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On possible psychophysical maps: I. Quadratic transformations

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A certain class of quadratic transformations can be used to map multidimensional physical spaces into multidimensional perceptual spaces; their numerical representations change under various coordinate transformations in both spaces.

The problem of relating two or more multidimensional spaces arises both in scaling and in factor analysis. The technology for fitting such transformations optimally in some sense (usually a least squares sense) is reasonably well understood for the group of affine transformations and its subgroups: similarities, rigid motions, and rotations ("Procrustes" problems, e.g., Ahmavaara, 1957; Borg, 1978; Cliff, 1966; Fischer & Roppert, 1964; Gower, 1975; Green, 1952; Kristof, 1964; Mosier, 1939; Schönemann, 1964, 1966; Schönemann & R. M. Carroll, 1970). A defining characteristic of affine transformations is that they carry parallel lines into parallel lines. More recently, a number of psychophysical studies have suggested the need for more general transformations that are not subject to this limitation. This paper summarizes some preliminary results on one type of such more general, "transaffine" mappings.

For a concrete illustration, consider the results of a recent psychophysical study by Krantz and Tversky (1975). They presented 17 rectangles of various shapes and sizes in pairs to a number of subjects. The experimental task was to indicate on a scale of 1-10 how "dissimilar" the two stimuli appeared in each presentation. The main objective of the study was apparently to test ordinally a so-called "decomposability" condition (see Beals, Krantz, & Tversky, 1968), which is a generalization of the well-known family of Minkowsky metrics. Krantz and Tversky tested it for two pairs of derived physical measures that were defined in terms of the height (h) and width (w) of the rectangles:

$$T_{q_1}: E \rightarrow S: u = \log h, v = \log w \tag{1}$$

$$T_{q_2}: E \rightarrow S: a = u + v, s = u - v, \tag{2}$$

where E denotes the physical, experimentally controllable space that is represented by a (possibly bounded) region in $Re \times Re = Re^2$, and $S \subset Re^2$ denotes the subjective space. The second pair, a,s, can be interpreted as log transforms of area (hw) and shape (h/w). Krantz and Tversky found decomposability violated for both Equations 1 and 2. This means that neither pair can give rise to Minkowsky metrics monotone with the observed dissimilarities. Krantz and Tversky then processed the observed dissimilarities with two multidimensional scaling programs. They accepted a fit of the INDSCAL model (J. D. Carroll & Chang, 1970) in two dimensions, but without testing it with any stringency. If the assumptions of the model are met, then the coordinate axes in S, that is, the "psychological dimensions," are uniquely identified by the data.[1] The main result of this scaling analysis was a distortion of the design configuration that seemed to corroborate the results of the ordinal analysis: Since the stimuli had been defined so that successive levels on both dimensions (h and w) corresponded to constant length ratios, the physical design configuration in E was a square, rotated through 45 deg. The square was distorted into a trapezoid in the subjective space S (Krantz & Tversky, 1975, Figure 1). This means that subjective shape varies as a combined function of physical area and shape.

As Schönemann (1977) pointed out, the results of this scaling analysis, if valid, also imply the definition of a third pair of subjective variables that must satisfy decomposability and must yield euclidean distances monotone with the observed similarities, as a direct consequence of the claimed successful euclidean embedding. The mapping cannot be linear (as Krantz & Tversky, 1975, implicitly assumed when they investigated Equations 1 and 2), because affine transformations preserve parallelism. Inspection of the two figures shows that the map can be written:

$$T_{q_3}: E \rightarrow S: a^* = a, s^* = s(1 + ca), \tag{3}$$

where c is a (small) constant. The "starred" quantities are the psychological dimensions expected to reproduce the order of the observed dissimilarities, and the "unstarred" quantities are the (derived) physical variables defined in Equation 2. This mapping reflects the main finding of the ordinal and scaling analysis: Perceived shape increases with area.

In the literature, one finds similar, although of course not identical, maps in other studies of this type. For example, in the experiments reported by Eissler and Roskam (1977) and Roskam (1979), the stimuli were vectors of various lengths and orientation. If they are interpreted as diagonals of rectangles, then the design configuration in the physical space is again square. This design is mapped into a general quadrilateral in the subjective perceptual space. The map from E to S is approximately of the form,

$$T_{q_4}: E \rightarrow S: u = ax + cxy; v = by + dxy, \tag{4}$$

where x and y are the independent physical variables, and u and v are the subjective perceptual dimensions.

QUADRATIC TRANSFORMATIONS

The psychophysical maps, Equations 3 and 4, are special cases of a larger class of transformations that can be written:

$$T_q: E \rightarrow S: y_k = \zeta' G_k \zeta, \ k = 1,q, \tag{5}$$

where G_k is a $(p+1) \times (p+1)$ symmetric matrix, p is the number of independent variables, q is the number of dependent variables, and

$$\zeta' = (\xi', 1), \text{ and } \xi' = (x_1, x_2, \ldots, x_p) \tag{6}$$

contains the p physical variables x_i (i = 1,p).

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Since the y_k are quadratic functions of the x_i, such maps will be called "quadratic transformations."[2] The exact type depends on the coordinates g_{ijk} in G_k. For fixed y_k, Equation 5 defines q quadrics. On setting some of the elements in G_k to zero, or by constraining them in some other way, a number of familiar special cases can be obtained, including translations, rotations, rigid motions, and affine transformations. For the present purpose, the most interesting case arises when the transformation from E to S is truly "transaffine," as it was in Equations 3 and 4.

For what follows, it will be convenient to distinguish between "original" physical variables x_j^*, which are those actually used in the expriment, and "derived" physical variables x_j, which result from the x_j^* by applying some suitably chosen linear transformation W so that T_q takes on a simpler, more easily interpreted form. With this notation, Equation 5 can be written, when $p = q = 2$:

$$y_k = a_{k_1}x_1^* + a_{k_2}x_2^* + b_{k_1}x_1^{*2} + 2c_kx_1^*x_2^* + b_{k_2}x_2^{*2} + d_k, \quad (k = 1,2) \tag{7}$$

Of particular interest are those cases in which a linear transformation $\xi = W\xi^*$ can be found that eliminates the squared terms, so that $b_{k_1} = b_{k_2} = 0$. In this case, T_q takes on the simpler form,

$$y_k = \alpha_k{}'\xi + \xi'Q_k\xi + d_k, \xi = W\xi^*, k = 1,2, \tag{8}$$

where

$$Q_k = (q_{ijk}), q_{ijk} = c_k \text{ (0) if } i{\neq}j \text{ } (i=j), \alpha_k{}' = (a_{k_1}, a_{k_2}), \tag{9}$$

and where $W = (w_{ik})$ is the linear transformation that carries the original physical measures, x_k^*, into the derived physical measures, x_k. For example, in the Krantz and Tversky (1975) study, the x_k^* are (log−) height and length and the x_k are (log−) area and shape. Equations 8 and 9 can be written:

$$\eta = A\xi + 2\gamma x_1x_2 + \delta, \xi = W\xi^*, \tag{10}$$

with $\eta' = (y_1,y_2)$, $A = (a_{kj})$, $\gamma' = (c_1,c_2)$, $\delta' = (d_1,d_2)$. The rest of this paper will be limited to such planar maps.

GEOMETRIC INTERPRETATION

The slightly different formulations (Equations 7-10) convey different geometrical aspects of such mappings, which are most easily described in the derived space. On forming the difference $c_2y_1 - c_1y_2$ in Equation 10, one finds:

$$c_2y_1 - c_1y_2 = p_1x_1 + p_2x_2 + k_1, \tag{11}$$

where

$$p_1 = c_2a_{11} - c_1a_{21}, p_2 = c_2a_{12} - c_1a_{22}, k_1 = c_2d_1 - c_1d_2. \tag{12}$$

Thus T_q establishes a 1:1 correspondence between the family of parallel lines $p_1x_1 + p_2x_2 =$ const in E, and the family of parallel lines $c_2y_1 - c_1y_2 =$ const in S. One further finds from Equation 7 that all lines parallel to the coordinate axes in E, $x_k =$ const, map into a family of straight lines with varying slopes and intercepts in S. The lines $y_k =$ const in S are images of hyperbolae in E. In general, lines in E will map into conics in S. As long as the elements c_k in γ are "small" relative to the a_{kj} in A, T_q will not depart too much from an affine transformation.

The quadratic map, T_q, is not unrestrictedly invertible. To find the singularities, we set the Jacobian to zero:

$$| \partial y_k/\partial\xi | = | \alpha_1 + 2Q_1\xi, \alpha_2 + 2Q_2\xi | = 0. \tag{13}$$

On expanding the determinant, one obtains the line,

$$p_1x_1 - p_2x_2 + |A| = 0, \tag{14}$$

as the locus of singular points. This lack of unrestricted invertibility does not vitiate the potential utility of such maps. The main virtues of the T_q are ease of computability and interpretation, and the availability of standard statistical tests. As will be shown elsewhere, they can be approximated by other maps that are more easily and unrestrictedly inverted but less easy to fit and to interpret.

STATISTICAL ESTIMATION AND TESTS

Equation 7 is the expected value of a bivariate curvilinear regression problem. A convenient and optimal method of estimating the parameters of T_q on the basis of N replications in $X^* = (x_{ik}^*)$, $Y = (y_{ik})$, $i = 1,N$ is, therefore, provided by the general linear model (GLM):

$$Y = X^{**}M + U, \tag{15}$$

where $U = (u_{ik})$ is the residual matrix, $M' = (A,\gamma,\delta)$ contains the parameters a_{kj}, b_k, c_k, d_k, and

$$X^{**} = (x_{i_1}^*, x_{i_2}^*, x_{i_1}^{*2}, 2x_{i_1}^*x_{i_2}^*, x_{i_2}^{*2}, 1) \tag{16}$$

is the design matrix. For M to be (statistically) estimable, X^{**} must be of full column rank. Under the well-known mild assumptions of the GLM, it is then possible to test linear hypotheses of the form,

$$H_0: Q'MP = 0, \tag{17}$$

for all conformable full column rank Q,P. In particular,

$$Q' = \begin{bmatrix} 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \end{bmatrix} \tag{18}$$

can be used to test the hypothesis that the simpler form, Equation 8, suffices to describe T_q. Since the mechanics of the GLM are well documented elsewhere (e.g., Seber, 1966), they need not be repeated here.

A minor complication in practice is the linear transformation, $\xi = W\xi^*$, which carries the original physical measures into the derived measures, permitting the simpler representation of T_q. If the x_k^* are physical measures, as assumed, then this transformation is not "admissible" in the sense of measurement theory. Rather, it is an empirically meaningful transformation that affects the outcome of the statistical test. Although some progress has been made in developing an optimal W for minimizing the contribution of the squared terms in Equation 16, the problem needs further investigation. Fortunately, the exact choice of W does not appear to be critical. To maximize the fit of the reduced transformation, Equations 8-10, X^* should be rotated so that corresponding sides of the configuration are approximately parallel. For example, for the Krantz and Tversky (1975) data,[3] a rotation of X^* through $\pi/2$, as Schönemann (1977, p. 163) suggested, led to a reduced T_q of the form of Equation 8, which explained 99.4% of the variance of subjective coordinates as quadratic functions of the physical coordinates. Moderate changes in W, within 20 deg around $\pi/2$, did not appreciably affect the c_k. This relative robustness means, on the other hand, that the derived physical measures, x_k, are only approximately defined by W in terms of the original x_k^*.

CANONICAL REDUCTIONS IN S

Since it is assumed that the coordinates Y in S result from a distance-based scaling analysis, they are subject to certain "admissible transformations," which arbitrarily change the numerical representation of Y, and hence, also of T_q. It is then reasonable to use these admissible transformations to simplify T_q as much as possible, so as to obtain a parsimonious and, hopefully, empirically stable and substantively revealing "canonical form" of the transformation T_q. The extent of possible simplification depends on the scaling model. Occasionally, Y will be determined within a translation and a central dilation. More typically, Y will be determined within rigid motions or similarities. Both Horan's (1969) model and the INDSCAL model leave the stimulus coordinates on two (unrelated) interval scales; that is, they preclude rotations, provided, of course, these models fit (see Footnote 1).

Although it is not possible to treat all details of such identifiability problems here, a few general observations will be made. The first problem to be dealt with is the multidimensional analogue of a question raised by Luce (1959) (see also Rozeboom, 1962): Does the general mathematical form of the mapping T_q reproduce itself under all admissible transformations of X and Y? The answer is affirmative for all affine transformations of ξ and η, since Equation 8 retains its form under affine transformations of ξ, and Equation 10 under affine transformations of η. This is more than needed. If one subjects the coordinates Y in S to affine transformations,

$$T: S \rightarrow S: \eta = R\eta^* + \beta, \text{ R nonsingular}, \beta' = (b_1, b_2), \quad (19)$$

then substitution into Equation 10 gives

$$\eta^* = R^{-1}A\xi + 2R^{-1}\gamma x_1 x_2 + R^{-1}(\delta - \beta), \quad (20)$$

which is again of the form of Equation 10. On choosing $R = A$, $\delta = \beta$, and writing $\gamma^* = R^{-1}\gamma$, one obtains

$$\eta^* = \xi + 2\gamma^* x_1 x_2 \quad (21)$$

as a canonical form for T_q, when the y_k are determined within affine transformations. This form has a very simple interpretation: Each physical variable corresponds directly to a subjective variable, and the c_k^* indicate the extent of the transaffine deformation. The same form can also be used if Y has been derived from euclidean distances, which define it within rigid motions. The change in S from η to $\eta^* = A^{-1}\eta$ can then be interpreted as a change from an orthogonal to an oblique coordinate system in S. To preserve the observed distances, a new metric, $A'A$, can be introduced, so that

$$d^2(\eta_i, \eta_j) = (\eta_i^* - \eta_j^*)'A'A(\eta_i^* - \eta_j^*) = (\eta_i - \eta_j)'(\eta_i - \eta_j). \quad (22)$$

Since only rigid motions are admissible, the new metric $A'A$ is empirically meaningful in this case. Such oblique transformations have been used in factor analysis to reduce factor patterns to "simple structure." Here, they can be used to simplify mappings.

On the other hand, if it is desired to preserve the original euclidean metric, then the transformations in S will usually be limited to similarities:

$$T: S \rightarrow S: \eta^* = cR\eta + \beta, R'R = I, c > 0 \text{ in Re}. \quad (23)$$

In this case, a number of options are available for simplifying T_q: (1) The singular value decomposition of A can be used to define a unique R that symmetrizes A, the linear part of T_q. (2) R can be used to introduce one zero in γ^*, thus eliminating the mixed term x_1x_2 for one of the subjective variables. (3) R can be used to rotate γ into $J = (1,1)'$, thus equalizing the contribution of the mixed term to each new subjective variable. (4) R can be used to triangularize A, so that either $a_{12}^* = 0$ or $a_{21}^* = 0$. The two choices in Options 2 and 4 are equivalent within permutations of the original variables.

Although the choice among the various options for simplifying T_q under similarities in S is ultimately a question of empirical utility, Options 2 and 4 seem most attractive at this time. If the physical variables x_k are on two unrelated ratio scales, then Option 2 can be written

$$\begin{aligned} y_1^* &= x_1 + a_{12}^* x_2 + d_1^* \\ y_2^* &= a_{21}^* x_1 + x_2 + 2c_2^* x_1 x_2 + d_2^*. \end{aligned} \quad (24)$$

Under the same assumptions, Option 4 can be written

$$\begin{aligned} y_1^* &= x_1 + 2c_1^* x_1 x_2 + d_1^* \\ y_2^* &= a_{21}^* x_1 + x_2 + 2c_2^* x_1 x_2 + d_2^*. \end{aligned} \quad (25)$$

If the x_k are on a joint ratio scale, only one of the a_{kk}^* can be set to unity in Equations 24 and 25.

If the y_k are on interval scales, Equation 20 reduces to

$$\eta^* = D_p^{-1}\xi + 2D_p^{-1}\gamma^* x_1 x_2, \quad (26)$$

where the free matrix $D_p = \text{diag}(p_1, p_2)$ leaves parameters in the same rows of (A^*, γ^*) on joint ratio scales.

REFERENCE NOTES

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NOTES

1. At the time their work was done, there apparently existed no practical tool for falsifying the critical (diagonality) assumption on which this claim rests. Such a tool has now been made available (Schönemann, James, & Carter, 1979). That the model proposed by Horan (1969) and the variant recommended by Carroll and Chang (1970) are falsifiable in principle had been shown by Schönemann (1972, Note 1) and Schulz (1971, Note 2).

2. This terminology is not standard. Older texts, for example, Graustein (1948, p. 403), use the term "quadratic transformations" for certain birational transformations.

3. Due to space limitations, detailed numerical graphical illustrations are not presented in this paper.

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