Reverse-engineering Reverse Mathematics

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Abstract

An important open problem in Reverse Mathematics ([16, 25]) is the reduction of the first-order strength of the base theory from $I\Sigma_1$ to $I\Delta_0 + \exp$. The system ERNA, a version of Nonstandard Analysis based on the system $I\Delta_0 + \exp$, provides a partial solution to this problem. Indeed, Weak König's lemma and many of its equivalent formulations from Reverse Mathematics can be 'pushed down' into ERNA, while preserving the equivalences, but at the price of replacing equality with '$\approx$', i.e. infinitesimal proximity ([19]). The logical principle corresponding to Weak König's lemma is the universal transfer principle from Nonstandard Analysis. Here, we consider the intermediate and mean value theorem and their uniform generalizations. We show that ERNA's Reverse Mathematics mirrors the situation in classical Reverse Mathematics. This further supports our claim from [19] that *the Reverse Mathematics of* ERNA *plus universal transfer is a copy up to infinitesimals of that of* WKL_0. We discuss some of the philosophical implications of our results.

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This paper is dedicated to the courage of everyone who experienced the unforgettable events of March 11, 2011 in Tohoku, Japan.

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1. Introduction: A copy of the Reverse Mathematics of WKL_0

Reverse Mathematics is a program in the Foundations of Mathematics founded in the Seventies by Harvey Friedman ([5, 6]). Stephen Simpson's famous monograph *Subsystems of Second-order Arithmetic* is the standard reference ([25]). The goal of Reverse Mathematics is to determine the *minimal* axiom system necessary to prove a particular theorem of ordinary Mathematics. Classifying theorems according to logical strength reveals the following striking phenomenon: *It turns out that, in many particular cases, if a mathematical theorem is proved from appropriately weak set existence axioms, then the axioms will be logically equivalent to the theorem* ([25, Preface]). This phenomenon is dubbed the 'Main theme' of Reverse Mathematics. The following theorem is a good instance ([25, p. 36]).

Theorem 1 (Reverse Mathematics for WKL_0)**.** *Within* RCA_0*, Weak König's Lemma* (WKL) *is provably equivalent to any of the following statements:*

1. *The Heine-Borel lemma: every covering of* $[0, 1]$ *by a sequence of open intervals has a finite subcovering.*
2. *Every continuous real-valued function on* $[0, 1]$ *is bounded.*
3. *Every continuous real-valued function on* $[0, 1]$ *is uniformly continuous.*
4. *Every continuous real-valued function on* $[0, 1]$ *is Riemann integrable.*
5. *The Weierstraß maximum principle.*
6. *The Peano existence theorem for differential equations* $y' = f(x, y)$*.*
7. *Gödel's completeness theorem for countable languages.*
8. *Every countable commutative ring has a prime ideal.*
9. *Every countable field* (*of characteristic* 0) *has a unique algebraic closure.*
10. *Every countable formally real field is orderable.*
11. *Every countable formally real field has a* (*unique*) *real closure.*
12. *Brouwer's fixed point theorem for* $[0, 1]^n$ *with* $n \geq 2$*.*
13. *The Hahn-Banach theorem for separable Banach spaces.*

Here, the theory WKL_0 is defined as RCA_0 plus Weak König's lemma. Similar theorems exist for the systems ACA_0, ATR_0 and Π^1_1-CA_0 (See [25, Theorem I.9.3, Theorem I.9.4 and Theorem I.11.5]). The aforementioned five theories make up the 'Big Five' systems and RCA_0 is called the 'base theory' of Reverse Mathematics. This is motivated by the surprising observation that, with very few exceptions, a theorem of ordinary mathematics is either provable in RCA_0 or equivalent to one of the other Big Five systems, given RCA_0. Moreover, each of the Big Five systems corresponds to a well-known foundational philosophy (See [25, Table 1, p.43]). We refer to Friedman-Simpson style Reverse Mathematics as 'classical' Reverse Mathematics.

An important open problem is whether Reverse Mathematics can be done in a *weaker* base theory (See e.g. [25, X.4.3], [7, 8], or [16, Section 6.1.2]. Indeed, RCA_0 has the first-order strength of $I\Sigma_1$ and some Reverse Mathematics results are proved in the base theory RCA_0^*, which has roughly the first-order strength of $I\Delta_0$+exp (See [25, X.4]). For ERNA, a version of Nonstandard Analysis based on $I\Delta_0$+exp, we have proved the following theorem. The latter contains several statements, translated from Theorem 1 and [25, IV] into ERNA's language, while preserving equivalence (See [19] for details).

Theorem 2 (Reverse Mathematics for ERNA+Π_1-TRANS)**.** *The theory* ERNA *proves the equivalence between* Π_1-TRANS *and each of the following theorems concerning near-standard functions:*

1. *Every S-continuous function on* $[0,1]$ *is bounded.*
2. *Every S-continuous function on* $[0,1]$ *is continuous there.*
3. *Every S-continuous function on* $[0,1]$ *is Riemann integrable.*
4. *Weierstraß' theorem: every S-continuous function on* $[0,1]$ *has, or attains a supremum, up to infinitesimals.*
5. *The strong Brouwer fixed point theorem: every S-continuous function* $\phi : [0,1] \to [0,1]$ *has a fixed point up to infinitesimals of arbitrary depth.*
6. *The first fundamental theorem of calculus.*
7. *The Peano existence theorem for differential equations* $y' \approx f(x,y)$*.*
8. *The Cauchy completeness, up to infinitesimals, of* ERNA*'s field.*
9. *Every S-continuous function on* $[0,1]$ *has a modulus of uniform continuity.*
10. *The Weierstraß approximation theorem.*

A common feature of the items in the previous theorem is that strict equality has been replaced with $\approx$, i.e. equality up to infinitesimals. This seems the price to be paid for 'pushing down' into ERNA the theorems equivalent to Weak König's lemma. For instance, item (7) from Theorem 2 guarantees the existence of a function $\phi(x)$ such that $\phi'(x) \approx f(x, \phi(x))$, i.e. a solution, up to infinitesimals, of the differential equation $y' = f(x,y)$. However, in general, there is no function $\psi(x)$ such that $\psi'(x) = f(x, \psi(x))$ in ERNA + Π_1-TRANS. In this way, we say that the Reverse Mathematics of ERNA + Π_1-TRANS is a *copy up to infinitesimals* of the Reverse Mathematics of WKL_0. This observation is important, as it suggests that the equivalences proved in Reverse Mathematics are *robust* in the sense this notion is used in the exact sciences. Robustness (i.e. stability under the variation of parameters) is a central notion in the exact sciences. This is discussed in greater detail in Section 7.1.

In this paper, we further explore the connection between classical Reverse Mathematics and ERNA's Reverse Mathematics. In particular, we consider the intermediate value theorem (IVT), the mean value theorem (MVT), and their 'sequential' or 'uniform' generalizations (See e.g. [25, Exercise IV.2.12] or Principles 20 and 29 below). By [25, Theorem II.6.6] and [11, Theorem 4], IVT and MVT can be proved in the base theory RCA_0, whereas the sequential generalizations are equivalent to WKL_0. In Sections 3 and 4, we show that ERNA proves IVT and MVT with '=' replaced with '$\approx$'. Moreover, we show that the 'sequential' or 'uniform' generalizations of IVT and MVT are equivalent to Π_1-TRANS. Inspired by these results, we obtain an entire class of similar results based on sequential generalizations in Section 5 and 6. We discuss some philosophical implications of our results in the concluding Section 7.

Thus, the situation in ERNA's Reverse Mathematics mirrors the situation in classical Reverse Mathematics, modulo the replacement of '=' by '≈'. In this way, we obtain further evidence to support the Main Theme of ERNA's Reverse Mathematics.

2. Preliminaries

For an introduction to ERNA, we refer the reader to [14,19,26]. In this section, we introduce some results concerning ERNA and some notation. Throughout this paper, we tacitly assume that all terms and formulas are free of ERNA's minimum operator. Also, lower and uppercase variables $n, m, k, l, i, j, \ldots$ are always assumed to run over the hypernatural numbers.

2.1. Transfer

In this paragraph, we recall the transfer principle for universal formulas and its properties (See [14,19]). This principle expresses Leibniz' law that the 'same laws' should hold for standard and nonstandard numbers alike.

Principle 3 (Π_1-TRANS)**.** *Let $\varphi(x) \in L^{st}$ be quantifier-free. Then*

$$(\forall^{st} x)\varphi(x) \to (\forall x)\varphi(x). \tag{1}$$

Here, L^{st} is ERNA's language L without the symbols ω, ε, $\approx$ and min. Note that standard parameters are allowed in the formula $\varphi(x)$.

Obviously, the scope of the above principle (also called 'universal transfer' or 'Π_1-transfer') is quite limited. Indeed, a formula cannot be transferred if it contains, for instance, ERNA's exponential function $e^x := \sum_{n=0}^{\omega} \frac{x^n}{n!}$ or similar objects not definable in L^{st}. This is quite a limitation, especially for the development of basic analysis. In [19], the scope of Π_1-transfer was expanded so as to be applicable to objects like ERNA's exponential. We briefly sketch these results here.

First, we label some terms which, though not part of L^{st}, are 'nearly as good' as standard for the purpose of transfer. As in [14, Notation 57] and Notation 12 below, the variable ω' in $(\forall \omega')$ runs over the infinite hypernaturals.

Definition 4. Let the term $\tau(n, \vec{x})$ be standard, i.e. not involve ω or $\approx$. We say that $\tau(\omega, \vec{x})$ is *near-standard* if ERNA proves

$$(\forall \vec{x})(\forall \omega')(\tau(\omega, \vec{x}) \approx \tau(\omega', \vec{x})). \tag{2}$$

An atomic inequality $\tau(\omega, \vec{x}) \leq \sigma(\omega, \vec{x})$ is called near-standard if both members are. Since $x = y$ is equivalent to $x \leq y \wedge x \geq y$, and $\mathcal{N}(x)$ to $\lceil x \rceil = \lfloor x \rfloor$, any internal formula $\varphi(\omega, \vec{x})$ can be assumed to consist entirely of atomic inequalities; it is called near-standard if it is made up of near-standard atomic inequalities.

In [19] and [24], several examples of near-standard terms and formulas are listed. In stronger theories of Nonstandard Analysis, near-standard terms would be converted to *standard* terms by the *standard part map* $\mathrm{st}(x)$ which satisfies $\mathrm{st}(x+\varepsilon) = x$, for $\varepsilon \approx 0$ and standard x. However, ERNA does not have such a map and hence functions of basic analysis, like $e^x := \sum_{n=0}^{\omega} \frac{x^n}{n!}$, are not allowed in Π_1-TRANS. Nonetheless, we can overcome this problem by expanding the scope of Π_1-transfer to near-standard formulas.

Notation 5. We write $a \ll b$ for $a \leq b \wedge a \not\approx b$ and $a \lessapprox b$ for $a \leq b \vee a \approx b$.

See [4, p. 15] for the definition of 'positive' and 'negative' sub-formulas.

Definition 6. Given a near-standard formula $\varphi(\vec{x})$, let $\overline{\varphi}(\vec{x})$ be the formula obtained by replacing every positive (negative) occurrence of a near-standard inequality $\leq$ with $\lessapprox$ ($\ll$).

Now consider the following principle, called 'bar transfer'.

Principle 7 ($\overline{\Pi}_1$-TRANS)**.** *Let $\varphi(x)$ be near-standard and quantifier-free. Then*

$$(\forall^{st} x)\varphi(x) \rightarrow (\forall x)\overline{\varphi}(x). \tag{3}$$

Despite its much wider scope, bar transfer is equivalent to Π_1-transfer.

Theorem 8. *In* ERNA*, the schemas* Π_1-TRANS *and* $\overline{\Pi}_1$-TRANS *are equivalent.*

Proof. For special Π_1-formulas, this was done in [15, §3] with a relatively easy proof. For general Π_1-formulas, the proof becomes significantly more involved (See [19, Theorem 9]). Ironically, we have to resort to ε-δ techniques. □

The following theorem guarantees that near-standard terms are automatically finite for finite arguments. This is surprising, since Definition 4 does not mention the (in)finitude of near-standard terms. Thus, near-standardness seems to be a natural property.

Theorem 9. *A near-standard term $\tau(\vec{x}, \omega)$ is finite for finite $\vec{x}$.*

Proof. This is immediate from Theorem 21 in [24] or Theorem 9 in [19]. □

2.2. Overflow

Here, we introduce the notions 'overflow' and 'underflow'.

Theorem 10. *Let $\varphi(n)$ be an internal quantifier-free formula.*

1. *If $\varphi(n)$ holds for every natural n, it holds for all hypernatural n up to some infinite hypernatural $\overline{n}$* ***(overflow)****.*
2. *If $\varphi(n)$ holds for every infinite hypernatural n, it holds for all hypernatural n from some natural $\underline{n}$ on* ***(underflow)****.*

Both numbers $\overline{n}$ and $\underline{n}$ are given by explicit ERNA*-formulas not involving* min.

Proof. Let ω be some infinite number. For the first item, define

$$\overline{n} := (\mu n \leq \omega)\neg\varphi(n+1), \tag{4}$$

if $(\exists n \leq \omega)\neg\varphi(n+1)$ and ω otherwise. Likewise for underflow. By [14, Theorem 58], the bounded minimum operator is available in ERNA. $\square$

Sometimes, we write $\overline{n}(\omega)$ instead of $\overline{n}$ to emphasize the dependence on ω. The following notations are necessary to keep track of the occurrences of ω in $\overline{n}(\omega)$.

Notation 11. The symbol 'ω' in $\tau(\vec{x}, \omega)$ represents *all* occurrences of ω in $\tau(\vec{x}, \omega)$, i.e. $\tau(\vec{x}, m)$ is $\tau(\vec{x}, \omega)$ with all occurrences of ω replaced by the new variable m.

In particular, let $\varphi(n, \omega)$ be as in Theorem 10 and consider (4). Then $\overline{n}(k)$ corresponds to $(\mu n \leq k)\neg\varphi(n+1, k)$. Similarly, we have the following notation.

Notation 12. The formula '$(\forall\omega)\varphi(\omega)$' is short for $(\forall n)[n \text{ is infinite} \rightarrow \varphi(n)]$. Similarly, '$(\exists\omega)\varphi(\omega)$' is short for $(\exists n)[n \text{ is infinite} \wedge \varphi(n)]$.

2.3. Continuity

In this paragraph, we formulate several notions of continuity inside ERNA and list some fundamental results.

Definition 13 (Continuity)**.** A function $f(x)$ is 'continuous over $[a, b]$' if

$$(\forall x, y \in [a, b])(x \approx y \rightarrow f(x) \approx f(y)). \tag{5}$$

A function $f(x)$ is 'S-continuous over $[a, b]$' if

$$(\forall^{st} k)(\exists^{st} N)(\forall^{st} x, y \in [a, b])(|x - y| < \tfrac{1}{N} \rightarrow |f(x) - f(y)| < \tfrac{1}{k}). \tag{6}$$

A sequence $f_n(x)$ is 'equicontinuous over $[a, b]$' if

$$(\forall^{st} k)(\exists^{st} N)(\forall^{st} n)(\forall^{st} x, y \in [a, b])(|x - y| < \tfrac{1}{N} \rightarrow |f_n(x) - f_n(y)| < \tfrac{1}{k}). \tag{7}$$

The attentive reader has noted that (5), (6) and (7) constitute the uniform versions of (non)standard continuity and equicontinuity. Let us motivate this choice. If we limit the variable x in (5) to $\mathbb{Q}$, the function $\frac{1}{x^2-2}$ satisfies the resulting formula, although this function is infinite in the interval $[-2, 2]$. Similarly, the function $g(x)$, defined as 1 if $x^2 < 2$ and 0 if $x^2 \geq 2$, satisfies (5) with x limited to $\mathbb{Q}$, but this function has a jump in its graph. The same holds for the pointwise ε-δ continuity and thus both are not suitable for our purposes. This explains the use of (5) and (6). We discuss ERNA's version of equicontinuity in more detail below.

Next, we study the connections between ERNA's various notions of continuity.

Theorem 14. *In* ERNA*, for an internal function $f(x)$, continuity, i.e.* (5)*, implies S-continuity, i.e.* (6)*.*

Proof. Assume that (5) holds for an internal function $f(x)$. Fix $k \in \mathbb{N}$ and consider the following internal formula

$$(\forall x, y \in [a,b])\big[(\|x,y\| \leq \omega \wedge |x-y| < 1/n) \to |f(x) - f(y)| < 1/k\big]. \tag{8}$$

By corollary [14, Corollary 53], the above formula is equivalent to a quantifier-free one. By assumption, (8) holds for all infinite n. Hence, by underflow, it holds for all $n \geq N$, for some $N \in \mathbb{N}$. From this, (6) follows immediately. □

Theorem 15. *In* ERNA*, for an internal sequence $f_n(x)$, continuity, i.e.* (5)*, for* all n*, implies equicontinuity, i.e.* (7)*.*

Proof. Assume that (5) holds for every element of the internal sequence $f_n(x)$. Fix $k \in \mathbb{N}$ and n and consider the following internal formula

$$(\forall x, y \in [a,b])\big[(\|x,y\| \leq \omega \wedge |x-y| < 1/m) \to |f_n(x) - f_n(y)| < 1/k\big]. \tag{9}$$

By corollary [14, Corollary 53], the previous formula is equivalent to a quantifier-free one. By assumption, (9) holds for all infinite m. Let $\overline{m}(k,n)$ be the finite number obtained by applying underflow to (9). Note that $\overline{m}(k,n)$ is finite for all n and all $k \in \mathbb{N}$. Let $\overline{m}(k)$ be $\max_{n \leq \omega} \overline{m}(k,n)$. By the previous, $\overline{m}(k)$ is finite for finite k and (9) holds for $m \geq \overline{m}(k)$ and $n \leq \omega$. From this, (7) follows immediately. □

Now consider the following continuity principles.

Principle 16 (Continuity principle)**.** *For a near-standard function $f(x)$, S-continuity implies continuity, i.e.* (6) *implies* (5)*.*

Principle 17 (Equicontinuity principle)**.** *For a near-standard sequence $f_n(x)$, equicontinuity implies continuity for* all n*.*

Theorem 18. *In* ERNA*, the Continuity principle is equivalent to* Π_1-TRANS*.*

Proof. See Theorem 43 in [19]. □

Theorem 19. *In* ERNA*, the Equicontinuity principle is equivalent to* Π_1-TRANS*.*

Proof. Easy adaptation of the proof of Theorem 43 in [19]. □

3. The intermediate value theorem

In this section, we study the well-known intermediate value theorem (IVT) inside ERNA's Reverse Mathematics. By [25, Theorem II.6.6], IVT is provable in RCA_0. Furthermore, the following 'sequential' or 'uniform' version of IVT is equivalent to WKL_0 (See [25, IV.2.12]) and [18].

Principle 20. *If ϕ_n, $n \in \mathbb{N}$, is a sequence of continuous real-valued functions on the closed unit interval $0 \leq x \leq 1$, then there exists a sequence of real numbers x_n, $n \in \mathbb{N}$, $0 \leq x_n \leq 1$ such that $(\forall n)(\phi_n(0) \leq 0 \leq \phi_n(1) \rightarrow \phi(x_n) = 0)$.*

Next, we show that ERNA proves IVT with equality '=' replaced with '≈'. Then, we introduce $\mathbb{IVT}$, ERNA's version of Principle 20, and show that it is equivalent to Π_1-transfer.

Theorem 21 (IVT)**.** *Let f be internal and S-continuous on $[0,1]$. If $f(0) \lessapprox 0$ and $f(1) \gtrapprox 0$, there is an $x_0 \in [0,1]$ such that $f(x_0) \approx 0$.*

Proof. Let f be as in the theorem. If either $f(0) \approx 0$ or $f(1) \approx 0$, we are done. Hence, we may assume $f(0) \ll 0$ and $f(1) \gg 0$. The S-continuity of f implies

$$(\forall^{st} k)(\exists^{st} N > k)(\forall x, y \in [0,1])(\|x,y\| \leq 2^N \wedge |x-y| < \tfrac{1}{N} \rightarrow |f(x)-f(y)| < \tfrac{1}{k}).$$

In the previous formula, replace the quantifiier '$\exists^{st} N$' by '$\exists N \leq \omega$'. By [14, Corollary 52], the resulting formula qualifies for overflow. Let $\overline{k}$ be the infinite number obtained in this way. This yields, for all $k \leq \overline{k}$, that

$$(\exists N \in [k,\omega])(\forall x, y \in [0,1])(\|x,y\| \leq 2^N \wedge |x-y| < \tfrac{1}{N} \rightarrow |f(x)-f(y)| < \tfrac{1}{k}). \tag{10}$$

For $k = \overline{k}$, let N_0 be a witness to the previous formula. Now define $x_i = \frac{i}{2^{N_0}}$ for $i \leq 2^{N_0}$. By the previous, we have $x_i \approx x_{i+1}$ and $f(x_i) \approx f(x_{i+1})$, for $i \leq 2^{N_0} - 1$. As $f(1) \gg 0$, there certainly are $j \leq 2^{N_0}$ such that $f(x_j) > 0$. Using ERNA's bounded minimum (See [14, Theorem 58]), define j_0 as the least $j \leq 2^{N_0}$ such that $f(x_j) > 0$. By definition, we have $f(x_{j_0-1}) \leq 0$, but also $f(x_{j_0}) \approx f(x_{j_0-1})$. Clearly, this implies $f(x_{j_0}) \approx 0$ and we are done. □

From the proof, it is clear that continuity, i.e. (5), is not necessary. Indeed, it suffices to have a grid of points x_i covering $[0,1]$ such that $x_i \approx x_{i+1}$ and $f(x_i) \approx f(x_{i+1})$. The existence of such a grid can be derived from the S-continuity of f.

As noted by Bishop in [3, Preface], there is a preference for uniform versions of continuity, convergence, differentiability, and other notions in constructive analysis. A similar preference seems present in ERNA's Reverse Mathematics. Indeed, comparing the items in Theorem 1 and Theorem 2, we observe that theorems in ERNA's Reverse Mathematics usually assume stronger conditions than their counterparts in the Reverse Mathematics of WKL_0. For instance, standard pointwise continuity is used in item (4) of Theorem 1, whereas item (3) in Theorem 2 uses standard *uniform* continuity. Thus, it should be no surprise that ERNA's version of Principle 20, considered next, requires a condition stronger than continuity, namely equicontinuity. Also note that both in ERNA and constructive analysis, only an approximate version of IVT is proved (See [3]). Other connections between ERNA and constructive analysis are observed in [19, Section 5], [21], [22] and Remark 41.

Now consider the following principle.

Principle 22 ($\mathbb{IVT}$)**.** *Let $f_n(x)$ be near-standard and equicontinuous on $[0,1]$. There exists $g(n) \in [0,1]$ such that $(\forall n)(f_n(0) \lessapprox 0 \lessapprox f_n(1) \to f_n(g(n)) \approx 0)$.*

We have the following theorem.

Theorem 23. *In* ERNA*,* $\mathbb{IVT}$ *is equivalent to* Π_1-TRANS*.*

Proof. For the direction from right to left, assume Π_1-TRANS and let $f_n(x)$ be as in $\mathbb{IVT}$. By Theorem 19, $f_n(x)$ is continuous on $[0,1]$, for each n. By Theorem 14, $f_n(x)$ is also S-continuous on $[0,1]$, for each n. By IVT, for all n, there is an $x_0 \in [0,1]$ such that $f_n(x_0) \approx 0$ if $f_n(0) \lessapprox 0 \lessapprox f_n(1)$. We define $g(n)$ as that $x \in [0,1]$ with $\|x\| \leq \omega$ such that $|f_n(x)|$ is minimal. By [14, Section 5.1 and Corollary 53], this function is available in ERNA. By the previous, $g(n)$ satisfies $f(g(n)) \approx 0$ if $f_n(0) \lessapprox 0 \lessapprox f_n(1)$, for all n.

For the forward direction, assume $\mathbb{IVT}$, let φ be as in Π_1-TRANS and let f_n be as in $\mathbb{IVT}$. Now suppose $\varphi(m)$ holds for all finite m and define the near-standard function $h_n(x)$ as follows:

$$h_n(x) = \begin{cases} f_n(x) & (\forall m \leq \|x, n\|)\varphi(m) \\ k(x) & \text{otherwise} \end{cases}. \tag{11}$$

Here, $k(x)$ is defined as $\frac{3}{4}$ if $x > \frac{1}{2}$ and $-\frac{1}{4}$ if $x \leq \frac{1}{2}$. Note that $k(x)$ satisfies $(\forall x \in [0,1])(k(x) \not\approx 0)$ and $k(0) \ll 0$ and $k(1) \gg 0$. For standard n and $x \in [0,1]$, we have $h_n(x) = f_n(x)$, by the definition of $h_n(x)$ and our assumption that $\varphi(m)$ holds for all finite m. Thus, $h_n(x)$ is also equicontinuous and $\mathbb{IVT}$ applies to this sequence. Let $g(n)$ be the sequence provided by the latter principle. If there were some m_0 such that $\neg\varphi(m_0)$, we would have $h_{m_0}(g(m_0)) = k(g(m_0)) \not\approx 0$, $h_{m_0}(0) = k(0) \ll 0$ and $h_{m_0}(1) = k(1) \gg 0$. However, by $\mathbb{IVT}$, $h_{m_0}(g(m_0)) \approx 0$. This yields a contradiction, implying that the number m_0 cannot exist. Hence, we have $\varphi(m)$ for *all* m. This implies Π_1-TRANS and we are done. □

Note that the number n in the final formula in $\mathbb{IVT}$ runs over *all* numbers, not just the standard ones. This is motivated by Corollary 24 below, provable in ERNA + Π_1-TRANS. By the former, $\mathbb{IVT}$ produces a *near-standard* term $f_n(g(n))$ if we have $(\forall^{st} n)(f_n(0) \lessapprox 0 \lessapprox f_n(1))$. By Theorem 8, such a term may appear in the formula φ in bar transfer. Hence, near-standard terms are the input *and* the output of $\mathbb{IVT}$, i.e. it is a 'closed circuit'. In ERNA, this cannot always be guaranteed, see e.g. the Bolzano-Weierstraß theorem in [20].

Corollary 24. *Let $f_n(x)$ be as in $\mathbb{IVT}$. If $(\forall^{st} n)(f_n(0) \lessapprox 0 \lessapprox f_n(1))$, then the term $f_n(g(n))$ is near-standard.*

Proof. Let $f_n(x)$ be as in $\mathbb{IVT}$ and assume $(\forall^{st} n)(f_n(0) \lessapprox 0 \lessapprox f_n(1))$. The latter formula implies $(\forall^{st} n, k)\big(f_n(0) \leq \frac{1}{k} \wedge f_n(1) \geq -\frac{1}{k}\big)$. As $f_n(x)$ is near-standard, we may apply bar transfer, implying $(\forall n, k)\big(f_n(0) \lessapprox \frac{1}{k} \wedge f_n(1) \gtrapprox -\frac{1}{k}\big)$ For $k = \omega$, this implies $(\forall n)\big(f_n(0) \lessapprox 0 \wedge f_n(1) \gtrapprox 0\big)$. Hence, by $\mathbb{IVT}$, there is a function $g(n)$ such that $f_n(g(n)) \approx 0$, for all n. Note that for any other choice of the infinite number ω, the term $f_n(g(n))$ still satisfies the latter formula. Hence, this term is near-standard and we are done. □

In light of Theorem 19, we can expect some flexibility in the conditions of $\mathbb{IVT}$. For instance, consider the following principle and reversal.

Principle 25 ($\mathfrak{B}$)**.** *Let m be infinite and $f_n(x)$ be near-standard and continuous on $[0,1]$ for $n \leq m$. There is $g(n) \in [0,1]$ s.t. $(\forall n)(f_n(0) \lessapprox 0 \lessapprox f_n(1) \to f_n(g(n)) \approx 0)$.*

Theorem 26. *In* ERNA*, $\mathfrak{B}$ is equivalent to* Π_1-TRANS.

Proof. For the reverse direction, let $f_n(x)$ be as in $\mathfrak{B}$. In particular, let m_0 be such that $f_n(x)$ is continuous on $[0,1]$ for $n \leq m_0$. Now fix $k \in \mathbb{N}$ and consider the following internal formula $\Phi(N)$

$$(\forall x, y \in [0,1])(\forall n \leq m_0)\big[(\|x,y\| \leq \omega \wedge |x-y| < 1/N) \to |f_n(x)-f_n(y)| < 1/k\big].$$

By the continuity of $f_n(x)$, $\Phi(N)$ is true for all infinite N. By [14, Corollary 53], $\Phi(N)$ qualifies for underflow. Using underflow, we obtain

$$(\forall^{st} k)(\exists^{st} N)(\forall^{st} x, y \in [0,1])(\forall^{st} n)\big[|x-y| < 1/N \to |f_n(x) - f_n(y)| < 1/k\big].$$

Hence, $f_n(x)$ is equicontinuous on $[0,1]$ and this case follows from Theorem 23.

For the forward direction, let φ be as in Π_1-TRANS and let f_n be as in $\mathfrak{B}$. In particular, let m_1 be an infinite number such that $f_n(x)$ is continuous for $n \leq m_1$. Now assume $(\forall^{st} m)\varphi(m)$ and apply overflow to obtain an infinite m_2 such that $\varphi(m)$ for $m \leq m_2$. Let m_0 be the least of m_1 and m_2 and let $h_n(x)$ be as in (11), but with '$\|x,n\|$' replaced by 'n'.

For $n \leq m_0$ and $x \in [0,1]$, we have $h_n(x) = f_n(x)$, by assumption. Thus, $\mathfrak{B}$ applies to $h_n(x)$ and let $g(n)$ be as provided by the former principle. If there were some n_0 such that $\neg\varphi(n_0)$, we would have $h_{n_0}(g(n_0)) = k(g(n_0)) \not\approx 0$, $h_{n_0}(0) = k(0) \ll 0$ and $h_{n_0}(1) = k(1) \gg 0$. However, this contradicts $h_{n_0}(g(n_0)) \approx 0$ and hence $\varphi(n)$ must hold for *all* n. This yields Π_1-TRANS and we are done. □

The previous theorem seems false if we replace '$n \leq m$' with '$n \in \mathbb{N}$' in $\mathfrak{B}$. However, we *can* replace continuity with S-continuity. Indeed, consider the following principle and reversal.

Principle 27 ($\mathfrak{C}$)**.** *Let m be infinite and let $f_n(x)$ be near-standard and S-continuous on $[0,1]$ for $n \leq m$. There exists $g(n) \in [0,1]$ such that $(\forall n)(f_n(0) \lessapprox 0 \lessapprox f_n(1) \to f_n(g(n)) \approx 0)$.*

Theorem 28. *In* ERNA*, $\mathfrak{C}$ is equivalent to* Π_1-TRANS.

Proof. The forward direction is essentially the same as in the proof of Theorem 26. For the reverse direction, let $f_n(x)$ and m be as in $\mathfrak{C}$. The S-continuity of the sequence f_n implies that for all $n \leq m$

$$(\forall^{st} k)(\exists^{st} N > k)(\forall x, y \in [0,1])(\|x,y\| \leq 2^N \wedge |x-y| < \tfrac{1}{N} \to |f_n(x)-f_n(y)| < \tfrac{1}{k}).$$

By [14, Theorem 58], there is a function $h(k,n)$ that computes the number N in the previous formula. Define $g(k)$ as $\max_{n\leq m} h(k,n)$. Note that $g(k)$ is finite for finite k, as $h(k,n)$ is finite for finite k and any $n \leq m$. We have

$$(\forall^{st} k)(\forall n \leq m)(\forall x, y \in [0,1])$$
$$(g(k) > k \wedge \|x,y\| \leq 2^{g(k)} \wedge |x-y| < \tfrac{1}{g(k)} \to |f(x) - f(y)| < \tfrac{1}{k}).$$

As $g(k)$ does not depend on n, the previous implies that for all finite k

$$(\exists^{st} N > k)(\forall n \leq m)(\forall x, y \in [0,1])(\|x,y\| \leq 2^N \wedge |x-y| < \tfrac{1}{N} \to |f_n(x) - f_n(y)| < \tfrac{1}{k}).$$

Finally, by weakening, we obtain that for all finite k, there is finite $N > k$ s.t.

$$(\forall^{st} n)(\forall x, y \in [0,1])(\|x,y\| \leq 2^N \wedge |x-y| < \tfrac{1}{N} \to |f_n(x) - f_n(y)| < \tfrac{1}{k}).$$

Now apply transfer and pull the quantifier $(\forall n)$ through the existential quantifier $(\exists^{st} N > k)$. Hence, we have, for all n,

$$(\forall^{st} k)(\exists^{st} N > k)(\forall x, y \in [0,1])(\|x,y\| \leq 2^N \wedge |x-y| < \tfrac{1}{N} \to |f_n(x) - f_n(y)| < \tfrac{1}{k}).$$

The rest of the proof is identical to that of Theorem 21, with the exception that the numbers $\overline{k}$, N_0, and j_0 now depend on n. However, the proof still goes through. □

A sketch of the previous proof is as follows: First of all, in the definition of continuity, bound the quantifier $(\forall x, y)$ using the condition $\|x,y\| \leq 2^N$ as in IVT. Secondly, push the quantifier $(\forall n \leq m)$ through $(\exists^{st} N > k)$. Thirdly, apply transfer to the former quantifier and pull it back out. Finally, the proof of IVT goes through for the resulting formula, for all n.

It seems that the condition on $f_n(x)$ in $\mathfrak{C}$ is weaker than equicontinuity, but we do not have a proof of this.

4. The mean value theorem

In this section, we study the well-known mean value theorem (MVT) inside ERNA's Reverse Mathematics. By [11, Theorem 4], MVT is provable in RCA_0. Furthermore, the following 'sequential' or 'uniform' version of MVT is equivalent to WKL_0. This is due to Takeshi Yamazaki, unpublished. In [12], a number of similar sequential principles are considered.

Principle 29. *Let ϕ_n be a sequence of functions, continuous on $[0,1]$, differentiable on $(0,1)$, and such that $(\forall n)(\phi_n(0) = \phi_n(1) = 0)$. There is a sequence x_n in $[0,1]$ such that $\phi_n'(x_n) \approx 0$, for all $n \in \mathbb{N}$.*

In ERNA, we will use the following definitions of differentiability, to be compared to [11, Definition 3] and [3, Definition 5.1]. We write '$\Delta_h f(x)$' for $\frac{f(x+h)-f(x)}{h}$.

Definition 30. [S-differentiability] A function f is 'S-differentiable over (a,b)' if there is a finite-valued function g such that for $a \ll c \ll d \ll b$

$$(\forall^{st} k)(\exists^{st} N)(\forall^{st} h)(\forall^{st} x \in [c,d])\big[0 < |h| < \tfrac{1}{N} \to |\Delta_h f(x) - g(x)| < \tfrac{1}{k}\big]. \quad (12)$$

Definition 31. [Differentiability] A function f is 'differentiable over (a,b)' if $\Delta_\varepsilon f(x) \approx \Delta_{\varepsilon'} f(x)$ is finite for all nonzero $\varepsilon, \varepsilon' \approx 0$ and all $a \ll x \ll b$.

Using underflow, it is easy to prove that differentiability implies S-differentiability. Moreover, using Π_1-transfer, S-differentiability implies differentiability. Thus, the function g in Definition 30 is called the 'derivative' of f and is denoted f'. In case of a differentiable function, f' can be taken to be any term $\Delta_\varepsilon f$ with $\varepsilon \approx 0$. Note that the derivative is only unique up to infinitesimals.

Before we can consider ERNA's version of MVT or Principle 29, we need to establish some properties of differentiable functions. As in Bishop's constructive analysis, ERNA uses *uniform* notions of differentiability. Hence, ERNA's derivative will have stronger properties, as witnessed by the following theorem. A function is said to be 'continuous over (a,b)' if it satisfies (5) for all $a \ll x, y, \ll b$.

Theorem 32. *If f is differentiable over (a,b), then $f'(x)$ is cont. over (a,b).*

Proof. Choose points $x \approx y$ such that $a \ll x < y \ll b$. If $|x-y| = \varepsilon \approx 0$, then

$$\Delta_\varepsilon f(x) = \tfrac{f(x+\varepsilon)-f(x)}{\varepsilon} = \tfrac{f(y)-f(y-\varepsilon)}{\varepsilon} = \tfrac{f(y-\varepsilon)-f(y)}{-\varepsilon} = \Delta_{-\varepsilon} f(y) \approx \Delta_\varepsilon f(y).$$

This implies $f'(x) \approx f'(y)$ and we are done. ☐

Since the derivative is only defined up to infinitesimals in ERNA, the statement $f'(x) > 0$ is not very strong, as $f'(x) \approx 0$ may also hold. Similarly, $f(x) < f(y)$ is consistent with $f(x) \approx f(y)$ and we need stronger forms of inequality to express meaningful properties of functions and their derivatives.

Definition 33. A function f is $\ll$-increasing over an interval $[a,b]$, if for all $x, y \in [a,b]$ we have $x \ll y \to f(x) \ll f(y)$. Likewise for $\ll$-decreasing.

Theorem 34. *If f is differentiable over (a,b), there is an $N \in \mathbb{N}$ such that*

1. *if $f'(x_0) \gg 0$, then f is $\ll$-increasing in $[x_0 - \frac{1}{N}, x_0 + \frac{1}{N}]$,*
2. *if $f'(x_0) \ll 0$, then f is $\ll$-decreasing in $[x_0 - \frac{1}{N}, x_0 + \frac{1}{N}]$,*

for all $a \ll x_0 \ll b$.

Proof. For the first item, $f'(x_0) \gg 0$ implies $f(y) > f(z)$ for all y, z satisfying $y, z \approx x_0$ and $y > z$. Fix an infinite number ω_1 and let $M \gg 0$ be $f'(x_0)/2$. By the previous, the following sentence is true for all infinite hypernaturals N:

$$(\forall y, z)\big[\|y,z\| \le \omega_1 \wedge y > z \wedge |x_0 - z| < \tfrac{1}{N} \wedge |x_0 - y| < \tfrac{1}{N} \to f(y) > f(z) + M(y-z)\big].$$

By [14, Corollary 53], the previous formula is equivalent to a quantifier-free one. Applying underflow yields the first item, as f is continuous over (a,b). Likewise for the second item. ☐

Now we are ready to prove ERNA's version of the mean value theorem. A function is said to be 'continuous at a' if (5) holds for $x = a$.

Theorem 35. *If f is differentiable over (a, b) and continuous in a and b, then there is an $x_0 \in [a, b]$ such that $f'(x_0) \approx \frac{f(b)-f(a)}{b-a}$.*

Proof. Let f be as in the theorem. First, we prove the particular case where $f(a) \approx f(b)$. By [19, Theorem 12], f attains its maximum (up to infinitesimals), say in x_0, and its minimum (idem), say in x_1, over $[a, b]$. If $f(x_0) \approx f(x_1) \approx f(a)$, then f is constant up to infinitesimals. By Theorem 34 we have $f'(x) \approx 0$ for all $a \ll x \ll b$. If $f(x_0) \not\approx f(a)$, then by Theorem 34 we have $f'(x_0) \approx 0$. The case $f(x_1) \not\approx f(a)$ is treated in a similar way. The general case can be reduced to the particular case by using the function $F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$. □

Theorem 36 (MVT)**.** *If f is S-differentiable over (a, b) and S-continuous in a and b, then there is an $x_0 \in [a, b]$ such that $f'(x_0) \approx \frac{f(b)-f(a)}{b-a}$.*

Proof. The proof of MVT is a straightforward, but long and tedious, adaptation of the proof of Theorem 35. Thus, we only provide a sketch.

First of all, we change (12) in the same way as (6) is changed in the proof of Theorem 21 using the bound 2^N. Then, we obtain a version of Weierstraß' extremum theorem for S-continuous functions where the weight of x in the conclusion is bounded. A similar theorem can be found for Theorem 34. Using these theorems, the proof of Theorem 35 can be adapted to suit S-differentiable functions. □

As noted before $\mathbb{IVT}$, the latter principle uses a stronger condition, namely equicontinuity, than Principle 20, the sequential version of IVT. Similarly, for ERNA's version of Principle 29, the sequential version of MVT, we need a stronger notion of differentiability.

Definition 37. [Equidifferentiability] A sequence $f_n(x)$ is 'equidifferentiable on (a, b)' if there is a finite-valued sequence g_n such that for $a \ll c \ll d \ll b$

$$(\forall^{st} k)(\exists^{st} N)(\forall^{st} h, n)(\forall^{st} x \in [c, d])\big[0 < |h| < \tfrac{1}{N} \to |\Delta_h f_n(x) - g_n(x)| < \tfrac{1}{k}\big]. \tag{13}$$

Surprisingly, this definition actually occurs in mathematical practice here and there (See [2,17,28]). As for equicontinuity, the equidifferentiability of f_n is equivalent to the differentiability of f_n, for all n. We can now formulate ERNA's version of Principle 29.

Principle 38 ($\mathbb{MVT}$)**.** *Let f_n be equidifferentiable over (a, b) and S-continuous in a and b. There exists $g(n) \in [a, b]$ s.t., for all n and $\varepsilon \approx 0$, $\Delta_\varepsilon f_n(g(n)) \approx \frac{f_n(b)-f_n(a)}{b-a}$.*

Theorem 39. *In* ERNA*,* $\mathbb{MVT}$ *is equivalent to* Π_1-TRANS*.*

Proof. For the reverse direction, assume Π_1-TRANS and let $f_n(x)$ be as in $\mathbb{MVT}$. It is easy to show that $f_n(x)$ is differentiable for all n. By Theorem 35, for every n, there is some $z \in [a,b]$ such that $f_n'(z) \approx \frac{f_n(b)-f_n(a)}{b-a}$. Put $x_i = \frac{i}{\omega}$, fix $\varepsilon \approx 0$ and define $g(n)$ as that $i \leq \omega$ such that $\left|\Delta_\varepsilon f_n(x_i) - \frac{f_n(b)-f_n(a)}{b-a}\right|$ is minimal. By [14, Section 5.1 and Corollary 53], this function is available in ERNA. By the previous, $g(n)$ satisfies the required condition.

For the forward direction, assume $\mathbb{MVT}$, let φ be as in Π_1-TRANS and let f_n be as in $\mathbb{MVT}$. For simplicity, put $a = 0$, $b = 1$, and $f_n(0) \approx f_n(1) \approx 0$ for all n. Now suppose $\varphi(m)$ holds for all finite m and define the near-standard function $h_n(x)$ as follows:

$$h_n(x) = \begin{cases} f_n(x) & (\forall m \leq \|n, x\|)\varphi(m) \\ z(x,n) & \text{otherwise} \end{cases}. \tag{14}$$

Here, $z(x,n)$ is any function which is not differentiable at $x = \frac{1}{2}$ for infinite n. For instance, $z(x,n)$ could be a certain instance of the well-known Koch curve at some limit stage, i.e. for some infinite n. Note that $h_n(x)$ is continuous at 0 and at 1, and that $z(x,n)$ is not differentiable for $x = \frac{1}{2}$. For standard n and $x \in [0,1]$, we have $h_n(x) = f_n(x)$, by the definition of $h_n(x)$ and our assumption that $\varphi(m)$ holds for all finite m. Thus, $h_n(x)$ is also equidifferentiable and $\mathbb{MVT}$ applies to this sequence. Let $g(n)$ be such that $h_n'(g(n)) \approx 0$, for all n. If there were some m_0 such that $\neg\varphi(m_0)$, we would have $h_{m_0}(g(m_0)) = z(g(m_0), m_0)$. However, by $\mathbb{MVT}$, $h_{m_0}'(g(m_0)) \approx 0$. This yields a contradiction, implying that the number m_0 cannot exist. Hence, we have $\varphi(m)$ for *all* m, not just the finite numbers. This implies Π_1-TRANS and we are done. □

Let $\mathfrak{D}$ (resp. $\mathfrak{E}$) be $\mathbb{MVT}$ with 'equidifferentiable over (a,b)' replaced by 'differentiable over (a,b) for $n \leq m$, for some infinite m' (resp. 'S-differentiable for $n \leq m$ over (a,b), for some infinite m'). As for IVT, we have the following theorem.

Theorem 40. *In* ERNA*,* $\mathbb{MVT}$ *is equivalent to* $\mathfrak{D}$ *and to* $\mathfrak{E}$*.*

Proof. Similar to the proofs of Theorems 26 and 28. □

We end this section with a note on differentiability and a preliminary conclusion.

Remark 41. In the weaker theories of Reverse Mathematics, the notion of differentiability can be quite subtle. For instance, the existence of $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ does not guarantee the existence of the derivative $f'(x)$ in RCA_0. In particular, for continuously differentiable functions, the existence of $f'(x)$ is equivalent to ACA_0 ([27, Theorem 3.8]). For ERNA, consider the following natural candidate for a definition of differentiability.

$$(\forall^{st} k)(\exists^{st} N)(\forall^{st} h, h')(\forall^{st} x \in [c,d])\big[|h - h'| < \tfrac{1}{N} \to |\Delta_h f(x) - \Delta_{h'} f(x)| < \tfrac{1}{k}\big]. \tag{15}$$

Nonetheless, it seems difficult to extract a derivative $f'(x)$ in ERNA from the previous formula. Moreover, the statement *a function satisfying* (15) *is differentiable* is equivalent to Π_1-TRANS, implying that $\Delta_\varepsilon f(x)$ is not a good derivative in ERNA. Thus, we choose (12), inspired by Bishop's definition ([3, Definition 5.1]). Finally, in [21], we pointed out a connection between Π_1-TRANS and ACA_0. The above indicates a further correspondence.

In the previous two sections, we showed that ERNA's Reverse Mathematics mirrors the situation in classical Reverse Mathematics when it comes to IVT, MVT and their uniform generalizations. These are examples of the following schema.

Schema 42. Let T be a theorem of ordinary mathematics asserting the existence of a solution x to a problem P. Let $\mathbb{T}$ be the statement that there is a certain *sequence* x_n of solutions to the sequence of problems P_n. If T is provable in the base theory, then $\mathbb{T}$ is equivalent to the next system[1] of Reverse Mathematics.

In the following sections, we observe several other examples of this schema in ERNA's Reverse Mathematics.

5. The integral mean value theorem

In this section, we investigate the integral mean value (IMV) theorem (See [13, Theorem 21.96]) inside ERNA's Reverse Mathematics. In particular, we show that IMV conforms to the situation described in Schema 42. For details concerning integration in ERNA, we refer to [19, Section 3.1].

First of all, we prove the following theorem inside ERNA.

Theorem 43 (IMV)**.** *On $[a,b]$, let f be continuous and let g be integrable. If g is non-negative on $[a,b]$, there exists $c \in [a,b]$ such that*

$$\textstyle\int_a^b f(x)g(x)\,dx \approx f(c)\int_a^b g(x)\,dx.$$

Proof. By ERNA's version of the Weierstraß extremum theorem ([19, Theorem 12]), there exists $c, d \in [a,b]$ such that $f(c) \lessapprox f(x) \lessapprox f(d)$, for all $x \in [a,b]$. This implies

$$f(c)J \lessapprox \int_a^b f(x)g(x)\,dx \lessapprox f(d)J, \tag{16}$$

where $J = \int_a^b g(x)\,dx$. If $J \approx 0$, the theorem follows, as $f(c)$ and $f(d)$ are finite. If $J \not\approx 0$, then (16) implies

$$f(c) \lessapprox \frac{1}{J}\int_a^b f(x)g(x)\,dx \lessapprox f(d).$$

By IVT, there exists $e \in [a,b]$ such that $f(e) \approx \frac{1}{J}\int_a^b f(x)g(x)\,dx$. □

[1] Here, the 'next system' is meant in terms of increasing logical strength.

As an aside, we prove the following reversal. Let $\mathfrak{IMV}$ be IMV with 'continuous' replaced by 'S-continuous and near-standard'.

Corollary 44. *In* ERNA*,* $\mathfrak{IMV}$ *is equivalent to* Π_1-TRANS*.*

Proof. Immediate from Theorems 43 and 50 in [19]. □

In the context of classical Reverse Mathematics, it is easy to show that IMV, limited to uniformly continuous functions, is provable in RCA_0 and that IMV limited to pointwise continuous functions is equivalent to WKL_0.

We now define ERNA's sequential version of IMV.

Principle 45 ($\mathbb{IMV}$)**.** *On* $[a, b]$*, let the near-standard* f_n *be equicontinuous and let* g *be integrable. If* g *is non-negative on* $[a, b]$*, there exists* $h(n) \in [a, b]$ *such that*

$$(\forall n) \left[\int_a^b f_n(x)g(x)\,dx \approx f_n(h(n)) \int_a^b g(x)\,dx\right].$$

Theorem 46. *In* ERNA*,* $\mathbb{IMV}$ *is equivalent to* Π_1-TRANS*.*

Proof. The forward implication is immediate from [19, Theorem 50]. For the reverse direction, assume Π_1-TRANS and let $f_n(x)$ and $g(x)$ be as in $\mathbb{IMV}$. In particular, assume that g is non-negative on $[a, b]$. By Theorem 19, $f_n(x)$ is continuous on $[0, 1]$, for each n. By IVM, we have that for all n, there is an $z_0 \in [0, 1]$ such that $\int_a^b f_n(x)g(x)\,dx \approx f_n(z_0) \int_a^b g(x)\,dx$. We define $h(n)$ as that $z \in [0, 1]$ with $\|z\| \leq \omega$ such that $\left|\int_a^b f_n(x)g(x)\,dx - f_n(z) \int_a^b g(x)\,dx\right|$ is minimal. By [14, Section 5.1 and Corollary 53], this function is available in ERNA. By the previous, $h(n)$ satisfies $\int_a^b f_n(x)g(x)\,dx \approx f_n(h(n)) \int_a^b g(x)\,dx$, for all n. □

It should be straightforward to prove that a suitable version of $\mathbb{IMV}$ is equivalent to WKL_0. Moreover, let $\mathfrak{F}$ be $\mathbb{IMV}$ with 'equicontinous over $[a, b]$' replaced by 'continuous over $[a, b]$ for $n \leq m$, for some infinite m'. As for IVT and MVT, we have the following theorem.

Theorem 47. *In* ERNA*,* $\mathbb{IMV}$ *is equivalent to* $\mathfrak{F}$*.*

Proof. Similar to the proofs of Theorems 26 and 28. □

6. Et Sequentia

In this section, we consider several more theorems that conform to the situation described in Schema 42. Furthermore, we sketch an informal procedure for generating such theorems.

6.1. The Weierstraß Extremum Theorem

Here, we consider ERNA's version of the Weierstraß extremum theorem. By [19, Theorem 12], the following theorem is provable in ERNA.

Theorem 48 (WEI)**.** *If f is continuous over $[a, b]$, there is a number $c \in [a, b]$ such that for all $x \in [a, b]$, we have $f(x) \lessapprox f(c)$.*

The previous theorem yields the following principle.

Principle 49 ($\mathbb{WEI}$)**.** *Let f_n be equicontinuous on $[a, b]$ and near-standard. Then there exists $g(n) \in [a, b]$ such that $(\forall x \in [a, b])(|f_n(x)| \lessapprox |f_n(g(n))|)$, for all n.*

Theorem 50. *In* ERNA*,* $\mathbb{WEI}$ *is equivalent to* Π_1-TRANS.

Proof. For the forward implication, note that $\mathbb{WEI}$ reduces to the Weierstraß extremum principle for $n = 1$ (See [19, Principle 44]). By [19, Theorem 45], this principle is equivalent to Π_1-TRANS.

For the inverse implication, let f_n be as in $\mathbb{WEI}$. By Theorem 17, f_n is continuous over $[a, b]$, for all n. By WEI, we have that for all n, there is a $c \in [0, 1]$ such that $(\forall x \in [a, b])(|f_n(x)| \lessapprox |f_n(c)|)$. We define $g(n)$ as that $x \in [a, b]$ with $\|x\| \leq \omega$ such that $|f_n(x)|$ is maximal. By [14, Section 5.1 and Corollary 53], this function is available in ERNA. By the previous, we have $(\forall x \in [a, b])(|f_n(x)| \lessapprox |f_n(g(n))|)$, for all n. □

6.2. The Peano Existence Theorem

Here, we consider ERNA's version of the Peano existence theorem. By [19, Theorem 31], the following theorem is provable in ERNA.

Theorem 51 (PEA)**.** *Let $f(x, y)$ be continuous on the rectangle $|x| \leq a$, $|y| \leq b$, let M be a finite upper bound for $|f|$ there and let $\alpha = \min(a, b/M)$. Then there is a function ϕ, S-differentiable for $|x| \leq \alpha$, such that*

$$\phi(0) = 0 \text{ and } \phi'(x) \approx f(x, \phi(x)). \tag{17}$$

The previous theorem gives rise to the following principle.

Principle 52 ($\mathbb{PEA}$)**.** *Let $f_n(x, y)$ be near-standard and equicontinuous for $|x| \leq a$, $|y| \leq b$, let M_n be a finite upper bound for $|f_n|$ there and let $\alpha_n = \min(a, b/M_n)$. There is a sequence ϕ_n, S-differentiable for $|x| \leq \alpha_n$ and all n, such that*

$$\phi_n(0) = 0 \text{ and } \phi_n'(x) \approx f_n(x, \phi_n(x)). \tag{18}$$

We have the following reversal.

Theorem 53. *In* ERNA*,* $\mathbb{PEA}$ *is equivalent to* Π_1-TRANS.

Proof. For the forward implication, note that $\mathbb{PEA}$ reduces to the Peano existence theorem for $n = 1$ (See [19, Theorem 31]). By [19, Theorem 54], this principle is equivalent to Π_1-TRANS.

For the inverse implication, let f_n be as in $\mathbb{PEA}$. By Theorem 17, f_n is continuous over $[a, b]$, for all n. By PEA, we have that for all n, there is an S-differentiable function $\phi_n(x)$ such that $\phi_n(0) = 0$ and $\phi'_n(x) \approx f_n(x, \phi_n(x))$ for $|x| \leq \alpha_n$. Moreover, the function $\phi_n(x)$ is given by an explicit formula (See [19, Formula (22)]). □

Note the final sentence of the previous proof: Like in constructive analysis, the existence of a mathematical object in ERNA (in general) comes with a procedure to construct it.

6.3. A general schema

From the previous paragraphs, it should be clear that there is a general schema underlying the examples considered hitherto. Thus, we sketch a procedure for generating theorems that conform to Schema 42 in ERNA's Reverse Mathematics.

Procedure 54.

1. Find a theorem $T(=)$ of ordinary Mathematics that states the existence of a solution x to a problem $P(=)$ involving equality.
2. Replace equality '=' by '≈' to obtain $T(\approx)$.
3. If necessary, change the conditions of $T(\approx)$ to make it provable (or meaningful) in ERNA.
4. Let $\mathbb{T}$ be the sequential version of $T(\approx)$, i.e. the statement *there is a sequence x_n of solutions to $P(\approx)$*.
5. In $\mathbb{T}$, introduce equicontinuity or similar conditions.

Then ERNA proves that $\mathbb{T}$ is equivalent to Π_1-TRANS.

Now, it is an easy exercise to consider the Weierstraß approximation theorem (See [19, Section 4.5]) in this context.

To conclude this section, we list a possible interpretation of $\mathbb{IVT}$ and other theorems conforming to Schema 42. As mentioned in the latter, such theorems state the existence of a *sequence* of solutions x_n to a collection of problems P_n. In the case of ERNA, the objects x_ω are also solutions to P_ω for infinite ω. Classically, one would say that 'x_n still satisfies P_n after taking the limit $n \to \infty$'. Thus, a possible interpretation of $\mathbb{IVT}$, and similar principles, is that -under certain conditions- if ${}^*x$ and ${}^*P$, the limits of x_n and P_n for $n \to \infty$, are somehow meaningful, then ${}^*x$ is still a solution to ${}^*P$. In other words, as long as the limits ${}^*x$ and ${}^*P$ are meaningful, the limit $n \to \infty$ can be taken for x_n and P_n without problems. The latter is a typical example of the informal reasoning in Physics where operations such as limits are performed without much mathematical rigor, as long as the end result is physically meaningful.

7. Conclusion

In this section, we formulate some concluding remarks to this paper.

7.1. Robust Reverse Mathematics

The main goal of Reverse Mathematics is to identify the *minimal* axioms that prove a certain theorem of *ordinary* Mathematics. As Theorem 1 shows, in many cases, the minimal axioms are also equivalent to the theorem at hand, given some base theory. Historically, the framework of *second-order arithmetic* is used to formalize ordinary Mathematics and to carry out the program of Reverse Mathematics ([25]). While second-order arithmetic is generally agreed upon to be the *right* system to formalize (countable or countably dense) Mathematics, the question nonetheless remains whether the observations made in Reverse Mathematics (e.g. the Main Theme) depend somehow on the formalization or framework used.

In this paper, we have gathered evidence in support of the thesis that no such dependence exists. Indeed, by Theorem 2, many of the equivalences belonging to the Reverse Mathematics of WKL_0 remain valid when changing the framework to Nonstandard Analysis with ERNA as a base theory, provided the replacement of '$=$' by '$\approx$'. Thus, we observe similar equivalences in a framework very different from second-order arithmetic. From another point of view, these equivalences are even observed to be *robust*, i.e. stable under variations of parameters. Indeed, the introduction of an infinitesimal error does not change the essential meaning of the observation that many theorems of ordinary Mathematics are equivalent (either to WKL_0 or Π_1-TRANS). In this paper, we have demonstrated that ERNA's Reverse Mathematics mirrors classical Reverse Mathematics when it comes to IVT, MVT and their sequential generalizations, modulo the replacement of '$=$' by '$\approx$'. Thus, we have contributed to showing that the equivalences of Reverse Mathematics are indeed robust.

A subsequent natural question is whether it is possible to construct a general procedure that translates equivalences from classical Reverse Mathematics to ERNA's Reverse Mathematics (and vice versa). Although it is clear in many instances how to translate theorems while preserving equivalences between them, our experience and intuition suggest that no such procedure exists. We now discuss two reasons why this need not be problematic. Note that such discussion is inherently vague, but, in our opinion, meaningful to the above.

First of all, a similar observation can be made for Reverse Mathematics. Indeed, once a given kind of theorem $\mathcal{T}$ is established to be equivalent to some logical principle $\mathcal{A}$, it is usually a generic[2] exercise to find many similar theorems $\mathcal{T}'$, $\mathcal{T}''$,... which are also equivalent to $\mathcal{A}$. Nonetheless, there is no general procedure that takes a theorem of ordinary Mathematics as input and produces a

[2]Sometimes, it is colloquially said that *Once you've seen one reversal, you've seen them all.* Note that the author does not share this opinion.

proof of equivalence to some logical principle. The existence of such a procedure seems highly doubtful, as it would provide us with a kind of formal criterion concerning *ordinary Mathematics*, an inherently vague concept.

Secondly, the notion of robustness (i.e. invariance under variations of parameters) seems to involve syntax *and* semantics. While a syntactical translation is (more or less) a literal transposition, a robust translation connects two syntactical systems (different due to some variation in parameters) that still have (approximately) the same semantical behaviour. Thus, it seems doubtful that there might be a finite procedure (providing a syntactical translation) connecting classical and ERNA's Reverse Mathematics. We finish this paragraph with two clarifying examples.

As discussed above, the comparison of Theorems 1 and 2 provides us with an example of robust behaviour: although syntactically different, both theorems carry the same meaning: they express that theorems of ordinary Mathematics are equivalent to a logical principle. An example of *non-robust* behaviour is provided by [1]. In this paper, the authors construct a pair of computable random variables (X, Y) in the unit interval whose conditional distribution $P[Y|X]$ encodes the halting problem. However, they also show that the introduction of a small perturbation, such as independent absolutely continuous noise, results in a *computable* conditional distribution. Thus, the non-computability of $P[Y|X]$ is not a robust phenomenon: a small variation (the introduction of noise) breaks the non-computability. In other words, the introduction of noise causes a sharp *phase transition* in the semantical behaviour of the conditional distribution (i.e from non-computable to computable).

Finally, the previous example suggests the value of robust models in the exact sciences: if a robust model has a sudden change in its (semantical) behaviour, we can trust this happens *not* due to some artifact of our modelling, but due to a genuine *real-world* phenomenon (which we are trying to discover/study). In other words, robustness provides a 'no-false positives' guarantee.

7.2. Philosophical implications

In this paragraph, we consider our results from the point of view of Philosophy of Science.

The system ERNA was introduced by Richard Sommer and Patrick Suppes to *provide a foundation that is close to the mathematical practice characteristic of theoretical physics* (See [26, p. 2]). In [23], it is argued that several equivalent formulations of Π_1-transfer (e.g. the Continuity principle, the Dirac Delta theorem, and the Peano existence theorem) are essential to Physics. Here, we claim the same for sequential principles like $\mathbb{IVT}$ introduced in this paper. In particular, we argue that these principles are essential to a well-known renormalization technique from Physics called *dimensional regularization*.

In general, renormalization is a collection of techniques used to treat infinities arising in calculations in physical theories. A philosophical discussion of this topic can be found in [9]. An early example of dimensional regularization can be found in [10]. This technique provides a way of studying (physical) objects whose mathematical representation $\tau(z_0)$ is singular (i.e. infinite or undefined) in a certain physical theory. The first step to extracting information from $\tau(z_0)$ is to avoid the singularity z_0 by introducing a parameter $\varepsilon > 0$. Thus, $\tau(z_0+\varepsilon)$ is (mathematically) well-defined, but need not have physical meaning. Secondly, $\tau(z_0+\varepsilon)$ undergoes some mathematical manipulation, yielding a term $\sigma(z_0+\varepsilon)$ that behaves better around the singularity z_0. Thirdly, for the resulting object $\sigma(z_0+\varepsilon)$, the limit $\varepsilon \to 0$ is taken to obtain a (physically) meaningful term $\sigma(z_0)$. The properties of the latter yield new information about $\tau(z_0)$ and the corresponding physical object. It goes without saying that plenty of (mathematical *and* physical) objections can be raised with regard to dimensional regularization.

First of all, an essential part of this regularization technique is that the object $\sigma(z_0+\varepsilon)$ is 'well-behaved' in the limit $\varepsilon \to 0$. As such a limit is in general not even a function, this property is by no means a trivial requirement. Secondly, limits and other operations are applied in Physics without much care for mathematical detail as long as the end result somehow has (physical) meaning. As motivated at the end of Paragraph 6.2, both these considerations are reflected in sequential principles like $\mathbb{IVT}$: these principles express that, if the limits ${}^*x$ and ${}^*P$ of x_n and P_n for $n \to \infty$ are somehow meaningful, then ${}^*x$ is still a solution for ${}^*P$. Moreover, a *sequence* of objects is always given by the sequential principles. This is important, as in Physics, an existence statement concerning an object is usually accompanied by a procedure to approximate or determine this object.

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