Philosophical Aspects of Spontaneous Symmetry Breaking

Giacomo Schwarz*

May 4, 2012

Abstract

This essay expounds the algebraic framework describing general physical theories, within which the phenomenon of spontaneous symmetry breaking (SSB) makes its appearance in infinite quantum systems. This is in contrast with the fact that a large class of theories - both classical and quantum, finite and infinite - are termed, in the conventional account of classical and quantum mechanics, as exhibiting SSB. This discrepancy will be understood in the light of an interpretation that finds the symmetry breaking to be in some respects stronger in the algebraic account than is generally the case in the conventional picture.

The case of SSB in the standard account of quantum field theory (QFT) will then be discussed, and it will be argued that, although one would expect a connection with the algebraic account to be possible, this turns out to be problematic. Finally the role of the idealisation of infinite systems, crucial to algebraic SSB, will be discussed.

*Churchill College, University of Cambridge - giacomo.schwarz@cantab.net

Contents

1 Introduction

The general notion of spontaneous symmetry breaking "indicates a situation where, given a symmetry of the equations of motion, solutions exist which are not invariant under the action of this symmetry *without the introduction of any term explicitly breaking the symmetry* (whence the attribute 'spontaneous')" [4, p. 327]. Rather than questioning the legitimacy of the assumption that certain physical theories should display higher symmetries than their observed solutions,[1] this essay will focus on the implications of such an assumption.

The phenomenon of SSB in the general sense just mentioned is exhibited both by classical and quantum mechanical systems, but as one may expect the quantum mechanical case is more subtle. In the case of infinite quantum systems, peculiar mathematical structures will be shown to arise, which are completely unexpected in the standard approach to quantum mechanics. This calls for a new approach to the description of such systems, in which these structures will be understood both mathematically and physically. In the so-called algebraic formulation, it will be shown that these novel structures, namely unitarily inequivalent representations, will be responsible for what, in this specific context, is taken to be SSB. Once their physical meaning has been clarified, it will become apparent how the algebraic version of SSB embodies the general physical notion of symmetry breaking. The peculiar feature of this account of SSB will be argued to be that different symmetry-breaking states may be interpreted as being confined in different "closed worlds". This is a feature not contained in the general notion of SSB, which will in comparison carry a weaker sense of symmetry breaking.

The main goal of this essay is to study the deep issues arising in connection with SSB in infinite quantum systems. As the algebraic approach will be central in providing insight into this phenomenon, the starting point of this essay will be a review of this conceptual framework. Since the algebraic formulation offers a unifying description for general physical systems, before concentrating on the specific case of infinite quantum systems in §2.6, attention will be given in §2.3 and §2.4 to how this approach describes classical and finite quantum systems. This novel perspective will offer interesting physical insights, such as the "unique quantization" of finite quantum systems.

The infinite quantum-mechanical spin chain, as the simplest system exhibiting SSB in both its conventional and its algebraic description, will be discussed in some detail in §2.7. There, both the physical and the mathematical considerations done within the algebraic formulation will be shown to provide a rigorous understanding of what happens when a quantum mechanical system goes from being finite to being infinite.

Subsequently attention will be given to a very important class of infinite quantum systems, namely quantum fields. A general discussion of symmetries in this context will be necessary, before presenting the definition of what is meant by SSB in QFT, which can be found in §3.1. It will be argued in §3.2 and §3.3, with the aid of some examples, that these theories may be understood to some extent within the algebraic ap-

[1] An assumption which, as E. Castellani notes in [4], is worthy of thorough philosophical investigation.

proach, and that characteristic features of the algebraic version of SSB can be outlined. Nonetheless, due in part to mathematical inconsistencies within the standard account of QFT, an understanding of this subject within the rigorous mathematical framework of the algebraic formulation will turn out to be problematic.

Finally, §3.4 will be dedicated to the realisation that, since the algebraic version of SSB exists solely within infinite quantum systems, a reflection on the role of such an idealisation is necessary, and this will question the relevance of algebraic SSB to concrete physical systems.

2 Algebraic formulation

2.1 $C^\star$ algebras in physics

This section will take a relatively long path towards the goal of expounding SSB in the algebraic approach, but as a result it also offers some insight into the understanding of general physical theories in terms of their abstract mathematical structure; and the perspective gained will turn out to be in many ways beneficial towards obtaining new insights into the representation of physical systems. An overview of the general setting in which the algebraic version of SSB takes place will also show the importance of this peculiar phenomenon.

In the algebraic approach, the mathematical structure of a general physical theory is taken to be the following [23, p. 24]:

1. A physical system is defined by its $C^\star$ algebra[2] $\mathcal{A}$ of observables (with identity).
2. A state of the system is a normalized positive linear functional on $\mathcal{A}$.

The above assumption can be motivated in several ways. The first, an elaboration of which would be beyond the scope of this essay and may be found for example in [23, ch. 1.3], is based on considerations about operating on a general physical system. The starting point of this approach is to argue that, since the way we gain knowledge about a physical system is by performing experiments on it, it is natural to describe it operationally, by the outcomes of such experiments. From this point of view, a system is defined by the set of quantities (also known as observables) which can be measured on it and the possible states it may be in, that is to say the possible average values such quantities may be found to have[3]. Working with equivalence classes of observables and states spanning all possible measurement setups (thus separating the physical system in consideration from the measurement procedure) and proceeding with operational considerations, one can eventually identify in the observables and states the structures mentioned above.

[2] Appendix A clarifies the terms used here.

[3] After performing replicated measurements on identically prepared systems.

Although it is impressive how such operational considerations may lead to the identification of an overarching mathematical structure, there are various reasons to question the completeness of the picture obtained solely by such considerations. Firstly, the algebraic description of a physical system does not include information about the time evolution or the physical symmetries of such a system, which in the standard account of both classical and quantum systems is encoded in the system's Lagrangian. As R. Haag puts it in [11, p.300], "in the algebraic frame we have not understood the rôle of the Lagrangian in quantum theory". Taking these considerations into account, the $C^\star$ algebra and the algebraic states of a physical system may be seen as describing the constituents of such a system, but extra structure needs to be added in order to determine how these building blocks will behave, and as is apparent from Haag's comment above, a way to incorporate this into the algebraic formulation has not yet been found.

Another limitation is that by describing a system using measurement outcomes only, one excludes from such a description any unmeasurable elements. This obvious fact has the important consequence that gauge theories, as theories containing degrees of freedom without a physical counterpart, are outside the scope of the theories obtained by the operational considerations above. This puts the validity of such an approach in doubt, since many of the successful modern physical theories are gauge theories. One may argue that, for a system which is conventionally described by a gauge theory, the operational considerations would strip such a description bare, leaving only the physical, gauge-independent content; and thus although at first it seemed that such a system would defy the operationally identified mathematical structures, eventually it would not. This operationally motivated point of view may well be true, but again as noted by R. Haag, how gauge theories fit into the conceptual frame of the algebraic approach still needs to be understood [11, p.299].

Another way of motivating the algebraic approach is by identifying an algebraic structure in theories that have proven successful in describing known physics. As is argued in appendix A, such an identification is possible in the well-established theories of classical and quantum mechanics, which both present a $C^\star$-algebraic structure. Given the enormous success and range of these theories, the fact that both possess the same algebraic structure motivates the assumption that any physical theory should fall within the algebraic frame. The fact that both these theories and the operational approach agree on the relevant mathematical structures is further evidence in support of such an assumption.

As noted in appendix A, the major difference between the algebraic structures underlying classical mechanics and quantum mechanics is the fact that in the classical case the elements of the $C^\star$ algebra of observables commute, while in the quantum case they don't. This not only provides a way of distinguishing, at the algebraic level, classical from quantum theories but, as remarked by F. Strocchi in [23, ch. 2]: it also points to the feature that fundamentally sets classical and quantum systems apart, namely the Heisenberg uncertainty relations. For the position-momentum relation these take the familiar form [4]

$$(\Delta_\omega q_j)(\Delta_\omega p_j) \geq \frac{\hbar}{2}. \tag{1}$$

The fact that the standard deviation of two observables on a given state cannot both be arbitrarily small is a purely quantum-mechanical phenomenon, as classically there theoretically exists no limitation to the precision of measurements. More specifically: although, as noted by F. Strocchi in [23, p. 12], in a realistic preparation or detection of a state of any physical system a certain indeterminacy is unavoidable, in classical systems this undeterminacy could theoretically be made arbitrarly small by choosing ever better measurement and preparation devices.

This existence of uncertainty relations for certain pairs of observables can be related to the commutation relations on the $C^\star$ algebra describing the system by using the relation

$$(\Delta_\omega A)(\Delta_\omega B) \geq \frac{1}{2}|\omega([A, B])|, \tag{2}$$

derived in [23, ch. 2] for an algebraic state ω and any two elements A and B of the algebra satisfying $A^\star = A$ and $B^\star = B$. For systems described by abelian algebras this leaves the standard deviations unconstrained, which is consistent with the possibility of arbitrarily precise measurements on classical systems. On the other hand if the algebra of a given system is non-abelian, there will be at least one pair of observables satisfying an uncertainty relation with a non-zero lower bound. The behaviour of the system when measured for those observables will thus be quantum-mechanical, as opposed to classical.

It is now apparent how, with the $C^\star$ algebra of observables as a starting point, commutativity properties of the algebra of observables determine if the system will behave classically or quantum-mechanically. On the other hand, once the relevance of $C^\star$ algebras in general physical system has been recognised, one may want to determine the structure of the algebra describing a given physical system. In the case of quantum-mechanical systems, knowledge of the algebraic relation (2) would motivate the search for non-commutative algebras which satisfy the same uncertainty bounds that are found to hold experimentally. Thus the experimentally verified position - momentum uncertainty relation (1) for a particle motivates the following commutation relation for the position and momentum observables describing the particle:

$$q_j p_k - p_k q_j = i\hbar\delta_{jk}\mathbb{1}. \tag{3}$$

These are known as the *Heisenberg commutation relations*, and are usually taken as the starting point of the standard account of quantum mechanics.

In the rest of this section the algebraic approach will be followed, and a physical system will be defined by its algebra of observables and by its algebraic states. This description is abstract, and it is often useful to have more concrete mathematical objects to work with. Furthermore, in the case of quantum mechanics, the conventional Hilbert

[4] $\Delta_\omega A$ denotes the standard deviation of the observable A measured on the state ω.

space account includes some information that has so far not been given in the algebraic formulation, for example the transition amplitude between two states. It would thus be interesting to know what kind of connections exist between the algebraic and the Hilbert space account describing quantum mechanical systems.

As the next subsection describes, for every physical system (be it quantum-mechanical or not) and every one of its algebraic states there exists a unique (up to unitary equivalence) Hilbert space representation of the algebra of observables such that the expectations of the chosen algebraic state may be represented by a vector in the Hilbert space. This is the content of the GNS theorem, which forges the link between the abstract algebra and a concrete representation. Why this representation is relevant will be discussed in the following, as well as its physical implications.

2.2 The GNS theorem

Taking the $C^\star$ algebra generated by the observables of a physical system as a starting point, it is natural to wonder how this algebraic structure is related to the usual mathematical description of such a system, which in the quantum mechanical case is the Hilbert space representation, and in the case of a classical system is the phase space representation. The following theorem will be shown to provide such a connection:

Theorem 2.1 *(Gelfand, Naimark and Segal) Given a $C^\star$ algebra $\mathcal{A}$ (with identity) and a state ω, there is a Hilbert space $\mathcal{H}_\omega$ and a representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$,[5] such that*

(i) $\mathcal{H}_\omega$ contains a cyclic vector Ψ_ω, i.e. the vectors $\{\pi_\omega(A)\Psi_\omega \colon A \in \mathcal{A}\}$ are dense in $\mathcal{H}_\omega$,

(ii) $\omega(A) = \langle \Psi_\omega, \pi_\omega(A)\Psi_\omega \rangle$,

(iii) every other representation π in a Hilbert space $\mathcal{H}_\pi$ with a cyclic vector Ψ such that $\omega(A) = \langle \Psi, \pi(A)\Psi \rangle$ is unitarily equivalent to π_ω, i.e. there exists an isometry $U : \mathcal{H}_\pi \to \mathcal{H}_\omega$ such that $U\pi(A)U^{-1} = \pi_\omega(A)$, $U\Psi = \Psi_\omega$.

One refers to a representation with the above mentioned properties as a *GNS representation*.

The GNS theorem guarantees that the search for a Hilbert space representation of the algebra of observables will be successful, but there may be further representations that do not satisfy the requirements of the theorem, and in this case the question which representation one should consider would arise. One may find strong arguments in support of the fact that the cyclicity condition, which a GNS representation satisfies, is a desirable property. The fact that Fock representations, which are relevant in many physical theories, by definition contain a cyclic vector [12, p.174-175] - the vacuum - supports the importance of cyclic representations. Also, in the words of F. Strocchi [24, p. 73]:

[5] A *representation* of a $C^\star$ algebra $\mathcal{A}$ is a mapping $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ from the abstract algebra into the concrete algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ such that $\pi(\lambda A + \mu B) = \lambda\pi(A) + \mu\pi(B)$, $\pi(AB) = \pi(A)\pi(B)$, and $\pi(A^\star) = \pi(A)^\dagger$ for all $A, B, \in \mathcal{A}$ and all $\mu, \nu, \in \mathbb{C}$.

> The general lesson from the GNS theorem is that a state Ω on the algebra of observables (...) defines a realization of the system in terms of a Hilbert space $\mathcal{H}_\Omega$ of states with a reference vector Ψ_Ω which represents Ω as a cyclic vector (...). In this sense, a state identifies the family of states related to it by observables, equivalently accessible from it by means of physically realizable operations. Thus, one may say that $\mathcal{H}_\Omega$ describes a closed world, or phase, to which Ω belongs.

Thus, the cyclicity property ensures we are considering a single world of physically connectable states to which the given state belongs. This interpretation will be elaborated upon at various stages throughout this essay.

An example where the notion of the "closed world" a physical system belongs to is relevant is given by an idealized infinite ferromagnet, with all spins aligned in a given direction[6]. In this case the closed world associated with this state is the one given by the possible configurations obtained by changing the direction of a finite number of spins, which is a physically realizable process. On the other hand, a state given by all spins pointing in any other direction from the initial one belongs to a different closed world, since modifying an infinite number of spins would require an infinite energy and thus, if the ferromagnet is isolated, is not physically realizable. Of the (infinitely) many closed worlds the infinite ferromagnet may belong to, if such a ferromagnet were to exist, it would have to belong to a specific one. Thus, in a certain sense, talking of an "infinite ferromagnet" does not fully specify what one is referring to until one establishes what representation one is considering, since possible realizations of such a system will never be able to "escape" such a representation.

This example is closely linked to spontaneous symmetry breaking, both in the algebraic approach and in the way this system would conventionally be described, although in different ways. The algebraic version of SSB will be explained in section 2.6, but at this point it is important to note that it is a phenomenon that only concerns quantum-mechanical systems, which as argued above are described by non-abelian algebras. On the other hand the phenomenon that is usually referred to as SSB in the non-algebraic litterature[7] may take place both at the classical and at the quantum level, and is in S. Coleman's words when "we conjecture that the laws of nature may possess symmetries which are not manifest to us because the vacuum state is not invariant under them" [5, p. 116]. What Coleman takes to be SSB embodies the general physical notion mentioned in the introduction while focusing on a symmetry-breaking state of particular physical meaning, and can be summarised in the following definition:

Definition *of SSB (in the conventional approach)*: A symmetry of a physical theory is said to be spontaneously broken when the vacuum configuration (the state with lowest energy) is not invariant under such a symmetry.

Although this statement is clear in the context of classical and quantum mechanics, what the relevant symmetry-breaking vacuum state is in the case of QFT will need

[6] The spins considered here may either be quantum mechanical or simply classical magnetic dipoles.

[7] Which is what will be referred to as the conventional approach.

further clarification, thus giving rise to various approaches: some taking the relevant symmetry breaking to occur at the classical level and others at the quantum level. The case of QFT will be discussed in section 3.

The example of the infinite ferromagnet is a case of SSB in Coleman's sense, because the rotational invariance of space (which one expects to be a symmetry of the theory) is broken by the preferred direction chosen by the polarization of the vacuum state (the configuration with all spins pointing in a given direction), and this is independent of the classical or quantum nature of the spins. On the other hand, as will be shown in section 2.7, the infinite spin chain with quantum-mechanical spin variables is also a case of SSB in the algebraic formulation, since one may in this case identify the characteristic structures of SSB in this approach, which are unitarily inequivalent representations of the algebra of observables, linked by the broken symmetry.

Another fact that distinguishes SSB in the two approaches is the relevance of the fact that the ferromagnet should be taken to be infinite. In the conventional approach, there is no reason why one should not consider instead a finite ferromagnet. A polarized, finite ferromagnet is also a configuration of minimal energy, as well as breaking rotational invariance. On the other hand, algebraic SSB relies crucially on the existence of unitarily inequivalent representations of the algebra, and as will be pointed out in section 2.4, this is not possible for quantum systems with a finite number of canonical variables (which is what is meant by "finite quantum systems"). In the case of the ferromagnet, these variables are the spins at every site of the lattice, and their number will be infinite only if the spatial extention of the ferromagnet is.

Despite the fact that the two approaches offer different frameworks within which SSB is defined, once the algebraic notion of SSB is introduced, it will become apparent that both embody the notion that a physical realization of a system breaks (in some sense) a symmetry which the laws of the system are supposed to possess. The sense in which an algebraic symmetry is broken will be explained to be the lack of its unitary implementability.

So how come, one may ask, are so many more systems said to exhibit SSB in the conventional approach than in the algebraic one? Taking into account the considerations above about closed worlds, one may say the broken symmetries in the algebraic approach are broken in a more radical sense than in the conventional frame, since the former phenomenon concerns a symmetry connecting different closed worlds, while the latter may occur for a symmetry that connects configurations within a given closed world. This can be seen by again considering the ferromagnet in the finite-volume case: every symmetry-breaking vacuum configuration is related by a physically realizable rotation, and thus they all belong to a single closed world. On the other hand, as for the infinite ferromagnet, it may also happen that some systems exhibiting SSB according to the conventional approach are also confined to broken phases.

One might argue that quantum-mechanical tunneling effects should be possible between two closed worlds, thus allowing for transitions between them. But one may offer the following heuristic argument against this: since the tunneling amplitude depends on the energy barrier between two states, being zero when the barrier is infinite, the tunneling amplitude between closed worlds should be zero since they are separated by an infinite amount of energy. Thus the closed worlds should be stable, also quantum-mechanically.

Going back to the GNS theorem, it was argued above that it provides a concrete representation of the abstract algebra of observables that describes the closed world to which a given algebraic state "belongs", i.e. can be expressed as giving the expectation of the represented observables in terms of a given Hilbert space vector. Strocchi's interpretation assumes that all the vectors on such a Hilbert space analogously correspond to physical algebraic states, and that since operations by the represented observables will map vectors in the Hilbert space to other vectors within the same space, the world which contains the states the Hilbert space vectors stand for is closed. Thus, according to this interpretation, the GNS theorem is central in identifying the set of closed worlds a given physical system may belong to. More about the assumptions involved in this interpretation, that will be argued to be of quantum-mechanical nature, can be found in section 2.5.

Before continuing the investigation along this line, it is important to stop and consider classical systems (cf. §2.3). This is because, as will be argued in the next subsection, the closed world interpretation of the GNS representation does not hold in this case. Nonetheless, in the classical case the GNS representation is still relevant, but its usefulness is reduced to showing how the conventional picture may be recovered from the algebraic description of a classical system.

2.3 Classical systems

The $C^\star$ algebras describing classical-mechanical systems are abelian, while those describing quantum-mechanical systems ane non-abelian. This determines the kind of GNS representations these systems can have, and the following subsection is dedicated to considering the abelian case.

One can show [23, ch.2] that the irreducible GNS representations π_ω of an abelian $C^\star$ algebra are one-dimensional:

$$\pi_\omega(A) = \omega(A)\mathbb{1}. \tag{4}$$

The GNS representations of classical systems thus turn out to be rather trivial, with the represented observables acting as simple multiplicative operators. Furthermore an assumption fundamental to Strocchi's interpretation concerning closed worlds, namely that the action of the represented operators on the Hilbert space vectors may be interpreted as corresponding to a modification of the physical state the system is in, will be argued in section 2.5 to be sustainable only if the system in consideration is quantum-mechanical, and one identifies the GNS representation with the operators and vectors standard quantum mechanics would associate to the system. As the present case deals with classical systems, it follows that Strocchi's interpretation does not apply.

At this point one may ask what significance the GNS representaton retains in the classical case. As F. Strocchi points out in [23, ch. 1-2], it may be used to recover the standard picture for classical mechanics as follows: if one considers the family $\mathcal{F}$ of inequivalent irreducible GNS representations and uses the collection $\{\omega(A), \omega \in \mathcal{F}\}$ to construct the function $\tilde{A}(\omega) \equiv \omega(A)$, then one can show that $\mathcal{F}$ is a compact Hausdorff topological space, and that $\tilde{A}$ is continuous. A Hausdorff topological space is a generalization of coordinate space [23, p. 15], and thus the information contained in all the inequivalent GNS representations may be equivalently expressed in the conventional picture of continuous functions on (generalized) phase space.

If on the other hand one considers quantum-mechanical systems, the irreducible GNS representations need not be one-dimensional, and this allows for more complex structures to arise since, in a given representation, $A \in \mathcal{A}$ acts as more than simply a multiplicative operator. As argued in the next subsection, in this case the GNS representation of a system will be taken to coincide with the ordinary quantum-mechanical description of such a system.

2.4 Uniqueness Theorem for finite quantum systems

The following subsections are dedicated to considering the quantum case in close detail since, as anticipated above, it is within quantum theories that the algebraic version of SSB will make its appearance.

Let us for simplicity consider the quantization of classical theories with phase space $\mathbb{R}^{2n}$, where n is finite. In this case, the canonical observables p_i and q_i satisfy the familiar Heisenberg form of the canonical commutation relations (CCRs):

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = i\mathbb{1}\delta_{ij}. \tag{5}$$

It is mathematically more convenient to deal with the unitary operators obtained by exponentiating q and p, since in this way one obtains bounded operators everywhere defined on the Hilbert space of the quantized theory [19, p. 37]. By defining the so-called Weyl operators

$$U(a) = exp(-i\sum_{i=1}^{n} a_i q_i), \quad V(b) = exp(-i\sum_{i=1}^{n} b_i p_i), \quad a, b \in \mathbb{R}^n, \tag{6}$$

that generate the Weyl algebra $\mathcal{A}_{\mathcal{W}}$, one obtains equivalently to (5) the Weyl relations

$$U(a)V(b) = exp(-ia \cdot b)V(b)U(a). \tag{7}$$

The self-adjointedness of the position and momentum operators q and p naturally defines an antilinear $\star$ operation in $\mathcal{A}_{\mathcal{W}}$

$$U(a)^{\star} \equiv U(-a), \quad V(a)^{\star} \equiv V(-a), \tag{8}$$

which turns $\mathcal{A}_{\mathcal{W}}$ into a $\star$-algebra. By introducing the unique [24, part II, ch.1] norm $\|\cdot\|$ on $\mathcal{A}_{\mathcal{W}}$ with the property

$$\|A^{\star}A\| = \|A\|^2, \quad \forall A \in \mathcal{A}_{\mathcal{W}}, \tag{9}$$

the Weyl algebra becomes a $C^{\star}$ algebra.

The following theorem illustrates that there is, up to unitary equivalence, a unique way of representing the Weyl algebra $\mathcal{A}_{\mathcal{W}}$ on a Hilbert space:

Theorem 2.2 *(Stone-von Neumann) All regular*[8] *irreducible representations of the Weyl $C^{\star}$ algebra are unitarily equivalent.*

It is of interest to note that, as L. Ruetsche points out in [19, p.41-42], the Schrödinger representation of the Weyl relations is irreducible and strongly continuous, and thus from the above theorem it follows that any such representation is unitarily equivalent to the Schrödinger representation. This is sometimes referred to as the fact that there is a "unique quantization" for (finite) quantum-mechanical theories, a fact that in L. Ruetsche's words "assuages an anxiety a worrywart might have had about the Hamiltonian quantization recipe. The anxiety is that different physicists, each starting with the same classical theory and each competently following the recipe, could produce *different quantum theories*" [19, p. 41].

One may of course argue that one could go beyond such a uniqueness by considering representations that do not satisfy the premises of the theorem. Suspending continuity is discussed in [19], and the existence of unitarily inequivalent non-regular representations is expounded; the question if these are physical or not is left open. On the other hand, considering reducible representations would not add to the standard Schrödinger picture, since a reducible representation would - according to the above - decompose into a direct sum of subrepresentations unitarily equivalent to one-another and the Schrödinger picture, thus not adding any new structures.

The consequence of the above considerations for irreducible GNS representations of the Weyl $C^{\star}$ algebra is that they are all unitarily equivalent to the Schrödinger representation. It follows that these GNS representations now may be given a clear physical meaning, which carries over from the meaning of operators and vectors in the standard Hilbert space account of quantum mechanics. The fact that a GNS representation may be taken as being the representation standard quantum mechanics would assign to a given system in certain states is the tacit assumption which underlies Strocchi's arguments about closed worlds, where the action of the operators on the vectors of the GNS representation space is given the physical interpretation of changing the state the system is in[9]. The interpretation of the GNS representation as being the representation describing the system in the conventional quantum mechanical picture will make its

[8]The regularity condition means that the representations $\pi(U(a))$, $\pi(V(b))$ are assumed to be strongly continuous in a, b respectively.

[9]A more detailed discussion about this point can be found in section 2.5.

appearence also in the context of quantum theories where the Stone-von Neuman theorem no longer holds, as described below.

The Stone-von Neumann theorem concerns quantum theories where the CCR's are satisfied by a finite number of pairs of canonical variables. If one instead takes this number to be infinite, the premises of the theorem are no longer satisfied, and as J. Earman points out in [8, p. 340], the theorem breaks down. Unitarily inequivalent irreducible representations of the algebra of observables then become possible, as will be exemplified in section 2.7. The next subsections will show that this is the key factor in allowing for different closed worlds to exist for a given theory (where the concept of *folium*, defined in section 2.5, will be central) and, after algebraic SSB will have been defined in section 2.6, it will turn out that these phases are connected by the broken symmetries.

2.5 Algebraic states in the Hilbert space picture

In the algebraic picture, a physical theory is not only described by its algebra $\mathcal{A}$ of observables, but also by the set of states on it, that is to say normalized positive linear functionals on $\mathcal{A}$ that correspond to the expectations of the observables when the system is in a given physical state. When considering Hilbert space representations of a given theory, one thus also needs to find a way of incorporating the algebraic states.

For a given state ω, the GNS theorem provides a Hilbert space representation of the algebra on $\mathcal{H}_\omega$, with one special vector Ψ_ω that, according to the relation $\langle \Psi_\omega, \pi_\omega(A)\Psi_\omega\rangle = \omega(A)$, is in correspondence with the algebraic state ω. This does not necessarily mean that, for a GNS representation on $\mathcal{H}_\omega$, all the states on the algebra in consideration have this association with some vector in $\mathcal{H}_\omega$. This fact deserves some further considerations.

One may first of all note that any normalized vector Φ in $\mathcal{H}_\omega$ defines a normalized, positive linear functional on $\mathcal{A}$:

$$\phi(A) \equiv \langle \Phi, \pi_\omega(A)\Phi \rangle. \tag{10}$$

This is not only true for vectors (also known as *pure states*), but the same holds more generally for density matrices (or *mixed states*):

$$\phi(A) \equiv Tr(\rho_\phi \pi_\omega(A)). \tag{11}$$

This motivates the question of what subset of the algebraic states of a given physical system may be represented in the above way on a given GNS representation. The concept of folium of a state answers this question:

Definition Let ω be a state on a $C^\star$ algebra. ω's *folium* $\mathcal{F}_\omega$ is the set of all states expressible as density matrices on ω's GNS representation $(\pi_\omega, \mathcal{H}_\omega)$.

The definition of folium captures the concept of closed world discussed in the previous subsections. To see why this is the case one needs to go back to the passage by

F. Strocchi cited in section 2.2, where it is said that via the GNS representation, "a state identifies the family of states related to it by observables, equivalently accessible from it by means of physically realizable operations." What exactly is meant by this statement can be understood by the identification, mentioned above, of the GNS representation (in the case where the system in consideration is quantum-mechanical) with the Hilbert space and operators the standard form of quantum mechanics would associate with the physical system in consideration.

As explained in section 2.4, the fact that this identification is justified in the finite case is suggested by the fact that there are no further possible irreducible representations (up to unitary equivalence). This identification adds to the purely algebraic account some physical information such as the transition amplitude between two states (obtained from the inner product of the vectors representing those states) as well as predicting what state the system will be in after an observable has been measured on it. As pointed out in [25, p. 10], the standard account of quantum mechanics states that, in the simplifying case where the result of such a measurement is a simple eigenvalue, after the measurement has been performed the system will be in the state corresponding to the eigenvector to that eigenvalue (this is often referred to as the *projection postulate* or as *state reduction*). In this sense an observable "relates" a state (corresponding to a vector in the GNS representation space) to the set of states given by the eigenvectors of that observable, and the "physically realizable" operation that embodies this relation, changing the initial state into a final state in this set, is that of the measurement of the observable in the initial state.

The cyclicity condition of the GNS representation then implies that, starting from the state the GNS representation is associated to, all the states represented as vectors on this representation space can be reached by physical operations and no other states may in this way be reached, which is why such a representation is taken to describe a closed world. The algebraic states contained in this closed world are the ones expressible on this specific GNS representation, which correspond to the folium of the state the GNS representation is associated to.

As was noted above, the identification of the GNS representation with the representation space of a conventional quantum-mechanical theory adds certain physical interpretations to the Hilbert space states of the GNS representation. This is the kind of extra structure that, as was mentioned in section 2.1, needs to be added to the algebraic account that has so-far been developed in order to fully describe the physical system in consideration. As was pointed out there, it has not yet been possible to incorporate such elements into the purely algebraic setting, and indeed in this case they are introduced only once one has a concrete Hilbert space representation.

From these considerations one deduces that if different GNS representations exist with different folia, then these representations will describe the system in different phases, and a single GNS representation will not be enough to express all the physically possible scenarios that are encoded in the states on the algebra. For finite quantum-mechanical systems this variety is not possible, since from the Stone-von Neumann

Theorem there can only be one representation of the algebra up to unitary equivalence. Furthermore, as L. Ruetsche states in [19, p. 96], if two GNS representations π_ω and π_ϕ are unitarily equivalent, then their folia coincide: $\mathcal{F}_\omega = \mathcal{F}_\phi$. Thus for finite quantum-mechanical systems there exists a single folium. This folium will contain all the states on the algebra, since every algebraic state belongs to its own folium.

As mentioned above, the fact that the Stone-von Neumann theorem no longer applies for infinite quantum systems allows for unitarily inequivalent representations of the algebra of observables to exist, and thus the existence of a single folium is no longer guaranteed[10]. States belonging to different folia will belong to different closed worlds, and there will be no physically realizable operation linking the two. It is interesting to note that two irreducible GNS representations π_ω and π_ϕ are either unitarily equivalent, or the folia $\mathcal{F}_\omega$ and $\mathcal{F}_\phi$ have null intersection, which is to say they are disjoint [19, p.98]. In the latter case, no state expressible on one GNS representation is expressible on the other, and not only do the two GNS representations describe two different closed worlds, but starting from one state in each representation there is no common state which may be reached by physical operations; in other words they are disjoint, with no overlap. The study of the class of unitarily inequivalent irreducible GNS representations of a given algebra can thus be seen as the study of the physically possible, disjoint worlds the system described by the algebra may exist in.

The following subsection explains that the existence of unitarily inequivalent representations of the algebra also allows for algebraic symmetries (which will be defined there) to be non-unitarily implementable on the operators of a given GNS representation. This phenomenon is what will be defined to be SSB in the algebraic approach, and it will be shown to connect unitarily inequivalent GNS representations. It thus follows that, as was briefly mentioned in section 2.2, a spontaneously broken algebraic symmetry connects different closed worlds.

From the point of view of ordinary quantum mechanics, the fact that non-unitarily implementable symmetries may exist sounds suspicious, not only because for finite systems this does not occur, but also because Wigner's theorem resounds in any quantum physicist's mind as soon as the word "symmetry" is uttered, linking it to unitarity. Following [10], in quantum mechanics one usually takes a symmetry to be a map $S : \Pi(\mathcal{H}) \to \Pi(\mathcal{H}')$, $[\psi] \mapsto [\psi']$ from the set of unit rays[11] in a given Hilbert space $\mathcal{H}$ to the set of unit rays in another Hilbert space $\mathcal{H}'$, such that the transition probabilities are preserved: $|\langle\psi_1, \psi_2\rangle| = |\langle\psi_1{}', \psi_2{}'\rangle|$. This definition captures the notion of symmetry as "invariance with respect to a specified transformation group" [4, p. 322], since the transformation in consideration leaves invariant the structures encoded within the states describing the physical system in consideration. Wigner's theorem then states

[10] To simplify the following discussion, attention will only be given to irreducible GNS representations. Note that, as stated in [24, p. 71], a GNS representation π_ω is irreducible iff ω is *pure*, that is to say it cannot be decomposed as a convex linear combination of other states on the algebra.

[11] A unit ray is the equivalence class given by $[\psi] = \{\lambda\psi; \lambda \in \mathbb{C}, ||\psi|| = 1\}$. It contains the physical information encoded in a Hilbert space vector, since a global phase is irrelevant.

that such a transformation can only be implemented on the Hilbert state vectors by a linear isometry[12] which, in the case of the map S being bijective (which will be assumed), corresponds to a unitary transformation on the vectors. Thus symmetries are implemented quantum-mechanically by unitary operators.

Keeping the above arguments in consideration, how can it be that an algebraic symmetry which - if worthy of its name - should also leave the structure of the physical system in consideration invariant, is not unitarily implementable? As will be pointed out in the next subsection, the apparent contradiction arises from the frequent use of imprecise terminolgy in the literature: algebraic symmetries turn out to be unitarily implementable in Wigner's sense (on the Hilbert space vectors), but nonetheless in the broken case, non-unitarily implementable on the operators.

2.6 Spontaneous Symmetry Breaking

Although algebraic symmetries have been mentioned, a definition needs yet to be provided:

Definition An *algebraic symmetry* of an algebra $\mathcal{A}$ of observables is defined to be an automorphism β of $\mathcal{A}$ which preserves all algebraic relations, including the $\star$.

Such a transformation of $\mathcal{A}$ corresponds to a transformation of the physical system it describes which leaves its structure invariant, and is thus a symmetry according to the definition provided at the end of the previous subsection.

An important point to stress is that not every algebraic symmetry of $\mathcal{A}$ is a *physical* symmetry of the system it describes, and in the same way not every symmetry in Wigner's sense is a symmetry of the system being described in the conventional quantum-mechanical picture. The reason for this is that the structure of the theory describing a physical system is not all there is to it. One also needs to keep the dynamics in consideration, which in the standard quantum-mechanical picture is done by requiring that the action of the symmetry should leave the Hamiltonian of the system invariant. Since, as was pointed out in section 2.1, it has not yet been understood how to incorporate the source of the dynamics into the algebraic approach, in this case one must find a way, outside the scope of strictly algebraic considerations, of determining which symmetries are physical. In the case where the starting point for the analysis of a given system is a Hilbert space representation in the standard quantum-mechanical picture, one may take to be physical those algebraic symmetries (of the $C^\star$ algebra generated by the observables in the representation) that, acting on the observables, leave the Hamiltonian invariant. At this point one may discard the specific representation one started with, and consider the abstract algebra describing the system, with the symmetries that have been identified as physical. An example of this procedure is exemplified in section 2.7, following L. Ruetsche's exposition in [20].

[12] The isometry could, according to Wigner's theorem, also be antilinear (an antilinear operator A: $\mathcal{H} \to \mathcal{H}'$ is a map such that $A(\lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle) = \bar{\lambda}_1 A|\psi_1\rangle + \bar{\lambda}_2 A|\psi_2\rangle$; $\lambda_i \in \mathbb{C}$, $|\psi_i\rangle \in \mathcal{H}$). For simplicity this case will be omitted, as in most physical applications the linear case is the relevant one.

A given algebraic symmetry β on a GNS representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$ is said to be *unitarily implementable* if there is a unitary operator $U_\beta : \mathcal{H}_\omega \to \mathcal{H}_\omega$ such that

$$\pi_\omega(\beta(A)) = U_\beta \pi_\omega(A) U_\beta^{-1}. \tag{12}$$

On the other hand, one may consider the action of a symmetry β on the states as follows: if ω is a state on $\mathcal{A}$, then so is the symmetry-transformed state $\tilde{\beta}\omega$, defined by

$$(\tilde{\beta}\omega)(A) \equiv \omega(\beta(A)). \tag{13}$$

As J. Earman notes in [8, p.399], "If the automorphism represents time evolution, the difference between the two points of view amounts to the difference between the Heisenberg and Schrödinger pictures". Then, according to the GNS theorem, up to unitary equivalence:

$$\pi_\omega(\beta(A)) = \pi_{\tilde{\beta}\omega}(A), \tag{14}$$

and it follows that unitarily implementability of a symmetry β in π_ω is equivalent to the states ω and $\tilde{\beta}\omega$ having unitarily equivalent GNS representations, and thus belonging to the same folium. This means that in the unitarily implementable case, the physical description given by ω's GNS representation is β-symmetric, in the sense that the symmetry connects states expressible as density matrices on this GNS representation (β is a transformation in a closed world).

One can now introduce the concept of spontaneous symmetry breaking in the algebraic approach of this section.

Definition *of SSB*: A state ω is said to break a symmetry β, if β is not unitarily implementable in ω's GNS representation.

In this case ω and $\tilde{\beta}\omega$ have unitarily inequivalent GNS representations. The symmetry β connects these representations, in the sense that it can be seen as connecting the vector $\Psi_\omega \in \mathcal{H}_\omega$ corresponding to ω with the vector $\Psi_{\tilde{\beta}\omega} \in \mathcal{H}_{\tilde{\beta}\omega}$ corresponding to $\tilde{\beta}\omega$. If the GNS representations of ω and $\tilde{\beta}\omega$ are irreducible, it follows that in the spontaneously broken case $\mathcal{F}_\omega \cap \mathcal{F}_{\tilde{\beta}\omega} = \emptyset$, and the closed worlds connected by the symmetry are disjoint. As F. Strocchi points out [24, p. 120], if the initial state ω is pure, then also the symmetry transformed state $\tilde{\beta}\omega$ will be. It then follows that if the GNS representation of the initial state is irreducible, this will also hold for the transformed state's GNS representation, and thus the restriction of SSB to pure states deals with symmetries that connect disjoint folia.

As mentioned above, this essay will not go beyond dealing with the case of pure states/irreducible representations, and this will be sufficient to understand the examples which will be exposed in the course of this essay. This does not mean that further considerations about the general case of mixed states would not bring interesting results, and an elaboration in this direction can be found in [19]. There one finds a necessary and sufficient condition for the folia of two states to coincide, which is that of *quasi* *equivalence*, an understanding of which requires the introduction and discussion of von Neumann algebras, which this essay will not deal with.

In the previous subsection it was mentioned that the existence of non-unitarily implementable symmetries may potentially cause some confusion with respect to Wigner's theorem. In fact, in several occasions this has been the case, as for example in [20, p. 483], where it is stated that broken symmetries do not preserve transition probabilities; this cannot be the case, since it would contradict the very nature of symmetry, as is noted in [2, p. 7]. The important distinction that needs to be made is between unitary equivalence of representations and unitary implementability of symmetries as acting on Hilbert space states. There is no contradiction in the fact that the symmetry-transformed states are connected by a unitary mapping, while the representations acting on the Hilbert spaces related via this symmetry are unitarily inequivalent. This is indeed the case in algebraic SSB, as is explained in detail in [2].

As a quick aside, one should note that the algebraic form of SSB is a purely quantum-mechanical phenomenon, since - as noted above - the GNS representations of classical systems do not have the same interpretation as for quantum systems. Thus, although any symmetry connecting the folia of different pure states would be non-unitarily implementable in the classical case (since different pure states then define unitarily inequivalent irreducible representations with folia consisting of single states), this does not count as a case of SSB because in this case it is a purely mathematical phenomenon without a physical counterpart. Of physical relevance in the classical case is the unique, phase space representation.

The next subsection will present an example of how the algebraic considerations done so far can be employed to better understand a concrete physical model, which in this case will be a one-dimensional, infinite spin chain in the standard quantum formalism. It will be pointed out that in this model there exist different configurations of the system belonging to unitarily inequivalent representations, and it will be argued that the symmetry connecting such configurations, once lifted to the algebraic level, will turn out to be spontaneously broken. By understanding this model in the algebraic approach, one thus makes sense of the (from the conventional point of view) unusual mathematical phenomenon of the occurrence of unitarily inequivalent representations by identifying them to be a case of spontaneous symmetry breaking in the algebraic formulation. The unitary inequivalence of these representations thus gains physical significance, as one then understands it to distinguish between physical realisations of the system being described belonging to different closed worlds.

2.7 An example: the infinite spin chain

Consider a one-dimensional spin chain, with spin $\frac{1}{2}$ variables at the sites of a doubly infinite lattice labeled by the integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. As L. Ruetsche explains in [20, p. 478], to construct a quantum theory of this system one associates with each site k a self-adjoint operator $\sigma^k = (\sigma_x^k, \sigma_y^k, \sigma_z^k)$ satisfying the following algebraic relations:

$$[\sigma_x^k, \sigma_y^{k'}] = i\delta_{kk'}\sigma_z^k \quad \text{and cyclic exchange of x, y, z ;} \quad \sigma^k \cdot \sigma^k = 3\mathbb{1}. \tag{15}$$

These relations are equivalent to taking this variables at a given lattice point to satisfy the canonical anticommutation relations (CARs), and to commute at different lattice sites.

One way of constructing such a theory is by considering the set of sequences $S^{(+)} = \{s_k = \pm 1, k \in \mathbb{Z}, s_k \text{ has finitely many entries which differ from } s_k^{(+)}\}$ where $s_k^{(+)}$ is the sequence with all entries equal to $+1$, and by defining $\mathcal{H}^{(+)}$ to be the Hilbert space of square-summable functions on $S^{(+)}$, which is to say

$$\left\{ f : S^{(+)} \to \mathbb{C} | \sum_{s \in S^+} |f(s)|^2 < \infty \right\}, \tag{16}$$

with the inner product

$$\langle f, g \rangle^{(+)} = \sum_{s \in S^+} \bar{f}(s) g(s). \tag{17}$$

As G. Sewell points out in [21, p. 16], a complete orthonormal basis for this space is provided by the vectors $\{\phi_s^{(+)} | s \in S^{(+)}\}$, as defined by the formula

$$\phi_s^{(+)}(s') = \delta_{ss'} \quad \forall s, s' \in S^{(+)}. \tag{18}$$

The elements of this basis are in one-to-one correspondence with the elements in $S^{(+)}$.

One may now define the operators $\{\sigma^{k(+)} = (\sigma_x^{k(+)}, \sigma_y^{k(+)}, \sigma_z^{k(+)}) | k \in \mathbb{Z}\}$ in $\mathcal{H}^{(+)}$ in such a way that the action of $\sigma^{k(+)}$ on $\phi_s^{(+)}$ is the analogue of that given by the action of an isolated Pauli spin operator on the canonical basis of $\mathbb{C}^2$ as follows:[13]

$$\sigma^{k(+)} \phi_s^{(+)} = \left(s_n \phi_{\theta_n s}^{(+)}, i s_n \phi_{\theta_n s}^{(+)}, s_n \phi_s^{(+)} \right) \tag{19}$$

where θ_n is the transformation whose action on a configuration of $S^{(+)}$ changes the sign of the n-th component and leaves the rest of the sequence unchanged. It is now straightforward to see that the algebraic relations (15) are valid on the basis vectors $\phi_s^{(+)}$, and therefore the operators ${\sigma_{x,y,z}^k}^{(+)}$ form a representation of these relations in $\mathcal{H}^{(+)}$. As argued in [21, p. 16], this representation is irreducible, since the passage between any

[13] If one sets $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ with $\sigma_x, \sigma_y, \sigma_z$ the three Pauli matrices and one denotes $\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then the action of σ on ϕ_s is given by the relation $\sigma\phi_s = (\phi_{-s}, i s\phi_{-s}, s\phi_s)$ for $s = \pm 1$. This is generalized in the following considerations, where $\sigma^{k(+)}$ is taken to act only on the k-th site of the lattice analogously to the single-particle case, while leaving the other sites of the lattice unchanged.

two of the basis vectors ϕ_s and $\phi_{s'}$, which implies the reversal of a finite number of spins, is implemented by the action of a monomial in the operators $\sigma_x^{k\,(+)}$.

As $\mathcal{H}^{(+)}$ is spanned by sequences with only finitely many elements differing from $+1$, one would expect the polarization of every configuration of the infinite spin chain described by an element in this Hilbert space to point in the positive z-direction. This is indeed the case: the polarization observable is given by the limit as $N \to \infty$ of

$$m_N^{(+)} = \frac{1}{2N+1} \sum_{n=-N}^{N} \sigma^{n\,(+)}, \tag{20}$$

and the expectation value of this observable (for finite N) in a state $\phi_s^{(+)}$ is

$$\langle \phi_s^{(+)}, m_N^{(+)} \phi_s^{(+)} \rangle = \left(0, 0, \frac{1}{2N+1} \sum_{n=-N}^{N} s_n \right). \tag{21}$$

In the limit one thus obtains the polarization $\langle \phi_s^{(+)}, m_\infty^{(+)} \phi_s^{(+)} \rangle = (0,0,1)$ as expected. This result can be extended to the expectation of $m_\infty^{(+)}$ on any unit vector in $\mathcal{H}^{(+)}$ [21, p.17], and thus a polarization in the positive z-direction is a global property of this representation space.

Analogously to the above considerations one may construct a representation of the infinite spin chain on a Hilbert space $\mathcal{H}^{(-)}$ with a negative polarization in the z-direction as a global property, by considering the set of sequences that differ by finitely many elements from the sequence $s_k^{(-)} = -1$ for all k.

So far two representations of the algebraic relations (15) describing the infinite spin chain have been constructed, and it has been shown that they contain states with distinct physical properties, namely these representations capture the spin chain in configurations with opposite polarizations. This characteristic is by now familiar from the algebraic considerations done in the previous subsections. The algebraic formulation developed so far offers a context in which to understand the different Hilbert space descriptions that have been found for the infinite spin chain, which would with difficulty be understood in the conventional account of quantum systems, where one would expect a single Hilbert space to be sufficient to contain all the possible states of a system. The mathematical structures which, with the knowledge gained so far, one would expect to be responsible for such a multiplicity are unitarily inequivalent representations of the algebraic relations describing the spin chain. This is indeed the case, and can be seen as follows: if the representations $\sigma^{(\pm)}$ were unitarily equivalent, there would be a unitary mapping $U : \mathcal{H}^{(+)} \to \mathcal{H}^{(-)}$ such that $U\sigma^{k\,(+)}U^{-1} = \sigma^{k\,(-)}$ for all k, which would imply that

$$U m_N^{(+)} U^{-1} = m_N{}^{(-)}. \tag{22}$$

For $|\psi^+\rangle$ and $|\psi^-\rangle$, unit vectors in $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$ related by $|\psi^-\rangle = U|\psi^+\rangle$, it follows that

$$\langle \psi^+, m_N^{(+)} \psi^+ \rangle = \langle \psi^-, m_N^{(+)} \psi^- \rangle. \tag{23}$$

This leads to a contradiction, since in the limit as $N \to \infty$ the left- and right-hand sides tend to different values. One thus concludes that the quantizations represented on $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$ are not unitarily equivalent, as expected from the algebraic point of view. Some further considerations are still required though in order to fully describe this example in the algebraic approach.

The states in $\mathcal{H}^{(\pm)}$ are clearly asymmetric with respect to rotations, but for them to qualify as symmetry breaking (in any of the ways described until now in this essay) one first of all needs rotational invariance to be a physical symmetry of the system in consideration. So far one cannot say this is the case, as the spin chain has not been equipped with a Hamiltonian. This could be done for example by considering the infinite spin chain in the context of the Heisenberg model of ferromagnetism, where neighboring spins in the chain are supposed to interact via the Hamiltonian

$$H = -J \sum_k \sigma_k \cdot \sigma_{k+1}, \tag{24}$$

where J is a positive real number [20, p. 479]. This Hamiltonian is minimized by configurations with all spins aligned in the same direction, and thus $s^{(+)}$ and $s^{(-)}$ correspond to energetically equivalent ground states. Furthermore, H is invariant under rotations of the spin operators σ^k, and thus rotational invariance is a physical symmetry of this model. As a consequence, in the conventional approach, the rotational symmetry is spontaneously broken, with $s^{(+)}$ and $s^{(-)}$ symmetry-breaking ground states.

In the following, these ground states will also be shown to be symmetry-breaking in the algebraic sense, which is a fact one may already suspect at this point: various characteristic features of this phenomenon are readily recognizable in the infinite spin-chain as expounded so far, as for example the unitary inequivalence of Hilbert space representations describing this system (which one would expect to be connected by the map implementing a rotation from the positive to the negative z-direction), as well as the configurations in the inequivalent representations belonging to two different, disjoint closed worlds. In fact, analogously to the considerations done in section 2.2, an isolated infinite spin chain with the Heisenberg Hamiltonian, starting in a configuration in $\mathcal{H}^{(+)}$ cannot evolve to a configuration in $\mathcal{H}^{(-)}$, since this would involve the flipping of an infinite number of spins, which would require an infinite amount of energy.

A first step towards describing the infinite spin chain in the algebraic approach is determining the $C^\star$ algebra describing it. As argued in appendix A, with a Hilbert space account of such a system, it is straight-forward to do so (with the $\star$ operation being given by the Hilbert space adjoint operation). The $C^\star$ algebra thus obtained, satisfying the algebraic relations (15), will be called the *CAR algebra*, and the representations $\sigma^{(\pm)}$ can now be seen as representations of this algebra. The ground states in $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$, represented by Hilbert space vectors $|\Omega^{(+)}\rangle$ and $|\Omega^{(-)}\rangle$ respectively, are expected to correspond to algebraic states ω_+ and ω_-, as discussed in section 2.5.

In order to be understood in the light of the algebraic considerations done in this section, the representations on $\mathcal{H}^{(\pm)}$ would need to be GNS representations of the CAR $C^\star$ algebra. The lacking piece of information in order to be able to denote them as such is the existence of a cyclic Hilbert space vector. One would expect the vectors $|\Omega^{(\pm)}\rangle$ to be good candidates in $\mathcal{H}^{(\pm)}$, as starting from these it is possible to reach any one of the basis vectors $\phi_s^{(\pm)}$ by applying a combination of the operators $\sigma^{k(\pm)}$. This expectation is confirmed by [20, p. 482], and thus $\sigma^{k(+)}$ and $\sigma^{k(-)}$ are unitarily inequivalent GNS representations of ω_+ and ω_-.

In order for the states $\omega_\pm$ to qualify as symmetry breaking in the algebraic sense, one firstly needs to identify the (physical) algebraic symmetries of the spin chain. As was expressed in section 2.6, the algebraic frame is not automatically equipped with the concept of physical symmetry, and one must find another way of determining which symmetries are physical. As was suggested there (and as put in practice for this example by [20, p. 483]), in the case where the starting point is a quantum mechanical system described in the standard account, one may take to be physical those algebraic symmetries that, implemented on the operators in the Hilbert space representation describing the system, leave the Hamiltonian invariant.

In the specific case at hand, the spin reversal symmetry is physical because the Heisenberg dynamics are invariant under its action. This action can be seen as the implementation of the following automorphism θ of the CAR algebra:

$$\theta(\sigma_z^k) = -\sigma_z^k, \quad \theta(\sigma_y^k) = \sigma_y^k, \quad \theta(\sigma_x^k) = \sigma_x^k. \tag{25}$$

As one may expect, this algebraic symmetry, acting on the states, connects the algebraic states describing the two polarized ground states of the spin chain:

$$\omega_- = \tilde{\theta}\omega_+, \quad \omega_+ = \tilde{\theta}\omega_-. \tag{26}$$

Since, as pointed out in section 2.6, unitary implementability of an algebraic symmetry β in the GNS representation of a state ω is equivalent to the states ω and $\tilde{\beta}\omega$ having unitarily equivalent GNS representations, and due to the fact that the GNS representations of ω_+ and ω_- are unitarily inequivalent, it thus follows that the symmetry θ is spontaneously broken in the algebraic sense.

The careful reader may have objected, at the beginning of this example, that a number of unitarily inequivalent representations for the infinite spin chain would actually not be surprising from the point of view of standard quantum mechanics, as the Stone von-Neumann theorem only applies to systems described by the CCR algebra. The answer to this objection is that there exists an analogous theorem, known as the Jordan-Wigner Uniqueness Theorem, which applies to representations of the CAR algebra for a finite number of "spin" variables. As stated in [19, p.62], this theorem claims that

Theorem 2.3 *(Jordan-Wigner Uniqueness Theorem) For each finite n, every irreducible representation of the CARs is unitarily equivalent to every other.*

This assures that the occurence of unitarily inequivalent representations of the CARs is solely caused by taking the number of canonical variables in the algebra to tend to infinity, as is the case for the CCRs. This prompts an investigation of what exactly happens, in the Hilbert space picture, when taking the limit $N \to \infty$. One interesting aspect to consider is for example the following: in the conventional quantum-mechanical picture one would expect the vectors of a single (irreducible) Hilbert space representation to fully describe the states of a given physical system, so somehow this representation "splits up" in the infinite limit; how does this happen, and are there any physical consequences?

To consider this case, the starting point needs to be, rather than the infinite spin chain, the case with a finite number of lattice sites, ranging from $-N$ to N. One may then consider the set $S_N = \{s_k = \pm 1, k = -N, ..., N\}$ of sequences, and construct a representation for the CARs as above. The Hilbert space $\mathcal{H}_N$ thus obtained will contain all the possible configurations of the finite spin chain with $2N + 1$ lattice sites. One may do this construction for every $N < \infty$, and in the limit as $N \to \infty$, the representation spaces $\mathcal{H}^{(\pm)}$ will emerge as subrepresentations in the larger Hilbert space $\mathcal{H}_\infty$ generated by the construction based on all sequences in $S_\infty = \{s_k = \pm 1, k = -\infty, ..., \infty\}$.

Although in the limit $\mathcal{H}^{(\pm)}$ form irreducible representation spaces, for every $N < \infty$ the representation space $\mathcal{H}_N$ will be irreducible (since every vector can be reached by the action of the observables), and will thus fully describe the finite spin chain in the conventional picture. Keeping this in consideration, one may argue that $\mathcal{H}_\infty$ could be taken to represent the infinite spin chain, although not in the conventional picture, but as a Hilbert space containing all possible irreducible representations, and thus all possible information about the system. If one were to take this point of view, then one would gain additional physical information about the system, namely the fact that by calculating the overlap of different states within $\mathcal{H}_\infty$ one could quantify the transition amplitudes between these states. In the example of the subspaces $\mathcal{H}^{(\pm)}$ of $\mathcal{H}_\infty$, one can easily see that they are orthogonal, and thus no quantum transitions between them would be possible in this interpretation. This fact is the realization, in this example, of the expectation that in the infinite limit tunneling amplitudes between different broken phases should be zero.

Summarising, what has been shown so far is that in the limit as the spin chain becomes infinite the Hilbert space account of the system goes from one irreducible representation to many disjoint, irreducible representations (of which two specific ones were considered above), each of which is sufficient to describe a given physical realization of the infinite spin chain (a "closed world"), but the multitute of which stands for the number of different such physically possible realizations. Between these closed worlds there is no possible physical evolution, tunneling being also excluded.

The algebraic approach has proven useful in offering both mathematical and physical insight into the example of the infinite spin chain, which is an idealized, one dimensional model of what may, in the thermodynamic limit, describe a "real-world" ferromagnet. Apart from the thermodynamic limit of condensed-matter systems, there is another very important class of theories where an infinite number of canonical variables makes its appearance, and that is theories describing quantum fields. The next section will be dedicated to considering this case in some detail, although due to the nature and depth of this subject the discussion will not possibly be exhaustive nor fully rigorous. Furthermore, attention will be given solely to global symmetries, as in this case some analogies to the algebraic formulation may be drawn.

3 Heuristic QFT

3.1 Symmetries

In the canonical procedure of quantization of a field theory, which is the one this section will be based on, the starting point is a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$. Analogously to the quantization of classical systems with finite dimensional phase space, the canonical quantization procedure for fields "promotes" the variables ϕ_a and $\pi_a \equiv \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_0 \phi_a)}$ to operators satisfying, in the bosonic case, the equal time commutation relations

$$[\hat{\phi}_a(x,t), \hat{\pi}_b(x',t)] = i\delta_{ab}\delta^3(x - x') \tag{27}$$

$$[\hat{\phi}_a(x,t), \hat{\phi}_b(x',t)] = 0 \tag{28}$$

$$[\hat{\pi}_a(x,t), \hat{\pi}_b(x',t)] = 0. \tag{29}$$

From the algebraic perspective, this step corresponds to taking the field operators to be a representation of this algebra, which is a version of the CCR's exposed in equation (5) for the case where the number of canonical variables is infinite and indexed by a continuous label (the spacetime coordinates). As opposed to, for example, the algebra describing the infinite spin chain, this algebra is the result of two limiting processes: that of taking the spatial volume of the region where the fields are defined to be infinite, as well as taking spacetime to be a continuum.

What Hilbert space the commutation relations are represented on is not at all obvious. In the standard account of QFT, the representation space which one considers is the Fock space spanned by all the possible particle excitations of the *free* vacuum. This is obtained by considering the creation and annihilation operators $a_i^\star$ and a_i that diagonalise the quadratic part of the Hamiltonian (obtained from the Lagrangian density via Legendre transformation), postulating the existence of a vacuum state $|0\rangle$ which is annihilated by all a_i's, and then identifying the particle states with the states obtained from $|0\rangle$ by applying any combination of $a_i^\star$'s.

As long as the theory in consideration is free, that is to say the Hamiltonian is purely quadratic, this procedure poses no problems; but if one chooses this representation space for an interacting theory - as in the standard perturbative treatment of QFT - one encounters mathematical problems that show this procedure to be inconsistent. To be more specific, in the standard interaction picture one takes the Hilbert space the commutations relations (27-29) are represented upon to be the Fock space constructed for the free theory. But, as R. Haag argues in [11, p. 56-57], this cannot be a consistent procedure, since if one considers the simplest example of an interacting scalar field with coupling g as given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}g\phi^4, \tag{30}$$

different coupling constants g, g' must give rise to inequivalent representations. In particular if $g = 0$ (which corresponds to the free case), it follows that the representation, for any $g' \neq 0$, must be inequivalent to the Fock representation. Haag's conclusion, which is known as *Haags theorem*, is thus that the determination of the representation class of the equal times commutation relations is a "dynamical problem", which is to say that it not only depends on the form of the commutation relations, but also on the value of the coupling constant g, which influences the dynamics.

As mentioned above, the standard account of perturbative QFT, this inconsistency notwithstanding, uses the Fock representation independently of the value of the coupling constant. This is indicative of the nature of the standard approach to QFT, which contains some heuristic elements as the one just mentioned, whose justification is in the success such theories have had in describing physical processes. For this reason this account of QFT will sometimes be referred to in the following as *heuristic QFT*.

The examples of SSB in QFT presented in this section will, in line with the arguments given above, in some measure be based on heuristic assumptions, and a thorough understanding of them in the algebraic framework developed in this essay, as was done for the infinite spin chain, will not generally be possible. Nonetheless, there exist analogies between algebraic SSB and the different approaches of SSB in the conventional account of SSB in QFT, and in some cases it will be possible to highlight sound connections. Firstly, though, it is necessary to study symmetries in QFT, both at the classical level and in their implementation in the quantized theory.

In the context of QFT, global symmetries (that is to say symmetries which act independently of the space-time coordinates) of a physical system described by a given Lagrangian are identified already at the classical level, that is to say before canonical quantisation, as global transformations on the fields that leave the action $I = \int d^4x\mathcal{L}$ invariant. In the case where the symmetry is continuous, Noether's theorem then associates a conserved current j^μ and thus a conserved charge to this transformation:

$$Q = \int d^3xj^0, \quad \partial_0 Q = 0. \tag{31}$$

In addition to being conserved, the charges corresponding to a symmetry at the classical level furthermore *generate* the symmetry transformation, that is to say that the Poisson bracket of a charge and the field gives the corresponding infinitesimal change in the field:

$$\{Q, \phi_i\} = \left.\frac{\partial A_\xi(\phi_i)}{\partial \xi}\right|_{\xi=0}, \tag{32}$$

where $A_\xi(\phi_i)$ is the family of continuous transformations of the fields ϕ_i and $A_0(\phi_i)$ corresponds to the identity transformation.

This motivates the expectation that, in the quantum case, the quantized charge generates the analogous transformation on the fields:

$$-i[\hat{Q},\hat{\phi}_i] = \left.\frac{\partial A_\xi(\hat{\phi}_i)}{\partial \xi}\right|_{\xi=0}, \tag{33}$$

and that $A_\xi(\hat{\phi}_i)$ is a unitary transformation on the fields, in that it is given by the exponential of the (hermitian) charge $\hat{Q}$:

$$A_\xi(\hat{\phi}_i) = U_\xi \hat{\phi}_i U_\xi^{-1}, \quad U_\xi = e^{-i\xi\hat{Q}}. \tag{34}$$

A subtle point is that the quantized charge operator $\hat{Q}$ need not necessarily be well-defined. In fact, a theorem by Fabri and Picasso [1, p. 71] states that for the action of the quantized charge $\hat{Q}$ on the vacuum there are only two possibilities: either $\hat{Q}|0\rangle = 0$ or $\hat{Q}|0\rangle$ has infinite norm, thus not belonging to the representation space of the theory[14].

In the case where $\hat{Q}|0\rangle = 0$, and no other factors conspire to make $\hat{Q}$ ill-defined, one obtains via exponentiation a well-defined unitary operator as one would usually expect. On the other hand, if $\hat{Q}|0\rangle$ has infinite norm, this operator will be ill-defined, and one may wonder what exactly happens, where $\hat{Q}|0\rangle$ "goes to". As J. Earman notes in [8, p.342] the charge operator corresponding to a finite volume in space is well defined, while it is only the infinite volume limit which is not. This situation is in some respects similar to the case of the spin chain considered in section 2.7, and an analogy with that example may provide an intuition of the mechanisms at work here. For the finite spin chain, the action of a global rotation on the states of the representation space $\mathcal{H}_N$ (as for example flipping the sign of all the spins) would be a mapping back into $\mathcal{H}_N$, but in the limit $N \to \infty$, as the relevant representation space for the spin chain would become for example $\mathcal{H}^{(+)}$, the action of such a global rotation would map, for example, into the representation space $\mathcal{H}^{(-)}$. Thus, if one would attempt to describe it as a map from $\mathcal{H}^{(+)}$ into itself, this map would be ill-defined.

Fabri-Picasso's theorem, with its criterium for determining when the quantized charge generates a unitary transformation on the states of a given representation of the field algebra motivates Aitchison's definition in [1, p. 71] of SSB in QFT:

Definition *of SSB (Aitchison's version):* A symmetry in QFT is said to be spontaneously broken if $\hat{Q}|0\rangle \neq 0$, thus not belonging to the Hilbert space $\mathcal{H}$ of physical states.

As noted above, one expects $\hat{Q}$ to generate the symmetry transformation on the operators as in (33). If this expectation is satisfied, then in the case where $\hat{Q}|0\rangle \neq 0$ it

[14]Consider the norm of $\hat{Q}|0\rangle$: $\langle 0|\hat{Q}\hat{Q}|0\rangle = \int d^3x \langle 0|j^0(x)\hat{Q}|0\rangle = \int d^3x \langle 0|j^0(0)\hat{Q}|0\rangle$, where the last equality follows from $j^0(x) = e^{-iP\cdot x} j^0(0) e^{iP\cdot x}|0\rangle$ and $[\hat{Q}, P^\mu] = 0$ (since the symmetry is internal) where P^μ is the generator of spacetime translations. It then follows that either $\hat{Q}|0\rangle$ has infinite norm (which corresponds to the case where $\langle 0|j^0(0)\hat{Q}|0\rangle \neq 0$) or $\langle 0|j^0(0)\hat{Q}|0\rangle = 0$, from which it follows $\langle 0|\hat{Q}\hat{Q}|0\rangle = 0$ and thus $|\hat{Q}|0\rangle = 0$.

follows that the symmetry transformation is not unitarily implementable, and Aitchison's version of SSB then implies what R.F. Streater's opening comment in [22] takes to be SSB in QFT: "the term 'spontaneous breakdown of symmetry' (...) has come to mean a field theory whose Lagrangian is invariant under a certain transformation of the fields, whereas there exist solutions, i.e. realizations of the algebra of operators, that do not possess the symmetry as a unitary transformation". This prompts the following definition:

Definition *of SSB (Streater's version):* A symmetry transformation of the field operators in a given representation (orginating at the classical level from a symmetry of the Lagrangian) is said to be spontaneously broken if it is not unitarily implementable in that representation.

Neither of the two definitions provided so far is the one most commonly found in the literature. A standard textbook discussion of SSB in QFT, as found for example in Peskin and Schroeder [18, p. 348], takes it to the following case:

Definition *of SSB (Peskin and Schroeder's version):* A symmetry of a given QFT is said to be spontaneously broken if a given field configuration of the classical Lagrangian which minimizes the energy (i.e. a vacuum configuration) is not invariant under the action of the symmetry on the (classical) fields.

If one also keeps in consideration that quantum fields are systems where one would expect possible the algebraic version of SSB (as they are infinite quantum systems), then Baker and Halvorson's comment in [2, p. 1] that "the precise mathematical definition of spontaneous symmetry breaking (SSB) in quantum theory is somewhat up for grabs" may seem justified. Nonetheless, these different accounts of SSB all embody in one way or the other the general notion of broken symmetry: Peskin and Schroeder take the relevant reduction of symmetry of a theory to occur at the classical level, before the fields are quantised, while both Streater's and Aitchison's versions consider directly the quantized theory, and are thus closer to the algebraic formulation.

Since quantum fields are described by an infinite number of canonical variables, one may expect all these definitions to be more or less equivalent[15], this can hardly be proven, due to the heuristic nature of the standard account of QFT. For example, in order to relate Peskin and Schroeder's version to the others, one needs to specify how their account of SSB carries over to the quantized theory. This step, which is exemplified in section 3.3 in the Goldstone model, will be argued to rely on heuristic arguments, and its mathematical consistency still remains to be shown.

Connections can more readily be identified between Streater's definition of SSB and the algebraic approach, as in both versions unitarily inequivalent representations are the decisive factor in allowing for SSB. From the algebraic point of view a symmetry transformation on the field operators may be seen as the implementation of an algebraic

[15] As opposed to the case of SSB in classical and quantum systems in the standard account, where only certain cases of SSB - namely in infinite quantum systems - are expected to also qualify as SSB in the algebraic sense.

symmetry, with the field operators taken to be a representation of the defining algebra. If the representation in consideration is cyclic (as is for example the case for a Fock representation, as was noted above), then by identifying the expectations of the field operators on the cyclic vector with a physical algebraic state, the representation of the field operators becomes a GNS representation. Then, SSB in Streater's sense translates into SSB in the algebraic sense.

On the other hand one could consider a scenario where the field operators are a GNS representation of the abstract algebra, and an algebraic symmetry induces a transformation on the operators that corresponds to a global symmetry of the Lagrangian. In this case, if the symmetry is spontaneously broken in the algebraic sense it is also broken in Streater's sense.

It is important to note that, although straight-forward, these connections cannot be guaranteed to be mathematically rigorous, since as it has been pointed out above, heuristic QFT allows for mathematical inconsistencies in its formulation.

For most symmetries commonly treated in examples of QFT the transformation generated by $\hat{Q}$ is well-defined, and implements the symmetry transformation of the fields as expected, but as the considerations done so far have revealed, it need not necessarily be the case, and this can be confirmed by considering the following example.

3.2 A free example: the massless Klein-Gordon field

A free theory with a symmetry that is not unitarily implementable is given by the massless Klein-Gordon field, with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi. \tag{35}$$

It is clear that the transformation $\phi \to \phi + \xi$ with ξ a constant function is a symmetry of the Klein-Gordon theory. In the light of the Fabri-Picasso theorem, one would expect such a symmetry to be non-unitarily implementable, since the vacuum state is not annihilated by the quantized charge operator, and thus the charge does not generate a well defined transformation: the conserved current associated to this symmetry is $j^{\mu} = \partial^{\mu}\phi$, and thus the conserved charge is given by $Q = \int d^3x\partial^0\phi$. Upon quantization, this will contain both annihilation and creation operators, and thus will not annihilate the vacuum.

This expectation is satisfied: at the quantum level the action of the symmetry corresponds to the transformation $\hat{\phi} \to \hat{\phi} + \mathbb{1}\xi$ on the operators being a symmetry of the quantum theory. If it were implemented unitarily on a vacuum representation, there would be a unitary operator U_{ξ} such that

$$U_{\xi}\hat{\phi}U_{\xi}^{-1} = \hat{\phi} + \mathbb{1}\xi. \tag{36}$$

At this point one needs to take into account the fact that the group of Poincaré transformations Λ is taken to be represented as acting on the states of the theory by unitary operators U_Λ (an assumption motivated by Poincaré invariance and Wigner's theorem[16]), and that under these transformations the vacuum is the unique invariant vector [13, p. 5]. Since the symmetry transformation considered above is internal (that is it acts independently of the spacetime coordinates), it will commute with the Poincaré transformations, and it thus follows, as explained in [19, p.318], that

$$U_\xi|0\rangle = |0\rangle. \tag{37}$$

It now turns out that for the symmetry to be unitarily implementable is a contradiction: since the vacuum is taken to be a Fock space vacuum, it is orthogonal to all one-particle states $a_i^\star|0\rangle$, and thus so is the field operator $\hat{\phi}$, which is a linear combination of creation and annihilation operators: $\langle 0|\hat{\phi}|0\rangle = 0$. From this it follows that:

$$\langle 0|\hat{\phi} + \mathbb{1}\xi|0\rangle = \xi. \tag{38}$$

On the other hand, according to the assumption of unitary implementability

$$\langle 0|\hat{\phi} + \mathbb{1}\xi|0\rangle = \langle 0|U_\xi\hat{\phi}U_\xi^{-1}|0\rangle = \langle 0|\hat{\phi}|0\rangle = 0 \tag{39}$$

and thus the contradiction.

The symmetry in this example is spontaneously broken in Peskin and Schroeder's sense, as a given (classical) vacuum configuration is not invariant under the action of the symmetry. It is also spontaneously broken in Streater's sense, as the symmetry is not unitarily implementable and in Aitchison's sense as $\hat{Q}$ does not annihilate the vacuum. Finally, it is also spontaneously broken in the algebraic sense, as the considerations done at the end of the last subsection for going from SSB in Streater's sense to algebraic SSB apply here.

As one may have been expecting following Haag's theorem, in this non-interacting case no mathematical inconsistencies have been encountered in choosing the representation space to be a Fock space, and this has made it possible to frame this example also within the algebraic approach. In order to encounter these mathematical problems, as well as the heuristic arguments often used to motivate the quantisation of classical field theories with degenerate vacua, it will be necessary to consider an interacting case, as will be done in the following subsection.

3.3 An interacting example: the Goldstone model

After having seen a free example, this subsection will be dedicated to a brief review of an interacting example exhibiting SSB, known as the Goldstone model. The main purpose of this example will be to illustrate the kind of heuristic arguments used to

[16] Poincaré invariance is a symmetry of the theory, thus - by Wigner's theorem - one has a unitary operator associated with each such transformation. The fact that such operators form a representation can be deduced by the fact that the action of two subsequent Poincaré transformations should be equivalent to the action of the composite transformation.

motivate the multitude of representations which are expected to arise in this model, as well as point out the menace posed by mathematical inconsistencies in this heuristic approach.

The Goldstone model, as exposed in [1, p. 83-87], is given by the following Lagrangian density for a (classical) complex scalar field ϕ:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \frac{1}{2}\lambda^2 |\phi|^4 + \frac{1}{2}\mu^2 |\phi|^2. \tag{40}$$

$\mathcal{L}$ has the global $U(1)$ symmetry $\phi \to e^{-i\alpha}$, with constant α. In the case where $\mu^2 > 0$, it can easily be shown that the ground state configurations of the fields lie along the circle $Re(\phi)^2 + Im(\phi)^2 = \mu^2/\lambda^2$; these configurations are not left invariant by the symmetry, which is thus spontaneously broken in Peskin and Schroeder's sense.

It is then argued that the degeneracy of classical ground states suggests what L. Ruetsche [19, p. 323] calls a "'semi-classical' approximation" which consists of the expectation that, in the quantized case, the field theory vacuum should be associated with $|\phi|^2 = \text{constant} = \mu^2/\lambda^2$. That is to say: the vacuum vector in a given representation should be $|\omega\rangle$ such that

$$\langle \omega | \hat{\phi} | \omega \rangle = \frac{1}{\sqrt{2}} e^{i\omega} |\mu| / |\lambda|, \tag{41}$$

where ω is some phase angle.

As noted by Aitchison in [1, p. 85], a rigorous proof that the quantised version of the Lagrangian (39) with $\mu^2 > 0$ has indeed $|\omega\rangle$ as the vacuum state, and that it is consistent to require $\langle \omega | \hat{\phi} | \omega \rangle \neq 0$ seems not to be available. This is the kind of inconsistencies that was mentioned above.

Assuming that such a procedure is actually consistent, this model may be argued to exhibit Aitchison's version of SSB, since as he argues in [1, p. 85], in this case the charge operator is not well defined. One would thus also expect this to be a case of Streater's version of SSB. In this case, provided the representation of the field algebra containing a given vacuum vector $|\omega\rangle$ is cyclic, then this would be found to exhibit SSB also in the algebraic sense.

3.4 The idealisations involved in the infinite limit

The concept of infinite systems has often made its appearance in the course of this essay, and the crucial role played by an infinite number of canonical variables in making algebraic SSB possible has been stressed. Taking this into account, one may wonder if the algebraic form of SSB has any relevance whatsoever to the physical world, composed of physical systems of finite extent, where the infinite limit is an idealization of such systems. As J. Earman points out in [6, p. 191], "a sound principle of interpretation would seem to be that no effect can be counted as a genuine physical effect if it disappears when the idealizations are removed". The unique characteristics exhibited by algebraic SSB may thus, according to this point of view, simply not be manifest in the physical world. There are nonetheless, he argues, two possible ways of retaining the physical relevance of such a phenomenon. The first approach would be to consider the finite version of theories exhibiting algebraic SSB: one would expect them to exhibit the characteristic features of this phenomenon in some approximation, and thus the idealization would simply serve to crystalize these features. What one may for example expect, as he points out, would be for the symmetry, acting on a given state ω's GNS representation $\mathcal{H}_\omega$ unitarily, would send some Hilbert space states to others whose overlap would be as close to zero as desired as the volume is increased. Such states exist for example in the spin chain presented in section 2.7. These promising considerations are, on the other hand, stunted by the fact that the difference between the finite and the infinite case is clearly marked by the boundary between an automorphism being unitarily implementable and being non-unitarily implementable. If the unitary implementability of a symmetry turned out to be relevant to any of the observed features of SSB, he argues, then the infinite idealization could not be discarded.

The second approach Earman proposes is to claim that the infinite volume limit is not an idealization: assuming that all matter is described by QFT, and taking quantum fields to permeate all of space (which for this approach to hold must be of infinite extent), it follows that all physical systems are infinite, even though the states they are in appear to be spatially localized. From this point of view the situation is reversed, with the treatment of certain systems as being spatially finite objects as an idealization done for practical purposes.

G.G. Emch and C. Liu [16, p. 155-156] offer a point of view in some respects similar to Earman's first approach, while stressing the importance of idealizations as tools for highlighting qualitative aspects of the systems in consideration that exist prior to the idealization. G. Sewell's considerations are in line with this approach. He points out in [21, p. 4-5] that the idealization of macroscopic systems as ones possessing infinite numbers of degrees of freedom has long been essential to statistical thermodynamics, where "the characterisation of phase transitions by singularities in thermodynamic potentials necessitates a passage to the mathematical limit in which both the volume and the number of particles of a system tend to infinity in such a way that the density remains finite". This limit, he notes, "has served to replace the merely quantitative distinction between systems of 'few' and 'many' (typically 10^{24}) particles by the qualitative distinction between finite and infinite ones, and has thereby brought new physically relevant structures into the theory of collective phenomena". In the same way, he argues, in the algebraic approach to quantum theories, this qualitative distinction between macrostate[17] and microstate is achieved in the infinite limit, corresponding respectively to a given representation in the set of unitarily inequivalent ones, and a vector in the representation.

[17] What Sewell denotes as macrostate corresponds to a closed world in Strocchi's interpretation.

4 Conclusions

The algebraic framework presented in this essay brings a novel perspective on physical theories, and from this point of view one has a clear understanding of the peculiar phenomena that arise within infinite quantum systems. Although the algebraic approach proposes to fully encompass all physical theories within its frame, some of its limitations have been pointed out, and it is yet to be seen if the important case of heuristic QFT may be completely understood in the algebraic language. On the other hand, where it is possible to frame a physical system in the algebraic setting, as in the example of the infinite spin chain, the undertanding of SSB gained in the algebraic setting carries over to the conventional description of such systems, explaining structures that would else be puzzling. In these cases the relevant physical symmetry is also expected to be spontaneously broken in the standard account, but with an additional feature, namely that a concrete realisation of the system is confined to a broken phase.

As was argued in the second part of this essay, in heuristic QFT the possibility of mathematical inconsistencies makes an algebraic understanding problematic, but nonetheless analogies between the various accounts of SSB are possible, and the algebraic approach provides some intuition of the mechanisms that may be at work in the heuristic account.

The physical relevance of the considerations done in this essay is threatened by the crucial role played by the infinite idealization in allowing for the algebraic version of SSB to occur, and its status is debated in the final section of this essay.

5 Acknowledgments

I am particularly grateful to Nazim Bouatta and Jeremy Butterfield for detailed comments and helpful discussions on the present essay.

A $C^\star$ algebras

Definition A $C^\star$ algebra $\mathcal{A}$ is defined to have the following properties:

1. it is a linear associative algebra over the field of complex numbers, i.e. a vector space over $\mathbb{C}$ with an associative product linear in both factors;

2. it is a normed space, and the product is continuous with respect to the norm:

$$\|AB\| \leq \|A\|\|B\|. \tag{42}$$

   Furthermore $\mathcal{A}$ is a complete space with respect to the topology defined by the norm (thus $\mathcal{A}$ is a Banach algebra);

3. it is a $\star$-(Banach) algebra, i.e. there is an involution $\star : \mathcal{A} \to \mathcal{A}$ such that

$$(A+B)^\star = A^\star + B^\star, \quad (\lambda A)^\star = \bar{\lambda} A^\star, \quad (AB)^\star = B^\star A^\star, \quad (A^\star)^\star = A; \tag{43}$$

4. the following "$C^\star$-condition" holds:

$$\|A^\star A\| = \|A\|^2. \tag{44}$$

Firstly, the algebraic structure of a classical-mechanical physical system will be determined, and it will turn out to be that of an abelian $C^\star$ algebra. The considerations will be restricted to classical Hamiltonian systems, the states of which are described by a point in the phase space manifold (or more generally by a probability distribution), and the observables by continuous functions.

From the considerations done in chapter 2, one would expect the algebraic structure of such theories to be that of a $C^\star$ algebra, and this is indeed the case. Briefly, one can see this as follows: one should start by taking the elements of the algebra to be the functions, with the product on the algebra to be given by the pointwise multiplication, the identity given by the identity function, and the $\star$ operation defined by ordinary complex conjugation. These properties describe a $\star$-algebra with identity. Limiting the considerations to compact phase spaces, one can define the norm of a function to be the supremum of its absolute value; this norm then also satisfies the $C^\star$ condition, and thus the algebra of classical observables with the above norm is a $C^\star$ algebra.

Note that the commutativity of the ordinary pointwise product of two functions implies that this algebra is abelian, which is the fundamental property that sets it apart from algebras describing quantum mechanical systems, which are generally described by non-abelian $C^\star$ algebras, as will be outlined below.

In classical mechanics, every state determines the expectation of the observables on that state, and this is given by the functional

$$\omega(f) = \int f d\mu_\omega, \quad \omega(1) = 1 \tag{45}$$

where $d\mu_\omega$ is the probability measure associated with the state in consideration. From the definition of ω it is straightforward to see that the linearity condition

$$\omega(\lambda f_1 + \mu f_2) = \lambda\omega(f_1) + \mu\omega(f_2), \quad \forall f_1, f_2 \in A, \quad \lambda, \mu \in \mathbb{C} \tag{46}$$

and the positivity condition

$$\omega(f^\star f) \geq 0, \quad \forall f \in A \tag{47}$$

are both satisfied, and thus ω is a normed, positive linear functional on the algebra of observables. Thus, as expected, one can see how a classical system has the algebraic structure exposed in chapter 2.

The algebraic structure of systems described by quantum mechanics also turns out to be that of a $C^\star$ algebra, but in contrast with classical systems, the product on the algebra is noncommutative, and thus the algebra is non-abelian. In standard quantum mechanics observables are described by a subset of the bounded self-adjoint operators on a separable complex Hilbert space $\mathcal{H}$, and the states by density matrices on $\mathcal{H}$. From these postulates, and by analogous considerations as for the classical case, one may identify the underlying algebraic structure[18], which is indeed the one discussed in chapter 2. The reason why the algebra is non-abelian is the fact that the product on the algebra of observables is taken to be the product of operators, which in general does not commute.

[18] As for example done in chapter 2.4 for the CCR algebra.

References

[1] I.J.R. Aitchison. *An informal introduction to gauge field theories*, Cambridge University Press, 1982.

[2] D. Baker and H. Halvorson. *How is spontaneous symmetry breaking possible?* (Preprint).

[3] O. Brattelli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2*, Second Edition, Springer, 1997.

[4] E. Castellani. On the meaning of symmetry breaking. *Symmetries in Physics*, chapter 19, Cambridge University Press, 2003.

[5] S. Coleman. *Aspects of symmetry*, Cambridge University Press, 1985.

[6] J. Earman. Curie's Principle and spontaneous symmetry breaking, *International Studies in Philosophy of Science*, 18:2-3, 173-198.

[7] J. Earman. Laws, symmetry and symmetry breaking: Invariance, conservation principles, and objectivity. *Philosophy of Science*, 71 (2004), pp. 1227-1241.

[8] J. Earman. Rough guide to spontaneous symmetry breaking, in K. Brading and E. Castellani (eds.). *Symmetries in Physics: Philosophical Reflections*, Cambridge University Press, 2003.

[9] F. Englert. Broken symmetry and Yang-Mills theory. *Fifty years of Yang-Mills theory*, World Scientific Publishing Co., 2005.

[10] G.M. Graf. *Quantenmechanik I und II*, Lecture notes '10/'11, ETH Zurich.

[11] R. Haag. *Local Quantum Physics*, Springer, 1996.

[12] R.L. Hudson. A Stochastic Double Product in Non-Fock Quantum Stochastic Calculus. *Operator Algebras, Operator Theory and Applications*, Operator Theory Advances and Applications, Vol. 195, Birkhaeuser, 2010.

[13] A. Jaffe and E. Witten. Quantum Yang-Mills Theory. *The Millenium prize problems*, Clay Mathematics Institute, 2006.

[14] G. Jona-Lasinio. Cross fertilization in theoretical Physics: the case of condensed matter and particle physics. *Symmetries in Physics*, chapter 18, Cambridge University Press, 2003.

[15] P. Kosso. The Epistemology of Spontaneously Broken Symmetries. *Synthese*, Vol. 122, No. 3 (Mar., 2000), pp.359-376.

[16] C. Liu and G. Emch. Explaining quantum spontaneous symmetry breaking. *Studies in History and Philosophy of Modern Physics*, 36 (2005), pp. 137-163.

[17] M. Morrison. Spontaneous symmetry breaking: theoretical arguments and philosophical problems. *Symmetries in Physics*, chapter 21, Cambridge University Press, 2003.

[18] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory* , ABP, 1995.

[19] L. Ruetsche. *Interpreting Quantum Theories*, Oxford University Press, 2001.

[20] L. Ruetsche. Johnny's So Long an the Ferromagnet. *Philosophy of Science*, Vol. 73, No. 5, Proceedings of the 2004 Biennial Meeting of The Philosophy of Science Association.

[21] G. Sewell. *Quantum Mechanics and its Emergent Macrophysics*, Princeton University Press, 2002.

[22] R.F. Streater. Spontaneous breakdown of symmetry in axiomatic theory. *Proceedings of the Royal Society of London*, A, Vol. 287, pp. 510-518.

[23] F. Strocchi. *An introduction to the Mathematical Structure of Quantum Mechanics (2nd Edition)*, Advanced Series in Mathematical Physics Vol. 28, World Scientific, 2008.

[24] F. Strocchi. *Symmetry Breaking*, Lect. Notes Phys. 643 (Springer Berlin Heidelberg 2005).

[25] H.M. Wiseman and G.J. Milburn. *Quantum Measurement and Control*, Cambridge University Press, 2010.

[26] E. Witten. From Superconductors and four-manifolds to weak interactions. *Bulletin-American Mathematical Society* 44 (2007), pp.361-391.

[27] S. Weinberg. *The Quantum Theory of Fields*, Vol. II. Cambridge University Press 1996.