, the natural numbers are exactly the 0-finite numbers), we can obtain ACA^- (the comprehension schema for arithmetical formulas without set parameters) with a minimum of comprehension axioms. Indeed, if TRANS is available, the reduction theorem yields that every arithmetical formula with 0-finite parameters is equivalent to a Δ_0 -formula. Thus, comprehension for Δ_0 -formulas suffices to obtain ACA^- , if TRANS is available. The latter is not a strong requirement, as, by [10, Corollary 7.11], TRANS is not a strong schema in the context of ACA_0 . It should be noted, however, that in order to work in second-order arithmetic, we have to adopt Robinson’s external view of nonstandard mathematics.

4 Approaching Peano arithmetic

In this section, we obtain lower bounds for the strength of Σ_n -TRANS. First, we prove that Σ_n -TRANS, when added to $I\Delta_0 + \text{NS}$, makes the resulting theory at least as strong as $I\Sigma_n$. Thus, TRANS takes us all the way up from bounded arithmetic to Peano arithmetic.

In arithmetic, the basic operations $+$ and \times are introduced in Robinson’s theory Q . To obtain stronger theories, different flavours of induction can be added, like the following schema (see [1, 4]). The set Φ contains formulas in the language L of arithmetic.

Axiom schema 4.1 (Φ -IND) For every formula $\varphi \in \Phi$, there holds

$$[\varphi(0) \wedge (\forall n)(\varphi(n) \rightarrow \varphi(n+1))] \rightarrow (\forall n)\varphi(n). \quad (12)$$

The theory $Q + \Sigma_n$ -IND is usually denoted $I\Sigma_n$. The union of all these theories is called Peano arithmetic, or PA for short.

Theorem 4.2 *The theory $I\Delta_0 + \text{NS} + \Sigma_n$ -TRANS proves Σ_n -IND.*

Proof. Let φ be a Σ_n -formula in the language of arithmetic and assume the antecedent of Σ_n -IND holds for this formula, i.e. we have

$$\varphi(0, \vec{y}) \wedge (\forall n)(\varphi(n, \vec{y}) \rightarrow \varphi(n+1, \vec{y})). \quad (13)$$

To increase readability, we suppress the parameters \vec{y} in this proof. It is an elementary verification that we may do this without loss of generality. Also, it is easily proved that Δ_0 -MIN is available in $I\Delta_0$ (see e.g. [1]). Thus, we can calculate the least n such that $\phi(n)$, if such there are, for all $\phi \in \Delta_0$.

Now suppose there is an n_0 such that $\neg\varphi(n_0)$. By theorem 3.1, there is a Δ_0 -formula $\psi(n)$ such that $\neg\varphi(n)$ is equivalent to $\psi(n)$ for $n \leq n_0$. Let n_2 be the least $n \leq n_0$ such that $\psi(n)$. Thus, there holds $\psi(n_2)$ and also $\neg\psi(n_2 - 1)$ if $n_2 > 0$. But $\psi(n_2)$ is equivalent to $\neg\varphi(n_2)$ and by (13), there holds $\varphi(0)$. This implies $n_2 > 0$ and hence we have $\neg\psi(n_2 - 1)$, which is equivalent to $\varphi(n_2 - 1)$. But then there holds $\varphi(n_2 - 1) \wedge \neg\varphi(n_2)$, which contradicts (13). Hence, $\varphi(n)$ must hold for all n and we have proved (12) for Φ equal to Σ_n . \square

Besides induction, there are other ways of axiomatizing arithmetic. In particular, the so-called ‘collection’ or ‘replacement’ axiom schemas yield a series of theories similar to $I\Sigma_n$.

Axiom schema 4.3 (Φ -REPL) For every formula $\varphi \in \Phi$, there holds

$$(\forall x \leq t)(\exists y)\varphi(x, y) \rightarrow (\exists z)(\forall x \leq t)(\exists y \leq z)\varphi(x, y). \quad (14)$$

The theory $I\Delta_0 + \Sigma_n$ -REPL is usually denoted $B\Sigma_n$. It is well-known that $I\Sigma_{n+1}$ implies $B\Sigma_{n+1}$ and that the latter implies $I\Sigma_n$ (see e.g. [1]). Thus, the theories $B\Sigma_n$ also form a hierarchy of Peano arithmetic. Together with these facts, theorem 4.2 implies that $I\Delta_0 + \text{NS} + \Sigma_{n+1}$ -TRANS proves Σ_{n+1} -REPL. The following theorem proves this directly.

Theorem 4.4 *The theory $I\Delta_0 + \text{NS} + \Sigma_{n+1}$ -TRANS proves Σ_{n+1} -REPL.*

Proof. Let φ be a Σ_{n+1} -formula and assume the antecedent of Σ_{n+1} -REPL holds for this formula, i.e. we have $(\forall x \leq t)(\exists y)\varphi(x, y)$. Again, we suppress most parameters (but not t) to increase readability. Assume $\varphi(x, y)$ is of the form $(\exists x_1)(\forall x_2) \dots (Qx_{n+1})\phi(x, y, x_1, \dots, x_{n+1})$, where $\phi \in \Delta_0$. Fix c_1, \dots, c_{n+1} such that $x \ll c_1 \ll \dots \ll c_{n+1}$. By theorem 3.1, for all $x \leq t$, the formula $(\exists y)\varphi(x, y)$ is equivalent to

$$(\exists y \leq c_1)(\exists x_1 \leq c_1)(\forall x_2 \leq c_2) \dots (Qx_{n+1} \leq c_{n+1})\phi(x, y, x_1, \dots, x_{n+1}),$$

where $t \ll c_1 \ll \dots \ll c_{n+1}$. Thus, for all $x \leq t$, there are $y', x'_1 \leq c_1$ such that

$$(\forall x_2 \leq c_2) \dots (Qx_{n+1} \leq c_{n+1})\phi(x, y', x'_1, x_2, \dots, x_{n+1}).$$

By the reduction theorem for $x = c_1$, this formula is equivalent to

$$(\forall x_2) \dots (Qx_{n+1})\phi(x, y', x'_1, x_2, \dots, x_{n+1}),$$

which yields the consequent of Σ_{n+1} -REPL with $z = c_1$. \square

Using the appropriate maximization axioms it is possible to make the bound z a t -standard number. It is well-known that such axioms are available in $I\Delta_0$.

5 Reducing transfer to the reduction theorem

In the third section, we showed that Σ_n -transfer suffices to obtain the Σ_n -reduction theorem. Interestingly, the former is also equivalent to the latter, by theorem 5.3 below. However, we need the following nonstandard tool, provable in $I\Delta_0 + \text{NS}$. Note that x -infinite parameters are allowed in the formula φ .

Theorem 5.1 (Stratified Overflow and Underflow) *Assume $\varphi \in \Delta_0$.*

1. *If $\varphi(n)$ holds for all x -finite n , it holds for all n up to some x -infinite \bar{n} . (**overflow**).*
2. *If $\varphi(n)$ holds for all x -infinite n , it holds for all n from some x -finite \underline{n} on. (**underflow**).*

Proof. For the first item, assume $\varphi(n) \in \Delta_0$ holds for all x -finite n . Then calculate the least n_0 such that $\neg\varphi(n_0)$, which must be x -infinite. Define \bar{n} as $n_0 - 1$. Likewise for the second item. \square

Corollary 5.2 *Assume $\varphi \in \Delta_0$. If $\varphi(n)$ holds for all x -infinite $n \leq n_0$, with n_0 x -infinite, it holds for all $n \leq n_0$ from some x -finite \underline{n} on.*

Proof. Define $\psi(n)$ as $\varphi(n) \vee n \geq n_0$ and apply underflow. \square

In the following, the previous corollary is also referred to as ‘underflow’.

Theorem 5.3 *In $I\Delta_0 + \text{NS}$, the Σ_n -reduction theorem is equivalent to the transfer principle Σ_n -TRANS.*

Proof. By theorem 3.1, the inverse implication is immediate. For the forward implication, we proceed by induction on n . For better readability, we suppress the x -standard parameters \vec{y} in both Σ_n -TRANS and the Σ_n -reduction theorem.

For the case $n = 1$, let φ be as in Σ_1 -TRANS and assume $(\exists x_1)\varphi(x_1)$. By the reduction theorem, we have $(\exists x_1 \leq c_1)\varphi(x_1)$, for all $c_1 \gg x$. By underflow with c_1 as n_0 , there holds $(\exists^{x-st} x_1)\varphi(x_1)$. This proves the downward implication in Σ_1 -TRANS, i.e. that (1) implies (2) for $n = 1$. The upward implication is trivial and this case is done.

For the case $n = 2$, let φ be as in Σ_2 -TRANS and assume $(\exists x_1)(\forall x_2)\varphi(x_1, x_2)$. By the reduction theorem, we have $(\exists x_1 \leq c_1)(\forall x_2 \leq c_2)\varphi(x_1, x_2)$, for all $c_2 \gg c_1 \gg x$. Fix c'_2 and c'_1 such that $c'_2 \gg c'_1 \gg x$. For all x -infinite $d \leq c'_1$, there holds $(\exists x_1 \leq d)(\forall x_2 \leq c'_2)\varphi(x_1, x_2)$. By underflow, there is an x -finite \underline{d} such that $(\exists x_1 \leq \underline{d})(\forall x_2 \leq c'_2)\varphi(x_1, x_2)$. As $c'_2 \gg x$, this implies $(\exists^{x-st} x_1)(\forall^{x-st} x_2)\varphi(x_1, x_2)$. This proves the downward implication in Σ_2 -TRANS, i.e. that (1) implies (2) for $n = 2$. The upward implication is easily proved using Σ_1 -TRANS, obtained earlier.

For the case $n > 2$, let φ be as in Σ_n -TRANS and assume (1) holds. By the Σ_n -reduction theorem, (9) follows, for all c_1, \dots, c_n such that $x \ll c_1 \ll \dots \ll c_n$. Now fix c'_1, \dots, c'_n such that $x \ll c'_1 \ll \dots \ll c'_n$. For all x -infinite $d \leq c'_1$, there holds

$$(\exists x_1 \leq d)(\forall x_2 \leq c'_2) \dots (Qx_n \leq c'_n)\varphi(x_1, \dots, x_n),$$

and underflow implies $(\exists^{x-st} x_1)(\forall x_2 \leq c'_2) \dots (Qx_n \leq c'_n)\varphi(x_1, \dots, x_n)$. Fix suitable x -finite x'_1 such that for all x -finite x'_2 , we have

$$(\exists x_3 \leq c'_3)(\forall x_4 \leq c'_4) \dots (Qx_n \leq c'_n)\varphi(x'_1, x'_2, x_3, x_4, \dots, x_n). \quad (15)$$

By the Σ_{n-2} -reduction theorem, (15) becomes

$$(\exists x_3)(\forall x_4) \dots (Qx_n)\varphi(x'_1, x'_2, x_3, x_4, \dots, x_n). \quad (16)$$

By the induction hypothesis, the Σ_{n-2} -reduction theorem yields Σ_{n-2} -TRANS, and Σ_{n-2} -transfer applied to (16) yields

$$(\exists^{x-st} x_3)(\forall^{x-st} x_4) \dots (Q^{x-st} x_n)\varphi(x'_1, x'_2, x_3, x_4, \dots, x_n).$$

This can be done for all x -standard x'_2 and thus we obtain (2). This settles the downward implication in Σ_n -TRANS, i.e. that (1) implies (2). The upward implication is easily proved using Σ_{n-1} -TRANS, which is available thanks to the induction hypothesis. \square

6 Arithmetical truth

In this section, we investigate the so-called ‘truth predicate’ or ‘truth definition’ \mathbb{T} in our stratified framework. This unary predicate has the property that

$$\psi \leftrightarrow \mathbb{T}(\ulcorner \psi \urcorner), \text{ for all sentences } \psi. \quad (\text{T})$$

Thus, the formula $\mathbb{T}(\ulcorner \psi \urcorner)$ simply expresses that ψ is true (or false). As truth is one of the fundamental properties of logic, such predicate \mathbb{T} is a most interesting object of study. For instance, in $I\Sigma_{n+1}$, there is a truth predicate for Σ_n -sentences which respects the logical connectives and this allows for a smooth proof of $I\Sigma_{n+1} \vdash \text{Con}(I\Sigma_n)$ (see [1, p. 137]). However, by Tarski’s well-known theorem on the undefinability of truth, there is no arithmetical formula \mathbb{T} with the property (T) for *all* arithmetical sentences. Nonetheless, by the reduction theorem, the truth of an arithmetical formula (with x -standard parameters) is equivalent to that of a bounded formula and the truth of the latter can be expressed quite easily. Based on this heuristic idea, we shall obtain an external, i.e. non-arithmetical, formula \mathbb{T} with the property (T) for all arithmetical sentences.

Theorem 6.1 *In $I\Delta_0 + \text{NS} + \text{TRANS}$, there is a truth definition for all arithmetical sentences.*

Proof. By theorem 4.2, $I\Delta_0 + \text{NS} + \Sigma_n\text{-TRANS}$ is at least as strong as $I\Sigma_n$ and thus the exponential function is available. Hence, we may assume without loss of generality that blocks of existential and universal quantifiers are coded into single quantifiers. In particular, if c is a code for a vector (c_1, \dots, c_n) , then the projection function $[x]_y$ is defined as $[c]_i = c_i$ for $1 \leq i \leq n$. Furthermore, following Buss' arithmetization of metamathematics (see [1, Chapter II]), we may assume that the predicate 'Form $_{\Sigma_n \cup \Pi_n}(x)$ ' which is true if and only if x is the Gödel number of either a Σ_n or Π_n -formula, is available. Now define the predicate $\mathbb{B}\mathbb{F}(x, y, c, n)$ as follows. If x is the Gödel number of the $\Sigma_n \cup \Pi_n$ -formula

$$(Qx_1)(Qx_2) \dots (Qx_k)\varphi(x_1, \dots, x_k, \vec{y}),$$

with $k \leq n$ and y is the Gödel number of a vector \vec{z} with the same length as \vec{y} , then $\mathbb{B}\mathbb{F}(x, y, c, n)$ is defined as true if

$$(Qx_1 \leq [c]_1)(Qx_2 \leq [c]_2) \dots (Qx_k \leq [c]_k)\varphi(x_1, \dots, x_k, \vec{z}).$$

Define $\mathbb{B}\mathbb{F}(x, y, c, n)$ as 'false' otherwise. As $I\Delta_0 + \text{exp}$ has a truth definition for Δ_0 -formulas (see e.g. [9, Corollary 52]), it is clear that the predicate $\mathbb{B}\mathbb{F}(x, y, c, n)$ is available. Now define the formula $\mathbb{T}(x, y)$ as

$$(\exists c)(\exists n)[\text{Form}_{\Sigma_n \cup \Pi_n}(x) \wedge y = [c]_0 \wedge (\forall i \leq n)([c]_i \ll [c]_{i+1}) \wedge \mathbb{B}\mathbb{F}(x, y, c, n)]. \quad (17)$$

By the reduction theorem, the arithmetical sentence $\psi(\vec{z})$ is true if and only if $\mathbb{T}(\ulcorner \psi \urcorner, \ulcorner \vec{z} \urcorner)$. \square

As formula (17) explicitly involves the predicate ' \ll ', Tarski's theorem does not contradict the previous corollary. Indeed, the reduction theorem does not apply to external formulas and thus the usual diagonalization argument does not go through.

In Latin, 'infinite' literally means 'the absence of limitation'. In the stratified framework, where the 'infinite' abounds, there is indeed no limitation to our knowledge of arithmetical truth.

7 Philosophical considerations

In the final section, we argue that the reduction theorem yields a formalization of Nelson's notion of impredicativity (see [13]). The latter is a key ingredient of Nelson's philosophy of mathematics, which is described by Buss as 'radical constructivism' (see [2]).

In Nelson's philosophy, there is no finished set of natural numbers. The only numbers that 'exist' for him, are numbers which have been constructed (thus, finitely many, at any given time). By rejecting the 'platonic' existence of the natural numbers as a finished totality, the induction principle also becomes suspect. This is best expressed in the following quote by Nelson himself ([13, p. 1])

The reason for mistrusting the induction principle is that it involves an impredicative concept of number. It is not correct to argue that induction only involves the numbers from 0 to n ; the property of n being established may be a formula with bound variables that are thought of as ranging over all numbers. That is, the induction principle assumes that the natural number system is given. A number is conceived to be an object satisfying every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question.

As an example, take Σ_1 -induction as in (12) where $\varphi(n)$ is $(\exists m)\psi(m, n)$, with $\psi \in \Delta_0$. Even if n only ranges over numbers that have been constructed so far, the existential quantifier $(\exists m)$ may refer to numbers that have not been defined at this point. For this reason, Σ_1 -induction is considered meaningless by Nelson. In general, any statement that potentially refers to numbers that have not been defined at that point, is called 'impredicative' and Nelson only deems predicative (i.e. not impredicative) mathematics to be meaningful. Next, we attempt to formalize this notion of impredicativity. As is to be expected, such formalization requires us to step outside of predicative mathematics.

We work in $I\Delta_0 + \text{NS} + \Sigma_1\text{-TRANS}$. According to Nelson, there are only finitely many numbers available at any given time. Thus, assume that all numbers that are available at this moment in predicative arithmetic are x -finite, for some x . Now consider the following induction axiom, which is essentially $\Sigma_1\text{-IND}$ for ψ , limited to x -finite numbers,

$$[(\exists n)\psi(n, 0) \wedge (\forall^{x\text{-st}} m)((\exists n)\psi(n, m) \rightarrow (\exists n)\psi(n, m+1))] \rightarrow (\forall^{x\text{-st}} m)(\exists n)\psi(n, m). \quad (18)$$

Here, ψ is in Δ_0 and the possible x -standard parameters have been suppressed. Fix a number $c \gg x$. In $I\Delta_0 + \text{NS} + \Sigma_1\text{-TRANS}$, formula 18 is equivalent to

$$[(\exists n \leq c)\psi(n, 0) \wedge (\forall^{x\text{-st}} m)((\exists n \leq c)\psi(n, m) \rightarrow (\exists n \leq c)\psi(n, m+1))] \rightarrow (\forall^{x\text{-st}} m)(\exists n \leq c)\psi(n, m).$$

Although induction for bounded formulas is acceptable in predicative arithmetic, the previous formula is not: the bound c used to bound ‘ $(\exists n)$ ’ is not x -finite and hence this number is not available in predicative arithmetic yet. Thus, we see that in $I\Delta_0 + \text{NS} + \Sigma_1\text{-TRANS}$, the limited Σ_1 -induction axiom (18) indeed refers to numbers which are not available at this point in predicative mathematics and as such, $\Sigma_1\text{-IND}$ is not acceptable in the latter. Again, we stress that the previous steps take us outside of predicative arithmetic, i.e. the formalization of impredicativity goes beyond predicative arithmetic.

Obviously, this generalizes to Σ_n -induction, for all $n \in \mathbb{N}$. However, Σ_{n+1} -induction is also impredicative (in the sense of Nelson) ‘relative’ to Σ_n -induction. Indeed, fix numbers $x \ll c_1 \ll \dots \ll c_{n+1}$. By the reduction theorem, a Σ_{n+1} -formula (with x -finite parameters) is equivalent to a Δ_0 -statement about numbers below c_{n+1} , whereas a Σ_n -formula (with x -finite parameters) is equivalent to a Δ_0 -statement about numbers below c_n . Hence, both $\Sigma_n\text{-IND}$ and $\Sigma_{n+1}\text{-IND}$, limited to x -standard numbers, can be written in a similar equivalent form as the previous centered formula. Thus, even if we regard this limited form of Σ_n -induction (and hence all numbers below c_n) as ‘basic’, the limited form of Σ_{n+1} -induction refers to numbers which are not basic, namely c_{n+1} .

In light of the above, we may also interpret $x \sqsubseteq y$ as ‘ x is available when y is’. This interpretation makes the impredicative character of induction apparent.

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