A Fuzzy Application of Techniques from Topological Supersymmetric Quantum Mechanics to Social Choice Theory: A New Insight on Flaws of Democracy

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Abstract

We introduce a new theorem in social choice theory built on a path integral approach which will show that, under some reasonable conditions, there is a unique way to aggregate individual preferences based on fuzzy sets into a social preference based on probabilities, and that this way is invariant under any permutation of alternatives. We then apply this theorem to the case of democratic decision making with data of the behaviour and voting preferences of voting agents and show that there is a tradeoff between fairness and efficiency and that no voting system can achieve both simultaneously.

1 Introduction

In this day and age, we often hear about how democracy is the best political system because of its promises such that it will give equal representation to the individuals who interact within its system. Even with all of these promises which appear to be good to the eyes of many, democracy faces many challenges and criticisms, including majority tyranny, electoral conflict, and the influence of finance and media on public opinion. It is in social choice theory, the field which examines how individual choices or collective decisions are made to produce decisions or outcomes, where we see some of the main criticisms of democracy. Social choice theory proposes that many choices or pathways of social selection are either going to turn out to be wrong or impossible under certain circumstances. Just as an example, Arrow's theorem states that no simultaneous voting can reach consensus, regardless of other non-uniform, non-controlling, and unrestricted alternatives. These are some of the necessary criteria for fair and effective political elections, but they are from social choice theory we see that these become inconsistent and one thence could question whether the framework of democracy is consistent. Here we propose a new theorem that questions the logical structure of democracy from a quantum social scientist's point of view through a fuzzy scope.

2 Quantum Framework

Theorem 1. *Let X be a compact Hausdorff space and F be a fuzzy set on X. Suppose that F is invariant under a quantum group action of G on X. Then there exists a unique probability measure μ on X such that $\mu(F) = 1$ and μ is also invariant under the action of G.*

Proof of Theorem 1. We will start the proof by using the path integral approach to supersymmetric quantum mechanics and topology. Consider the supersymmetric quantum mechanical model with target space X, Hilbert space $L^2\ (X)$, and Hamiltonian $H = -\Delta + V$, where Δ is the Laplace-Beltrami operator and V is a potential function. We assume that H commutes with the action of G, so that the Hilbert space decomposes into irreducible representations of G. We also assume that there exists a supersymmetry operator Q such that $Q^2 = H, Q^\dagger = Q$, and $[Q, H] = 0$. Then the partition function of the model is given by □

$$Z = \int_P e^{-S} \mathcal{D}x\mathcal{D}\psi \tag{1}$$

where P is the space of paths on X, $DxD\psi$ is the path integral measure, and S is the action functional. We can rewrite the action as

$$S = \int_0^1 dt \left(Q\psi^\dagger + \frac{1}{2}x - iQx \right)^2 + Q \int_0^1 dt \left(\psi^\dagger V(x) - iV(x)x \right) \tag{2}$$

By using the invariance of the measure under supersymmetry transformations, we are able to perform a mutation of variables in order to eliminate the quadratic term in the action. This leads to:

$$Z = e^{-W} \int_P e^{-QS_0} \mathcal{D}x\mathcal{D}\psi \tag{3}$$

where $W = -\log Z_0$ is the Witten index of the model, and S_0 is the topological action

$$S_0 = \int_0^1 dt \left(\psi^\dagger V(x) - iV(x)x \right) \tag{4}$$

By supersymmetric localisation, the path integral localises to the fixed points of Q, which are precisely the critical points of V. Let $\{x_i\}$ be the set of critical points of V, and let λ_i be their corresponding eigenvalues of H. Proceeding, we then acquire:

$$Z = e^{-W} \sum_i e^{\lambda_i} I_i \tag{5}$$

where I_i are one-loop determinants around the fixed points. By using the Atiyah-Bott localisation formula, we can then express these determinants as the following:

$$I_i = \frac{\text{Pfaff}\left(Q|_{T_{x_i}X}\right)}{\sqrt{\det'\left(H|_{T_{x_i}X}\right)}} \tag{6}$$

where Pfaff denotes the Pfaffian, and $\det'$ denotes the determinate that lacks zero modes. Since $Q^2 = H$, we acquire $\text{Pfaff}(Q|_{T_{x_i}X}) = (\det'(H|_{T_{x_t}X}))^{\frac{1}{2}}$, and thus we get:

$$I_i = 1 \tag{7}$$

Henceforth we will now use a fuzzy set function on X which will assign a degree of membership within a fuzzy set F to each point $x \in X$. We are to first hold the assumption that this function is invariant under the action of G, so that it defines a class function on the irreducible representations of G. In this way, we are able to subsequently define a twisted partition function with the insertion of this function into the path integral as follows:

$$Z_F = e^{-W} \int_P F(x(1)) e^{-QS_0} DxD\psi \tag{8}$$

With the same localisation argument that we have used prior, we can now write this as:

$$Z_F = e^{-W} \sum_i F(x_i) e^{-} \lambda_i \tag{9}$$

We should be interpreting this twisted partition function as a probability measure on the set of the critical points of V, where the probability of x_i is proportional to $F(x_i)e^{-}\lambda_i$. We can normalise this measure by dividing by Z, and thus we acquire the quotient:

$$\mu_F(x_i) = \frac{F(x_i)\, e^{-\lambda_i}}{\sum_j F(x_j)\, e^{-\lambda_j}}. \tag{10}$$

We now have a well-defined probability measure on the set of the critical points of V which also happens to satisfy $\mu_F(F) = 1$ by the construction. It is also good to note that while under the action of G, it is invariant, for since on the irreducible representations of G both F and $e^{-\lambda_i}$ are class functions. We should also extend this probability measure to the whole space X and to do that we will have to use the fuzzy integral which is the Lebesgue integral's generalisation and this allows us to integrate fuzzy sets with respect to fuzzy measures. The fuzzy set A with respect to a fuzzy measure v will have a fuzzy integral that is defined as:

$$\int A dv = \sup_{x \in X} \min(A(x), v(x)) \tag{11}$$

Now because of sup and min this means intuitively that the fuzzy integral is the largest value that both A and v may attain on X. With the usage of this fuzzy integral, we may define a probability measure μ on X by setting

$$\mu(A) = \int A d\mu_{F'} \tag{12}$$

where A is any Borel subset of X. This is a well-defined probability measure, since it satisfies the axioms of probability. It also agrees with μ_F on the set of critical points of V, since for any x_i, we have:

$$\mu(\{x_i\}) = \int \{x_i\}\, d\mu_F = \sup_{x \in X} \min(F(x), \mu_F(X)) = 1 \tag{13}$$

where the last equality follows from the fact that $\mu_F(F) = 1$. Finally, it is invariant under the action of G, since both F and μ_F are invariant under the action of G. Therefore, we have constructed a probability measure μ on X that satisfies the properties of fuzzy integrals. To prove the uniqueness of μ, suppose that there exists another probability measure v on X that satisfies $v(F) = 1$ and v under the action of G happens to be invariant. Then we have

$$v(A) = v(A \cap F) + v(A \cap F^c) = v(A \cap F) \tag{14}$$

where F^c denotes the complement of F. This now implies that $v(A) = 0$ for another Borel subset A of X that does not intersect F. Particularly, this means that $v(\{x_i\})$ for any critical point x_i that does not belong to F. On the other hand however, for any critical point x_i that belongs to F, we have:

$$v\left(\{x_i\}\right) = \frac{v\left(\{x_i\}\right)}{v(F)} = \frac{v\left(\{x_i\}\right)}{\sum_j F(x_j) e^{-\lambda_j}} \tag{15}$$

in which we have used the fact that $v(F) = Z_f$. When comparing this expression for $\mu\left(\{x_i\}\right)$, we are able to see that they must indeed be equal, for since they are both normalised probabilities on the same finite set. Therefore, we acquire $v(\{x_i\}) = \mu(\{x_i\})$ for all critical points x_i of V. Since we now see that v and μ are probability measures on X, they must therefore agree on every single Borel subset of our whole space X, as per the uniqueness of extension theorem. As such, we now finally hold that $v = \mu$ and thus the theorem has been proven.

3 Voting System Integration

Now that we have proven the first theorem, we are ready to move on and integrate a voting system which will have its foundations based on the first theorem. Firstly, we should make some assumptions such as that the set of alternatives X represents the possible outcomes of a democratic decision, such as electing a new leader, passing laws, or choosing a national policy. Also, the voters would have fuzzy preferences over the alternatives, meaning that they do not necessarily have a strict ranking of the alternatives rather assigning degrees of membership to each alternative that happens to be within a fuzzy set F. As the degree of membership becomes higher, so too does the alternative's preferability. Since we are constructing this voting system on the first theorem, the voting system in this context is now meant to be aggregating the individual preferences based on F into a social preference based on m, in which we recall that m is the probability measure which we constructed earlier in the theorem.

We will say that a democratic decision is fair if it is true that it respects the voters' preferences, that is, if the social preference agrees with the majority of the indivdiual preferences. We will also say that such a democratic decision is to be deemed efficient if it is also true that the expected utility of the society turns out to be maximised by that democratic decision, that is, if the social preference, with the highest probability according to μ, chooses the alternative. We can now begin our attempt which will show that democracy is flawed by its nature in due of which a trade-off does indeed exist between efficiency and fairness, and that there would be no possible voting system which is able to overcome this tradeoff.

Theorem 2. *Let X be a finite set of policy positions that exist within a left-right scale, and F be a fuzzy set function on X. Suppose that there is some voting system that exists in which anonymity, neutrality, unanimity, and the decision theory axom IIA, and that maps any profile of individual preferences based on F to a social preference based on μ, in which μ is the probability measure from Theorem 1. Then μ is also invariant under the action of S_n and it is also unique up to a multiplicative constant.*

Proof of Theorem 2. Let us first assume that the set of options X is finite and that it also consists of n alternatives, which would be denoted by $x_1, x_2, ...x_n$. Assume also that the quantum group G is the symmetric group S_n which, by the permutation of n alternatives, acts on X.

Definition 1. *A voting system is a function which maps a profile of individual preferences over X to a social preference over X. A profile of individual preferences is a list of preference oders over X, one for each voter. A preference order over X is a complete, transitive, and antisymmetric binary relation on X. A social preference over X is also a preference order of X.*

Definition 2. *A voting system satisfies anonymity if all voters have been treated equally, that is, the voters are permuted within a profile in which the social preference does not change.*

Definition 3. *A voting system satisfies neutrality if all alternatives are equally treated, so applying any permutation in S_n to both the profile and the social preference does not result in the social preference being changed*

Now note that anonymity implies that the fuzzy set function F is invariant under any permutation of the voters and also that neutrality implies that it is invariant under any permutation of the alternatives. By one of the assumptions we made, we then conclude that F is invariant under the action of S_n. In order for us to see this, let

σ be any permutation of the voters, and let τ be any permutation of the alternatives. Then for any voter v and any alternative x, we hold have the equation:

$$F(\sigma)(v), \tau(x)) = F(v, x) \quad (16)$$

since applying σ to v does not change their preference order over X, and applying τ to x does not change its position on the left-right scale. Secondly, note that unanimity implies that $\mu(F) = 1$, since if all voters have a preference for x_i to all the other alternatives, then $\mu(\{x_i\})$. Let's look at this a bit closer by first supposing that for some i, we have

$$F(v, x_i) = 1 \quad (17)$$

for all voters v. This would then mean that all voters assign full membership to x_i in their fuzzy set of preferred policies. By the definition of μ, we have:

$$\mu(\{x_i\}) = \frac{\sum_v F(v, x_i) e^{-\lambda_i}}{\sum_j \sum_v F(v, x_j) e^{-\lambda_j}} = \frac{e^{-\lambda_j}}{\sum_j e^{-\lambda_j}} \quad (18)$$

in which we used the fact that $\sum_v F(v, x_i) = 1$ and also that $\sum_v F(v, x_j) = 0$ for all $j \neq i$. Now since $\sum_j e^{-\lambda_j} > 0$, we have $\mu(\{x_i\}) = 1$. Thus by the definition of $\mu(F)$, we now have:

$$\mu(F) = \sup_{x \in X} (\sum_v (F(v, x) - F^c(v, x))) \quad (19)$$

in which the complement of F, that being F^c, assigns to each element its degree of non-membership in F. Since $F(v, x_i) = 1$ amd $F^c(v, x_i) = 0$ for all voters v, we then have:

$$\mu(F) \geq \sum_v (F(v, x_i) - F^c(v, x_i)) = 1 \quad (20)$$

On the contrary, because $\mu(F)$ is a probability measure, then we also have:

$$\mu(F) \leq 1 \quad (21)$$

We therefore now conclude with the fact that $\mu(F) = 1$. □

We must thirdly note that IIA implies that μ is only dependent on the eigenvalues λ_i of the Hamiltonian H because changing the relative ranking of two alternatives that are not x_i or x_j does not affect their eigenvalues. Suppose that for some voters v_1 and v_2 and some alternatives x_k and x_l, we have the following:

$$F(v_1, x_k) > F(v_1, x_l) \quad (22)$$

and

$$F(v_2, x_k) < F(v_2, x_l) \quad (23)$$

What this tells us it that the voter v_1 would prefer x_k to x_l while voter v_2 would instead prefer x_l to x_k. Suppose that their preferences are changed by us through swapping their degrees of membership for these two alternatives, that is in which we set:

$$F'(v_1, x_k) = F(v_2, x_k) \quad (24)$$

and

$$F'(v_1, x_l) = F(v_2, x_l) \tag{25}$$

and also vice versa. So now this means that voter v_1 would prefer x_l to x_k and voter v_2 now holds the preference of x_k to x_l. We keep their preferences for all other alternatives unchanges by setting:

$$F'(v_1, x_m) = F(v_1, x_m) \tag{26}$$

and

$$F'(v_2, x_m) = F(v_2, x_m) \tag{27}$$

for all $m \neq k, l$. This is a change in the relative ranking of two alternatives that are not x_i or x_j, where $i, j \neq k, l$. By IIA, this change should not be able to affect the relative ranking of x_i and x_j in the social preference. Therefore, the equation that we will have should be:

$$\mu'(\{x_i\}) = \mu(\{x_i\}) \tag{28}$$

and

$$\mu'(\{x_j\}) = \mu(\{x_j\}) \tag{29}$$

in which μ' is the probability measure that is constructed from F'. Although by the definition of μ we then have:

$$\mu'(\{x_i\}) = \frac{\sum_v F'(v, x_i) e^{-\lambda_i}}{\sum_m \sum_v F'(v, x_m) e^{-\lambda_i}} \tag{30}$$

and

$$\mu'(\{x_j\}) = \frac{\sum_v F'(v, x_j) e^{-\lambda_i}}{\sum_m \sum_v F'(v, x_m) e^{-\lambda_j}} \tag{31}$$

Due to the fact that we only changed the preferences of two voters for two alternatives, we are thus able to write these expressions as:

$$\mu'\left(\{x_i\}\right) = \frac{\sum_{v \neq v_1, v_2} F(v, x_i) e^{-\lambda_i} + F'(v_1, x_i) e^{-\lambda_i} + F'(v_2, x_i) e^{-\lambda_i}}{\sum_m \sum_{v \neq v_1, v_2} F(v, x_m) e^{-\lambda_m} + F'(v_1, x_m) e^{-\lambda_m} + F'(v_2, x_m) e^{-\lambda_m}} \tag{32}$$

along with

$$\mu'\left(\{x_j\}\right) = \frac{\sum_{v \neq v_1, v_2} F(v, x_j) e^{-\lambda_j} + F'(v_1, x_j) e^{-\lambda_j} + F'(v_2, x_j) e^{-\lambda_j}}{\sum_m \sum_{v \neq v_1, v_2} F(v, x_m) e^{-\lambda_m} + F'(v_1, x_m) e^{-\lambda_m} + F'(v_2, x_m) e^{-\lambda_m}} \tag{33}$$

Indeed because we have $F'(v_1, x_k) = F(v_2, x_k)$ and $F'(v_2, x_k) = F(v_1, x_k)$ for all k, we may proceed by simplifying those expressions to become the following:

$$\mu'\left(\{x_i\}\right) = \frac{\sum_v F(v, x_i) e^{-\lambda_i}}{\sum_m \sum_v F(v, x_m) e^{-\lambda_m}} \tag{34}$$

and

$$\mu'\left(\{x_j\}\right) = \frac{\sum_v F(v, x_j) e^{-\lambda_j}}{\sum_m \sum_v F(v, x_m) e^{-\lambda_m}} \tag{35}$$

Let us now suppose that five policy positions within a left and right scale exist. These will be denoted by the variables x_1, x_2, x_3, x_4, x_5, in which x_1 is the most left-wing while x_5 indicates the most right-wing. Suppose then that three voting agents exist: v_1, v_2, v_3. Their fuzzy preferences are given by the table:

These voters will hold different fuzzy preferences over the five different policy positions. The table that we have constructed represents the degree of membership for every policy position in the fuzzy set of preferred policies that corresponds to each voter. For an example, we see that voter v_1 holds a greater preference to policies considered left-wing rather than those considered right-wing, and thus assigns a high degree of membership to x_1 while assigning a low degree to x_5. Voter v_2 holds preferences that are considered moderate and so he assigns similar degrees of membership for all of the policy positions. Voter v_3 is more in preference of policies that are right-wing than left, so he assigns a high degree of membership to x_5 but a low degree of membership to x_1. We have the advantage that uncertainty and ambiguity within the voter preferences can be analysed more easily than by preferences that are ordinal or cardinal.

Suppose that the eigenvalues of the Hamiltonian H are given by $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $\lambda_4 = 0.4$, and $\lambda_5 = 0.5$. By using the formula for μ, we obtain:

$$\mu(\{x_1\}) = \frac{0.9e^{-0.1} + 0.6e^{-0.1} + 0.2e^{-0.1}}{Z} \approx 0.28 \tag{36}$$

$$\mu(\{x_2\}) = \frac{0.8e^{-0.2} + 0.7e^{-0.2} + 0.4e^{-0.2}}{Z} \approx 0.25 \tag{37}$$

$$\mu(\{x_3\}) = \frac{0.6e^{-0.3} + 0.8e^{-0.3} + 0.6e^{-0.3}}{Z} \approx 0.22 \tag{38}$$

$$\mu(\{x_4\}) = \frac{0.4e^{-0.4} + 0.7e^{-0.4} + 0.8e^{-0.4}}{Z} \approx 0.15 \tag{39}$$

$$\mu(\{x_5\}) = \frac{0.2e^{-0.5} + 0.6e^{-0.5} + 0.9e^{-0.5}}{Z} \approx 0.10 \tag{40}$$

where Z is the normalisation constant.

Suppose now that this voting system chooses the alternative with the highest probability according to μ. Then because of that, it will choose x_1 since . It is indeed the case that this decision is efficient for it maximises the expected utility of the society, but the problem with this case is that there is no fairness since $\mu = (\{x_1\}) > \mu = (\{x_2\}) > \mu = (\{x_3\}) > \mu = (\{x_4\}) > \mu = (\{x_5\})$. The preferences of the voters is not respected. In fact, only one voter (v_1) holds a preference for x_1 to all other alternatives, while two voters (v_2, v_3) would rather prefer x_3 to x_1, and there

Table 1 Voter-Policy Preference

Voter	x_1	x_2	x_3	x_4	x_5
v_1	0.9	0.8	0.6	0.4	0.2
v_2	0.6	0.7	0.8	0.7	0.6
v_3	0.2	0.4	0.9	0.8	0.9

is just one voter (v_3) who prefers x_5 to x_1. Therefore, the majority rule is violated by this decision. By contrast, suppose that the alternative with the highest degree of membership in F is chosen by the voting system. Then it will choose x_2, since $F(x_2) = 0.8 > F(x_1) = F(x_3) = 0.6 > F(x_4) >= F(x_5) = 0.4$. The decision made by the voting system here is indeed fair because the voters' preferences are respected, but efficiency has not been satisfied. That is because the society's expected utility has not been maximised. Since also $\mu = (\{x_2\}) < \mu = (\{x_1\})$, choosing x_2 would then mean that its probability of being optimal than choosing x_1 would be lower. Thus, the welfare of the society for the sake of the satisfaction of the preferences of voters has been sacrificed. We see indeed that the trade-off occurs and both fairness and efficiency cannot be guaranteed as both desirable properties for a social choice by the democratic framework.

4 Reconceptualisation

It is possible that we may consider fuzzy preferences that do not hold invariancy under the action of G, but rather having a dependence on some variables or parameters that G happens to act upon. We could consider, for example, fuzzy preferences that depend on the location, social status, or the time of the voting agents, and assume that G is a group of transformations that has the ability to mutate these parameters or variables. What could be done then is finding a probability measure m that is invariant under the action of G and that which also satisfies further conditions, such as $\mu(F) = 1$ or $\mu(A) = \int A d\mu_F$. This would be a more realistic and also flexible way for the modelling of preferences that vary with different circumstances and factors. In regards to both of our theorems, it might be better to consider various types of fuzzy sets and fuzzy measures, like intuitionistic fuzzy sets, hesistant fuzzy sets, type-2 fuzzy sets, and so forth. Such extensions of the classical fuzzy sets make it more easy for ambiguity and uncertainty to occur within the degrees of membership and the fuzzy measures. Intuitionistic fuzzy sets assign, not just a degree of membership, but a degree non-membership as well to each element and those may not add up to 1. Type-2 fuzzy sets assign fuzzy sets of degrees of memberships rather than a singular value to each element. Hesitant fuzzy sets, on the other hand, assign a set of possible degrees of membership, instead of a single value. Such variations of fuzzy sets and measures have the ability to make observations about even more complex or nuanced preferences and judgements about objects. Let us now see what we may do with these variations in regards to both of the theorems that we have proven. First of all, we would have to modify the definitions and assumptions that both of the original theorems adhere to. In Theorem 1, the classical fuzzy set F should be replaced by the intuistionic fuzzy set F^*, in which two membership functions μ_F and v_F define it, such that for each $x \in X$, we have:

$$0 \leq \mu_F(x) \leq 1 \tag{41}$$

$$0 \leq v_f(X) \leq 1, \mu_F(X) + v_F(X) \leq 1 \tag{42}$$

$$\mu_F(X) + v_F(X) \leq 1 \tag{43}$$

We see here that $\mu_F(x)$ represents the degree of membership associated to x in F^* whereas $v_F(X)$ represents the degree of non-membership associated to x in F^*. We would now also need to assume that F^* holds invariancy under the Hopf algebra action of H on X, meaning that for any $h \in H$ and any $x \in X$, we have:

$$\mu_F(h \cdot x) = \mu_F(x) \tag{44}$$

$$v_F(h \cdot x) = v_F(x) \tag{45}$$

We would then need to find a probability measure μ on X that is able to satisfy some conditions, such as $\mu(F') = 1$, where $\mu(F^*)$ is defined as

$$\mu(F^*) = \sup_{x \in X}(\mu_F(x) - v_F(x)) \tag{46}$$

which generalises the classical definition of $\mu(F)$

Now, regarding Theorem 2, we would in a similar way replace the classical fuzzy set function F with an intuitionistic fuzzy set function F^*, where the two membership functions μ_F and v_f define it, such that for each voter v and each alternative x, we are now in acquisition of:

$$0 \leq \mu_F(v, x) \leq 1 \tag{47}$$

$$0 \leq v_F(v, x) \leq 1 \tag{48}$$

$$\mu_F(v, x) + v_F(v, x) \leq 1 \tag{49}$$

Where $\mu_F(v, x)$ represents the degree of membership of x in $F^*(v)$ while $v_F(v, x)$ represents the degree of non-membership of x in $F^*(v)$. An assumption should be held in which some criteria for a voting system, such as anonymity, neutrality, unanimity, and IIA, are satisfied by F^*. A probability measure μ on X that satisfies conditions such as $\mu(F^*) = 1$ should be found, where $\mu(F^*)$ is defined as:

$$\mu(F^*) = \sup_{x \in X}\left(\sum_v (\mu_F(v, x) - v_f(v, x))\right) \tag{50}$$

which generalises the classical definition of $\mu(F)$.

5 Discussion

In this paper, we investigated whether there is a unique and invariant way to aggregate individual preferences based on fuzzy sets into a social preference based on probabilities. We found that, under some reasonable conditions, such a way exists.

Our results support the idea that fuzzy sets can provide a more realistic and flexible representation of individual preferences than ordinal or cardinal utilities. They also show that the path integral approach, which is inspired by techniques from topological supersymmetric quantum mechanics, can be applied to social choice theory and yield interesting insights. We have extended the results of Arrow (1951) and Sen (1970) by relaxing some of their assumptions and allowing for more diversity and uncertainty in the preferences.

The results that we have found also indicate that probabilistic social preferences have the ability to analyse uncertainty and information much more easily than social preferences that are deterministic, but they may also be more difficult to elicit and communicate. Furthermore, our findings raise some ethical questions about how to respect individual preferences while ensuring social welfare.

The study, however, also has some limitations that need to be acknowledged. We used a path integral approach that relies on some mathematical assumptions and approximations that may not hold in all cases. We also did not consider the effects of strategic behavior, manipulation or external influences on the preferences or the voting outcomes.

6 Conclusion

We have presented a new theorem in social choice theory that has indeed shown that there is a unique and invariant way to aggregate individual preferences based on fuzzy sets into a social preference based on probabilities, and that this way is given by a path integral approach. Thence we also observed the tradeoff within the democratic system in action. Our study also has practical implications for designing and evaluating voting systems that can account for more information and uncertainty in the preferences.

Based on what we have gathered through our results and an analysis on the limitations that we approached, we recommend several directions for future research. First, it would be interesting to test our theorem with a large amount of data from surveys or experiments, and to compare the outcomes of different voting systems based on fuzzy sets and probabilities. It would also be useful to explore other ways to construct fuzzy sets from individual preferences, such by the utilisation of linguistic variables or membership functions. It would be important to investigate how to elicit and communicate probabilistic social preferences in a clear and transparent way, and to examine how they affect the behavior and satisfaction of voting agents.

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