Logic and **L**ogical **P**hilosophy
Volume 21 (2012), 415–438
DOI: 10.12775/LLP.2012.019

Fabien Schang

ABSTRACT LOGIC OF OPPOSITIONS

Abstract. A general theory of logical oppositions is proposed by abstracting these from the Aristotelian background of quantified sentences. Opposition is a relation that goes beyond incompatibility (not being true together), and a question-answer semantics is devised to investigate the features of oppositions and opposites within a functional calculus. Finally, several theoretical problems about its applicability are considered.

Keywords: INRC Group, negation, opposite, opposition, proto-negation.

1. The philosophical background

Despite a special attention paid to dichotomy (a dialectical process of contradiction) in Plato's philosophy [20], the latter cannot be considered as the primary development of a genuine theory of opposition. Rather, the forerunner of opposition is clearly Aristotle. In a good number of his works ([1, 2, 3, 4]), Aristotle used this theory as a crucial complement for his theory of demonstration or *syllogistics*. After assuming that each sentence is characterized by its quality (affirmative, or negative) and quantity (universal, or particular), a combination of these two parameters yields four sorts of sentences including both a subject S and a predicate P: universal+affirmative (S**a**P), universal+negative (S**e**P), particular+affirmative (S**i**P), particular+negative (S**o**P). Then the distinction between affirmed premises and their denials resulted in a variety of valid conclusions, and the theory of opposition helped to deduce theorems (imperfect moods, e.g. Baroco) from axioms (perfect moods, e.g. Barbara) by putting constraints on the set of logical consequences for given premises. Opposition is closely related to *negation*,

Special Issue: *Logic, Reasoning and Rationality*
Guest Editors: Erik Weber, Dietlinde Wouters and Joke Meheus
 ISSN: 1425-3305

since each sentence occurs as the negation of another one within the set {SaP, SeP, SiP, SoP}. There are different ways of negating a sentence, however; hence several sorts of negation and resulting oppositions.

The nature of a logical opposition is far from being clear. According to Aristotle ([4]: 63b21-30),

> Verbally four kinds of opposition are possible, viz. universal affirmative to universal negative, universal affirmative to particular negative, particular affirmative to universal negative, and particular affirmative to particular negative; but really there are only three: for the particular affirmative is only verbally opposed to the particular negative. Of the genuine opposites I call those which are universal *contraries*, e.g. "every science is good", "no science is good"; the others I call *contradictories*.

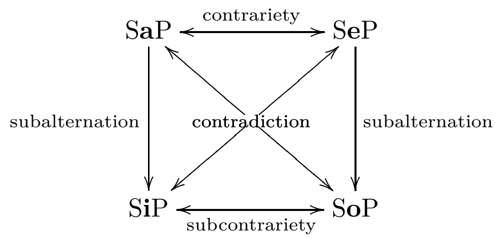
Only one pair of sentences can express a relation of contrariety, whereas two pairs express contradiction among the three "genuine" oppositions; as for the fourth pair, it stands for a "verbal" relation of subcontrariety. In symbols (where $\between$ denotes the general relation of opposition), this yields a subset of six genuine oppositions (1)–(6) augmented by two verbal ones (7)–(8):

(i) Contrariety
(1) SaP $\between$ SeP (2) SeP $\between$ SaP
(ii) Contradiction
(3) SaP $\between$ SoP (4) SoP $\between$ SaP
(5) SeP $\between$ SiP (6) SiP $\between$ SeP
(iii) Subcontrariety
(7) SiP $\between$ SoP (8) SoP $\between$ SiP

A famous logical tool had been devised to display these relations inside a visual object, namely: the *logical square*. Although it used to be called the "Aristotelian" square, let us recall that Aristotle never made use of it since he was only concerned with contrariety and contradiction. The complete figure is obtained (see Figure 1) with the addition of a fourth relation of subalternation.

(iv) Subalternation
(9) SaP $\between$ SiP (10) SiP $\between$ SaP
(11) SeP $\between$ SoP (12) SoP $\between$ SeP

It is worth noting that the theory of opposition has been undermined by two main difficulties: the so-called "existential import" on the one hand; and the lexicalization of opposite statements in natural language on the



other. These troubles largely contributed to the decline of the theory, but they can be overcome.

For one thing, the existential import is to the effect that the additional relation of *subalternation* between universals and particulars of the same quality does not hold whenever the statements include *empty terms*, i.e. terms that fail to refer to an existing thing. This difficulty has been treated at length in the literature, but without any definite result; now it has been recently shown [11] that the square of opposition can be saved even when including empty terms.

Furthermore, the lexicalization of the square is closely related to the previous problem. A crucial precondition to the validity of the square is to avoid any existential commitment with the *particular* statements. This means that none of the statements S**i**P and S**o**P should be lexicalized as “Some S is P” and “Some S is not P” but, rather, as “Not every S is not P” and “Not every S is P”. The logical difference between “Not every ... is ...” and ”Some ... is not ...” is justified in [11], thus saving the square.

Two questions result from this primary presentation: why a square, instead of any other geometrical polygon for the logical relations? An answer to this depends upon another one, namely: when can a relation be considered as a relation of opposition? Subcontrariety has been said to be a “verbal” (not genuine) opposition by Aristotle, whereas subalternation has not even been mentioned as a verbal opposition in his work. Moreover, the square can be devised by introducing subalternation on the strength of a “technical” definition of opposition. Thus Keynes stated that

> Two propositions are technically said to be opposed to each other when they have the same subject and predicate respectively, but differ in quantity and quality or both. [14: 109]

This technical definition includes subcontraries as differing in quality, whereas subalterns and superalterns differ in quantity. Now a main feature that is missing in subcontraries makes these differ from genuine opposites, namely: *incompatibility*. For two sentences are said to be genuinely opposed to each other if and only if they cannot be true together. Then the various oppositions seem to proceed as combinations of compossible relations between the truth-values of sentences, so that we obtain four sorts of opposition including (I) genuine ones, (II) a verbal one and (III) a merely technical one:

(I) Two "genuine" or incompatible relations of opposition between sentences
  (i) contraries cannot be true together and can be false together
  (ii) contradictories cannot be true together and cannot be false together
(II) A "verbal" relation between weakly compatible sentences
  (iii) subcontraries can be true together and cannot be false together
(III) A "technical" relation between strongly compatible sentences
  (iv) subalterns and superalterns can be true together and can be false together.

A difficulty arises with this purely combinatorial approach. It is misleading, in the sense that it cannot account for the *asymmetry* of the fourth relation of subalternation: the superaltern can be false whenever the subaltern is so, but the subaltern cannot be false whenever the superaltern is true. Hence it cannot be said that a relation of subalternation safely admits its relata to be true or false together, and it cannot be said in turn that these cannot be true or false together irrespective of their place in the relation. Something is missing for an exhaustive definition of logical oppositions, therefore. Can we provide a general view of opposition that includes the asymmetric relation of subalternation along with the other three plainly symmetric relations? To answer this, we propose in what follows a logical background for the theory of opposition while abstracting it from its historical roots.

2. The logical background

2.1. The logical status of opposition

Let α_1, α_2, ... be finitely many elements of a sentential language L, and v a valuation function from L to the domain of values $\mathrm{V}^2 = \{\mathrm{F}, \mathrm{T}\}$ with F for "false" and T for "true". Opposition can be viewed as a sentential t-ary relation $\mathrm{Op}(\alpha_1, \ldots, \alpha_t)$ between the truth-values of t sentences (with $t \geq 2$), where Op is one arbitrary element from the set of oppositions {CT, CD, SCT, SB} (CT for contrariety, CD for contradiction, SCT for subcontrariety, and SB for subalternation). These single oppositions can be defined as follows, with $t = 2$ and $\alpha_2 = \psi$:

$$\begin{array}{rcl} \mathrm{CT}(\alpha, \psi) & \equiv_{\mathrm{df}} & v(\alpha) = \mathrm{T} \Rightarrow v(\psi) = \mathrm{F} \\ \mathrm{CD}(\alpha, \psi) & \equiv_{\mathrm{df}} & v(\alpha) = \mathrm{T} \Leftrightarrow v(\psi) = \mathrm{F} \\ \mathrm{SCT}(\alpha, \psi) & \equiv_{\mathrm{df}} & v(\alpha) = \mathrm{F} \Rightarrow v(\psi) = \mathrm{T} \\ \mathrm{SB}(\alpha, \psi) & \equiv_{\mathrm{df}} & v(\alpha) = \mathrm{T} \Rightarrow v(\psi) = \mathrm{T} \end{array}$$

Two things are to be noted about the "technical" oppositions.

First, subcontrariety. It is a complex relation between sentences, in contrast with the two simple relations of contrariety and contradiction. This complexity is stated by Jean-Yves Béziau in *functional* terms:

> Let us recall that Aristotle does not introduce explicitly the notion of "subcontraries", but refers to them only indirectly as "contradictories of contraries". [6: 224]

While contraries correspond to the relation $\mathrm{CT}(\alpha, \psi)$, what contradictories of contraries amount to is unclear. Is their logical form something like $\mathrm{CD}(\mathrm{CT}(\alpha, \psi))$, $\mathrm{CT}(\mathrm{CD}(\alpha, \psi), \mathrm{CD}(\alpha, \psi))$, or even $\mathrm{CT}(\mathrm{CD}(\alpha)$, $\mathrm{CD}(\psi))$? None of these, given that each element Op is a binary relation and not a function. An alternative definition of subcontrariety is in order to account for Béziau's functional expression, accordingly.

Next, subalternation. Just as the former case of subcontrariety, subalternation is a relation to be defined by means of a complex (but this time singular and double) function. For if ψ is the subaltern of α, then ψ is the "contradictory of the contrary of" α.

A way to introduce functions into the theory of opposition is to supplement it with *opposites* as opposition-forming operators. Thus, for any opposition we have the general equation

$$\mathrm{Op}(\alpha, \psi) = \mathrm{Op}(\alpha, \mathrm{O}(\alpha))$$

between the mutual opposites α and ψ, where O stands for an opposition-forming operator such that its application to the first relatum α yields the second one $\mathrm{O}(\alpha) = \psi$. Op proceeds like the unary connectives of classical affirmation and negation, as a homomorphism from V to V; but a difficulty arises for the suggested theory of opposites: only contradiction is an effective mapping of classical logic, since it maps from V^2 to V^2. Letting O be one arbitrary element from the set of opposites {ct, cd, sct, sb}, it is well known that

$$\text{if } \mathrm{cd}(\alpha) = \psi, \text{ then } v(\alpha) = \mathrm{T} \Leftrightarrow v(\psi) = \mathrm{F}.$$

Apart from this unambiguous valuation for contradictoriness, it is equally well known that the other opposites are *intensional* operators that don't proceed truth-functionally.

α	ct(α)	cd(α)	sct(α)	sb(α)
T	F	F	?	F
F	?	T	T	?

A solution to this second difficulty requires another semantics for the operators $\mathrm{O}(\alpha)$. For this purpose, an alternative logic for oppositions of opposites is proposed including alternative logical values.

2.2. A calculus of oppositions

Three ways have been proposed thus far to extend the Aristotelian theory of opposition.

Firstly let us look at a *syntactic* extension. The Aristotelian initial quantified sentences can be superseded by other sorts of sentences. Blanché's modalities [7], Bocheński's connectives [8] and Béziau's non-classical negations [6] resulted in a group of sentences establishing that the theory of opposition is a pure theory of logical structures abstracted from the historical framework of Aristotle's syllogistics.

Secondly let us consider a *geometrical* extension. As a corollary of the preceding, the new sorts of sentences augment the square with a greater number of relations $\mathrm{Op}(\alpha, \psi)$ and new logical polygons. Thus Blanché and Czeżowski's hexagons ([7], [12]), Buridan's octagon of alethic modalities [13] or Pellissier's tetraicosahedron [19] are different sets of related sentences with more than the 6 basic relations of the square: 15 for hexagons, 28 for octagons, and 120 for tetraicosahedrons. This greater number of instances does not imply that there are more than four kinds of opposition in the end, however, but the fourth case of subalternation is to be replaced by a larger category of non-contradictoriness that includes the former (see [28], and section 2.3). The geometry of oppositions has been investigated at length by Moretti [18], and the present paper will focus on the properties of any related terms of the polygons. The result is an *algebraic* theory in two senses of the word, namely: an algebraic, many-valued semantics based on Boolean algebra.

Finally there is a *categorial* extension. The set of oppositions might be supplemented with further elements, i.e. further sorts of opposition. A case in point is Sion [26], who completed the four usual oppositions with two additional sorts: implicance, and unconnectedness. Although the last two instances have been questioned elsewhere [23], it can be seen that such a categorial extension challenges the traditional theory of opposition by altering the "transcendental" number of elements in {CT, CD, SCT, SB}.

This leads one to another problem to solve. How many sorts of opposition can there be? A semantic analysis is used for this purpose, i.e. a logical framework where the sentential values are not Fregean values.

Let us introduce an alternative theory of meaning: Question-Answer Semantics (hereafter: **QAS**). It is a non-classical model that emphasizes the role of dialogue and its dialectical process upon our daily-life reasonings. By a *dialectical* process is meant the basic game of oppositions between at least two speakers in a given sequence of arguments. Assuming that the Principle of Bivalence is too strong a constraint upon the normal rules of communication, we prefer a set of questions-answers to determine the meaning of sentences and rule what can be accepted or not in a given dialogue. Unlike the Fregean tradition, the reference of a given sentence α in **QAS** is not a single truth-value $v(\alpha)$ but a set of ordered answers $\mathbf{A}(\alpha) = \langle \mathbf{a}_1(\alpha), \ldots, \mathbf{a}_n(\alpha) \rangle$ to corresponding closed questions $\langle \mathbf{q}_1(\alpha), \ldots, \mathbf{q}_n(\alpha) \rangle$ about it. By doing so, the meaning of a sentence is not given any more by ontological values like the True, $v(\alpha) = \mathrm{T}$, or the False, $v(\alpha) = \mathrm{F}$. Rather, it is pragmatically defined by its use in a given context, and this use is rendered by a finite ordered set of questions. It results in a finite many-valued logic which extends the classical (bivalent) framework into a combination of basic answers. A philosophical motivation of this algebraic semantics has to do with alternative rationalities and their various conditions for truth-ascription. For example, this general framework helps to model various types of information expressed by speech acts ([23], [25]) and to account for allegedly "irrational" reasonings from the ancient skeptic (the Greek *ou mallon*, the Indian tetralemma) and relativist tradition (the Jain non-one-sided theory of predication)([24]).

Moreover, such a semantics echoes with what Moretti [18] called an "Aristotelian P^Q-Semantics" (P for the number of answers, Q for the number of questions). Moretti's question-answer game for logical oppositions consists of 2 questions about the compossible truth-values of the opposed sentences α and ψ. Let $\Phi = \mathrm{Op}(\alpha, \psi)$, with the two ensuing questions "Can α and ψ be true together?" and "Can α and ψ be false together?". In symbols, where $\Diamond$ stands for possibility:

$$\mathbf{q}_1(\Phi)\text{: "}\Diamond(v(\alpha) = v(\psi) = \mathrm{T})\text{?"}$$
$$\mathbf{q}_2(\Phi)\text{: "}\Diamond(v(\alpha) = v(\psi) = \mathrm{F})\text{?"}$$

Following the Fregean view that the meaning of a sentence relies upon both its sense and reference, the sense of Φ is conveyed by $n = 2$ basic questions about it: $\mathbf{Q}(\Phi) = \langle \mathbf{q}_1(\Phi), \mathbf{q}_2(\Phi)\rangle$. The reference of Φ is the corresponding pair of answers $\mathbf{A}(\Phi) = \langle \mathbf{a}_1(\Phi), \mathbf{a}_2(\Phi)\rangle$, every yes-answer being symbolized by 1 and each no-answer by 0. We obtain a $P^Q = 2^2$-semantics, with a set of $2^2 = 4$ complex values that obey Boolean algebra (0 and 1 are the basic values) but clearly differ from the Fregean simple truth-values T and F. That is:

$$\begin{array}{rcl}
\mathrm{CT}(\alpha,\psi) = \mathrm{T} & \text{iff} & \mathbf{A}(\mathrm{CT}(\alpha,\psi)) = \mathbf{A}(\Phi_1) = \langle 0, 1\rangle \\
\mathrm{CD}(\alpha,\psi) = \mathrm{T} & \text{iff} & \mathbf{A}(\mathrm{CD}(\alpha,\psi)) = \mathbf{A}(\Phi_2) = \langle 0, 0\rangle \\
\mathrm{SCT}(\alpha,\psi) = \mathrm{T} & \text{iff} & \mathbf{A}(\mathrm{SCT}(\alpha,\psi)) = \mathbf{A}(\Phi_3) = \langle 1, 0\rangle \\
\mathrm{SB}(\alpha,\psi) = \mathrm{T} & \text{iff} & \mathbf{A}(\mathrm{SB}(\alpha,\psi)) = \mathbf{A}(\Phi_4) = \langle 1, 1\rangle
\end{array}$$

Recalling the preceding first difficulty, it appears clear that this semantics accounts for the cardinality of four oppositions but cannot account for the asymmetry of subalternation: the value $\langle 1, 1\rangle$ does not bring out that ψ cannot be false whenever α is true in $\mathrm{SB}(\alpha, \psi)$.

A solution to this problem is to go through a more accurate question-answer game for t-ary sentences (connected by a logical constant of arity t), in the vein of Piaget's theory of binary connectives [20]. Considering any complex sentence $\Theta = \alpha \bullet \psi$ as the connection of two arbitrary sentences α and ψ by a binary connective $\bullet$, each instantiation of Θ characterizes both the meaning of $\bullet$ and a Disjunctive Normal Form. The sense of Θ is given by the 4 following questions:

$$\mathbf{q}_1(\Theta)\text{: "}v(\alpha) = v(\psi) = \mathrm{T}\ \text{?"}$$
$$\mathbf{q}_2(\Theta)\text{: "}v(\alpha) = \mathrm{T} \text{ and } v(\psi) = \mathrm{F}\ \text{?"}$$

$\mathbf{q}_3(\Theta)$: "$v(\alpha) = \mathrm{F}$ and $v(\psi) = \mathrm{T}$?"
$\mathbf{q}_4(\Theta)$: "$v(\alpha) = v(\psi) = \mathrm{F}$?"

The result is a set of $2^4 = 16$ logical values, thereby characterizing the 16 binary connectives from the bivalent classical logic:

$\mathbf{A}(\bot) = \mathbf{A}(\Theta_1) = \langle 0000 \rangle$
$\mathbf{A}(\mathrm{p}\wedge\mathrm{q}) = \mathbf{A}(\Theta_2) = \langle 1000 \rangle$
$\mathbf{A}({\sim}(\mathrm{p}\rightarrow\mathrm{q})) = \mathbf{A}(\Theta_3) = \langle 0100 \rangle$
$\mathbf{A}({\sim}(\mathrm{p}\leftarrow\mathrm{q})) = \mathbf{A}(\Theta_4) = \langle 0010 \rangle$
$\mathbf{A}({\sim}(\mathrm{p}\vee\mathrm{q})) = \mathbf{A}(\Theta_5) = \langle 0001 \rangle$
$\mathbf{A}(\mathrm{p}) = \mathbf{A}(\Theta_6) = \langle 1100 \rangle$
$\mathbf{A}({\sim}(\mathrm{p}\leftrightarrow\mathrm{q})) = \mathbf{A}(\Theta_7) = \langle 0110 \rangle$
$\mathbf{A}({\sim}\mathrm{p}) = \mathbf{A}(\Theta_8) = \langle 0011 \rangle$
$\mathbf{A}(\mathrm{p}\leftrightarrow\mathrm{q}) = \mathbf{A}(\Theta_9) = \langle 1001 \rangle$
$\mathbf{A}(\mathrm{q}) = \mathbf{A}(\Theta_{10}) = \langle 1010 \rangle$
$\mathbf{A}({\sim}\mathrm{q}) = \mathbf{A}(\Theta_{11}) = \langle 0101 \rangle$
$\mathbf{A}(\mathrm{p}\vee\mathrm{q}) = \mathbf{A}(\Theta_{12}) = \langle 1110 \rangle$
$\mathbf{A}({\sim}(p\wedge\mathrm{q})) = \mathbf{A}(\Theta_{13}) = \langle 0111 \rangle$
$\mathbf{A}(\mathrm{p}\rightarrow\mathrm{q}) = \mathbf{A}(\Theta_{14}) = \langle 1011 \rangle$
$\mathbf{A}(\mathrm{p}\leftarrow\mathrm{q}) = \mathrm{A}(\Theta_{15}) = \langle 1101 \rangle$
$\mathbf{A}(\top) = \mathbf{A}(\Theta_{16}) = \langle 1111 \rangle$

Albeit restricted by Piaget to the binary sentences of classical logic, **QAS** can be equally applied to afford the meaning of other sentences including quantifiers, modalities, or even non-classical constants. As an example, Smessaert has shown with his Quantified Modal Algebra [11] that a special question-answer game is in position to restate the theory of generalized quantifiers and turn modalities into scalar degrees of truth or falsity.

By now, two things are to be noted about Piaget's theory.

For one thing, each single answer $\mathbf{a}_i(\Theta)$ of a logical value $\mathbf{A}(\Theta)$ corresponds to a row of a classical truth-table, so that $\mathbf{a}_i(\Theta) = 1$ (or 0) iff $v(\alpha \bullet \psi) = \mathrm{T}$ (or F) in the i^{th} row.

α	ψ	$\Theta_i = \alpha \bullet \psi$	$\mathbf{A}(\Theta_i)$
$v(\alpha)$	$v(\psi)$	$v(\alpha \bullet \psi)$	$v(\alpha) = v(\psi) = \mathrm{T}$?
$v(\alpha)$	$v(\psi)$	$v(\alpha \bullet \psi)$	$v(\alpha) = \mathrm{T}$ & $v(\psi) = \mathrm{F}$?
$v(\alpha)$	$v(\psi)$	$v(\alpha \bullet \psi)$	$v(\alpha) = \mathrm{F}$ & $v(\psi) = \mathrm{T}$?
$v(\alpha)$	$v(\psi)$	$v(\alpha \bullet \psi)$	$v(\alpha) = v(\psi) = \mathrm{F}$?

A case in point is conjunction, with the logical value $\mathbf{A}(\alpha \wedge \psi) = \langle 1000 \rangle$.

α	ψ	$\Theta_2 = \alpha \wedge \psi$	$\mathbf{A}(\Theta_2)$
T	T	T	$\mathbf{a}_1(\Theta_2) = 1$
T	F	F	$\mathbf{a}_2(\Theta_2) = 0$
F	T	F	$\mathbf{a}_3(\Theta_2) = 0$
F	F	F	$\mathbf{a}_4(\Theta_2) = 0$

Furthermore, equating the Disjunctive Normal Form of a sentence with its logical value helps to introduce a functional calculus of oppositions through Piaget's theory of reversibility. Let denial be a function $'$ of a Boolean algebra, such that $\alpha' = 0$ iff $\alpha = 1$; then Piaget's Group {I, N, R, C} is a set of transformations $\otimes$ upon an arbitrary logical value.

$$\begin{array}{ll}
\text{Identity: } \mathrm{I}(\alpha) & \mathbf{A}(\mathrm{I}(\alpha)) = \langle \mathbf{a}_1(\alpha), \ldots, \mathbf{a}_n(\alpha) \rangle \\
\text{Inversion: } \mathrm{N}(\alpha) & \mathbf{A}(\mathrm{N}(x)) = \langle \mathbf{a}_1(\alpha)', \ldots, \mathbf{a}_n(\alpha)' \rangle \\
\text{Reciprocity: } \mathrm{R}(\alpha) & \mathbf{A}(\mathrm{R}(\alpha)) = \langle \mathbf{a}_n(\alpha), \ldots, \mathbf{a}_1(\alpha) \rangle \\
\text{Correlation: } \mathrm{C}(\alpha) & \mathbf{A}(\mathrm{C}(\alpha)) = \langle \mathbf{a}_n(\alpha)', \ldots, \mathbf{a}_1(\alpha)' \rangle
\end{array}$$

These are applied to conjunction as follows, with $n = 4$ and $x = \alpha \wedge \psi = \Theta_2$.

$$\begin{array}{lll}
\text{Identity: } \mathrm{I}(\Theta_2) & \mathbf{A}(\mathrm{I}(\Theta_2)) & = \langle 1000 \rangle \\
\text{Inversion: } \mathrm{N}(\Theta_2) & \mathbf{A}(\mathrm{N}(\Theta_2)) & = \langle 1'0'0'0' \rangle = \langle 0111 \rangle \\
 & & = \mathbf{A}({\sim}(\alpha \wedge \psi)) = \mathbf{A}(\Theta_{13}) \\
\text{Reciprocity: } \mathrm{R}(\Theta_2) & \mathbf{A}(\mathrm{R}(\Theta_2)) & = \langle 0001 \rangle = \mathbf{A}({\sim}(\alpha \vee \psi)) = \mathbf{A}(\Theta_5) \\
\text{Correlation: } \mathrm{C}(\Theta_2) & \mathbf{A}(\mathrm{C}(\Theta_2)) & = \langle 0'0'0'1' \rangle = \langle 1110 \rangle \\
 & & = \mathbf{A}(\alpha \vee \psi) = \mathbf{A}(\Theta_{12})
\end{array}$$

This means that the 4 functions $\otimes$ are applied to a binary sentence $(\alpha \bullet \psi) = \Theta_i$ and lead to another one Θ_j, so that $\otimes(\Theta_i) = \Theta_j$. The result is a matrix of such transformations, and the relation between each of these paired sentences corresponds to a relation of opposition $\mathrm{Op}(\Theta_i, \otimes(\Theta_i)) = \mathrm{Op}(\Theta_i, \Theta_j)$.

	Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	Θ_6	Θ_7	Θ_8
I	Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	Θ_6	Θ_7	Θ_8
N	Θ_{16}	Θ_{13}	Θ_{14}	Θ_{15}	Θ_{12}	Θ_8	Θ_9	Θ_6
R	Θ_1	Θ_5	Θ_4	Θ_3	Θ_2	Θ_8	Θ_7	Θ_6
C	Θ_{16}	Θ_{12}	Θ_{15}	Θ_{14}	Θ_{13}	Θ_6	Θ_9	Θ_8

	Θ_9	Θ_{10}	Θ_{11}	Θ_{12}	Θ_{13}	Θ_{14}	Θ_{15}	Θ_{16}
I	Θ_9	Θ_{10}	Θ_{11}	Θ_{12}	Θ_{13}	Θ_{14}	Θ_{15}	Θ_{16}
N	Θ_7	Θ_{11}	Θ_{10}	Θ_5	Θ_2	Θ_3	Θ_4	Θ_1
R	Θ_9	Θ_{11}	Θ_{10}	Θ_{13}	Θ_{12}	Θ_{15}	Θ_{14}	Θ_{16}
C	Θ_7	Θ_{10}	Θ_{11}	Θ_2	Θ_5	Θ_4	Θ_3	Θ_1

Just like the classical function of *affirmation*, I is a redundant operator that leaves its operand unchanged. And just like the classical function of *negation*, N is an operator that turns its operand α_i into another one α_j by denying each of its basic answers $\mathbf{a}(\alpha_i)$ in the order. More generally, the 3 nontrivial operators N, R and C are opposite-forming operators that behave like sentential negations turning any value into another one. One of these is a purely *extensional* operator, namely: the inversion operator N, which turns any sentence into its contradictory so that

$$\mathrm{CD}(\alpha, \psi) = \mathrm{CD}(\alpha, \mathrm{cd}(\alpha)) = \mathrm{CD}(\alpha, \mathrm{N}(\alpha))$$

At the same time, the remaining two operators R and C are *intensional* operators that don't proceed as bijective functions: there can be more than one opposition formed by a sentence α and its opposite $\mathrm{R}(\alpha)$ or $\mathrm{C}(\alpha)$, because the value of $\mathrm{O}(\alpha)$ relies upon the value of α.

Turning again to the functional characterization of subcontrariety, it is worthwhile to note some striking properties of the opposite-forming operators O. For any natural integers i, j, k:

$$\mathrm{O}_i(\mathrm{O}_i(\alpha)) = \mathrm{O}_j(\mathrm{O}_j(\alpha)) = \mathrm{O}_k(\mathrm{O}_k(\alpha)) = \mathrm{I}(\alpha) \tag{O.1}$$

$$\mathrm{O}_i(\mathrm{O}_j(\alpha)) = \mathrm{O}_j(\mathrm{O}_i(\alpha)) = \mathrm{O}_k(\alpha) \tag{O.2}$$

$$\mathrm{N}(\alpha) = {\sim}\alpha \tag{O.3}$$

This helps to explain why subcontraries are contradictories of contraries. Indeed, such a functional definition can be parsed into the following statement:

$$\mathrm{SCT}(\alpha, \psi) = \mathrm{CT}(\mathrm{cd}(\alpha), \mathrm{cd}(\psi)).$$

PROOF: by induction upon the set of subcontrary relations.

Taking $\mathrm{SCT}(\Theta_{12}, \Theta_{13})$ as an instance, we obtain

$$\mathrm{CT}(\mathrm{cd}(\Theta_{12}), \mathrm{cd}(\Theta_{13}) = \mathrm{CT}(\mathrm{N}(\Theta_{12}), \mathrm{N}(\Theta_{13})) = \mathrm{CT}(\Theta_5, \Theta_2).$$

(The remaining instances are left to the reader.) □

As for the relation of subalternation, it can be similarly established that every subaltern (of a given sentence α) is the contradictory of a contrary (of α). That is: $\mathrm{sb}(\Theta) = \mathrm{cd}(\mathrm{ct}(\Theta)) = \mathrm{ct}(\mathrm{cd}(\Theta))$ (by O.2). For instance, it is the case that $\mathrm{sb}(\Theta_2) = \mathrm{cd}(\mathrm{ct}(\Theta_2) = \mathrm{cd}(\Theta_5) = \Theta_{12}$.

Moreover, subalternation is usually taken to be a counterpart of *entailment*. Assuming that entailment has to do with the basic relation of logical consequence, let us see now if logical consequence can be seen as a single case of opposition within a general theory of oppositions and opposites.

2.3. Consequence in opposition

A cornerstone of modern logic is the semantic concept of truth, insofar as it helps to define the basic relation of consequence. Can one construct logics without assuming what Tarski came to characterize as a truth-preserving relation? The answer is affirmative, given that Tarski began to define consequence as a syntactic operator; moreover, an inferentialist would claim that the logical constants needn't be interpreted by truth-conditions to make sense. Whether it be about truth or not, consequence means in the following that any logical conclusion preserves affirmative answers from the premises to the conclusion. The independence of consequence with respect to truth is endorsed by **QAS** and its non-Fregean logical values, where truth is only one possible semantic predicate in question-answer games.

That $\mathrm{SB}(\alpha, \psi)$ is asymmetric can be taken for granted by depicting subalternation as an inclusive relation between the supaltern α and its subaltern ψ. Following the older view of logical consequence as a relation of entailment or, better, *containment* between premises and a conclusion, Blanché reworded this point as an inclusion of "truth-cases". Indeed,

> The indeterminates are implied by the determinates whose truth-case they contain: since each has three truth-cases, the latter is thus implied by three determinates [...]. [8, p. 137]

This means in **QAS** that, for any sentences X and Y, a similar view of the Tarskian relation of logical consequence can be found without using the semantic predicate of truth in the basic questions:

$$\text{X entails Y iff } v(\mathrm{Y}) = \mathrm{T} \text{ whenever } v(\mathrm{X}) = \mathrm{T}$$
$$\text{X entails Y iff } \mathbf{a}_i(\mathrm{Y}) = 1 \text{ whenever } \mathbf{a}_i(\mathrm{X}) = 1.$$

Blanché confined his approach to Piaget's theory of binary connectives including the case of "indeterminates", i.e. those binary sentences Y with 3 yes-answers such that $\mathbf{a}_i(\mathrm{Y}) = 1$. But the definition of entailment equally holds with any other sort of sentences and irrespective of their logical values. An instance of entailment is the relation between conjunction and disjunction, namely: $\mathrm{SB}(\Theta_2, \Theta_{12})$. For $\mathbf{A}(\Theta_2) = \langle 1000\rangle$ and $\mathbf{A}(\Theta_{12}) = \langle 1110\rangle$, so that it is the case that $\mathbf{a}_1(\Theta_{12}) = 1$ whenever $\mathbf{a}_1(\Theta_2) = 1$. By contrast, the relation of contradiction is to be defined in **QAS** by never preserving the same value from one relatum to another one. Thus

> as to the fourth determinate, i.e. the one whose unique truth-case coincides with the unique falsity-case of the indeterminate, it forms the alternative with it by standing for the exact negation of the other one. ([8, p. 137]

That is:

$$\psi = \mathrm{cd}(\alpha) \qquad \text{iff, for every } \mathbf{a}_i, \qquad \mathbf{a}_i(\alpha) = 1 \Leftrightarrow \mathbf{a}_i(\psi) = 0$$

Taking disjunction as a case of indeterminacy, it means that its contradictory is the binary sentence with only one truth-case. Taking $\mathbf{A}(\Theta_{12}) = \langle 1110 \rangle$, we obtain $\mathrm{CD}(\Theta_{12}, \Theta_5) = \mathrm{CD}(\langle 1110 \rangle, \langle 0001 \rangle)$ where the opposite term can be formed by the Piagetian operator of inversion N.

More generally, each relation of opposition can be defined in Boolean algebra without ever talking about truth and falsity. Following the classical matrices of the binary sentences $\alpha \bullet \psi$, Op is a set of relations with various constraints upon its relata.

All of this can be summarized by means of Boolean algebra and its basic operations of meet $\cap$ and join $\cup$. It is a bivalent calculus that maps each sentence onto the two-valued set of answers $\{0, 1\}$, where 0 is the minimal value and 1 is the maximal value. Moreover, every answer which is not positive is negative (and conversely): $\mathbf{a}(\alpha) \neq 1 \Leftrightarrow \mathbf{a}(\alpha) = 0$. One value is assigned from $\{0, 1\}$ to every componing answer $\mathbf{a}_i(\Phi)$, such that $\mathbf{a}_i(\alpha) \cap \mathbf{a}_i(\psi) = min(\alpha, \psi)$ and $\mathbf{a}_i(\alpha) \cup \mathbf{a}_i(\psi) = max(\alpha, \psi)$. Hence the following set of clauses for the logical oppositions:

$\mathrm{CT}(\alpha, \psi) \equiv_{df} \mathbf{a}_i(\alpha) = 1 \Rightarrow \mathbf{a}_i(\psi) = 0$ and $\mathbf{a}_i(\alpha) = 0 \not\Rightarrow \mathbf{a}_i(\psi) = 1$
$\mathbf{A}(\alpha) \cap \mathbf{A}(\psi) = \langle 0000 \rangle$ and $\mathbf{A}(\alpha) \cup \mathbf{A}(\psi) \neq \langle 1111 \rangle$
$\mathrm{CD}(\alpha, \psi) \equiv_{df} \mathbf{a}_i(\alpha) = 1 \Rightarrow \mathbf{a}_i(\psi) = 0$ and $\mathbf{a}_i(\alpha) = 0 \Rightarrow \mathbf{a}_i(\psi) = 1$.
$\mathbf{A}(\alpha) \cap \mathbf{A}(\psi) = \langle 0000 \rangle$ and $\mathbf{A}(\alpha) \cup \mathbf{A}(\psi) = \langle 1111 \rangle$
$\mathrm{SCT}(\alpha, \psi) \equiv_{df} \mathbf{a}_i(\alpha) = 1 \not\Rightarrow \mathbf{a}_i(\psi) = 0$ and $\mathbf{a}_i(\alpha) = 0 \Rightarrow \mathbf{a}_i(\psi) = 1$
$\mathbf{A}(\alpha) \cap \mathbf{A}(\psi) \neq \langle 0000 \rangle$ and $\mathbf{A}(\alpha) \cup \mathbf{A}(\psi) = \langle 1111 \rangle$

$\mathrm{NCD}(\alpha, \psi) \equiv_{df} \mathbf{a}_i(\alpha) = 1 \not\Rightarrow \mathbf{a}_i(\alpha) = 0$ and $\mathbf{a}_i(\alpha) = 0 \not\Rightarrow \mathbf{a}_i(\psi) = 1$
$\mathbf{A}(\alpha) \cap \mathbf{A}(\psi) \neq \langle 0000 \rangle$ and $\mathbf{A}(\alpha) \cup \mathbf{A}(\psi) \neq \langle 1111 \rangle$

Note that the fourth relation NCD is not subalternation but a larger relation of mere *non-contradiction*, stating that no special constraint is imposed upon the values of its relata. While subalternation and non-contradiction are wrongly merged into each other within the limited case of the logical square, an extension of the square to further polygons like the hexagon helps to show that two opposed relata may not be contradictory to each other without being in a relation of subcontrariety or subalternation. Indeed, an exhaustive theory of technical oppositions can be seen as a partition of two basic sets, i.e. incompatible ("genuine") and compatible (merely "technical") sentences.

As for the usual relation of subalternation, which does not appear in the above group of oppositions, it is a special case of non-contradictoriness which puts one constraint upon its relata and cannot be fully expressed in terms of compossible truth-values:

$$\mathrm{sb}(\alpha, \psi) \equiv_{df} \mathbf{a}_i(\alpha) = 1 \Rightarrow \mathbf{a}_i(\psi) = 1$$
$$\mathbf{A}(\alpha) \cap \mathbf{A}(\psi) = \mathbf{A}(\alpha) \text{ and } \mathbf{A}(\alpha) \cup \mathbf{A}(\psi) = \mathbf{A}(\psi)$$

2.4. Abstract oppositions

A number of problems can be addressed about oppositions once the theory is abstracted from its historical background, namely: *what* its essential meaning is, *how* it proceeds, and *how many* oppositions there can be.

2.4.1. What?

Borrowing from Plato's method of definition by *dichotomy* ([21]), which is nothing but a general form of contradiction, it appears that the four traditional oppositions are not on a par: Aristotle's genuine oppositions are all the incompatible relations such that their relata cannot be accepted at once, whereas the technical oppositions are all the compatible relata whose contents can be accepted together. As for subalternation, its meaning in **QAS** requires a further question in addition to the initial two ones about the compossibility of truth-values: beyond Moretti's question-answer semantics about what can be true or false together, subalternation requires the second relatum to be true whenever the first one is. Actually, SB extends the questioning from Moretti's P^{Q}-semantics ($\mathbf{Q}(\Phi) = \langle \mathbf{q}_1(\Phi), \mathbf{q}_2(\Phi)\rangle$) to Piaget's theory of binary connectives ($\mathbf{Q}(\Theta) = \langle \mathbf{q}_1(\Theta), \mathbf{q}_2(\Theta), \mathbf{q}_3(\Theta), \mathbf{q}_4(\Theta)\rangle$). Correspondingly, whoever accepts such an extension should also introduce Sion's implicance into the set of oppositions Op. While Sion's relation of unconnectedness cannot make sense only with Moretti's value $\langle 1, 1\rangle$, our point is that not every such alleged opposition stems from the same questioning but does result in a muddled set of sentential relations with various constraints.

Assuming a possible extension from Moretti's questions to those of Piaget, the following depicts how the arbitrary sentences α and ψ can be related to each other insofar as Moretti's questions $\mathbf{Q}(\Phi)$ are just fragments of Piaget's ones $\mathbf{Q}(\Theta)$: they talk about compossibilities only, the denial of which amounts to various entailment relations between their relata.

Extended questions	Extended answers:
$\mathbf{q}_1(\Phi)$: "$\Diamond(v(\alpha)=v(\psi)=\mathrm{T})$?"	$\mathbf{a}_1(\Phi)=0$ iff $v(\alpha)=\mathrm{T} \Rightarrow v(\psi)=\mathrm{F}$
$\mathbf{q}_2(\Phi)$: "$\Diamond(v(\alpha)=v(\psi)=\mathrm{F})$?"	$\mathbf{a}_2(\Phi)=0$ iff $v(\alpha)=\mathrm{F} \Rightarrow v(\psi)=\mathrm{T}$
+	+
$\mathbf{q}_3(\Phi)$: "$\Diamond(v(\alpha)=\mathrm{T}$ & $v(\psi)=\mathrm{F})$?"	$\mathbf{a}_3(\Phi)=0$ iff $v(\alpha)=\mathrm{T} \Rightarrow v(\psi)=\mathrm{T}$
$\mathbf{q}_4(\Phi)$: "$\Diamond(v(\alpha)=\mathrm{F}$ & $v(\psi)=\mathrm{T})$?"	$\mathbf{a}_4(\Phi)=0$ iff $v(\alpha)=\mathrm{F} \Rightarrow v(\psi)=\mathrm{F}$

Correspondence between Moretti's questions and Piaget's questions:
$\mathbf{q}_1(\Phi) = \mathbf{q}_1(\Theta)$; $\mathbf{q}_2(\Phi) = \mathbf{q}_4(\Theta)$; $\mathbf{q}_3(\Phi) = \mathbf{q}_2(\Theta)$; $\mathbf{q}_4(\Phi) = \mathbf{q}_3(\Theta)$;
$\mathbf{Q}(\Theta) = \langle \mathbf{q}_1(\Theta), \mathbf{q}_2(\Theta), \mathbf{q}_3(\Theta), \mathbf{q}_4(\Theta)\rangle = \langle \mathbf{q}_1(\Phi), \mathbf{q}_3(\Phi), \mathbf{q}_4(\Phi), \mathbf{q}_2(\Phi)\rangle$

According to the below partition between several kinds of constraints upon the sentential relations, it appears that our general view of opposition largely extends the Aristotelian cases of contradiction and contrariety. It also turns out that subalternation is a more particular relation encompassed in the class of non-contradiction, in the sense that being non-contradictory means the possibility of the sentences being true or false together.

Assuming that only Moretti's questions $\mathbf{q}_1(\Phi)$ and $\mathbf{q}_2(\Phi)$ properly characterize the relation of opposition, this should entail that any relation whose meaning essentially relies upon $\mathbf{q}_3(\Phi)$ or $\mathbf{q}_4(\Phi)$ should not be considered as a relation of opposition. Compatible relations can be added to the incompatible ones, for this reason; at the same time, subalternation and implicance are specified by the last two questions and should not characterize Op as such.

Related sentences
$f(\alpha,\psi)$

Incompatibility $\mathbf{a}_1(\Phi) = 0$ Compatibility $\mathbf{a}_1(\Phi) = 1$

Contradiction	Contrariety	Subcontrariety	Non-contradiction
$\mathbf{a}_1(\Phi) = 0$	$\mathbf{a}_1(\Phi) = 0$	$\mathbf{a}_1(\Phi) = 1$	$\mathbf{a}_1(\Phi) = 1$
$\mathbf{a}_2(\Phi) = 0$	$\mathbf{a}_2(\Phi) = 1$	$\mathbf{a}_2(\Phi) = 0$	$\mathbf{a}_2(\Phi) = 1$

Subalternation	Mere non-contradiction
$\mathbf{a}_1(\Phi) = 1$	$\mathbf{a}_1(\Phi) = 1$
$\mathbf{a}_2(\Phi) = 1$	$\mathbf{a}_2(\Phi) = 1$
$\mathbf{a}_3(\Phi) = 0$	$\mathbf{a}_3(\Phi) = 1$

Implicance	Mere subalternation
$\mathbf{a}_1(\Phi) = 1$	$\mathbf{a}_1(\Phi) = 1$
$\mathbf{a}_2(\Phi) = 1$	$\mathbf{a}_2(\Phi) = 1$
$\mathbf{a}_3(\Phi) = 0$	$\mathbf{a}_3(\Phi) = 0$
$\mathbf{a}_4(\Phi) = 0$	$\mathbf{a}_4(\Phi) = 1$

The above branching tree shows that the 4 standard oppositions are not on a par: subalternation is a special subcase of non-contradictoriness, while the meaning of NCD is the logical value $\mathbf{A}(\Phi) = \langle 1, 1\rangle$ that merely relies upon $\mathbf{q}_1(\Phi)$ and $\mathbf{q}_2(\Phi)$. Regarding implicance, the crucial role of $\mathbf{q}_3(\Phi)$ or $\mathbf{q}_4(\Phi)$ in its definition might lead to a rejection of its oppositional nature: rather, implicance IM is an identity relation R that can be formed by the trivial opposite-forming operator I in

$$\mathrm{IM}(\Phi) = \mathrm{R}(\alpha, \mathrm{I}(\alpha))$$

and means that not every relation R between an arbitrary sentence α and its transformation $\otimes(\alpha)$ results in a proper opposition; only the non-trivial operators N, R and C are opposite-forming operators, accordingly. No wonder I is not an opposite-forming operator, indeed: I has been shown to be the direct product of any element $\otimes$ by itself, and the fact that these elements behave like sentential negations means that I amounts to an "exact" (involutive) double negation or affirmation.

The same holds for sb, since $\mathrm{sb}(\alpha) = \mathrm{cd}(\mathrm{ct}(\alpha)) = \mathrm{N}(\mathrm{R}(\alpha)) = \mathrm{C}(\alpha)$, with the crucial difference that the double negation $\mathrm{NR}(\alpha)$ does not lead to a proper affirmation $\mathrm{I}(\alpha)$ but a sort of weak affirmation $\mathrm{C}(\alpha)$ implied by its superaltern α.

Here is a matrix of the five oppositional relations between binary sentences Θ, including two classes of incompatible relations (CT + CD) and compatible relations (SCT + NCD) in addition to the subcase SB $\subseteq$ NCD. The blanks are identity relations $\mathrm{I}(\alpha) = \mathrm{R}(\alpha, \mathrm{I}(\alpha))$ that don't belong to the range of $\mathrm{Op}(\alpha, \mathrm{O}(\alpha))$, again, and the two extremes cases

Θ_1 and Θ_{16} distinguish themselves by standing into a double relation of contradiction and subalternation.

	Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	Θ_6	Θ_7	Θ_8
Θ_1		SB	SB	SB	SB	SB	SB	SB
Θ_2	SB		CT	CT	CT	SB	CT	CT
Θ_3	SB	CT		CT	CT	SB	SB	CT
Θ_4	SB	CT	CT		CT	CT	SB	CT
Θ_5	SB	CT	CT	CT		CT	CT	SB
Θ_6	SB	SB	SB	CT	CT		NCD	CD
Θ_7	SB	CT	SB	SB	CT	NCD		SB
Θ_8	SB	CT	CT	SB	SB	CD	NCD	
Θ_9	SB	SB	CT	CT	SB	NCD	CD	NCD
Θ_{10}	SB	SB	CT	SB	CT	NCD	NCD	NCD
Θ_{11}	SB	CT	SB	CT	SB	NCD	NCD	NCD
Θ_{12}	SB	SB	SB	SB	CD	SB	SB	SB
Θ_{13}	SB	CD	SB	SB	SB	SCT	SB	SB
Θ_{14}	SB	SB	CD	SB	SB	SCT	SCT	SB
Θ_{15}	SB	SB	SB	CD	SB	SB	SCT	SCT
Θ_{16}	CD, SB	SB	SB	SB	SB	SB	SB	SB

	Θ_9	Θ_{10}	Θ_{11}	Θ_{12}	Θ_{13}	Θ_{14}	Θ_{15}	Θ_{16}
Θ_1	SB	SB	SB	SB	SB	SB	SB	CD, SB
Θ_2	SB	SB	CT	SB	CD	SB	SB	SB
Θ_3	CT	CT	SB	SB	SB	CD	SB	SB
Θ_4	CT	SB	CT	SB	SB	SB	CD	SB
Θ_5	SB	CT	SB	CD	SB	SB	SB	SB
Θ_6	NCD	NCD	NCD	SB	SCT	SCT	SB	SB
Θ_7	CD	SB	NCD	SB	SB	SCT	SCT	SB
Θ_8	SB	NCD	NCD	SCT	SB	SB	SCT	SB
Θ_9		NCD	NCD	SCT	SCT	SB	SB	SB
Θ_{10}	NCD		CD	SB	SCT	SB	SCT	SB
Θ_{11}	NCD	CD		SCT	SB	SCT	SB	SB
Θ_{12}	SB	SB	SB		SB	SB	SB	SB
Θ_{13}	SCT	SCT	SB	SCT		SCT	SCT	SB
Θ_{14}	SB	SB	SCT	SCT	SCT		SCT	SB
Θ_{15}	SB	SCT	SB	SCT	SCT	SCT		SB
Θ_{16}	SB	SB	SB	SB	SB	SB	SB	

To sum up, our abstract theory of opposition embraces the opposed relations of *consequence* and *rejection* between arbitrary sentences α and ψ.

Firstly, the semantic relation of consequence $\alpha \vdash \psi$ or $\mathrm{Cn}(\alpha, \psi)$ is a relation of truth-preservation that exclusively refers to the compatible relation SB.

$$\alpha \vdash \psi\colon \text{ for every } \mathbf{a}_i,\ \mathbf{a}_i(\alpha) = 1 \text{ only if } \mathbf{a}_i(\psi) = 1.$$

Secondly, the opposite relation of rejection $\alpha \dashv \psi$ or $\mathrm{Cn}^{-1}(\alpha, \psi)$ ([26]) is a relation of truth-nonpreservation that refers to incompatibility (CD and CT) while including the relation of falsity-preservation ([27]).

$$\alpha \dashv \psi\colon \text{ for every } \mathbf{a}_i,\ \mathbf{a}_i(\alpha) = 1 \text{ only if } \mathbf{a}_i(\psi) = 0.$$

Finally, the relation of opposition $\alpha \between \psi$ or $\mathrm{Op}(\alpha, \psi)$ encompasses consequence and rejection by proceeding as a common theory of truth-inversion. That is:

$$\alpha \between \psi\text{: for some } \mathbf{a}_i,\ \mathbf{a}_i(\alpha) = 1 \text{ and } \mathbf{a}_i(\psi) = 0.$$

Then every opposition proceeds by inversion, whether globally or locally ([17]): global inversion obtains if $\mathbf{a}_i(\alpha) \neq \mathbf{a}_i(\psi)$ for every answer $\mathbf{a}_i$, whereas local inversion means that $\mathbf{a}_i(\alpha) \neq \mathbf{a}_i(\psi)$ for some $\mathbf{a}_i$.

2.4.2. How?

Opposition constitutes a pluralist theory of negation, where the latter occurs through some opposite-forming operators. Apart from the trivial operator of identity $\mathrm{I} = \mathrm{O}(\mathrm{O})$, the Piagetian operators of reversibility lead to the whole relation $\mathrm{Op}(\alpha, \psi) = \mathrm{Op}(\alpha,\ \mathrm{O}(\alpha))$ by proceeding as *difference*-forming operators. In other words, any two sentences are opposed to each other whenever they differ in meaning, and this matches with the view that the sole sentential relation departing from the range of Op is equivalence (Sion's implicance IM).

Just as there is a variety of plausible meanings for the logical connectives, the non-trivial operators of reversibility N, R, C are different *sentential negations*. The most famous instance of these is N, which usually proceeds as a classical or contradictory-forming negation $\mathrm{cd}(\alpha)$. However, this current comparison between inversion and contradictoriness does not mean that every Piagetian operator corresponds to exactly one relation of opposition or one sentential negation. The situation is indeed more intricated between the three overlapping classes of functions O, $\otimes$, and sentential negations.

We have already said that, unlike N, the operators R and C are not extensional functions but intensional unary operators that variously play the role of either negations or affirmations (depending upon the value of their operands). By generalizing some results of Béziau ([6]), it can be proved that

R behaves like *paracomplete* negation $\neg$ when $\mathrm{Op}(\alpha,\ \mathrm{R}(\alpha)) = \mathrm{CT}(\alpha, \psi)$

Example. Let $\alpha = \Theta_2$; then $\mathrm{R}(\Theta_2) = \Theta_5$, and $\mathrm{Op}(\Theta_2,\ \Theta_5) = \mathrm{CT}(\alpha, \psi)$.

More generally:

$$\mathrm{Op}(\alpha,\ \mathrm{R}(\alpha)) = \mathrm{Op}(\alpha,\ \neg\alpha)$$

whenever α is a determinate sentence, i.e. $\mathbf{a}_i = 1$ for one $\mathbf{a}_i$.

R behaves like *classical* (complete and consistent) negation $\sim$ when $\mathrm{Op}(\alpha,\, \mathrm{R}(\alpha)) = \mathrm{CD}(\alpha, \psi)$. □

Example. Let $\alpha = \Theta_6$; then $\mathrm{R}(\Theta_6) = \Theta_8$, and $\mathrm{Op}(\Theta_6, \Theta_8) = \mathrm{CD}(\alpha, \psi)$. More generally:

$$\mathrm{Op}(\alpha,\, \mathrm{R}(\alpha)) = \mathrm{Op}(\alpha,\, {\sim}\alpha)$$

whenever α is a semi-determinate sentence, i.e. $\mathbf{a}_i = 1$ for two $\mathbf{a}_i$. R behaves like *paraconsistent* negation, when $\mathrm{Op}(\alpha, \mathrm{R}(\alpha)) = \mathrm{SCT}(\alpha, \psi)$. □

Example. Let $\alpha = \Theta_{12}$; then $\mathrm{R}(\Theta_{12}) = \Theta_{13}$, and $\mathrm{Op}(\Theta_{12}, \Theta_{13}) = \mathrm{SCT}(\alpha, \psi)$. More generally:

$$\mathrm{Op}(\alpha,\, \mathrm{R}(\alpha)) = \mathrm{Op}(\alpha, -\alpha)$$

whenever α is an indeterminate sentence, i.e. $\mathbf{a}_i(\alpha) = 1$ for three $\mathbf{a}_i$.

C behaves like a *weak double* negation (an iteration of $\sim$, $\neg$, or $-$), since $\mathrm{C} = \mathrm{NR}$. □

2.4.3. How many?

What is the cardinality of the set of oppositions, assuming that there is only one such set? Op can reasonably go beyond the four Aristotelian oppositions by extending the questioning of $\mathbf{Q}(\Phi)$ in **QAS**. Whereas we previously claimed that some attempted extensions are wrong in relying upon different questions, nothing prevents Moretti's questions $\mathbf{Q}(\Phi) = \langle \mathbf{q}_1(\Phi), \mathbf{q}_2(\Phi)\rangle$ from including further questions $\mathbf{q}_i(\Phi)$ about extra truth-values like indeterminacy I beyond T and F. Not even the historical background of Aristotle's logic of propositions, where any sentence about future events was said to be neither true nor false; despite this, the theory of opposition has always been confined to a bivalent framework. Given this unjustified restriction, an abstract logic of oppositions is entitled to go beyond this philosophical objection.

Since every question $\mathbf{q}_i(\Phi)$ characterizing an opposition is a question about a truth-value, there can be only two such questions for a theory of classical oppositions where $\mathrm{V}^2 = \{\mathrm{F},\ \mathrm{T}\}$. And given that the whole m^n oppositions result from n questions and m corresponding sorts of answers, any extension of n or m should extend the number of oppositions.

Let us take some examples of many-valued logics, e.g. three-valued logics ([10, 15, 16, 22]). There should be $n = 3$ questions about their opposed sentences, while maintaining $m = 2$ corresponding yes- or no-answers:

$$\mathbf{q}_1(\Phi)\text{: “}\Diamond(v(\alpha) = v(\psi) = \mathrm{T})\text{?”}$$
$$\mathbf{q}_2(\Phi)\text{: “}\Diamond(v(\alpha) = v(\psi) = \mathrm{F})\text{?”}$$
$$\mathbf{q}_3(\Phi)\text{: “}\Diamond(v(\alpha) = v(\psi) = \mathrm{I})\text{?”}$$

Does it mean that any such truth-functional semantics uniquely results in a set of $m^n = 2^3 = 8$ different oppositions $\mathbf{A}(\Phi) = \langle \mathbf{a}_1(\Phi), \mathbf{a}_2(\Phi), \mathbf{a}_3(\Phi)\rangle$ mapping from Φ to $\{0, 1\}$; and if so, how are they related to each other? Matters are not so easy with many-valuedness, for at least two reasons.

On the one hand, there may be an ambiguity between the meaning of the gappy truth-value I, "neither true nor false", and the content of classical oppositions. As any set of questions $\mathbf{Q}(\Phi)$ is about whether related sentences $\mathrm{Op}(\alpha, \psi)$ can be true or false together, an *ontological* reading of $v(\alpha) = \mathrm{I}$ (as in [16]) is to the effect that α is neither true nor false for the time being but will come to be true or false afterwards. In other words, that $v(\alpha) = \mathrm{I}$ seems to imply that α and ψ may be true (or false) together when $v(\alpha) = \mathrm{T}$ (or F), respectively. If so, then no difference occurs between an opposition arising from non-classical or classical valuations, and the non-standard answer $\mathbf{a}_3(\Phi)$ collapses into a standard one $\mathbf{a}_1(\Phi)$ or $\mathbf{a}_2(\Phi)$. At the same time, an *epistemological* reading of I (as in [15]) does not mean that something indeterminate will come to be either true or false as previously, thus making the answering game more complicated. The same conceptual trouble arises with the paradoxical or glutty readings of I as "meaningless" and "both true and false", respectively ([10], [22]), thus leaving undecided the cardinality of oppositions for many-valued logics.

On the other hand, another way to account for many-valued oppositions would consist in extending the number of answers rather than questions. Then **QAS** can turn the non-classical value I into a non-bivalent answer about classical values; that is: assigning the truth-value I to a sentence may mean that it can be true or false but not definitely, so that a proper answer to $\mathbf{q}_1(\Phi)$ and $\mathbf{q}_2(\Phi)$ could be "maybe" (1/2) instead of "yes" (1) or "no" (0). It results in an alternative set of logical values from $m = 3$ answers and $n = 2$ questions, i.e. $m^n = 3^2 = 9$ different oppositions $\mathbf{A}(\Phi) = \langle \mathbf{a}_1(\Phi), \mathbf{a}_2(\Phi)\rangle$ mapping Φ onto $\{0, 1/2, 1\}$. In addition to the ambiguous meanings of I, there is a patent gap here between Moretti's 2^3- and 3^2-semantics: what about the 9^{th} value in the former? Such difficulties need to be settled before going to more complex many-valued systems, e.g. the four-valued FDE system combining the gappy and glutty interpretations of I ([5]). Whatever the result may be, the point is that any set of oppositions comes from a combination of questions and answers about the compossible properties of sentences.

3. Conclusion: results and open problems

Our algebraic theory of opposition generates five main results.

1. Opposition is a n-ary relation; its 2-ary version $\mathrm{Op}(\alpha, \psi) = \mathrm{Op}(\alpha, \mathrm{O}(\alpha))$ includes an opposition-forming operator of opposite O such that $\mathrm{O}(\alpha) = \psi$ is an opposite of α.

2. The asymmetry of subalternation is due to its special status within the range of oppositions. Unlike the two basic questions characterizing an opposition in Moretti's 2^2-semantics, SB means that, for every answer $\mathbf{a}_i$ in Moretti's semantics, $\mathbf{a}_i(\alpha) = 1$ only if $\mathbf{a}_i(\psi) = 1$; but the converse need not hold.

3. A functional calculus of oppositions can be devised within an alternative logical framework: Question-Answer Semantics, turning the Fregean truth-values into non-Fregean logical values and expressing the meaning of an opposition Φ by means of ordered answers.

4. The number of the standard oppositions is justified by the number of answers characterizing oppositions, namely: $2^2 = 4$ combinations of yes-no answers to their compossible truth-values.

5. Oppositions can be viewed as a set of compossible properties for sentences; although the usual property at hand is truth and falsity (i.e. non-truth) in a bivalent frame, the concept of truth can be kept apart and replaced by a combination of answers 1 and 0 that makes logical consequence (or entailment) a mere special case of opposition as subalternation.

The present paper has not exhausted all the various topics included into the theory of opposition. One of these is to investigate the logical properties of the general operator of opposition, beyond those of consequence (as subalternation) and rejection (as incompatibles). Another one is a functional calculus of opposite-forming operators and their application to philosophical logics for speech acts: assertions and denials can be turned into more fine-grained attitudes, and our operators O pave this way with different sorts of denials like mere opposition (contradiction), strong opposition (contrariety), qualification (subcontrariety), or concession (weak double affirmation).

Despite these positive results, two kinds of main difficulties are to be settled before going on:

I. A *computational* difficulty, in connection with many-valued oppositions and the algebraization of mixed sentences.

Many-valued oppositions have been considered in the preceding section, where the problem is whether the introduction of further truth-values beyond truth and falsity leads to new oppositions or not.

The algebraization of mixed sentences has not been approached in the present paper, but it cannot be eluded if we are to achieve a comprehensive theory of logical oppositions. The problem is how to construct a common question-answer game including different sorts of sentences with quantifiers, modalities, and the like. Our present state of research presented various algebraic semantics for a limited syntax, so that modal sentences or binary sentences are not merged there into more complex expressions like modal binary sentences.

II. A *representational* difficulty, concerning the translation of non-standard logics within **QAS**.

A Boolean algebra has been assumed throughout the paper, but it might be too much limited to render the relations of non-standard logics like non-monotonic patterns. An extension of oppositions to such areas could call for non-Boolean algebras. This case has been already (only slightly) considered in **QAS**, with the new answer "maybe" beyond the standard yes-no answers; but it need to be completed, in order to test the relevance of this abstract logic of oppositions.

Acknowledgement: I want to thank the anonymous referees for their helpful remarks and corrections.

References

[1] Aristotle, *Categories*.
[2] Aristotle, *Metaphysics*.
[3] Aristotle, *On Interpretation*.
[4] Aristotle, *Prior Analytics*.

[5] Belnap, N. D., "A useful four-valued logic", pages 8–37 in: *Modern uses of multiple-valued logic*, G. Epstein and J. Dumm (eds.), D. Reidel Publishing Company, Dordrecht, 1977.

[6] Béziau, J.-Y., "New light on the square of opposition and its nameless corner", *Logical Investigations* 10 (2003): 218–233.

[7] Blanché, R., "Sur l'opposition des concepts", *Theoria* 19 (1953): 89–130.

[8] Blanché, R., *Structures intellectuelles (essai sur l'organisation systématique des concepts)*, Vrin, Paris, 1966.

[9] Bocheński, J. M., *Formale Logik*, Freiburg-München, 1956.

[10] Bochvar, D. A., "On a three-valued calculs and its application to analysis of paradoxes of classical extended functional calculus", *Matematičceskij Sbornik* 4 (1938): 287–308.

[11] Chatti, S., & F. Schang, "A cube to save the square", submitted.

[12] Czeżowski, T., "On certain peculiarities of singular propositions", *Mind* 64, 255 (1955): 392–395.

[13] Hughes, G. E., "The modal logic of John Buridan", pages 93–111 in: *Atti del Convegno internazionale di storia della logica: le teorie delle modalitŕ*, Editrice Cooperativa Libraria Universitaria Editrice, Bologna, 1989.

[14] Keynes, J. N., *Studies and Exercises in Formal Logic*, Macmillan and Co., London, 1884.

[15] Kleene, S., *Introduction to Metamathematics*, North-Holland Publishing Co. and P. Noordhoff N. V., Amsterdam and Groningen, 1952.

[16] Łukasiewicz, J., "O logice trójwartościowej", *Ruch Filozoficzny* 5 (1920), 170–171.

[17] Marcos, J., "Opposition from the viewpoint of universal logic", talk presented at the 3rd Workshop N.O.T., University of Nice, 22–23 June 2010.

[18] Moretti, A., "The geometry of logical opposition", University of Neuchâtel, 2009.

[19] Pellissier, R., "'Setting' *n*-opposition", *Logica Universalis* 2 (2008), 235–262.

[20] Piaget, J., *Traité de logique (essai de logistique opératoire)*, Armand Colin, 1972 (1st ed. 1949).

[21] Plato, *Sophist*.

[22] Priest, G., "The logic of paradox", *Journal of Philosophical Logic* 8 (1979): 219–241.

[23] Schang, F., "Questions and answers about oppositions", pages 283–314 in: *New Perspectives on the Square of Opposition*, J.-Y. Béziau and G. Payette (eds.), Peter Lang, Bern, 2011.

[24] Schang, F., "Two Indian dialectical logics: *saptabhaṅgī* and *catuṣkoṭi*", *Journal of the Indian Council of Philosophical Research* 27 (2011): 45–75.

[25] Schang, F., "Relative charity", *Revista Brasileira de Filosofia* 233 (2009): 159–172.
[26] Sion, A., *Future Logic*, Geneva, 1996.
[27] Skura, T., "A refutation theory", *Logica Universalis* 3 (2009): 239–302.
[28] Słupecki, J., "The theory of rejected propositions", *Studia Logica* 29, 1 (1971): 75–115.
[29] Smessaert, H., "On the 3D Visualisation of logical relations", *Logica Universalis* 3 (2009): 212–231.

FABIEN SCHANG
LHSP Henri Poincaré,
Université de Lorraine,
Nancy, France
schang.fabien@voila.fr