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On possible psychophysical maps: II. Projective transformations

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Projective transformations can be used to map multidimensional physical spaces into multidimensional perceptual spaces; they can be related to the quadratic transformations discussed in an earlier paper.

This is a continuation of earlier work on transaffine psychophysical maps (Schönemann, 1977, 1979). "Transaffine maps" means certain transformations that include, but are not restricted to, affine transformations (i.e., transformations of the type $\eta = T\xi + \gamma$). A defining characteristic of affine maps is that they preserve parallelism. A number of recent psychophysical studies suggest that parallelism may not be a psychophysical invariant and, thus, indicate the need for more general maps. For example, Krantz and Tversky (1975) scaled the dissimilarity judgments for pairs of rectangles, which had been constructed in terms of (log−) height and width so that the design configuration in the "physical space," E, was square. They found that this configuration was distorted in the "subjective space," S, into a trapezoid. Similar results have been reported by Borg and Lingoes (1979), Eissler and Roskam (1977), Roskam (1979), Eissler (Note 1), Noma and Johnson (Note 2), Pachella, Somers, and Hardzinski (Note 3), and others. In all these cases, the mapping from E to S cannot be linear or affine, because parallelism is destroyed. However, since the departure from an affine mapping seems to be slight and systematic, it might be described by a somewhat more general class of transformation which includes affinities as a special case.

In Schönemann (1979), one class of possible psychophysical maps was introduced to meet this requirement. They were called "quadratic," since they can be written as quadratic forms:

$$T_q\colon E \rightarrow S\colon y_k = \zeta' G_k \zeta, \quad k = 1,q, \tag{1}$$

where G_k is a $p+1$ by $p+1$ symmetric matrix, q is the number of dependent subjective, psychological variables, and

$$\zeta' = (\xi',1), \text{ and } \xi = (x_1, x_2, \ldots, x_p) \tag{2}$$

contains the p physical variables x_i ($i = 1,p$).

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Of particular interest are those special cases of Equations 1 and 2, where a linear transformation $\xi = W\xi^*$ exists that eliminates the squared terms. In the planar case, T_q then takes on the simpler form

$$y_k = \alpha_k'\xi + \xi' Q_k \xi + d_k, \ \xi = W\xi^*, \ k = 1,2, \tag{3}$$

where

$$Q_k = (q_{ijk}), \ q_{ijk} = c_k \ (0) \text{ if } i \neq j \ (i=j), \ \alpha_k' = (a_{k_1}, a_{k_2}), \tag{4}$$

and where $W = (w_{ik})$ is the linear transformation that carried the original physical measures, x_k^*, into the derived physical measures, x_k. For example, in the Krantz and Tversky (1975) study, the x_k^* are (log−) height and length, and the x_k are (log−) area and shape. Alternatively, such maps can be written

$$\eta = A\xi + \gamma x_1 x_2 + \delta, \ \xi = W\xi^*, \tag{5}$$

where $\eta' = (y_1,y_2)$, $A = (a_{kj})$, $\gamma' = (c_1 c_2)$, $\delta' = (d_1,d_2)$.

These quadratic transformations were discussed in more detail in Schönemann (1979). It was shown how they can be fitted to two coordinate matrices, X (in E) and Y (in S), in a least-squares sense, how certain hypotheses about the parameters of such maps can be tested statistically, and how their representation changes under various coordinate transformations in E and S. For example, if one subjects the coordinates Y in S to affine transformations,

$$T\colon S \rightarrow S\colon \eta = R\eta^* + \beta, \ R \text{ nonsingular}, \ \beta' = (b_1 b_2), \tag{6}$$

then one obtains

$$\eta^* = \xi + \gamma^* x_1 x_2 \tag{7}$$

as a canonical form for T_q under affine transformations of the y_k. This form has a very simple interpretation: Each physical variable x_k corresponds

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directly to a subjective variable y_k, and the c_k^* in γ indicates the extent of the transaffine deformation. In Schönemann (1979, p. 331), a number of other canonical forms were derived under more restricted admissible transformations in S.

The relative ease with which the T_q can be fitted and interpreted is one of the major strengths of this class of transaffine maps. Another is the availability of standard statistical tests (see Schönemann, 1979, for details). On the other hand, a drawback is that they are not unrestrictedly invertible. As was shown in the previous report, the locus of their singular points is a straight line in the planar case.

In the present paper, a second class of transaffine maps will be discussed that are not subject to this limitation and that, moreover, preserve collinearity throughout. These maps arise in (and, in a sense, define) projective geometry. Hence, they are called "projective transformations." Since they preserve collinearity, they are also called, in the planar case, "collineations." The planar case will be sufficiently general to illustrate the basic ideas, and the remainder of this discussion will be limited to it. In the plane, these maps can be written

$$T_p: E \rightarrow S: y_k = (b_{k_1}x_1 + b_{k_2}x_2 + t_k)/(s_1x_1 + s_2x_2 + 1), \quad k = 1,2, \quad (8)$$

or, in matrix notation,

$$T_p: E \rightarrow S: \eta = (B\xi + \tau)/(\sigma'\xi + 1), \quad (9)$$

where $\eta' = (y_1,y_2)$, $B = (b_{kj})$, $\xi' = (x_1,x_2)$, $\tau' = (t_1,t_2)$, and $\sigma' = (s_1,s_2)$.

HOMOGENEOUS COORDINATES

On the surface, these maps appear to be more complicated and more difficult to work with than the quadratic transformations, T_q, discussed previously. However, they can be given a more pleasing appearance by a suitable change of the coordinate system used to describe them. In this case, they reemerge as linear transformations in a three-dimensional space, where they are more easily dealt with computationally. To bring about this change in representation, let $\xi' = (x_1,x_2)$ be the conventional coordinate vector of a point in the plane relative to some chosen basis. They are usually called "inhomogeneous" to distinguish them from the more convenient "homogeneous" coordinates,[1] which are defined by the map,

$$R^\circ: \xi \rightarrow \xi^\circ: \xi^{\circ\prime} = (x_1^\circ,x_2^\circ,x_3^\circ), \quad (10)$$

where $x_1^\circ = kx_1$, $x_2^\circ = kx_2$, $x_3^\circ = k$, $\forall k(\neq 0) \in Re$.

Thus, each pair of inhomogeneous coordinates (x_1x_2) is replaced by a triple of homogeneous coordinates $(x_1^\circ,x_2^\circ,x_3^\circ)$, and any two triples that are proportional to each other, for example,

$$\xi^{\circ\prime} = (x_1^\circ,x_2^\circ,x_3^\circ) \text{ and } c\xi^{\circ\prime} = (cx_1^\circ,cx_2^\circ,cx_3^\circ), \quad (11)$$

are equivalent descriptions of the same point, as long as $c \neq 0$. The totality of all triples of this form, with the sole exception of (0,0,0), constitutes the "projective plane." In particular, triples of the form

$$(x_1^\circ,x_2^\circ,0) \quad (12)$$

denote "points at infinity," or "ideal points," the totality of which forms a "line at infinity," or "ideal line," in the projective plane. All other points in this plane will be called "finite." Since c is arbitrary, it is convenient to set x_3° to unity when it is nonzero. With this choice, the homogeneous coordinates of any finite point with inhomogeneous coordinates $\xi' = (x_1,x_2)$ become

$$\xi^{\circ\prime} = (x_1, x_2, 1). \quad (13)$$

Geometrically, this change in representation can be visualized as passing the plane $x_3^\circ = 1$ through the third coordinate axis. All finite points of this plane then have the homogeneous coordinates (Equation 13), while the points at infinity are obtained on setting $x_3^\circ = 0$.

COLLINEATIONS

A "projective transformation" or "collineation" is a mapping that carries the projective plane into itself. In homogeneous coordinates, it can be written

$$T_p: \xi^\circ \rightarrow \eta^\circ: \eta^\circ = kH^\circ\xi^\circ, \; k(\neq) \in Re, \quad (14)$$

where $H^\circ = (h_{ij}^\circ)$ is a 3 by 3 nonsingular matrix. Thus, if $X = (\xi_i')$ contains the inhomogeneous coordinates of N stimuli in the physical space E and $Y = (\eta_i')$ contains the inhomogeneous coordinates of the same stimuli in the subjective space S, then we can define two associated matrices of homogeneous coordinates X°, Y° for the subset of finite points ($x_3^\circ \neq 0$, $y_3^\circ \neq 0$) as

$$X^\circ = (\xi_i^{\circ\prime}) = (\xi_i',1) = (X,J);$$

$$Y^\circ = (\eta_i^{\circ\prime}) = (\eta_i',1) = (Y,J) \quad (15)$$

and represent the transformation (Equation 9) from E to S as

$$Y^\circ = D_k X^\circ \begin{bmatrix} B' & \sigma \\ \tau' & 1 \end{bmatrix} = D_k X^\circ H^{\circ\prime}, \quad (16)$$

where $D_k = \text{diag}(k_i)$ contains the unknown proportionality coefficients k_i needed to normalize the finite points, as in Equation 13. These points can then be graphically represented by plotting the first two columns of $Y°$ as inhomogeneous coordinates of points in an affine plane. Of particular interest are finite points in S that correspond to ideal points in E. For example, in the Krantz and Tversky (1975) study, there is one such point, namely, the finite intersection in S of lines that were parallel in E.[2] The first two rows of T_p', (B', σ), give the homogeneous coordinates of the two points at infinity in E, (1,0,0), (0,1,0).

If $\sigma = 0$, then T_p reduces to an affine transformation. Depending on B and τ, it may further simplify to a rigid motion, a similarity, or a translation. T_p can be reduced to a more convenient and interpretable canonical form under the admissible transformations in E and S, as already discussed at some length in Schönemann (1979) for T_q.

The vector σ in T_p measures the departure from an affine transformation. It, therefore, corresponds roughly to the vector γ of the T_q discussed previously. More explicit relations between T_q and T_p will be given shortly.

LEAST-SQUARES FIT

In contrast to the quadratic transformations, T_q, which can be fitted to fallible data with standard regression programs (see Schönemann, 1979), the projective transformations, T_p, require some additional computational effort. Although it is always possible to fit them exactly to four points (no three of which are collinear) in the plane, it will be useful, for the applications envisioned here, to obtain a somewhat more robust least-squares solution that evens out the errors of fit when there are more than four points in X and Y.

The defining equation of this least-squares problem is

$$Y° = D_k X° H°' + E, \qquad (17)$$

where $X°$ and $Y°$ contain the N homogeneous coordinates of the N stimuli in E and S, respectively, D_k contains the N unknown multipliers k_i, and $H°'$ represents the projective transformation relative to the bases defining $X°$ and $Y°$. Since not only $H°$, but also D_k is unknown, the loss function $\mathrm{tr}E'E$ must be differentiated w.r.t. both $H°'$ and D_k. On setting the matrix derivatives (see, e.g., Schönemann, Note 4) equal to zero,

$$\partial\,\mathrm{tr}E'E/\partial H°' = \phi,\ \partial\,\mathrm{tr}E'E/\partial D_k = \mathrm{diag}(\phi), \qquad (18)$$

one obtains

$$H°' = (X°'D_k^2X°)^{-1}(X°'D_k)Y° = (D_kX°)^{+}Y°, \qquad (19)$$

where "+" denotes the Moore-Penrose inverse, and

$$D_k = \mathrm{diag}(X°H°'Y°')\mathrm{diag}^{-1}(X°H°'H°X°'), \qquad (20)$$

as implicit solutions for the least-squares estimates of $H°$ and D_k. To solve Equations 19 and 20, an "alternating least-squares" approach can be used (see, e.g., Schönemann, Bock, & Tucker, Note 5, p. 15), which, in this case, happens to converge fairly rapidly, usually within 10 iterations.

RELATIONS BETWEEN QUADRATIC AND PROJECTIVE MAPS

It is clear that the two types of transaffine maps, T_q and T_p, are not strictly equivalent because T_p preserves collinearity throughout, while T_q does not; and T_p is invertible throughout, while T_q is not. Nevertheless, it is possible to approximate one by the other locally, as long as the "transaffine deformation," which is reflected in the vectors γ and σ, is not too large. In psychophysical applications, we are usually not interested in the whole plane Re^2, but only in a certain subregion of it. We might consider it an achievement to be able to model such a region, even though the regularities cannot be extrapolated beyond it. Weber's law is a case in point. If we were able to approximate one mapping, say T_p, by the other, T_q, in the region of interest, then we could use the two interchangeably in this region and capitalize on their respective advantages. Specifically, we may work with the T_p when we are interested in the inverse map, because it is simply given by $H°^{-1}$. On the other hand, we may wish to work with T_q when we are interested in testing hypotheses about the map and also when we wish to interpret it psychologically, because T_q renders the relationship between the physical and the subjective variables more transparent than does T_p.

To find the approximate relationship between T_p and T_q, we can attempt to solve

$$B\xi + \tau = (\sigma'\xi + 1)(A\xi + \gamma x_1 x_2 + \delta) \qquad (21)$$

at different values of $\xi' = (x_1, x_2)$ for the parameters of one of the maps in terms of those of the other. Specifically, at the corners of the rectangle defined by $\xi' = (x_1, x_2)$:

$$\xi_1' = (x_1, 0),\ \xi_2' = (0, x_2),\ \xi' = (x_1, x_2),\ \xi_0' = (0,0), \qquad (22)$$

one obtains:

$$\delta = \tau \qquad (23)$$

$$A = (B - \tau J'D_1)(I + D_1 D_2)^{-1}, \qquad (24)$$

where $D_1 = \text{diag}(s_1,s_2)$, $D_2 = \text{diag}(x_1,x_2)$, $J' = (1,1)$, and

$$\gamma = [(B\xi + \tau)/(\sigma'\xi + 1) - (A\xi + \delta)]/x_1x_2, \quad (25)$$

with A as in Equation 24. Although A depends on ξ, the solution should be relatively robust as long as the products x_is_i remain small. This conjecture was borne out empirically, as will be shown in more detail elsewhere.[2]

Alternatively, $T_q = (A,\gamma,\delta)$ can be computed as

$$T_q' = X^{**-1}Y, \quad (26)$$

where X^{**} is the "design matrix" associated with the ξ_j in Equation 22 (see Schönemann, 1979, Equation 16) and Y contains the associated (inhomogeneous) subjective coordinates predicted from Equation 9. Equation 24 implies that even if $\gamma = \delta = 0$, $A \neq B$, although A will be diagonal iff B is, in this case.

Similarly, the parameters of T_p, in terms of those of T_q, are given at the quadruple (Equation 22) for $\xi = J$:

$$\tau' = \delta', \quad (27)$$

$$\sigma' \begin{bmatrix} 0\,1\,1\,0 \\ 1\,0\,1\,0 \end{bmatrix} T_q' = (0\ -1\ -1\ 0)T_q', \quad (28)$$

$$B' = \begin{bmatrix} 1\,0\,0\,1 \\ 0\,1\,0\,1 \end{bmatrix} T_q' D_1 + \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{bmatrix} T_q', \quad (29)$$

all of which have exact algebraic solutions. Among other things, these relations can be used to obtain approximate and interpretable inverses of T_q in the region in which they are defined.

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NOTES

1. This language is motivated by the fact that the equation of a line, which in inhomogeneous coordinates might be written $m_1x_1 + m_2x_2 + m_3 = 0$, takes on the simpler, homogeneous, form $m_1^\circ x_1^\circ + m_2^\circ x_2^\circ + m_3^\circ x_3^\circ = 0$, or $\mu^{\circ\prime}\xi^\circ = 0$. This notation also reflects a duality relation between points and lines that is characteristic of projective geometry. Projective transformations that map points into lines are called "correlations." For readable introductions to projective geometry from an analytic point of view, see any of the classic texts (e.g., Graustein, 1948; Levy, 1961; Meserve, 1955; Veblen & Young, 1938). For some previous applications in psychology, see, for example, Johansson (1978) and Levine (1974).

2. Due to space limitations, detailed numerical and graphical illustrations will be presented in another paper.

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