Steiner versus Wittgenstein: Remarks on Differing Views of Mathematical Truth

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ABSTRACT: Mark Steiner criticizes some remarks Wittgenstein makes about Gödel. Steiner takes Wittgenstein to be disputing a mathematical result. The paper argues that Wittgenstein does no such thing. The contrast between the realist and the demonstrativist concerning mathematical truth is examined. Wittgenstein is held to side with neither camp. Rather, his point is that a realist argument is inconclusive.

Key words: Mark Steiner, Gödel, Wittgenstein, Juliet Floyd, realism, mathematical truth, provability.

1. Introduction

Clearly having in mind Gödel's theorem about the existence of undecidable sentences in arithmetic, Wittgenstein writes:

> I imagine someone asking my advice; he says: "I have constructed a proposition (I will use 'P' to designate it) in Russell's symbolism, and by means of certain definitions and transformations it can be so interpreted that it says: 'P is not provable in Russell's system'. Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable." (Wittgenstein , p. 50e).

Here Wittgenstein is considering a familiar argument that there are arithmetical truths that are unprovable. Call this argument "Argument A".

To this argument Wittgenstein responds:

> Just as we ask: "'provable' in what system?", so we must also ask: "'true' in what system?" 'True in Russell's system' means, as was said: proved in Russell's system; and 'false in Russell's system' means: the opposite has been proved in Russell's system. —Now what does your "suppose it is false" mean? *In the Russell sense* it means 'suppose the opposite is proved in Russell's system'; *if that is your assumption*, you will now presumably give up the interpretation that it is unprovable. And by 'this interpretation' I understand the translation into this English sentence. —If you assume that the proposition is provable in Russell's system, that means it is true *in the Russell sense*, and the interpretation "P is not provable" again has to be given up. If you assume that the proposition is true in the Russell sense, *the same* thing follows. Further: if the proposition is supposed to be false in some other than the Russell sense, then it does not contradict this for it to be proved in Russell's system. (What is called "losing" in chess may constitute winning in another game.) (Wittgenstein, p. 51e)

Call this argument "Argument B". Argument B is based on two claims: (i) proof in mathematics is a system-relative notion; (ii) truth in mathematics is a system-relative notion.

The first claim does not appear to be controversial. For example, a classical Peano proof is based on the Peano axioms and classical logic. A non-classical Peano proof is based on the Peano axioms and non-classical logic. But Argument B also contains a premise saying that truth in mathematics is a system-relative notion.

A theory, which I shall call the provability theory, gets its inspiration from this premise. This theory holds that the right way to regard pure mathematics is as a discipline that deals with no existent objects whatsoever, and which doesn't intend to. Mathematics is precisely what it looks to be: a system of notation within which we carry out proofs. What gets established in establishing mathematical existence is just a certain type of mathematical sentence, and its truth is nothing more than its provability.

The key point is that in mathematics proof establishes nothing beyond itself.

It is natural to see this as wrong, to see a proof as always showing that something is true, but not as itself making for the truth of what it shows to be true. The idea is that a proof can show some statement is true, but the statement is not true in virtue of its having a proof, but in virtue of things being as it says they are.

The provability theorist affirms that this holds only outside of pure mathematics. It is agreed that what ordinarily establishes a sentence is not what makes it true. What makes for the truth of a sentence like 'There are whales' is how the world is, not the sequence of sentences we set forth as establishing this sentence. But this familiar distinction lacks application within pure mathematics. Proof inside mathematics is completely different from proof outside mathematics. In the non-mathematical cases, proof establishes something more — truth. In mathematics, being true *consists in* having a proof.

My understanding of Wittgenstein is that he is not pushing the provability theory. Rather, his purpose is to show that Argument A is not as persuasive as it appears because it presupposes that mathematical truth is a system independent notion. Argument B is intended by Wittgenstein to show that if this presupposition is rejected one does not end up with the conclusion that some mathematical truths are unprovable — the conclusion of argument A. Wittgenstein himself takes no position on whether mathematical truth is system independent or system relative. He certainly did not take himself to be refuting Gödel or any mathematical result.

This conclusion is reinforced by Juliet Floyd who compiles historical evidence that Wittgenstein understood and accepted the Gödel result. (Floyd, pp.281-286)

2. Mark Steiner

In a recent paper Mark Steiner contends that "It is a mathematical theorem (of set theory) that the Gödel sentence *P* is true in Tarski's sense if and only if it is unprovable." (Steiner, p.267) Steiner adds, "… Argument *A* is a mathematical theorem, it is clear that no refutation will be forthcoming, and in particular Wittgenstein's response [Argument B] is ineffectual." (Steiner, p.267) So, for Steiner, Wittgenstein was in fact challenging a mathematical result whether or not that was his intention.

A point similar to this could be made about standard models for arithmetic and truth in such models. By a model for a language L I mean the usual sort of structure: a domain set D and a function v, standardly defined, assigning elements and sets of n-tuples of elements of the domain set to expressions of L.

The common model theoretic account proceeds by describing a certain model (e.g., by saying that its domain set is to be the set of natural numbers, that v('0') is to be zero, that v('s') is to be the successor function, and so on) and then calling a model standard just in case it is isomorphic to the described model. Let K be the described model. We then have the following definition:

Df Where L is the language of arithmetic and M is a model for L: M is *standard* if and only if M is isomorphic to K.

On the account of truth in a standard model of arithmetic it is unproblematic that Gödel sentences are true in such a model and unprovable if arithmetic is consistent.

I shall defend Wittgenstein against Steiner's criticism of him.

3. Criticism of Steiner

I agree with Juliet Floyd who writes, "[Wittgenstein] would have questioned whether Tarski's model-theoretic account of truth definitions for formalized languages yields a philosophical account of our notion of *mathematical truth* (let alone truth in general)." (Floyd, p. 304)

Wittgenstein would never have denied that "the Gödel sentence *P* is true in Tarski's sense if and only if it is unprovable." He would never have denied that *P* is true in a standard model of arithmetic if and only if it is unprovable. But is truth in Tarski's sense or truth in a standard model of arithmetic mathematical truth *simpliciter*?

As I see matters, the point of Argument B is that unless you can justify a yes answer to this question Argument A does nothing to show that there is a difference between mathematical truth and mathematical provability.

4. Mathematical Truth and Model-Theoretic Truth

In response to something like the criticism just made, Steiner writes, "… to make a sharp boundary (on Wittgenstein's behalf) between 'true' and 'Tarski true', is to overlook the organic relation between the two which is illuminated brightly by Gödel's theorem. Namely, the mathematician who has proved Gödel's theorem, and who wants to extend the 'Russell notion' of truth (truth as provability in *Principia Mathematica,* or, as in our treatment, in PA) to cover the undecidable sentence P, has no choice whatever — he *must* adopt Tarski truth as the extension of 'true' in light of Gödel's theorem!" (Steiner, p. 268)

Against this, I contend that a person who accepts the provability thesis need not reject the notion of truth in a model. Such a person rejects only the alleged link between arithmetical truth and truth in a model.

But one might here ask: How could one admit *truth in a model* and not admit that *truth* is bivalent? (If mathematical truth is bivalent, then it follows directly from the Gödel result that mathematical truth does not coincide with mathematical provability.)

Here a certain *picture* forces itself upon us — the denotational picture of what *truth* consists in. We slide from *truth* to *truth in a model*, and mistake the former for the latter.

A reason can be given for this slide: After all, we are inclined to think, semantics pertains to truth, and truth in a model provides the framework of our semantics.

To resist this assimilation of concepts we must soberly concentrate on the concept *truth in a model* and rigorously focus on what kind of concept it is, on the work it gets done. Its task is semantical. It serves to define the logic of the language of arithmetic, *not* to establish the metaphysics of arithmetical truth. The semantical concept, strictly construed as such, is *altogether neutral* about the metaphysics of arithmetical truth.

My point of view can be expressed as follows: Model theory is a mathematical exercise. It is a part of, and thus is internal to, mathematics. Model theory speaks of sets, relations among sets and so forth in the usual way. Thus, no one who works *within* the discipline — including one who accepts the provability theory — need deny the concepts of model theory, including the concept of truth in a model as constructed within mathematics.

Still, when we say in our model theory that in a model '2' denotes 2 we *seem to* connect up word and object. But in fact our assertion is on all fours with the assertion $3^2 > 2^3$. Once we see this clearly we will no longer be tempted to suppose that a concept internal to mathematics and strictly mathematical in content — the concept of truth in a model — is itself a concept apt for illuminating the concept of arithmetical truth. For what reason do we have for supposing that the concept of arithmetic truth is a concept of mathematics? In fact, there is no reason. The concept *mathematical truth* is no more a concept of *mathematics* than is the concept of *mathematical necessity*.

5. Bivalence

The philosophical issue before us can also be put in terms of bivalence because it is an obvious result of the Gödel result that there are unprovable truths if arithmetic is bivalent.

To bring the general situation into perspective, compare the following two philosophers. The first is a pre-Gödelian philosopher who believes that each arithmetic proposition is true or false in virtue of how things stand with mathematical objects. In addition, she believes that each arithmetical truth has a proof within some single system of proof.

The second is a pre-Gödelian philosopher who does not believe there are any mathematical objects. He reasons thus: Since there are no mathematical objects in virtue of which arithmetical propositions are truth-valued, there cannot be anything more to arithmetic truth than provability. He too will believe that each arithmetic truth has a proof within some single system of proof.

Both philosophers accept the thesis that, for any mathematical proposition φ, φ is an arithmetical truth if and only if φ is provable. But they accept this in different ways. For the first philosopher arithmetical truth *coincides with* provability. But it is a mathematical reality that makes for arithmetical truth. For the second philosopher there is no such reality. Arithmetical truth *consists in* provability.

When the Gödel result becomes known these philosophers draw different morals. The first concludes that the provability thesis is untenable and asserts, what now seems obvious, that arithmetic truth is not axiomatizable. The second does not conclude this. The Gödel result does not convince him that there *is* some mathematical reality that fixes a truth-value for each arithmetical sentence and so he remains wedded to the provability theory and concludes that there is no system of proof relative to which each arithmetic proposition is truth-valued. He then draws the moral that arithmetic is not bivalent.

For the philosopher of the first sort the *primary* conviction is that each sentence of arithmetic is truth-valued. And so long as it goes without question that arithmetical truth coincides with provability, the provability thesis will seem to her to provide a way of construing truth and falsity for arithmetic that avoids realism about mathematical objects. For such a philosopher the Gödel result can only show that arithmetical truth does not coincide with provability.

The philosopher of the second sort does not have it as a *primary* conviction that the sentences of arithmetic are one and all truth-valued. Rather, the starting point for such a philosopher is a distrust of mathematical realism. And, of course, the Gödel result is, *on its own*, quite incapable of convincing anyone that a realist view of mathematics is correct. On the other hand, the view that all the sentences of arithmetic are truth-valued seems to inexorably lead to mathematical realism granted the Gödel result. Thus the philosopher of the second sort concludes that the Gödel result shows that not all the sentences of arithmetic are truth-valued.

If this survey of the situation is accurate, then the dividing issue is not the significance of the Gödel result — it says what it says — but the correctness of the assumption that each arithmetical sentence is truth-valued. And the Gödel result cannot on its own decide *that* issue.

6. Final Remarks

Mathematical realism is characteristically connected with a conception of mathematical meanings on which it is possible for us to associate such meanings with sentences without thereby providing methods sufficient for justifying their acceptance or rejection. This conception typically thinks of mathematical meanings in terms of truth-conditions, so that what is possible for us to do is to associate truth-conditions with mathematical sentences without thereby providing methods for deciding those truth-values. Opposed to realism is the provability view. On this view we endow our sentences with mathematical meanings not by systematically associating them with truth-conditions of a certain sort, but by providing methods of justification for the language of mathematics. Mathematics itself says nothing about which of these two we ought to accept or whether we should accept either one. Nor did Wittgenstein.[1]

REFERENCES

Floyd, Juliet (2001). "Prose Versus Proof: Wittgenstein on Gödel, Tarski, and Truth," *Philosophia Mathematica*, 280-307.

Steiner, Mark (2001). "Wittgenstein as His Own Worst Enemy," *Philosophia Mathematica*, 257-279.

Wittgenstein, Ludwig (1956). *Remarks on the Foundations of Mathematics,* Basil Blackwell. Edited by G.H. von Wright, R. Rhees, G.E.M. Anscombe. Translated by G.E.M. Anscombe.

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