Peirce's Topical Continuum: A "Thicker" Theory

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Abstract
Although Peirce frequently insisted that continuity was a core component of his philosophical thought, his conception of it evolved considerably during his lifetime, culminating in a theory grounded primarily in topical geometry. Two manuscripts, one of which has never before been published, reveal that his formulation of this approach was both earlier and more thorough than most scholars seem to have realized. Combining these and other relevant texts with the better-known passages highlights a key ontological distinction: a collection is bottom-up, such that the parts are real and the whole is an *ens rationis*, while a continuum is top-down, such that the whole is real and the parts are *entia rationis*. Accordingly, five properties are jointly necessary and sufficient for Peirce's topical continuum: rationality, divisibility, homogeneity, contiguity, and inexhaustibility.

Keywords: Charles S. Peirce; collection; connection; continuity; continuum; dimensionality; limit; part; portion; whole.

Charles Sanders Peirce made no secret of his conviction that continuity was a central tenet—perhaps *the* central tenet—of his overall philosophical system, but he struggled over the course of several decades to work out an adequate conception of it. Scholars have identified four (Potter and Shields 1977) or five (Havenel 2008) broad stages of that effort, the most significant of which are the last two—the "Kantian" or "supermultitudinous" period, beginning around 1895 or 1897, and the "post-Cantorian" or "topological" period, beginning around 1908.

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Most of the recent secondary literature has dealt with the super-multitudinous conception (Havenel 2008, 104–117), which Peirce attempted to develop within his "quasi-Cantorian theory of collections" (Moore 2013, 103). Moore even calls it "*the* Peircean continuum" despite conceding that this is "not strictly accurate," because it constitutes "a mathematical analysis of continuity that matched up with the philosophical motivations" (2007a, 425). Nevertheless, he concludes that "the errors in Peirce's thinking about continuity are serious and they do serious damage" (2013, 117).

There is a sense in which this focus is misplaced, since Peirce ultimately preferred the topological conception (Havenel 2008, 117–125). Moore takes it up separately, calling it the "topical" theory in accordance with Peirce's own nomenclature, in which the relevant branch of mathematics is "geometrical topics" or "topical geometry" rather than "topology" (Moore 2015a, 1056). I will adopt the same practice, especially since modern point-set topology embraces the Cantorian approach to continuity that Peirce explicitly rejected (Havenel 2008, 124).

Moore distinguishes two kinds of theory—"thin" for "a more or less partially worked out guess," or "thick" for "a fully articulated and at least partially substantiated account"—and in his estimation, "Peirce does have a topical theory of continuity, thinly speaking, but does not have one, thickly speaking" (2015a, 1057). He adds, "All other things being equal, a clear definition is a powerful thickening agent, whose absence virtually guarantees thinness. To state and explicate such a definition is to go most, if not all the way, to a theory in the thicker sense" (*ibid.*, n6). Evidently his opinion is that Peirce never supplied a clear topical definition of continuity.

In this essay, I argue otherwise, quoting and discussing two manuscripts that past authors have never cited. Taking these into account, I then reinterpret several more familiar passages and ultimately propose what I believe qualifies as a "thicker" version of Peirce's topical theory.

On General Topic

The first overlooked manuscript is R 144, titled "On General Topic" by Peirce and undated by Robin (1967).[1] Analysis of the handwriting suggests that he composed it around 1900, and certainly no later than 1906,[2] so it precedes by several years the alleged beginning of the "topological" period (Havenel 2008, 117). In fact, Peirce admits within the same time frame that true continua cannot be treated mathematically as if they were infinite collections:

> The hypotheses of mathematics relate to systems which are either finite collections, infinite collections, or true continua; and the modes of reasoning about these three are quite distinct. These, then,

> constitute three orders. The last and highest kind of mathematics, consisting of topical geometry, has hitherto made very little progress; and the methods of demonstration in this order are, as yet, little understood. (CP 1.283, 1902)[3]

Further advancement in topical geometry was needed, and perhaps R 144 was an attempt to spur it along. Since it is unpublished and just four pages long,[4] I will quote it in its entirety and provide interspersed commentary.

> *Art*. 1. *Topic* is the geometrical theory of continuous connections. It is divisible into *general* and *special topic*. *General topic* treats of the general principles of the subject in their application to continua of each dimensionality. *Special topic* treats of particular problems such as the number and description of topically regular nets, the number of colors requisite to distinguish regions on a map and the like. (R 144:1)

This is a rare case where Peirce utilizes "topic" in the singular, and perhaps the only occasion where he splits it into general and special branches. The first statement comports with at least two of his previous definitions: "It is the study of the continuous connections and defects of continuity of loci which are free to be distorted in any way so long as the integrity of the connections and separations of all their parts is maintained" (CP 4.219, 1897); and "[t]hat which topic treats of is the modes of connection of the parts of continua" (RLT 246, 1898).[5]

> *Art*. 2. Topic considers Time and Space. This Time and this Space are mathematical hypotheses not necessarily coinciding in their properties with the Time and Space of the real world. The words *Time* and *Space* in this sense are proper names and as such are properly written with capital initial. (R 144:1)

Although inconsistent about it in the remainder of the text, Peirce capitalizes *Time* and *Space* to signal that they are being "used, not as vernacular, but as terms defined" (CP 6.452, EP 2:434, 1908).[6] They refer to "mathematical hypotheses," rather than "the Time and Space of the real world," although this does not preclude them from accurately representing the latter.

> Time and Space have certain properties, which will be explained, to which the name of *continuity* is applied. Anything having continuity is called *continuous* or a *continuum*. The first property of continuity is that everything continuous has parts. The parts of Time are called *times*. A time, in the sense of a part of Time, being a common noun, is written without a capital. The parts of Space are called *places*. (By a *space*, in the terminology here employed, a particular kind of place is meant.) (R 144:1)

Continuity is the attribute possessed by anything that is continuous, which is properly called a continuum; Peirce does not otherwise differentiate the three terms. He identifies four associated properties in this manuscript, the first of which "is that everything continuous has parts," whose nature he subsequently explains in terms of the other three. The parts of Time and Space are referred to by common nouns, *times* and *places*. Peirce never gets around to specifying what "particular kind of place is meant" by a *space*.

> The second property of a continuum is that it can be made up (in how many ways will be stated below) of two continuous parts, having no such parts in common. A continuous time is called a *lapse* of time. A continuous place is called a *room*. Every lapse of time is composed of two lapses of times which have no lapse of time in common. Every room is composed of two rooms. It is usually said that of two different times neither of which is a part of the other one is *later* the other *earlier*. Putting aside for the moment a vagueness which will be found to attach to this notion, we may provisionally adopt the statement. If any time, A, is earlier than any other time, B, it is earlier than every time, C, which is later than B. (R 144:1–2)

The first two properties, taken together, amount to a restatement of "Kant's definition that a continuum is that of which every part has itself parts of the same kind" (CP 6.168, PMSW 138, c. 1903–1904).[7] One sense in which at least some parts of a continuum are "of the same kind" is that they are likewise continuous, rather than discrete. A continuous part of Time is a *lapse*, and a continuous part of Space is a *room*. Peirce says nothing further about the latter, instead establishing the directionality of Time.

Portions and Limits

There is another kind of part that a continuum must have:

> The third property is that in order to make up a continuum, two continua must have something in common, but their common part need not be like them in complexity of its composition. By a *portion*, in the terminology of this memoir, is meant a part of like complexity of composition of its whole. A *limit* between two portions of a continuum having no common portion is the part of lower complexity of composition. (R 144:2)

The second property requires "two continuous parts" of a continuum to have "no such parts in common," but the third property requires them to "have something in common." This calls for an explanation, and Peirce obliges by distinguishing a *portion*, which is "a part of like complexity of composition of its whole," from a *limit*, which is "the part of lower complexity" between two adjacent portions. He then offers a thoroughly topical definition of *dimensionality*:

> The *dimensionality* of a continuum is the number which measures the complexity of its composition. If the limit between two portions of a continuum having no common portion is not continuous, that continuum is said to have its dimensionality equal to one, or to have *one dimension*. If the limit between two portions of a continuum that have no common portion is, at highest, of dimensionality, N, that continuum is said to have its *dimensionality* equal to N+1, or to have N+1 *dimensions*. (R 144:2)

Rather than the minimum number of coordinates to specify a point within a continuum, here dimensionality "measures its complexity of composition." Every portion has the same dimensionality as the continuum itself, and if the limit that adjacent portions have in common is *discrete* (e.g., a point), then the continuum that it divides has one dimension (e.g., a line). Similarly, the dimensionality of a more complex continuum is one greater than that of the highest-dimensional *continuous* limit that can divide it into two non-overlapping portions.

> The fourth property is that there is no multitude of limits which embraces all the possible limits in a continuum. By a *multitude* is meant the character by which one collection of distinct objects is greater than another. One collection, A, is said to be *greater* than another, B, if there is a possible one-to-one correspondence of all the subjects* of B to some of the subjects of A, but no possible one-to-one correspondence of all subjects of A to some subjects of B.
>
> *I prefer to speak of the *subjects* of a collection rather than its objects, as being an expression more in harmony with the general terminology of logic. They are certainly subjects of the character which defines the collection. (R 144:2)

This might seem like a summary of the supermultitudinous conception, but it is the possible *limits* in a continuum that Peirce claims are beyond all multitude, not its potential *portions*. He then acknowledges that multitude is only meaningful when comparing "one collection of distinct objects" to another. The standard definition in terms of "one-to-one correspondence" follows, and the accompanying footnote alludes to the fact that a collection "always must embrace whatever there may be in the universe that has a certain character" (CP 4.649, 1908).

> I have proved (*Monist* _____) that of any two collections one is greater than the other, or else there is a possible one-to-one correspondence between all the subjects of the one and all the subjects of the other.

> I have also proved (Ibid. p. ___), what is very important, that the collection of all the possible ways of separating a collection into two parts, is, in every case, greater than the collection, A. (R 144:2)

Although he left the citation and page number blank, Peirce is referring to his article in the January 1897 issue of *The Monist*, "The Logic of Relatives" (CP 3.456–552), which includes his versions of the Generalized Continuum Hypothesis and Cantor's Theorem as restated here. Their combination is what Moore calls the Step Lemma (2007a, 426; 2007b, 62–63).

> But the possible limits between portions even of a unidimensional continuum exceed any infinite multitude. And the possible ways of separating them into parts, could they be regarded as so many distinct things, would be no greater [in multitude] than the limits themselves. But they do not form a collection, because they are not distinct, but merge into one another. (R 144:2–3)

Again, what "exceed any infinite multitude" are "the possible limits between portions," which are of lower dimensionality than the continuum itself. In the case of "a unidimensional continuum," such as a line or a lapse of time, those limits—no matter how numerous—are necessarily discrete and indivisible; but in order to form so vast a collection that there is no greater multitude, they must somehow "merge into one another." This is a fundamental problem with the supermultitudinous conception, and the topical theory furnishes a solution that Peirce does not yet notice: the potential portions of a continuum "do not form a collection, because they are not distinct."

> According to this conception, the system [of] real quantities, that is, of all rational fractions together with all possible limits of convergent infinite series of rational fractions do not constitute a continuum. For those quantities form a collection equal in multitude to the collection of possible ways of separating the collection of all whole numbers into two parts. And, being a collection, it is not so great as the collection of possible ways of separating *it* into two parts, and so on *ad infinitum*. (R 144:3)

Contrary to Cantor, Peirce denies that the real numbers "constitute a continuum." Although infinite, they are distinct subjects of a collection whose multitude is the next greater than that of "the collection of all whole numbers." Consequently, there is another collection of next greater multitude, and another greater than that, "and so on *ad infinitum*."

Instants and Moments

However, that is not the case for *instants* as the possible limits between adjacent portions of Time:

> The instants, or limits between possible complementary portions of a lapse of time, are not exceeded by anything. This can be so, owing to the instants merging into one another and not existing as distinct objects. In the ordinary way of reasoning upon the subject there is a *petitio principii* consisting in assuming that the "assignable" instants are *all* the instants. According to the present conception of time, whether it agrees with the properties of real time or not, if instants were assigned for all the rational quantities in their order, there would be for each limit of a convergent infinite series of such quantities, not a single instant, which would thus be "assignable," but a continuous time containing instants corresponding to all the instants of any continuous time. Such a time whose parts, according to the imagined system of measurement, would not be distinguishable by distinct quantities, and so would not be separately "assignable," I would term a *moment*. In another system of measurement, some of its instants would be separately assignable, and another system of "moments" would result. (R 144:3)

Just as Cantor treats the aggregate of points corresponding to the real numbers as if it were a continuous line, "the ordinary way of reasoning" about Time treats the "assignable" instants as if they were "*all* the instants." This begs the question by presupposing—wrongly, in Peirce's view—that a continuum, like a collection, consists of discrete elements. His alternative is to define a *moment* as the continuous time that is divided by an "assignable" instant in accordance with an "imagined system of measurement," such as the rational or real numbers. Unlike lapses, each moment shares a common portion with the immediately earlier and later moments.

A moment is thus "an infinitesimal duration" (CP 6.111, EP 1:315, 1892) or "timelet," the temporal equivalent of a "microsegment" in synthetic differential geometry, also known as smooth infinitesimal analysis (Bell 2006, 287–288). This category-theoretic approach is perhaps the most promising contemporary candidate for rigorous mathematical treatment of a Peircean continuum (Herron 1997, 621–623; Moore 2007, 468n45; Havenel 2008, 99–100, 111–112).

> All this is not a conjecture. It is demonstrable that such a hypothesis involves no contradiction; but at present, having another purpose in view, I do not stop to give the demonstration. It was, however, important to state the property, because it follows that a lapse of time does not necessarily have a final and an initial instant. If two lapses of time make up together a lapse of time, no instant can be omitted. The limiting instant must be there. And since an instant does not exist by

> itself in a lapse of time, but only by virtue of being continuously led up to, so an instant cannot by itself be taken away from a lapse of time. Hence, the two lapses cannot form a lapse, unless the limiting instant is in both. (R 144:3)

Demonstrating that hypothesizing infinitesimals "involves no contradiction" was requisite during Peirce's lifetime, because it was contrary to the reigning consensus.[8] Glossing over this initially, he restates the second and third properties of Time as a continuum: any lapse can be made up of two lapses that have a common instant between them, although it need not "have a final and an initial instant." For example, the present is "assignable" as the "limiting instant" between the past, which has no initial instant, and the future, which has no final instant. However, we *experience* the present as an indefinite moment (see CP 7.649–657, 1903) rather than as a durationless instant, and the text concludes with a different example that highlights the resulting logical difficulty:

> If a proposition is true up to an instant and from that instant ceases to be true, it is at that instant either both true and not true, or neither true nor not true. And if this is absurd, it shows that the proposition, as supposing a sudden change, which is an absurd phrase, involves an absurdity in a continuum, from which absurdity it can be freed by so modifying [it] as to make it possible for it at once to be in form both true and false, or else neither true nor false. For example, suppose the proposition to be "It is not yet noon." Since no instantaneous noon exists as a distinct part of time, if it is true quite up to noon itself then it must have been true already by the time noon is reached. If this is absurd, the absurdity may be rectified by changing the proposition to "In all this neighborhood of time noon is future," of which the denial is "In all this neighborhood of time noon has past." For since an instant does not exist separately, nothing is true except by being true throughout some neighborhood of time. (R 144:3–4)

Any perceived absurdity stems from "supposing a sudden change," which would be necessary in a succession of discrete instants conforming to classical logic but is impossible if Time is a true continuum. Instead, for a proposition to be true at all, it must be "true throughout some neighborhood of time." In the manuscript, the first two occurrences of this new term replace the crossed-out word "moment," implying rough equivalence. The instant assigned to precisely noon "does not exist separately" from its surrounding moment, and neither the proposition of interest ("noon is future") nor its denial ("noon has past") is true during the *entire* noon moment.

Peirce is right that there is no contradiction here, but the principle of excluded middle does not hold while a continuous change is in progress. More than a decade before anyone else (Fisch and Turquette 1966, 72), he outlined a non-classical system of logic that could accommodate such indeterminacy:

> Triadic Logic is that logic which, though not rejecting entirely the Principle of Excluded Middle, nevertheless recognizes that every proposition, S is P, is either true, or false, or else S has a lower mode of being such that it can neither be determinately P, nor determinately not-P, but is at the limit between P and not P. (R 339:624[344r], 1909 Feb 23)

The limit between two portions of a continuum corresponds to the limit between truth and falsity, so "the *natural* logic associated to the connecting modes of the *continuum* is really an intermediate logic—the intuitionistic logic" (Zalamea 2012, 26). This is precisely the logic of synthetic differential geometry and smooth infinitesimal analysis (Bell 2006, 294–297).

Logical Graphs

The second overlooked manuscript is RS 30, untitled by Peirce and comprising groups of pages that he labeled as Copies S–U and W–Z. Until very recently, just one paragraph from Copy T was published,[9] as a footnote to CP 4.564; the editors date it "c. 1906," calling it "one of a number of fragmentary manuscripts designed to follow the present article," which is "Prolegomena to an Apology for Pragmaticism" (CP 4.530–584, 1906). However, Roberts states that it actually "was an early draft" of that paper (1973, 36n8), and Robin's date is 1905 (1971, 50).

As the latter's title of "Logical Graphs" suggests, the primary subject matter is Peirce's system of existential graphs (EGs).[10] A brief account of continuity becomes necessary when he characterizes two EGs—the blank sheet representing coexistence, and the line of identity representing an individual—as *continuous* because "any portion of an Instance of such a Graph is itself an Instance of the same Graph" (RS 30:11[Copy T:5]):

> In the first place, then, I do not call a line, or a surface, or anything else, continuous unless every part of it that is homogeneous in dimensionality with the whole and is marked off in the simplest way is, in respect to the connexions of its parts, precisely like every other such part; although, if the whole has but a finite number of interruptions, I do call it "continuous in its uninterrupted portions." (RS 30:11–12[Copy T:5–6])

This echoes R 144 by describing the two different kinds of parts that a continuum must have: portions that are "homogeneous in dimensionality with the whole" and remain continuous between "interruptions," and limits that serve as the "connexions" between such parts. Each portion, in respect to the limits by means of which it is "marked off in the simplest way," is "precisely like every other such part." Accordingly, it seems more perspicuous—and more consistent with Peirce's later terminology—to reserve the term *part* strictly for a portion and refer to a limit as a *connection*.

> In the next place, I conceive that a Continuum has, IN ITSELF, no definite parts, although to endow it with definite parts of no matter what multitude, and even parts of lesser dimensionality down to absolute simplicity, it is only necessary that these should be marked off, and although even the operation of thought suffices to impart an approach to definiteness of parts of any multitude we please.*
>
> *This indubitably proves that the possession of parts by a continuum is not a real character of it. For the real is that whose being one way or another does not depend upon how individual persons may imagine it to be. It shows, too, that Continuity is of a Rational nature. (RS 30:12–13[Copy T:6–7])[11]

This reflects Peirce's eventual realization that the parts of a continuum are *indefinite*, rather than distinct, except where they are "marked off" by introducing connections between them—i.e., "parts of lesser dimensionality." Since there are no restrictions on the number or arrangement of such limits, there are also no restrictions on the number or arrangement of the resulting portions. Moreover, these divisions must be *deliberately* inserted; Peirce goes on to say of the two continuous EGs, "they and their Instances can be separated into parts of any multitude we like, whenever we like, and with such boundaries as we choose to impose" (RS 30:14[Copy T:8]). As he wrote elsewhere:

> On the whole, therefore, I think we must say that continuity is the relation of the parts of an unbroken space or time. … This must not be confounded (as Kant himself confounded it) with infinite divisibility, but implies that a line, for example, contains no points until the continuity is broken by marking the points. In accordance with this it seems necessary to say that a continuum, where it is continuous and unbroken, contains no definite parts; that its parts are created in the act of defining them and the precise definition of them breaks the continuity. (CP 6.168, PMSW 138–139, c. 1903–1904)
>
> In a continuum there really are no points except such as are marked; and such interrupt the continuum. It is true that the capability of being marked gives to the points the beginnings of *potential being*, but only the beginnings. It should be called a *conditional being*, since it depends upon some will's being exerted to complete it. (R 1041:13, 1906)

These statements about points—parts of "absolute simplicity"—are equally valid for any lower-dimensional limit that "breaks the continuity" to create *definite* portions, whose own parts are also indefinite unless additional connections are "marked" within them. The dependence of all such limits "upon some will's being exerted" to mark them by an "operation of thought" entails that possessing parts "is not a real character of" a continuum.

Familiar Definitions

We can now revisit Peirce's later and better-known discussions of continuity in light of the two overlooked manuscripts. There are three versions of an addendum to his 1908 article for *The Monist*, "Some Amazing Mazes (Conclusion): Explanation of Curiosity the First" (CP 4.594–641), all prompted by "reading the proofs" and prepared within two days of each other. The first does not go much beyond reaffirming that the real numbers do not "constitute a continuum," and instead "should be called a *pseudo-continuum*" (PMSW 218, 1908 May 24). The second is more detailed:

> A perfect continuum belongs to the genus, of a whole all whose parts without any exception whatsoever conform to one general law to which same law conform likewise all the parts of each single part. *Continuity* is thus a special kind of *generality*, or conformity to one Idea. More specifically, it is a *homogeneity*, or generality among all of a certain kind of parts of one whole. Still more specifically, the characters which are the same in all the parts are a certain kind of relationship of each part to all the coördinate parts; that is, it is a *regularity*. (CP 7.535n6, PMSW 225, 1908 May 24)

Continuity involves generality, homogeneity, and regularity. This is nothing new for Peirce—he affirms it even before writing R 144, in a manuscript with a similar title, "On Topical Geometry, in General":

> That continuity is only a variation of regularity, or, if we please so to regard it, that regularity is only a special case of continuity, will appear below, when we come to analyze the conception of continuity. It is already quite plain that any continuum we can think of is perfectly regular in its way as far as its continuity extends. No doubt, a line may be say an arc of a circle up to a certain point and beyond that point it may be straight. Then it is in one sense continuous and without a break, while in another sense, it does not all follow one law. But in so far as it is continuous, it everywhere follows a law; that is, the same thing is true of every portion of it; while in the sense in which it is irregular its continuity is broken. In short, the idea of continuity is the idea of a homogeneity, or sameness, which is a regularity. (CP 7.535, 1899)

All the parts of a *perfect* continuum "conform to one general law" or "to one Idea," including "all the parts of each single part." This pertains to "all of a certain kind of parts of one whole"; namely, the portions (R 144) that are "homogeneous in dimensionality with the whole" (RS 30). The second addendum continues:

> The step of specification which seems called for next, as appropriate to our purpose of defining, or logically analyzing the Idea of continuity, is that of asking ourselves what kind [of] relationship between parts it is that constitutes the regularity [of] a continuity; and the first, and therefore doubtless the best answer for our purpose, not as the ultimate answer, but as the proximate one, is that it is the relation or relations of *contiguity*; for continuity is unbrokenness (whatever that may be,) and this seems to imply a *passage* from one part to a contiguous part. What is this "passage"? This passage seems to be an act of turning the attention from one part to another part; in short an actual event in the mind. (CP 7.535n6, PMSW 225, 1908 May 24)

Peirce's tentative designation for the regular "relationship between parts" is *contiguity*, signifying "unbrokenness." This pertains to the connections of parts, namely, the limits (R 144) that are "of lesser dimensionality" (RS 30) and that adjacent portions have in common upon their introduction by "an actual event in the mind." Finally, the third version is what appeared with the published article:

> If in an otherwise unoccupied continuum a figure of lower dimensionality be constructed—such as an oval line on a spheroidal or anchor-ring surface—either that figure is a part of the continuum or it is not. If it is, it is a topical singularity, and according to my concept of continuity, is a breach of continuity. If it is not, it constitutes no objection to my view that all the parts of a perfect continuum have the same dimensionality as the whole. (Strictly, all the *material*, or *actual*, parts, but I cannot now take the space that minute accuracy would require, which would be many pages.) That being the case, my notion of the essential character of a perfect continuum is the absolute generality with which two rules hold good, first, that every part has parts; and second, that every sufficiently small part has the same mode of immediate connection with others as every other has. (CP 4.642, PMSW 215, 1908 May 26)

Peirce's "two rules" are more accurate topical reformulations of *Kanticity*, meaning "infinite intermediety, or divisibility," and *Aristotelicity*, describing "that whose parts have a common limit" (CP 4.121–122, 1893).[12] If any such "figure of lower dimensionality" is constructed as "a part of the continuum," then "it is a topical singularity" and "a breach of continuity." The alternative is to treat each limit as a connection between parts—specifically, an *immediate* connection—so that the *mode* of connection, contiguity, is everywhere the same. That leaves only portions as parts, since "all the parts of a perfect continuum have the same dimensionality as the whole."

Peirce qualifies this by confining it to "all the *material*, or *actual*, parts." The term "actual parts" does not appear elsewhere in his writings, and there are no known pages where he explains clearly what he has in mind, let alone the many "that minute accuracy would require." Based on RS 30, a plausible hypothesis is that it refers to the "uninterrupted portions" of a continuum that remain after "a finite number of interruptions" have been "marked off in the simplest way," i.e., the *definite* portions between limits "of lesser dimensionality."

Parts and Wholes

By contrast, Peirce addresses "material parts" at some length in a manuscript written just two months earlier:[13]

> Whatever is continuous has *material parts*. I begin by defining these thus: The *material parts* of a thing or other object, *W*, that is composed of such parts, are whatever things are, firstly, each and every one of them, other than *W*; secondly, are all of some one internal nature (for example, are all places, or all times, or all spatial realities, or are all spiritual realities, or are all ideas, or are all characters, or are all relations, or are all external representations, etc.); thirdly, form together a collection of objects in which no one occurs twice over and, fourthly, are such that the Being of each of them together with the modes of connexion between all subcollections of them, constitute the being of *W*. Almost everything which has material parts has different sets of such parts, often various *ad libitum*. … It will be seen that the definition of Material Parts involves the concept of *Connexion*, even if there be no other connexion between them than co-being; and in case no other connexion be essential to the concept of *W*, this latter is called a *Collection* … . (CP 6.174, PMSW 208–209, 1908)[14]

Every continuum has material parts, but not everything that has material parts is a continuum; a collection has them, as well. Such parts "are all of some one internal nature," and the being of the whole consists in the being of those parts "together with the modes of connexion between all subcollections of them." For a collection, there is "no other connexion between them than co-being," but for a continuum, the mode of connection is contiguity through the limits that adjacent portions have in common. There are typically "different sets of such parts, often various *ad libitum*": for a collection, the distinct combinations that are of next greater multitude, and for a continuum, the parts that are indefinite until marked off as *actual* parts by an exertion of some will.

The upshot is that a collection is *bottom-up*, such that the parts are real and the whole is an *ens rationis* (see CP 6.382, 1902), while a continuum is *top-down*, such that the whole is real and the parts are *entia rationis*. Havenel (2015, 104) similarly contrasts "the anteriority of elements over relations, or bottom-top approach" with "the top-bottom approach that emphasizes the anteriority of relations, the anteriority of continuity over the discrete." Zalamea likewise opposes "Cantor's analytical object" to "Peirce's *continuum*, as a synthetical concept" (2012, 8), and Moore calls the latter "*prepunctual* because it insists that continua are ontologically prior to their points and not the other way around" (2015b, 128).

Peirce himself expresses this in Aristotelian terms: "Efficient causation is that kind of causation whereby the parts compose the whole; final causation is that kind of causation whereby the whole calls out its parts" (CP 1.220, 1902). Moreover, "Rationality is being governed by final causes" (CP 2.66, 1902) and "Continuity is of a Rational nature" (RS 30). The final cause of any continuum is the "one general law" or "one Idea" to which all its portions—and all their portions, and so on—conform.

Consequently, any attempt to assemble a continuous whole from its parts will be unsuccessful: "The problem is that the construction of the [supermultitudinous] continuum proceeds by pasting together a bunch of discrete point sets, which means that it is going to be a neat trick to make out that the points are not after all prior to the continuum" (Moore 2015a, 1068). An unrelated example from Peirce serves as a helpful illustration:

> But let us compare it rather with a painting,—with an impressionist seashore piece,—then every Quality in a Premiss is one of the elementary colored particles of the Painting; they are all meant to go together to make up the intended Quality that belongs to the whole as whole. That total effect is beyond our ken; but we can appreciate in some measure the resultant Quality of parts of the whole,—which Qualities result from the combinations of elementary Qualities that belong to the premisses. (CP 5.119, EP 2:194, 1903)

The "total effect" of "an impressionist seashore piece" is the aggregate of the separate effects of the "colored particles" that comprise it. Beyond a certain viewing distance, the gaps between the latter effectively disappear. From the bottom-up standpoint, if the individual particles are of large enough infinite multitude (Cantor) or exceed all multitude (Peirce), then the painting itself is a continuum. By contrast, the top-down perspective recognizes that at best the painting approximately *represents* a hypothetical instantaneous state of a continuum—the seashore itself, where the real situation is constant motion. The parts with their "elementary Qualities" are created by the artist for the purpose of *simulating* "the intended Quality that belongs to the whole."

Along the same lines, the real numbers are a useful *model* of a continuum for many—perhaps most—mathematical and practical purposes. However, they cannot constitute a continuum themselves, and neither can a supermultitudinous collection.

Conclusion

The challenge issued by Moore is to formulate "a clear definition" of a continuum in accordance with Peirce's topical theory, which would then qualify as "a fully articulated and at least partially substantiated account" (2015a, 1057). Zalamea outlines three "global properties," from which three others follow (2012, 9–23):

- Genericity: "free of particularizing attachments … a law or regularity beyond the merely individual" (11); which implies
- Supermultitudeness: "its size must be fully generic, and cannot be bounded by any other actually determined size" (14).
- Reflexivity: "any of its parts possesses in turn another part similar to the whole … the whole can be reflected in *any* of its parts" (16); which implies
- Inextensibility: "the *continuum* cannot be composed by points … not possessing other parts than themselves" (16).
- Modality: "while points can 'exist' as discontinuous marks … the 'true' and steady components … are generic and *indefinite* neighborhoods" (20); which implies
- Plasticity: "the 'transit' of modalities, the 'fusion' of individualities, the 'overlapping' of neighborhoods" (22).

This nomenclature is a bit inscrutable, especially for non-mathematicians, and does not reflect Peirce's own usage. Moreover, supermultitudeness as a characteristic "size" is a vestige of the bottom-up/analytical/collection-theoretic approach that Peirce ultimately abandoned in favor of the top-down/synthetical/geometric conception. As an alternative, I propose that the following five properties—arranged to match up loosely with the four that he identified in R 144, and employing the terminology of portions as (material/actual) parts and limits as (immediate) connections—are jointly necessary and sufficient for Peirce's topical continuum:

- Rationality: every portion conforms to one general law or Idea, which is the final cause by which the ontologically prior whole calls out its parts.
- Divisibility: every portion is an indefinite *material* part, unless and until it is deliberately marked off with a limit to become a distinct *actual* part.
- Homogeneity: every portion has the same dimensionality as the whole, while every limit between portions is a topical singularity of lower dimensionality.

- Contiguity: every portion has a limit in common with each adjacent portion, and thus the same mode of immediate connection with others as every other has.
- Inexhaustibility: limits of any multitude, or even exceeding all multitude, may always be marked off to create additional actual parts within any previously uninterrupted portion.

I believe that this "thicker" theory is fully consistent with "the common-sense idea of continuity" (CP 6.168, PMSW 138, 1903 Sep 18) that Peirce persistently sought to capture.[15]

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NOTES

1 Citations given as R or RS with manuscript number are from Peirce 1839–1914 in accordance with Robin 1967 or Robin 1971, respectively. Page numbers correspond to the microfilm sequence as reproduced in the scanned images made available online by the Digital Peirce Archive (*https://rs.cms.hu-berlin.de/peircearchive/*) and the Scalable Peirce Interpretation Network (*https://fromthepage.com/collection/show?collection_id=16*), followed by Peirce's own handwritten page numbers [in square brackets] where different.

2 André De Tienne, e-mail correspondence, September 13, 2019.

3 Citations given as CP with volume and paragraph number(s) are from Peirce 1931–1958.

4 As revealed by De Tienne, the fourth page, written on paper half the size of the other three, is among the various fragments in RS 64.

5 Citations given as RLT with page number(s) are from Peirce 1992.

6 Citations given as EP with volume and page number(s) are from Peirce 1992–1998.

7 Citations given as PMSW with page number(s) are from Peirce 2010.

8 Peirce offers support for his claim at CP 4.125–127, 1893, and CP 3.565–569, 1900.

9 A complete transcription appears in Vol. 3 of Peirce 2020.

10 Explaining EGs is beyond the scope of this essay. I recommend Pietarinen 2015 for a concise introduction, Roberts 1973 for a detailed exposition, and Peirce 2020 for a comprehensive compilation of the relevant manuscripts. In addition, Oostra 2012 proposes minor adjustments to EGs that facilitate their use in accordance with intuitionistic logic, rather than classical logic.

11 Peirce's footnote continues but does not elaborate further on his conception of continuity, instead drawing a religious implication from it:

> But it conveys no gleam of evidence that Continuity itself is Unreal, an opinion against which there rise Alps, and Andes, and heaven-touching Atlases of insuperable objection. To my humble intelligence, the Rationality of Continuity, the chief character of the foundation stones of the real universe, adds another to the hundred already interpretable revelations of our Super-august and Gracious Father. (By "super-august" I mean having the majesty of that silent voice, *sibilus aurae tenuis* [a gentle whisper], that Elijah (I Kings xix.12) heard, too sublimely august for any admixture of the belittling insistence upon recognition that clings to the humanly august, with its comical *Majestäts-beleidigung* [offense against the monarchy].) As for my Anthropomorphism, after what F. C. S. Schiller has written, it is needless for me to say that it belongs to the essence of Pragmatism. (RS 30:13[Copy T:7])

12 Peirce elsewhere recognizes Kant's mistake in equating infinite divisibility with continuity (e.g., CP 3.569, 1900, and CP 6.168, PMSW 138, 1903 Sep 18) and consistently explains Aristotelicity in terms of points. However, adjacent parts having a lower-dimensional limit in common is how Aristotle himself discriminates the continuous from the discrete:

> Consider the parts of a number. You find there is no common limit at which they may join or unite. For example, two fives will make ten. These, however, are wholly distinct; there is no common limit whatever at which these two fives coalesce. And the same with the parts three and seven. And, indeed, in the case of all numbers you never will find such a boundary, common to any two parts, for the parts remain ever distinct. Thus is number discrete, not continuous. …
>
> A line is, however, continuous. Here we discover that limit of which we have just now been speaking. This limit or term is a point. So it is with a plane or a solid. Their parts also have such a limit—a line in the case of the former, a line or a plane in the latter. (Aristotle 1938, 37; *Categories*, I.6, 4b25–5a5)

13 The manuscript (R 300) is dated 1906 by the CP editors and 1905 by Robin 1967, but subsequent investigation places its writing "around March of 1908" (Moore 2015, 1059n10).

14 The text continues: "concerning which I have merely to say that my reflexions on [Kempe 1886] … have led me to believe it to be indecomposable." Moore understandably associates this remark with the word "*Collection*," characterizing it

as “a despairing aside” (2013, 103) and a “complaint” that “reflects Peirce’s mounting frustration” with the supermultitudinous conception (2015, 1068). However, inspection of the manuscript page (R 300:85[43]) indicates that before various insertions and deletions, the sentence referred to “the concept of *Connexion*” as “an indecomposable Concept.” Parker (1998, 63) notes that “irreducible three-term relations embody *connection*, or *combination*,” which “is the basis of the principle of continuity”; and in Peirce’s own words, “the general idea of a combination must be an indecomposable idea. For otherwise it would be compounded, and the idea of combination would enter into it as an analytic part of it” (EP 2:364, 1905).

15 The Peirce-L e-mail list (*http://www.iupui.edu/~arisbe/peirce-l/peirce-l.htm*) served as a valuable sounding board for much of my thinking that culminated in this essay. I am especially grateful to Gary Richmond and Gary Fuhrman for their feedback and encouragement upon reading an early draft. I also appreciate the helpful comments from Robert Lane and an anonymous reviewer regarding my initial submission.