Weyl's search for a difference between 'physical' and 'mathematical' automorphisms

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Abstract

During his whole scientific life Hermann Weyl was fascinated by the interrelation of physical and mathematical theories. From the mid 1920s onward he reflected also on the typical difference between the two epistemic fields and tried to identify it by comparing their respective automorphism structures. In a talk given at the end of the 1940s (ETH, Hs 91a:31) he gave the most detailed and coherent discussion of his thoughts on this topic. This paper presents his arguments in the talk and puts it in the context of the later development of gauge theories.

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Introduction

The English translation of *Philosophy of Mathematics and Natural Science* (Weyl 1949), in the following quoted as PMN, gave Weyl the opportunity to reflect once more on the relation between mathematical and physical automorphisms (often called symmetries) and its meaning for understanding the basic features of nature. This topic had occupied Weyl during his whole scientific life. In a talk *Similarity and congruence: a chapter in the epistemology of science* preserved in his Zürich Nachlass (Weyl Hs 1948/49) we find a coherent balance of Weyl's thoughts on this theme at the time 1948/49.[1] Some parts of this manuscript are identical with passages in PMN, others (but less) are close to formulations which Weyl later used in his book on *Symmetry* (Weyl 1952). All in all we have here a kind of bridge text between his two famous books. Many aspects discussed in the lecture are also present in PMN at different places and therefore not necessarily new for the informed reader, but the concentration on the topic of mathematical versus physical automorphisms gives the manuscript a uniqueness and coherence which justifies a separate discussion of its content. With regard to *Symmetry* the manuscript can be read as an epistemologically and physically deeper reflection of the same considerations which lay at the base of (Weyl 1952). The text of the lecture is going to be published in the forthcoming third edition of the German translation (Weyl 1955).

The lecture concentrates on a topic which was of great interest to Weyl: the question of how to distinguish between mathematical and physical automorphisms in 20th century physics. It is mentioned in (Weyl 1952) and more extensively discussed in PMN.[2] In the talk it is presented as a whole and discussed with even more details than in PMN. Weyl argued more clearly for the necessity to distinguish between the automorphism structures of mathematical and physical theories. He accentuated the difference by even speaking of the "group of automorphisms of the physical world". Although this sounded like an ontological claim, Weyl's arguments basically pursued an epistemological interest (indicated already in the title of the paper).

This contribution presents Weyl's arguments in favour of taking automorphisms seriously for characterizing objectivity (section 1), his distinction between mathematical and physical automorphisms of classical geometry and physics (section 2) and the shifts arising from general relativity (GR) and "early" (pre 1950) quantum theory (sections 3, 4). Section 5 proposes an interpretation of how we may express Weyl's distinction in more general terms; section 6 makes a first step into discussing later developments from this perspective. At the end of his talk Weyl reflected the special nature of the Lorentz group (generalizing the "rotations" of classical geometry) which

[1] The lecture is undated, but during the talk Weyl mentioned that his proposal for a unified field theory was made "just 30 years ago".

[2] (Weyl 1949, 72f., 82–84)

had survived the transition from classical to relativistic physics (section 7). A short outlook on later developments, not yet to be seen by Weyl at the end of the 1940s, follows and relates Weyl's arguments to more recent discussion on symmetries in physics and its philosophy (section 8).

1 Automorphisms and objectivity

The talk shows how Weyl intended to demarcate the distinction between mathematical and physical automorphism of physical theories. In classical geometry and physics, physical automorphisms could be based on the material operations used for defining the elementary equivalence concept of congruence ("equality and similitude" in the old terminology). But Weyl started even more generally, with Leibniz' explanation of the *similarity* of two objects, "two things are similar if they are indiscernible when each is considered by itself (Hs, p. 1)",[3] and remarks on the Clarke-Leibniz correspondence regarding the indiscernibility of the regions in space. Here, like at other places, Weyl endorsed this Leibnzian argument from the point of view of "modern physics", while adding that for Leibniz this spoke in favour of the unsubstantiality and phenomenality of space and time. On the other hand, for "real substances" the Leibnizian monads, indiscernability implied identity. In this way Weyl indicated, prior to any more technical consideration, that similarity in the Leibnizian sense was the same as *objective equality*. He did not enter deeper into the metaphysical discussion but insisted that the issue discussed in his talk "is of philosophical significance far beyond its purely geometric aspect".

After some remarks on historical shifts of what was considered as "objective" properties, e.g. the vertical direction for Democritus or absolute space for Newton,[4] Weyl had good news: "...we can say today in a quite definite manner what the adequate mathematical instrument is for the formulation of this idea (objectivity, E.S.). It is the notion of group" (Hs, p. 4). Weyl did not claim that this idea solves the epistemological problem of objectivity once and for all, but at least it offers an *adequate mathematical instrument* for the *formulation* of it. He illustrated the idea in a first step by explaining the automorphisms of Euclidean geometry as the structure preserving bijective mappings of the point set[5] underlying a structure satisfying the axioms of "Hilbert's classical book on the Foundations of Geometry" (Hs, p. 4f.). He concluded that for Euclidean geometry these are the *similarities*, not the congruences as one might expect at a first glance (see section 2). In the mathematical sense, we then "come to *interpret objectivity as the invariance*

[3] A similar passage is also to be found in (Weyl 1952, 127f.). Page quotes referring to the manuscript (Weyl Hs 1948/49) are abbreviated by (Hs, p. xx).

[4] For the last point see also (Weyl 1949, 100).

[5] Weyl preferred to avoid the language of sets and used the the terminology of a "point-field".

under the group of automorphisms" (Hs., p.6, emphasis in the original).

But Weyl warned to identify mathematical objectivity with that of natural science, because once we deal with *real space* "neither the axioms nor the basic relations are given" (Hs, 6). As the latter are extremely difficult to discern, Weyl proposed to turn the tables and to take the *group* Γ *of automorphisms*, rather than the 'basic relations' and the corresponding relata, *as the epistemic starting point*:[6]

> Hence we come much nearer to the actual state of affairs if we start with the group Γ of automorphisms and refrain from making the artificial logical distinction between basic and derived relations. Once the group is known, we know what it means to say of a relation that it is objective, namely invariant with respect to Γ . . . (Hs, 6)

By such a well chosen constitutive stipulation it becomes clear what objective statements are, although this can be achieved only at the price that ". . . we start, as Dante starts in his Divina Comedia, in mezzo del camin" (Hs. p. 7). A phrase characteristic for Weyl's later view follows:

> It is the common fate of man and his science that we do not begin at the beginning; we find ourselves somewhere on a road the origin and end of which are shrouded in fog (Hs, p. 6).

Weyl's juxtaposition of the *mathematical* and the *physical* concept of *objectivity* is worthwhile to reflect upon. The mathematical objectivity considered by him is relatively easy to obtain by combining the axiomatic characterization of a mathematical theory (Hilbert) with the epistemic postulate of invariance under a group of automorphisms (Klein). Both are constituted in a series of acts characterized by Weyl in PMN as *symbolic construction*, which is free in several regards. For example, the group of automorphisms of Euclidean geometry may be expanded by "the mathematician" in rather wide ways (affine, projective, or even "any group of transformations").[7] In each case a specific realm of mathematical objectivity is constituted. With the example of the automorphism group Γ of (plane) Euclidean geometry in

[6]In this context the "basic relations" might be interpreted as the *laws of nature*. For readers who prefer to use the more concrete sounding word *symmetries* to "automorphisms", Weyl's proposal may be read as the advice to take the symmetries as epistemic starting point and the laws/equations invariant under this group as epistemically secondary (derived). Note that Weyl did *not* pose the question in terms of which of the two levels is more *fundamental* (see the quote below).

[7]For Weyl "any group" was usually constrained by the condition of differentiability. This was not necessarily so for other mathematicians, see e.g. F. Hausdorff's argument for the lack of meaning of an "absolute" objective structure of space and time in his philosophical essay *Das Chaos in kosmischer Auslese* in which arbitrary point transformation (bijections) are considered as the most general case (Hausdorff 1898); see (Epple 2006).

mind Weyl explained a little later how, through the use of Cartesian coordinates, the automorphisms of Euclidean geometry can be *represented* by *linear transformations* "in terms of reproducible numerical symbols" (Hs, p.9).

For natural science the situation is quite different; here the freedom of the constitutive act is severely restricted. Weyl described the constraint for the choice of Γ at the outset in very general terms:

> The physicist will question Nature to reveal him her true group of automorphisms (Hs, p. 7).

This is a striking, even surprising remark. Different to what a philosopher might expect, Weyl did not mention, at this place, the subtle influences induced by theoretical evaluations of empirical insights on the constitutive choice of the group of automorphisms for a physical theory. He even did not restrict the consideration to the range of *a* physical theory but aimed at *Nature* as a whole. Still in 1948/49, after several turns of his own views and radical changes in the fundamental views of theoretical physics, Weyl hoped for an insight into *the true* group of automorphisms *of Nature* without any further specifications. Of course he did not stop with this general characterization. In the following parts of the talk Weyl explored in much more detail how the "true group" of physical automorphisms was shaped with the increasing and deepening empirical and theoretical knowledge of nature.

2 Physical and mathematical automorphisms of classical geometry

Looking at classical geometry and mechanics, Weyl followed Newton and Helmholtz in considering *congruence* as the basic relation which lay at the heart of the "art of measuring" by the handling of that "sort of bodies we call rigid" (Hs, p. 9). In a short passage he explained how the local congruence relations established by the comparison of rigid bodies can be generalized and abstracted to congruences of the whole space. In this respect Weyl followed an empiricist approach to classical physical geometry, based on a theoretical extension of the material practice with *rigid bodies* and their *motions*. Even the mathematical abstraction to mappings of the whole space carried the mark of their empirical origin and was restricted to the group of *proper congruences* (orientation preserving isometries of Euclidean space, generated by the translations and rotations) denoted by him as Δ^+. This group seems to express "an intrinsic structure of space itself; a structure stamped by space upon all the inhabitants of space" (Hs, p. 10). From a historical perspective, Δ^+ could serve as the group of physical automorphisms of space until the early 19th century. As we shall see in a moment, Weyl argued that during the 19th century it would be extended to the *group of all congruences* Δ which also includes the orientation reversing isometries (point symmetries, reflections).

But already on the earlier level of physical knowledge, so Weyl argued, the *mathematical automorphisms* of space were larger than Δ. Even if one sees "with Newton, in congruence the one and only basic concept of geometry from which all others derive" (Hs, p. 10), the group Γ of automorphisms in the mathematical sense turns out to be constituted by the *similarities*.

The structural condition for an automorphism $C \in \Gamma$ of classical congruence geometry is that any pair (v_1, v_2) of congruent geometric configurations is transformed into another pair (v_1^*, v_2^*) of congruent configurations ($v_j^* = C(v_j)$, $j = 1, 2$). For evaluating this property Weyl introduced the following diagram:

$$\begin{tikzcd} v_1 \arrow[r, "(T)"] & v_2 \arrow[d, "(C)"] \\ v_1^* \arrow[u, "(C^{-1})"] & v_2^* \end{tikzcd}$$

Because of the condition for automorphisms just mentioned the maps CTC^{-1} and $C^{-1}TC$ belong to Δ^+ whenever T does. By this argument he showed that the *mathematical automorphism group* Γ *is the normalizer of the congruences* Δ^+ in the group of bijective mappings of Euclidean space.[8]

This argument contained the *mathematical* reason for Weyl's decision in 1918 to consider the similarities as the structure determining morphisms of his *purely infinitesimal geometry* (Weyl 1918*b*). More generally, it also explains the reason for his characterization of generalized similarities in his *analysis of the problem of space* in the early 1920s. In 1918 he translated the relationship between physical equivalences as congruences to the mathematical automorphisms as the similarities/normalizer of of the congruences from classical geometry to special relativity (Minkowski space) and "localized" them (in the sense of physics), i.e., he transferred the structural relationship to the infinitesimal neighbourhoods of the differentiable manifold characterizing spacetime (in more recent language, to the tangent spaces) and developed what later would be called *Weylian manifolds*, a generalization of Riemannian geometry.[9] In his discussion of the problem of space he generalized the same relationship even further by allowing any (closed) subgroup of the general linear group as a candidate for characterizing generalized congruences at every point.

[8]The same argument is given in (Weyl 1949, 79).

[9]In 1918 Weyl even hoped that the extension to the similarities was not only relevant from a mathematical point of view, but also plays a crucial role for physics (by incorporating electromagnetism into the geometrical structure field and leading to a unified field theory). In his terminology of 1948/49 he had hoped that his generalization of Riemannian geometry to Weylian manifolds of 1918 corresponded to an extension of the physical automorphisms of general relativity. In the transition to the "new" quantum mechanics in the 1920s he gave up this hope; see his corresponding remarks below.

Moreover, Weyl argued that the enlargement of the physico-geometrical automorphisms of classical geometry (proper congruences) by the mathematical automorphisms (similarities) sheds light on *Kant's riddle* of the "incongruous counterparts". Weyl presented it as the question: Why are "incongruous counterparts" like the left and right hands intrinsically indiscernible, although they cannot be transformed into another by a proper motion? From his point of view the intrinsic indiscernibility could be characterized by the mathematical automorphisms Γ. Of course, the congruences Δ including the reflections are part of the latter, $\Delta \subset \Gamma$; this implies indiscernibility between "left and right" as a special case. In this way Kant's riddle was solved by a Leibnizian type of argument. Weyl very cautiously indicated a philosophical implication of this observation

> And he (Kant, E.S.) is inclined to think that only transcendental idealism is able to solve this riddle. No doubt, the meaning of congruence and similarity is founded in spatial intuition. Kant seems to aim at some subtler point. But just this point is one which can be completely clarified by general concepts, namely by subsuming it under the general and typical group-theoretic situation explained before ... (Hs. p. 7).

Weyl stopped here without discussing the relationship between group theoretical methods and the "subtler point" Kant aimed at more explicitly. But we may read this remark as an indication that he considered his reflections on automorphism groups as a contribution to the transcendental analysis of the conceptual constitution of modern science. For Weyl this meant, of course, modern science in the sense of the 20th century, i.e., taking general relativity and quantum physics into account. A little later, in his book on *Symmetry*, he went a tiny step further. Still with the Weylian restraint regarding the discussion of philosophical principles he stated: "As far as I see all a priori statements in physics have their origin in symmetry" (Weyl 1952, 126).

To prepare for the following, Weyl specified the subgroup $\Delta_o \subset \Delta$ with all those transformations that fix one point ($\Delta_o = O(3, \mathbb{R})$, the orthogonal group in 3 dimensions, $\mathbb{R}$ the field of real numbers). In passing he remarked:

> In the four-dimensional world the Lorentz group takes the place of the orthogonal group. But here I shall restrict myself to the three-dimensional space, only occasionally pointing to the modifications, the inclusion of time into the four-dimensional world brings about. (Hs, p. 13f.)

Keeping this caveat in mind (restriction to three-dimensional space) Weyl characterized the "*group of automorphisms of the physical world*", in the sense of classical physics (including quantum mechanics) by the combination (more technically, the semidirect product $\rtimes$) of translations and rotations, while the mathematical automorphisms arise from a normal extension:

– physical automorphisms $\Delta \cong \mathbb{R}^3 \rtimes \Delta_o$ with $\Delta_o \cong O(3)$, respectively $\Delta \cong \mathbb{R}^4 \rtimes \Delta_o$ for the Lorentz group $\Delta_o \cong O(1,3)$,

– mathematical automorphisms $\Gamma = \mathbb{R}^+ \times \Delta$ ($\mathbb{R}^+$ the positive real numbers with multiplication).

In Weyl's view the *difference between mathematical and physical automorphisms* established a fundamental distinction between mathematical geometry and physics.

> *Congruence, or physical equivalence*, is a geometric concept, the meaning of which refers to the laws of physical phenomena; the congruence group Δ is essentially the group of physical automorphisms. If we interpret geometry as an abstract science dealing with such relations and such relations only as can be logically defined in terms of the one concept of congruence, then the group of *geometric automorphisms* is the normalizer of Δ and hence *wider* than Δ. (Hs, p. 16f., emphasis E.S.)

He considered this as a striking argument against what he considered to be the Cartesian program of a reductionist geometrization of physics (physics as the science of *res extensa*):

> According to this conception, Descartes's program of reducing physics to geometry would involve a vicious circle, and the fact that the group of geometric automorphisms is wider than that of physical automorphisms would show that such a reduction is actually impossible." (Hs, p. 16f., similar in (Weyl 1949, 83))

In this Weyl alluded to an illusion he himself had shared for a short time as a young scientist. After the creation of his gauge geometry in 1918 and the proposal of a geometrically unified field theory of electromagnetism and gravity he believed, for a short while, to have achieved a complete geometrization of physics.[10]

[10] In the third German edition of *Space - Time - Matter* (1919) Weyl included his proposal for a unified field and matter theory. At the end of the book he drew the following conclusion:

> We have realized that physics and geometry collaps to one, that the world metric is a physical reality, and even the only one. In the end, the whole physical reality appears to be a mere form; not geometry has become physicalized, but physics has turned into geometry. (Weyl 1919, 263, my translation, E.S.)

("Wir hatten erkannt, daß Physik und Geometrie schließlich zusammenfallen, daß die Weltmetrik eine, ja vielmehr die physikalische Realität ist. Aber letzten Endes erscheint

He gave up this illusion in the middle of the 1920s under the impression of the rising quantum mechanics. In his own contribution to the new quantum mechanics groups and their linear representations played a crucial role. In this respect the mathematical automorphisms of geometry and the physical automorphisms "of Nature", or more precisely the automorphisms of physical systems, moved even further apart, because now the physical automorphism started to take non-geometrical material degrees of freedom into account (phase symmetry of wave functions and, already earlier, the permutation symmetries of n-particle systems).

But already during the 19th century the physical automorphism group had acquired a far deeper aspect than that of the mobility of rigid bodies:

> In physics we have to consider not only points but many types of physical quantities such as velocity, force, electromagnetic field strength, etc. ...

All these quantities can be represented, relative to a Cartesian frame, by sets of numbers such that *any orthogonal transformation* T performed on the coordinates keeps the basic physical relations, the physical laws, invariant. Weyl accordingly stated:

> *All the laws of nature are invariant under the transformations thus induced by the group* Δ. Thus physical relativity can be completely described by means of a group of transformations of space-points. (Hs. p. 14, emphasis in orginal)

By this argumentation Weyl described a deep shift which ocurred in the late 19th century for the understanding of physics. He described it as an *extension of the group of physical automorphisms.* The laws of physics ("basic relations" in his more abstract terminology above) could no longer be directly characterized by the motion of rigid bodies because the physics of fields, in particular of electric and magnetic fields, had become central. In this context, the motions of material bodies lost their epistemological primary status and the physical automorphisms acquired a more abstract character, although they were still completely characterizable in geometric terms, by the full group of Euclidean isometries. The indistinguishability of left and right, observed already in clear terms by Kant, acquired the status of a physical symmetry in electromagnetism and in crystallography.[11]

so diese ganze physikalische Realität doch als eine bloße Form; nicht die Geometrie ist zur Physik, sondern die Physik zur Geometrie geworden.")
In the next years, and already in the following editions of *Space - Time - Matter*, Weyl withdrew step by step from this geometrization of physics perspective. In his Rouse Ball lecture 1930 he likened it with premature "geometrical jumps into the air (geometrische Luftsprünge)" which had lost contact with the "solid ground of physical facts" (of quantum physical observations) (Weyl 1931, 343).

[11]In geometrical crystallography the point inversion symmetry played a crucial role

Weyl thus insisted that in classical physics the physical automorphisms could be characterized by the group Δ of Euclidean isometries, larger than the physical congruences (proper motions) Δ^+ but smaller than the mathematical automorphisms (similarities) Γ.[12]

This view fitted well to insights which Weyl drew from recent developments in quantum physics. He insisted – differently to what he had thought in 1918 – on the consequence that "length is not relative but absolute" (Hs, p. 15). He argued that *physical length* measurements were no longer dependent on an arbitrary chosen unit, like in Euclidean geometry. An "absolute standard of length" could be fixed by the quantum mechanical laws of the atomic shell:

> The atomic constants of charge and mass of the electron atomic constants and Planck's quantum of action h, which enter the universal field laws of nature, fix an *absolute standard of length*, that through the *wave lengths of spectral lines* is made available for practical measurements. (Hs, 15, emphasis E.S.)

This statement was important for Weyl; he repeated the passage in (Weyl 1949, 83) and (Weyl 1952, 129) in similar words. It demarcates a crucial difference of Weyl's mature view of the physical metric from his earlier ones (1918 until about 1924). In the terminology of his 1918 debate with Einstein, Weyl came now to accept that the laws of quantum physics and the constitution of the atom establish a kind of "universal bureau of standards (Eichamt)", contrary to what pure field physics made him expect in 1918.[13]

3 The 'shock of relativity'

"So far so good. But now comes the shock of general relativity theory. It taught us that the group of physical automorphisms is much larger than we had assumed so far ..." (Hs, p. 16). With these words Weyl turned towards

already a short time after Kant's death and was developed during the 19th century in both, the atomistic and the dynamistic, programs of crystallography. With the transition to group theoretic descriptions of crystallographic symmetries the extension of proper motions by orientation reversing isometries was made explicit between Camille Jordan's paper of 1869 *Mémoire sur les groupes de mouvements* and Sohncke's adaptation in crystallographic studies to Schoenflies' papers on crystallographic groups in 1888ff. under the influence of F. Klein (Scholz 1989, chap. I).

[12]Similar arguments can be found in (Weyl 1949, 83) and (Weyl 1952, 129). The argument was apparently meant to hold, mutatis mutandis, also for special relativistic physics ("...only occasionally pointing to the modifications, the inclusion of time into the four-dimensional world brings about").

[13]In this context it is interesting to see that the International System of Standards (SI) is now substituting the pragmatic convention of the Paris *urmeter* by implementing a fixed standard of length on the basis of the ground state hyperfine splitting frequency of the caesium 133 atom and reference to a collection of natural constants (velocity of light c, Planck constant $\hbar$, elementary charge e, Boltzmann, Avogadro and Rydberg constants).

the shift general relativity brought about for the understanding of mathematical and physical automorphisms of geometry, or even "Nature" as such. He described the mathematical structure of differentiable and Riemannian manifolds by means of coordinate systems and characterized the tangent spaces by point dependent "Cartesian" (orthonormal) vector bases. This description of orthonormal frames was a simplification of a method for generalizing Dirac's electron theory to the general relativistic context (Weyl 1929).[14]

Weyl described the automorphism group of general relativity verbally, i.e., without formulae, as containing "all transformations (satisfying certain continuity or differentiability restrictions)" (Hs, p. 16), i.e., the diffeomorphisms of the space(-time) manifold M. A little later he characterized this part of the physical automorphism group (sloppily) by its expression in coordinates and talked about

> ......the group Ω of all coordinate transformations, which expresses the generally relativistic molluscous nature of space as the 'field of possible coincidences' (Hs, p. 18).

In addition, the different choices of orthonormal frames at each point of the manifold are associated to point dependent "rotations" of type Δ_o (orthogonal, or Lorentz transformations). They have to be taken into account for transforming between different pictures of a general relativistic field constellation. Thus:

> The laws of nature are independent of the arbitrariness involved in these two acts. In other words, they are invariant (1) with respect to arbitrary continuous (or rather differentiable) coordinate transformations, (2) with respect to any rotation of the Cartesian frame at P, a rotation that may depend in an arbitrary manner on the point P. (Hs, p. 18)

On a first reading two, or even three, features of Weyl's characterization of the repercussions of the "shock of relativity" on the perception of the physical automorphisms may appear puzzling. He did not specify, at least not in general terms, what the "independence of the arbitrariness" under the respective choices of coordinates and frames precisely meant for the natural laws. At a superficial glance his characterization may seem to be subject to Kretschmann's criticism of the meaning of general covariance for general relativity. But Weyl made it quite clear that he understood the *invariance of the laws of nature* under the physical automorphisms in a *strong sense*, i.e, without assuming an additional background structure (e.g. the Euclidean metric in the case of Newtonian mechanics) which would have to be transformed concomitantly with the dynamical quantities appearing in

[14] Similarly in (Weyl 1949, 88), while the following discussion of physical automorphisms is much more detailed in the talk (Hs, pp. 16ff.).

the natural laws.[15] Following Einstein in this regard, Weyl expressed this idea for general relativity in clear terms:

> The metric structure, and the inertial structure derived from it, exert a powerful influence upon all physical phenomena. But what acts must also suffer. In other words, the metric structure must be conceived as something variable, like matter and like the electromagnetic field, which stands with all other physical quantities in the commerce of interaction: it acts and suffers reactions. Only by admitting the metrical field as a variable physical entity among the other physical quantities can the principle of general relativity be carried through. (Hs, p. 16)

The second puzzling feature is Weyl's rather generous quid pro quo of differentiable coordinate transformations and diffeomorphisms of the manifold, mentioned already above. But this was a general way of expressing diffeomorphisms by Weyl. He did not like transfinite sets, and therefore tended to avoid the description of manifolds by locally Euclidean Hausdorff spaces endowed with what later would be called an atlas of coordinate systems. He rather preferred a definition by equivalences of coordinate domains, because he considered this a more constructive approach to the concept of manifold. This being said, we need not bother much about this peculiar mode of expression in our context.

The third puzzling feature relates to Weyl's treatment of the localized, point dependent operation of the (Lorentz) orthogonal group on the tangent spaces. This aspect deserves more attention.

4 Automorphisms of general relativity as a gauge group

Weyl dealt her with what later would be called a *gauge group* or, more precisely, a *gauge automorphism group* over spacetime M with structure group $G = \Delta_o$. The situation is complicated by different uses of the term *gauge group* in the present literature. In fibre bundle language, referring to a principal fibre bundle $G \circlearrowright P \rightarrow M$ over the base manifold M with structure group G, some authors consider the group $\mathcal{G}_P$ of fibrewise ("vertically") operating bundle automorphisms as the gauge group of P. The elements of $\mathcal{G}_P$ are called (global) *gauge transformations*.

In another, more general, view the group $\mathcal{G}(P)$ of all (equivariant) bundle automorphisms is considered as the gauge group of P. In order to disambiguate the terminology, the elements of $\mathcal{G}(P)$ might better be called (bundle)

[15]This difference is sometimes considered as the crucial difference between *symmetries* of the laws and *covariance of the equations* of motion, see (Giulini 2009) or the commentary by the same author in the 3rd edition of (Weyl 1955).

automorphisms,[16] and $\mathcal{G}(P)$ the *group of gauge automorphisms*. Different from the first definition, the elements of $\mathcal{G}(P)$ allow diffeomorphic transformations of the base. In our context the latter definition is more appropriate, because it expresses Weyl's description of the automorphism group of general relativity in modernized terms.

To make things even more involved, one often speaks of gauge transformations for describing the local *changes of trivialization* of a given fibre bundle, including their operation on the local representatives of connections. This last reading of gauge transformations corresponds to a local realizations of the vertical gauge transformations in the sense of $\mathcal{G}_P$, just like the differentiable coordinate transformations correspond to diffeomorphisms of a manifold.

The physical automorphism group of general relativity, intuitively described by Weyl as composed by two components (Ω and the family of point dependent Δ_o-s) would have been difficult to formulate at the end of the 1940s. But already a few years later, in the early 1950s, the next generation of mathematicians learned to describe such a group $\mathcal{G}(P)$ by a principle fibre bundle P over M, $P \longrightarrow M$ with structure group $G \cong \Delta_o$.[17] The latter describes both, the fibres of P and their allowable transformations (G operates fibrewise on P). As $\mathcal{G}(P)$ consists of the diffeomorphisms of the bundle P, which induce diffeomorphisms on the base manifold M and map the fibres in such a way that the group operations are respected, it formalizes Weyl's intuition of the liberty of choice of a local reference system (orthogonal frame) at every point quite well.[18] An element of $\mathcal{G}(P)$ induces a local diffeomorphism of the base manifold M and, moreover, it specifies an orthogonal transformation at each point of M. This corresponds to a point dependent change of frames in Weyl's description. Therefore $\mathcal{G}(P)$ is a well adapted modernized (and only minimally anachronistic) global expression for Weyl's automorphism group of general relativity.

Weyl compared the new group with the automorphisms of special relativity or even classical physics. The translations of the classical automorphism group were generalized by the diffeomorphism group Ω and the "rotations" (Lorentz transformations) became point dependent.

But this was only a first step into modernity, not yet the final word. For the generalization of the Dirac theory to general relativity it turned out to

[16]This is the terminology in, e.g., (Bleecker. 1981, 46). I thank an anonymous referee for the literature hint.

[17]The Strasbourg group of differential geometers about Charles Ehresmann played a crucial role for this development. Ehresmann was a student of E. Cartan and had been in close contact with Weyl during his time at Göttingen (1930/1931) and again in Princeton from 1932 to 1934.

[18]After a choice of a local frame a numerical representation of the (Lorentz) orthogonal is specified, and the operation of the orthogonal group is "trivialized". A change of frames leads to another representation arising from the first one by conjugation (in modernized terminology a local change of trivialization).

be necessary (and in Weyl's view also natural) to extend the structure group Δ_o by a complex phase factor to $\tilde{\Delta}_o \cong \Delta_o \times U(1)$. Weyl indicated this step by the transition to a (2-component) spinor representation of the rotation group (resp. Lorentz group) and added that the latter was underdetermined by a complex factor of norm 1:

> The two components ψ_1, ψ_2 of the electronic wave field have the peculiarity that they are determined by the local frame $\mathbf{f}$ only up to an arbitrary factor $\alpha = e^{i\lambda}$ of modulus 1. The real λ could be described as a common shift of phase in the two complex quantities ψ_1, ψ_2. This gauge factor α adds one more parameter to the representing group of transformations ... (Hs, p. 20)

This extension leads to a general relativistic theory of the electron field, which had been proposed independently by Weyl and V. Fock in 1929.[19] In modernized description Weyl finally considered the gauge automorphisms $\mathcal{G}(P)$ of a bundle

$$\Delta_o \times U(1) \cong \tilde{\Delta}_o \circlearrowright P \to M \tag{1}$$

constructed over spacetime M (considered as a Lorentz manifold) by extending the orthogonal frame bundle as the *physical automorphisms* of (phase extended) general relativity.

He indicated that the new automorphism group corresponds to important *conservation principles* and/or *structure theorems* of physics: conservation of energy and momentum correspond to the coordinate transformations of spacetime, symmetry of the energy tensor to the point-dependent "rotations" (HS, p. 19). The first part of this remark should not be taken literally; it is (overly) simplified. While the second part of the statement (symmetry of energy tensor) is a consequence of the rotational symmetries of the automorphism group, the conservation of energy/momentum is more problematic. It makes sense in special relativity and under restriction to the Poincaré group, or more generally under strong homogeneity conditions of spacetime and a restriction to adapted frames of reference; in the general case the conserved quantities exist mathematically (Noether currents) but do not allow a coordinate and observer independent physical interpretation. Weyl had discussed this point more precisely in his 1929-paper; here he simplified it, probably for didactical reasons, a bit too strongly.[20]

The justification of the attribute "physical" for the diffeomorphism component of the automorphism group of general relativity is therefore more subtle than admitted here by Weyl. But he had given another necessary criterion for their *physicality* earlier (Hs. p. 16): the invariance of the laws

[19] (Vizgin 1994, Straumann 2001, Scholz 2005).

[20] Weyl touched the problematics of the Noether theorems, without quoting Noether. For establishment, reception and philosophical discussion of the Noether theorems see (Kosmann-Schwarzbach 2011, Rowe 1999, Brading 2005, Sus 2016).

of nature, which included the invariance of the Lagrangian density for a field theory like Einstein gravity, *without any covariant non-dynamical background structure* (see above) – a subtle characterization of physicality indeed.

The simplified discussion of conservation of energy/momentum made it easier to emphasize the physcial role of the phase gauge invariance of electromagnetism. After the rather formal introduction of the phase extension of the structure group $\tilde{\Delta}_o \cong \Delta_o \times U(1)$, Weyl announced:

> The law of conservation of charge corresponds to it [phase invariance, E.S.] in the same manner as the law of the conservation of energy-momentum corresponds to the invariance under coordinate transformations. (Hs, p. 20)

This feature was of particular importance for Weyl's 1929 extension of the automorphism group of general relativity. It is, in fact, less problematic than the energy conservation statement mentioned above.[21]

On the other hand, the "gauging" of phase (the choice of a local trivialization in mathematical language) relied on a completely abstract, symbolical choice; the same holds for the localized phase transformations. While gauging the (Lorentz) orthogonal group could still be understood as a choice of frames, i.e. as a result of a choice of observer systems with point dependent relative motions, and the gauging of Weyl's 1918 scale group could be understood as as point dependent choice of units of measurement, the gauging of phase was emptied of any direct empirical content, it became "descriptive fluff" (Earman 2004). This poses the question in which respect Weyl's talk of "physical automorphisms" deals with more than a bunch of transformations of descriptive fluff.

The answer was already indicated by Weyl by emphasizing the *invariance of physical laws* (without non-dynamical covariant background structure) and the reference to *conserved quantities* or *structural consequences* of what would later become known as conserved Noether currents. In the language of fibre bundles we may rephrase Weyl's argument by calling to attention that the gauge automorphism group $\mathcal{G}(P)$ operates on the whole system of dynamical variables in a way which allows to deal with the invariance of the "laws of nature" (in particular the Lagrangian densities in case of Lagrangian field theories) in a mathematically precise way without introducing new non-dynamical background features. This allows to decide whether a given group consists of automorphisms of the theory and gives a necessary condition for the latter's "physicality" (background independence); but it does not yet give a sufficient criterion. We may still be able to extend the bundle of a given physically meaningful structure by a new symmetry *and* a hypothetical new dynamical field which avoids the appearance of a non-dynamical background

[21]The reason lies in the specific structure of the phase gauge extension; see (Brading 2002).

structure.[22] The crucial question is as to whether or not the extension of the *structure as a whole*, not every single field in isolation, leads to *new physical insight*. This may be of any kind, conservation laws are only the most prominent example.

Weyl's proposal of 1918 for introducing a metrical (point-dependent) scale gauge invariance as a fundamental symmetry of electromagnetism was substituted by a point-dependent phase choice. He discussed the transition in some detail and emphasized that in the "old theory" the unit of the electromagnetic potential was related to Einstein's cosmological constant, while in the new theory the potential was "measured not in an unknown cosmological, but in a known atomic unit" (Hs, p. 21). Moreover, quantum theory seemed to speak for the existence of an "absolute standard of length" (by fixing the frequency of atomic clocks, see above); he therefore concluded:

> I have no doubt that my old speculative theory has to be given up in favor of the quantum mechanical principle of phase invariance, that rests on sound empirical foundations. The facts of atomism teach us that length is not relative but absolute, and that the origin for the standard of length must be sought not in the cosmos as a whole, but in the elementary material particles. The additional group parameter is not geometric dilatation, but electronic phase shift. I was on the right track in 1918 as far as the formalism of gauge invariance is concerned. But the ψ's on which to hang the gauge factor α were utterly unknown at that time, and so I wrongly hitched it on to Einstein's gravitational potentials g_{ik}. (Hs, p. 21f.)

Weyl definitely no longer considered the scale invariance of 1918 to be of physical relevance. After 1929 he saw its status reduced to being part of the mathematical automorphisms of general relativity, while the physical automorphisms were extended to include a "gauged" (point-dependent) phase symmetry.

5 Weyl's distinction in the context of gauge structures

In this passage Weyl attributed different epistemic qualities to two possible extensions $\tilde{\Delta}_o = \Delta_o \times H$ of the structure group (here Δ_o the orthogonal group) by abelian factors $H = U(1)$ (phase) or $H = \mathbb{R}^+$ (scale). The distinction sheds light on the central topic of the talk because the first one lead to an extension of the *physical automorphisms* of general relativity, while the

[22] A good example is the reintroduction of Weyl's scale symmetry into gravity by Utiyma and Dirac in the early 1970s. They added a hypothetical scale covariant scalar field coupled to the Hilbert term, similar to the Jordan-Brans-Dicke theory, and endowed it with a kinetic term of its own (Weyl geometric scalar tensor theory).

second one appeared as an extension of the *mathematical* automorphisms only. This was a nice analogy to the relationship between *congruences* and *similarities* in the classical case. But Weyl avoided to give an explicit and general description of the demarcation between the physical and the mathematical automorphisms of "Nature".

On the other hand, Weyl developed different aspects and criteria for the distinction he was after in his case by case discussion of three different phases in the development of modern physics (early modern, late 19th century, early 20th century). In the following we try to distill the principles underlying the case-bound criteria which served Weyl for his distinction in in a more general and explicit form. Then we can see whether they tell us something about other theoretical developments, partly contemporary to Weyl but not dicussed in his talk, like Einstein-Cartan gravity, but also about later ones like Sciama-Kibble and Utiyma-Dirac gravity. Maybe they even can enrich the philosophical debate on the gauge theories of the standard model of elementary particle physics.

I propose the following generalization and interpretation: In principle any normal extension $\mathfrak{A}_m$ of a given automorphism group $\mathfrak{A}_o$ may constitute a new level of *mathematical objectivity* for "the mathematician", if it leaves some interesting structure invariant. For a theory aiming at *physical objectivity*, on the other hand, the automorphisms have to be constrained to the largest subgroup $\mathfrak{A}_p \subset \mathfrak{A}_m$ which satisfies the following criteria:

(i) Basic physical relations ("laws of nature") are invariant under $\mathfrak{A}_p$.

(ii) There are no non-dynamical background structures (no "absolute" elements).

(iii) All degrees of freedom of $\mathfrak{A}_p$ have some physically meaningful, perhaps even striking, consequence for the theory as a whole. Such consequences may consist in a crucial heuristic role of $\mathfrak{A}_p$ for determining the principles underlying the "laws of nature", e.g. in decisive constraints for the Lagrange density.

Criterion (iii) has been vaguely formulated, because it may be realized by quite different types of consequences. The most prominent ones, beside the symmetry constraint for the Lagrangian, are the Noether equations of the respective symmetries which may lead to empirically relevant conserved quantities (*Noether charge paradigm*). But this type is not the only one; there may be other consequences of a more *structural nature*.

In Weyl's discussion this was the case for the rotational degrees of freedom of his physical automorphism (1); these implied the symmetry of the energy-momentum tensor rather than a conserved charge. In addition, Weyl's argument for the physical character of the diffeomorphism component in the base manifold of (1) by hinting at the conservation of energy and momentum has to be taken *cum grano salis*, because the conserved Noether charges of the diffeomorphism degrees of freedom cannot be given an empirically relevant meaning without assuming special conditions, e.g. asymptotic flatness (cf. footnote 20). On the other hand, the postulate of diffeomorphism invariance constrains the choice of the Lagrangian for "the metric structure, and the inertial structure derived from it" (Hs, p. 16, quote above); it leads to the Hilbert action plus a constant term which may, but need not, vanish. Therefore the diffeomorphisms are clearly part of the physical automorphism group of general relativity.

6 A side glance at other gauge theories

From a more recent point of view, we may add that in the standard model of elementary particle physics the most important physical insight of the underlying gauge structures, in addition to the symmetry constraint for the Lagrangians, consists in the property of *renormalizability*. The Noether equations do not lead to empirically relevant conserved quantities; but under quantization they develop a structural effect ("Slavnov-Taylor identies") which is crucial for the renormalizability of the theories. They thus establish a crucial precondition for the empirical relevance of the theory. Weyl often insisted on the necessity for comparing a theory with the respective segment of "the world" only as a whole. For him, the "physicality" of automorphisms could be established by features of a much more general nature than one might expect from a strictly empiricist point of view. Perhaps his conception may shed light on the discussion in the present literature on philosophy of physics why, or even whether, gauge transformations may be of physical significance, if taking into account that gauge transformations seem to deal with nothing more than "descriptive fluff".

Before we come to the specific Weylian point, we have to remember that gauge transformations appear of primarily descriptive nature only if we consider them in their function as changes of local (in the mathematical sense) changes of trivializations. In this function they are comparable to the transformations of the coordinates in a differentiable manifold, which also seem to have a purely "descriptive" function. But the coordinate changes stand in close relation to (local) diffeomorphisms, like in Weyl's argumentation above. Therefore the postulate of coordinate independence of natural laws, or of the Lagrangian density, can and is being restated in terms of diffeomorphism invariance in general relativity. Similarly, the local changes of trivializations may be read as local descriptions of elements of $\mathfrak{G}_P$ in the notation above, i.e., as fibrewise operations of gauge automorphisms. $\mathfrak{G}_P$ is a subgroup of the more general gauge automorphism group $\mathfrak{G}(P)$ which includes transformations of the base like in Weyl's discussion; it thus reflects an important part of the structural features of the bundle $G \circlearrowright P \to M$.

The question as to whether or not the automorphisms of $\mathfrak{G}_P$, or even of $\mathfrak{G}(P)$, express crucial physical properties (item (iii) above) has nothing to do with the specific gauge nature of the groups, but hinges on the more overarching question of physical adequateness and physical content of the theory. The question of whether or why gauge symmetries can express physical content is not much different from the Kretschmann question of whether or why coordinate invariance of the laws, respectively coordinate covariance description of a physical theory, can have physical content. In the latter case the answer to the question has been dealt with in the philosophy of physics literature in great detail. Weyl's contribution to both levels of the debate, the original Kretschmann question *and* the gauge symmetry question has been described above; his answer is contained in his thoughts on the distinction of physical and mathematical automorphisms. In our rephrasing they are basically covered by (iii).

Let us shed a side-glance at gravitational gauge theories not taken into account by Weyl in his talk. In Einstein-Cartan gravity, which later turned out to be equivalent to Kibble-Sciama gravity, the localized rotational degrees of freedom lead to a conserved spin current and a non-symmetric energy tensor.[23] This is a structurally pleasing effect, fitting roughly into the Noether charge paradigm, although with a peculiar "crossover" of the two Noether currents and the currents feeding the r.h.s of the dynamical equations, inherited from Einstein gravity and Cartan's identification of *translational* curvature with *torsion*. The rotational current, spin, feeds the dynamical equation of translational curvature; the translational current, energy-momentum, feeds the rotational curvature in the (generalized) Einstein equation.[24] According to the experts it may acquire physical relevance only if energy densities surpass the order of magnitude 10^{38} times the density of neutron stars.[25] By this reason the current cannot yet be considered a physically striking effect. It may turn into one, if gravitational fields corresponding to extremely high energy densities acquire empirical relevance. For the time being, the rotational current can safely be neglected, Einstein-Cartan gravity reduces effectively to Einstein gravity, and Weyl's argument for the symmetry of the energy-momentum tensor remains the most "striking consequence" in the sense of (iii) for the rotational degrees of freedom.

On the other hand, the translational degrees of freedom give a more direct expression for the Noether currents of energy-momentum than the diffeomorphisms. The physical consequences for the diffeomorphism degrees of freedom reduce to the invariance constraint for the Lagrangian density for Einstein gravity considered as a special case of the Einstein-Cartan theory (with effectively vanishing spin). Besides these minor shifts, it may be more interesting to realize that the approach of Kibble and Sciama agreed nicely

[23](Trautman 2006, Hehl 2016).

[24](Hehl 2016, sec. 9, 10)

[25](Blagojević/Hehl 2012, p. 108), (Trautman 2006, 194).

with Weyl's methodological remark that for understanding nature we better "start with the group Γ of automorphisms and refrain from making the artificial logical distinction between basic and derived relations ..." (Hs. 6, see above, sec. 1). This describes quite well what Sciama and Kibble did. They started to explore the consequences of localizing (in the physical sense) the translational and rotational degrees of freedom of special relativity. Their theory was built around the generalized automorphism group arising from localizing the Poincaré group.

7 The enduring role of the Lorentz group

At the end of his talk Weyl pondered on the reasons why the structure group of the physical automorphisms still contained the "Euclidean rotation group" (respectively the Lorentz group, E.S.) in such a prominent role:

> The Euclidean group of rotations has survived even such radical changes of our concepts of the physical world as general relativity and quantum theory. What then are the peculiar merits of this group to which it owes its elevation to the basic group pattern of the universe? For what 'sufficient reasons' did the Creator choose this group and no other?" (Hs, 22)

He reminded his audience that Helmholtz had characterized $\Delta_o \cong SO(3,\mathbb{R}))$ by the "fact that it gives to a rotating solid what we may call its just degrees of freedom" of a rotating solid body;[26] but this method "breaks down for the Lorentz group that in the four-dimensional world takes the place of the orthogonal group in 3-space" (Hs, p. 22). In the early 1920s he himself had given another characterization living up to the new demands of the theories of relativity in his mathematical analysis of the problem of space.[27]

But now, twenty years later, he wanted to go further. A bit earlier in his talk he mentioned the idea that the Lorentz group might play its prominent role for the physical automorphisms because it expresses deep lying matter structures; but he strongly qualified the idea immediately after having stated it:

> Since we have the dualism of invariance with respect to two groups and Ω certainly refers to the manifold of space points,

[26] Weyl explained this metaphorical description by what is now being called *simple flag transitivity*: "any incident set of $1-$, $2-$, ..., $(n-1)-$ dimensional directions can be carried into any other such set by a suitable but uniquely determined element of the group" (Hs. p. 22).

[27] "I have given another characterization free from this blemish by showing that the group of linear transformations that leave a non-degenerate quadratic form invariant is the only one that ties affine connection to metric in the manner so characteristic for Riemannian geometry and Einsteinian gravitation." (Hs, p. 22) Cf. (Bernard 2015, Scholz 2016*a*)

> it is a tempting idea to ascribe Δ_o to *matter* and see in it a characteristic of the localizable elementary particles of matter. I leave it undecided whether this idea, the very meaning of which is somewhat vague, has any real merits. (Hs, p. 19)

Coming closer to the end of his talk he indicated another, rather more mathematical idea, even more abstract than his approach in the mathematical analysis of the space problem

> ... But instead of analysing the structure of the orthogonal group of transformations Δ_o, it may be wiser to look for a characterization of the group Δ_o as an *abstract* group. Here we know that the homogeneous n-dimensional orthogonal groups form one of 3 great classes of *simple Lie groups.* This is at least a partial solution of the problem. (Hs, 22)

He left it open why it ought to be "wiser" to look for abstract structure properties in order to answer a natural philosophical question. Could it be that he wanted to indicate an open-mindedness toward the more structuralist perspective on automorphism groups, preferred by the young algebraists around him at Princetion in the 1930/40s? Today the classification of simple Lie groups distinguishes 4 series, A_k, B_k, C_k, D_k. Weyl apparently counted the two orthogonal series B_k and D_k as one. The special orthogonal groups in even complex space dimension form the series of simple Lie groups of type D_k, with complex form $SO(2k, \mathbb{C})$ and real compact form $SO(2k, \mathbb{R})$. The special orthogonal group in odd space dimension form the series type B_k, with complex form $SO(2k+1, \mathbb{C})$ and compact real form $SO(2k+1, \mathbb{R})$.[28]

But even if one accepted such a general structuralist view as a starting point there remained a question for the specification of the space dimension of the group inside the series.

> But the number of the dimensions of the world is 4 and not an indeterminate n. It is a fact that the structure of Δ_o is quite different for the various dimensionalities n. Hence the group may serve as a clue by which to discover some cogent reason for the dimensionality 4 of the world. What must be brought to light, is the distinctive character of one definite group, the four-dimensional Lorentz group, either as a group of linear transformations, or as an abstract group. (Hs, p. 22f.)

The remark that the "structure of Δ_o is quite different for the various dimensionalities n" with regard to even or odd complex space dimensions (type D_k,

[28] For the classification simple Lie algebras see any book on Lie algebras/groups; for the complex case, e.g., (Knapp 1996), thms. 2.84, 2.111 (abstract classification and existence), for a survey of the simple real Lie algebras chap. VI.10. A lucid exposition (in German) gives (Brieskorn 1985, 501–513). Weyl had essentially contributed to the theory in his book on the *Classical Groups* (Weyl 1938).

resp. B_k) strongly qualifies the import of the general structuralist characterization. But already in the 1920s Weyl had used the fact that for the (real) space dimension $n = 4$ the universal covering of the unity component of the Lorentz group $SO(1,3)^o$ is the realification of $SL(2,\mathbb{C})$. The latter belongs to the first of the A_k series (with complex form $SL(k+1,\mathbb{C})$). Because of the isomorphism of the initial terms of the series, $A_1 \cong B_1$, this does not imply an exception of Weyl's general statement. We even may tend to interpret Weyl's otherwise cryptic remark that the structuralist perspective gives a "at least a partial solution of the problem" by the observation that the *Lorentz group in dimension* $n = 4$ is, in a rather specific way, the *realification of the complex form of one of the three most elementary non-commutative simple Lie groups* of type $A_1 \cong B_1$.[29] Its compact real form is $SO(3,\mathbb{R})$, respectively the latter's universal cover $SU(2,\mathbb{C})$.[30]

Weyl stated clearly that the answer cannot be expected by structural considerations alone. The problem is only "partly one of pure mathematics", the other part is "empirical". But the question itself appeared of utmost importance to him

> We can not claim to have understood Nature unless we can establish the uniqueness of the four-dimensional Lorentz group in this sense. It is a fact that many of the known laws of nature can at once be generalized to n dimensions. We must dig deep enough until we hit a layer where this is no longer the case. (Hs, p. 23)

In 1918 he had given an argument why, in the framework of his new scale gauge geometry, the "world" had to be of dimension 4. His argument had used the construction of the Lagrange density of general relativistic Maxwell theory $\mathcal{L}_f = f_{\mu\nu} f^{\mu\nu}\sqrt{|detg|}$, with $f_{\mu\nu}$ the components of curvature of his newly introduced scale/length connection, physically interpreted by him as the electromagnetic field. $\mathcal{L}_f$ is scale invariant only in spacetime dimension $n = 4$.[31] The shift from scale gauge to phase gauge undermined the importance of this argument. Although it remained correct mathematically, it lost its convincing power once the scale gauge transformations were relegated from physics to the mathematical automorphism group of the theory only.

Weyl's talk ended with the words:

> Our question has this in common with most questions of philosophical nature: it depends on the vague distinction between es-

[29] The other "most elementary" ones belong to the types C_1, D_2.

[30] See, e.g., (Brieskorn 1985, 512f.).

[31] Scale invariance of this Lagrange density presupposes that the lifting of indices for f compensates the change of the $\sqrt{|detg|}$ under rescaling. Counting the scale weight of the metric $g_{\mu\nu}$ as 2, scale invariance of $\mathcal{L}_f$ holds if and only if $-4 + \frac{1}{2}2n = 0 \longleftrightarrow n = 4$, (Weyl 1918*a*, p. 31 (p. 37 in the English version)). Similarly for *any Yang-Mills Lagrangian* with Lie algebra valued 2-form A, $\mathfrak{L}_{YM} = -\frac{1}{2}tr(A \wedge *A) = -\frac{1}{4}A_{\mu_1\mu_2}A^{\mu_1\mu_2}\sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n$.

> sential and non-essential. Several competing solutions are thinkable; but it may also happen that, once a good solution of the problem is found, it will be of such cogency as to command general recognition.

This kind of consideration was typical in its openness for his later, mature way of reflecting.

8 Later developments and discussion

In his talk Weyl spanned a huge arc from classical geometry and physics until the 20th century. By surveying this development he hoped to be able to identify criteria for the distinction between physical and mathematical aspects in the automorphisms of the respective theories. In the first phase material operations of solid bodies could be used for establishing the elementary basis of congruence. They constituted the fundamental precondition for the possibility of stable trans-subjective measurements in the time from Newton to about Helmholtz (section 2). With the shift towards the viewpoint of invariance transformations of physical laws in the last third of the 19th century, a weak extension by including orientation reversing isometries became necessary. Although such transformations could no longer be realized by motions of bodies, an aspect which had been observed in relational form already by Kant, there still remained a chance for an indirect material re-modelling of the additional transformations by mirror transformations and/or point inversions.

On the other hand, mathematical automorphisms of classical geometry were generalized in different versions during the 19th century. Roughly speaking they could be "any", depending on the structure mathematicians intended to study. Among these, so Weyl argued, the normalizer of physical automorphisms played a distinguished role. Normalizer "in which group"? Weyl may have thought of the general linear transformations or of the differentiable bijective point transformations (diffeomorphisms) of space. In the light of our context, the last interpretation seems more likely. But one might also consider any normal extension appearing natural or promising, by some criterion, as candidate for mathematical automorphisms associated to a given physical automorphism group. The reason would be the same as the one Weyl gave for similarities with regard to the congruence relation. A normal extension is "conservative" with regard to the relations typical for the physical automorphisms.

After the "shock of relativity" the structure of the physical automorphisms became that of a gauge group $\mathcal{G}(P)$ of a principal bundle over spacetime $P \to M$ with structure group Δ_o "read off" from the then best available theories of matter and its dynamics. For Weyl this was general relativity (gravity and electromagnetism) combined with the Dirac theory of the electron; then $\Delta_o \cong SO(1,3;\mathbb{R}) \times U(1)$ or $SL(2,\mathbb{C}) \times U(1)$. The invariance transformations of the physical automorphism group were, in general, no longer representable by material operations, in particular the diffeomorphisms of spacetime. They now took on the form of mathematical transformations of the theoretical structure designed for representing natural laws. A part of the transformations, the "point dependent rotations", however, could still be interpreted physically in a wider sense, as transitions between families of observer systems. In any case, the former clear distinction between mathematical and physical automorphisms became blurred. This point was not discussed by Weyl. Most important for him was the difference which he could now state between his 1918 scale gauge theory ("only mathematical") and 1929 phase gauge theory (part of "true" physics).

In hindsight Weyl's 1948/49 discussion of the physical automorphisms as a gauge group may appear as an anticipation of what was to become a central paradigm for field physics in the last third of the 20th century. Gauge theories gained wide recognition with the rise of the standard model (SM) of elementary particle physics, from the 1970s onward. The physical automorphisms of the SM is here again a gauge group $\tilde{\mathcal{G}}(P_{SM})$, but now considered over Minkowski space $\mathbb{M}$ of special relativity, $P_{SM} \longrightarrow \mathbb{M}$. In comparison with Weyl's general relativistic group, the morphisms of the base are reduced to the Poincaré transformations in $\mathbb{M}$. In this way the structure group of the SM is both, reduced and enlarged, in comparison with the physical automorphisms of GR considered by Weyl:

- The morphisms of the base $M = \mathbb{M}$ are no longer the full diffeomorphism group but are restricted to the linear transformations of the Poincaré group. As a result the Lorentz transformations are no longer "localized" but are considered as part of the "global" transformations of $\mathbb{M}$.

- On the other hand and most importantly, the remaining part of Weyl's structure group $U(1)$ is extended to $\tilde{\Delta}_o \cong SU(2) \times U(1)_Y \times SU(3)$ (with the electromagnetic $U(1)$ as residuum after $SU(2)$ symmetry breaking).

Moreover, the distinction between physical and mathematical aspects of automorphisms have been undermined even deeper than in Weyl's discussion: There remains no chance for a material interpretation of the gauge transformations of the SM, whereas for Weyl the change of orthonormal frames was still interpretable – not realizable – as change between families of "local" inertial observers.[32]

[32]Compare the discussion in present philosophy of physics on the "ontology" (the physical interpretation) of gauge transformations and the Higgs mechanism (Giulini 1996, Earman 2002, Earman 2004, Lyre 2004, Healey 2007, Rovelli 2014, Friederich 2014b, Friederich 2014a).

There remains the problem of unifying the viewpoints of GR and the standard model of elementary particles in one coherent theoretical framework. One of the different approaches to achieve this goal envisages a first step towards an integration without a full quantization of gravity: the strategy to formulate SM fields and their dynamics on "curved spaces" (i.e. Lorentz manifolds).[33] Although it is not explicitly discussed by the authors, their work contains an underlying automorphism group of the theory $\mathcal{G}(P)$ with full diffeomorphism group of the base manifold M. If one formulates their theory in the orthonormal frame approach, the structure group becomes the Lorentz group, or its universal covering, normally extended by the structure group of the SM

$$G \cong SL(2,\mathbb{C}) \times SU(2) \times U(1)_Y \times SU(3) \tag{2}$$

This perspective may even open a new look at Weyl's scale invariance. The Lagrangian of the standard model is basically scale invariant, its fields are formulated scale covariantly and consistent with their import to scale covariant GR. The only exception is the dimensionful coefficient μ of the quadratic term $\mu\,\Phi\Phi^*$ of the Higgs field Φ. But it is not difficult to bring the latent scale covariance of the SM fields into the open by introducing a second, gravitational and real valued, scalar field ϕ of correct scale weight such that $\mu = \tilde{\mu}\phi^2$ with *numerical* coefficient $\tilde{\mu}$. With regard to "global" scale symmetries over Minkowski space this is being done, e.g., by (Shapovnikov/Zenhäusern 2009). The scale symmetries are "localized" in *conformal approaches* to SM fields like in and in those working in the framework of *Weyl geometry*[34]. Inherent in this research is an extension of the automorphism group of the theory to $\tilde{\mathcal{G}}(P)$ like above, but here with structure group

$$\tilde{G} \cong R^+ \times SL(2,\mathbb{C}) \times SU(2) \times U(1)_Y \times SU(3) \tag{3}$$

But has not such an approach already been refuted by Weyl's argument that quantum physics establishes an "absolute" standard of measurement? – The answer is "no", because the gravitational scalar field ϕ allows to form scale invariant proportions of quantities which give rise to the measurement values once the unities have been fixed. Moreover, in a particular scale gauge (the one in which the norm of the scalar field is constant) measurement values are expressed directly, up to a global normalization according to the chosen unit. If one prefers one may describe it, metaphorically, as a kind of "symmetry breaking" of scale symmetry by the gravitational scalar field.[35]

[33]E.g., (Fredenhagen e.a. 2007, Bär e.a. 2009).

[34]For the conformal view see, e.g., (Meissner/Nicolai 2009, Bars 2014), for the integrable Weyl geometric one (Nishino/Rajpoot 2011, Nishino/Rajpoot 2007, Quiros 2014, Almeida e.a. 2014, Scholz 2016*b*), for a non-integrable scale connection (Ohanian 2016).

[35]"Metaphorical" because no phase transition is involved (as far as we can see at the moment) like in the usual understanding of "spontaneaous symmetry breaking", cf. (Friederich 2011).

At the moment, the recent research line based on the Weyl geometric approach explores mathematical models of gravity with or without standard model fields. As long as there are no striking physical consequences deducible in this framework, which remain unexplained on the basis of Riemann/Einstein gravity, one has to consider the group $\tilde{\mathcal{G}}(P)$ of this theory as part of the *mathematical automorphisms* in Weyl's language. But what, if important empirical phenomena are better explained in this framework than within the underlying Riemann geometric structure with automorphism group $\mathcal{G}(P)$? – Then we will be in a situation where the physical automorphisms of gravity plus SM fields are extended from $\mathcal{G}(P)$ to $\tilde{\mathcal{G}}(P)$ (with structure group (3)) and Weyl's scale extension, the point dependent "similarities", become part of the *physical automorphism group* again.

This might even give an unexpected and intriguing twist to Weyl's final question: Why does the Lorentz group play such a prominent role in the structure group Δ_o, and why $n = 4$? As outlined above, his argument of 1918 for the 4-dimensionality of the "world", the spacetime manifold of gravity plus electromagnetism, did not depend on his specific physical interpretation of the scale connection (Weyl's φ_k) as the potential and the scale curvature (the $f_{\mu\nu}$) as the components of the Maxwell field. It is a *structural property of the scale invariance of the Lagrangian density*, not only for the scale curvature but for all Yang-Mills type Lagrange densities (see fn. 31). In particular, the specification of dimension $n = 4$ by the scale invariance condition of the Lagrange density works independently of the physical interpretation of the φ_k. If there are reasons to include the (localized) scale transformations in the physical automorphism group again, Weyl's argument for $n = 4$ will acquire new strength. Together with his observation of the striking simplicity of the Lorentz group as one of the simplest simple Lie groups we may have the impression that in following Weyl's hints we can come a bit closer to having "understood Nature" .

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