Automata on Ordinals and Automaticity of Linear Orders*

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Abstract. We investigate structures recognizable by finite state automata with an input tape of length a limit ordinal. At limits, the set of states which appear unboundedly often before the limit are mapped to a limit state. We describe a method for proving nonautomaticity and determine the optimal bounds for ranks of linear orders recognized by such automata.

Keywords: Automatic structures, linear orders

1 Introduction

Let us consider a class of structures such as linear orders, partial orders, or graphs. The structures with a simple algorithmic presentation often have simpler algorithmic and structural properties than arbitrary structures. For example, every ordinal recognized by finite automata is below ω^ω [5] and every linear order recognized by finite automata has finite Cantor-Bendixson rank [7]. In this paper, we compute similar bounds for

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the ranks of linear orders recognized by automata with an input tape of length a limit ordinal. Büchi [2], Choueka [4], Wojciechowski [11], and Bruyere-Carton [1], for instance, studied various types of automata indexed by ordinals. Let us briefly describe the type of automaton studied here. Consider a finite state automaton which has read all letters with finite index in an ordinal-indexed input word. To extend the run of the automaton to the infinite ordinals, we need to determine the limit state depending on the sequence of the previous states.

Definition 1. Suppose Σ, Ξ are finite alphabets. An ordinal automaton consists of

– a finite set S of states,
– an initial state,
– a set of accepting states,
– a successor transition function $S \times (\Sigma \cup \{\diamond\}) \times (\Xi \cup \{\diamond\}) \to S$, and
– a limit transition function $\mathcal{P}(S) \to S$.

The input of an ordinal automaton is a word $w : \gamma \to \Sigma \cup \{\diamond\}$ of length some limit ordinal $|w| = \gamma$. The input words for a given automaton will always have a fixed length γ and in this case we will speak of a γ-automaton. In addition, the automaton may process a parameter (or oracle) $p : \gamma \to \Xi \cup \{\diamond\}$, where $p = \diamond^\gamma$ if no parameter is specified. The automaton successively reads the letters of the input word and (simultaneously) the oracle. At any time α, the next state is determined via the successor transition function by the current state, the input letter in place α, and the oracle in place α. At any limit $\lambda \leq \gamma$, the state at λ is determined via the limit transition function by the set of states appearing unboundedly often before λ. The input is accepted if the state at γ

is accepting and rejected otherwise. Thus the limit rule resembles that of a Muller automaton rather than a Büchi automaton. We would like to thank Sasha Rubin for suggesting to study automata with oracles.

We will also consider automata reading input words $w : [\beta, \gamma) \to \Sigma \cup \{\diamond\}$, where γ is a limit ordinal and $\beta < \gamma$. In this case we will speak of a $[\beta, \gamma)$-automaton.

Example 2. *Consider the following* $(n+1)$*-state* ω^n*-automaton. We go into state* 0 *at every successor. We go into state* $m+1$ *at a limit* $\lambda < \omega^n$ *if the maximal state appearing unboundedly often before* λ *is* m*.*

This automaton detects the limit type of the current step, in the sense that the state in any step of the form $\omega^n k_n + \omega^{n-1} k_{n-1} + \ldots + \omega^m k_m$ with $k_m \neq 0$ is exactly $m < n$. This can be used to convert any ω^n-automaton into an ω^n-automaton recognizing the same words, whose state records the limit type of the current step.

Suppose γ is a limit ordinal and $p : \gamma \to \Xi \cup \{\diamond\}$ is a parameter. A *γ-p-automatic presentation* of a relational structure M in a finite language is a structure $N \cong M$ whose domain consists of γ-words together with γ-automata for the domain and each relation in N accepting exactly the words in the domain or the respective relation of N with oracle p. A γ-word is *finite* if all but finitely many of its letters are $\diamond$.

Definition 3. Suppose γ is a limit ordinal and $p : \gamma \to \Xi \cup \{\diamond\}$ is a parameter. A structure is *finite word γ-p-automatic*, or *γ-p-automatic* for short, if it is has an γ-p-automatic presentation whose domain consists of finite γ-words. We omit p if $p = \diamond^\gamma$.

This is a straightforward generalization of automatic structures (see [6]). We will also consider automatic presentations with (finite) input words

$w : [\beta, \gamma) \to \Sigma \cup \{\diamond\}$ and a parameter $p : \gamma \to \Sigma \cup \{\diamond\}$. A structure defined by this process is called $[\beta, \gamma)$-p-automatic. Note that if necessary, we can mark the end of every finite string in the domain of a presentation by attaching an extra symbol, and thus obtain another α-automatic presentation of the same structure. The finite word ω-automatic structures are exactly the automatic structures, i.e. structures with a presentation by finite automata which halt at the end of the finite input word.

Note that some decidability properties of automatic structures have analogous proofs in our setting. A proof of the decidability of the emptyness problem for automata on countable scattered linear orderings can be found in [3].

Comparison of the current input letter with the limit type yields an ω^n-automatic presentation of the following set.

Example 4. *Consider the set of finite ω^n-words with the letter m or $\diamond$ at each place $\omega^{n-1}n_{n-1} + \omega^{n-2}n_{n-2} + \ldots + \omega^m n_m$ with $n_m \neq 0$.*

A natural question is: what is the supremum of ordinals β so that $(\beta, <)$ is α-automatic? Delhommé [5] proved that the supremum of the automatic ordinals is ω^ω. We first conjectured that the supremum of the α-automatic ordinals is ω^α, however this is false for any ϵ with $\omega^\epsilon = \epsilon$. Since $(\alpha, <)$ is α-automatic and any finite product of α-automatic structures is again α-automatic, the supremum is at least α^ω.

Example 5. *Suppose γ is a limit ordinal and $p, q : \gamma \to \omega$ are partial functions with finite domain. Let $p <^* q$ if $\max(dom(p) < \max(dom(q))$, or $\max(dom(p) = \max(dom(q))$ and there is $\beta \leq \max(dom(p)$ with $p(\beta) < q(\beta)$ and $p(\alpha) = q(\alpha)$ for all $\alpha < \beta$.*

Here we define $\diamond < \alpha$ for all $\alpha \in Ord$. This is an example of a wellordering of type ω^γ. We can represent it as a $\omega \cdot \gamma$-automatic structure by representing each $p(\alpha) \neq \diamond$ by the ω-word $0^{p(\alpha)}1\diamond^\omega$ and each $p(\alpha) = \diamond$ by the ω-word $\diamond^\omega$.

Consider the simplest case for an upper bound: proving that ω^{ω^2} is not ω^2-automatic. In the ω-automatic case [5,6], the domain of the structure is split into finitely many pieces, parametrized by words of a fixed length. Since in this setting the words have infinite length, the domain is split into infinitely many pieces and the argument breaks down. However, there are only finitely many possibilities, or types, in which two pieces can be arranged relative to each other. This leads to a product of structures discussed in the next section.

2 Finite-type products

Let us consider a product of arbitrary structures which naturally occurs in γ-automatic representations. For a partial function $f : A \times B \to C$ and $(a,b) \in dom(f)$, let $f_a(b) = f^b(a) = f(a,b)$. Let $p_1(D)$ and $p_2(D)$ denote the projections of a set $D \subseteq A \times B$ to its coordinates. We consider relational structures in a finite signature τ. If A, C are τ-structures and $f : A \to C$ is a partial function, we define $tp(f,g)$ as the isomorphism type of the two-sorted structure $(A, range(f) \cup range(g), f, g, \boldsymbol{r})$, where $\boldsymbol{r}$ denotes the family of the restrictions of the relations of C to $range(f) \cup range(g)$ together with the relations of A and a constant for each element of A.

Definition 6. Suppose A, B, C are τ-structures. A *partial isomorphism* f between τ-structures is an isomorphism $f : dom(f) \to range(f)$. A partial function $f : A \times B \to C$ has *finite type* if $f_a : B \to C$ and

$f^b : A \to C$ are partial isomorphisms[3] for all $a \in p_1(dom(f))$ and $b \in p_2(dom(f))$ and the sets

$$Tp_1(f) = \{tp(f_a, f_{a'}) : a, a' \in p_1(dom(f))\},$$

$$Tp_2(f) = \{tp(f^b, f^{b'}) : b, b' \in p_2(dom(f))\}$$

are finite.[4]

Lemma 7. *Suppose $f : A \times B \to C$ is a finite-type partial function and $A' \subseteq A$, $B' \subseteq B$. Then $f \restriction A' \times B'$ has finite type.*

Proof. Since $tp(f_a, f_{a'})$ uniquely determines $tp(f_a \restriction B', f_{a'} \restriction B')$ for all $a, a' \in p_1(dom(f))$, $|Tp_1(f \restriction A' \times B')| \leq |Tp_1(f)|$.

Definition 8. Suppose E, F are finite sets of pairwise disjoint τ-structures and C is a τ-structure. A finite-type partial function $f : \cup E \times \cup F \to C$ is *faithful* with respect to E, F if $dom(f_a) \in F$ and $dom(f^b) \in E$ for all $a \in \cup E$, $b \in \cup F$.

Definition 9. A τ-structure C is a *finite-type product* of τ-structures A and B if there is a finite-type partial function of $A \times B$ onto C. A τ-structure C is a *faithful* finite-type product of finite sets E, F of τ-structures if there is a faithful finite-type partial function of $\cup E \times \cup F$ onto C.

[3] Except for the faithful maps defined below, partial isomorphisms could be replaced by partial homomorphisms for the purpose of this paper.

[4] Please note that the corresponding [10, Definition 11] is incorrect, since the condition that Tp_1 is finite is missing.

The finite-type product is a refinement of the box-augmentation of [5]. The commutative product of ordinals[5] (see [8]) is a special case of the finite-type product.

Example 10. *[10, Lemma 9] For $\alpha, \beta \in Ord$, $\alpha \otimes \beta$ is the maximal order type of finite-type products of $(\alpha, <)$ and $(\beta, <)$.*

We will decompose a γ-automatic structure as a finite-type product of structures, similar to [5, Proposition 1.2]. Let $\#\boldsymbol{v}$ denote the length of a tuple $\boldsymbol{v}$.

Proposition 11. *Suppose γ is a limit ordinal and A is a γ-p-automatic structure. Then for every formula $\varphi(x, \boldsymbol{y})$ which is a boolean combination of atomic formulas, there is some $m \in \omega$ such that for every $\alpha < \gamma$, there is a set E of $[\alpha, \gamma)$-p-automatic structures with $|E| \leq m$ satisfying the property:*

If $\boldsymbol{b}_0, ..., \boldsymbol{b}_n \in A^{\#\boldsymbol{y}}$ with $n \in \omega$ and $|(\boldsymbol{b}_i)_j| \leq \alpha < \gamma$ for all $i \leq n$ and $j < \#\boldsymbol{y}$, then for all $i \leq n$ the reduct of A to $A_\varphi^{\boldsymbol{b}_i} := \{a \in A : A \vDash \varphi(a, \boldsymbol{b}_i)\}$ is a disjoint union of an α-p-automatic structure and a faithful finite-type product of E with a finite set of α-p-automatic structures.

Proof. The words of length below α in A form an α-p-automatic structure. Consider an automaton for deciding $\varphi(x, \boldsymbol{y})$. For each word $|v|$ of length α, we record the states $s(v)$ of all automata on inputs tuples in $\{v\} \cup \{(\boldsymbol{b}_i)_j :$

[5] Recall that for $\alpha = \Sigma_i \omega^{\alpha_i}$ and $\beta = \Sigma_j \omega^{\beta_j}$ in Cantor normal form, the *commutative sum* $\alpha \oplus \beta$ is defined as the ordinal sum of all ω^{α_i} and ω^{β_j} arranged in decreasing order. The *commutative product* $\alpha \otimes \beta$ is defined as the commutative sum of all $\omega^{\alpha_i \oplus \beta_j}$. Note that the commutative sum and product are strictly increasing in both arguments.

$i \leq n,\ j < \#\boldsymbol{y}\}$. Let m be the number of possible combinations of these states. Let $A_v = \{w : vw \in A\}$. The isomorphism type of A_v depends only on $s(v)$, since $vw \mapsto v'w$ is an isomorphism between A_v and $A_{v'}$ if $s(v) = s(v')$. Let E contain one A_v for each isomorphism type. Similarly, the isomorphism type of $B_w = \{v : |v| = \alpha,\ vw \in A\}$ depends only on the set of states at α with a transition to an accepting state at $|vw|$. Let F contain one B_w for each isomorphism type. Each A_v is $[\alpha, \gamma)$-p-automatic, B_w is α-p-automatic, and the set of words in A of length at least α is a faithful finite-type product of E and F.

Corollary 1. *If in the same situation $p = \diamond^\gamma$ and γ is additively closed, then there is a finite set E of γ-automatic structures such that for any $\boldsymbol{b} \in A^{<\omega}$, the reduct of A to $A_\varphi^{\boldsymbol{b}} := \{a \in A : A \vDash \varphi(a, \boldsymbol{b})\}$ is the disjoint union of a $< \gamma$-automatic structure and a faithful finite-type product of E with a $< \gamma$-automatic structure.*

Proof. If $p = \diamond^\gamma$ and γ is additively closed, then the set E in the previous proof only depends on the states of the automata after reading the corresponding words of length $|(\boldsymbol{b}_i)_j|$.

3 Applications

We will use the finite-type product to bound the ranks of γ-automatic linear orders following [7]. The linear orders with no suborder isomorphic to $(\mathbb{Q}, <)$ are called *scattered*. For any linear order L, the *finite condensation function c_{FC}* forms a quotient of L by identifying elements with only finitely many elements in between. Let c_{FC}^α be the α^{th} iterate of c_{FC}.

Definition 12. The *rank* $rk(L)$ is the least α so that $c_{FC}^\alpha(L)$ does not contain a convex suborder isomorphic to ω or the reversed order ω^*.

Note that $c_{FC}^{rk(L)+1}(L) = c_{FC}^{rk(L)+2}(L)$. If L is a linear order and $(L_i : i \in L)$ is a family of linear orders, the L-sum of $(L_i : i \in L)$ is defined as the lexicographic order on pairs (i, j) where $i \in L$ and $j \in L_i$. It is easy to see that for any scattered linear order L, $rk(L) = 0$ if L is finite, and $rk(L) \leq \alpha$ if L is a $\mathbb{Z} \cdot n$-sum of linear orders of rank below α for some n.

Lemma 13. *Suppose A is a scattered linear order and E is a finite partition of A with $rk(B) \leq \alpha$ for all $B \in E$. Then $rk(A) \leq \alpha$.*

Proof. This follows by induction on α.

Lemma 14. *Suppose C is a scattered linear order and C is a finite-type product of A and B. Then $rk(C) \leq rk(A) \oplus rk(B)$.*

Proof. Let f be a finite-type partial function of $A \times B$ onto C. There are finite partitions of A and B into $\mathbb{Z}$-sums of linear orders of smaller rank. We further partition these $\mathbb{Z}$-sums by the finite-type property, so that on each piece for all $b \neq b'$ either

(i) $range(f^b)$ and $range(f^{b'})$ are cofinal and coinitial, i.e. $\sup(range(f^b)) = \sup(range(f^{b'}))$ and $\inf(range(f^b)) = \inf(range(f^{b'}))$.

(ii) $range(f^b)$ and $range(f^{b'})$ do not overlap, i.e. $range(f^b) < range(f^{b'})$ or $range(f^b) > range(f^{b'})$ pointwise, or

Moreover, the partitions can be refined so that the restrictions of f_a and f^b to each piece are total by Lemma 7. It suffices to show $rk(f(D \times E) \leq rk(D) \oplus rk(E)$ for all pieces $D \subseteq A$, $E \subseteq B$ by Lemma 13.

If case (i) occurs for all $b, b' \in E$, then for all $c < c'$ in $f(D \times E)$, there is an interval $[d, d'] \subseteq D$ such that $[c, c']$ is a subset of a finite-type product of $[d, d']$ with E. Thus $rk([c, c']) \leq rk([d, d']) \oplus rk(E) < rk(D) \oplus rk(E)$ by the inductive hypothesis and hence $rk(f(D \times E)) \leq rk(D) \oplus rk(E)$.

If case (ii) occurs for some $b, b' \in E$, let us partition $f(D \times E)$ into three intervals $C_0 < C_1 < C_2$ such that for all $b \in E$, $range(f^b) \subseteq C_i$ for some $i \leq 2$. Note that we may assume that $C_0, C_2 \neq \emptyset$. Let us look at intervals $[c, c'] \subseteq C_i$ for each $i \leq 2$. For all $c < c'$ in C_1, there is $[e, e'] \subseteq E$ such that $[c, c']$ is a subset of a finite-type product of D with $[e, e']$. If there is a least $\inf(range(f^b))$, we may choose C_0 so that $\inf(range(f^b))$ and $\sup(range(f^b))$ do not depend on b, for b with $range(f^b) \subseteq C_0$. We can then argue as in case (i). If there is no least $\inf(range(f^b))$, then for all $c < c'$ in C_0, there is $[e, e'] \subseteq E$ such that $[c, c']$ is a subset of a finite-type product of D with $[e, e']$. The argument for $c < c'$ in C_2 is analogous. In each case $rk(C_i) \leq rk(D) \oplus rk(E)$ and hence $rk(f(D \times E)) \leq rk(D) \oplus rk(E)$.

This can be used to determine the supremum of ranks of α-automatic scattered linear orders. Note that it is false for arbitrary products.

Proposition 15. *Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$ and $p : \alpha \to \Xi$. Then $\beta \cdot \omega$ is the supremum of ranks of α-p-automatic linear orders.*

Proof. Suppose L is a linear order of rank $\beta \cdot \omega$ with an α-p-automatic presentation. Since L is a dense sum of scattered linear orders, it is sufficient to prove that the ranks of scattered intervals are bounded below $\beta \cdot \omega$. Suppose $[u_n, v_n]$ is an interval with rank $\beta \cdot n$ for each $n \in \omega$. Let $\varphi(x, u, v)$ be the formula $u \leq x \leq v$. There are $m \in \omega$, α' with $|u_n|, |v_n| \leq \alpha' < \alpha$ for each $n < m + 1$, and E for α' with $|E| \leq m$ as in Proposition 11. All α'-p-automatic linear orders have ranks below β by the inductive hypothesis. Each $[u_n, v_n]$ is a union of a scattered linear order with rank below β and a faithful finite-type product of E with a scattered linear order with rank below β by Proposition 11. Then $rk([u_n, v_n])$ can take at most m different values of the form $\beta \cdot k$ by Lemma 14.

We directly obtain

Corollary 2. *Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$ and $p : \alpha \to \Xi$. Then $\omega^{\beta \cdot \omega}$ is the supremum of the α-p-automatic ordinals. Hence for $n < \omega \leq \gamma$,*

(a) ω^{ω^n} is the supremum of the ω^n-automatic ordinals, and
(b) $\omega^{\omega^{\gamma+1}}$ is the supremum of the ω^γ-automatic ordinals.

Proof. This follows from Example 5 and Proposition 15.

Is every linear order (partial order) γ-p-automatic for some ordinal γ and some parameter p?

4 Conclusion

We extended the methods from [5,9,10] to prove nonautomaticity. It has to be seen if finite-type products occur in tree automatic structures. Moreover, we do not yet know if the results are applicable to other structures, for example to compute bounds for the ranks of α-p-automatic well-founded partial orders.

References

1. Véronique Bruyère and Olivier Carton. Automata on linear orderings. *J. Comput. System Sci.*, 73(1):1–24, 2007.
2. Julius Richard Büchi. On a decision method in restricted second order arithmetic. In *Proc. International Congress on Logic, Method, and Philosophy of Science 1960*, volume 2420, pages 1–12. Stanford, 1962.

3. Olivier Carton. Accessibility in automata on scattered linear orderings. In *MFCS2002*, volume 2420 of *Lect. Notes in Comput. Sci.*, pages 155–164. La Jolla, CA, 2002.
4. Yaacov Choueka. Finite automata, definable sets, and regular expressions over ω^n-tapes. *J. Comput. System Sci.*, 17(1):81–97, 1978.
5. Christian Delhommé. Automaticité des ordinaux et des graphes homogènes. *C. R. Math. Acad. Sci. Paris*, 339(1):5–10, 2004.
6. Bakhadyr Khoussainov and Mia Minnes. Three lectures on automatic structures. In *Logic Colloquium 2007*, volume 35 of *Lect. Notes Log.*, pages 132–176. Assoc. Symbol. Logic, La Jolla, CA, 2010.
7. Bakhadyr Khoussainov, Sasha Rubin, and Frank Stephan. Automatic linear orders and trees. *ACM Trans. Comput. Log.*, 6(4):675–700, 2005.
8. Hilbert Levitz. The Cartesian product of sets and the Hessenberg natural product of ordinals. *Czechoslovak Math. J.*, 29(104)(3):353–358, 1979.
9. Alexander Rabinovich and Sasha Rubin. Finite-set interpretations in scattered trees. submitted.
10. Philipp Schlicht and Frank Stephan. Automata on ordinals and linear orders. *Seventh conference on Computability in Europe, CiE 2011*, pages 252–259, 2011.
11. Jerzy Wojciechowski. Finite automata on transfinite sequences and regular expressions. *Fund. Inform.*, 8(3-4):379–396, 1985.