Fraenkel's axiom of restriction: Axiom choice, intended models and categoricity

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1 Introduction

A recent debate has focused on different methodological principles underlying the practice of axiom choice in mathematics (cf. Feferman et al., 2000; Maddy, 1997; Easwaran, 2008). The general aim of these contributions can be described as twofold: first to clarify the spectrum of informal justification strategies retraceable in the history of mathematical axiomatics. Second, to evaluate and to philosophically reflect on the actual reasoning involved in the introduction of new axioms in mathematical practice such as large cardinal axioms in set theory.

The most extensive treatment of these matters for the case of set theory can be found in (Maddy, 1997). Her philosophical discussion of axiom choice covers both of the mentioned approaches, i.e., it is both descriptive in reconstructing the justification types in early axiomatic set theory as well as normative in devising "methodological maxims" for the evaluation of present set theoretic axiom candidates. More specifically, Section 1.3 of her book provides a historical survey of the arguments given for ZFC by Zermelo, von Neumann, and Fraenkel (among others) with the intention "to explicate and analyze its distinctive modes of justification" given there (Maddy, 1997, p. 72). In the final two sections of her book (Sections 3.5 and 3.6), Maddy in turn devises "a naturalistic program" for discussing more recent axiom candidates (starting from Gödel's axiom of constructibility to large cardinal or determinacy axioms) intended to model the "justificatory structure of contemporary set theory" (Maddy, 1997, p. 194).

In this paper I attempt to take up Maddy's historical discussion by drawing attention to a historical episode from early axiomatic set theory centered on Abraham Fraenkel's *axiom of restriction* ("*Beschränktheitsaxiom*") (in the following AR). The axiom candidate was first introduced by Fraenkel in the early 1920s and can be considered as a minimal axiom devised to

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express a restriction clause, more specifically a minimal condition for any set model satisfying ZFC. His attempts to devise such a restriction clause on set models varied in the course of his intellectual development and eventually led to different, partly independent versions of restrictive axioms for set theory. Now, all of these axiom candidates are considered in retrospect as "ad hoc devices" due to their "vague, metatheoretic" character without any real, remaining significance in modern axiomatic set theory (cf., e.g., Kanamori, 2004, p. 515) (cf. Section 5). Nevertheless, I will argue that Fraenkel's attempts to introduce such a minimal axiom remain interesting from an historical point of view since the axiom takes a central and so far neglected place in a broader discussion on the (non-)categoricity of set theory and its role as a foundational discipline in mathematics. Moreover, closer study of it will also prove to be instructive for the general methodology of axiom choice given the specific justifications that Fraenkel provides for his axiom candidate.

The paper has two main aims. The first is to reconstruct the different arguments for AR in light of Maddy's account of extrinsic (in contrast to intrinsic) justification. Given Fraenkel's case, I show that one can expand Maddy's analysis by a new type of extrinsic argument that concerns the metatheoretic property of categoricity of the resulting axiomatization. The second aim is to analyze AR in terms of Maddy's "methodological maxims", namely Unify and Maximize, devised for axiom choice in contemporary set theory (cf. Maddy, 1997, pp. 208–215). I argue that AR deserves closer attention since—being a minimizing principle for the set theoretic domain—it is *prima facie* diametrically opposed to Maddy's second principle that calls for a maximization of the set theoretic domain. This—given the overall viability of Maddy's principles - could be taken as an additional argument against the legitimacy of AR in axiomatic set theory. However, I argue that a direct evaluation of Fraenkel's axiom candidate in terms of Maddy's maxims is problematic since they are motivated by different conceptions of set theory as a foundational enterprise.

The paper is organized as follows: I present a brief overview of different types of axiom justification described in (Maddy, 1997), focusing on types of extrinsic, non-epistemic arguments she identifies in the early axiomatization of set theory (Section 2). Her account of extrinsic evidence is compared to the different lines of argumentation Fraenkel develops for his axiom candidate (Section 3). His main motivation for AR is a metatheoretic consideration, i.e., to restrict the set theoretic universe to his intended model of ZFC and thereby to render his axiom system categorical (Section 3.1). I discuss different proposed versions of AR intended to achieve this categorical axiomatization (Section 3.2). Further, I suggest that Fraenkel seems to develop his views on minimal models and the intended effect of

AR in close analogy to Dedekind's approach to defining sets via closure principles (Section 4). In Section 5 a number of objections directed against AR by Baldus, von Neumann, and Zermelo from the late 1920s will be discussed that eventually resulted in a fundamental shift in Fraenkel's own understanding of his axiom candidate. Fraenkel's response to these objections and the resulting new versions of AR will be discussed (Section 6). Finally, I compare Fraenkel's different versions of AR with Maddy's naturalistic method of axiom choice, specifically with her maxims UNIFY and MAXIMIZE, and discuss whether such an evaluation can be justified (Section 7).

2 Maddy on extrinsic justification

Maddy's historical survey of the axiomatization of set theory and the different motivations for the axioms of ZFC is based on the distinction (anticipated in Gödel, 1964) between two types of justification. In the case of extrinsic justification, an axiom is assessed in terms of its theoretical fruitfulness, i.e., with an eye to its intended consequences for the resulting theory. In the case of intrinsic justification, an axiom is defended in terms of the intuitive nature of the properties it is supposed to express (cf. Maddy, 1997, pp. 36–37). The two kinds of arguments are commonly associated with different types of mathematical axioms: *structural axioms* of the "working mathematician" (e.g., the axioms of rings, groups etc.) and *foundational axioms* concerning structures that "underlie all mathematical concepts" (e.g., the Peano axioms for arithmetic and ZFC for set theory) (cf. Feferman, 1999, p. 3). Structural axioms are often considered to be justifiable on extrinsic grounds comparable to the experimental testing of hypotheses in the natural sciences. Foundational axioms in turn are attributed an entirely different status. Their justification is often based on intrinsic considerations, either by reference to certain epistemic norms (such as those of intuitiveness, obviousness, immediacy, and naturalness) or by reference to a pre-axiomatic conception of the subject matter, i.e., the mathematical structure the axiom in question is supposed to capture.[1]

However, Maddy succeeds in showing that the assumed link between foundational axioms and intrinsic arguments is not exclusive. Her survey of the practice of axiom choice in set theory identifies a number of genuinely extrinsic considerations laying the ground for the foundational axioms of

[1]Numerous examples from the history of axiomatic mathematics suggest such a relation between foundational axioms and intrinsic arguments. In discussing the (epistemological) primacy of the Peano axioms over ZF, Skolem argues that in contrast to the latter the former are "immediately clear, natural and not open to questions" (Skolem, 1922, p. 299). In opposition to this, Gödel—in a well known passage—describes a faculty of "intuition" as a sufficient "criterion of truth" for the set theoretic axioms. (Gödel, 1964, p. 271)

ZFC.[2] Moreover, she points out that the extrinsic arguments given there are explicitly non-epistemic in character. In several cases (most prominently in Zermelo's defense of the axiom of choice, cf. (Maddy, 1997, p. 56) the motivation for the acceptance of an axiom does not depend on its intuitiveness but rather on its theoretical consequences for mathematics.[3] Naturally, since ZFC is primarily considered a foundational theory, these consequences have to be closely tied to what Maddy describes as its "foundational goal" within mathematics. Her specific understanding of this goal is clearly non-epistemological: in contrast to a stronger "foundationalist" reading of ZFC in terms of an ontological reduction that "reveals the true identities of [...] mathematical objects" or the reduction to an epistemologically secure basis, set theoretic axioms in her "modest" version of foundations share no "preferred epistemological status" (cf. Maddy, 1997, pp. 24–25).[4] Instead, they provide a fruitful codification of all other branches of mathematics by allowing a set theoretic "representation" of all other mathematical entities and structures (cf. Maddy, 1997, pp. 25–26). By this,

> [...] vague structures are made more precise, old theorems are given new proofs and unified with other theorems that previously seemed quite distinct, similar hypotheses are traced at the basis of disparate mathematical fields, existence questions are given explicit meaning, unprovable conjectures can be identified, new hypotheses can settle old problems, and so on. (Maddy, 1997, pp. 34–35)

For Maddy, it is the sum of these theoretical virtues that amount to the foundational goal of ZFC. Concerning the question of axiom justification, she argues that the capacity of a particular axiom to contribute to these theoretical objectives (and thus to the overall success of the foundational discipline) can be taken as direct extrinsic evidence for it: "[...] I see the effectiveness of an axiom candidate at helping set theoretic practice reach its foundational goal as a sound extrinsic reason to adopt it as a new axiom".[5]

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[2]Cf. (Feferman et al., 2000) for an interesting discussion between Feferman and Maddy on the intrinsic/extrinsic distinction and its bearing on Feferman's classification of mathematical axioms mentioned above (Feferman et al., 2000, pp. 416–419).

[3]Maddy refers to Russell (1973) for an early methodology of axiom choice along similar lines. In fact, in his lecture, Russell proposes a "regressive method" for justifying logical axioms (such as the axiom of reducibility) without any direct intrinsic support by a kind of probabilistic confirmation through its "obvious" consequences: "Hence we tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true." (Russell, 1973, pp. 273-274)

[4]For a comparable account of (higher-order) logic as a non-foundationalist foundation for mathematics, cf. (Shapiro, 1991).

[5](Feferman et al., 2000, p. 418); there is a certain tension between Maddy's account of the "foundational goal" of ZFC and Feferman's account of foundational axioms. For her own evaluation of this relation (cf. Feferman et al., 2000, pp. 417–419).

One central criterion of success that allows the assessment of an individual axiom concerns set theory's unifying power. Its strong unifying role is due to the creation of a single domain of discourse to which all of mathematics is reducible:[6]

> The force of set-theoretic foundations is to bring (surrogates for) all mathematical objects and (instantiations of) all mathematical structures into one arena—the universe of sets—which allows the relations and interactions between them to be clearly displayed and investigated. (Maddy, 1997, p. 26)

Note that one central implication of this picture of set theoretic unification through a "unified arena" is that a specific conception of the domain of set theory, i.e., the intended universe of sets, becomes a central issue in the foundational goal. Now, Maddy's historical discussion of extrinsic argument types for the axioms of set theory focuses on the standard axioms of ZFC. What is not mentioned in her survey, however, is that there was already a strong and ongoing debate throughout in 1920s on how to conceive this universe of sets and characterize it axiomatically. In the course of different attempts to fix a domain of set theory that is capable of providing such a "unified arena" for mathematics, one specific axiom candidate, namely Fraenkel's axiom of restriction stands out as the most prominent contribution. In the remaining sections of the paper I will focus on this specific episode in the early history of the axiomatic set theory in general and Fraenkel's axiom candidate in particular. It will be shown that one can identify an extrinsic argument in his remarks on set theoretic restriction based on a similar motivation for unification not discussed in (Maddy, 1997).

3 Fraenkel's axiom of restriction

In the early 1920s, Fraenkel suggested two axioms to be added to the axiom system presented in (Zermelo, 1908b): the axiom of replacement, now a standard axiom of ZF, as well as the lesser known axiom of restriction (AR). The latter was basically devised to express a restriction clause, more specifically a minimal condition for any set model satisfying the axioms set up by Zermelo. In what follows I will give a brief reconstruction of the evolution of Fraenkel's thought on the notion of restriction during this period.

The first mention of AR can be found in an article titled "Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre" in *Mathematische Annalen*

[6]Compare also the following passage: "One methodological consequence of adopting the foundational goal is immediate: if your aim is to provide a single system in which all objects and structures of mathematics can be modeled or instantiated, then you must aim for a single, fundamental theory of sets." (Maddy, 1997, pp. 208–209)

from 1922 (Fraenkel, 1922b). Fraenkel's motivation for adding the axiom candidate is mainly pragmatic and explicitly concerns set theory as a foundational discipline. He states that "Zermelo's concept of set is more comprehensive than seems to be necessary for the needs of mathematics [...]." (Fraenkel, 1922b, p. 223) Fraenkel goes on to mention two types of possible sets in the "domain" ("*Grundbereich*") of set theory that are consistent with the existing axioms, but are irrelevant for mathematical purposes. The first are "non-conceptual" sets consisting of physical elements. The second are non-well-founded sets, i.e., sets with infinite membership chains originally specified by Mirimanoff (1917). From their possibility within Zermelo's axiomatization, Fraenkel draws an interesting consequence for its general status:

> Whereas sets of the first as of the second kind are not necessary for set theory considered as a mathematical discipline, it in any case follows from the fact that they have a place within Zermelo's axiomatization that the axiom system [...] does not have a "categorical character", that is to say it does not determine the totality of sets completely. (Fraenkel, 1922b, p. 234)

Categoricity is understood here as a "complete" characterization of the domain of sets. In an added footnote, Fraenkel refers to one of his earlier works on number theory, more specifically on different axiomatizations of p-adic numbers, for an informal definition of categoricity based on Veblen's notion of a "categorical set of postulates" (Fraenkel, 1911, p. 76).[7] A more structured presentation of his arguments for AR can be found in the second edition of his monograph *Einleitung in die Mengenlehre* (Fraenkel, 1924). Here, the introduction of the additional axiom leads to a "simplification of the set theoretic edifice" by ruling out non-well-founded numbers without losing its significance for mathematics due to the fact that "all mathematically relevant sets can [...] be saved with such a restricted axiomatization." (Fraenkel, 1924, p. 218) As a second independent argument the property of categoricity is mentioned: "Moreover, without such a restriction it is not within reach that our axiom system captures the totality of admissible sets *completely* as is desirable for the construction of every axiomatization." (Fraenkel, 1924, p. 218) Two short remarks are in order here. First, one can identify at least two related but non-identical objections against Zermelo's original axiomatization here: the non-eliminability of extraordinary sets that are redundant for the formalization of mathematics on one hand. On the other hand, the non-categoricity of Zermelo's proposed axiom system is considered as a general theoretical deficiency of any axiomatization.[8]

[7]On Veblen's understanding of categoricity and a closer comparison to the modern notion, cf. (Awodey and Reck, 2002, pp. 22–25).

[8]This second point is further highlighted in a passage in his published lectures from

Note second that the two mentioned issues, i.e., the applicability of set theory to mathematics and the axiomatic property of categoricity, are treated independently here. One can find no explicit remark about the possible implications of the categoricity of the extended axiom system $\mathsf{ZF} + \mathsf{AR}$ for its foundational goal in mathematics. I will return to this point in the last section.

3.1 The (non-)categoricity of set theory

In the second edition of *Einleitung* we also find an explicit definition of the notion of categoricity as one type of completeness of an axiom system referred to in the argument above:[9]

> According to it an axiomatic system is called complete, if it determines uniquely the mathematical objects governed by it, including the basic relations between them, in such a way that between any two interpretations of the basic concepts and relations one can effect a transition by means of a 1–1 and isomorphic correlation. (Fraenkel 1924, quoted from Awodey and Reck 2002, p. 30)

For the specific case of set theory the following explication is given:

> If the axiom system is complete and one has chosen in two distinct ways, each in accord with the axioms, an interpretation of the concept of set—in particular also its extension—and of the basic relation $a \in b$, then it has to be possible to maintain a correlation between the sets of the one interpretation and those of the other such that first, to each set of the first interpretation corresponds one and only one [...] set of the other interpretation and vice versa and that secondly, if $a \in b$ is a valid relation in the first interpretation [...] then the relation $a' \in b'$ also holds for the sets a' and b' that have been assigned to a and b in the other interpretation and vice versa. (Fraenkel, 1928, p. 228)

This is probably the first application of the concept of categoricity via isomorphism to axiomatic set theory. Nevertheless, his presentation remains sketchy compared to modern standards. The central concept used in these remarks about the conditions of categoricity for set theory is the notion of an isomorphic correlation between set models. In modern terminology such

1925: "It means more than a mere flaw of our axiom system that the totality of all possible sets is not unequivocally fixed but that instead there are always narrower and more comprehensive interpretations of the concept of set that remain compatible with our axiom system." (Fraenkel, 1927, p. 101)

[9] For Fraenkel's treatment of three types of completeness, i.e., semantic completeness, syntactic completeness and categoricity, as well as their relationship and later reception, mainly by Carnap, cf. (Awodey and Reck, 2002).

In the third edition of *Einleitung* the equivalence of the type of completeness given in the quotation with the notions of "categorical" (Veblen) and "monomorph" (Feigl-Carnap) is explicitly stated (cf. Fraenkel, 1928, p. 349).

a correlation is taken as a 1–1 mapping between two set models M and N that is structure-preserving, i.e., as a function f mapping M one-to-one onto N such that for two binary relations F and G (on M and N respectively), for all members a and b of M, $F(a,b)$ iff $G(f(a), f(b))$. However, in the 1920s, Fraenkel does not yet provide a comparable notion of isomorphism for set theory.[10] Nor does he get more explicit on the kind of the formal background language in which his axiom candidate and the notion of isomorphism should be cast. It should be stressed here that Fraenkel's remarks on AR and the categoricity of axiom systems from this period are in general presented informally. There is no attempt to provide a formulation of the axiom candidate in a formal symbolism such as Russell's type theoretic language commonly used at that time. This lack of formalization makes Fraenkel's claim about the categoricity of his axiomatization ZF + AR debatable from a modern point of view. If the axiom system ZF (specifically the axiom of replacement) is thought to be presentable in first order logic, then the expanded ZF + AR fails to be categorical due to the Löwenheim- Skolem theorems. Fraenkel's claim is only valid if a second-order axiomatization is assumed.[11] However, this fact is simply not noticed in his writings on the categoricity of set theory from that time. Despite discussing the Skolem paradox in the second and third edition of *Einleitung*, Fraenkel seems to be simply ignorant of its impact on his own project of providing a categorical axiomatization.[12] (I will return to this point in Section 6.2)

It is also in the third edition of *Einleitung* (Fraenkel, 1928) that one can find an interesting remark concerning his understanding of the concept of isomorphism. Following a more general discussion of the categoricity of axiom systems, he adds in a footnote:

> The expression "isomorphic" has a considerably more general sense than is usually common [...]. In fact the isomorphism is applicable to arbitrary relations, not only to those tertiary and n-ary relations denoted as "operations".[13]

[10] An alternative notion of isomorphism for sets had already been introduced some years before Fraenkel's version in (Mirimanoff, 1917). His definition is based on the simple notion of equivalence between sets and does not take into account a correlation between set models (cf. Mirimanoff, 1917, p. 41). For an early discussion of this definition, cf. (Sierpiński, 1922).

[11] Compare (Shapiro, 1991, pp. 85–86).

[12] van Dalen and Ebbinghaus (2000) retrace the different receptions of the Skolem paradox by Zermelo, von Neumann, and Fraenkel in the 1920s. For the latter's case they state that "the role of logic in set theory was not quite clear to Fraenkel". They in my mind correctly conclude that the impact of logical formalization on his categoricity claim transcended his "expertise" on logical matters in that period (van Dalen and Ebbinghaus, 2000, p. 148).

[13] (Fraenkel, 1928, p. 349); it was due to Rudolf Carnap who seems to have followed Fraenkel's informal remarks on a generalized concept of isomorphism to develop a formal

Irrespective of this, Fraenkel holds that Zermelo's axiomatization from 1908 is non-categorical in the sense specified above. This was a commonly acknowledged position in the 1920s shared by such eminent figures such as Skolem, von Neumann and Zermelo himself. Subject to debate were the possible reasons for this fact and whether Zermelo's original axiomatization could be rendered categorical by adding additional axioms.[14] As we have seen, according to Fraenkel's view anno 1924, the non-categoricity of Z is mainly due to the non-eliminability of "extraordinary sets" by the existing axioms. This in turn is due to the fact that the existential axioms, i.e., the empty set axiom and the axiom of infinity, do not restrict the domain of sets whereas the restrictive axioms like the axiom of separation are not restrictive enough to yield an "unequivocal specification" of the concept of set. As a solution to this Fraenkel proposes to introduce his AR which is described in analogy to Hilbert's completeness axiom in geometry:

> [...] as is the case there, the mentioned deficiencies can be remedied by setting up a [...] last axiom, the "axiom of restriction" that imposes on the concept of set, or more appropriately the domain [of sets], the smallest extension compatible with the remaining axioms. (Fraenkel, 1928, p. 234)

An alternative definition of the axiom can be found in (Fraenkel, 1924): "Aside from the sets imposed by the axioms [of Zermelo (1908)] there exist no further sets." (Fraenkel, 1924, p. 219) Now, the underlying motivation for introducing AR is clearly extrinsic in Maddy's more general spirit. The intention behind both versions of the axiom is evident: to rule out non-intended and non-well-founded sets by restricting either the interpretation of the concept of set or the domain of set and, by doing so, to render the axiom system categorical.

3.2 Versions of restriction

Fraenkel's early elucidations of the intended effect of AR do not go beyond the level of informal remarks. The most detailed exposition can be found in the article "Axiomatische Begründung der transfiniten Kardinalzahlen" (Fraenkel, 1922a) in which he develops an axiomatization for cardinal numbers. Here Fraenkel formulates two versions of AR that prove to be instructive for the case of standard set theory. According to the first, restriction is considered as a minimality condition on sets: There exist no sets apart from the sets implied by the given axioms. The second reading is more interesting, since it sketches the intended effect of the axiom. According to

definition of a "n-stage isomorphism correlator" for a type-theoretic language in his works on a general methodology of axiomatics. Cf. (Bonk and Mosterín, 2000; Carnap and Bachmann, 1936).

[14] Compare, e.g., (Kanamori, 2004, pp. 515–516) and (Shapiro, 1991, pp. 184–189).

it AR can be viewed as imposing a minimal model for the axiom system: "If the domain (*Grundbereich*) B contains a smallest submodel (*Teilbereich*) B_0 satisfying the axioms [...], then B is identical with such a smallest submodel B_0." (Fraenkel, 1922a, p. 163) This in effect rules out the existence of any possible submodel of B_0 that also satisfies the axiom system. The second definition is followed by a footnote concerning the method of constructing such a minimal model:

> As is usual, a smallest submodel of the indicated character is to be understood as a model that is the intersection of all submodels of B with the property in question and that also possesses the property itself. (Fraenkel, 1922a, p. 163)

Two claims are made here: first that a minimal model for Z can be conceived as the intersection of all possible models satisfying the axioms. Second, if such a minimal model exists, the extended axiom system ZF + AR, i.e., the Zermelo axioms plus replacement and restriction, is categorical. Now, Fraenkel does not get more explicit about his conception of the domain or the models of set theory. How are these notions conceived? In approaching this question it will prove to be fruitful to take into consideration Fraenkel's intellectual background. Specifically, a closer look at Richard Dedekind's methodological innovations concerning set formation and mapping in *Was sind und was sollen die Zahlen* from 1888 will be helpful for the understanding of how Fraenkel's ideas behind restriction might have evolved.

4 Dedekind's heritage

Two interpretive issues concerning AR are in need of further consideration. First, how exactly did Fraenkel conceive the intended restrictive effect of his axiom on the possible set models satisfying ZF? Second, how is it supposed to constitute the categoricity of the axiom system? I argue that in order to address both questions a closer look at Dedekind's methodological work in the foundations of arithmetic will be instructive. More specifically, I show that there is a striking similarity between Fraenkel's scattered remarks about his understanding of AR and Dedekind's theory of chains ("*Kettentheorie*") introduced in 1888 that suggests that Fraenkel actually modeled his idea of restriction based on Dedekind's approach.

Concerning the first question, we can find an insightful remark in *Einleitung* from 1928 about the "special character" of the axiom compared to the "existential" and "relational axioms" of ZF. Here AR is described as similar in effect to Peano's induction axiom. Fraenkel states that "in both versions [of AR], the inductive moment is essential." (Fraenkel, 1928, p. 355) What is his intuition about this "inductive character" underlying AR? As we have already seen, the concept of intersection plays a central role for the intended effect of the axiom. It is supposed to impose a minimal model as the intersection of all possible models satisfying ZF. From a methodological point of view, this is essentially a paring down approach of defining a specific minimal structure by taking the intersection of all closed subsets of a given set. This method was first introduced by Dedekind in 1888 and used for fixing the standard model of arithmetic. One could conjecture that Fraenkel's idea of a minimal model for set theory was shaped in direct analogy to Dedekind's strategy of defining the natural numbers as a minimal set closed by induction. Now, there is no immediate textual evidence that Fraenkel was directly guided by Dedekind's method in his thinking about set theoretic restriction. However, I will present a number of points that strengthen the plausibility of this relation of influence. In the next section a short presentation of the central concepts developed in (Dedekind, 1888) will be presented that seem of relevance for Fraenkel's axiom candidate.

4.1 The theory of chains

Dedekind's project of developing an "unambiguous foundational conception" of the natural numbers in 1888 is based on a number of methodological results concerning the central concepts that allow the reduction of numbers to a logical basis (Dedekind, 1888, p. 351). Here the idea of an isomorphism based on a 1–1 mapping ("*ähnliche Abbildung*") between elements of two systems is expressed formally for the first time. Systems that are isomorphic in this sense are terminologically fixed as "classes of similar systems".[15] A second newly introduced concept allowing Dedekind to devise the sequence $\mathbb{N}$ of the natural numbers is that of a *chain* (relative to mapping function φ and a system S): in modern terminology, a subsystem B of S is called a chain if it is closed under a mapping φ (Dedekind, 1888, p. 352). Subsequently, a system A_0 is defined as the *chain of A* ("*Kette des Systems A*") if and only if A_0 is the intersection of all chains containing A (Dedekind, 1888, p. 353). The way Dedekind conceives A_0 as the intersection of closures implies that it is also the smallest chain containing A, i.e., the smallest subset of S closed under φ. Again, in modern terminology, this effectively says that A_0 is the minimal closure of A under φ.[16]

[15](Dedekind, 1888, p. 351); compare (Sieg and Schlimm, 2005) for a systematic presentation of the evolution of the concept of mapping in Dedekind's foundational work.

[16]Compare (Sieg and Schlimm, 2005) on this fact: "A_0 obviously contains A as a subset, is closed under the operation φ; and is minimal among the chains that contain A, i.e., if $A \subseteq K$ and $\varphi(K) \subseteq K$ then $A_0 \subseteq K$." (Sieg and Schlimm, 2005, p. 145)

Dedekind (1888) himself is not explicit about the minimality property of *chains of A*. There exists, however, as Sieg and Schlimm have pointed out, a note in Dedekind's earlier manuscript "Gedanken über Zahlen" from the Nachlass in which this issue is explicitly mentioned: "(A) [i.e., the chain of A] is the "smallest" chain that contains the system A". (Quoted from Sieg and Schlimm, 2005, p. 144). I would like to thank Dirk Schlimm for drawing my attention to this passage.

There is an obvious similarity between the idea of minimal chains developed here, i.e., the method of building minimal closures of a given base set and a specific operation via intersection, and Fraenkel's remarks on AR throughout the 1920s. A number of additional points can be mentioned that further highlight this affinity. First, both positions are strikingly similar in their motivations for imposing a minimal condition on the intended model. In Fraenkel's case, as we have seen, the aim is to restrict the model to well-founded and abstract sets, thereby keeping out all types of non-standard and extraordinary sets. A comparable account can also be found in Dedekind's writings, most explicitly in his famous letter to Keferstein from 1890. After a short discussion of his basic concepts used for expressing $\mathbb{N}$ he states:

> [...] however, these facts are still far from being adequate for completely characterizing the nature of the number sequence $\mathbb{N}$. All these facts would hold also for every system S that, besides the number sequence $\mathbb{N}$, contained a system T, of arbitrary additional elements t, to which the mapping φ could always be extended while remaining similar and satisfying $\varphi(T) = T$. [...] What, then, must we add to the facts above in order to cleanse our system S again of such alien intruders t as disturb every vestige of order and to restrict it to $\mathbb{N}$. (Dedekind, 1890, p. 100)

To exclude such non-standard elements from the interpretation in question can thus be considered a common motivation behind the method of devising a minimal model. In Fraenkel's case this restriction is imposed by his AR. In Dedekind's proto-axiomatic presentation of the natural numbers it is required by his clause four of a "simple infinite system" stating, in modern terms, that A_0 is the smallest set containing A and closed under φ (cf. Dedekind, 1888, p. 352).[17]

This immediately leads to a second observation concerning Fraenkel's original conception of the intended model of Z which seems to be modeled based on this idea of closure. In Dedekind's account of the natural numbers 1 is the base element and the sequence $\mathbb{N}$ the intersection of all sets containing 1 and closed under the successor operation. Accordingly, Fraenkel's intended set model is understood as the intersection of all set models that share the properties of (a) containing the empty set and the infinite set $\mathbb{Z}$ and (b) being closed under the operations specified in the Zermelo axioms, i.e., pairing, union, power set, etc.. This is essentially an understanding of models as "algebraic closures" (cf. Kanamori, 2004, p. 515). One can find additional textual evidence for this conception in Fraenkel's work from that time, mainly in the context of building different set models satisfying certain

[17] As pointed out by Awodey and Reck (2002, pp. 8–9), the latter effectively corresponds to Peano's (second order) axiom of induction.

restricted versions of Z—e.g., as sets closed under the operations of power set or union—used for independence proofs (cf., e.g., Fraenkel 1922b, p. 233; also Fraenkel 1922a, pp. 165–171). Here, as well as in the third edition of *Einleitung*, he gives an informal sketch of his account of the standard model ("*Normalbereich*") of Z as a system closed under all operations specified in the axioms. Adding the AR to ZF would impose the following effect:

> This will probably result in the fact that only the empty set functioning as the primary building block for all sets is set up as the initial point. Then only those sets are admissible which emerge from the empty set and the sets imposed by [the axiom of infinity] by an arbitrary but certainly finite application of the individual axioms. (Fraenkel, 1928, p. 355)

Even though Dedekind's notion of chains is not explicitly mentioned in Fraenkel's remarks on model building, it seems obvious that AR can be understood here as a "restriction clause for closures" (Kanamori, 2004, p. 515), i.e., for a universe of sets conceived in direct analogy to Dedekind's method of constructing minimal systems.

4.2 Categoricity results

As I have mentioned before, there is no direct indication in Fraenkel's writings of Dedekind's influence on his conceptualization of models and AR. In the first edition of *Einleitung* from 1919, Dedekind is mentioned only for his existence proof of infinite systems and his definition of a finite system given in 1888. In the concluding remarks of the second edition there is a single reference to his theory of chains that, as Fraenkel writes, has received a "general and fundamental significance in set theory." [18] No connection is made to his concept of restriction. There exists, however, a passage in his lectures from 1925 that allows one to draw a direct link between Dedekind's minimal closures and his own approach of devising a minimal model for set theory. In a section on the "non-predicative" methods in mathematics, more specifically the debate between Poincaré and Zermelo on the indispensability of non-predicative proofs in mathematics, there is an interesting footnote mentioning Dedekind's theory:

> In a series of important and thoughtful proofs in set theory especially due to Dedekind and Zermelo [...], deductions of the following kind take center stage: a set M is considered whose elements are all sets of a specific property E exclusively characteristic for it; M is thus the set of all sets sharing the property E. For the cases in question it is

[18] (Fraenkel, 1924, p. 244); most important in this respect is of course Zermelo's adaptation of Dedekind's "closure approach" in terms of chains in his second proof of the well-ordering theorem (as well as in his proof of the Schroeder-Bernstein theorem) in (Zermelo, 1908a). For further details compare (Kanamori, 2004, pp. 501–503 and pp. 510–511).

> then shown that the sum s and the intersection d of all elements of M themselves share the property E; therefore s and d—which exist by virtue of the definition as *sum* and *intersection* respectively—also belong to the set M and can be characterized as the sets the *most comprehensive* and the *most limited in size* sharing the property E. Due to this characterization s and d play a decisive role in the concerned proof. (Fraenkel, 1927, p. 29; notation slightly changed)

The approach described here essentially follows the proof strategy introduced by Dedekind in 1888 to prove the categoricity of arithmetic. And it is precisely this idea—here formulated in Fraenkel's own words—that also seems to lie behind Fraenkel's own understanding of AR. To interpret his tacit assumptions underlying restriction in this way also sheds further light on the second unsolved issue mentioned above, namely how to understand the claim that the addition of AR to Z would render the resulting axiomatization categorical. Fraenkel's remarks alone are not conclusive on this intended effect. Here a glance at Dedekind's categoricity proofs will be instructive to show how Fraenkel might have conceived a similar categoricity result for set theory.

Dedekind's well-known metatheoretic results (cf. Dedekind, 1888, §10) can be qualified as instances of a categoricity based on minimal models according to which a theory is categorical if and only if it has a minimal model and any two minimal models are isomorphic.[19] His proofs (in remarks 132 and 133) that the simple infinite system N can be captured completely, i.e., up to isomorphism by the conditions equivalent to the Peano axioms, strongly depends on his idea of minimal chains (cf. Dedekind, 1888, pp. 376–377). This connection follows from Dedekind's definition of a *mapping of a number sequence by induction* used in his subsequent proofs.[20] The equivalence $\psi(\mathbb{N}) = \theta_0(\omega)$ proved in remark 126 makes explicit the central link between minimal chains and the mapping of a simple infinite system via induction that plays a central role in Dedekind's categoricity proofs.

Now, unlike Dedekind, Fraenkel did not develop an actual proof of the categoricity of ZF + AR nor does he make any remarks how such a proof based on AR might be built. Besides this fact, the presentation of the central concepts of his theory (most importantly those of restriction, the

[19](cf. Grzegorczyk, 1962, p. 63); a minimal model can be defined as a model M satisfying a theory T such that for every submodel N of M that also satisfies T, N is isomorphic to M. Cf. (Grzegorczyk, 1962, p. 63).

[20]In remark 126 he shows that there is one and only one mapping of $\mathbb{N}$ into any system Ω via a function ψ that satisfies the conditions that (i) the closure of $\mathbb{N}$ is a subset of Ω, that (ii) $\psi(1) = \omega$, where ω is an element of Ω and that (iii) for any number n, $\psi(n') = \theta\psi(n)$, where "n'" stands for the successor of "n" and θ is a function on Ω (cf. Dedekind, 1888, pp. 370–371). In remark 128, Dedekind then proves that there exists an equivalence between such an inductive mapping $\psi(\mathbb{N})$ and a minimal closure $\theta_0(\omega)$ of Ω that contains ω, i.e., $\psi(\mathbb{N}) = \theta_0(\omega)$ (Dedekind, 1888, p. 372).

set universe and minimal models) is not comparable in technical rigor to Dedekind's foundational work in arithmetic. Nevertheless, given the textual evidence above as well as his various informal remarks on the effect of the axiom candidate as imposing a minimal model, on its "inductive character" as well as on his conception of the intended set model as a minimal closure, it seems at least a plausible interpretation that AR was conceived by Fraenkel in close analogy to Dedekind's method developed in 1888.

5 Objections to AR

A number of serious objections were raised against Fraenkel's axiom candidate shortly after its first presentation in print in 1922 that led to general skepticism concerning the validity of AR as a set theoretical axiom and eventually prevented it from being added to the canonical list of ZF. In what follows I will first briefly present the main arguments adduced against AR. Then, Fraenkel's reaction to the objections and its subsequent impact on his own conception of set theoretic restriction will be discussed.

5.1 Baldus on meta-axioms

One point of criticism against AR was first put forward by the German mathematician Richard Baldus during a discussion of Hilbert's completeness axiom in geometry (Baldus, 1928). It concerns the general metatheoretic character of Hilbert's and related axioms, later terminologically clustered as "extremal axioms" (cf. Carnap and Bachmann, 1936). Baldus argues that unlike the other axioms in Hilbert's axiomatization of geometry (e.g., the axioms of order), the completeness axiom makes an assertion "*not only over the thought things* [of an interpretation] *but actually over all conceivable things*" (Baldus, 1928, p. 331). This assumption of the *non-extensibility* ("*Nicht-Erweitungsfähigkeit*") of the basic elements of the domain involves generalizing over the individuals in all models. Baldus correctly indicates a methodological doubt about the validity of such quantification over models:

> In order to preserve the completeness axiom's status as an axiom, one would have to allow as axioms also assertions over other things than those thought in the respective interpretation of the axiom system, which would extend the concept of axioms in geometry in a precarious and superfluous way. (Baldus, 1928, p. 331)

In an attached footnote, Baldus explicitly mentions Fraenkel's AR in this respect expressing a direct critique of it based on similar grounds:

> At a meeting in Kissingen Mr. A. Fraenkel has suggested that set theory can in no other way be rendered monomorphic than by a "postulate" [...], namely by an axiom of restriction, against which similar objections can be raised as against the axiom of completeness. (Baldus, 1928, p. 331)

Baldus' criticism of the problematic semantic character of the axiom has meanwhile become a standard argument against extremal axioms in general and Fraenkel's axiom candidate in particular. The basic objection is that AR imposes no condition on sets as the individuals of set theory, but on set models, thus conflating "formal languages with their model-theoretic semantics".[21]

5.2 von Neumann's subsystems

A second and somewhat related objection raised specifically against Fraenkel's axiom candidate is found in (von Neumann, 1925). Von Neumann presents an alternative axiomatization of set theory based on the primitive notions of functions (II-objects), arguments (I-objects), and objects that can be both arguments and functions (I-II-objects).[22] Furthermore, two primitive operations $[x, y]$ denoting the value of a function x for an argument y and (x, y) expressing an ordered pair of arguments are given (von Neumann, 1925, pp. 397–398). Given his specific axiomatization of set theory based on these terms, von Neumann provides the first formalized version of AR intended to capture Fraenkel's original intention of imposing a minimal model for the resulting theory. In von Neumann's terminology, a subsystem of a given system is minimal if and only if it contains no subsystem that also satisfies the axioms:

> Let Σ be the system of I-objects and II-objects. Let Σ' be a subsystem of Σ. Let $I_{\Sigma'}$-objects and $II_{\Sigma'}$-objects be the I-objects and II-objects, respectively, that are in Σ'. Let $[x, y]_{\Sigma'}$ (where x is an $II_{\Sigma'}$-object and y an $I_{\Sigma'}$-object) mean $[x, y]$; let $(x, y)_{\Sigma'}$ (where x and y are $I_{\Sigma'}$-objects) mean (x, y); let $A_{\Sigma'}$ be A and let $B_{\Sigma'}$ be B. Now if these $I_{\Sigma'}$-objects and $II_{\Sigma'}$-objects, the operations $[x, y]_{\Sigma'}$ and $(x, y)_{\Sigma'}$ and the objects $A_{\Sigma'}$ and $B_{\Sigma'}$ also satisfy our axioms, we say for short that Σ' satisfies our axioms. Then the axiom of restriction just mentioned simply requires that besides Σ itself no other subsystem Σ' of Σ shall satisfy Axioms I–V. (von Neumann, 1925, p. 404)

A and B (and their respective correlates $A_{\Sigma'}$ and $B_{\Sigma'}$) are both arguments, i.e., I-objects. A subsystem in this sense is thus a collection of I-objects and II-objects resulting from a restriction of the original system. The operations of $[x, y]$ and (x, y) in turn are restricted to the (two types of) elements of the subsystem. Given this formal presentation of AR, von Neumann then

[21](Shapiro, 1991, p. 185); compare also (Ferreirós, 1999) who states that: "Formulated as above [as a minimal condition on set models], the axiom is unacceptable it is no condition on sets but on models of set theory, i.e., it is not an axiom but a metaaxiom." (Ferreirós, 1999, p. 369)

[22](von Neumann, 1925, pp. 399–402); von Neumann's later class/set distinction is clearly anticipated here: those II-objects that are not I-II-objects have to be treated as classes. Compare (von Neumann, 1925, p. 401).

presents two "serious objections" against the axiom that are "equally true in Fraenkel's system." (von Neumann, 1925, p. 404). According to the first, AR presupposes notions of "naive set theory", most importantly that of a submodel that is not precisely definable in his own theory of sets.[23] The resulting regression to informal set theory would make the whole process of axiomatizing set theory circular. A possible remedy for this is to assume a "higher set theory" and a corresponding expanded domain P in which the original domain Σ can be properly defined as a class of P (and the subsystems Σ' of Σ as subclasses of P).[24] However, this additional "hypothesis" implies a second, even greater difficulty for expressing a restriction clause for his axiomatization. Von Neumann argues that Fraenkel's proposed method of devising a model via the intersection of all possible models need not necessarily lead to a single, unique minimal model satisfying the other axioms (and thus to a categorical axiomatization) (von Neumann, 1925, p. 405). This is due to the fact that the range of the generalization over all (sub-)models of a theory involved in the intersection approach strongly depends on which higher background set theory is assumed. Different higher systems might allow different ranges of submodels. Since Fraenkel's minimal model is defined via the intersection of all possible models (as in Dedekind's approach), the method may lead to different results when different systems are assumed as the background theory.[25]

A second and more technical argument against the intersection approach presented by von Neumann is closely related to his own axiomatization and the satisfaction conditions he devises for subsystems of a given system. One problem about relativizing functions and arguments of Σ to Σ' concerns the fact that in order for Σ' to satisfy von Neumann's axioms, additional satisfaction conditions stronger than the axioms have to be devised. For example, axiom III.2 states that there is a II-object a as such that for all I-objects x: $[a, x] = x$. Von Neumann argues that this imposes the undesirable effect for a possible subsystem Σ' of Σ that there is a II-object a in Σ' such that for all II-objects x in Σ': $[a, x] = x$. In order to evade this problem, a stronger condition is added that the subsystem has to satisfy in order to

[23](von Neumann, 1925, p. 404); for his distinction between sets and classes that plays a central role in his argumentation (cf. von Neumann, 1925, p. 403).

[24](von Neumann, 1925, p. 404); this remark can in fact be taken as a first and informal expression of the conception of the set theoretic universe as a infinite progression of "higher set theories" in which lower systems are submodels of the higher theories. This is eventually fully spelled out in (Zermelo, 1930) (cf. the next section). One can also find a reference in von Neumann's remarks on the set theoretic hierarchy to Russell's type theory that seems to anticipate Gödel's later work on set theory as a cumulative and transfinite hierarchy of types (cf. Section 6): "The idea is partly the same as the one upon which Russell's "hierarchy of types rests"" (von Neumann, 1925, p. 405).

[25]Compare (Shapiro, 1991, p. 186) on this objection to Fraenkel's paring down approach for AR.

satisfy the axiom system: condition 3 states that there is a II-object a in Σ' such that for all I-objects x in Σ: $[a, x] = x$ (von Neumann, 1925, p. 406). Here (as in similar cases for other axioms postulating the existence of sets) the quantification over all objects of the subsystem that is considered to be improper is substituted by the quantification over all objects of the original system. In the case of axiom III.2 the improper quantification $\forall x_{II} \in \Sigma'$ is replaced by $\forall x_I \in \Sigma$.[26] The resulting conditions are thus stronger than the axioms with the effect that they are sufficient, but no longer necessary conditions for Σ' to satisfy the axioms. In addition, von Neumann points out that there exists a smallest submodel that satisfies the given additional conditions and that can be constructed via intersection, but not necessarily a minimal model for the axioms (cf. von Neumann, 1925, pp. 405–408). Given this set of objections, the following strong conclusion is drawn for Fraenkel's axiom candidate:

> For these reasons we believe that we must conclude, first, that the axiom of restriction absolutely has to be rejected and, second, one cannot possibly succeed in formulating an axiom to the same effect. (von Neumann, 1925, p. 405)

According to von Neumann, this fact, together with the existence of "inaccessible sets" (such as "descending sequences of sets") that lie "outside the system" in question are the main sources of the non-categoricity of set theory (von Neumann, 1925, p. 405).

5.3 Zermelo on set models

A third and more general objection against Fraenkel's axiom candidate was expressed in Zermelo's seminal paper "On boundary numbers and set-domains" (Zermelo, 1930). Zermelo introduces a new conception of the set universe as an open and unbounded sequence of set models (*Normalbereiche*) of increasing size that satisfy his axiomatization.[27] We have mentioned that a comparable view of a sequence of larger and larger set models had already been presented but not further developed in (von Neumann, 1925). Zermelo, in giving a formal explication of a cumulative hierarchy of sets, also provides a definitive clarification of the semantic notions of set models and submodels that can be found both in Fraenkel and in von Neumann. According to Zermelo, each set is decomposable into *layers* and cumulative *sections* that include all sets formed at earlier layers in the set theoretic hierarchy.[28] Set models $\mathbf{V}_{\kappa}^{Q}$ in turn are treated as sets that can be specified by

[26] For a comparable notion of relativization (cf. also Gödel, 1940, p. 76).

[27] Zermelo's proposed axiom system can be considered as a second order version of ZF since it contains a second order formulation of the axiom of replacement.

[28] (Zermelo, 1930, pp. 32–33); for the technical details of this early version of an iterative conception of sets (cf. Kanamori, 2004, pp. 521–524).

two numbers, a *base* Q—the cardinality of its first rank, i.e., the base set of individuals—and a *characteristic* or *boundary number* κ as the least ordinal greater than all ordinals contained in the model. From this it follows that each model can act as a submodel of a set model with a higher boundary number (cf. Zermelo, 1930, p. 31). Thus, the universe $\mathbf{V}$ is composed of a "boundless progression" of set models (Zermelo, 1930, p. 29).

This conception of the set theoretic universe as an unlimited sequence of models obviously differs substantially from Fraenkel's static conception of a closed and fully describable universe of sets. This divergence also results in an opposing view on the issue of the (non-)categoricity of set theory. Whereas Zermelo shows that his axiomatization captures set models of a given base and boundary number up to isomorphism—the main results of his article in fact are three "isomorphism theorems" and their respective proofs[29]—an absolute concept of categoricity in Fraenkel's meaning of capturing a unique model is not possible due to the boundlessness of the set theoretic universe, i.e., the "existence of a unlimited sequence of boundary numbers" (cf. Zermelo, 1930, pp. 40–41).

Given Zermelo's picture of $\mathbf{V}$, talk about the single intended model captured by ZF is shown to be inadequate. This insight also underlies Zermelo's more general critique of restrictive axioms. We have seen that von Neumann holds the assumption that for set theory there always exists a larger domain, a higher set theory in which the original model is definable as a set and in which a restriction for the lower theory yielding categoricity could at least in principle be formulated. Zermelo's theory of relative or quasi-categoricity essentially conforms to this view. Nevertheless, for him a domain restriction will never be desirable from a practical point of view, because it decisively delimits the functional role of set theory as a foundational discipline:

> Our axiom system is non-categorical which in this case is not a disadvantage but rather an advantage, for on this very fact rests the enormous importance and unlimited applicability of set theory. (Zermelo, 1930, p. 45)

Here lies the central objection to Fraenkel's account of restriction. Its effect is not considered a theoretical virtue of the axiomatization, but rather as a deficiency in a practical sense: it restricts set theory in its proper foundational goal, i.e., in the task of formalizing mathematics. Zermelo explicitly

[29]Briefly, the first holds that two models A and B with the same base cardinality and the same characteristic are isomorphic to each other. The second theorem states that a model A that shares the same base cardinality with model B but has a different characteristic is isomorphic to a certain cumulative rank of B. Theorem 3 holds that if A and B share the same characteristic but have different bases, then one is isomorphic to a submodel of the other model. Cf. (Kanamori, 2004, p. 527) for a more detailed account.

refers to Fraenkel's axiom candidate in order to underline the difference between their conceptions. He remarks that,

> Naturally one can always force categoricity artificially by the addition of further 'axioms', but always at the cost of generality. Such postulates, like those proposed by Fraenkel [...] do not concern set theory as such, but rather only characterize a quite special model chosen by the author concerned. [...] the applicability of set theory has to be given up. (Zermelo, 1930, p. 45)

Note that Zermelo's actual objection against restrictive axioms like Fraenkel's primarily concerns the fruitfulness of set theory as foundational discipline. Any deliberate restriction of the set universe negatively affects the "full generality" of set theory, i.e., its "unlimited applicability" to mathematics (Zermelo, 1930, p. 45). As we shall see in Section 7, this objection anticipates Maddy's recent critique of what she terms "restrictiveness" concerning set theoretic structures in questions of axiom choice. For present purposes, it suffices to point to the fact that Zermelo, in his motivation for a cumulative universe claims to be more attentive to this pragmatic ideal of the foundational goal than Fraenkel in his call for a categorical axiomatization.

6 Fraenkel's reaction

Fraenkel's reaction to the presented objections against the axiom of restriction in his subsequent work is instructive in several ways. First, it better illustrates his own tacit understanding of the concepts involved in his earlier presentation of the axiom. Second, it also highlights substantial shifts in his understanding of the axiom as a direct result of these criticisms.

6.1 Vindications of AR

As far as I know Fraenkel never responded in print to Baldus' legitimate doubts about the metatheoretic character of extremal axioms and their semantic implications. Even though he acknowledged the "special character" of the axiom in comparison to the other axioms of ZF he never seemed to become aware of the problem that the axiom in fact requires a generalization over set models.[30] More generally, as mentioned above, Fraenkel—in his writings on set theory in the 1920s—seems to have been indecisive concerning questions of the adequate logical presentation of his axiomatization, and in particular of his AR. The question of the proper formalization of the axiom candidate is eventually taken up by Fraenkel almost three decades later in his *Foundations of Set Theory* (Fraenkel and Bar-Hillel, 1958). Here

[30] As Shapiro points out this fact is probably due to the circumstance that a clearly delineated syntax/semantic-distinction was far from being standard by the time Fraenkel developed his theory. Compare (Shapiro, 1991, p. 184).

the authors refer to different attempts to formalize AR as a minimal axiom in a higher order logical language by Carnap:

> True, recently [...] Carnap proposed a vindication of this axiom of restriction, and Carnap formulated it symbolically, as an axiom of a minimal model [...]. (Fraenkel and Bar-Hillel, 1958, p. 90)

Carnap, in his (Carnap, 1954) formally expressed Fraenkel's account of a minimal property codified in AR by the use of higher-level binary relations (representing membership relations) and the notion of a partial relation: "There exists no proper partial relation of E that also satisfies the properties stated in axioms A1 to A8 [i.e., ZF]." (Carnap, 1954, p. 154). His formal version depends on the (higher order) universal quantification over (partially defined) membership relations (Carnap, 1954, p. 154):

$$(H)[(x)(y)(Hxy \supset Exy).\mathrm{Kon}(H) \supset (x)(y)(Hxy \equiv Exy)]$$

"Kon" stands for the union of all axioms of ZF; x and y are individual variables ranging over sets. The axiom in this logical formulation basically states that E is the minimal interpretation for set theoretic membership consistent with ZF.[31] Fraenkel, in an attached footnote to the passage cited above stresses this notion of a restriction condition in Carnap's formal presentation as an adequate version of his own informal treatment of AR: "The pith of the axiom is then the demand that no "partial relation" ε should fulfill the conditions expressed by the other axioms".[32]

Whereas comments on the logical status of AR are limited to his later writings, Fraenkel immediately reacted to the objections leveled against the axiom by von Neumann. This might seem surprising at first sight because it is far from obvious that the latter's critique actually meets Fraenkel's informal presentation of the restriction on set models. First, it seems more reasonable that the technical objections against the AR rather concern von Neumann's own non-standard axiomatization of set theory (and specifically his formalization of the AR) than Fraenkel's preliminary and informal ideas. Second, the general validity of his critical remarks against restriction can be challenged by drawing attention to a number of inconsistencies in his own treatment of set models. As Shapiro (1991) has shown, von Neumann's set theory does not allow a consistent presentation of models (as classes containing its subclasses) due to the fact that proper classes cannot be conceived as

[31] Compare (Carnap and Bachmann, 1936) for a slightly different formalization of AR as a minimal model axiom.

[32] (Fraenkel and Bar-Hillel, 1958, p. 90); it remains debatable—in light of modern model theoretic semantics—if Carnap's approach actually vindicates Fraenkel's AR from objections along the lines of Baldus' remarks. For interesting related remarks concerning the axiom (cf. Shapiro, 1991, p. 186).

elements of either sets or classes in his theory.[33] Nevertheless, Fraenkel in grosso modo seems to have acknowledged von Neumann's critique. Already in 1927 he considers it "very doubtful" whether his version of restriction can be attributed "a sound meaning". He states that,

> One seriously has to take the eventuality into consideration that the possible realizations of the axiom system that differ in their size do not have a smallest common subpart that would also satisfy all the axioms. Also the previously given instruction for a "construction" of such a smallest model [...] need not lead to a definite result since the axioms IV–VI [i.e., the axioms of power set, separation and choice] themselves do have a purely constructive character. This is a serious and so far not satisfactorily solved problem from which possibly the natural necessity of a certain "boundlessness" and also a certain vagueness (so to speak at the boundaries) of the yet legitimate concept of set will follow. (Fraenkel, 1927, p. 102)

The first remark essentially rephrases von Neumann's critique. The second remark concerning the "boundlessness" of the set concept seems already—i.e., three years before Zermelo (1930)—to indicate doubts about his conception of the set universe as an (algebraic) closure.[34] Despite the acknowledged criticism, Fraenkel remains optimistic about the practical usefulness and general correctness of a restrictive axiom.[35]

6.2 New axioms of restriction

Fraenkel's subsequent discussion of the axiom candidate is marked by a number of substantial modifications. By 1958, in *Foundations of Set Theory*, both the intended character of AR and its extrinsic justification change substantially. One can witness here a more tolerant attitude concerning different additional axioms candidates for set theory, including stronger infinity axioms that are—as Fraenkel puts it—"antithetical" to AR (Fraenkel and Bar-Hillel, 1958, p. 87). Fraenkel acknowledges that, in contrast to prior belief, inaccessible numbers have "significance not only for the foundations of set theory but also for some applications." (Fraenkel and Bar-Hillel, 1958, p. 87) Following a short presentation of different kinds of axioms that

[33] Cf. (Shapiro, 1991, p. 186); Shapiro suggest a modification of this system in order to vindicate the axiom of restriction, to the effect that "sets and proper classes of the original theory (can be treated as) as elements, i.e., as sets" thereby allowing to treat models for a theory T as the subclasses of a higher-level theory T'. The effect on restriction would then be that "[...] one can state in the higher theory that a given class has no proper subclasses that are models of ordinary set theory." (Shapiro, 1991, p. 186)

[34] Similar remarks along these lines can be found in (Fraenkel, 1928).

[35] In (Fraenkel, 1928) he concludes his discussion of the axiom by stating that: "Nevertheless I like to believe that the mentioned doubts can be resolved and that the axiom of restriction can be maintained—and then considered as a very central part of the axiomatization! if only its formulation can be made more precise." (Fraenkel, 1928, p. 355).

postulate inaccessible cardinalities, he returns to a discussion of his axiom of restriction, however, not without providing a new motivation for it. The main intention behind AR is no longer to rule out non well-founded sets but to be restrictive in another sense, namely to "yield just all ordinals less than the least inaccessible ordinal number" (Fraenkel and Bar-Hillel, 1958, p. 88). Whereas the well-foundedness of the sets composing the set theoretic universe is now secured by the axiom of foundation (as proposed in von Neumann 1925 and Zermelo 1930), the main rationale for introducing the axiom of restriction is to secure the "non-existence of inaccessible numbers" (Fraenkel and Bar-Hillel, 1958, p. 88). The former version of AR is thus functionally divided into an independent axiom of foundation and a kind of accessibility axiom introduced to exclude all inaccessible numbers. Interestingly, Fraenkel's original paring down method for imposing a minimal model as the intersection of all "realizations" is still upheld as an adequate means of achieving this new and structurally different set theoretic restriction: "Then the non-existence of inaccessible numbers or of the corresponding sets, as well as of extraordinary sets [...] could be proved." (Fraenkel and Bar-Hillel, 1958, p. 89)

Fraenkel's change in perspective concerning the nature of AR calls for two comments. First, recall Zermelo's critical remarks on restrictive axioms like AR and the fact that inaccessible cardinals ("*Grenzzahlen*") were first introduced in his 1930 paper as a direct consequence of his dynamic account of the set theoretic universe as a progression of set models (delimited by such boundary numbers). Given this, it is natural to interpret the shift in Fraenkel's conception of set theoretic restriction as a direct reaction to the theory of set models proposed in (Zermelo, 1930). Second, notice also the underlying change from Fraenkel's original understanding of the set theoretic universe as a static structure that becomes evident here. In *Foundations of Set Theory*, Fraenkel must, at least to a certain point, have acknowledged Zermelo's conception of the set theoretic universe. Otherwise this new reading of AR as an accessibility axiom that rules out higher natural models would simply make no sense. Nevertheless, he still argues for a restriction of the universe to a minimal model. His motivation outlined in 1958 seems to be a somewhat unjustified disapproval of Zermelo's presentation of the set models in general and the idea of relative categoricity results for set theory in specific. In a remark he critically comments on Zermelo's results from 1930:

> The cardinal of the basis and the ordinal α together are an invariant characteristic of the intended domain of sets. The first leads to the domain of finite sets, the second to the domain of sets up to the first inaccessible number. However, Zermelo's proof that this invariant guarantees the monomorphism (categoricalness) of the domain

> can hardly be considered stringent, and even the concepts used, e.g., "cardinal of the basis" are objectionable. (Fraenkel and Bar-Hillel, 1958, p. 92).

Zermelo's different categoricity results for his axiomatization, especially his version of categoricity in a given power seems to be in conflict with Fraenkel's principal aim of providing a single categorical axiomatization of set theory. This adherence to the axiomatic ideal of absolute categoricity is probably the main reason why Fraenkel not only did not accept Zermelo's 1930 objection against his axiom candidate but in effect redefined AR with the motivation to rule out the existence of Zermelo's higher boundary numbers. The extrinsic motivation of gaining a categorical axiomatization via this new version of a "limitative axiom" is again clearly stated in 1958:

> A suitable axiom of restriction should enable us to prove that all models of the axiom system are isomorphic, i.e., admit a one-to-one mapping which preserves the ∈-statements. (Cf. also Gödel's postulate of constructibility [...]). (Fraenkel and Bar-Hillel, 1958, p. 89)

Fraenkel's reference to Gödel's axiom **V**=**L** (or alternatively *axiom A*) in the passage above deserves closer attention here. As has been pointed out by Maddy (1997), the axiom **V**=**L** also imposes a kind of minimal model on ZFC. Without going into further details here, one can understand the constructible universe **L** as the minimal inner model of ZFC,[36] since for all inner models M of ZFC, the constructible model $\mathbf{L}^M$ (as the class of all constructible sets in M) is identical to **L**, i.e., $\mathbf{L}^M = \mathbf{L}$ due to the absoluteness of **L**.[37] In Maddy's own terms: "[...] if M is a transitive model of ZFC containing all ordinals, then the constructible sets of M are the real constructible sets, and thus, $\mathbf{L} \subseteq M$." (Maddy, 1997, p. 73) Thus any extension of **L** "will contain sets different from all relevant constructible sets." (Maddy, 1997, p. 73)

Now, it is worth noting that for both accounts, **V**=**L** and AR, the respective minimal models can be constructed—despite von Neumann's original objections—by the kind of intersection approach first outlined by Fraenkel. One can understand the model **L** as the intersection of all inner models of ZFC.[38] In contrast, the model satisfying ZF plus Fraenkel's AR (as an accessibility axiom) can be constructed via the intersection of all natural models with an empty base. Concerning the question of the categoricity of

[36] The layers of **L** are constructed in similar fashion to the cumulative hierarchy in **V**, however, as elements of a level $\alpha + 1$ only of the (first order) definable subsets of $\mathbf{L}_\alpha$ are admitted. Compare (Maddy, 1997, p. 65).

[37] Cf. (Gödel, 1940, pp. 68–78) for a formal presentation of **L** and on the notion of absoluteness. Compare also (Maddy, 1997, p. 73) and, for a more detailed discussion of Gödel's set theory, cf. (Kanamori, 2007).

[38] Compare (Jech, 2002, p. 187).

axiomatic set theory, there is also a point of continuity between Fraenkel's AR and Gödel's axiom. What is especially worth noting in this respect is that on both accounts categoricity can be achieved by (different types of) set theoretic minimization (again given that ZF is second-order). In a recent paper, Kanamori explicitly points out this methodological affinity between Fraenkel and Gödel:

> Gödel's axiom A, that every set is constructible, can be viewed as formally achieving this sense of categoricity [i.e., "Fraenkel's idea of a minimizing, and hence categorical axiomatization"] since [...] in axiomatic set theory **L** is a definable class, containing all the ordinals, that, together with the membership relation restricted to it, is a model of set theory, and **L** is a submodel of every other such class. (Kanamori, 2004, p. 539)

Obviously, the (minimal) set models characterized up to isomorphism differ substantially in the two cases. In Fraenkel's 1958 version of restriction, $\mathsf{ZFC} + \mathsf{AR}$ yields the model $\mathbf{V}^0_\kappa$, where 0 signifies the empty basis and κ the first inaccessible rank of $\mathbf{V}$. In contrast, $\mathsf{ZFC} + \mathbf{V}{=}\mathbf{L}$ does not rule out any limit numbers but characterizes the least inner model of ZFC.

Beside the reference to Gödel's axiom in the passage cited above, Fraenkel does not get more explicit on how his AR relates to the axiom of constructibility. However, there exists an interesting aftermath to Fraenkel's own conceptualization of AR that can be found in the second and revised edition of *Foundations*, (Fraenkel et al., 1973), published after Fraenkel's death in 1966. In an expanded section devoted to axioms of restriction, Lévy proposes two new candidates, both reflective of Fraenkel's own considerations given in 1958 (Fraenkel et al., 1973, pp. 113–119). It is argued that both new versions of AR can be "equated with" Fraenkel's original conception but can be formulated "within our axiomatic theory", thus evading the objections against the former axiom's metatheoretic character (Fraenkel et al., 1973, p. 114). The first axiom candidate, AR_1, is conceived as the conjunction of an axiom of foundation and an accessibility axiom (as already suggested in the first edition of *Foundations* by Fraenkel himself). For the second and stronger restrictive axiom candidate, AR_2, Lévy seems to have taken into account Fraenkel's analogy with Gödel's axiom of constructibility. AR_2 is thus defined as the conjunction of $\mathbf{V}{=}\mathbf{L}$ and a principle ruling out transitive sets as models of ZF (Fraenkel et al., 1973, p. 116).

6.3 An "intuitive" consideration

Overall, the presentation of the restrictive axioms in (Fraenkel et al., 1973) is close in spirit to Fraenkel's considerations in the first edition. In particular the extrinsic motivations stated for the two axiom candidates, namely to rule out non-well-founded sets and sets of inaccessible ordinals, are identical to his arguments from 1958. Lévy, in 1973, also mentions the fact that the addition of either one of the two axioms to a second order version of ZF would lead to a categorical axiomatization.[39] However, one can also detect a significant change in the second edition compared to Fraenkel's earlier remarks. We have seen that Fraenkel, in his writings from the 1920s onwards, provides exclusively extrinsic motivations for AR. There is to my knowledge not a single remark in his work indicating a kind of intrinsic motivation for the axiom candidate based on mere reflection on his underlying concept of set universe.[40] In contrast, in the second edition of *Foundations* a discussion of "the desirability of restriction in general" as a basis for evaluating the axioms AR_1 and AR_2 is given. The extrinsic arguments for the axiom candidates mentioned above are complemented here by a new type of reflection. Lévy makes the following remark concerning a necessary "intuitive justification" of the axioms of restriction:

> In the case of the axiom of induction in arithmetic and the axiom of completeness in geometry, we adopt these axioms not only because they make the axiom systems categorical or because of some metamathematical properties of these axioms, but because, once these axioms are added, we obtain axiomatic systems which perfectly fit our intuitive ideas about arithmetic and geometry. In analogy, we shall have to judge the axioms of restriction in set theory on the basis that the set theory obtained after adding these axioms fits our intuitive ideas about sets. (Fraenkel et al., 1973, p. 117)

This passage documents a substantial shift in the methodology of axiom choice compared to Fraenkel's original approach. Part of the new evaluation strategy proposed here is clearly intrinsic in character in Maddy's sense: not in terms of an allusion to a set-theoretic intuition or self-evidence, but explicitly based on a prior explanation of the set universe underlying the axiomatization (cf. Fraenkel et al., 1973, pp. 87–88). What is effectively called for here is that the justification of an axiom has to depend on a combination of extrinsic and intrinsic considerations. The latter are necessary to confirm (so to speak in a second loop) the choice originally made on extrinsic grounds. An acceptance solely based on extrinsic considerations—such as on "metamathematical properties" of the resulting axiomatization—do not provide sufficient evidence unless backed up by intrinsic reflection whether

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[39] In contrast to Fraenkel's own perception, Lévy makes clear in the revised version that the possible categoricity results for set theory are those "essentially contained in Zermelo [1930]." (Fraenkel et al., 1973, p. 115).

[40] It is usually Zermelo that is credited with providing the first genuinely intrinsic justification of his axioms based on the cumulative hierarchy of set. Compare Kanamori on this point in Zermelo's 1930 paper: "In a notable inversion, what has come to be regarded as the underlying iterative conception became a heuristic for motivating the axioms of set theory generally." (Kanamori, 2004, p. 521)

the axioms actually fit or comply with the underlying conception of the universe they are intended to characterize. Turning to the cases of AR (without focusing on a specific version), Lévy draws a rather pessimistic conclusion for Fraenkel's axiom candidate:

> To restrict our notion of set to the narrowest notion which is compatible with the axioms of ZFC just for the sake of economy is appropriate only if we have absolute faith that the axioms of ZFC (and the statements which they imply) are the only mathematically interesting statements about sets. It is difficult to conceive of such absolute faith in the sufficiency of the axioms of ZFC [...]. Even if one had such faith in the axioms of ZFC, it is likely that he would settle rather for something like an axiom of completeness, if there were some reasonable way of formulating it. (Fraenkel et al., 1973, p. 117)

This can only be understood as an intrinsically based claim against minimizing principles in set theory. Unfortunately, no supporting argument showing exactly why AR does not match the cumulative hierarchy of sets is given. However, the underlying idea here seems to mirror Zermelo's objection in his 1930 paper that any kind of minimizing principle artificially restricts the cumulative hierarchy of sets, thereby ruling out potentially mathematically interesting structures.[41]

7 Maddy's MAXIMIZE

Fraenkel's axiom candidate is not mentioned in (Maddy, 1997). Nevertheless, Lévy's negative conclusion concerning "the desirability of restriction" as a set theoretic principle also seems to be in line with her more systematic approach of evaluating axiom candidates in terms of two "methodological maxims", UNIFY and MAXIMIZE (cf. Maddy, 1997, pp. 208–212). We have seen that Fraenkel shares with Maddy an understanding of set theory as a foundational discipline and thus the motivation for assuring its "foundational goal" through unification, i.e., through providing a single arena in which mathematics can be presented.[42] Therefore, AR fits perfectly with

[41]Note that one can also identify an interesting affinity to Gödel's changing attitude towards his axiom **V**=**L** as a minimizing principle here. In a footnote in (Gödel, 1939), an intrinsic motivation for his axiom A is presented: "In order to give A an intuitive meaning, one has to understand by sets' all objects obtained by building up the simplified hierarchy of types on an empty set of individuals (including types of arbitrary transfinite orders)." (Gödel, 1939, p. 29) By 1964, in contrast, Gödel provides a direct intrinsic argument against a "minimum property" such as expressed by **V**=**L**: "Note that only a maximum property would seem to harmonize with the concept of set [...]." (Gödel, 1964, pp. 262–263) Compare also (Maddy, 1997, p. 84) on this shift in Gödel's view.

[42]Compare Maddy on this point: "One methodological consequence of adopting the foundational goal is immediate: if your aim is to provide a single system in which all objects and structures of mathematics can be modeled or instantiated, then you must aim for a single fundamental theory of sets." (Maddy, 1997, pp. 208–209)

Maddy's first maxim, Unify. The tension with her program clearly arises in relation to the more extensively discussed second principle of axiom choice, Maximize. The general idea underlying this principle, namely to defend axioms on the basis whether they maximize the set theoretic structure, i.e., secure new isomorphism types not reachable from the given axioms, looks antithetical to Fraenkel's intentions behind AR.[43] In fact, Maddy proposes a notion of Maximize based on a formal "criterion of restrictiveness" for axiom systems (cf. Maddy, 1997, III.6). According to it, a theory "T is restrictive iff there is a consistent T' that strongly maximizes over T" (Maddy, 1997, p. 224). T' in turn strongly maximizes over T iff (i) it delivers new isomorphism types not provable from T , (ii) it is inconsistent with T and (iii) there is no T'' that maximizes over T' (cf. Maddy, 1997, p. 224). Given this explication, we can see in what sense the different versions or AR are restrictive in terms of this formal criterion. Maddy is clear in pointing out that Fraenkel's first version of the axiom candidate as a kind of axiom of foundation is not restrictive (in relation to an axiom system including an "anti-foundation axiom") since the addition of non well-founded sets does not provide any additional isomorphism types compared to the models of ZFC (Maddy, 1997, p. 217). What about Fraenkel's 1958 understanding of AR as an accessibility axiom? Here again, Maddy's criterion would not rule out the axiom candidate. As Maddy shows in Section III.6., the axiom system ZFC + "there exists an inaccessible cardinal" (IC) does not maximize over ZFC in a proper sense, even though it clearly allows new isomorphism types. The reason is that it simply does not satisfy condition (ii) of being inconsistent with ZFC. Given this, ZFC is not restrictive in relation to ZFC + IC (Maddy, 1997, p. 222). By the same token, ZFC + "there exist no inaccessible cardinals" is not restrictive relative to ZFC since the latter does not "inconsistently maximize" over the former. However, compared to higher cardinal axioms like IC, the AR is in fact restrictive in the formal sense. The case remains where AR is conceived as a kind of axiom of constructibility (as pointed out by Lévy in Fraenkel et al. 1973). Here again, Maddy shows that ZFC + $\mathbf{V}{=}\mathbf{L}$ is restrictive in the strong sense relative to different axiom systems imposing higher isomorphism types like ZFC + "$0^{\#}$ exists", ZFC + $\mathbf{V}{\neq}\mathbf{L}$ or ZFC + MC (a measurable cardinal axiom) (Maddy, 1997, pp. 223–224).

One can interpret these results (provided that one accepts Maddy's formal criterion of restrictiveness as a viable principle for axiom choice) as a direct and strong extrinsic argument against Fraenkel's AR. However, a

[43]Cf. again Maddy: "[...] if set theory is to play the hoped-for foundational role, then set theory should no impose any limitations of its own: the set theoretic arena in which mathematics is to be modeled should be as generous as possible; the set theoretic axioms [...] should be as powerful and fruitful as possible." (Maddy, 1997, pp. 210–211)

direct evaluation in terms of Maddy's second maxim would seem somewhat misplaced, for several reasons. First, it would be insensitive to the historical context of axiomatic set theory prior to 1930 in which the axiom candidate was originally devised. It would fail to take into account the general understanding of the axiomatic project in mathematics around that time that guided Fraenkel in setting up his axiom candidate in the first place. We have seen that one of the main motivations of introducing AR was to yield the "completeness of an axiom system" in the sense of its categoricity which was conceived as a central theoretic ideal for any axiomatization, irrespective of its subject matter.[44] Second, modern higher set theory in general and the various higher cardinal axioms in specific which form Maddy's actual field of investigation for her methodological principles certainly transcend Fraenkel's original horizon of set theory as a foundational discipline.[45] His attempt to meet the foundational goal was to propose an axiomatization that would allow a "clean and secure construction of the foundations of all mathematical sciences", without further detailing the spectrum of mathematics referred to here (Fraenkel, 1928, p. 393). In his discussion of AR he explicitly states the belief that "likely all mathematically significant sets, e.g., sets of numbers and points, can [...] be secured within the thus restricted axiomatic [system]." (Fraenkel, 1928, p. 355) Fraenkel assumed—and correctly so—that the axiomatization $\mathsf{ZFC} + \mathsf{AR}$ sufficiently fulfilled the foundational goal for all relevant mathematics faced with in this period.[46]

Given this, one could be inclined to argue that even though Fraenkel's project might have been consistent with the mathematical knowledge of his time, it is simply outdated as a foundational enterprise from a modern point of view given the progress in mathematics since 1930 (and specifically the recognition of higher set theory as a proper and autonomous mathematical discipline). But here again, a final judgment depends on the specific role ascribed to set theory. If it is considered as a proper mathematical discipline, Maddy's maximizing maxim is a perfectly reasonable principle for the choice of its axioms. Note, however, that set theory then can hardly be called a foundational discipline in the original strict sense.[47] If, in contrast,

[44] Compare (Corcoran, 1980).

[45] Fraenkel, in 1928, already discusses the possibility of "special existence axioms" postulating higher cardinals (than those secured by the axioms of infinity and replacement) and then concludes: "However, theses problems lie in the most remote regions of the theoretical science and have so far barely a connection to the questions raised by the scientific demands of the present." (Fraenkel, 1928, p. 310)

[46] The model characterized by $\mathsf{ZFC}_2 + \mathsf{AR}_1$ is sufficiently strong to represent the real and complex numbers, as well as function over real numbers etc. used in classical mathematics. For different models of second order ZF compare, e.g., (Uzquiano, 1999, p. 290).

[47] In this case, it is difficult to provide a sound argument why unification should be a desirable maxim (in contrast to allowing a series of possibly incompatible set theories designed for different tasks).

it is primarily understood as a foundational enterprise used for modeling "everyday mathematics" (and not as a proper mathematical field), it seems less clear that MAXIMIZE is actually a reasonable methodological guideline, since it is simply not necessary from a mathematical point of view. One can sense here a certain tension in Maddy's naturalistic account between the two principles she attributes to axiomatic set theory.[48] A similar point was recently stressed by Friedman in (Feferman et al., 2000). He argues that MAXIMIZE as a maxim for set theoretic axiom choice is not relevant for the working mathematician exclusively interested in set theory as a tool for "a more or less standard set theoretic interpretation of mathematics, with ZFC generally accepted as the current gold standard for rigor".[49] Not only is ZFC then perfectly sufficient for supplying a foundation for ordinary mathematics, one can also quite naturally limit it to more restricted versions along the lines of Fraenkel's suggestions. Concerning Maddy's objections to **V**=**L**, Friedman remarks: "However, for the normal mathematician, since set theory is merely a vehicle for interpreting mathematics so as to establish rigor, and not mathematically interesting in its own right, the less set theoretic difficulties and phenomena the better." (Feferman et al., 2000, p. 436) He puts forward an informal principle inverse to Maddy's MAXIMIZE for foundational set theory: "more is less and less is more" (Feferman et al., 2000, p. 437)

One is inclined to view Fraenkel's motivation for introducing his AR in exactly this sense, i.e., in securing a streamlined concept of set (or of the set theoretic universe) that allows for the reconstruction of standard mathematics and rules out anything nonessential for this task. Given this, Fraenkel's motivations for AR have to be considered as fully rational (in Maddy's own naturalistic sense). Moreover, as Friedman's remarks underline—given a modest understanding of set theory's foundational task—it remains far from evident that minimizing principles like AR are less reasonably grounded in mathematical practice than Maddy's principle for set theoretic maximization.

[48] Maddy explicitly mentions a possible tension between UNIFY and MAXIMIZE if taken as complementary maxims for some cases of axiom choice (cf. Maddy, 1997, pp. 211–212). The point I want to make here is more general and concerns the rationality of MAXIMIZE if one presupposes a foundational goal for set theory.

[49] (Feferman et al., 2000, pp. 434–435); a similar point is made by Feferman in his discussion of Maddy's naturalistic project. He states that "there is not a shred of evidence so far that we will need anything beyond ZFC—or even much weaker systems—to settle outstanding combinatorial problems of interest to the working mathematician." (Feferman et al., 2000, p. 407)

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