Decoherence Induced Equilibration

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Abstract A pair of harmonic oscillators come in contact and then separate. This could be a model of an atom encountering an electromagnetic field. We explore the coherence properties of the resulting state as a function of the sort of initial condition used. A surprising result is that if one imagines a large collection of these objects repeatedly coming in contact and separating, the asymptotic distribution functions are not Boltzmann distributions, but rather exponentials with the *same* rate of dropoff.

1 Introduction

Decoherence usually represents a loss of information; as such one may expect general principles like the second law of thermodynamics to govern the states that emerge from decohering processes. An example is the phenomenon noted in [1]. Particles of different masses having Gaussian wave packets scatter off one another. In general, momentum conservation alone leads one to expect that their wave functions should become entangled. Nevertheless, there is a particular value of wave function spread such that a scattering event does *not* induce kinematic entanglement. The relation is [spread]$^2 \approx 1/$[mass]. If a box of particles begins in a state that does not satisfy this relation, successive scatterings lead to a density matrix for which this holds. In other words, the system attains a state in which it has lost as much information as possible, after which momentum-conservation induced entanglement ceases. In this case the von Neumann entropy has been maximized [2].

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In this article I examine the effect of decoherence on another kind of interacting system, pairs of harmonic oscillators. The simplest physical case would be an atom, whose first few levels are approximated as a harmonic oscillator, that scatters a photon. Suppose that before the scattering the atom is in its ground state, while afterwards it has some (non-unit) probability of being excited. One can ask whether the atom (considered on its own) should be described by a mixed or pure state, in other words, are the ground and excited states coherent after the scattering? Another context to which our considerations applies is pairs of atoms, each approximated as an oscillator, and the effect of their scattering and subsequent separation.

In all cases (ignoring recurrences and such), the separation should allow one to treat the separated systems as decoherent. It will be seen that the effect of this decoherence can vary. In some instances, successive interactions of a strong field (whose mode is one of the oscillators under consideration) with an atom (the other oscillator) can bring the field into a state whose density matrix is Poisson distributed, like that of a coherent state. Under other circumstances the oscillator density matrices approach an exponential form (distinct from the Poisson-distributed coherent state). Remarkably, although the oscillators have different frequencies, the dropoff in their probability distributions is the same. As a result the equilibrium that is reached is different from that associated with the Boltzmann distribution at some temperature.

In Sect. 2 is a short statement of oscillator properties, included to establish notation. Following that, in Sect. 3 I note the properties of various oscillator states with respect to entangling. The way in which repeated contact can make an initial number operator eigenstate into a Poisson distributed density matrix is shown in Sect. 4. Section 5 is concerned with a collection of oscillators that repeatedly are brought into brief contact, ultimately reaching *the same* exponential distribution. In Sect. 6 I take up entropy changes subsequent to various scattering events. Finally, some of the results are discussed in Sect. 7, and I also mention why the non-Boltzmann distribution does not lead to conflict with the laws of thermodynamics.

2 Formalism

Two systems, $\mathcal{A}$ and $\mathcal{B}$, both satisfy harmonic oscillator equations of motion. Physically they may be modes of an electromagnetic field, of a phonon field or levels of an atom that has been approximated as harmonic. They are brought into temporary contact. The Hamiltonian when they are in contact is

$$H = \omega_a a^\dagger a + \omega_b b^\dagger b + \gamma(a^\dagger b + b^\dagger a), \tag{1}$$

with the usual notation. The coupling, γ, is only turned on for a time T and is assumed constant during that time. Letting

$$
\begin{aligned}
A &= a\cos\theta + b\sin\theta, \\
B &= -a\sin\theta + b\cos\theta,
\end{aligned}
\tag{2}
$$

the Hamiltonian takes the diagonal form

$$H = \omega_A A^\dagger A + \omega_B B^\dagger B, \tag{3}$$

with

$$\omega_A \equiv \omega_a \cos^2 \theta + \omega_b \sin^2 \theta + 2\gamma \cos \theta \sin \theta$$
$$\omega_B \equiv \omega_b \cos^2 \theta + \omega_a \sin^2 \theta - 2\gamma \cos \theta \sin \theta, \tag{4}$$

and

$$\tan 2\theta = \frac{2\gamma}{\omega_a - \omega_b}. \tag{5}$$

3 Entangling and its Absence for Various Initial Conditions

3.1 Coherent State Initial Conditions

We first establish the well-known fact [3] that when the systems begin in coherent states the temporary coupling does *not* induce entanglement.

Let

$$\psi(0) = |z_a, z_b\rangle = \exp[-|z_a|^2/2 + -|z_b|^2/2 + z_a a^\dagger + z_b b^\dagger]|0, 0\rangle, \tag{6}$$

for $z_a, z_b \in \mathbb{C}$, and the two zeros in the final ket refer to the $\mathcal{A}$ and $\mathcal{B}$ ground states respectively. We are interested in $\psi(t) = \exp(-iHt)\psi(0)$. This involves several standard steps. The operators $a^\dagger$ and $b^\dagger$ in (6) are written in terms of $A^\dagger$ and $B^\dagger$ (which are obtained by inverting (2), and which commute) and the Hamiltonian is applied. Observing that $e^x f(y)e^{-x} = f(e^x y e^{-x})$, the time evolution is completely determined using $\exp(-it\omega_A A^\dagger A)A^\dagger \exp(it\omega_A A^\dagger A) = \exp(-it\omega_A)A^\dagger$, with a similar relation for B. Define $u_A \equiv \exp(-i\omega_A t)$ and $u_B \equiv \exp(-i\omega_B t)$. A short calculation shows that the operators $a^\dagger$ and $b^\dagger$ become

$$a^\dagger \to (U_{11}a^\dagger + U_{12}b^\dagger)$$
$$b^\dagger \to (U_{21}a^\dagger + U_{22}b^\dagger), \tag{7}$$

with

$$U = \begin{pmatrix} \cos^2 \theta u_A + \sin^2 \theta u_B & \cos \theta \sin \theta (u_A - u_B) \\ \cos \theta \sin \theta (u_A - u_B) & \cos^2 \theta u_B + \sin^2 \theta u_A \end{pmatrix} \tag{8}$$

U is unitary (but not the most general element of $U(2)$, since it is symmetric). The determinant of U is $u_A u_B$. Substituting (7) in (6) shows that

$$\psi(t) = |z_a', z_b'\rangle, \qquad \text{with} \begin{pmatrix} z_a' \\ z_b' \end{pmatrix} = U \begin{pmatrix} z_a \\ z_b \end{pmatrix} \tag{9}$$

where use has been made of the symmetry and unitarity of U. *This is an unentangled state.*

Defining $e^{i\phi} \equiv u_B u_A^*$, U can also be written

$$U = e^{i\phi/2} \begin{pmatrix} \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos 2\theta & i \sin \frac{\phi}{2} \sin 2\theta \\ i \sin \frac{\phi}{2} \sin 2\theta & \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \cos 2\theta \end{pmatrix} = e^{\frac{i\phi}{2}(\mathbf{1} + \vec{\sigma} \cdot \hat{n})}, \tag{10}$$

where $\hat{n} \equiv \sin 2\theta \hat{x} + \cos 2\theta \hat{z}$ and the σ's the Pauli spin matrices.

3.2 Number Operator Eigenstates of the Oscillators

Now consider initial states that are eigenfunctions of the number operators. Thus

$$\psi(0) = |n_a, n_b\rangle = \frac{(a^\dagger)^{n_a}}{\sqrt{n_a!}} \frac{(b^\dagger)^{n_b}}{\sqrt{n_b!}} |0,0\rangle, \tag{11}$$

where number operator eigenfunctions are distinguished (from coherent states) by the use of the letters n and m (rather than z). By the same reasoning as above

$$\psi(t) = \frac{1}{\sqrt{n_a!}\sqrt{n_b!}} (U_{11}a^\dagger + U_{12}b^\dagger)^{n_a} (U_{21}a^\dagger + U_{22}b^\dagger)^{n_b} |0,0\rangle. \tag{12}$$

This is a complicated sum of states, $|m_a, m_b\rangle$, and is in general not a product state.

The physical situation that I have in mind is that systems $\mathcal{A}$ and $\mathcal{B}$ interact, subsequent to which they go their separate ways. If one is interested in the later state of (say) $\mathcal{A}$, one should trace over the coordinates of $\mathcal{B}$ and arrive at a density matrix for $\mathcal{A}$, which we designate ρ_a. The (m_a, m_a') component of this matrix is[1]

$$\rho_a(m_a, m_a') = \langle m_a | \left[\sum_{m_b} \langle m_b | \psi(t)\psi^\dagger(t) | m_b\rangle \right] | m_a'\rangle. \tag{13}$$

Because the interaction conserves total boson number, $(n_a + n_b)$, once m_a is given, $\psi(t)$ will only have a component with $m_b = n_a + n_b - m_a$. Therefore in (13) one necessarily has $m_a = m_a'$ and ρ_a is diagonal. Therefore *unless there is only one nonzero term in ρ_a, the reduced density matrix describes a mixed state*. In a typical situation there will be several significant contributions for different m_a values to the sum, so the state is far from pure.

By expanding the various powers in (12) we obtain

$$\rho_a(m_a, m_a) = |\psi_{m_a m_b}|^2, \tag{14}$$

with

$$\psi_{m_a m_b} = \sqrt{\frac{m_a! m_b!}{n_a! n_b!}} \sum_k C_k^{n_a} U_{11}^k U_{12}^{n_a-k} C_{m_a-k}^{n_b} U_{21}^{m_a-k} U_{22}^{n_b-m_a+k}, \tag{15}$$

where the sum over k is appropriately limited, reflecting the range of acceptable indices in (13).

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[1]Equation (13) suppresses some mathematical niceties. $\psi\psi^\dagger$ lives on $\mathcal{H}_A \otimes \mathcal{H}_B$, the full Hilbert space. The state $|m_b\rangle$ is $(b^\dagger)^{m_b}|0_b\rangle/\sqrt{m_b!}$, and is in $\mathcal{H}_B$, so that the trace of $\psi\psi^\dagger$ over $\mathcal{H}_B$ leaves it an operator on $\mathcal{H}_A$. Finally the matrix element with respect to $|m_a\rangle = (a^\dagger)^{m_a}|0_a\rangle/\sqrt{m_a!}$ and $|m_a'\rangle$ is a complex number.

4 Successive Scatterings for a Field that Begins in a Number Operator Eigenstate

I next consider a monochromatic field, $\mathcal{B}$, in a high number operator state ("Fock state") that successively scatters off many atoms ($\mathcal{A}$'s) all in their ground state. After each such scattering we trace out the coordinates of the atoms and follow the state of $\mathcal{B}$.

Since $n_a = 1$, (12) now involves only a single binomial expansion. After taking into account the several factorials that enter, one gets

$$\psi(t) = \sum_{k=0}^{N} \sqrt{C_k^N} U_{11}^{N-k} U_{12}^k |N-k,k\rangle, \tag{16}$$

where I have let $N \equiv n_b$. To evaluate ρ_b, a trace over the $\mathcal{A}$ variables is taken. Since (as above, by conservation of total boson number) fixing the $\mathcal{B}$ level fixes the $\mathcal{A}$ level, ρ_b is diagonal and only the square of the absolute value of each term in (16) appears. $|U_{11}|^2$ and $|U_{12}|^2$ sum to unity, so I can write $\sin \chi = |U_{11}|$ and $\cos \chi = |U_{12}|$, for some angle χ.

Because ρ_b is diagonal it is characterized by an $(N+1)$-component vector, which I call y. At time-0, $y(0)_k = \delta_{kN}$. Because the result of an interaction with an atom (or more precisely, an oscillator) in its ground state can only decrease the excitation level, no more than $N+1$ quantities are needed. Thus after repeated scatterings the density matrix can still be written $\rho_b = \sum_{k=0}^{N} y_k |k\rangle\langle k|$. From (16) it follows that

$$y(1)_k = C_k^N \cos^{2k} \chi \sin^{2(N-k)} \chi. \tag{17}$$

For the subsequent time evolution I assume that new exemplars of the $\mathcal{A}$ oscillators (all in the ground state) are encountered. For convenience I will also assume the parameters T and γ are the same, although this does not affect the conclusions. Thus I take each state $|k\rangle\langle k|$ and treat it as an initial condition for the process previously studied—except that now the initial state has k ($\leq N$) bosons, rather than N. I then add the results with weight y_k. If y is considered to be a column vector, a scattering event multiplies it by a matrix whose $v + 1^{\text{st}}$ column is given by

$$\mathcal{L}(\chi)_{\cdot v} = \begin{pmatrix} C_0^v \sin^{2v} \chi \\ C_1^v \sin^{2(v-1)} \chi \cos^2 \chi \\ \cdots \\ C_\ell^v \sin^{2(v-\ell)} \chi \cos^{2\mu} \chi \\ \cdots \\ C_v^v \cos^{2v} \chi \\ 0 \\ \cdots \\ 0 \end{pmatrix}. \tag{18}$$

This can also be written as

$$\mathcal{L}_{\mu\nu} = \begin{cases} C_\mu^\nu (\sin^2 \chi)^{\nu-\mu} (\cos^2 \chi)^\mu & \mu \le \nu, \\ 0 & \mu > \nu, \end{cases} \tag{19}$$

for $\mu, \nu = 0, 1, \ldots, N$.

What I will show is that when $\mathcal{L}$ acts repeatedly on an initial state that is an eigenstate of the number operator with eigenvalue N, the resulting density matrix approaches that of a diagonal coherent state. The density matrix for a coherent state, $e^{-|z|^2/2} \exp(zb^\dagger)|0\rangle$, has as its mth diagonal component $|z|^{2m} \exp(-|z|^2)/m!$, and is a Poisson distribution.

The eigenvalues of $\mathcal{L}$ are its diagonal matrix elements, since $\mathcal{L}$ is upper triangular. Thus $\lambda_n = (\cos^2 \chi)^n$. The associated eigenvector is found to be $u_k^{(n)} = C_k^n (-1)^{n-k} \Theta(n-k)$, where $\Theta(\ell)$ is 1 for $\ell \ge 0$. Note that this $u^{(n)}$ is not normalized by the sum of its squares being one, but rather I have taken its last nonzero component (the nth) to be unity. That this is an eigenvector can be verified directly.

The quantity of interest is $y(t) \equiv (\mathcal{L})^t y_0$. To this end expand y_0 in eigenfunctions of $\mathcal{L}$. This yields

$$y_0 = \sum_0^N a_n u^{(n)}, \quad \text{with } a_n = C_n^N. \tag{20}$$

This assertion can be checked by substitution. Writing $e^{-\Gamma} \equiv \cos^2 \chi$, the full time-dependence of y is

$$y(t) = \sum_{n=0}^N u^{(n)} C_n^N e^{-\Gamma nt} \tag{21}$$

and the kth component of y is

$$y_k(t) = C_k^N e^{-\Gamma kt} (1 - e^{-\Gamma t})^{N-k}. \tag{22}$$

This is a binomial distribution and is clearly normalized (sum over k to get 1). So to show its relation to the Poisson distribution I can drop overall factors. Up to factors independent of k, we have

$$y_k(t) = \frac{1}{k!} \left[\frac{Ne^{-\Gamma t}}{(1 - e^{-\Gamma t})} \right]^k \frac{N!}{N^k (N-k)!}. \tag{23}$$

If $k \ll N$, the last factor is approximately 1 and

$$y_k(t) \approx \frac{1}{k!} \left[\frac{Ne^{-\Gamma t}}{(1 - e^{-\Gamma t})} \right]^k. \tag{24}$$

This is a Poisson distribution with parameter $\frac{Ne^{-\Gamma t}}{1-e^{-\Gamma t}}$, giving the same density matrix as a coherent state. However, the all-important coherence properties of the coherent state (allowing zero entropy increase, as in Sect. 3), are not present, so that scattering off the mixed state described by $y_k(t)$ does increase entropy.

5 A "Gas" of $\mathcal{A}$'s and $\mathcal{B}$'s: the Exponential Distribution

Equation (12) gives the new state after the scattering of two Fock states. I will use this to compute the density matrix of a collection of $\mathcal{A}$ and $\mathcal{B}$ oscillators that repeatedly interact. That is, pairs of them come in contact, then separate, then go on to meet other oscillators. For simplicity I assume $\mathcal{A}$ oscillators do not interact with other $\mathcal{A}$'s and similarly for the $\mathcal{B}$'s. The investigation is numerical.

Many specimens of oscillator types $\mathcal{A}$ and $\mathcal{B}$ are begun in Fock states with excitation levels n_a and n_b. (In practice, the oscillator types are distinguished only by their frequencies, ω_a and ω_b.) With repeated interaction they most likely find themselves in states between 0 and $n_a + n_b$. However, there is some probability that a system will climb to higher and higher states (although the average should remain $n_a + n_b$). Therefore in a numerical study it is necessary to impose a cutoff for the operators that represent $a^\dagger$ and $b^\dagger$. Moreover, these operators are not simply raising operators (with the sequencing of the vector y that we use above they actually move down the column). Rather they are Kronecker products of the form $a^\dagger \otimes \mathbf{1}$ and $\mathbf{1} \otimes b^\dagger$. As a result high dimension is reached quickly, although the matrix representations of these operators are sparse. The numerical process was as follows. After the first scattering, for each pair the wave function has the form (see (12))

$$\psi(t) = \frac{1}{\sqrt{n_a!}\sqrt{n_b!}}(U_{11}a^\dagger + U_{12}b^\dagger)^{n_a}(U_{21}a^\dagger + U_{22}b^\dagger)^{n_b}|0,0\rangle. \qquad (25)$$

One must project from this the wave function components associated with each particular value of the new $\mathcal{A}$ and $\mathcal{B}$ quantum numbers. Again because of conservation of excitation number the $\mathcal{A}$ value fixes the $\mathcal{B}$ value. One then calculates the new density matrices for $\mathcal{A}$ and $\mathcal{B}$, which are diagonal but no longer represent pure states.

For the given n_a and n_b the new ρ_a and ρ_b are related. In particular, once ρ_a has been calculated, ρ_b is found from $\rho_b(n_a + n_b - \mu) = \rho_a(\mu)$, provided the argument of ρ_b is non-negative.

Following the first scattering, subsequent steps use the density matrices at any particular stage. One takes a pair m_a and m_b, calculates as above, and adds them to the new density matrix with weight $\rho_a(m_a)\rho_b(m_b)$.

The result of this process does *not* give a Poisson distribution, but rather an exponential, exactly what one gets by maximizing the entropy with the given constraints (recall that ρ_a and ρ_b are not independent).[2] The remarkable feature is that *both exponentials are the same. There is no equipartition.* Taking the form of the exponential to be $\rho(n) = (1-r)r^n$, r is given by $\frac{\bar{n}}{1+\bar{n}}$, with $\bar{n} = (n_a + n_b)/2$. See Fig. 1.

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[2]For $S = -\sum p_n \log p_n$ with the requirements $\sum p_n = 1$ and $\bar{n} = \sum p_n n$, maximizing S subject to these constraints yields $p_n = (1-r)r^n$, with $r = \bar{n}/(1+\bar{n})$. In practice we have two distributions, ρ_a and ρ_b, but as shown in the text they are directly related and not independent.
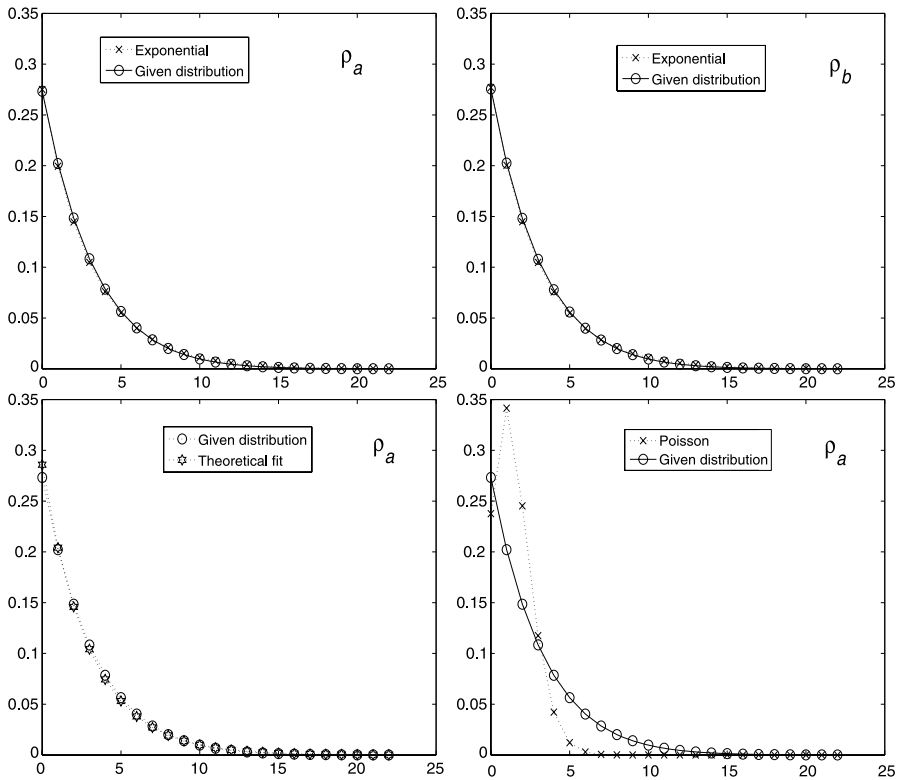
Fig. 1 Probability distribution after 6 scattering events. Fits to exponential and Poisson distributions. Note the equality of the exponential rates. The dropoff from the exponential fit for ρ_a is 0.322, from ρ_b is 0.324 and from the theoretical value of the average initial excitation (based on $\bar{n} = \frac{1}{2}(n_a + n_b) = 2.5$) is 0.337

6 Entropy Changes

The von Neumann entropy, S_{vN}, is defined as the minimum over the choice of basis vectors of $-\operatorname{Tr} \rho \log \rho$. For the process $|z_a, z_b\rangle \to |z_a', z_b'\rangle$ there is no change in S_{vN}. They are pure states both before and after and $S_{vN} = 0$.

For the process $|0, N\rangle \to (\rho_a, \rho_b)$, there is the following change. Initially, $S_{vN} = 0$. Subsequently

$$\rho_a = \sum_{k=0}^{N} C_k^N \sin^{2k} \chi \cos^{2(N-k)} \chi |k\rangle\langle k|$$

and

$$\rho_b = \sum_{k=0}^{N} C_k^N \cos^{2k} \chi \sin^{2(N-k)} \chi |k\rangle\langle k|.$$

These are binomial distributions and in fact have the same entropy. Thus $p_k = C_k^N u^k (1-u)^{(N-k)}$ (with $u = \cos^2 \chi$) and $S_{vN} = -2 \sum_{k=0}^{N} p_k \times \log p_k$. For large N and u away from 0 or 1 this gives

$$S(u) \approx [\log N + 1 + \log(2\pi u(1-u))] = \left[\log N + 1 + \log\left(\frac{\pi}{2}\sin^2 2\chi\right)\right]. \quad (26)$$

On the other hand, for small u (or small $1-u$) the distribution resembles the Poisson distribution in the scaled variable uN. The entropy is then given by (30). Note though that once this scaled variable exceeds 3 the entropy for the binomial and Poisson distributions (essentially) agree.

For the process $|n_a, n_b\rangle$ (in a "gas") going over to a pair of exponentials, there is the following change. Initially the von Neumann entropy is zero. Subsequently it is given by a pair of exponential distributions both of which have the average value $\bar{n}$, with $\bar{n} = (n_a + n_b)/2$. The entropy (by (31)) is

$$S(r) = 2\left[(\bar{n}+1)\log(\bar{n}+1) + \bar{n}\log\bar{n}\right]. \quad (27)$$

Compared to the other distributions, for which the entropy is order $\log N$, with N the photon or excitation level, this is a much larger entropy, namely of order $N \log N$. This is because the exponential is largest near zero, even if the expectation is large, while the binomial or Poisson distributions have a peak near the maximum, with spread roughly $\sqrt{N}$.

7 Discussion

The most remarkable finding in the present paper is the failure to achieve equipartition in the course of repeated scattering. Preliminary investigation of the spin-boson model shows this feature to be present in that model as well, although there are other features there that require separate consideration. In any case, this feature is not an artifact of the boson number conserving feature of the Hamiltonian equation (1). One can in fact arrive at this conclusion (for this particular model, anyway) by maximizing the entropy over the set of possible distributions. The conservation law then connects the $\mathcal{A}$ and $\mathcal{B}$ distributions and one gets a single exponential with a decay law that forces the total expected number of bosons to be the sum of those present in the initial state.

It should be noted that with this single exponential the temperature that one would assign to the $\mathcal{A}$ and $\mathcal{B}$ systems is different (assuming their frequencies are different). One might imagine that a mixed gas of these substances could be separated using a field not relevant to their mutual interaction, thereby creating a temperature gradient and violating the laws of thermodynamics. The flaw in this reasoning is that turning off and on the coupling between them (the γ term of (1)) takes energy, and one avoids producing a perpetuum mobile for the same reasons that were discussed in [4].

In Sect. 4 there is an explicit analytical calculation showing that a field interacting with many oscillators in their ground states goes over to a binomial distribution, which in some cases is well approximated by a Poisson distribution. This result is known in the quantum optics context [5], where the $\mathcal{A}$ "atoms" are beam splitters.

Appendix Entropic Properties of the Poisson, Binomial and Exponential Distributions

For a density matrix having a Poisson distribution and diagonal in the number operator, the von Neumann entropy is immediate. The Poisson distribution with parameter R is

$$p(n; R) = \frac{R^n}{n!} e^{-R}. \tag{28}$$

Since ρ is diagonal in this basis (hence the infimum of $-\mathrm{Tr}\,\rho \log \rho$ is given by the usual expression) the von Neumann entropy is

$$S(R) = -\sum_{n=0}^{\infty} p_n \log p_n. \tag{29}$$

This can be approximated by

$$S(R) \sim \begin{cases} R(1 - \log R) + O(R^2) & \text{for } R <\sim \frac{1}{2} \\ \frac{1}{2} \log(2\pi R) + \frac{1}{2} & \text{for } R >\sim 3 \end{cases} \tag{30}$$

The exponential distribution is $p_n = (1 - r) r^n$, for $0 \le r < 1$ and $n = 0, 1, \ldots$. The entropy given by

$$S(r) = -\frac{1}{1 - r}[(1 - r) \log(1 - r) + r \log r] = (\bar{n} + 1) \log(\bar{n} + 1) + \bar{n} \log \bar{n}, \tag{31}$$

where $\bar{n} = \langle n \rangle_p = \frac{r}{1-r}$.

The binomial distribution, for a given N, is $p_n = C_n^N u^n (1 - u)^{(N-n)}$ with $0 \le n \le N$ and $0 \le u \le 1$. For large enough N and u not too close to 0 or 1, this can be approximated by a Gaussian,

$$p_n \approx \sqrt{\frac{1}{2\pi N u(1 - u)}} \exp\left[-\frac{N(x_n - u)^2}{2u(1 - u)}\right],$$

with $x_n = n/N$. Using this approximation, the entropy is

$$S(u) \approx \frac{1}{2}[\log N + 1 + \log(2\pi u(1 - u))]. \tag{32}$$

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