Truth-functionality

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Abstract: It is shown that the standard definitions of truth-functionality, though useful for their purposes, ignore some aspects of the usual informal characterisations of truth-functionality. An alternative definition is given that results in a stronger notion which pays attention to those aspects.

Truth-functionality is a basic notion at the heart of classical logic. Hence, it is desirable to have a precise account of what it consists in. Many of the definitions ordinarily given are, however, either imprecise or yield only a rather weak notion of truth-functionality (see section 1). But even the strongest notion definable along the standard lines fails to capture certain aspects of the informal characterisations of truth-functionality that are usually given (see section 2). So, an alternative definition that takes them into account will be developed (see section 3).

1. Some Standard Accounts of Truth-functionality

An *n*-ary sentential operator or connective is any expression whose combination with *n* arbitrary sentences yields a complex sentence. Such an operator may consist of a single word or sign ('and', '¬'), or it may be rather complex ('it is a well-known fact that', 'the mayor of London said that either … or ---').

Truth-functionality is a property of sentential connectives, often characterised as follows:

> A *sentence connective is truth-functional* iff whether or not any resultant sentence it forms is true or false is determined completely by, and only by, whether its components are true or false. (Sainsbury 2001: 60)

Talk about something *determining* something is itself in need of clarification. In the present context, it is usually taken in a thin reading in which it only aims at the mathematical notion of a function or a mapping. Thus, Quine also first characterises truth-functionality in terms of *determining* and then goes on to spell it out as follows:

> More precisely: a way of forming compound statements from component statements is *truth-functional* if the compounds thus formed always have matching truth-value as long as their components have matching truth-value. (Quine 1982: 8)

A *truth-function* is any mapping from the truth-values T and F (or tuples of those values) to the truth-values T and F.[1] So, Quine's characterisation amounts to calling an operator ζ *truth-functional* iff there is a truth-function corresponding to it, i.e. if the truth-value of a complex sentence formed by combining ζ with the appropriate number of sentences is a function of the truth-values of those sentences.[2]

Call this the *Simple Proposal*. It can be framed into a more concise definition by using some formalisms. For ease of presentation, let me use different types of variables ranging over different things:

'*c*' ranges over contexts of evaluation.

'S_i' ranges over sentences of a given language *L* (and possible extensions of *L*).[3]

'*f*' ranges over truth-functions.

Let me moreover use the functors 'V' and 'CONC' as follows:

V (*x*)	:=	the truth-value of *x*.
CONC ($x_1 \dots x_n$)	:=	the concatenation of $x_1 \dots x_n$.

So, 'V ('snow is white') = T, and 'CONC ("¬", "snow is white")' = '¬ snow is white'.

Now, the following precise statement of the Simple Proposal can be given:

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1 For sake of simplicity bivalence is assumed throughout this essay. However, what will be said applies *mutatis mutandis* to many-valued logics as well.

2 Some characterisations of truth-functionality use an *epistemic* notion of determining; thus, Newton-Smith (1985: 16) writes a truth-functional sentential operator is such that '[…] given the truth-values of the sentences concatenated with [it] we can determine on the basis of that information alone the truth-value of the resulting complex sentence.' Sometimes, epistemic and non-epistemic characterisations get conflated; see Cauman (1998: 74). But the epistemic component (*we* can know or determine the truth-value of the complex sentence from those of the embedded sentences) does not follow from the characterisation in terms of a truth-function. The truth-value of the concatenation of the sentential operator:

Δ Goldbach's conjecture is true or …

with some sentence *S* is a function of the truth-value of *S*. But since we do not know whether Goldbach's conjecture is true or false, we do not know *which* truth-function is relevant: either both T and F are mapped to T, or only T is mapped to T and F to F. So, *we* cannot determine, on the basis of the truth-value of *S*, whether the concatenation of Δ and *S* is true or false, even though the truth-value *is* determined by the value of *S*. I take it that truth-functionality is not an epistemic notion and henceforth ignore the epistemic dimension.

3 If one only talks about the sentences of a language *L*, without allowing possible extensions of *L*, one may end up classifying operators as truth-functional merely because the language to which they belong is limited in some peculiar way. If, for instance, a very small language *L* is considered that accidentally contains only true sentences, all operators would come out truth-functional in *L*. Since this would be an unintended result, one should also take into account sentences belonging to an extension of a language.

Df. TF_{Simple} An *n*–ary sentential connective ζ of a language *L* is truth-functional $\leftrightarrow_{df.}$

$$\exists f\ \forall S_1 \ldots S_n: \quad V(CONC(\zeta, S_1 \ldots S_n)) = f\ (V(S_1), \ldots V(S_n)).$$

This characterisation, however, has the unwelcome effect of making the truth-functionality of some operators dependent upon arbitrary contingent facts. For assume Jeanne d'Arc never uttered any English sentence. Then the following complex operator qualifies as truth-functional in the above sense:[4]

Δ_1 If Jeanne d'Arc ever said in English that …, then it is true that …

As a matter of fact, this operator only generates true sentences, so there is a truth-function corresponding to it and it is truth-functional by the standards of (Df. TF_{Simple}). But this is only due to the contingent assumption that Jeanne d'Arc did not utter any false English sentence. Had she done it, there would be no truth-function corresponding to the operator, because combined with *some* false sentences (i.e. those uttered by Jeanne d'Arc) it would generate false sentences, while combined with *other* false sentences (those not uttered by Jeanne d'Arc) it would generate true ones.

This is an undesirable result. A quotation from Edgington shows how to avoid it:

> [Operators are truth-functional iff] in any possible circumstance, the truth value of the complex sentence is fixed by the truth value(s) of the simple sentence(s). (Edgington 1995: 241)

Following Edgington's characterisation, one should modify the above definition and take non-actual contexts of evaluations into consideration by adding a universal quantifier for them. But where? Due to the resulting differences in scope, the different possible choices will lead to two concepts that differ extensionally:[5]

Df. TF_{ws} An *n*–ary sentential connective ζ of a language *L* is truth-functional $\leftrightarrow_{df.}$

$$\forall \boldsymbol{c}\ \exists f\ \forall S_1 \ldots S_n: \quad V(CONC(\zeta, S_1 \ldots S_n), c) = f\ (V(S_1, c) \ldots V(S_n, c)).$$

Df. TF_{ns} An *n*–ary sentential connective ζ of a language *L* is truth-functional $\leftrightarrow_{df.}$

$$\exists f\ \forall \boldsymbol{c}\ \forall S_1 \ldots S_n: \quad V(CONC(\zeta, S_1 \ldots S_n), c) = f\ (V(S_1, c) \ldots V(S_n, c)).$$ [6]

As can be seen from the following example, the first of these definitions yields a weaker notion of truth-functionality than the second:

Δ_2 If France won the World Cup in 1978, then …

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4 In what follows, 'if … then …' is treated like the material arrow '$\rightarrow$'. Nothing hinges on that, because the examples would equally work with the arrow instead of 'if … then …'.

5 For this point cp. Humberstone (1986) (but notice that his exposition differs from the above).

6 Of course, there is a third possible position for the placement of '$\forall$c', but it is equivalent to the second position.

This sentential operator is truth-functional according to the first definition. The truth-value of the concatenation of Δ_2 and any arbitrary sentence is T, thus there is a truth-function corresponding to the operator with respect to the actual world, namely:

f: $f(\mathrm{T}) =_{df.} \mathrm{T}$; $f(\mathrm{F}) =_{df.} \mathrm{T}$.

Now imagine a counterfactual situation in which France *did* win the World Cup in 1978. Evaluated with respect to such a context c, the concatenation of Δ_2 and some sentence S will take on the truth-value T if S is true with respect to c, and it will take on F otherwise. Hence, there is a truth-function corresponding to the operator with respect to c, namely:

f^*: $f^*(\mathrm{T}) =_{df.} \mathrm{T}$; $f^*(\mathrm{F}) =_{df.} \mathrm{F}$.

But f^* and f are not the *same* function. So, although for every context a truth-function can be specified that corresponds to the operator in that context, it is not possible to specify *one and the same* function for all such contexts. This is why Δ_2 is not truth-functional according to the second definition.

Which of these definitions one should choose depends on the purposes that the defined notion of truth-functionality is supposed to serve. One may want a notion of truth-functionality such that the meaning of a truth-functional operator corresponds to a certain truth-function, independently of contingent facts. Contrary to (Df. TF_{ns}), (Df. TF_{ws}) manages to filter out just those operators, and whoever finds the described feature a desirable one for truth-functionality therefore should prefer the latter definition over the former.

2. Limitations of the Standard Accounts

The following operators are both truth-functional according to all definitions given so far:

negation it is not the case that …

Δ_3 it is a proposition that …

Yet, one may sense an important difference between them related to ideas about truth-functionality. The difference is not only that contrary to negation, Δ_3 corresponds to a *constant* truth-function. To see this, compare the following operator:

$\top$ it is either true or it is false that …

Just as Δ_3, $\top$ corresponds to the constant truth-function that maps both values to T. However, the operators differ in that $\top$ is responsive to the truth-value of the embedded sentence in a way in which Δ_3 is not. Figuratively speaking, the meaning of $\top$ takes into account the truth-value of the argument sentence and calculates a new truth-value, while the calculation results in the same value if the input is T and if it is F. The meaning of Δ_3, however, does not in *any* intuitive way take into account the truth-value of the embedded sentence at all. This observation might also be described by saying that while in both cases the truth-value of the complex sentence is settled once the value of the embedded sentence is settled, only in the case of ⊤ the truth-value of the complex sentence is really *determined by* the other value, that only ⊤ *reacts* to the truth-value of the embedded sentence, really *operates* on it, etc.

The intuition I count on corresponds to another yet unmentioned feature of some informal characterisations of truth-functionality. Truth-functionality is sometimes described in terms of *dependence*:

> Being truth-functional means that the truth value of a complex sentence built up using these connectives depends on nothing more than the truth-values of the simpler sentences from which it is built. (Barwise & Etchemendy 1992: 35)

Now in the case of Δ_3, the truth-value of the complex sentence is strongly independent of the truth-value of the embedded sentence: it just has nothing to do with it at all. On the other hand, ⊤ is not equally independent of the truth-value of the embedded sentence. It is only weakly independent in the sense that it corresponds to a constant function; but the result *has* something to do with the input.

Of course, all the descriptions I have given of the difference between Δ_3 and ⊤ are rather metaphorical; however, I will now show how they can be given a more precise basis.

3. An Alternative Account

I shall motivate my proposal via the talk about *dependence*. Ontological investigations often make use of notions of existential dependence, which are classically spelled out in modal terms:

x existentially depends on *y* ↔ necessarily, if *x* exists, so does *y*.[7]

But as Fine and Lowe pointed out, this account fails to capture some important intuitions one may have on dependence relations.[8] For instance, given some assumptions on modal set theory, Socrates and singleton Socrates necessarily coexist. On the classic approach to dependence, they therefore bilaterally depend upon each other. But Fine and Lowe have the reasonable intuition that the set of Socrates should depend on Socrates in a way in which Socrates does *not* depend upon the set. As we have seen, truth-functionality is sometimes described in terms of dependence; in so far as the above definitions are meant to take this characterisation into account, they would correspond to a *modal* notion of dependence: the truth-value of

(1) It is a proposition that snow is white.

only in so far depends on that of ‘snow is white’, as it is true that

(i) necessarily, if ‘snow is white’ is true, then (1) is true, and

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[7] See for instance Simons (1987: ch. 8).

[8] See Fine (1995) and Lowe (1998: ch. 6).

(ii) necessarily, if ‘snow is white’ is false, then (1) is true.

In reaction to Fine’s and Lowe’s criticisms of the traditional account to dependence, other explications were suggested that do not basically rely on modal terms. Some of them are rather formulated in terms of the explanatory connective ‘because’.[9] The idea can be illustrated with the above example: on the one hand, singleton Socrates exists *because* Socrates exists. On the other hand, it is not true that Socrates exists *because* singleton Socrates does. The grounds for the truth of ‘Socrates exists’ are of an entirely different kind.

A few words on ‘because’ are in order: firstly, notice that here and henceforth, I ignore merely *evidential* uses of ‘because’, as exemplified by ‘He must be there, because his lights are on.’[10] Moreover, in employing the connective ‘because’ and in talking about explanations, I presuppose that an *objective* account of explanation can be given: propositions expressed by ‘because’-statements can be true or false, and their truth or falsity is not a *subjective* but an *objective* matter which is constituted by some mind-independent facts.[11] How the details of an objective account of explanation should look like is a matter of dispute which cannot be settled here.[12]

Now I can define a notion of truth-functionality in terms of ‘because’ as follows:

Df. TF_{expl} An *n*–ary sentential connective ζ of a language *L* is truth-functional $\leftrightarrow_{df.}$

$\forall c\ \forall S_1 \dots S_n$: whatever truth-value CONC (ζ, $S_1 \dots S_n$) has with respect to *c*, it has that value *because of* the truth-values that $S_1 \dots S_n$ have with respect to *c*.[13]

To understand how (Df. TF_{expl}) works, take a look at some standard logical connectives first: if a conjunction of two sentences *S* and *S** is true, then it is true because both *S* and *S**

[9] See Lowe (1998: 145f.), Correia (2005: ch. 4), and Schnieder (2006a).

[10] On such uses of ‘because’ see Morreal (1979).

[11] For the general idea of an objective account of explanation cp. Kim (1988). Acceptance of an objective account of ‘because’-statements is compatible with holding that a comprehensive theory of explanation will also have to deal with many pragmatic features of the usage of ‘because’; cp. Lewis (1986).

[12] For outlines of proposals that would fit my purposes see Correia (2005: ch. 4), Schnieder (2006b), and Tatzel (2002).

[13] An alternative formulation would be:

ζ is truth functional $\leftrightarrow_{df.}$ $\forall S_1 \dots S_n$: CONC (ζ, $S_1 \dots S_n$) has in *c* the truth-value it has, because $S_1 \dots S_n$ have the truth-values they have in *c*.

This way of formulating the definition, though concise, may to some extent cloud its underlying logical form. So, it should be pointed out that the definition is equivalent to a conjunction of 2^{n+1} conditionals, each of which has the form

if v(CONC (ζ, $S_1 \dots S_n$), *c*) = VAL_0 & v(S_1, *c*) = VAL_1 & … & v(S_n, *c*) = VAL_n, then v(CONC (ζ, $S_1 \dots S_n$), *c*) = VAL_0, *because* (S_1, *c*) = VAL_1 & … & v(S_n, *c*) = VAL_n.

The conditionals will vary only with respect to the instances of ‘VAL_i’, and they will correspond to the possible assignments of truth-values in *c* to CONC (ζ, $S_1 \dots S_n$), S_1, …, S_n.

are true. (It is not assumed that there can only be one correct explanation for a given *explanandum*; there may be other correct explanations for the truth of a certain conjunction, but the explanation by recourse to the truth of the conjuncts is always correct and available.) Similarly, if the negation of a sentence *S* is true, then it is true because *S* is false. And, if a sentence *S* is true, then *because of that* any disjunction of *S* and another sentence is true. As can be seen from disjunctions, there can be cases of explanatory overdetermination: if both disjuncts *S* and *S** of a disjunction *D* are true, then it is both correct that *D* is true because *S* is true, and that *D* is true because *S** is true – and, moreover, that *D* is true because both *S* and *S** are true.

The ordinary connectives qualify as truth-functional according to (Df. TF_{expl}). Now let us return to the examples of Δ_3 and $\top$. To start with the latter, take sentence

(2) It is either true or it is false that snow is white.

(2) is true, and it is moreover correct to say that (2) is true *because* ‘snow is white’ is true. Now imagine a counterfactual situation *c* in which snow is not white. With respect to *c*, (2) it still true: not because ‘snow is white’ is true with respect to *c* (which it is not), but rather because ‘snow is white’ is false with respect to *c*.

The case is different, though, with Δ_3. It is not the case that sentence

(1) It is a proposition that snow is white.

is true *because* ‘snow is white’ is true. And had snow not been white, then (1) would still have been true, but *not because* ‘snow is white’ would have been false. The above definition classifies$\top$ as truth-functional but not Δ_3 and thereby captures the intuitions mobilized in the foregoing section. Here is another example that is truth-functional by the standards of the ordinary definitions, but not by the standards of (Df. TF_{expl}):

Δ_4 It is expressible in English that …

Whenever Δ_4 is combined with a sentence *S*, the result will be true because the English sentence *S* just expresses what is said to be expressible in English. So, Δ_4 corresponds to a constant truth-function. But the result of combining Δ_4 with a true (false) sentence *S* will not be true (false) *because S* is true (false). It is expressible in English both that snow is white and that snow is black, but neither is the first expressible because ‘snow is white’ is true, nor the second because ‘snow is black’ is false. So, Δ_4 is not truth-functional according to (Df. TF_{expl}).

A final example can be generated from the operator Δ_1 which I used to criticise the *Simple Proposal*, i.e. classic non-modal characterisation of truth-functionality. Let me modify the operator by adding an ‘actually’:

Δ_5 If Jeanne d’Arc *actually* ever said in English that …, then it is true that …

Now, given the assumption that Jeanne d’Arc actually never said anything in English, Δ_5 proves to be truth-functional according to the traditional account. The concatenation of Δ_5 with any sentence *S* will be true, regardless of whatever truth-value *S* has, and regardless of the context of evaluation. So, the operator strongly corresponds to the constant truth-function that maps everything to T. However, this semantic behaviour of Δ_5 is not due to the fact that Δ_5 takes the truth-value of an embedded sentence into account (in any intuitive sense). The proposed definition of truth-functionality in terms of 'because' accordingly does not classify Δ_5 as truth-functional: the sentence 'If Jeanne d'Arc *actually* ever said in English that 2+3=6, then it is true that 2+3=6' has the value T, but it does not have this value *because of* the truth-value of '2+3=6'. Thus, Δ_5 is not truth-functional by the standards of my definition.

It is worth mentioning that, given certain assumptions on truth-making, the above account can partly be restated in terms of truth-makers: whatever the truth-maker of 'snow is white' is, is also the truth-maker of (2). But whatever the truth-maker of (1) may be, it will not be the same as the truth-maker of 'snow is white'. [14] Similarly, one may say that (2) is true *in virtue of* the fact that snow is white, while it would have been true in virtue of the fact that snow is not white, if snow had not been white. However, it is not in virtue of the fact that snow is white that (1) is true.

Summing up, it is possible to define a notion of truth-functionality in terms of the explanatory connective 'because' which proves to be stronger than the ordinary definitions and which can account for certain differences between sentences that cannot be accounted for by the ordinary definitions. I do not claim that therefore the ordinary accounts are *defective* in any way. Since the term 'truth-functional' is a genuine term of the art, I would rather regard those definitions as impeccable in themselves (because they are consistent), and I admit that they serve certain purposes. But if we want a notion of truth-functionality to take into account intuitions about dependency as described in section 2, the proposal in explanatory terms achieves what the ordinary proposals cannot.[15]

A final note on my proposal: its core idea is to employ the connective 'because' to define some semantic vocabulary, instead of the weaker notions (be they extensional or intensional) that are usually relied upon. I think it likely that this idea can fruitfully be applied to definition of other semantic notions as well (it can, for instance, easily be applied to a definition of *extensional contexts* in general).

[14] That truth-maker formulations serve the current purpose can be easily explained if talk about truth-makers is at bottom talk about *explanations* of truth; for this idea, see McFetridge (1977), Künne (2003: 148–75), Hornsby (2005), and Schnieder (2006b). Admittedly, not all aspects of the above account can be restated in terms of truth-makers; the point about simple negation (if 'p' is false, then because of that '¬p' is true) would, for instance, require *false-makers* which are usually not acknowledged by truth-maker theorists.

[15] I am deeply indebted to Miguel Hoeltje for discussions about the idea developed in this article. Further thanks for inspiration, comments, and/or corrections go to Dale Jacquette, Timothy Williamson, and an anonymous referee of this journal.

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