Friedrich Waismann's philosophy of mathematics

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1.

In February 1929 Moritz Schlick succeeded in arranging a first meeting with Ludwig Wittgenstein, whose *Tractatus Logico-Philosophicus* he greatly admired. Wittgenstein agreed to further meetings to which Schlick brought his protégé and unofficial assistant Friedrich Waismann, who later, from 1929 to 1931 took shorthand notes of their meetings (now published as *WVC*). In 1929 it was agreed that Waismann, because of his great ability for 'lucid representation' (Carnap), would produce a systematic exposition of Wittgenstein's philosophy intended to be the first volume in the series of publications of the Vienna Circle *Schriften zur wissenschaftlichen Weltauffassung*. The book, entitled *Logik, Sprache, Philosophie*, to which Wittgenstein himself had contributed dictations and typescripts, was finished by 1938 (published in 1976; English translation in 1965). Of Wittgenstein's key ideas in the philosophy of mathematics Waismann gave a brief survey at a conference in Köngisberg in 1930 and a more detailed account in his 1936 book *Einführung in das mathematische Denken* (Baker 1979 & 2003).

When Waismann first arrived in Cambridge in 1937 he was an eloquent proponent of Wittgenstein's philosophy and, in particular, of his middle-period views on mathematics. But meanwhile Wittgenstein's thinking had developed further and when lecturing in Cambridge in 1938 he did not want his students to listen at the same time to his earlier ideas presented in Waismann's lectures (Ayer 1977, 132; Baker 2003, xxi). Waismann was understandably embittered and soon left Cambridge, finding another employment in Oxford. Material from his Oxford lectures on topics in the philosophy of mathematics was published posthumously in 1982 under the title *Lectures on the Philosophy of Mathematics.*

Personally Waismann had moved away from Wittgenstein in the last two decades of his life. In conversation he described him as 'the greatest disappointment of his life' (Grassl 1982, 10). And repeatedly in his 1950s lectures he expresses his disagreement with Wittgenstein. How far then did his views in the philosophy of mathematics move away from his earlier Wittgensteinian position? Did Waismann succeed in developing his own distinctive philosophical position in this area? Considering his published lecture materials, and in particular the points where Waismann indicates his disagreement with Wittgenstein, we shall conclude that the answer to the latter question is no. However much he came to dislike Wittgenstein personally, his views on mathematics remained fairly close to those of his former intellectual guide.

2.

Four major tenets in Waismann's mature philosophy of mathematics can easily be traced back to Wittgenstein's ideas, namely his views on (i) existence in mathematics, (ii) the meaning of mathematical concepts, (iii) equations and tautologies, and (iv) infinity.[1] We shall first explain the parallels, and minor disagreements, in those four areas in turn, before turning (in section 3) to the two important points on which Waismann diverges from Wittgenstein's thought.

(i) Existence in mathematics.

In 'On the Notion of Existence in Mathematics', Waismann argues that there are different meanings of 'existence' as this concept is applied in mathematics. The different meanings of 'existence' are anticipated by Wittgenstein, who suggests that the concept of existence is directly related to what counts as an 'existence proof', arguing that Intuitionists wrongly try to circumscribe the latter concept with claims about what is to count as a legitimate proof and what isn't. This would only serve to give a new definition of 'existence', rather than capturing the multiple ways in which the concept is actually used (given how it relates to the various sign-structures that are called 'existence proofs') (MS 111, 155). Although Wittgenstein does at times in the intermediate period, for at least a couple of reasons, suggest that various sign-structures that are called 'proofs' aren't rightly so-called (*PG* 408-14), he ultimately comes to understand this concept as a family resemblance concept (and thus gives up attempting any regimentation of this concept) (*PG*, 299-300). He spends some time clarifying the concept of proof generally (*BT* 614-659) as well as the role and use of specific types of proofs (*BT* 650-656). Indeed, reflections on inductive proof are likely to have importantly contributed to Wittgenstein's development of the family resemblance concept (see *PR* 193-205; *PG* 395-426). Given this is the case, Waismann's principal claims regarding 'existence' are derived from Wittgenstein's work.

To illustrate the first meaning of 'existence', Waismann uses two examples: 'There exists a prime number between 31 and 41' and 'There exists a root to any given algebraic equation'. That there exists such a number in the first case is determined by a method for determining whether a number is prime or not. One need only use the appropriate method on each of the numbers of the interval to determine there is. In the second case, a method determines, at least in principle, a specific value. The 'at least in principle' is meant to allow for the possibility of real numbers: while these may not be calculable to a final place (because they are unending), they can be, using a rule that determines the infinite series of digits, determinable to an ever more precise number of places.

The aforementioned Waismann lecture also suggests that the different meanings of 'existence' are given by their different proofs, respectively, and he outlines six possibilities (a summary can be found in *LPM* 40-41). For example, he outright states the meaningfulness of infinite constructions (e.g. the possible never-ending construction which is given by rules that 'point beyond themselves'), even though he does not mention infinite proof procedures specifically (e.g. induction). Other examples he gives include non-constructive proofs such as indirect proof (his third example is *reductio ad absurdum*).

Wittgenstein, too, rejects the idea that a proof must be 'finite', if this is taken to mean that it can't prove something about an infinite set (although, of course, this must be understood with proper Wittgensteinian qualifications – it doesn't give credence to the actual infinite and Wittgenstein argues for a particular interpretation of a 'proof' insofar as it relates to an infinite set or series). Like Waismann, Wittgenstein is clearly aware of the legitimacy of infinite proof procedures and non-constructive methods of proof.

(ii) The Meaning of Mathematical Concepts.

In 'Number' and 'The Structure of Concepts', Waismann suggests that both formalist definitions (e.g. of the natural numbers) as well as formalist axioms (e.g. of geometrical concepts), respectively, are insufficient to fully determine the meaning of mathematical concepts. Ultimately, Waismann suggests, it is the use of these concepts *outside* mathematics that makes them into fully meaningful concepts (*LPM* 56; 135-136).

In order to argue for this position, Waismann distinguishes between formal properties of a concept and material ones (also referred to as the concept's 'structure' and the meaning it derives from a specific 'interpretation', respectively). The formal properties help determine a concept's meaning, but fall short of specifying its *full* meaning. Waismann uses geometrical concepts as an example of this. For example, axioms constrain the meaning of various geometrical terms. 'Point', 'straight line', etc. are constrained in different ways depending on the different systems of axioms corresponding to the different geometries to which they could possibly belong. A given system of axioms constrains the possible meaning of its terms to a certain set of possible interpretations. The possible interpretations that could satisfy a concept of a given axiomatic system is referred to as that concept's 'structure'. It is only through specifying a specific interpretation through appeal to specific empirically determinable concepts (e.g. interpreting 'straight line' to mean 'path of a light ray') that a concept obtains its full meaning. In this way, a formal characterization is supplemented by interpretations in the form of concepts that are themselves determined, although not necessarily in a straight-forward manner, by experience (Waismann suggests this ultimately importantly relates to ostensive definitions) (*LPM* 135-136).

Waismann uses this distinction in addressing a couple of philosophical problems. First, he examines measuring time, and claims, similar to Wittgenstein (*BB* 26), that the problem involves confounding spatial ideas of measurement with temporal ones. In giving a more detailed analysis, he uses the formal/material distinction to bring out what is shared and unique to the concept of 'equal' when applied to spatial and temporal domains (i.e., they have shared formal properties but different material ones; Waismann says an additional criterion is required to give them meaning) (*LPM* 138-139). Secondly, on a related topic, Waismann discusses the problem of measuring lines in physical space in comparison to visual space. This leads him to contrast the formal properties of 'equal' in the two domains. Moreover, in the course of the examination he gives an example also given by Wittgenstein (it concerns a segment of a circle *appearing* straight when it is known *to be* curved) (*LPM* 141; MS 107, 164-165; *RFM* I, §§96-8) and ultimately even speaks of the 'inexactitude' or 'blurredness' that is an 'element' of the visual field (*LPM* 142). Wittgenstein similarly addresses problems that arise with not having recourse to a distinction between appearance and reality when describing the visual field (MS 107, 29), and argues for the 'inexactness' (*Unbestimmtheit*) that is part of the logic of the visual field (MS 107, 171). It is in the context of examining the possibility of the phenomenological language that Wittgenstein examines these problems and comes to see them as insurmountable obstacles to the phenomenological project. Ultimately, Wittgenstein goes farther than Waismann and argues that the vagueness or inexactitude of the visual field, without the possibility of reference to an external standard (such as ordinary discourse has recourse to), is what makes the project of creating a phenomenological notation impossible (whereas Waismann suggests *some* formal language could still possibly capture the vagueness – *LPM* 141-142).[2] While Waismann's discussion of these topics does not take place in the context of developing a phenomenological language, Wittgenstein's influence on Waismann when it comes to these topics is readily apparent.

In the early intermediate period, culminating in *The Big Typescript*, Wittgenstein sees the meaning of symbols employed in mathematics as determined wholly by their employment in the rules of the respective calculus to which they belong. This 'calculus conception' of mathematics importantly influences his philosophy of language: words and sentences are seen also to belong to a calculus and their meanings also determined in this way. Beginning with his reflections in the intermediate period (*BT* 533; 566)[3] and continuing on into his later work (*RFM* 257; *LFM* 33), Wittgenstein comes to think that in the case of mathematics an extra-systemic application is required to give mathematical symbols meaning. Beginning in the intermediate period, reflections on primitive languages forced Wittgenstein to rethink the calculus conception in the philosophy of language (Engelmann 2013, 154-160; Schroeder 2013, 155-160), and reflections on rule-following and mathematical rule-following in particular undermine the calculus conception in the philosophy of mathematics (Rodych 2000, 300-302). A calculus is only fully-meaningful *mathematics* when its symbols (numbers, function symbols, etc.) are employed in empirical descriptions and calculations. Without this empirical application, the calculus, while still consisting of rules, is akin to a game; what makes it essentially *mathematical* is lost. This clearly anticipates Waismann's position.

(iii) Equations and Tautologies

In his lecture 'Equation and Tautology', Waismann argues for the distinctness of equations and tautologies and argues that the former can't be reduced to the latter. Indeed, the proof of a tautology serving as the translation of an equation is dependent on the equation itself and thus does not serve as proof of the truth of the equation (*LPM* 64-65). The different uses of equations and tautologies and a difference in their 'operational aspect' clearly display the impossibility of translating or reducing one to the other (*LPM* 70-71). The distinction between equations and tautologies was already being made by Wittgenstein in the *Tractatus*: equations were considered 'pseudo-propositions'

(*TLP* 6.2), whereas tautologies were considered limiting cases of a proposition ('lacking sense') (*TLP* 4.466; 4.461; for further details see Frascolla 1994, 27-28). In the intermediate period, Wittgenstein becomes more concerned to emphasize their distinctness as well as the former's irreducibility to the latter (e.g. *PR* 142). In particular, he argues against the logicist attempt to express equations as tautologies (*WVC* 35, fn.1). Thus, in this respect, Waismann's point originates with Wittgenstein. Moreover, Wittgenstein also argues for equations being substitution rules, in the context of discussing the role of equations and rules more generally (*WVC* 156).

(iv) Infinity.

Since Waismann deals with the topic of infinity in considerable detail and with notable clarity in his lecture 'Infinity and the Actual World', it is worth devoting more space to discussion of this lecture. In what follows we shall examine some of the important similarities and minor differences between Waismann's and Wittgenstein's approach and arguments as they relate to the topic of infinity. At this point, as we shall see in more detail below, it is apt to note that Wittgenstein's influence can be seen, sometimes in pretty subtle ways, throughout this lecture; for example, discussion of whether 'infinite' and 'finite' can be profitably defined (*LPM* 99-101), concern about understanding an infinite set as a totality (*LPM* 100-101), reflections on whether an unending rows of stars can be justifiably asserted to be infinite (*LPM* 113-115), and problems of 'time and paper' in relation to listing the members of an infinite set (*LPM* 118), are all, among other things, anticipated by Wittgenstein (*PR* 158; *PR* 146-147, 166; *PR* 169; *PR* 160-161, respectively). Sometimes, as in the case of Waismann's consideration of unending rows of stars, Waismann adds his own insights or distinct way of tackling the issue.

Unlike Wittgenstein, Waismann, in his discussion of the topic, addresses some of the actual physical theories related to infinity and time and space, even dealing with some of the more technical mathematical components of the theories. For example, in the context of discussing whether space and time are infinitely divisible, Waismann discusses numerous topics including, but not limited to: the (im)possibility of division of matter, electricity and energy, primary and secondary qualities and how this relates to the history of physics (up until Waismann's time), and how the aforementioned topics can be understood by appealing to some of the technical elements of relativity theory. In the course of the lecture, he also discusses matters of cosmology and empirical work dating everything from the age of rocks on earth, to the age of the earth itself, meteorites, the solar system, elements, the galaxy, and the universe (*LPM* 74-78). Wittgenstein rarely discusses empirical facts such as these. Waismann also gives a brief presentation of some of the technical details of set theory and considers Zeno's famous paradox involving the tortoise. Unlike Wittgenstein, Waismann often does not take a clear stand on the theories or ideas he is presenting, but merely lays them out, often in an *explanatory manner*, as possible (often opposing) answers to questions related to the infinite and finite.

When considering the possibility that the universe is infinite, Waismann points out that this would require that there are questions which are in principle undecidable. And this would call into question the Law of Excluded Middle. Waismann does not go into what specific questions he has in mind, but seemingly any empirical claim about the infinite would do. And the reason for their undecidability would come from the fact that such statements cannot be empirically verified (even though this is not explicitly stated by Waismann). Thus, according to Waismann, the infinite nature of the universe would clash with 'ordinary logic' (*LPM* 77-78).

This shares similarities with, but is different from, Wittgenstein's position. Wittgenstein's position in the intermediate period is that 'infinite' does not refer to a quantity (*WVC* 228; *PR* 157, 162; *PG* 463). It is categorially distinct from anything finite (*WVC* 228; *PR* 157-158) and remains the property of a law (*PR* 313-314). By examining a variety of seemingly empirical statements involving the concept of the infinite, Wittgenstein concludes that, in all of the cases, there would be no experience that would verify them (*PR* 167-168; 306-307). The meaning of a proposition being its method of verification (*WVC* 47; *PR* 200), this means that such a proposition is senseless (*WVC* 227; *PR* 306). Wittgenstein does allow the possibility of meaningful natural laws that involve the infinite (e.g. the Law of Inertia), but this is precisely because such statements have a different logical role from empirical ones; they are akin to rules. Moreover, he does make reference to the Law of Excluded Middle, but not as itself a reason for rejecting *a priori* a claim about the infinite nature of the universe, but rather as supplementary to the verification principle and as a way of delimiting types of propositions. The fact that a proposition can be falsified but not verified shows it is a proposition in a different sense of the word (*PR* 307).

This brings us to the point concerning the inapplicability of the Dedekind definition of infinity to the physical world. Waismann explains the Dedekind definition of the infinite as follows: 'a set is infinite if it is reflexive, i.e. if it can be mapped onto a subset that is not identical with the whole set' (*LPM* 112).[4] According to Waismann, the inapplicability is because of a fundamental difference between 'a proof' and 'a verification'. This distinction is of central importance to Wittgenstein, too, although it does not occupy a central place in Wittgenstein's thought when discussing the Dedekind definition of infinity. An essential part of Wittgenstein's thought from his early through to his later work is the categorial distinction between mathematics and empirical disciplines. Mathematics is invention whereas the empirical disciplines involve discovery. Insofar as mathematics is invention, to be able to give a 'description' of a proof is just to give the proof itself. This is in contrast to a discipline that importantly involves discovery about the world, where a description of something does not mean that is indeed how the world is (or that any 'description' is indeed verified).[5]

The inapplicability of the Dedekind definition to reality is indeed anticipated in Wittgenstein although the distinction between the *a priori* methods of mathematics and the *a posteriori* methods of the empirical scientific disciplines is *at best* only *part* of what Wittgenstein has in mind when he criticizes the Dedekind definition of the infinite or elements of Cantor's set theory. Rather, it would appear that it is the categorial divide between the infinite and the finite and, in turn, the impossibility of understanding one in terms of the other (*WVC* 70; 232; *PR* 158) or of being able to avoid circularity in one's definition (what Waismann also discusses – *LPM* 99-102) (*PR* 151; *PG* 464) that limits the definition's usefulness to either domain. Connected with this, it is useless to try to use the definition as a decision procedure (*PG* 464). Waismann is certainly right that the definition can't adequately apply to reality, but he could have gone farther, as Wittgenstein did, and suggest that the definition takes one no further in understanding the notions of 'finite' and 'infinite' in *either* domain. Indeed, the definition presupposes the very understanding of what it is meant to explain. Insofar as the definition can even be understood in mathematics it is not because it importantly explains something opaque (i.e., it does not involve a discovery), but instead, in the context of these mathematical practices, it can be reinterpreted using mathematical techniques (e.g. functions, 1-to-1 correlation, and the diagonal method). In this way, any meaningful mathematical definition is making reference to new concepts and not thereby escaping the various problems connected with offering definitions of the concepts of the infinite and the finite that Wittgenstein explains.

Waismann also calls into question the axiom of infinity. After consideration of some technical matters related to the axiom and interpretations of the axiom, he rightly concludes that the axiom of infinity can't be understood as an empirical claim; however, motivated by an incorrect view about the natural number series (criticized in section 3.(i) below), he concludes that 'what is behind the existence of infinity is the desire to…refer to the objective existence of a set, no matter whether we can or cannot construct it' (*LPM* 121). Wittgenstein, of course, would agree that the axiom is not dependent on the world. But, regardless of the details of the mathematical use to which the axiom can be put (which Waismann considers – *LPM* 117-122), it is apparent that Wittgenstein would reject the idea that the axiom of infinity 'refer[s] to the *objective existence* of a set' (*LPM* 121). Wittgenstein eschews Platonism and thus, whether taken empirically or as an *a priori* claim, the 'objective existence of a set' giving meaning to, or being importantly related to, the axiom of infinity. Instead, the axiom does simply relate to the possibility of a construction: there is an infinite set *in so far* as no matter how many sets are constructed further can still be constructed (in accordance with the rule, which defines the set). And, as this can be seen as a translation of the set of natural numbers into set theory, like the set of natural numbers itself, it has no last member. 'Last member of the set' has no meaning. This would obviously be in line with Wittgenstein's general criticisms of mathematics as a descriptive activity, his criticism of the actual infinite, and his interpretation of the legitimate use of 'infinite' (as relating to the possibility of endless construction in accordance with a rule).

Finally, Waismann, in the concluding section of the lecture, in order to examine the concept of time in relation to the infinite, imagines a machine that calculates $\sqrt{2}$ in reverse from an infinite time in the past. That is, starting at an infinite time in the past, the machine finishes calculating $\sqrt{2}$ in reverse with the first digits of the series ending the calculation. In the context of this general discussion, he makes several points that are clearly inspired by Wittgenstein. For example, he distinguishes between physical and logical possibility (*LPM* 124), notes that infinity is not a number (*LPM* 123), and argues that an infinite sequence is *constructed* and must be constructed in a certain direction (is 'unidirectional') (*LPM* 123-24). He also uses this example seemingly to object to Wittgenstein. Waismann's objection is a clear reference to a passage of Wittgenstein:

> Let's imagine a man whose life goes back for an infinite time and who says to us: 'I'm just writing down the last digit of π, and it's a 2'. Every day of his life he has written down a digit, without ever having begun; he has just finished.
>
> This seems utter nonsense, and a *reduction ad absurdum* of the concept of an infinite *totality*. [*PR* 166]

Waismann objects to the idea that the machine having never started serves as a *reductio ad absurdum* of the actual infinite.[6] However, neither Waismann's argument against his own example nor his subsequent considerations of the infinite in relation to time serve as refutations of Wittgenstein.

First, it should be noted that Waismann's example is not identical to Wittgenstein's. In Waismann's case, the machine 'never started' in the infinite past calculating $\sqrt{2}$ in reverse. Wittgenstein's example involves a man whose life goes back an infinite amount of time and who every day has written down a digit of π and finishes without ever having begun. In his case, Wittgenstein is objecting, at least in part, to the absurdity of the idea that π should end (made comical with the idea that the last digit is a 2). This would obviously be a factor in Waismann's example too (i.e., that 'infinite' doesn't mean the reverse calculation began with the final digit of the series – there is no such thing), although, as Waismann himself points out, in his case, it is not even intelligible to imagine the machine calculating in reverse at all. In Waismann's example, the unidirectional nature of the algorithm creates the problem for the meaningfulness of the example.[7] Aside from the specific example, Waismann's subsequent consideration of the concept of the infinite in relation to time calls into question the successfulness of Wittgenstein's argument (albeit, we think, unconvincingly).

In addition to the absurdity of the idea of reaching the final digit of the series, Wittgenstein wishes to emphasize that in order to make the example intelligible it is necessary to imagine a person requiring an infinite amount of time to undertake the calculation. This agrees with another of Wittgenstein's claims that the infinite can only be understood in terms of itself (*PR* 158). But if the person started an infinite time in the past, then he would never have begun. Waismann, at the end of the lecture, in his characteristic fashion, *wonders* whether the idea of the infinite past is intelligible and suggests, even if it is, that it then must be something essentially different from the infinite future. The infinite past, if intelligible, must be a 'closed totality'. We think it apparent that, regardless of whether the infinite past is unintelligible for Wittgenstein, Wittgenstein would have rejected the idea that it can be understood as a 'closed totality'. And this certainly makes the idea of something having *begun* an infinite time in the past unintelligible which, in turn, insofar as it is necessary, makes the notion of an infinite series (conceived as a completed totality, with recourse to the infinite past), even one constructed in the right direction, unintelligible. Waismann presents no decisive argument against this.

Waismann largely agrees with Wittgenstein regarding the concept of infinite. As has been shown, Wittgenstein's influence is felt in both the overall view of the infinite Waismann presents, as well as the details of his presentation (which include ideas or arguments he objects to in Wittgenstein's thought). Aside from some relatively small disagreements with Wittgenstein's views, the distinctiveness of Waismann's approach is seen in his exposition of at least some of the technical details of work in physics and mathematics, as well as his more tentative presentation style, especially when presenting conflicting views or theories.

3.

Waismann, in his later years, explicitly disagrees with Wittgenstein on two points. He objects to the view that mathematics is conventional and he protests that the meaning of a mathematical proposition cannot be due to its proof since mathematical conjectures are patently not meaningless. Let us consider the two issues in turn.

(i) Conventionalism

In his *Introduction to Mathematical Thinking* (ch.16) Waismann emphatically propounded Wittgenstein's view that a mathematician is an inventor, not a discoverer (*RFM* I-168). In 1954 he is more inclined to hold the opposite view (*LPM* 29-34). Considering the elementary example of natural numbers, he explicitly contradicts Wittgenstein's conventionalist ideas and says, in a Platonist vein:

> We *generate* the numbers, yet we have not choice to proceed otherwise. There is already something there that *guides* us. [*LPM* 33]

Again:

> The endlessness of the number series, far from being the result of adopting an arbitrary convention, is one of the first and most significant discoveries made right at the very beginning of mathematics. [*LPM* 32]

At this point he applauds Brouwer's idea of a fundamental intuition, but prefers to call it an *insight:* 'the insight, namely, that there is an open, endless possibility of going on' (*LPM* 33).

Moreover, Waismann reflects that mathematics is essentially incomplete, always pointing beyond itself towards further developments. And he finds mathematicians' Platonist feelings supported by the surprising interrelations cropping up between different parts of mathematics (*LPM* 30-31).

Waismann reverts to his criticism of Wittgenstein's conventionalist views in 1959, with reference to a discussion of possible language games of numbering, with finite or infinite number systems, in the *Brown Book* (just come out in 1958). Waismann writes that:

> the assumption of an endless number series is not merely a convention, an arbitrary rule, like a rule in a game of cards which we are free to accept or reject just as we please. [*LPM* 121]

Rather, a child learning to form the series 1, 1+1, 1+1+1, … makes the *discovery* of the potential infinity of the series (*LPM* 119), after which the opposite becomes unimaginable:

> If I try to visualise a situation in which the numbers come to an end, I feel dizzy: it is as if I try to think something that is unthinkable. [*LPM* 121]

However, this attack on Wittgenstein's conventionalist ideas is hardly convincing. For one thing, conventions need not be *arbitrary* rules, for it may be practically necessary to have some rules fulfilling a certain purpose. Thus the Highway Code is undoubtedly a system of conventions, but far from being arbitrary its rules are largely determined or justified by their purpose to facilitate the safety and efficiency of road traffic. For another thing, even with an arbitrary (aspect of a) convention (e.g., to drive on the left, rather than the right side of the road) individuals do not find themselves 'free to accept or reject them just as they please'. In a given social situation, conventions are often experienced as inexorable facts. What Waismann describes is not the introduction of our system of natural numbers, but an individual being taught that system. Given that one has learnt that system, one can of course be said to *discover*, or work out for oneself, its implications, most notably, its openness: that one can go on and on adding 1 without ever coming to an end. Similarly, a young chess player may discover that one can mate with only a rook, but not with only a knight. Yet that does not mean that the rules of chess have been discovered, rather than stipulated. And the same applies to the meanings of our number words. Again, given our infinite number system, it is of course impossible to make sense of the idea of a last number, but that is not to say that a finite number system is inconceivable. Indeed, some such finite systems are sketched by Wittgenstein in the *Brown Book* (*BB* 91ff.).

Is the construction of an endless number system the result of a 'fundamental insight'? That would presumably be the insight that an operation (such as *+1*) can be repeated again and again and again — as long as one likes; and added to that the consideration that — abstracting from human agency — the possibility of further steps will *never* be blocked. Although every number is finite, numbers never run out. However, that kind of potential infinity is not a distinctive feature of counting: it doesn't require the natural numbers. A knock can be repeated again and again; and abstracting from human agency, I can imagine that every knock could be followed by another knock. Thus, whereas a practical number system may well be finite (e.g. an abacus with 20 beads (*BB* 91)), Waismann's 'fundamental insight' of potential infinity could even be had by someone who hasn't learnt any number concepts yet. The 'insight' (or 'intuition') about the potentially endless continuation of a series can also be illustrated by a row on non-numerical marks, such as: ########... (cf. *PI* §214).

Waismann objects:

> Of course, I can write down as many 2's, or as many crosses as I like, but there is nothing *in* the series which, when I stop somewhere, points *beyond itself:* and what assures me in this case that I can go on forever is not anything connected with the *formation* of the series, but something *extraneous* to it — for instance, the infinity of space. [*LPM* 119-20]

Not so, both cases are on all fours. Of course a sequence such as '1, 1+1, 1+1+1' can be said to 'point beyond itself' in so far as we easily agree on the most natural continuation, but so we do in the case of an iterating series such as '2, 2, 2', or even '###'. In both kinds of cases, a simple formation rule can be perceived 'in' the series; extraneous factors don't come into it.

As Wittgenstein discussed in great detail in his rule-following considerations, it does indeed feel as if there was something to guide us — 'rails invisibly laid to infinity' (*PI* §218) —, but Platonism is only a naïve metaphorical expression of our experience, not a plausible explanation (*PI* §219). Nor does the unforeseen dovetailing of different parts of mathematics support Platonism. Why should a mathematical technique and its results not occasionally be found to fit fruitfully with other mathematical techniques?

(ii) Mathematical conjectures

It is not at all unreasonable for Waismann to balk at Wittgenstein's stark 1929 assertion that only proof gives meaning to a mathematical proposition (MS 105, 59; *PR* 183, 192). Surely, the implication that mathematical conjectures are altogether meaningless is absurd. Mathematicians that consider and have an opinion on Goldbach's conjecture do not talk nonsense (*LPM* 37). However, here Waismann's disagreement with Wittgenstein only anticipates the latter's own subsequent qualms and qualifications.

Considering Wittgenstein's 1940s remarks on the issue, a much more plausible picture emerges.[8] Wittgenstein holds on to the idea that proof is the principal source of mathematical meaning, but (in line with his move to the general view that meaning is use) he now regards legitimization by proof as an aspect of our use of mathematical propositions. Consequently, his position has the flexibility to give a plausible account of conjectures as well. The following points offer a persuasive response to Waismann's concern.

(a) Proofs explain how *a proposition is true.* Wittgenstein compares a mathematical proof to a jigsaw puzzle (MS 122, 49v). Indeed, sometimes he regards actual jigsaw puzzles as mathematical problems (*RFM* 55-7, *LFM* 53-5). In such a case, the conjecture to begin with would be something like: 'These 200 pieces can be assembled to form a rectangular picture of a mountain'. Here it is obvious that the proof — putting all the pieces together in the right way — would do more than establish the truth of the conjecture. It would not only convince us *that* the pieces can be put together to form a picture of a mountain, it would show us *how* they fit together (cf. *RFM* 301; 308). Thus the proof does not only verify a proposition, one can say that it gives us a much fuller understanding of it, showing us what exactly that proposition means.

(b) Proofs account for mathematical necessity. Mathematical propositions are characterised by a necessity that must be established by a demonstration. If that is correct, then the proposition that there is no greatest prime should be rendered more appropriately as: 'There *can't* be a greatest prime' —indicating the necessity we attribute to a proposition when we take it as a piece of mathematics. Then, of course, the meaning of the modal verb in that sentence needs to be understood. One is entitled to ask: 'What do you mean by "can't"?' And the answer that gives meaning to the 'can't' is that it follows from such-&-such considerations — the proof — that there is no greatest prime.

In this way, the mathematical proposition, when taken as such: as a demonstrably necessary truth, refers us to its proof (cf. *RFM* 309).

(c) Only proof shows a conjecture to be consistent and hence, ultimately, meaningful. Consider that, for all we know, a mathematical conjecture could be proven false (*RFM* 314d), that is, shown up to be inconsistent. Yet if something is inconsistent, or contradictory, it doesn't make sense: it cannot be understood: there is nothing to be understood. But then, given that we cannot even know whether a mathematical conjecture is fully *understandable* (and not nonsense), then *a fortiori* we cannot claim to *understand* it. A sentence that as far as we know may be inconsistent, i.e. nonsense, can hardly be said to have a clear sense for us. This, again, vindicates Wittgenstein's view that a proof gives meaning to a mathematical proposition.

(d) Proof affords normative legitimacy, which is a crucial part of the meaning of a mathematical proposition. One of Wittgenstein's key ideas is that mathematical propositions are akin to grammatical norms (*RFM* 162, 169, 199, 320). For a proposition to have mathematical sense it must not only have the contents, but also the *normative status* that characterises mathematics (*RFM* 425): it must be acknowledged as a grammatical rule, which obviously an unproven conjecture is not. Nothing unknown can fulfil a normative function (*PR* 143; 176). Therefore, even if we assume that it is possible to find a proof for Goldbach's conjecture — that the potential for such a proof is already there —, until it has actually been produced Goldbach's conjecture will not be accorded the status of a grammatical rule. That is, until then it cannot be accorded the full status of a mathematical proposition.

(e) Mathematical conjectures (or problems) can have a fairly clear sense, but it's not a genuinely mathematical sense. In as much as we understand their content, we take them as *empirical* propositions, corresponding to, but crucially different from, the mathematical proposition we would like to establish by proof. For example, in some sense we understand the idea of the construction of a heptagon with ruler and compass (which is impossible). But that is only because we have a clear empirical idea of a heptagon, that is, we can easily think of a 7-sided figure whose sides and angles when measured come out as all the same. So we are inclined to understand the problem as that of drawing such a figure. But in fact that is not the mathematical problem. The mathematical problem is that of finding a mathematical *construction* of a heptagon, analogous to the way one can give a mathematical construction of, say, a pentagon. The result of such a construction would of course also fulfil the empirical criteria (that measurement shows 7

sides and angles, all roughly equal), but that is not enough. As a solution to a geometrical problem, it is essential that the figure be arrived at, step by step, in a regular, repeatable and teachable way, using only ruler and compass. We are looking not just for a shape, but for a very specific way of producing it. Yet this specific way of producing such a shape is something we are unable to describe. We have no idea of such a geometrical construction; and therefore, our talk of such a construction — the conjecture of such a construction — has no clear mathematical sense; even though it has a very straightforward empirical sense, derived from empirical measurements of drawn figures (cf. *LSP* 572).

As another example, consider Goldbach's conjecture that every even number is the sum of two primes. Don't we understand that? — Again, Wittgenstein's response is that without a proof we have of course some understanding of it, but only as an empirical generalisation; meaning that for any even numbers we will ever consider we will be able to find two primes adding up to that number. That is an empirical hypothesis inductively supported by our evidence to date; but not a mathematical proposition (cf. *RFM* 280-1).

Infinity in mathematics is always the endless applicability of a law (cf. *PR* 313-14; *RFM* 290b). Hence, where (as yet) we have no law, no mathematical rules that can be understood to have an endless applicability, we cannot meaningfully speak of mathematical infinity. So we cannot as yet make sense *of the infinite scope* of Goldbach's conjecture.

4.

In spite of his personal disenchantment with Wittgenstein, in mathematics Waismann remained very much a Wittgensteinian philosopher. Most of the ideas he propounded in his 1950 lectures can be traced back to Wittgenstein's philosophy of mathematics. In some cases, Waismann used his considerable skills as a lucid writer and analytical thinker in order to present Wittgensteinian ideas in a more systematic manner (as he had already done in *LSP*), occasionally fleshing them out in more illuminating detail. Only on two substantive issues Waismann argues expressly against Wittgenstein. His later misgivings about Wittgenstein's conventionalism, however, are not very well supported, whereas his rejection of Wittgenstein's 1929 and early 1930s 'meaning through proof' doctrine is perfectly sensible, but (as we argued) not in conflict with Wittgenstein's more carefully qualified views from the 1940s.

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[1] Cp. Grassl 1982, 21-23, for a slightly different list of the main tenets of Waismann's philosophy of mathematics.
[2] For more details about this project and its ultimate failure, see Engelmann 2013, 28-43.
[3] The seeds of this idea can already be seen in the early intermediate period when Wittgenstein talks of a calculus being 'serious business' because of its possible application(s) to 'everyday life' (*WVC* 170). This stands in contrast, however, to his more common discussion (at this time) of a mathematical calculus being an 'application of itself' (and seemingly fully meaningful) (*PR* 130-132).
[4] Wittgenstein similarly explains the Dedekind definition of an infinite class as 'saying that it is a class which is similar to a proper subclass of itself' (*PG* 464).
[5] It would seem that only Waismann employs the term 'verification' (as opposed to 'method of verification') in order to make *this* distinction. Wittgenstein employs the verification principle to make logical distinctions between types of propositions and does not use the term 'verification' to only refer to empirical verification. According to this understanding, mathematical propositions also have 'verifications' (i.e., different types of proofs). What is true: the method of verification of an empirical statement relates importantly to the world, whereas the method of verification of mathematical propositions does not. A large part of Wittgenstein's work in the intermediate period and onwards is devoted to arguing for this point (often without any use of the term 'verification' at all – indeed the idea of 'methods of verification' *at most* complements the idea of categorially different methods employed by mathematics and the empirical disciplines). Despite the different use of terminology, Wittgenstein's work obviously anticipates Waismann's thought on this point also.
[6] Aside from a remarkable affinity between the wording of the two arguments, it also should be noted that the arguments are presented as arguments against the actual infinite (even though Waismann rejects this conclusion). Thus, it is virtually impossible that Waismann was not following Wittgenstein's ideas in this context.
[7] Although Waismann would deny it, it is not clear to us that his argument cannot also serve as a refutation of the actual infinite. For if one imagines his example as one involving an algorithm constructed in the right way, it would appear that in order to speak of a completed infinite series one must have recourse to the infinite past. Yet this itself is unintelligible (it conflicts with other conceptual truths regarding calculation – e.g. our use of 'begun'). And, in addition, the idea of an infinite series as a completed whole is undermined by numerous other comments made by Wittgenstein (e.g. *PR* 164; 167).
[8] For a more detailed account, see Schroeder 2012.