

Aboutness Paradox*

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Abstract

The present work outlines a logical and philosophical conception of propositions in relation to a group of puzzles that arise by quantifying over them: the Russell-Myhill paradox, the Prior-Kaplan paradox, and Prior's Theorem. First, I motivate an interpretation of the Russell-Myhill paradox as depending on *aboutness*. Aboutness informs the notion of propositional identity, of which I will offer two formalizations, depending on choices that have to be made about the syntax of propositional variables. I then extend to propositions a predicative response to the paradoxes presented in Linnebo (2013). On this approach, modal operators are used to uncover the implicit relation of dependence that characterizes propositions that are about propositions, and which Russell-Myhill shows to be of logical significance. Thus, the justification for predicativity is found in two ideas: (i) propositions are, in some sense, language-dependent entities; (ii) there is a distinction between what a sentence *says* (its semantic value) and what it *expresses* (a proposition). The propositions that can be expressed outstrip those that are, so to speak, "available" for reference and quantification. A modal abstraction principle for propositions formalizes this conception, and its benefits are shown by application to other intensional puzzles. The resulting view is an alternative to the plenitudinous metaphysics of unconstrained comprehension principles defended by Bacon et al. (2016), among others.

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The notion of aboutness has been around for some time, and has recently received some attention in relation to the logic of hyperintensionality.¹ Intuitively, sentence (1) is about Sal.

1. Sal is running.

Let us stipulate that the proposition expressed by a sentence is about whatever the sentence is about.

Perhaps we can agree on a couple of intuitively plausible principles. For example, for any object n , there is a proposition about n . This is plausible because it does not seem too hard to form a thought about a given object. For example, for any n , there is the proposition that n is running, or the proposition that n is a number. Another principle that seems plausible is that propositions about distinct objects are distinct. Since $2 \neq 3$, the proposition that $2 > 0$ is distinct from the proposition that $3 > 0$, even though these are both necessarily true propositions.

We could thus agree with the following Existence and Identity claims, relating the notion of aboutness, a domain of objects, and propositions about its members:²

- (E) For every m , there is a proposition about m ;
- (I) For every m and n , every proposition about m is about n , and conversely, only if $m = n$.

There seems to be no reason why versions of E and I should not hold in the higher-order domain, where m and n are variables over sets. Indeed, it is natural to think that the proposition expressed by (2) is about (the set of) sailors.

2. Some sailors are running.

¹Among others, see Gilbert Ryle, “About,” *Analysis*, 1, 1 (1933): 10–12; David Lewis, “Statements Partly About Observation,” *Philosophical Papers*, xvii, 1 (1988): 1–31; John Perry, “Possible Worlds and Subject Matter,” in J. Perry, ed., *The Problem of the Essential Indexical and Other Essays* (Palo Alto, CA: CSLI Publications, 1989): pp. 145–60. On hyperintensionality, see Daniel Nolan, “Hyperintensional Metaphysics,” *Philosophical Studies*, CLXXI, 1 (2014): 149–60; Stephen Yablo, *Aboutness* (Princeton, NJ: Princeton University Press, 2014); Kit Fine, “Angelic Content,” *Journal of Philosophical Logic*, XLV, 2 (2016): 199–226; for an overview, see Peter Hawke, “Theories of Aboutness,” *Australasian Journal of Philosophy*, xcvi, 4 (2018): 697–723.

²The relevant relation is that of a proposition being *entirely* about something (including, say, propositions entirely about pluralities).

for (2) says of (the set of) sailors that some (of its members) are running. If propositions can be about sets, some propositions belong to the set they are about. For example, a proposition about the set of abstract objects belongs to the set it is about, assuming that propositions are abstract objects. And so E and I set up an injection into the set of propositions of the set of all sets of propositions. A familiar diagonal argument brings up the inconsistency with Cantor's theorem, as Russell explains (using 'class' instead of 'set'):

If m be a class of propositions, the proposition 'every m is true' may or may not be itself an m . But there is a one-one relation of this proposition to m : if n be different from m , 'every n is true' is not the same proposition as 'every m is true.' Consider now the whole class of propositions of the form 'every m is true,' and having the property of not being members of their respective m 's. Let this class be w , and let p be the proposition 'every w is true.' If p is a w , it must possess the defining property of w ; but this property demands that p should not be a w . On the other hand, if p be not a w , then p does possess the defining property of w , and therefore is a w . Thus the contradiction appears unavoidable.³

This is the Russell-Myhill paradox (henceforth, RM; also known as Russell propositional paradox, or Appendix B paradox).⁴ Russell considers propositions of the form 'every m is true,' which he calls logical products. As noted by Harold Hodes, this syntax is unnecessary.⁵ The argument is unaffected by considering propositions of the form 'some m is true,' ' m is greater than 0,' or whatever. If (2) is about the set of sailors, so 'every m is true' is about the set of ms . The important thing about logical products is that they are propositions about sets, for then the motivation for asserting the premises becomes compelling. The premises are:

- (E) the existence claim implicit in the use of the definite article: for every m , Russell considers 'the proposition "every m is true"';
- (I) the identity claim stated contrapositively: 'if n be different from m , "every n is true" is not the same proposition as "every m is true".'

³Bertrand Russell, *The Principles of Mathematics* (Cambridge: Cambridge University Press, 1903), at Appendix B, Section 500.

⁴For an accessible introduction to the Russell-Myhill paradox and some helpful background, see Kevin Klement, "Russell-Myhill Paradox," *Internet Encyclopedia of Philosophy*, <https://iep.utm.edu/par-rusm/>.

⁵Harold Hodes, "Why Ramify?," *Notre Dame Journal of Formal Logic*, LVI, 2 (2015): 379–415.

The rest of RM is logic and set theory. It can be understood as a *paradox of aboutness*.⁶

This is not, and not intended to be, a reconstruction of Russell’s thought. Part of my goal in this paper is to place RM within larger contemporary debates on propositions, the logic of attitudes, and hyperintensionality. By reconstructing RM as a paradox of aboutness, we uncover an argumentative structure that is closely related to that of Russell’s more famous set-theoretic paradox. This will help to appreciate what sort of formal theories RM shows to be inconsistent. In the second part of the paper, I will present a modal account of RM inspired by analogous accounts of the set-theoretic paradoxes.⁷ The resulting view is articulated by an abstraction principle for propositions, in which, in order to restore consistency, we use modal operators to capture the logic of the dependence relation implicit in the claim that some propositions are *about* propositions. In the final section, some benefits of the modal abstractionist conception of semantic objects developed in the paper are found in the treatment of other intensional puzzles: the Prior-Kaplan paradox and Prior’s Theorem.⁸ The resulting universe of propositions is not as rich as impredicative quantification would sanction, but allows for hyperintensional distinctions, and for a unified pattern of response to the intensional puzzles.

1 Possible solutions

Frege dismissed RM, alleging that Russell blurred the distinction between sense and reference.⁹ On point of fact, Frege was right, as Russell’s reasoning cannot be reproduced as

⁶According to Tucker and Thomason (2011), RM is neither truth-theoretic (like the Liar), nor set-theoretic (like Burali-Forti), and it falsifies Ramsey’s famous classification of the antinomies in two classes. See Dustin Tucker and Richmond Thomason, “Paradoxes of Intensionality,” *The Review of Symbolic Logic*, IV, 3 (2011): 394–411; and Frank P. Ramsey, *The Foundations of Mathematics and Other Logical Essays* (London: Routledge and Kegan Paul, 1925). I would suggest that RM is an aboutness-theoretic paradox, hence still semantic, though not truth-theoretic. RM falsifies Ramsey’s distinction to the extent that all semantic paradoxes are expected to be taken care of by revising the logic of truth. RM is hardly set-theoretic, as the claim that a set is “about” its members is no part of our conception of set. But it’s hardly truth-theoretic, because disquotation principles play no apparent role in the reasoning.

⁷Among others, see Øystein Linnebo, “The Potential Hierarchy of Sets,” *The Review of Symbolic Logic*, VI, 2 (2013): 205–28; and Geoffrey Hellman, *Mathematics Without Numbers: Towards a Modal-Structural Interpretation* (New York, NY: Oxford University Press, 1989). For a similar approach to the intensional paradoxes, see Andy Yu, “A Modal Account of Propositions,” *Dialectica*, LXXI, 4 (2017): 463–88.

⁸Arthur Prior, *Objects of Thought* (Oxford: Clarendon Press, 1971); David Kaplan, “A Problem in Possible World Semantics,” in D. Raffman, W. Sinnott-Armstrong, N. Asher, eds., *Modality, Morality, and Belief: Essays in Honor of Ruth Barcan Marcus* (Cambridge: Cambridge University Press, 1995): pp. 41–52.

⁹Gottlob Frege, *Philosophical and Mathematical Correspondence*, in G. Gabriel, H. Hermes, F. Kambar-tel, C. Thiel, A. Veraart, B. McGuinness, H. Kaal, eds., (Oxford: Blackwell, 1980), at pp. 149ff.

it stands in a framework that reinforces the distinction. He was also correct, I think, on a more general level: the semantic value of a sentence (its referent, in Fregean terminology) shouldn't be confused with what the sentence expresses (its sense, or thought). However, it is well known that if one formalizes Frege's implicit commitments to a hierarchy of senses, one is able to formulate a version of RM. So it seems that Frege's project, broadly understood, that is, so as to include his semantic theory, is vulnerable to RM.¹⁰

Harold Hodes expresses some misgivings concerning Russell's use of bound variables in sentence position – a topic reminiscent of Frege's worries.¹¹ The legitimacy of such binding, however, is orthogonal to the possibility of formulating Russell's reasoning in a rigorous way. To illustrate this point, I will present two derivations below: a "Fregean" one in which propositional variables only occur in name position, and another, more "Russellian," which allows variable binding in sentence position. There is a conceptual issue at the bottom of RM that is not to be eliminated by a sanitized syntax.

I shall focus on two possible proposals about the significance of RM. (The set of solutions suggested in the literature, or merely conceivable, is much broader.) These two options amount to rejecting E, or rejecting I. The latter option consists in curtailing how much hyperintensionality one can consistently have. Russell assumes that propositions about distinct sets are distinct. I said that a more general commitment is intuitively plausible: propositions about distinct objects are distinct. However, according to some authors RM shows that, as one might say, there are more distinctions in Heaven and Earth than are conceivable in thought.¹² More precisely, propositions about different objects could be identical. According to some of these authors, the loss of hyperintensionality might be compensated by unrestricted generality.

Rejection of E is (one way of implementing) a predicative restriction on the comprehension principle governing which sets of propositions there are. There is good philosophical motivation for this proposal, coming from the idea that not all expressible propositions can always be referred to, or quantified over. Intuitively, reference to and quantification over propositions that are about a set depends on the prior availability of the members

¹⁰John Myhill, "Problems arising in the Formalization of Intensional Logic," *Logique et Analyse*, 1, 1 (1958): 78–83; for an overview, see Kevin Klement, *Frege and the Logic of Sense and Reference* (New York NY: Routledge, 2002).

¹¹Hodes, "Why Ramify?," *op. cit.*

¹²Among others, see Gabriel Uzquiano, "A Neglected Resolution of Russell's Paradox of Propositions," *The Review of Symbolic Logic*, VIII, 2 (2015): 328–44; Cian Dorr, "To Be F Is To Be G," *Philosophical Perspectives*, XXX, 1 (2016): 39–134; Jeremy Goodman, "Reality is Not Structured," *Analysis*, LXXVII, 1 (2017): 43–53.

of that set. This is because reference and quantification allow us to express further and further thoughts, making more propositions available. In this respect, a version of predicativism finds a natural application in the present setting, and supports a conception of the metaphysics of propositions justified by these intuitive remarks. Similar remarks on the stability of reference can be found in recent work by Sean Walsh, and perhaps in the noble origins of predicativism in mathematics: the work of Poincaré, Russell, and Weyl.¹³

2 Synonymy

In the passage quoted above, Russell appeals without definition to the notion of a proposition being “the same” as another. This notion is central to principle I, and to the idea that propositions about distinct sets are distinct. I proceed to a precise definition, building on the work of Alonzo Church, who formalized the notion of propositional identity implicit in *Principia Mathematica*.¹⁴

Quine once said that propositions are ‘shadows of sentences.’¹⁵ He meant that propositions should be dismissed from serious ontology. On the other hand, it is intriguing to think that the cardinal numbers are ‘shadows’ of equinumerosity relations among concepts, though this is no reason to deny that there are numbers. Crispin Wright says:

pure abstract objects [...] are no more than *shadows* cast by the syntax of our discourse.¹⁶

With apologies to Quine, I take the direction not of nominalism but of a thin ontology.

¹³Sean Walsh, “Predicativity, the Russell-Myhill Paradox, and Church’s Intensional Logic,” *Journal of Philosophical Logic*, XLV, 3 (2015): 277–326. In several respects, the present approach is similar to the latter. A macroscopic difference is that Walsh works within the Logic of Sense and Denotation (Alonzo Church, “A Formulation of the Logic of Sense and Denotation,” in P. Hellen, H. Kallen, S. Langer, eds., *Structure, Method, and Meaning: Essays in Honor of Henry M. Sheffer* (New York, NY: Liberal Arts Press, 1951), pp. 3–24). A detailed comparison between various types of predicative systems is left for another occasion. Further work on the Russell-Myhill, which I do not discuss here for reasons of space, includes most notably the work of C. Anthony Anderson. See C. Anthony Anderson, “Some New Axioms for the Logic of Sense and Denotation: Alternative (0),” *Noûs*, XIV, 2 (1980): 217–34; and C. Anthony Anderson, “Semantical Antinomies in the Logic of Sense and Denotation,” *Notre Dame Journal of Formal Logic*, XXVIII, 1 (1987): 99–114.

¹⁴See Alonzo Church, “Russellian Simple Type Theory,” *Proceedings and Addresses of the American Philosophical Association*, XLVII (1973): 21–33; Alonzo Church, “Comparison of Russell’s Resolution of the Semantical Antinomies with that of Tarski,” *Journal of Symbolic Logic*, XLI, 4 (1976): 747–60; and Alonzo Church, “Russell’s Theory of Identity of Propositions,” *Philosophia Naturalis*, XXI, 2/4 (1984): 513–22.

¹⁵Willard V. O. Quine, *Philosophy of Logic* (Cambridge, MA: Harvard University Press, 1970), at p. 10.

¹⁶Crispin Wright, *Truth and Objectivity* (Cambridge, MA: Harvard University Press, 1992), at p. 181.

Propositions are given to us, in some sense, by the sentences that express them, as numbers are (perhaps) given by equinumerosity relations.

Propositional identity, on this proposal, is defined by synonymy, understood as an equivalence relation on sentences. Formally, we expect propositional identity to suffice for substitutional reasoning, that is reasoning by Leibniz’s Law:

$$\forall x \forall y (x = y \rightarrow \forall F (Fx \rightarrow Fy)) \quad \text{LL:}$$

Fortunately, we won’t need a full list of properties of the relation synonymy, for it turns out that the assumptions needed for the paradox are very modest. So, even if a full analysis of the concept of synonymy is perhaps beyond our grasp, we can say enough for assessing RM.

It is convenient to work in the following types:

1. e is a type;
2. for $0 \leq i \leq j$, if ν_i, \dots, ν_j are types, (ν_i, \dots, ν_j) is a type;

and nothing else is a type. We call e the type of individuals. Note that by setting $0 = i = j$, the second clause gives a type $()$, which we may regard as the type of propositions. A type (ν_i, \dots, ν_j) , with $0 < i \leq j$, is the type of functions from objects of type ν_i, \dots, ν_j , to an object of type $()$. So, for example, (e) is the type of functions from one individual to a proposition, such as the function expressed by ‘is greater than 0.’ Syntactically, I assume we have variables for objects of any type, using a special syntax for propositional variables.

1. individual variables x, y, \dots of type e ;
2. predicate variables X, Y, \dots of type (ν_i, \dots, ν_j) for $0 < i \leq j$;
3. propositional variables p, q, \dots of type $()$.

There are two ways to understand propositional variables. They might be a special case of second-order variables for predicates of zero arguments.¹⁷ Alternatively, they might be first-order variables ranging over the special sort of propositions. In this case,

$$\exists p \psi$$

¹⁷Prior, “Objects of Thought,” *op. cit.* See also Timothy Williamson, “Everything,” *Philosophical Perspectives*, xvii, 1 (2003): 415–65.

is just an abbreviation of

$$\exists x(Px \wedge \psi)$$

where P is the predicate ‘is a proposition.’ The latter option seems to be often implicit in current work in metaphysics.¹⁸ The two options correlate with different attitudes toward the possibility of variable binding in sentence position, which is a source of philosophical controversy.¹⁹ In order not to compromise any issues, I shall discuss both options. For present purposes, it turns out that the choice is really stylistic, since paradoxical reasoning may be reproduced either way.

I assume the familiar syntax of second-order logic. This includes a comprehension axiom which, intuitively, converts a predicate of propositions or individuals into an object in the range of the second-order quantifiers. Note that this allows to generalize only over a relatively small part of the full type-theoretic universe. This is not an ideological choice, but one whose purpose is to keep the discussion focused: only relatively modest resources are needed for RM.

$$\exists X \forall x_g \forall p_h (X \langle x_g, p_h \rangle \leftrightarrow \psi \langle x_g, p_h \rangle) \quad \text{Comprehension:}$$

where x_g, p_h are sequences of g, h -many variables of type e and $()$, respectively. As usual, X does not occur in ψ .

I take identity to be an undefined primitive in type e : a reflexive relation for which LL holds. The notion of identity in higher types is defined. In type (ν_i, \dots, ν_j) , for $0 < i \leq j$, it is defined extensionally.

$$\forall X \forall X' (\forall x_g \forall p_k (X \langle x_g, p_h \rangle \leftrightarrow X' \langle x_g, p_h \rangle) \leftrightarrow X = X') \quad \text{Extensionality:}$$

Strictly speaking, we should use a different symbol for identity here, but conventions are sufficiently well-established, and context will disambiguate. An indiscernibility principle analogous to LL for predicate variables can be established by induction in the metatheory.

Identity in type $()$ is also defined, albeit by a different principle. Extensionality is vacuous since propositions have no arguments. Conceptually, propositions are not extensional objects, whose identity conditions are defined by their members. Instead, as I said, proposi-

¹⁸Among others, see Jeffrey King, Scott Soames, and Jeff Speaks, *New Thinking About Propositions* (Oxford: Oxford University Press, 2014).

¹⁹Nicholas K. Jones, “Propositions and Cognitive Relations,” *Proceedings of the Aristotelian Society*, CXIX, 2 (2019): 157–78.

tional identity relies on synonymy: an informal and notoriously slippery notion. Following Church, I introduce a symbol for *hyperintensional equivalence*, to express synonymy in the object language.²⁰ This is the first of two notational innovations I shall add to ordinary second-order logic, in order to formalize RM.

A formula ' $\psi \Leftrightarrow \varphi$ ' says that ψ is hyperintensionally equivalent to φ , or, roughly, that the sentences ψ and φ are synonymous. These glosses convey what is meant, I believe, at least as a first pass, although they wrongly suggest some kind of semantic ascent. This is not a reason for concern: it is just as convenient to express what ' $a = b$ ' means by saying that a and b are co-referential – even though there is nothing metalinguistic about identity. Similarly, hyperintensional equivalence is not metalinguistic: it has the syntax of an ordinary connective. Moreover, I assume that it has at least the following properties:

- Reflexivity: $\psi \Leftrightarrow \psi$
- Symmetry: $\psi \Leftrightarrow \varphi \rightarrow \varphi \Leftrightarrow \psi$
- Transitivity: $(\psi \Leftrightarrow \varphi \wedge \varphi \Leftrightarrow \phi) \rightarrow \psi \Leftrightarrow \phi$
- Hyp1: $\forall x \forall y (\psi x \Leftrightarrow \psi y \rightarrow x = y)$
- Hyp2: $\forall X \forall Y (\psi X \Leftrightarrow \psi Y \rightarrow X = Y)$

Reflexivity, Symmetry, and Transitivity, ensure that hyperintensional equivalence is indeed an equivalence. The last two conditions ensure that synonymy is a genuinely hyperintensional notion. Let a, b be metavariables for variables of extensional types (that is, individual or predicate variables, provided a and b are of the same type). Then we can state Hyp1 and Hyp2 as a single formula:

$$\forall a \forall b (\psi a \Leftrightarrow \psi b \rightarrow a = b) \quad \text{Hyp:}$$

Hyp is plausible, at least insofar as one finds principle I plausible. A sentence obtained by predicating ψ of a is intuitively about a , as (1) and (2) suggest.²¹ If so, then Hyp

²⁰See in particular Church, "Russell's Theory of Identity of Propositions," *op. cit.* There is more than one notion of propositional identity in Church's work. The notion of 'strict equivalence' is an amendment of the earlier notion of 'intensional' or 'synonymous isomorphism,' and the logical core of Alternative (0): the one among Church's proposals that underlies the metaphysics of Neo-Russellian propositions. The notion of hyperintensional equivalence is modeled on strict equivalence, although I depart from Church at various junctures. See also Alonzo Church, "Intensional Isomorphism and Identity of Belief," *Philosophical Studies*, v, 5 (1954): 65–73; and Alonzo Church, "Outline of a Revised Formulation of the Logic of Sense and Denotation (Part II)," *Noûs*, VIII, 2 (1974): 135–56.

²¹So much is explicitly assumed by John Perry, in his discussion of aboutness. See Perry, "Possible Worlds and Subject Matter," *op. cit.* Depending on further details, ψa may also be about whatever ψ is about, but with such details we need not be concerned.

asserts (the contrapositive of the claim) that sentences about distinct things are not synonymous. Hyperintensional equivalence thus incorporates the fine degree of distinctions among thoughts that is motivated by aboutness considerations.

Further assumptions about synonymy would be desirable to regiment the behaviour of this relation. For example, Church comments that ‘if sentences differ only by alphabetic change of bound variables, the corresponding propositions are the same,’ which seems plausible.²² Still more axioms could be added for a full logic of synonymy. However, the resources required by RM are quite weak, and indeed are limited to Reflexivity and Hyp – more precisely, Hyp2. Since the goal here is not to work out the logic of synonymy, I shall not officially assume anything beyond Hyp, and the axioms for equivalence.

The second and final notational innovation I will introduce is a term-forming operator $[\cdot]$, which, syntactically, takes a well-formed formula and yields a singular term. Intuitively, ‘ $[\psi]$ ’ is a singular term denoting the proposition expressed by ψ . In English, this operator roughly corresponds to the expression ‘that,’ or ‘the proposition that,’ which converts a sentence ‘grass is green’ into the proposition-denoting expression ‘the proposition that grass is green.’

I can now define propositional identity in terms of synonymy. This can be a definition by abstraction, adopting for present purposes the sophisticated formal and philosophical technology developed in the debate over Neo-Fregean philosophy of arithmetic. Abstraction is a well-investigated way to supply a thin ontology by means of definitions of identity. We can use it to explore the idea that propositions are ‘shadows of sentences.’²³

The general form of a definition by abstraction is that of a definition of a class of objects designated by the singular terms on the left-hand side of Abstraction, by taking equivalence classes of objects quantified over on the right-hand side.

$$\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta \qquad \text{Abstraction:}$$

It is a matter of controversy whether such definitions are acceptable, especially from a Fregean perspective.²⁴ If they are, perhaps on more general grounds, they might be suitable

²²Church, “Russell’s Theory of Identity of Propositions,” *op. cit.*, at p. 516.

²³On Neo-Fregeanism, see Crispin Wright, *Frege’s Conception of Numbers as Objects* (Aberdeen: Aberdeen University Press, 1983); Bob Hale and Crispin Wright, *The Reason’s Proper Study* (Oxford: Clarendon Press, 2001). On definition of identity by abstraction, see Agustín Rayo, *The Construction of Logical Space* (New York, NY: Oxford University Press, 2013); and Øystein Linnebo, *Thin Objects: An Abstractionist Account* (Oxford: Oxford University Press, 2018).

²⁴Frege notoriously considered and rejected a proposal to define numbers by abstraction in his *Die*

for present purposes. We can then try defining propositional identity as follows:

$$[\psi] = [\varphi] \leftrightarrow \psi \leftrightarrow \varphi \quad \text{PropAbst:}$$

This is a schematic principle of the form of Abstraction. Each of its instances asserts that the proposition expressed by ‘ ψ ’ is identical to the proposition expressed by ‘ φ ’ just in case ψ and φ are synonymous. As I remarked with regards to Extensionality, it is not a given that the symbol ‘=’ can occur in PropAbst. The principle, however, is adequate at least in the following sense: if p and q are the propositions expressed by ψ and φ respectively, then ψ and φ are synonymous only if q and p have the same properties.

$$\forall p \forall q (p = [\psi] \wedge q = [\varphi] \rightarrow (\psi \leftrightarrow \varphi \rightarrow \forall F (Fp \rightarrow Fq)))$$

That is, synonymy is a congruence over propositions (this can be shown in the metatheory). Similarly, LL states that identity of individuals is a congruence over individuals.

3 Two versions of Russell-Myhill

The semantics of $[\cdot]$ varies depending on whether propositional variables are sentential or nominal. Let us begin by assuming that propositions are special-sorted first-order objects. If so, the operator $[\cdot]$ picks an object of type $()$, that is, a proposition, given the semantic value of a sentence. In order to reconstruct RM, it is not necessary to be specific about the semantic values of sentences. It is possible, for example, to take sentences to designate truth-values, which might as well be two special individuals of type e , namely 0 and 1. In this case, the operator $[\cdot]$ maps truth-values to propositions. It is also possible to enrich the background type theory, and take the semantic values of sentences to be sets of possible

Grundlagen der Arithmetik: eine logisch mathematische Untersuchung über den Begriff der Zahl (Breslau: W. Koebner, 1884), translated by J. L. Austin as *The Foundations of Arithmetic: a logico-mathematical enquiry into the Concept of Number* (Oxford: Blackwell, 1974). Frege’s misgivings are related to the infamous ‘Caesar Problem.’ Besides controversy on what the problem is, there is controversy on which strategies are available in the Neo-Fregean program for avoiding it. For an introduction to this vast literature, see Hale and Wright, “The Reason’s Proper Study,” *op. cit.*, Richard Kimberly Heck, “The Julius Caesar Objection,” in R. K. Heck, ed., *Language, Thought, and Logic: Essays in Honour of Michael Dummett* (Oxford: Oxford University Press, 1997), pp. 273–308; Peter Sullivan and Michael Potter, “Hale on Caesar,” *Philosophia Mathematica*, v, 2 (1997): 135–52; Fraser MacBride, “Speaking with Shadows: A Study of Neo-Logicism,” *British Journal for the Philosophy of Science*, LIV, 1 (2003): 103–63; Fraser MacBride, “The Julius Caesar Objection: More Problematic Than Ever,” in F. MacBride, ed., *Identity and Modality* (Oxford: Oxford University Press, 2006), pp. 174–202.

worlds. Then the operator $[\cdot]$ maps sets of worlds to propositions. In all this, propositions are the senses of sentences, and need not be identified with the (extensional or intensional) semantic values of sentences. We thus draw a distinction between what a sentence says (its semantic value), and what a sentence expresses (a proposition).

Initially, I shall assume that we can infer $\exists xFx$ from a statement of the form Ft . Since the only singular terms we need to worry about are those designating propositions, this assumption may be captured by the following claim.

$$F[\psi] \rightarrow \exists pF_{p/[\psi]} \quad \text{Existence1:}$$

Note that a singular term $[\psi]$ is replaced by a bound propositional variable, which is consequently in name position.

Under these assumptions, the core observation to be drawn from RM is this:

Theorem. *PropAbst, Hyp, Comprehension, and Existence1 are jointly inconsistent in second-order logic with propositional variables, understood as special sorted first-order variables.*

The derivation proceeds as follows. Consider the open formula $\exists X(p = [fX] \wedge \neg Xp)$. This is satisfied by a proposition p just in case there is a set X such that p is the proposition that X is f but p is not itself an X . Here f can be anything of the right type: Russell used ‘every ... is true.’ By Comprehension, there exists a set R such that $\forall p(Rp \leftrightarrow \exists X(p = [fX] \wedge \neg Xp))$. Consider now the proposition expressed by ‘ fR ,’ which is designated by ‘ $[fR]$.’ By contraposing Existence1, $R[fR] \leftrightarrow \exists X([fR] = [fX] \wedge \neg X[fR])$. Suppose $R[fR]$ for *reductio*. Then $\exists X([fR] = [fX] \wedge \neg X[fR])$. So there is some H such that $[fR] = [fH] \wedge \neg H[fR]$. By the first conjunct and PropAbst left-to-right, $fR \Leftrightarrow fH$. By Hyp, $R = H$. By Extensionality, $\forall p(Rp \leftrightarrow Hp)$. By Existence1, $R[fR] \leftrightarrow H[fR]$. Since $\neg H[fR]$, we contradict the *reductio* hypothesis and conclude $\neg R[fR]$. By the reflexivity of identity, $[fR] = [fR] \wedge \neg R[fR]$,²⁵ and so $\exists X([fR] = [fX] \wedge \neg X[fR])$. Thus, $R[fR]$. Contradiction.

This derivation assumes, besides PropAbst, Hyp, Comprehension, and Existence1, the Reflexivity of synonymy and Extensionality for sets. These last two principles are hardly controversial. The argument assumes, however, that propositions are special first-order objects – and this might be suspicious. To avoid such assumption, we can allow for variable

²⁵The reflexivity of identity among proposition-designating singular terms follows from PropAbst right-to-left and Reflexivity of hyperintensional synonymy.

binding in sentence position and interpret such variables as second order. We should then make some changes. First, we reformulate Existence1:

$$\psi(\varphi) \rightarrow \exists p\psi_{p/\varphi} \quad \text{Existence2:}$$

In Existence2, the bound propositional variable replaces a sentence, which is a syntactic constituent of the complex sentence in the antecedent. Secondly, predicates of propositions are systematically converted into *prejectives*, if they have a nominal argument, or else connectives.²⁶ That is, we reformulate an expression $R[fR]$ as RfR , with fR a sentential constituent of the latter. Finally, the sentential argument of the operator $[\cdot]$ can now be bound by a propositional quantifier. The simplest way to understand this is, semantically, to take $[\cdot]$ to be redundant. Syntactically, however, it does the same job as above: it converts a sentence into a singular term for the proposition it expresses. This allows for singular terms designating propositions to occur besides the identity relation (which presumably takes nominal arguments).²⁷ With these changes, we can show that:

Theorem. *Hyp, Comprehension, and Existence2 are jointly inconsistent in second-order logic with propositional variables, understood as second-order variables in sentence position.*

The derivation is similar. By Comprehension, but choosing a different comprehension formula, there exists a set R such that $\forall p(Rp \leftrightarrow \exists X(p \leftrightarrow fX \wedge \neg Xp))$. By Existence2, $RfR \leftrightarrow \exists X(fR \leftrightarrow fX \wedge \neg XfR)$. Suppose RfR for *reductio*. Then $\exists X(fR \leftrightarrow fX \wedge \neg XfR)$. So there is some H such that $(fR \leftrightarrow fH) \wedge \neg HfR$. By the first conjunct and Hyp, $R = H$. By Extensionality, $\forall p(Rp \leftrightarrow Hp)$. Hence $RfR \leftrightarrow HfR$. Since $\neg HfR$, we contradict the *reductio* hypothesis and conclude $\neg RfR$. By Reflexivity of synonymy, $(fR \leftrightarrow fR) \wedge \neg RfR$, and so $\exists X((fR \leftrightarrow fX) \wedge \neg XfR)$. Thus, RfR . Contradiction.

Both derivations have their merits, setting aside what one thinks of propositional variables. The second derivation is more faithful to Russell's reasoning in *Principles of Mathematics*. The first is helpful in order to understand what kind of paradox RM is: it can be seen as a paradox of abstraction, much like Russell's more famous paradox from Basic

²⁶See Prior, "Objects of Thought," *op. cit.* and Robert Trueman, "The Prejective View of Propositional Content," *Synthese*, CXCv, 4 (2018): 1799–825.

²⁷ Further degrees of freedom are possible. Church, "Russell's Theory of Identity of Propositions," *op. cit.*, who in turn follows Alfred N. Whitehead and Bertrand Russell, *Principia Mathematica* (Cambridge: Cambridge University Press, 1927), allows for the identity predicate to take propositional variables as arguments. This way, hyperintensional equivalence collapses on identity among variables.

Law V. This observation might help us articulating a response, insofar as we have a view about what could be wrong with Basic Law V.²⁸

As for Russell’s set-theoretic paradox, it is not mandatory to state the abstraction principle as a premise in order to show the inconsistency: there, it suffices to consider the set purportedly designated by the term ‘ $\{x \mid x \notin x\}$,’ and to assume that sets are extensional. Likewise, PropAbst need not be stated as a premise, so long as one assumes that ‘ $[fR]$ ’ designates a proposition, and that hyperintensionally equivalent propositions are identical. The second derivation brings out that what’s really inconsistent with Cantor’s theorem is Hyp (in fact, Hyp2): Hyp establishes an injection of the set of sets of propositions into the set of propositions.²⁹ Each set of propositions X is correlated to a proposition $[fX]$. Moreover, for two distinct sets of propositions, $X \neq Y$, Hyp entails that $\neg(\psi X \Leftrightarrow \psi Y)$, hence by PropAbst $[\psi X] \neq [\psi Y]$. Thus different sets of propositions are correlated to distinct propositions. But by Cantor’s theorem there can be no relation between propositions and their sets that is both functional and onto.

4 Too much hype?

According to Jeremy Goodman, rejection of Hyp ‘implies that reality is not structured in the manner of the sentence we use to talk about it.’ Gabriel Uzquiano agrees that RM ‘is best viewed as a constraint on propositional granularity’ that ‘appears to put pressure on the Russellian conception of propositions.’ This lesson is also endorsed by Cian Dorr, in an argument against the thesis that propositions are individuated as finely as the sentences that express them. According to these authors, RM is a *reductio* of Hyp. This diagnosis allows us to retain classicality and impredicativity.³⁰ There are two points to make.

First, the paradox does disappear if one thinks of propositions as coarse-grained, not just technically but conceptually. If propositions are sets of possible worlds, for example,

²⁸Russell did not regard abstraction principles as legitimate at the time he wrote the *The Principles of Mathematics*. There he says: ‘of the three kinds of definition admitted by Peano – the nominal definition, the definition by postulates, and the definition by abstraction ... I recognize only the nominal’ (*ibid.*, p. 112). Perhaps this explains, in part, why he did not present RM as a paradox of abstraction.

²⁹This observation is clearly stated by Goodman, “Reality is Not Structured,” *op. cit.*, in whose paper Hyp is called Structure.

³⁰Goodman, “Reality is Not Structured,” *op. cit.*, at p. 46; Uzquiano, “A Neglected Resolution of Russell’s Paradox of Propositions,” *op. cit.*, at p. 343; Dorr, “To Be F Is To Be G,” *op. cit.*, at pp. 63–64. Some possible strategies for rescuing the theory of structured propositions from this analysis are discussed by Peter Fritz, Harvey Lederman, and Gabriel Uzquiano, “Closed Structure,” *Journal of Philosophical Logic*, forthcoming.

there is no good reason why Hyp should be true. After all, sentences that are about distinct things need not express distinct sets of possible worlds. The proposition that $2 > 0$ is identical to the proposition that $3 > 0$, both being necessarily true propositions, even though $2 \neq 3$. However, nobody thinks that this is a good feature of possible world theories of propositions. Even more puzzling is to say that this bad consequence is handed down to us by logic alone.

Secondly, and relatedly, I do not wish to dispute what seems to be a robust consensus in the historical scholarship, on why Russell himself took the premises of RM to be plausible. He seems to have assumed that ‘identical propositions must have identical constituents.’³¹ By contemporary lights, this claim is controversial. So it is to assume that aboutness should be understood mereologically, as constituency in a structured proposition.³² Goodman, Uzquiano, and Dorr, take RM to rule out accounts of propositions such as those recently defended by Nathan Salmon, Scott Soames, Peter Hanks, and others.³³ If RM is valid, it does put pressure on theories of structured propositions. However, that is not the end of it. RM purports to rule out any account of propositions on which propositional identity is sensitive to aboutness considerations. While the source of such considerations may be related to sentential structure, the theory of aboutness may be separated from the metaphysics of structured propositions, and it still provides enough motivation for the premises.

For example, contemporary theories of aboutness such as those of Stephen Yablo or Kit Fine take propositional identity to imply sameness of topic. So they validate versions of Hyp. Should we be prepared to extend such theories, which are stated in first-order languages, to the higher-order domain (an extension against which I see no motivation), Yablo and Fine’s theories would likewise be shown inconsistent by RM.³⁴ Contrary to Goodman,

³¹Bernard Linsky, *Russell’s Metaphysical Logic* (Stanford, CA: CSLI Publications, 1999), at p. 155.

³²Joshua Armstrong and Jason Stanley, “Singular Thoughts and Singular Propositions,” *Philosophical Studies*, CLIV, 2 (2011): 205–22.

³³Nathan Salmon, *Frege’s Puzzle* (Cambridge, MA: MIT Press, 1986); Scott Soames, *Rethinking Language, Mind, and Meaning* (Princeton, NJ: Princeton University Press, 2015); Peter Hanks, *Propositional Content* (New York, NY: Oxford University Press, 2015).

³⁴See Yablo, “Aboutness,” *op. cit.* and Fine, “Angelic Content,” *op. cit.* It suffices to show that these theories satisfy Hyp, contradiction then follows from Comprehension and Existence1 or Existence2. Suppose for *reductio* that $a \neq b$ and let F be such that, at a world/state w , Fa and $\neg Fb$. Suppose moreover that Fa expresses the same proposition as Fb . (i) The Yablovian ‘thick proposition’ expressed by Fa is an ordered pair of the intension $|Fa|$ and the subject matter of Fa , which is a set of verifiers $|Fa|_V^+$ and falsifiers $|Fb|_V^-$. So we assume that $\langle |Fa|, \{|Fa|_V^+, |Fa|_V^- \} \rangle = \langle |Fb|, \{|Fb|_V^+, |Fb|_V^- \} \rangle$. It follows that $|Fa| = |Fb|$, contradicting the assumption. (ii) The Finean ‘angelic content’ of Fa is an ordered pair of the set of verifiers $|Fa|_F^+$ and falsifiers $|Fb|_F^-$. So if Fa and Fb express the same proposition, by the same

Uzquiano, and Dorr, it is not just a possibly controversial metaphysics of structured propositions that rests on Hyp, but the logic of aboutness itself. Taking away Hyp puts severe limits on our ability to study hyperintensionality in a formal language. The move against Hyp comes at a very high cost.

5 Dynamic Abstraction

A different approach seems worth exploring. Seeing the striking parallel between RM and Russell’s paradox from Basic Law V, a natural way to proceed is to drop the problematic existence assumption, Existence1 or Existence2 depending on preference. Standard set theory has that the problematic Russellian set of non-self-membered sets does not exist. Not all predicates determine a set. Rejection of Existence1 or Existence2 amounts to denying the existence of the problematic Russellian proposition. In particular, the Russellian sentence ‘ fR ’ does not determine a proposition available for reference and quantification.

There are various proposals for a “bottom up” conception of abstraction. One advantage is that grounded abstraction lets us formulate all kinds of intuitively plausible abstractions, including the infamous Basic Law V, without contradiction. A second advantage is that we avoid restricting the logic of aboutness for exogenous reasons. The proposal outlined below is a version of the strategy defended by Øystein Linnebo.³⁵

I proceed by officially dropping Existence1, and replace it with its *free* counterpart. (Here and below, there are two versions of the revised theory, according to the choice concerning propositional variables discussed above. For simplicity, official statements are given only for the first-order version.)

$$F[\psi] \rightarrow (\exists![\psi] \rightarrow \exists p F_{p/[\psi]}) \quad \text{Free Existence:}$$

As usual, $\exists![\psi]$ abbreviates an existence statement such as $\exists p(p = [\psi])$. We can now allow for failure of reference for terms such as $[\psi]$, that purport to designate the sense of an expression.

reasoning as above it follows that $|Fa|_F^+ = |Fb|_F^+$. So the sets of verifying states are coextensive. Hence in particular $w \in |Fa|_F^+$ and $w \in |Fb|_F^+$, contradicting the assumption. So both for Fine and for Yablo, if Fa expresses the same proposition as Fb , then $a = b$, so long as F discriminates between a and b at some w .

³⁵Linnebo, “Thin Objects,” *op. cit.* For alternatives, see James Studd, “Abstraction Reconceived,” *British Journal for the Philosophy of Science*, LXVII, 2 (2016): 579–615; Hannes Leitgeb, “Abstraction Grounded: A Note on Abstraction and Truth,” in P. Ebert and M. Rossberg, eds., *Abstractionism: Essays in the Philosophy of Mathematics* (Oxford: Oxford University Press, 2016), pp. 269–82.

PropAbst must be weakened accordingly. PropAbst implicitly assumes that the grammaticality of a sentence suffices for reference to the proposition it expresses. For example, ‘ fR ’ is grammatical according to the type theory, but there is no proposition it expresses that we can refer to or quantify over. This lesson may sound incongruous: we use ‘ fR ’ in proof, so there must be something it says. On my view, it is best to think that the sentence does express a proposition, but that such proposition lies outside the initial domain of discourse. In other words, grammaticality suffices merely for the possible existence of the proposition expressed by a sentence: existence that depends, in some sense, on the objects the sentence is about.³⁶ Two claims are important:

- (i) Propositions are, in some sense, language-dependent: they are abstracts of sentences.
- (ii) What is expressed by sentences may not be what is referred to by singular terms that purport to designate propositions.

Suppose that whatever we can refer to is collected in a domain D . Given the objects in D , we may express a proposition about them, though that proposition may fail to be in D , lest we would be able to refer to it. This would blur the line between what can be referred to and what can be expressed. We could, of course, consider an extended domain D' which includes the propositions expressed by sentences that are about elements of D . However, sentences about the elements of D' would express propositions that are not in D' . And so on.³⁷

These considerations suggest that PropAbst is better replaced by a modal principle, which for clarity I state as the combination of two components, about the existence and the identity of propositions respectively:

$$\text{PAbst1: } \diamond \exists p(p = [\psi])$$

$$\text{PAbst2: } \Box \forall p \forall q((p = [\psi] \wedge q = [\varphi]) \rightarrow (p = q \leftrightarrow \psi \leftrightarrow \varphi))$$

³⁶The claim that grammaticality doesn’t suffice for actual existence doesn’t fit well with a Neo-Fregean interpretation of abstraction principles, and appears to be in tension with the so-called Syntactic Priority Thesis. This seems to imply that, more generally, the reading I am advocating for abstraction doesn’t fit well with the Neo-Fregean program.

³⁷There are various options concerning the notion of dependency in (i), one of which is contingentism. See Robert Stalnaker, “Merely Possible Propositions,” in B. Hale and A. Hoffmann, eds., *Modality* (Oxford: Oxford University Press, 2010), pp. 21–34; Peter Fritz, “Propositional Contingentism,” *The Review of Symbolic Logic*, IX, 1 (2016): 123–42. A non-metaphysical interpretation of the modality is presented by Kit Fine, “Relatively Unrestricted Quantification,” in A. Rayo and G. Uzquiano, eds., *Absolute Generality* (New York, NY: Oxford University Press, 2006), pp. 20–44; further discussion could benefit from Øystein Linnebo and Stewart Shapiro, “Actual and Potential Infinity,” *Noûs*, LIII, 1 (2019): 160–91. I remain open to these various options.

PAbst1 is a schema. Any of its instances states that it is possible that there is a proposition expressed by ψ . Any instance of schema PAbst2 states that necessarily for any p and q if p is the proposition expressed by ψ and q is the proposition expressed by φ , then p is q just in case ψ and φ are synonymous. Following Linnebo, the modality is constrained by the S4.2 axioms.³⁸

The notion of hyperintensional equivalence is as above, and so is the unrestricted comprehension principle. Of course, comprehension remains non-modal. The logic is overall predicative, since RM can be understood as a *reductio* of the modal impredicative comprehension statement $\exists X \Box \forall p (Xp \leftrightarrow \psi p)$. However, such statement is implausible: there is no reason to think that there is a set of all propositions that could possibly be expressed, and that satisfy ϕ . There is, of course, a set of all propositions that exist, and that satisfy ψ , but collections of “all possible propositions” are blocked. The present work belongs to a tradition of modal approaches to the set-theoretic and the semantic paradoxes, versions of which are explored also by Geoffrey Hellman and Roy Cook, among others.³⁹

We may think of the initial domain D as containing absolutely everything in the range of the first-order individual variables (of type e), together with all sets of them. The combination $\diamond \exists$ may then be understood as introducing more things to such domain: in particular, the propositions expressed by sentences about objects in D . These extra things are, of course, not individuals, nor sets of them, but abstracts of sentences. These can all be collected into sets, together with the objects that were already there. By abstraction, further propositions about this wider domain can be introduced. And so on.

6 Prior’s Troubles

The predicative conception of propositions I outlined relies on a distinction between propositions expressed by sentences and propositions that can be referred to by the use of a singular term, at any given stage in the abstraction process, and on the idea that propositions about a given domain of objects depend on those objects. This conception provides an account of various puzzles about propositions. We might look at these further consequences

³⁸Linnebo, “The Potential Hierarchy of Sets,” *op. cit.* Besides the rule of Necessitation, and the familiar laws K, M, and 4, namely $\vdash \Box(\psi \rightarrow \phi) \rightarrow (\Box\psi \rightarrow \Box\phi)$, $\vdash \Box\psi \rightarrow \psi$, and $\vdash \Box\psi \rightarrow \Box\Box\psi$ respectively, the logic includes axiom G: $\vdash \Box\diamond\psi \rightarrow \diamond\Box\psi$. These conditions are intended to isolate a plausible notion of possibility that is of service to modalized abstraction principles.

³⁹Geoffrey Hellman, “On the Significance of the Burali-Forti Paradox,” *Analysis*, LXXI, 4 (2011): 631–37; Roy Cook and Geoffrey Hellman, “Extendability and Paradox,” in R. Cook and G. Hellman, eds., *Hilary Putnam on Logic and Mathematics* (Cham: Springer, 2018), pp. 51–75.

to assess the fruitfulness of the predicative conception.

Consider, for example, the Prior-Kaplan paradox, also known as the Intensional Liar.⁴⁰ The paradox is best formulated by means of second-order propositional variables,⁴¹ and thus assuming Existence2. Consider the claim that any proposition materially equivalent to the proposition that $2 + 2 = 4$ is itself necessary. This claim is of course not true, since the proposition that grass is green is contingent and materially equivalent to $2 + 2 = 4$. But, perhaps, at some possible world a proposition is necessary if and only if it is materially equivalent to $2 + 2 = 4$. More generally, it seems intuitively possible that any proposition materially equivalent to some proposition ψ has property Q . This is Kaplan's principle. David Kaplan defends its intuitive plausibility, and argues that if it is false, it should be falsified by our metaphysical commitments, not by logic itself.⁴²

$$\Diamond \forall q (Qq \leftrightarrow (\psi \leftrightarrow q)) \quad \text{KP:}$$

KP is schematic in ψ and Q . Consider the proposition that nothing that is believed by Epimenides is the case: $\forall p (Qp \rightarrow \neg p)$, where Q now stands for 'is believed by Epimenides.' For brevity, let us shorten it ' γ .' Suppose that γ is true for *reductio*. An instance of KP is that $\Diamond \forall q (Qq \leftrightarrow (\gamma \leftrightarrow q))$. So, at some world, $\forall q (Qq \leftrightarrow (\gamma \leftrightarrow q))$. By Existence2, $Q\gamma \leftrightarrow (\gamma \leftrightarrow \gamma)$, hence $Q\gamma$. Moreover from the *reductio* assumption and Existence2, it follows that $Q\gamma \rightarrow \neg\gamma$. Hence $\neg\gamma$ and we conclude the *reductio*. By DeMorgan, $\exists p (Qp \wedge p)$, that is, something believed by Epimenides is the case. Let p' be such proposition. Then $Qp' \leftrightarrow (\gamma \leftrightarrow p')$, and so γ is the case. Contradiction.⁴³

The proof of the Prior-Kaplan paradox doesn't go through if we are required to be careful about which propositions are quantified over, at any given world, for the simple reason that the world at which the proposition expressed by $\forall p (Qp \rightarrow \neg p)$ exists, by PAbst1, may not be the world at which, by KP, everything materially equivalent to that proposition is believed by Epimenides. Intuitively, the proposition expressed by the sentence 'Nothing believed by Epimenides is the case' possibly exists. It is also possible that anything be-

⁴⁰Prior, "Objects of Thought," *op. cit.*; Kaplan, "A Problem in Possible World Semantics," *op. cit.*; Andrew Bacon, John Hawthorne, and Gabriel Uzquiano, "Higher-Order Free Logic and the Prior-Kaplan Paradox," *Canadian Journal of Philosophy*, XLVI, 2 (2016): 493–541.

⁴¹This allows us to avoid the use of a truth predicate. Since so much has been written on the truth predicate in connection with the paradoxes, its use would be a distraction. However, the truth predicate is irrelevant for RM.

⁴²Kaplan, "A Problem in Possible World Semantics," *op. cit.*

⁴³The proof is strictly classical, following Prior's original. Kaplan's version requires stronger principles and is formulated in possible world semantics. There is also an intuitionistic version.

lied by Epimenides is materially equivalent to the proposition that nothing believed by Epimenides is the case. But these two possibilities are distinct, and it is a simple modal fallacy to conflate them. Of course, should KP be strengthened to $\Box\forall q(Qq \leftrightarrow (\gamma \leftrightarrow q))$ to avoid the impasse, we would then recognize it as a necessitated instance of comprehension. The Prior-Kaplan paradox would then provide additional evidence, besides RM, that necessitated comprehension is false.

Prior discusses another argument, and the last I will consider in this paper. Consider again the proposition γ , that nothing believed by Epimenides is the case. Suppose that Epimenides believes γ , that is, believes that $Q\forall p(Qp \rightarrow \neg p)$, and that γ is the case, for *reductio*. Then by Existence2, $Q\gamma \rightarrow \neg\gamma$. By modus ponens and *reductio*, $\neg\gamma$. Therefore $Q\gamma \wedge \neg\gamma$, hence $\exists p(Qp \wedge \neg p)$ by Existence2. On the other hand, since $\neg\gamma$, by DeMorgan $\exists p(Qp \wedge p)$. Prior thus concludes what is known as Prior's Theorem:

$$Q\forall p(Qp \rightarrow \neg p) \rightarrow \exists p(Qp \wedge \neg p) \wedge \exists p(Qp \wedge p) \quad \text{PT:}$$

If Epimenides believes that nothing he believes is the case, then he believes something false, and something true as well. As Williamson remarks, this just looks to be straightforwardly false.⁴⁴ Notice that, in the reasoning, we do conclude that $\neg\gamma$. Indeed, that's a false proposition that Epimenides believes. We are left puzzling about what might be the true proposition he believes. On an impredicative conception of propositions, the universe of propositions is, as it were, *oddly* plenitudinous.

Prior himself had little to comment about PT but to quip that ‘there are surprises in logic.’⁴⁵ Thus there might be comfort in the observation the derivation doesn't go through on a predicative conception of propositions. In particular, even if we assume that nothing believed by Epimenides is the case, $\forall p(Qp \rightarrow \neg p)$, the proposition expressed by such sentence is not in its own domain of quantification. Even if Epimenides believes that nothing he believes is the case, and it is true that nothing he believes is the case, it doesn't follow that there is something true that he believes.

⁴⁴Timothy Williamson, “Reply to Bacon, Hawthorne, and Uzquiano,” *Canadian Journal of Philosophy*, XLVI, 4/5 (2016): 542–47, at p. 542.

⁴⁵Prior, “Objects of Thought,” *op. cit.*, at p. 80.

There is, however, a modally strengthened version of PT that does follow:⁴⁶

$$\Box Q\forall p(Qp \rightarrow \neg p) \rightarrow \Diamond \exists p(Qp \wedge \neg p) \wedge \Diamond \exists p(Qp \wedge p) \quad \Box\text{PT:}$$

Yet this formula hardly deserves the honorific title of theorem. It says that if Epimenides believes, of necessity, that nothing he believes is the case, then he could believe a truth and he could believe a falsehood. Since there is no plausible sense of necessity on which someone, including Epimenides, necessarily believes that nothing he believes is the case, $\Box\text{PT}$ is true simply because it is a material conditional with a false antecedent.

I do not expect that these results should strike anyone as the obvious ones to get concerning these puzzles. At this point, however, it is interesting to compare the two conceptions of the metaphysics of propositions I have contrasted. In the Prior-Kaplan paradox, someone who wishes to maintain Existence1 or Existence2 would have to deny KP. *A fortiori*, it is impossible that only one proposition be the unique bearer of a property. Indeed, PT comes out a logical truth.⁴⁷ However, even a plenitudinous universe is no relief with the Russell-Myhill paradox. There, as we discussed, one has to be denied that propositions are hyperintensional. The impredicative universe of proposition is therefore rich, perhaps oddly so, but also coarse-grained. In contrast, from a predicative perspective, a general approach yields a unified pattern of response that is shown to be robust across a range of puzzles. The universe is not as plenitudinous, but distinctions in reality may be sharply reflected in thought.

7 Conclusion

The main topic of discussion has been the Russell-Myhill paradox, which I presented as a paradox of aboutness. I then considered a recent interpretive line, according to which the paradox shows that propositions are not structured objects. I have argued that this reply goes too far: it must deny that propositions about distinct objects are distinct. This position is incompatible with a substantial notion of aboutness, on which the identity of

⁴⁶Let γ be $\forall p(Qp \rightarrow \neg p)$. Suppose $\Box Q\gamma$. By PAbst1, $\Diamond \exists p([p] = \gamma)$. Consider a world w in which the proposition expressed by γ exists. Suppose that γ is the case at w for *reductio*. By Free Existence, $Q\gamma \rightarrow \neg\gamma$. By $\Box Q\gamma$, we have $Q\gamma$ at w . Hence $\neg\gamma$, and we conclude the *reductio*. Therefore, at w , both $Q\gamma$ and $\neg\gamma$. Thus $\exists p(Qp \wedge \neg p)$ at w . Moreover by $\neg\gamma$, from DeMorgan, $\exists p(Qp \wedge p)$ at w . Closing the modal and conditional reasoning yields $\Box\text{PT}$.

⁴⁷Andrew Bacon and Gabriel Uzquiano, “Some Results on the Limits of Thought,” *Journal of Philosophical Logic*, XLVII, 6 (2018): 991–99.

propositions is sensitive to aboutness distinctions.

I have then reconstructed the Russell-Myhill paradox as one engendered by abstraction, and argued that it is not the result of a confusion between use and mention. Once a choice about propositional variables is made, the paradox follows – unless, that is, one maintains a conceptual distinction between propositions that can be referred to and quantified over and propositions that can be expressed by doing so. This distinction offers a good motivation to apply a version of the general strategy to deal with impredicativity outlined, for example, in the work of Linnebo.

In a way, Frege was right in thinking that his distinction between sense and reference undermines the Russell-Myhill paradox. Something like Frege’s distinction can be enforced in a modal setting, if we allow that anything expressed at a stage can be referred to at the next. Further puzzles about propositions are explained away by the same proposal, yielding a unified and well-motivated picture of semantic reality.

Appendix: Semantics

The framework described here can be interpreted in a positive free logic with dual domain semantics within a Kripke frame. The semantics I shall describe is intended for illustrative purposes, and to show consistency. There are two versions of the framework discussed above.

Version 1: First-order propositional variables

Let $E \neq \emptyset$, the sort of individuals, and P_n , the sort of propositions, be disjoint sets. A model M_n is a structure $\langle E, P_n, a_n^\sigma \rangle$, with a an interpretation function relative to a variable assignment σ . For brevity, for every linguistic expression f , I write ‘ f^a ’ for $a_n^\sigma(f)$. E is the (fixed) domain of type e , P_n is the domain of type $()$ in M_n .

A frame \mathbf{M} is a list of models $\langle M_0, M_1, \dots \rangle$ ordered by the relation \leq , with the following properties:

Reflexivity: $n \leq n$

Antisymmetry: if $n \leq m$ and $m \leq n$ then $n = m$

Transitivity: if $n \leq m$ and $m \leq n'$ then $n \leq n'$

Convergence: if $n \leq m$ and $n \leq m'$ then there is a n' , $m \leq n'$ and $m' \leq n'$

Monotonicity: if $n \leq m$ then $P_n \subseteq P_m$

The first four properties interpret the S4.2 axioms. Each model $M_n \in \mathbf{M}$ has a domain $D_n := E \cup P_n$. Note that if $n \leq m$ then $D_n \subseteq D_m$. Let $\mathbf{D} := \bigcup_n D_n$ be the so-called ‘outer domain’ of \mathbf{M} . The interpretation function works as follows:

$$\begin{aligned} x^a &\in E \\ p^a &\in P_n \\ X^a &\in Pow(\mathbf{D}) \end{aligned}$$

The set of terms t includes first-order individual variables, propositional variables, second-order variables, and singular terms designating propositions. Terms are interpreted as usual, and in addition, $([\psi])^a \in (\mathbf{D} - E)$. Let $\{0, 1\} \subseteq E$ be two special objects of type e . Formulas are interpreted on $\{0, 1\}$:

$$\begin{aligned} (Xt_1, \dots, t_n)^a = 1 &\quad \text{iff } \langle (t_1)^a, \dots, (t_n)^a \rangle \in X^a \\ (\neg\psi)^a = 1 &\quad \text{iff } \psi^a = 0 \\ (\psi \rightarrow \varphi)^a = 1 &\quad \text{iff } \psi^a = 0 \text{ or } \varphi^a = 1 \\ (x = y)^a = 1 &\quad \text{iff } x^a = y^a \\ (\psi \Leftrightarrow \varphi)^a = 1 &\quad \text{iff } ([\psi])^a = ([\varphi])^a \\ (\exists x\psi)^a = 1 &\quad \text{iff there is a } d \in E, \psi^{a[x/d]} = 1 \\ (\exists p\psi)^a = 1 &\quad \text{iff there is a } d \in P_n, \psi^{a[x/d]} = 1 \\ (\exists X\psi)^a = 1 &\quad \text{iff there is a } d \in Pow(\mathbf{D}), \psi^{a[x/d]} = 1 \\ a_n^\sigma(\diamond\psi) = 1 &\quad \text{iff there is a } M_m \in \mathbf{M}, n \leq m \text{ and } a_m^\sigma(\psi) = 1 \end{aligned}$$

Note that (i) since $\psi^a \in \{0, 1\}$ for all formulas ψ , the relation $[\cdot]$ takes an object in $\{0, 1\}$ and assigns it to an element of $\mathbf{D} - E$. (ii) It would be straightforward, however, to interpret formulas relative to a set W of possible worlds. Of course \diamond , defined as above, wouldn’t thereby range over elements of W . (iii) There are no propositional variables defined over \mathbf{D} , and *a fortiori* no unrestricted propositional quantifiers.

Hence, the metalanguage is strictly stronger than the object language. This is perhaps philosophically disturbing.⁴⁸ A proper discussion of this topic is both beyond the aims of this paper and too complex to be settled here.

⁴⁸G. Aldo Antonelli, “Proto-Semantics for Positive Free Logic,” *Journal of Philosophical Logic*, XXIX, 3 (2000): 277–94.

Version 2: Second-order propositional variables

We define $P_n := T_n \cup F_n$, with $T_n \cap F_n = \emptyset$. Intuitively, T_n and F_n are the set of true and false propositions respectively. The notions of model, frame, domain, and outer domain remain the same, with the additional requirement that if $n \leq m$ then $T_n \subseteq T_m$ and $F_n \subseteq F_m$. Let the set of n -ary propositional functions be $S := \{s : \mathbf{D}^n \mapsto (\mathbf{D} - E)\}$, with \mathbf{D}^n the n -fold Cartesian product of \mathbf{D} . The interpretation function may now send predicates to propositional functions:

$$\begin{aligned} x^a &\in E \\ p^a &\in P_n \\ X^a &\in S \end{aligned}$$

The denotation of terms is defined as above. Formulas, which now designate propositions, are interpreted by allocating the propositions expressed into T_n or F_n , for each model.

$$\begin{aligned} \psi^a \in T_n & \quad \text{iff it is not the case that } \psi^a \in F_n \\ (Xt_1, \dots, t_n)^a \in T_n & \quad \text{iff } X^a \langle (t_1)^a, \dots, (t_n)^a \rangle \in T_n \\ (\neg\psi)^a \in T_n & \quad \text{iff } \psi^a \in F_n \\ (\psi \rightarrow \varphi)^a \in T_n & \quad \text{iff } \psi^a \in F_n \text{ or } \varphi^a \in T_n \\ (x = y)^a \in T_n & \quad \text{iff } x^a = y^a \\ (\psi \Leftrightarrow \varphi)^a \in T_n & \quad \text{iff } ([\psi])^a = ([\varphi])^a \\ (\exists x\psi)^a \in T_n & \quad \text{iff there is a } d \in E, \psi^{a[x/d]} \in T_n \\ (\exists p\psi)^a \in T_n & \quad \text{iff there is a } d \in P_n, \psi^{a[x/d]} \in T_n \\ (\exists X\psi)^a \in T_n & \quad \text{iff there is a } d \in S, \psi^{a[x/d]} \in T_n \\ a_n^\sigma(\diamond\psi) \in T_n & \quad \text{iff there is a } M_m \in \mathbf{M}, n \leq m \text{ and } a_m^\sigma(\psi) \in T_m \end{aligned}$$

Note that in this case the relation $[\cdot]$ is the identity function on $\mathbf{D} - E$. A valuation $v_n : Form \mapsto \{0, 1\}$ can be defined, with $Form$ the set of formulas, and 0, 1 two designated individuals in E . It is enough to stipulate that $v_n(Xt_1, \dots, t_n) = 1$ iff $(Xt_1, \dots, t_n)^a \in T_n$. Then v_n can be extended to all formulas in accordance with the clauses above.