End of the Square?

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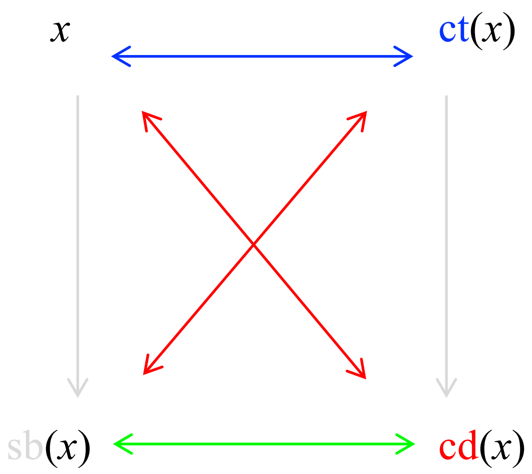
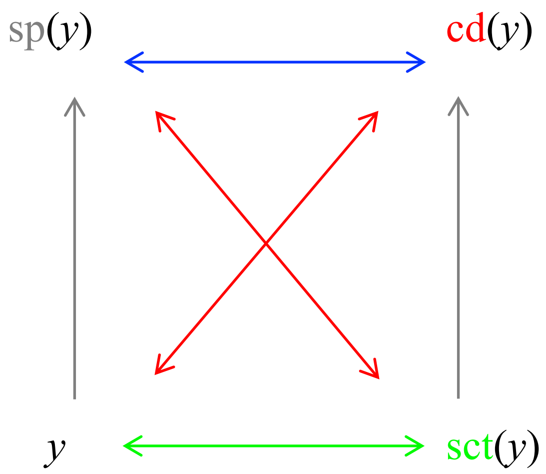
Oppositions with the square

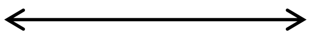
The Square of Opposition: General Structure





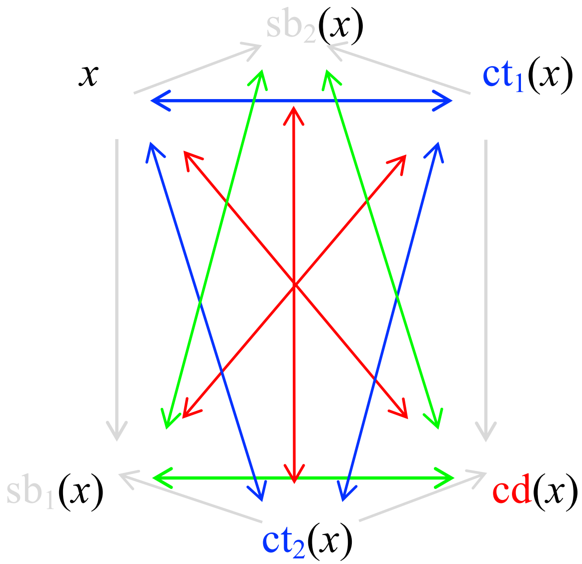
ct(x): “contrary of x”
cd(x): “contradictory of x”
sb(x): “subaltern of x”

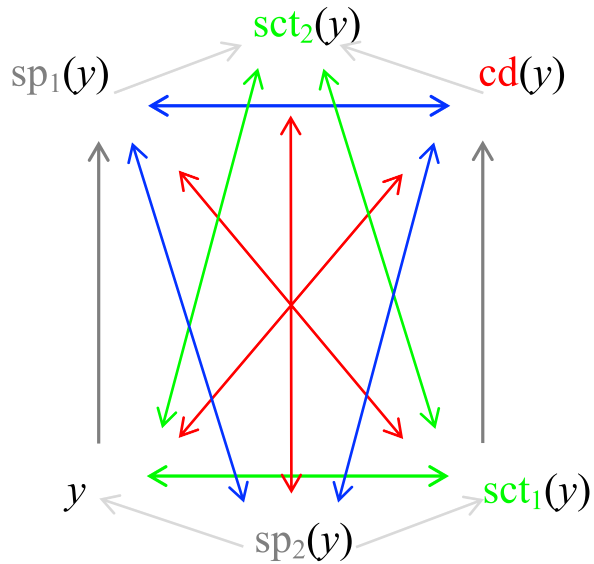
sct(y): “subcontrary of y”
cd(y): “contradictory of y”
sp(y): “superaltern of y”

symmetrical relations 

non-symmetrical relations 

The Hexagon of Oppositions: General Structure





$\mathrm{ct}_1(x)$	:	"1st contrary of x"
$\mathrm{ct}_2(x)$	:	"2nd contrary of x"
$\mathrm{sb}_1(x)$	:	"1st subaltern of x"
$\mathrm{sb}_2(x)$	:	"2nd subaltern of x"
$\mathrm{cd}(x)$	:	"contradictory of x"

$\mathrm{sct}_1(y)$	:	"1st subcontrary of y"
$\mathrm{sct}_2(y)$	:	"2nd subcontrary of y"
$\mathrm{sp}_1(y)$	:	"1st superaltern of y"
$\mathrm{sp}_2(y)$	:	"2nd superaltern of y"
$\mathrm{cd}(y)$	:	"contradictory of y"

2
Oppositions without the square

End of the Square? Costa-Leite's line segment

"Consider a question: is there a way to represent oppositions without two-dimensional objects such as squares or objects of higher dimensions? The answer is ***yes****."* (Costa-Leite, "Oppositions in a line segment": 2)

Let $\mathbb{Z}^*$ be a set of non-null integers, $\mathbb{Z}_+$ a set of positive integers, $\mathbb{Z}_-$ a set of negative integers, and $\mathbb{Z}' = \{-r, -q, q, r\} \subseteq \mathbb{Z}$

Let $\mathcal{C}$ be a set of a categorical statements {A,E,I,O}

i a function on $\mathcal{C}$ s.t. i: $\mathcal{C} \longmapsto \mathbb{Z}'$

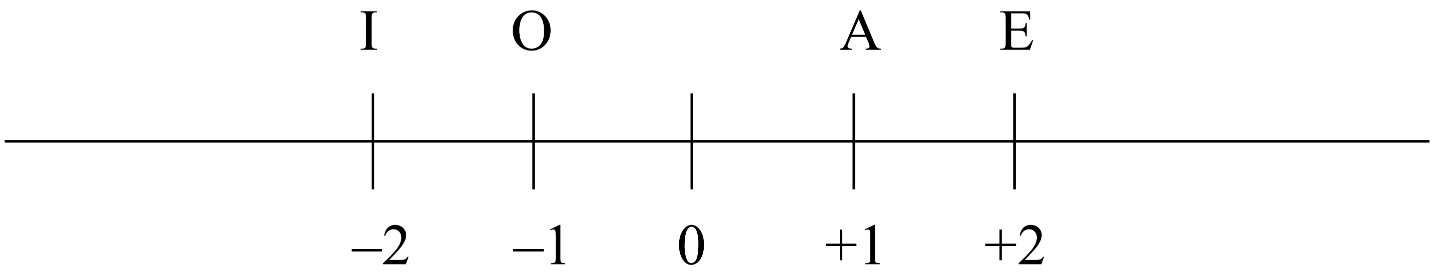
$j \in \mathbb{Z}^*_+$ iff $j \in$ {A,E} (universal sentences)

$j \in \mathbb{Z}^*_-$ iff $j \in$ {I,O} (particular sentences)

Then for every $\alpha, \beta \in \mathcal{C}$:

$i(\alpha)$ and $i(\beta)$ are *contraries*	iff	$i(\alpha), i(\beta) \in \mathbb{Z}^*_+$
$i(\alpha)$ and $i(\beta)$ are *contradictories*	iff	$i(\alpha) + i(\beta) = 0$
$i(\alpha)$ and $i(\beta)$ are *subcontraries*	iff	$i(\alpha), i(\beta) \in \mathbb{Z}^*_-$
$i(\beta)$ is the *subaltern* of $i(\alpha)$	iff	$i(\alpha) \neq i(\beta)$ and $i(\beta) \in \mathbb{Z}^*_-$

Segment Line of Oppositions: Categorical statements (Costa-Leite)



$+2 = ct(+1)$
$-1 = cd(+1)$
$-2 = sb(+1)$

$-1 = sct(-2)$
$+2 = cd(-2)$
$+1 = sp(-2)$

End of the Square? Costa-Leite's line segment

Problem: the above definitions fail with the hexagon of oppositions.

$\mathbb{Z}'' = \{-s, -r, -q, q, r, s\} \subseteq \mathbb{Z}$

$\mathcal{C}' = \{A, U, E, O, Y, I\}$

U = A or E, Y = I and O

$i(U) = i(A) + i(E)$

$i(Y) = i(I) + i(O)$

Let $i(A) = +1$, $i(U) = +3$, $i(E) = +2$, $i(O) = -1$, $i(Y) = -3$, $i(I) = -2$,

Y = ct(A)

now $i(Y) + i(A) = -3 + 1 = -2$, therefore $i(\alpha) + i(\beta) \notin \mathbb{Z}^*_+$

U = sct(I)

now $i(U) + i(I) = +3 - 2 = +1$, therefore $i(\alpha) + i(\beta) \notin \mathbb{Z}^*_-$

U = sb(A)

now $i(U) = +3$, therefore $i(U) \notin \mathbb{Z}^*_-$

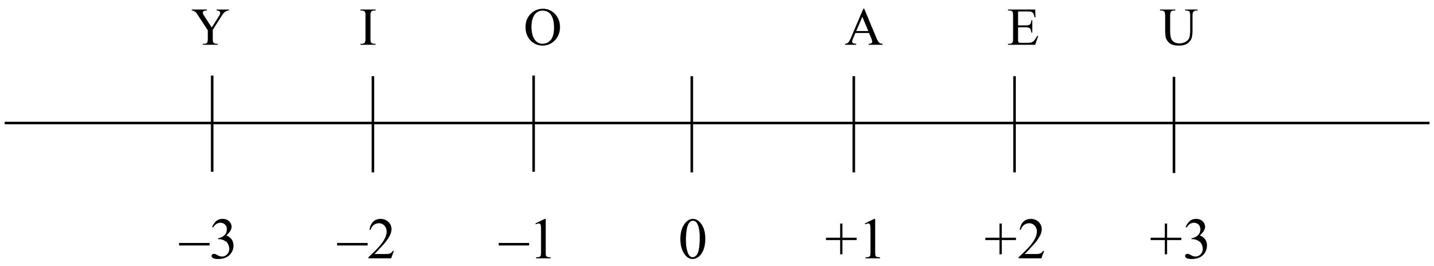
End of the Square? Costa-Leite's line segment

New definitions:

For every $\alpha, \beta, \gamma \in \mathcal{C}$:

$i(\alpha)$ and $i(\beta)$ are *contraries*	iff	$i(\alpha) + i(\beta) + \mathrm{i}(\gamma) = 0$ and $\mathrm{i}(\gamma) \in \mathbb{Z}^*_-$
$i(\alpha)$ and $i(\beta)$ are *contradictories*	iff	$i(\alpha) + i(\beta) = 0$
$i(\alpha)$ and $i(\beta)$ are *subcontraries*	iff	$i(\alpha) + i(\beta) + \mathrm{i}(\gamma) = 0$ and $\mathrm{i}(\gamma) \in \mathbb{Z}^*_+$
$i(\beta)$ is the *subaltern* of $i(\alpha)$	iff	$i(\alpha) \neq i(\beta)$ and $i(\beta) \in \mathbb{Z}^*_-$
		or $i(\alpha) \neq i(\beta)$ and (a) $i(\beta) > i(\alpha) \in \mathbb{Z}^*_+$
		and (b) $i(\beta) > i(\alpha) \in \mathbb{Z}^*_-$

Segment Line of Oppositions: Categorical statements (Costa-Leite)



+2 = ct(+1)
–1 = cd(+1)
–2 = sb(+1)

–1 = sct(–2)
+2 = cd(–2)
+1 = sp(–2)

End of the Square? Costa-Leite's line segment

Problem:

The new definitions seem to be *ad hoc* (hold for $\mathbb{Z}''$ only).

What of the extensions $\mathbb{Z}'^{\cdots}{}'$, for any set $\mathcal{C}'^{\cdots}{}'$ of 2^n elements?

"*There are, notwithstanding, some problems which remain open: the question to determine whether the same procedure can also be applied to solids and higher dimensions, as well as to* ***more than four oppositions****, are very complicated and still have to investigated in detail.*" (Costa-Leite, *ibid*.: 9)

For any family $\mathcal{C}'^{\cdots}{}'$, there is a maximal number of 2^n elements

Solution:

An alternative formal semantics based on oppositions

Cf. Sommers & Englebretsen's "Term-Functor Logic" (TFL)

3 kinds of opposition: C-oppositions, Q-oppositions, P-oppositions

3
Oppositions with another square

A formal semantics of oppositions

$L_{op} = \langle \mathfrak{L}, \mathbf{Q}, \mathbf{A}, S, \cap, \cup, Op, op \rangle$

$\mathfrak{L} = \{x, y, \ldots\}$
Q: question-forming function on x, s.t. $\mathbf{Q}(x) = \langle \mathbf{q}_1(x), \ldots, \mathbf{q}_n(x) \rangle$
A: answer-forming function on x, s.t.
$\mathbf{A}(x) = \langle \mathbf{a}_1(x), \ldots, \mathbf{a}_n(x) \rangle$
$\mathbf{a}(x) \longmapsto \{1,0\}$ (1: yes-answer, 0: no-answer)

S: set of bitstrings, i.e. ordered values of x s.t. Card(S) = 2^n (with n ordered bits)

Op(x,y) reads “x and y are opposed to each other”
$Op(x,y) = Op(x, op(x))$

A formal semantics of oppositions

$L_{op} = \langle \mathfrak{L}, \mathbf{Q}, \mathbf{A}, S, \cap, \cup, Op, op \rangle$

$\mathfrak{L} = \{x, y, \ldots\}$
Q: question-forming function on x, s.t. $\mathbf{Q}(x) = \langle \mathbf{q}_1(x), \ldots, \mathbf{q}_n(x) \rangle$
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$\mathbf{A}(x) = \langle \mathbf{a}_1(x), \ldots, \mathbf{a}_n(x) \rangle$
$\mathbf{a}(x) \longmapsto \{1,0\}$ (1: yes-answer, 0: no-answer)

S: set of bitstrings, i.e. ordered values of x s.t. Card(S) = 2^n (with n ordered bits)

op(x) reads as "opposite to x"
is a *multifunction* s.t. op(x): $S \longmapsto \wp(S)$

Multifunction: to any value of S corresponds zero, one, or several elements of S
: function taking its values in the set of the subparts of S, $\wp(S)$

A Boolean calculus of oppositions (with binary P-oppositions)

For every $\mathbf{a}_i(x)$ and $\mathbf{a}_i(y)$ and every opposite-forming operator op(x) on x:

ct(x) = y	iff	$\mathbf{a}_i(x) = 1 \Rightarrow \mathbf{a}_i(y) = 0$
cd(x) = y	iff	$\mathbf{a}_i(x) = 1 \Leftrightarrow \mathbf{a}_i(y) = 0$
sct(x) = y	iff	$\mathbf{a}_i(x) = 0 \Rightarrow \mathbf{a}_i(y) = 1$
sb(x) = y	iff	$\mathbf{a}_i(x) = 1 \Rightarrow \mathbf{a}_i(y) = 1$
sp(x) = y	iff	$\mathbf{a}_i(x) = 0 \Rightarrow \mathbf{a}_i(y) = 0$

Examples:

ct(**1**000) = **0**001

cd(**1000**) = **0111**

sct(111**0**) = 011**1**

sb(**1**000) = **1**110

sp(111**0**) = 100**0**

Questions about categorical statements $\Theta = \mathrm{S}x\mathrm{P}$

$\mathbf{Q}(\Theta) = \langle \mathbf{q}_1(\Theta), \mathbf{q}_2(\Theta), \mathbf{q}_3(\Theta) \rangle$

$\mathbf{q}_1(\Theta) = \mathrm{S}\boldsymbol{a}\mathrm{P}$

$\mathbf{q}_2(\Theta) = \overline{\mathrm{S}\boldsymbol{a}\mathrm{P}} \cap \overline{\mathrm{S}\boldsymbol{e}\mathrm{P}}$

$\mathbf{q}_3(\Theta) = \mathrm{S}\boldsymbol{e}\mathrm{P}$

Answers to questions about categorical statements $\Theta = \mathrm{S}x\mathrm{P}$

$\mathbf{A}(\Theta) = \langle \mathbf{a}_1(\Theta), \mathbf{a}_2(\Theta), \mathbf{a}_3(\Theta) \rangle$

$\mathbf{A}(\mathrm{S}\boldsymbol{a}\mathrm{P}) = 100$

$\mathbf{A}(\mathrm{S}\boldsymbol{a}\mathrm{P} \text{ or } \mathrm{S}\boldsymbol{e}\mathrm{P}) = 100 \cup 001 = 101$

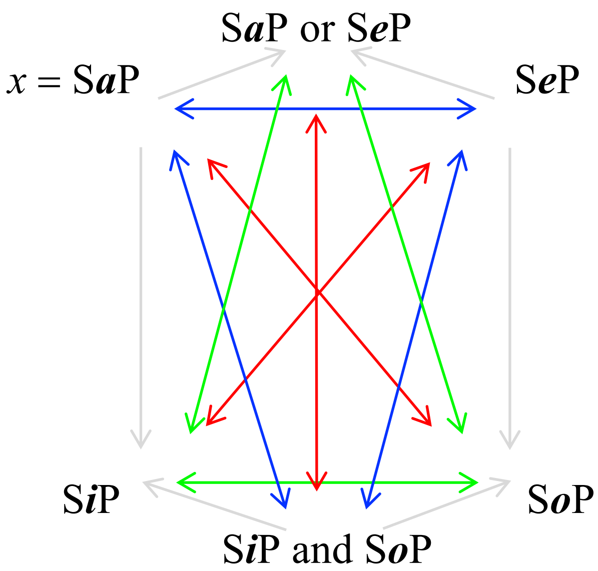
$\mathbf{A}(\mathrm{S}\boldsymbol{e}\mathrm{P}) = 001$

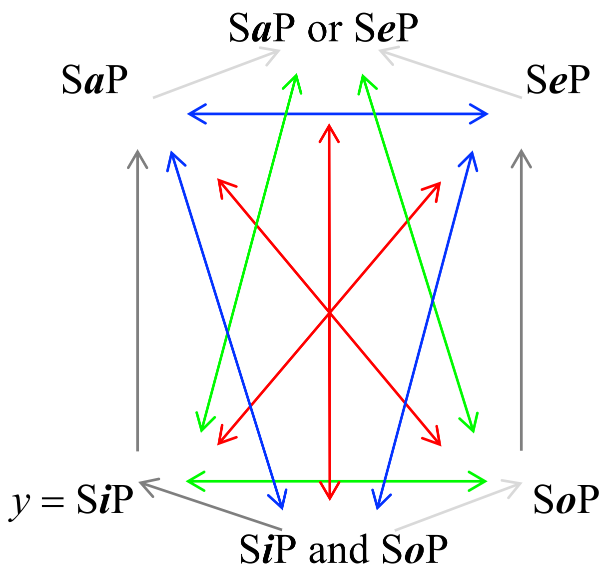
$\mathbf{A}(\mathrm{S}\boldsymbol{o}\mathrm{P}) = 011$

$\mathbf{A}(\mathrm{S}\boldsymbol{i}\mathrm{P} \text{ and } \mathrm{S}\boldsymbol{o}\mathrm{P}) = 110 \cap 011 = 010$

$\mathbf{A}(\mathrm{S}\boldsymbol{i}\mathrm{P}) = 110$

The Hexagon of Opposition: Categorical Statements (Aristotle)





$ct_1(100) = 001$
$ct_2(100) = 110 \cap 011 = 010$
$sb_1(100) = 110$
$sb_2(110) = 010$
$cd(100) = 011$

$sct_1(110) = 011$
$sct_2(110) = 100 \cup 001 = 10$
$sp_1(110) = 100$
$sp_2(110) = 110 \cap 011 = 010$
$cd(110) = 001$

Questions about modal sentences $\Pi = \blacksquare\varphi$

$\mathbf{Q}(\Pi) = \langle \mathbf{q}_1(\Pi), \mathbf{q}_2(\Pi), \mathbf{q}_3(\Pi) \rangle$

$\mathbf{q}_1(\Pi) = \Box\varphi$

$\mathbf{q}_2(\Pi) = \overline{\Box\varphi} \cap \overline{\Box\overline{\varphi}}$

$\mathbf{q}_3(\Pi) = \Box\overline{\varphi}$

Answers to questions about S5 modal statements $\Pi = \blacksquare\varphi$

$\mathbf{A}(\Pi) = \langle \mathbf{a}_1(\Pi), \mathbf{a}_2(\Pi), \mathbf{a}_3(\Pi) \rangle$

$\mathbf{A}(\Box\varphi) = 100$

$\mathbf{A}(\Box\varphi \vee \Box\neg\varphi) = 100 \cup 001 = 101$

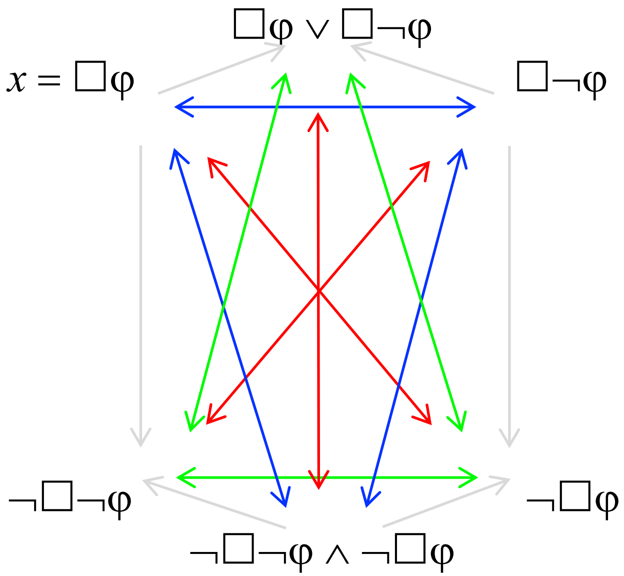
$\mathbf{A}(\Box\neg\varphi) = 001$

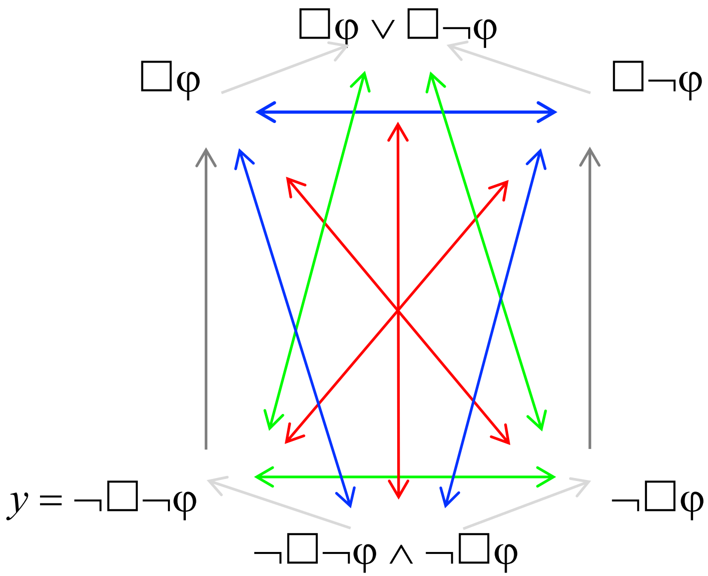
$\mathbf{A}(\neg\Box\varphi) = 011$

$\mathbf{A}(\neg\Box\neg\varphi \wedge \neg\Box\varphi) = 110 \cap 011 = 010$

$\mathbf{A}(\neg\Box\neg\varphi) = 110$

The Hexagon of Opposition: Modal sentences (Blanché)





$ct_1(100) = 001$
$ct_2(100) = 110 \cap 011 = 010$
$sb_1(100) = 110$
$sb_2(110) = 100 \cup 001 = 101$
$cd(100) = 011$

$sct_1(110) = 011$
$sct_2(110) = 100 \cup 001 = 101$
$sp_1(110) = 100$
$sp_2(110) = 110 \cap 011 = 010$
$cd(110) = 001$

Questions about bivalent binary propositions $\Phi = p \bullet q$

$\mathbf{Q}(\Phi) = \langle \mathbf{q}_1(\Phi), \mathbf{q}_2(\Phi), \mathbf{q}_3(\Phi), \mathbf{q}_4(\Phi) \rangle$

$\mathbf{q}_1(\Phi) = p \cap q$

$\mathbf{q}_2(\Phi) = \overline{p} \cap q$

$\mathbf{q}_3(\Phi) = p \cap \overline{q}$

$\mathbf{q}_4(\Phi) = \overline{p} \cap \overline{q}$

Answers to questions about bivalent binary propositions $\Phi = p \bullet q$

$\mathbf{A}(\Phi) = \langle \mathbf{a}_1(\Phi), \mathbf{a}_2(\Phi), \mathbf{a}_3(\Phi), \mathbf{a}_4(\Phi) \rangle$

$\mathbf{A}(p \wedge q) = 1000$

$\mathbf{A}((p \wedge q) \vee (\neg p \wedge \neg q)) = 1000 \cup 0001 = 1001$

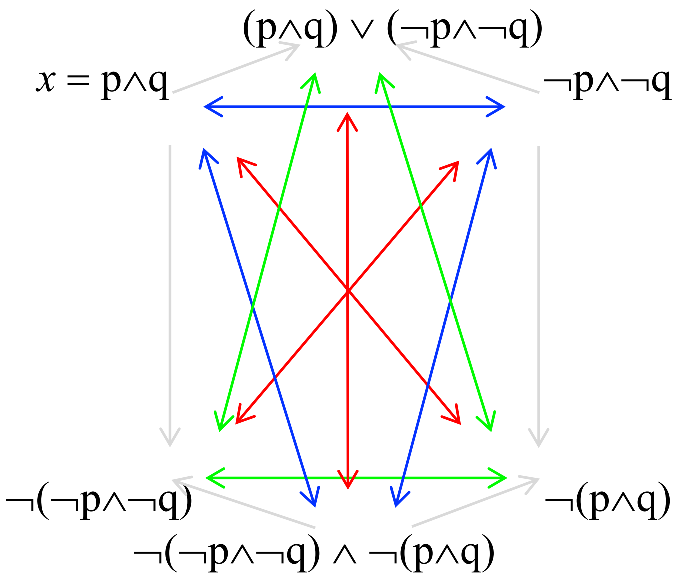
$\mathbf{A}(\neg p \wedge \neg q) = 0001$

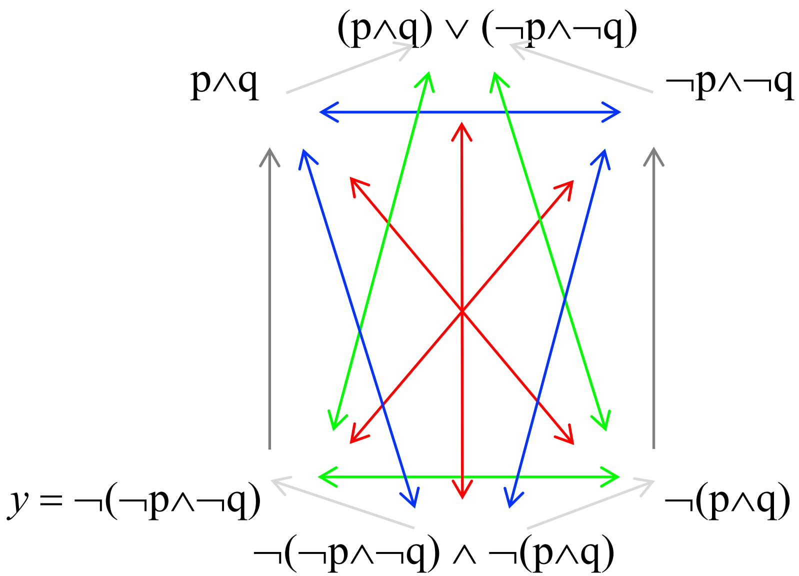
$\mathbf{A}(\neg(p \wedge q)) = 0111$

$\mathbf{A}((p \wedge q) \wedge (\neg p \wedge \neg q)) = 1000 \cap 0001 = 0110$

$\mathbf{A}(\neg(\neg p \wedge \neg q)) = 1110$

The Hexagon of Opposition: Binary sentences (Piaget)





$ct_1(1000) = 0001$

$ct_2(1000) = 1110 \cap 0111 = 1001$

$sb_1(1000) = 1110$

$sb_2(1110) = 0110$

$cd(1000) = 0111$

$sct_1(1110) = 0111$

$sct_2(1110) = 1000 \cup 0001$

$sp_1(1110) = 1000$

$sp_2(1110) = 1110 \cap 0111 = 0110$

$cd(1110) = 0001$

Questions about singular terms Ω = S is/is not P/not-P

$\mathbf{Q}(\Omega) = \langle \mathbf{q}_1(\Omega), \mathbf{q}_2(\Omega), \mathbf{q}_3(\Omega), \mathbf{q}_4(\Omega) \rangle$

$\mathbf{q}_1(\Omega) = \text{S is absolutely P}$

$\mathbf{q}_2(\Omega) = \overline{\text{S is absolutely P}} \cap \overline{\text{S is absolutely } \overline{\text{P}}}$

$\mathbf{q}_3(\Omega) = \text{S is absolutely not } \overline{\text{P}}$

Answers to questions about singular terms Ω = S is/is not P/not-P

$\mathbf{A}(\Omega) = \langle \mathbf{a}_1(\Omega), \mathbf{a}_2(\Omega), \mathbf{a}_3(\Omega) \rangle$

A(S is P) = 100

A(S is P or not-P) = 100 ∪ 001
= 101

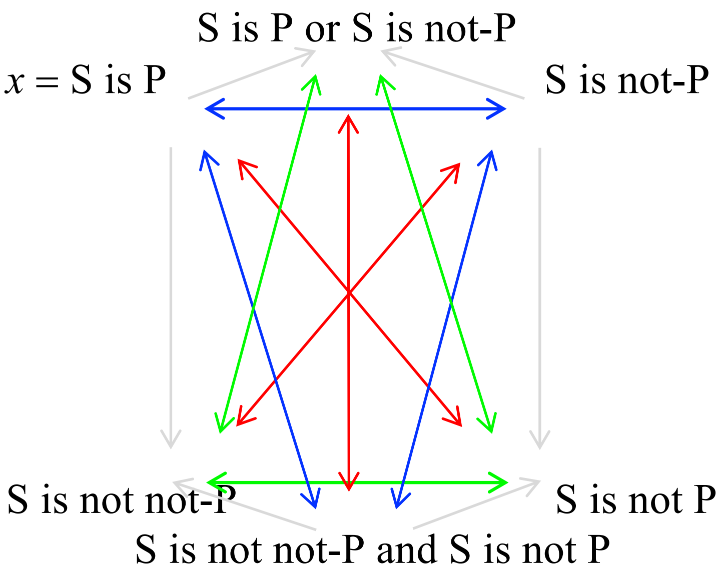
A(S is not-P) = 001

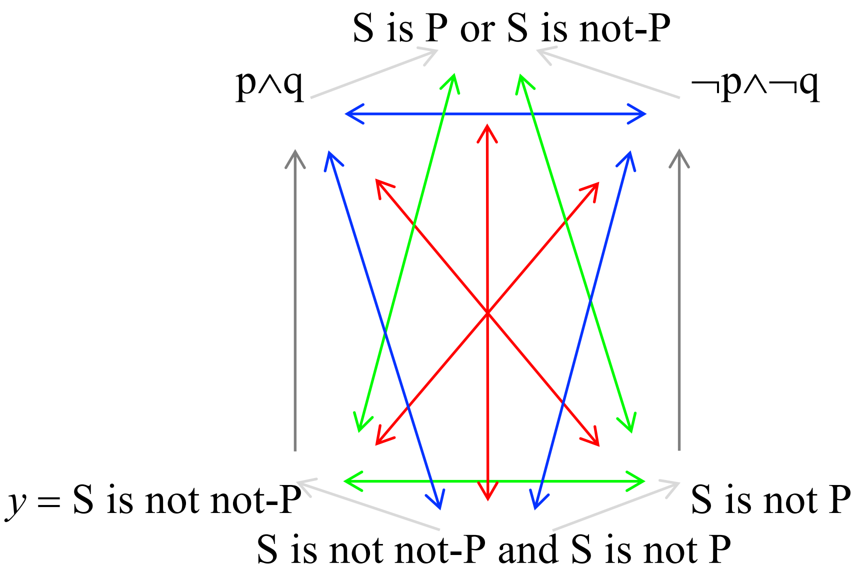
A(S is not P) = 011

A(S is not P and not not-P) = 110 ∩ 011
= 010

A(S is not not-P) = 110

The Hexagon of Opposition: Term logic (Aristotle, Englebretsen)





$ct_1(100) = 0001$

$ct_2(100) = 110 \cap 011 = 101$

$sb_1(100) = 110$

$sb_2(110) = 010$

$cd(100) = 011$

$sct_1(110) = 011$

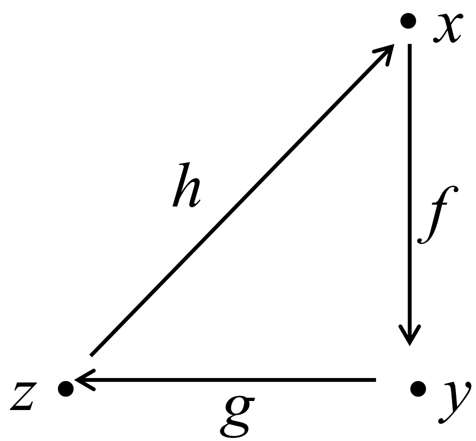
$sct_2(110) = 100 \cup 001$

$sp_1(110) = 100$

$sp_2(110) = 110 \cap 011 = 010$

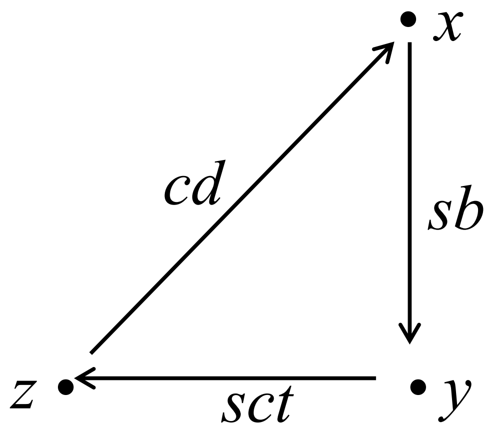
$cd(110) = 001$

Graphs: how to determine the values(s) of the multifunction op?



$y = f(x)$
$z = g(y) = g(f(x))$
$x = h(z) = h(g(f(x)))$

Graphs: how to determine the values(s) of the multifunction op?



A(y) = 0111 = sb(0001)
A(z) = 1110 = sct(0111) = sct(sb(0001))
A(x) = 0001 = cd(1110) = cd(sct(sb(0001)))

Definitions. For every x:

$\mathrm{op}(x) \neq x$ (non self-difference)

$\mathrm{cd}(\mathrm{cd}(x)) = x$ (contradictoriness)

$\mathrm{op}(x) = \mathrm{op}_i(\mathrm{op}_j(x))$ iff $\mathrm{op}^{-1}(x) = \mathrm{op}_j(\mathrm{op}_i(x))$ (converse)

$\mathrm{op}_i(\mathrm{op}_j^{-1}(x)) = \mathrm{op}_j(\mathrm{op}_i(x))$ (converses: sb/sp)

$\mathrm{sp}(y) = x$ iff $x = \mathrm{sb}(y)$

$\mathrm{cd}(x) = \mathrm{sb}(\mathrm{ct}(x)) = \mathrm{ct}(\mathrm{sp}(x))$ (contradictoriness)

$\mathrm{ct}(x) = \mathrm{cd}(\mathrm{sb}(x)) = \mathrm{sp}(\mathrm{cd}(x))$ (contrariety)

$\mathrm{sct}(x) = \mathrm{cd}(\mathrm{sp}(x)) = \mathrm{sb}(\mathrm{cd}(x))$ (subcontrariety)

$\mathrm{sb}(x) = \mathrm{cd}(\mathrm{ct}(x))$ (subalternation)

For every bitstring $\mathbf{A}(x)$ of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of *yes*- and *no*-answers in any $\mathbf{A}(x)$. Then:

Proposition 1
The number of *contraries* of x is
$\text{Card}(\text{ct}(x)) = 2^{\mu(x)} - 1$.

Examples: let $\mathbf{A}(x) = 1001$
$\mu(x) = 2$
Hence $\text{Card}(\text{ct}(x)) = 2^2 - 1 = 3$

let $\mathbf{A}(y) = 111111$
$\mu(x) = 0$
Hence $\text{Card}(\text{ct}(x)) = 2^0 - 1 = 0$

Proof: See Schang, F.: “Logic in Opposition”.

For every bitstring **A**(x) of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of *yes*- and *no*-answers in any **A**(x). Then:

Proposition 2
The number of *subalterns* of x is
$\mathrm{Card}(\mathrm{sb}(x)) = 2^{\mu(x)} - 1$.

Example: let **A**(x) = 1001
$\mu(x) = 2$
Hence $\mathrm{Card}(\mathrm{sb}(x)) = 2^2 - 1 = 3$

Proof: $\mathrm{sb}(x) = \mathrm{cd}(\mathrm{ct}(x))$
For every x, $\mathrm{Card}(\mathrm{cd}(x)) = \mathrm{Card}(x) = 1$
Hence $\mathrm{Card}(\mathrm{sb}(x)) = \mathrm{Card}(\mathrm{cd}(\mathrm{ct}(x)) = \mathrm{Card}(\mathrm{ct}(x)) = 2^{\mu(x)} - 1$ □

For every bitstring **A**(x) of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of *yes*- and *no*-answers in any **A**(x). Then:

Proposition 3
The number of *superalterns* of x is
$\text{Card}(\text{sp}(x)) = 2^{\nu(x)} - 1$.

Example: let **A**(x) = 1001
$\nu(x) = 2$
Hence $\text{Card}(\text{sp}(x)) = 2^2 - 1 = 3$

Proof: $\text{sp}(x) = \text{ct}(\text{cd}(x))$
For every x, $\text{Card}(\text{cd}(x)) = n - \mu(x) = \nu(x)$
Hence $\text{Card}(\text{sp}(x)) = \text{Card}(\text{ct}(\text{cd}(x)) = 2^{\nu(x)} - 1$. □

For every bitstring **A**(x) of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of *yes*- and *no*-answers in any **A**(x). Then:

Proposition 4
The number of *subcontraries* of x is
$\text{Card}(\text{sct}(x)) = 2^{\nu(x)} - 1$.

Example: let **A**(x) = 1001
$\nu(x) = 2$
Hence $\text{Card}(\text{sct}(x)) = 2^2 - 1 = 3$

Proof: $\text{sct}(x) = \text{cd}(\text{sp}(x))$
For every x, $\nu(\text{cd}(x)) = \nu(x)$.
Hence $\text{Card}(\text{sct}(x)) = \text{Card}(\text{cd}(\text{sp}(x)) = \text{Card}(\text{sp}(x)) = 2^{\mu(x)} - 1$. □

For every bitstring **A**(x) of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of *yes*- and *no*-answers in any **A**(x). Then:

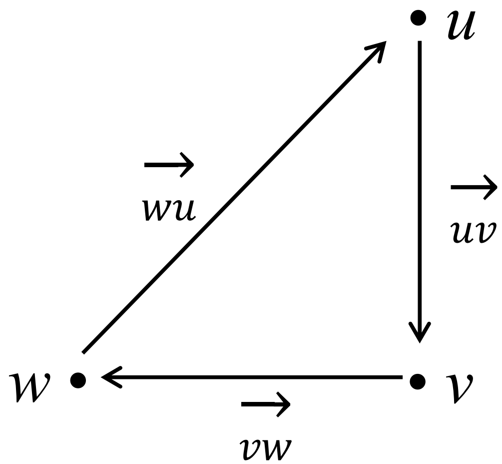
Proposition 5
The number of *indeterminates* of x is
$\text{Card}(\text{id}(x)) = (2^n - 1) - \text{Card}(\text{d}(x))$.

Examples: let **A**(x) = 1001
$\text{Card}(\text{d}(x)) = \text{Card}(\text{ct}(x) + \text{cd}(x) + \text{sct}(x) + \text{sb}(x) + \text{sp}(x)) = 11$
Hence $\text{Card}(\text{id}(x)) = 2^4 - 1 - 11 = 4$

Proof: Determinates are the disjoint union of ct(x), cd(x), sct(x), sb(x), sp(x).
For every x, $\text{Card}(\text{ct}(x) \cap \text{sp}(x)) = \text{Card}(\text{sct}(x) \cap \text{sb}(x)) = 1$.
Hence $\text{Card}(\text{d}(x)) = \text{Card}(\text{ct}(x) + \text{cd}(x) + \text{sct}(x) + \text{sb}(x) + \text{sp}(x)) - 2$. □

Vector theory

How to determine the values(s) of op?



$$\overrightarrow{uv} + \overrightarrow{vw} = \overrightarrow{uw}$$
$$\overrightarrow{vw} + \overrightarrow{wu} = \overrightarrow{vu}$$
$$\overrightarrow{wu} + \overrightarrow{uv} = \overrightarrow{wv}$$

An arithmetization of oppositions: bitstrings as base-2 integers

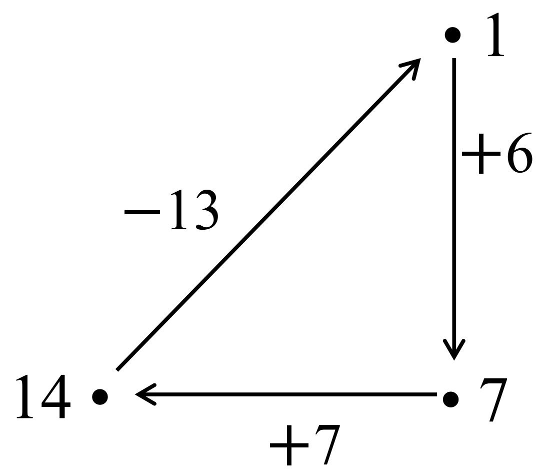
- base-2 integers are turned into base-10 integers with a function σ: S $\longmapsto \mathbb{N}$
- bitstrings are turned into integers, s.t.:
  $\sum(x) = \langle \sigma_1(x) + \ldots + \sigma_n(x) \rangle$, with $\sigma_k(x) = 2^{n-k} \times \mathbf{a}_k(x)$
  Example: $\sum(1101) = 8 + 4 + 0 + 1 = 13$
- opposite-forming operators are turned into arithmetic operators $\pm\sigma$, s.t.:
  $\pm(\sum(x)) = \sum(y)$

For every *x*,*y*:

x and *y* are *contradictories*	iff	$\sigma(x) \neq 0 \Leftrightarrow \sigma(y) = 0$
x and *y* are *contraries*	iff	$\sigma(x) \neq 0 \Rightarrow \sigma(y) = 0$
x and *y* are *subcontraries*	iff	$\sigma(x) = 0 \Rightarrow \sigma(y) \neq 0$
x is *subaltern* of *y*	iff	$\sigma(x) \neq 0 \Rightarrow \sigma(y) \neq 0$

Example: $\mathbf{A}(x) = 0111$, $\mathbf{A}(y) = 0001$
$\sigma(y) \neq 0 \Rightarrow \sigma(x) \neq 0$, therefore $\mathrm{Op}(x,y) = \mathrm{SB}(x,y)$

How to determine the value(s) of op?

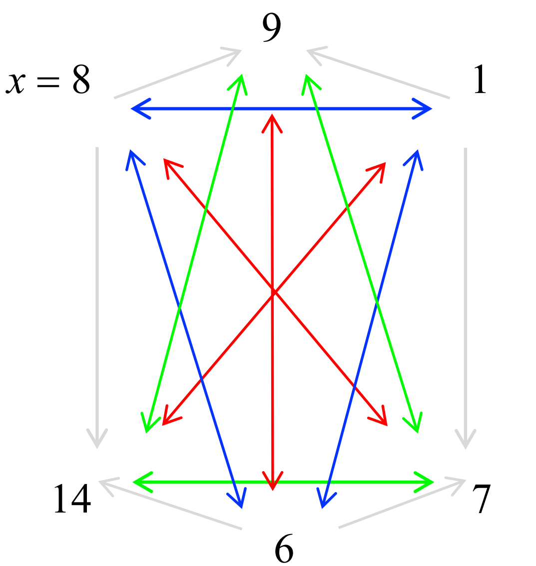


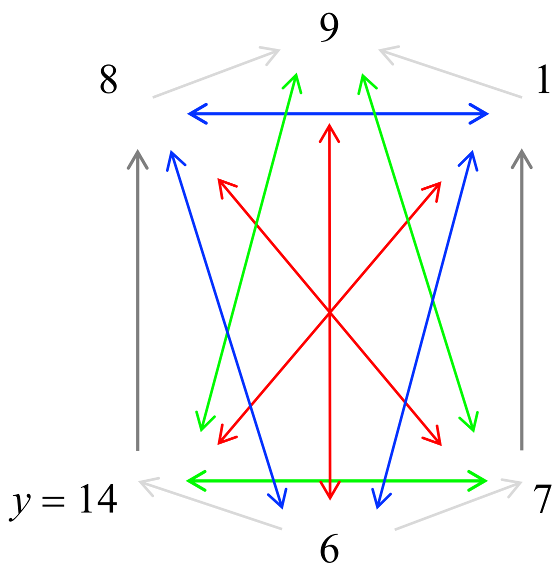
$+6(1) = 7$

$+13(1) = +7(+6(1)) = 14$

$\pm 0(1) = +7(+6-13(1))) = 1$

The Hexagon of Oppositions: General Structure





$1 = (-7)8 = 8 - 7$
$6 = (-2)8 = 8 - 2$
$14 = (+6)8 = 8 + 6$
$9 = (+1)8 = 8 + 1$
$7 = (-1)8 = 8 - 1$

$7 = (-7)14 = 14 - 7$
$9 = (-5)14 = 14 - 5$
$8 = (-6)14 = 14 - 6$
$6 = (-8)14 = 14 - 8$
$1 = (-13)14 = 14 - 13$

PROBLEMS:

- Costa Leite’s segments hold for limited diagrams only
- the vectorial behavior of oppositions holds with 2D diagrams only
  it is lost with, e.g., hypercubes (n = 3), tetraicosahedrons (n = 4), etc.

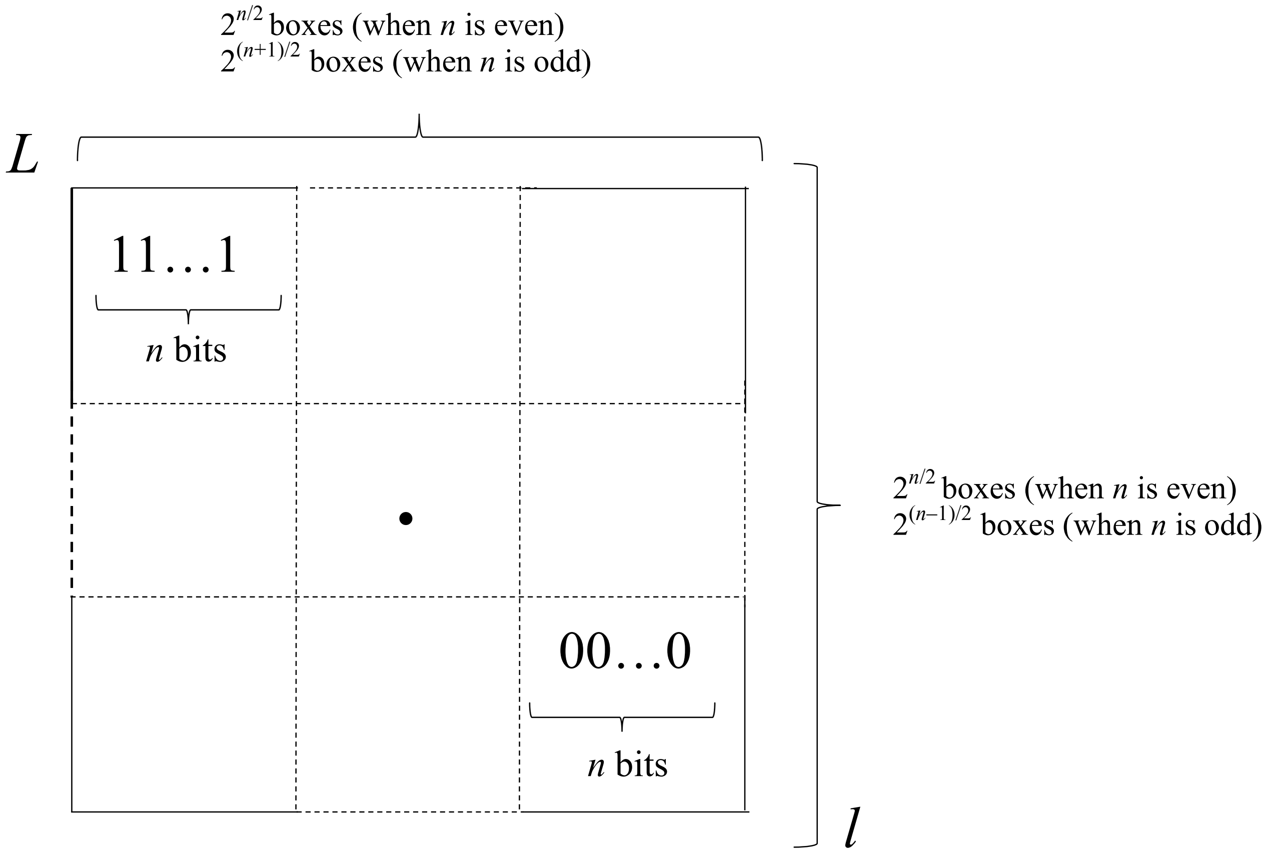
SOLUTION:

- a general diagram for oppositions of any structural complexity
- replacing vertices with areas in a diagram of n-chotomies

Diagrams with areas (rather than vertices) of *n*-bitstrings (*n* = length)

S is a square
if $L = l$

S is a rectangle
if $L \neq l$



Diagrams with areas (rather than vertices) of n-bitstring (n = length)

1	0

$n = 1$

$L = 2^{(1+1)/2} = 2^{2/2} = 2^1 = 2$
$l = 2^{(1-1)/2} = 2^{0/2} = 2^0 = 1$

Diagrams with areas (rather than vertices) of *n*-bitstring (*n* = length)

11	01
10	00

$n = 2$

$L = 2^{2/2} = 2^1 = 2$
$l = 2^{2/2} = 2^{2/2} = 2^1 = 2$

Diagrams with areas (rather than vertices) of n-bitstring (n = length)

111	110	011	010
101	100	001	000

$n = 3$

$$L = 2^{(3+1)/2} = 2^{4/2} = 2^2 = 4$$
$$l = 2^{(3-1)/2} = 2^{2/2} = 2^1 = 2$$

Diagrams with areas (rather than vertices) of n-bitstring (n = length)

1111	1101	0111	0101
1110	1100	0110	0100
1011	1001	0011	0001
1010	1000	0010	0000

$n = 4$

$L = 2^{4/2} = 2^2 = 4$
$l = 2^{4/2} = 2^2 = 4$

Diagrams with areas (rather than vertices) of n-bitstring (n = length)

11111	11110	10111	10110	01111	01110	00111	00110
11101	11100	10101	10100	01101	01100	00101	00100
11011	11010	10011	10010	01011	01010	00011	00010
11001	11000	10001	10000	01001	01000	00001	00000

$n = 5$

$$L = 2^{(5+1)/2} = 2^{6/2} = 2^3 = 8$$
$$l = 2^{(5-1)/2} = 2^{4/2} = 2^2 = 4$$

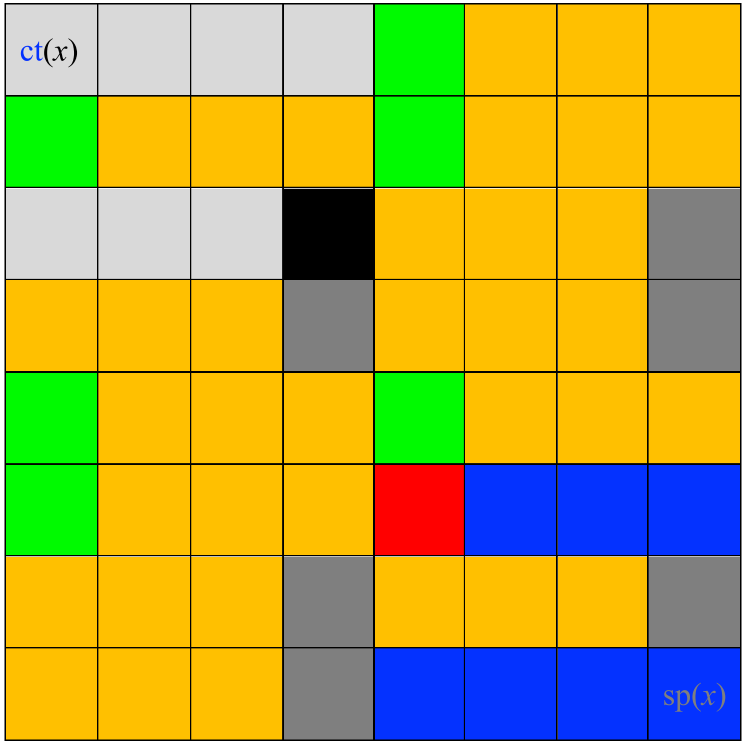
Diagrams with areas (rather than vertices) of n-bitstring (n = length)

111111	111101	101111	101101	011111	011101	001111	001101
111110	111100	101110	101100	011110	011100	001110	001100
111011	111001	101011	101001	011011	011001	001011	001001
111010	111000	101010	101000	011010	011000	001010	001000
110111	110101	100111	100101	010111	010101	000111	000101
110110	110100	100110	100100	010110	010100	000110	000100
110011	110001	100011	100001	010011	010001	000011	000001
110010	110000	100010	100000	010010	010000	000010	000000

$n = 6$

$$L = 2^{6/2} = 2^3 = 8$$
$$l = 2^{6/2} = 2^3 = 8$$

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

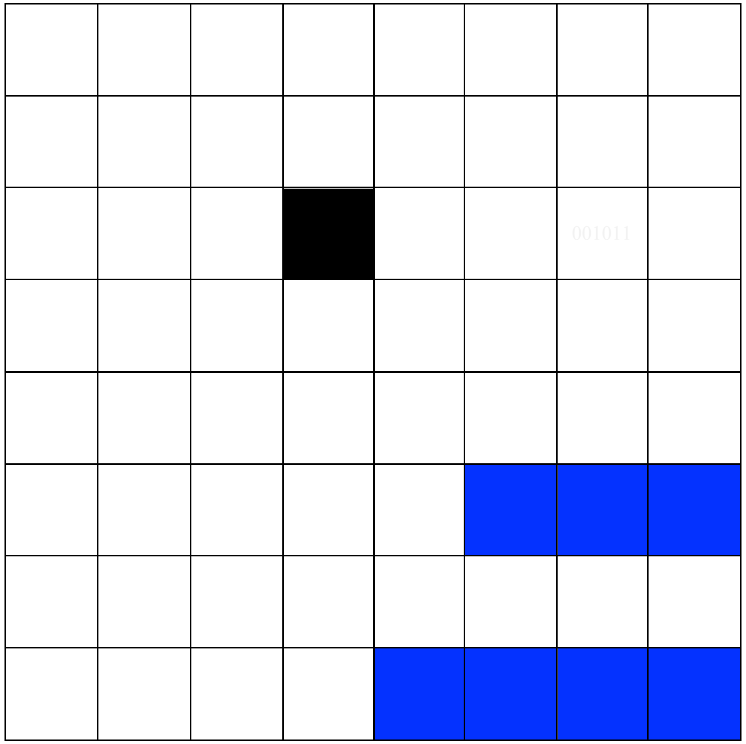
Card(d(x)) = $29 - 2 = 27$

Card(id(x)) = $64 - 1 - 27 = 36$

A(x) = 101001

$d(x)$ = {ct(x),cd(x),sct(x),sp(x),sb(x)}

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

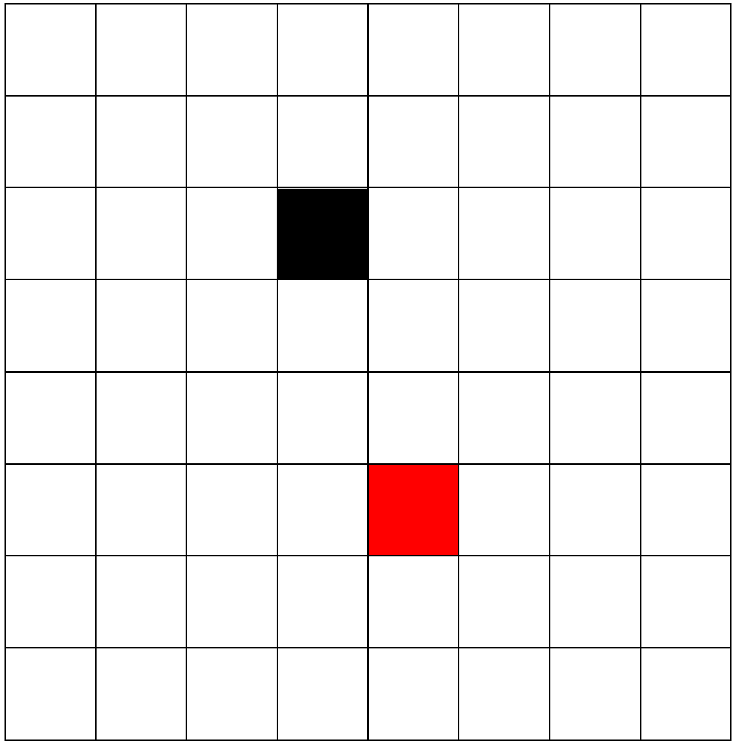
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

A(x) = 101001

ct(x) = {000000, 010000, 000100, 000010, 010100, 010010, 000110}

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

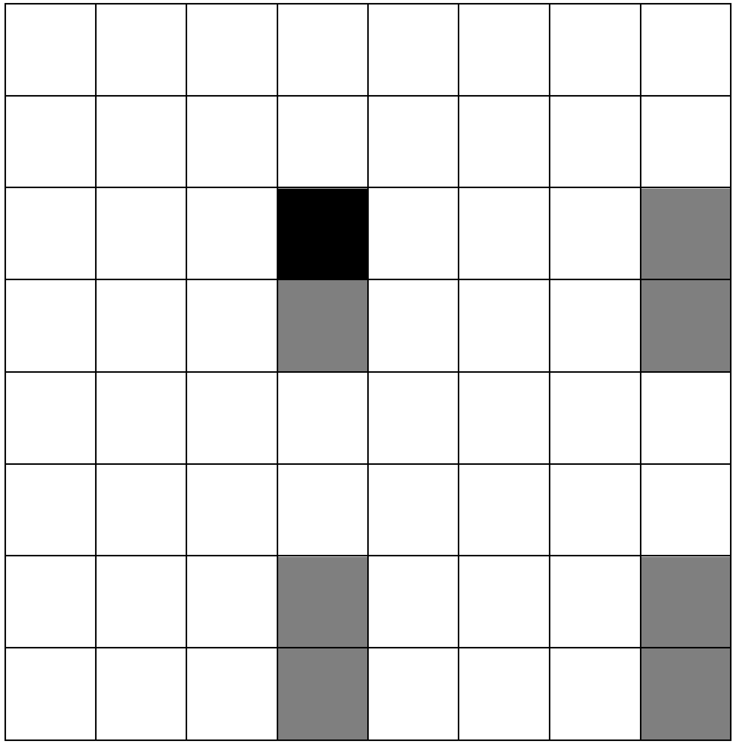
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

A(x) = 101001

cd(x) = {010110}

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

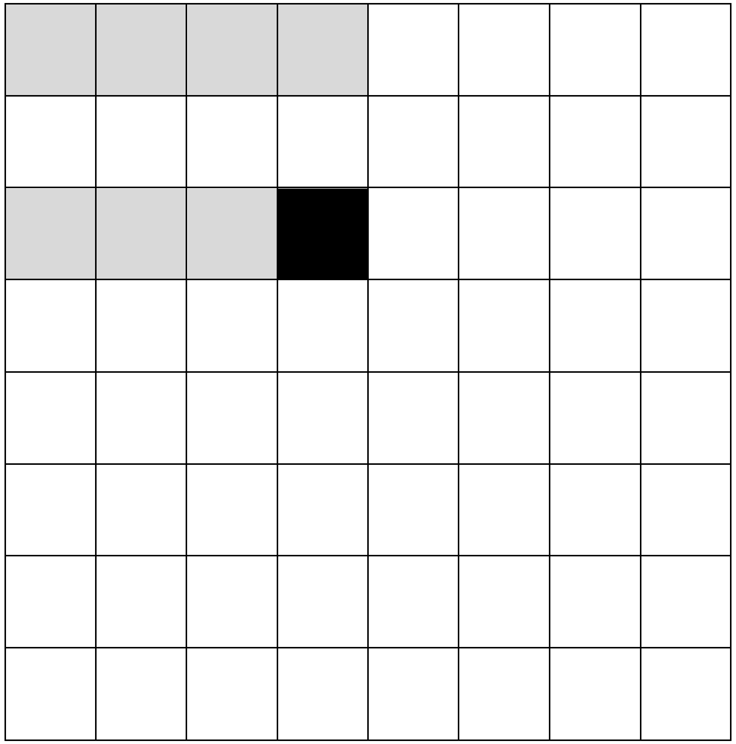
Card(d(x)) = $29 - 2 = 27$

Card(id(x)) = $64 - 1 - 27 = 36$

A(x) = 101001

sp(x) = {000000, 100000, 001000, 000001, 101000, 100001, 001001}

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

$\mathrm{Card}(\mathrm{ct}(x)) = 2^3 - 1 = 7$

$\mathrm{Card}(\mathrm{cd}(x)) = 1$

$\mathrm{Card}(\mathrm{sct}(x)) = 2^3 - 1 = 7$

$\mathrm{Card}(\mathrm{sb}(x)) = 2^3 - 1 = 7$

$\mathrm{Card}(\mathrm{sp}(x)) = 2^3 - 1 = 7$

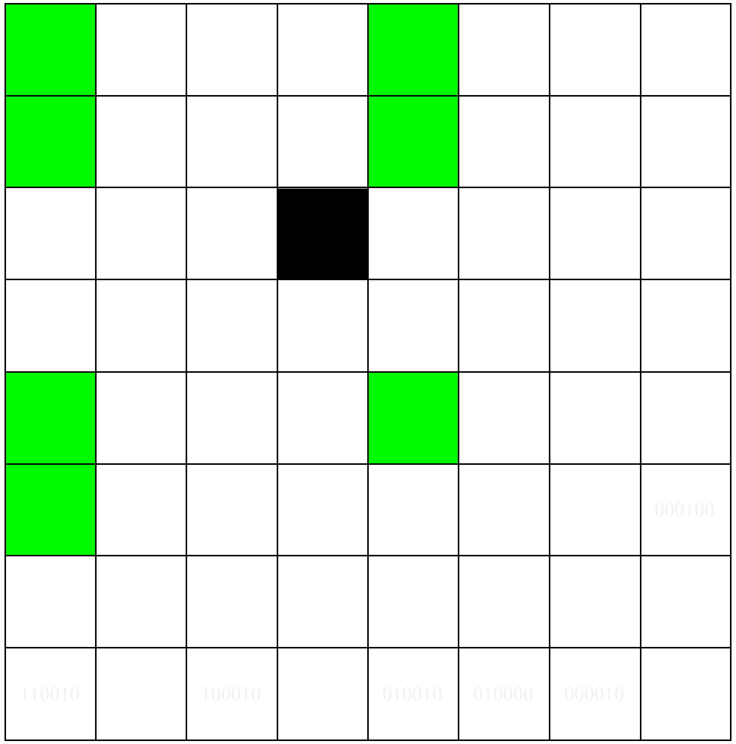
$\mathrm{Card}(\mathrm{d}(x)) = 29 - 2 = 27$

$\mathrm{Card}(\mathrm{id}(x)) = 64 - 1 - 27 = 36$

$\mathbf{A}(x) = 101001$

$\mathrm{sb}(x) = \{111111, 101111, 111011, 111101, 101011, 101101, 111001\}$

Diagrams with areas (rather than vertices) of *n*-bitstring (*n* = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

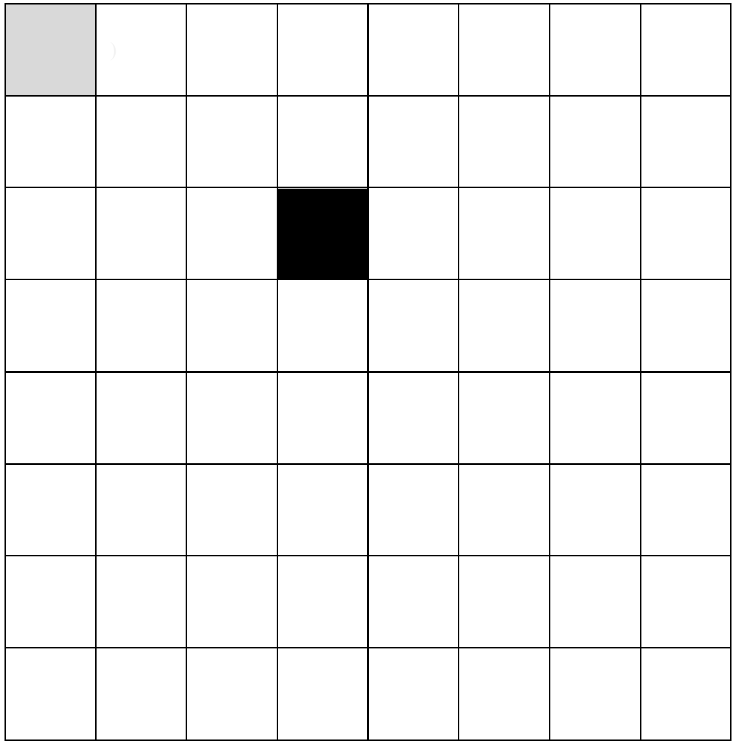
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

A(x) = 101001

sct(x) = {111111, 011111, 110111, 111110, 010111, 011110, 110110}

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x) ∩ sb(x)) = 1

Card(sp(x)) = $2^3 - 1 = 7$

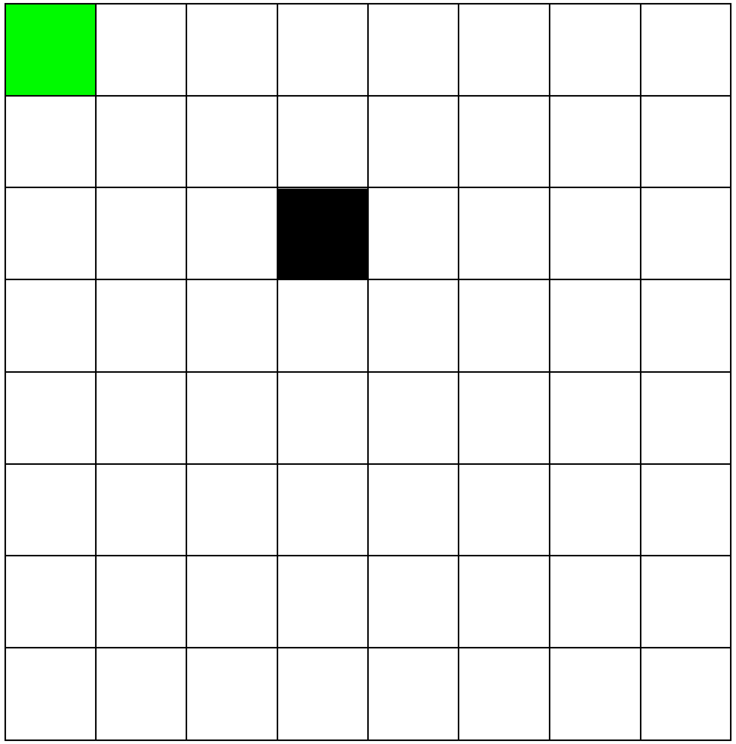
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

$n = 6$

111111 = sb(x)

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x) ∩ sb(x)) = 1

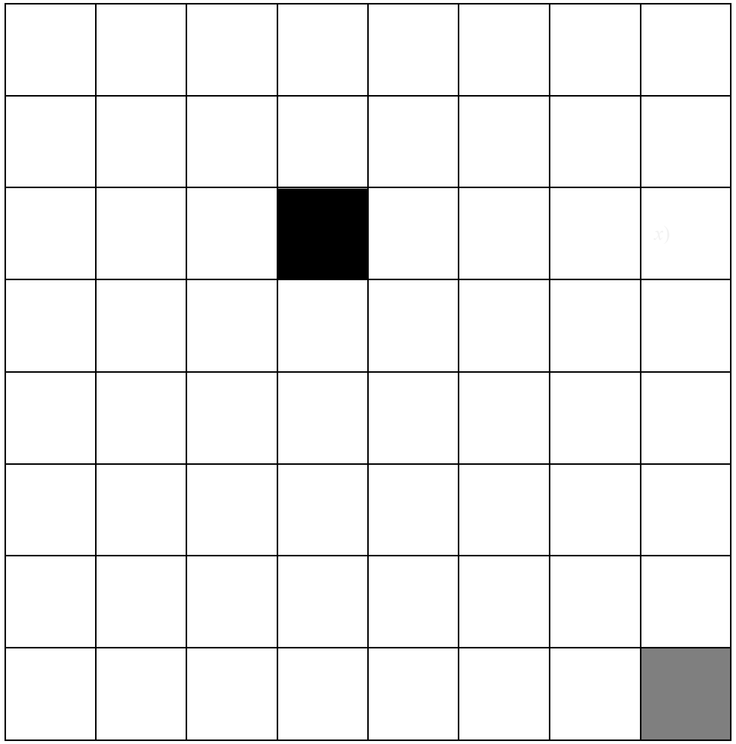
Card(sp(x)) = $2^3 - 1 = 7$

Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

111111 = sct(x)

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

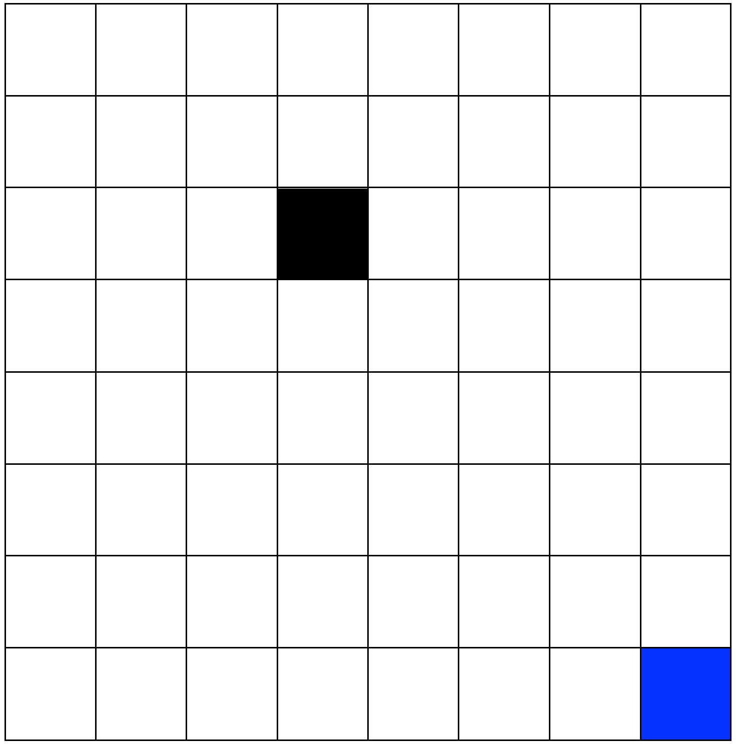
$000000 = \mathrm{sp}(x)$

$\mathrm{Card}(\mathrm{ct}(x) \cap \mathrm{sp}(x)) = 1$

$\mathrm{Card}(\mathrm{d}(x)) = 29 - 2 = 27$

$\mathrm{Card}(\mathrm{id}(x)) = 64 - 1 - 27 = 36$

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

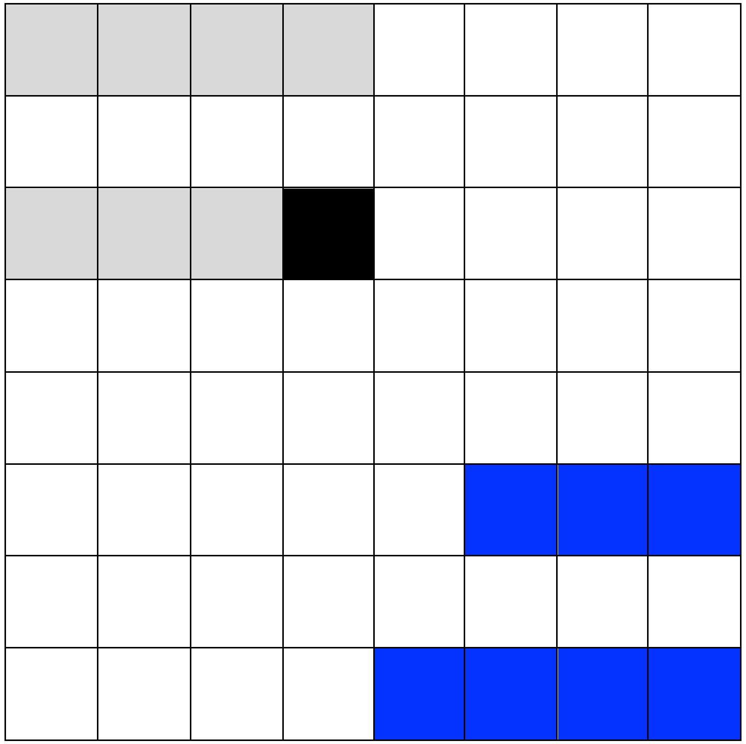
Card(ct(x) ∩ sp(x)) = 1

Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

000000 = ct(x)

Diagrams with areas (rather than vertices) of *n*-bitstring (*n* = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

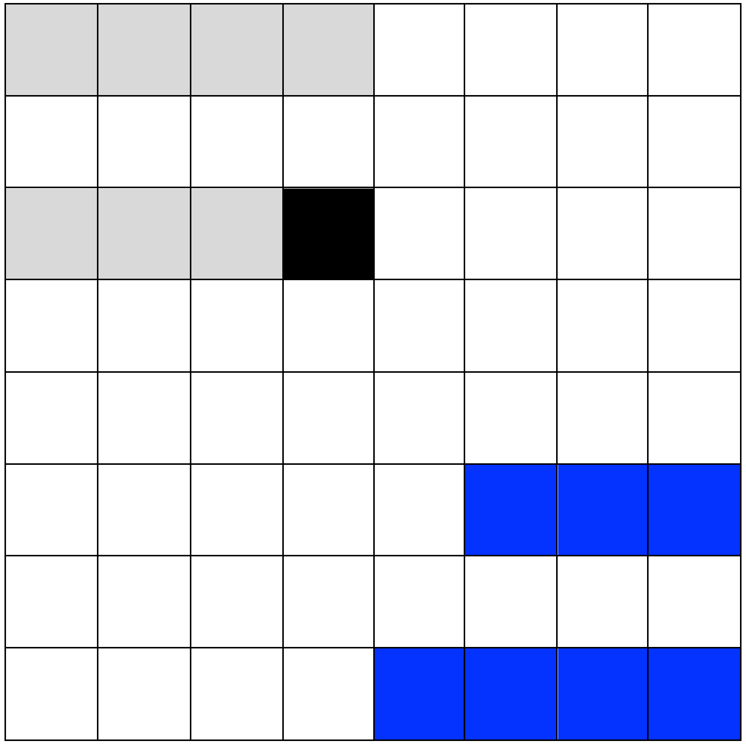
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

Subalterns are contradictories of contraries.

sb(x) = cd(ct(x))

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

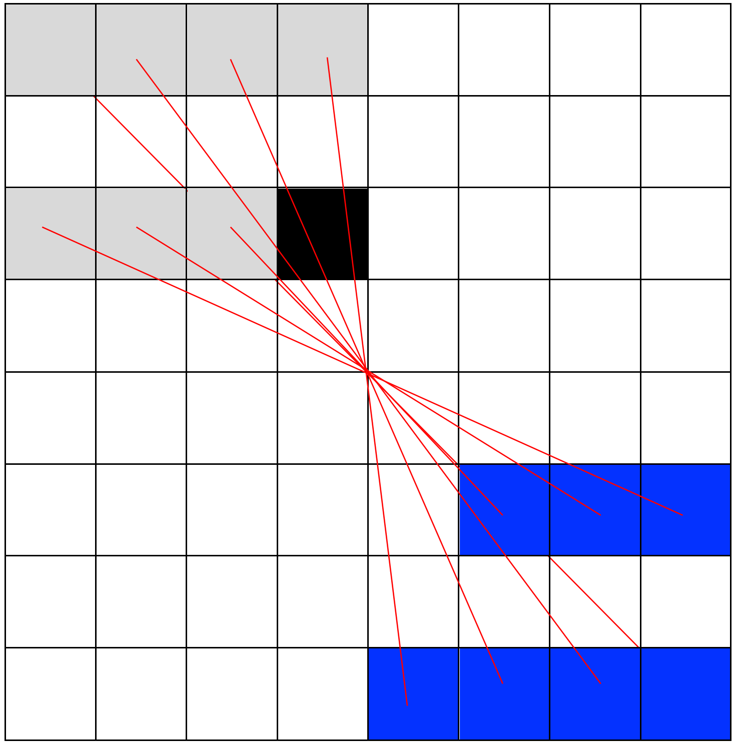
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

Subalterns are contradictories of contraries.

sb(x) = cd(ct(x))

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

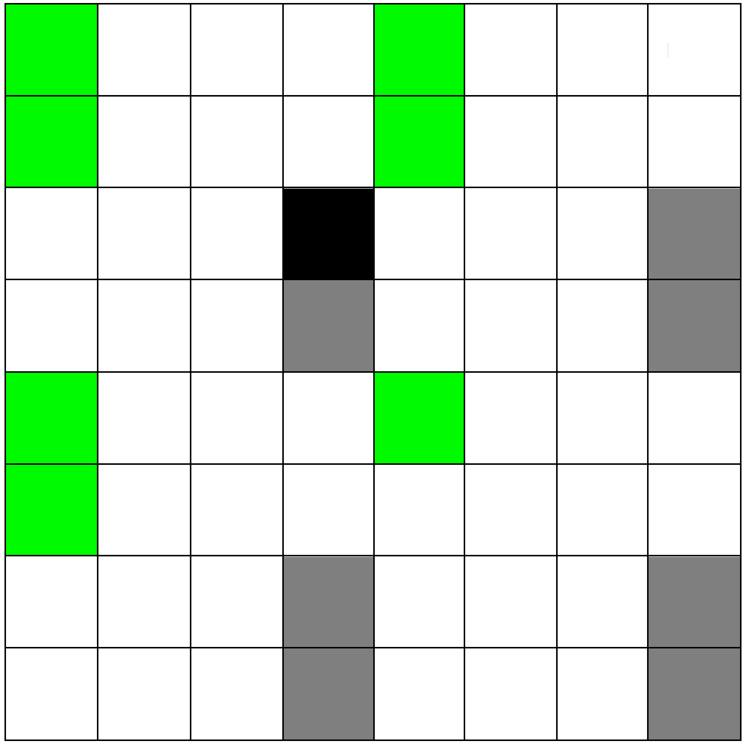
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

Subalterns are contradictories of contraries.

sb(x) = cd(ct(x))

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$\text{Card}(\text{ct}(x)) = 2^3 - 1 = 7$

$\text{Card}(\text{cd}(x)) = 1$

$\text{Card}(\text{sct}(x)) = 2^3 - 1 = 7$

$\text{Card}(\text{sb}(x)) = 2^3 - 1 = 7$

$\text{Card}(\text{sp}(x)) = 2^3 - 1 = 7$

$\text{Card}(\text{d}(x)) = 29 - 2 = 27$

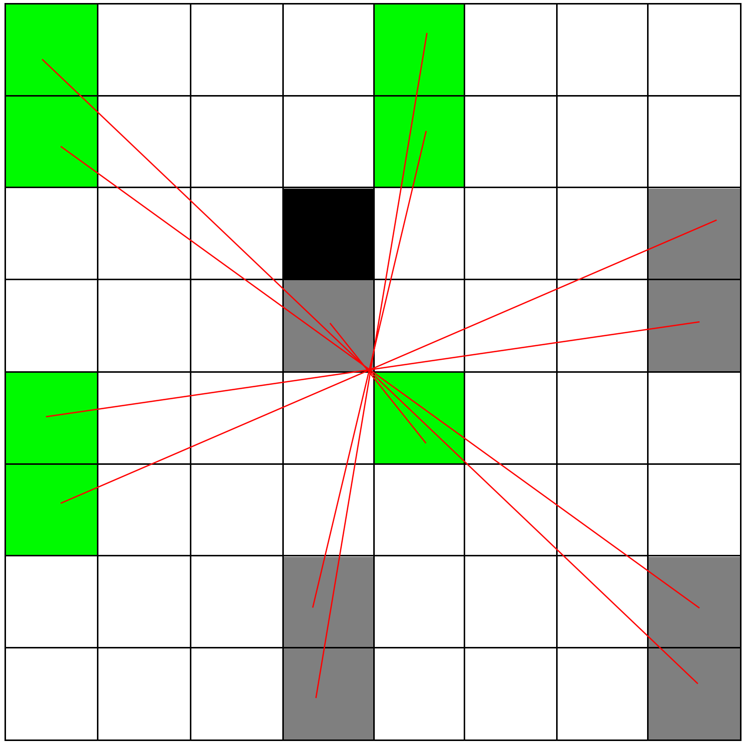
$\text{Card}(\text{id}(x)) = 64 - 1 - 27 = 36$

$n = 6$

Superalterns are contradictories of subcontraries.

$\text{sp}(x) = \text{cd}(\text{sct}(x))$

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

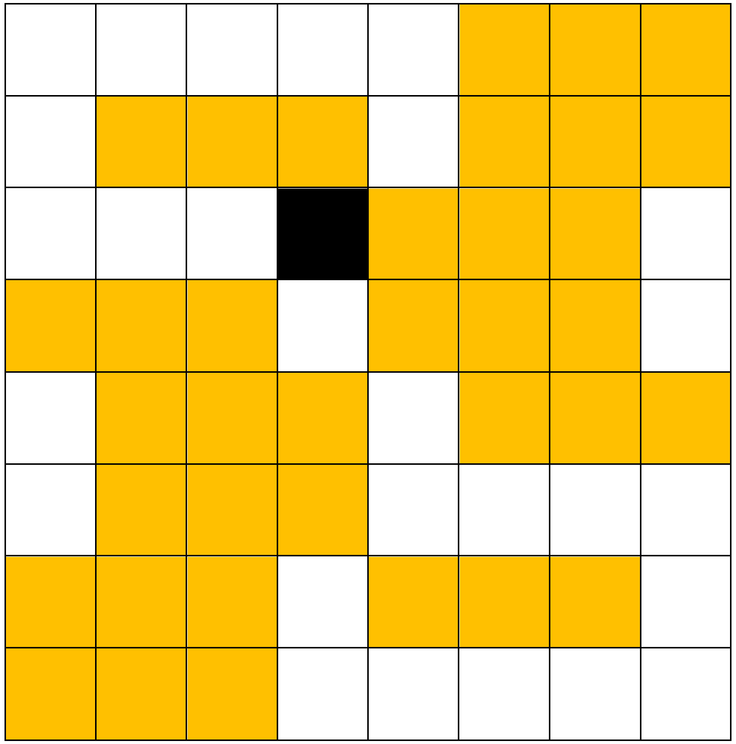
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

Superalterns are contradictories of subcontraries.

sp(x) = cd(sct(x))

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

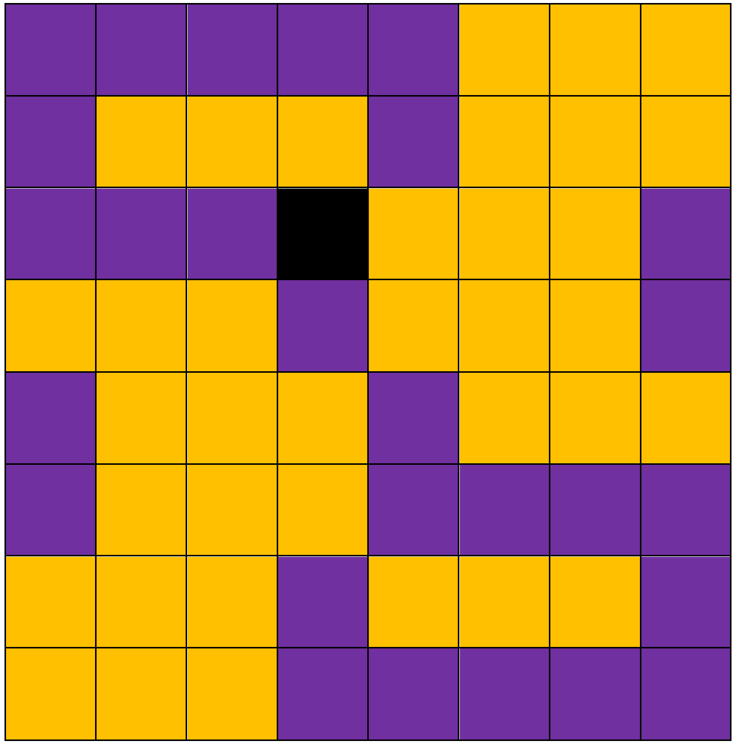
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

Indeterminates with respect to x

id(x)

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

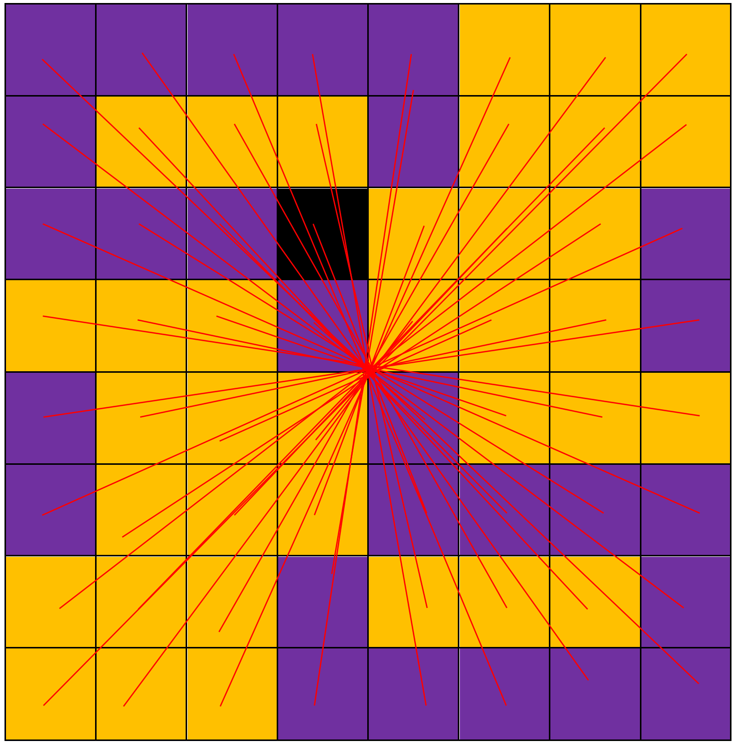
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

Indeterminates are contradictories of determinates d(x)

d(x) = {cd(x),ct(x),sct(x),sb(x),sp(x)}

Diagrams with areas (rather than vertices) of n-bitstring (n = length)



$n = 6$

Card(ct(x)) = $2^3 - 1 = 7$

Card(cd(x)) = 1

Card(sct(x)) = $2^3 - 1 = 7$

Card(sb(x)) = $2^3 - 1 = 7$

Card(sp(x)) = $2^3 - 1 = 7$

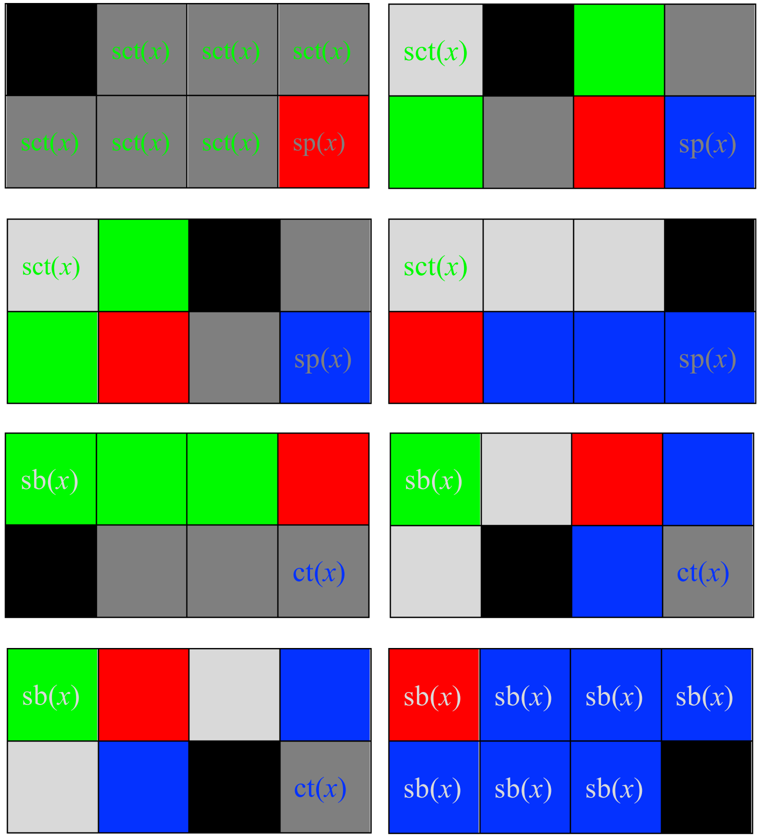
Card(d(x)) = 29 – 2 = 27

Card(id(x)) = 64 – 1– 27 = 36

Indeterminates are contradictories of determinates d(x)

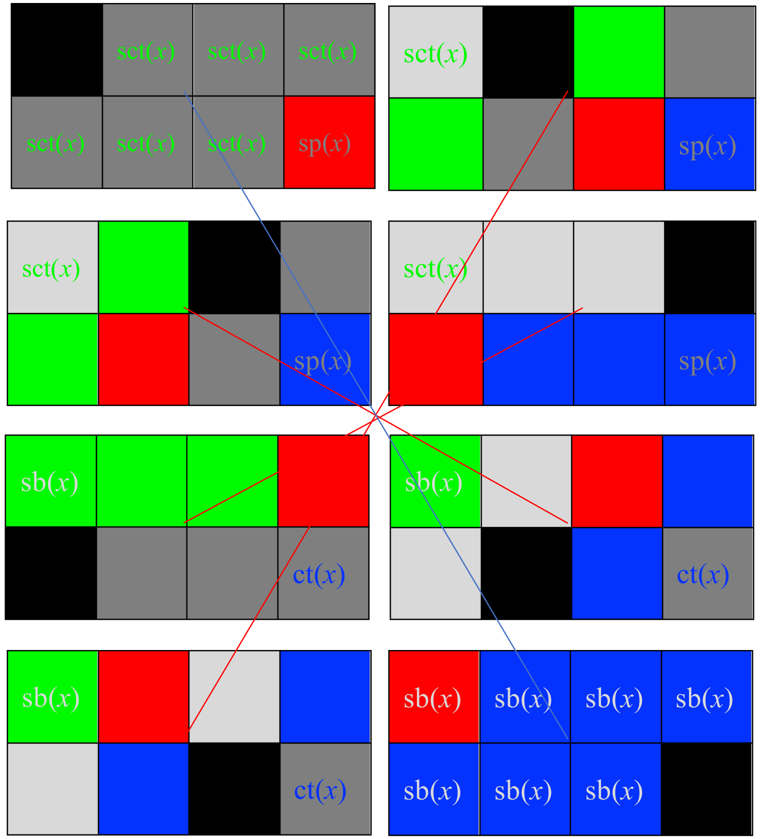
id(x) = cd(d(x))

1 with the square 2 without the square 3 another square



111	110	011	010
101	100	001	000

1 with the square   2 without the square   3 another square

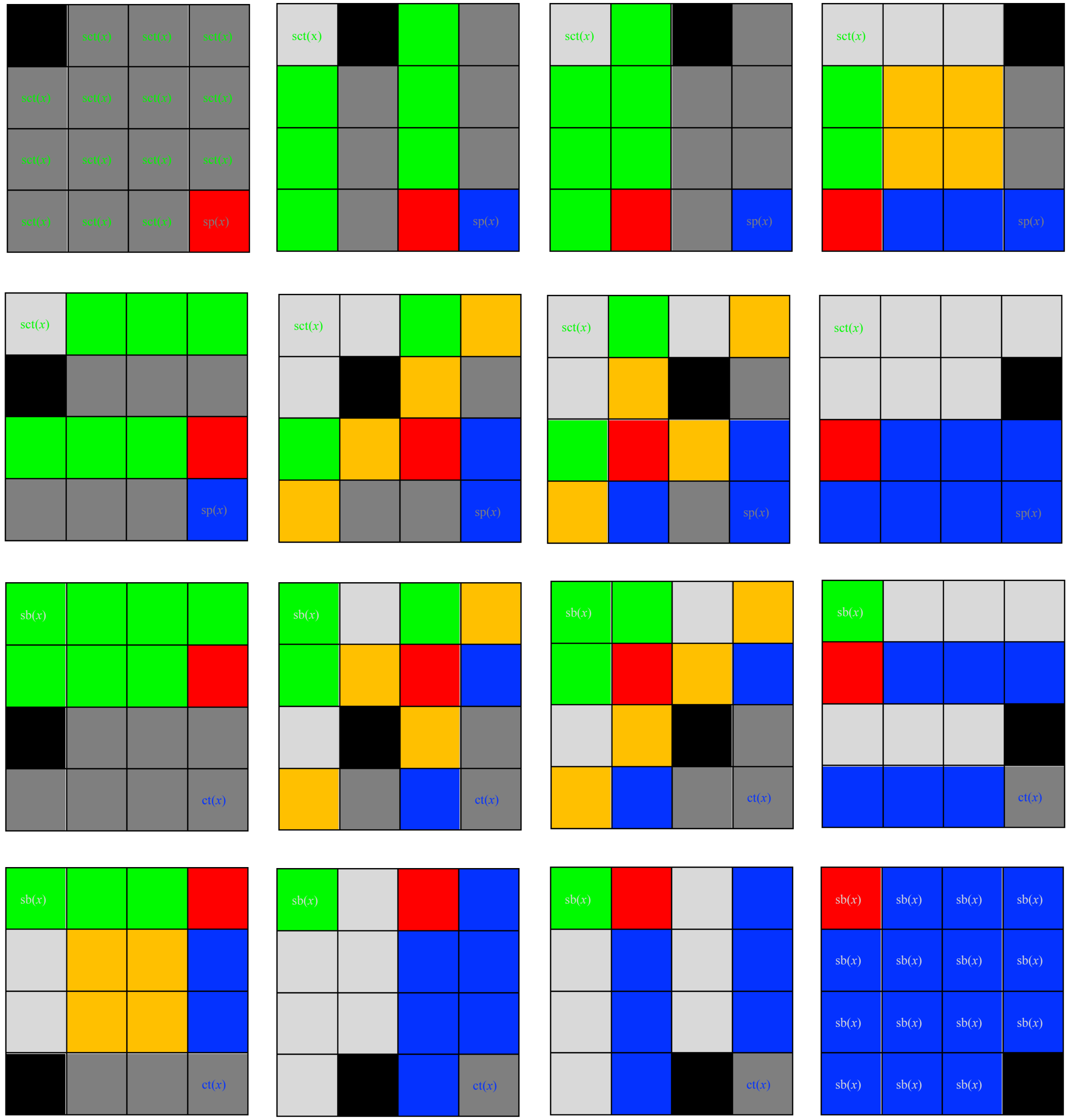


111	110	011	010
101	100	001	000

1 with the square

2 without the square

3 another square

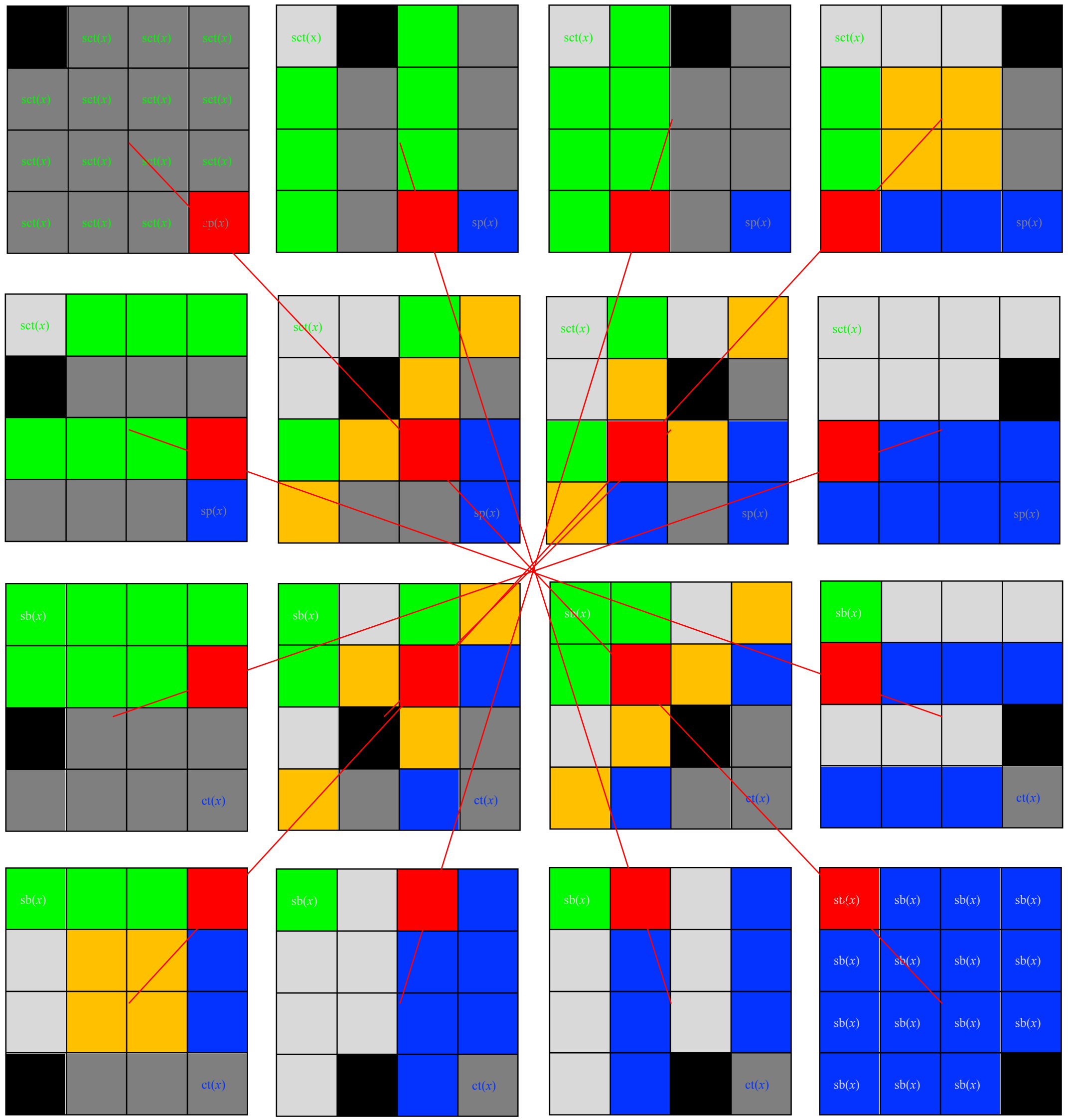


1111	1110	0111	0110
1101	1100	0101	0100
1011	1010	0011	0010
1001	1000	0001	0000

1 with the square

2 without the square

3 another square



1111	1110	0111	0110
1101	1100	0101	0100
1011	1010	0011	0010
1001	1000	0001	0000

End of the square?
Standard diagrams are diagrams with vertices + oriented graphs (sb/sp)
Open questions: how many diagrams/kinds of oppositions can there be?

Towards another square
A functional calculus of opposites helps to:

- determine complete structures of oppositions with 2^n elements (*n*-bitstrings)
- deal with logical oppositions as opposite-forming multifunctions
- device new diagrams of oppositions with areas + colored diagrams

Extended works
Towards a 3-dimensional theory of meaning though 3 kinds of oppositions:

- C-opposition: individual objects *x* are sets of sets of properties
- Q-opposition: quantified properties over time, space, individuals
- P-opposition: answers to ordered questions (cf. Schang 2017)

Towards a generalized theory of logical values: ***Partition Semantics***.

References

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Muito obrigado.