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# Extension of Crisp Functions on Neutrosophic Sets

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**Abstract**. In this paper, we generalize the definition of Neutrosophic sets and present a method for extending crisp functions on Neutrosophic sets and study some properties of such extended functions.

**Keywords:** Neutrosophic set, Multi-fuzzy set, Bridge function..

## 1 Introduction

L-fuzzy sets constitute a generalization of the notion of Zadeh's [26] fuzzy sets and were introduced by Goguen [8] in 1967, later Atanassov introduced the notion of the intuitionistic fuzzy sets [1] Gau and Buehrer [7] defined vague sets. Bustince and Burillo [2] showed that the notion of vague sets is the same as that of intuitionistic fuzzy sets. Deschrijver and Kerre [5] established the interrelationship between the theories of fuzzy sets, L-fuzzy sets, interval valued fuzzy sets, intuitionistic fuzzy sets, intuitionistic L-fuzzy sets, interval valued intuitionistic fuzzy sets, vague sets and gray sets [4].

## 2 Preliminaries

**Definition 2.1.** [26] Let $X$ be a nonempty set. A fuzzy set $A$ of $X$ is a mapping $A : X \to [0, 1]$, that is,
$A = \{(x, \mu_A(x)) : \mu_A(x)$ is the grade of membership of $x$ in $A$, $x \in X\}$. The set of all the fuzzy sets on $X$ is denoted by $\mathcal{F}(X)$.

**Definition 2.2.** [8] Let $X$ be a nonempty ordinary set, $L$ a complete lattice. An $L$-fuzzy set on $X$ is a mapping $A : X \to L$, that is the family of all the $L$-fuzzy sets on $X$ is just $L^X$ consisting of all the mappings from $X$ to $L$.

**Definition 2.3.** [1] An Intuitionistic Fuzzy Set on $X$ is a set

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x) \in [0, 1]$ denotes the membership degree and $\nu_A(x) \in [0, 1]$ denotes the non-membership degree of $x$ in $A$ and

$$\mu_A(x) + \nu_A(x) \leq 1, \forall x \in X.$$

The neutrosophic set (NS) was introduced by F. Smarandache [22] who introduced the degree of indeterminacy (i) as independent component in his manuscripts that was published in 1998.

Multi-fuzzy sets [12, 13, 16] was proposed in 2009 by Sabu Sebastian as an extension of fuzzy sets [8, 26] in terms of multi membership functions. In this paper we generalize the definition of neutrosophic sets and introduce extension of crisp functions on neutrosophic sets.

**Definition 2.4.** [22]A Neutrosophic Set on $X$ is a set

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T_A(x) \in [0, 1]$ denotes the truth membership degree, $I_A(x) \in [0, 1]$ denotes the indetermi-nancy membership degree and $F_A(x) \in [0, 1]$ denotes the falsity membership degree of $x$ in $A$ respectively and

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3, \forall x \in X.$$

For single valued neutrosophic logic $(T, I, F)$, the sum of the components is: $0 \leq T+I+F \leq 3$ when all three components are independent; $0 \leq T + I + F \leq 2$ when two components are dependent, while the third one is independent from them; $0 \leq T + I + F \leq 1$ when all three components are dependent.

**Definition 2.5.** [12, 13, 16]Let $X$ be a nonempty set, $J$ be an indexing set and $\{L_j : j \in J\}$ a family of partially ordered sets. A **multi-fuzzy set A** in $X$ is a set :

$$\mathbf{A} = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in X,\ \mu_j \in L_j^X,\ j \in J\}.$$

The indexing set J may be uncountable. The function $\mu_{\mathbf{A}} = (\mu_j)_{j\in J}$ is called the membership function of the multi-fuzzy set $\mathbf{A}$ and $\prod_{j\in J} L_j$ is called the value domain.

If J = {1, 2, ..., n} or the set of all natural numbers, then the membership function $\mu_{\mathbf{A}} = \langle \mu_1, \mu_2, ...\rangle$ is a sequence.

In particular, if the sequence of the membership function having precisely $n$-terms and $L_j = [0, 1]$, for $J = \{1, 2, ..., n\}$, then $n$ is called the dimension and $\mathbf{M^nFS}(X)$ denotes the set of all multi-fuzzy sets in X.

Properties of multi-fuzzy sets, relations on multi-fuzzy sets and multi-fuzzy extensions of crisp functions are depend on the order relations defined in the membership functions. Most of the results in the initial papers [12, 13, 15, 16, 18] are based on product order in the membership functions. The paper [21] discussed other order relations like dictionary order, reverse dictionary order on their membership functions.

Let $\{L_j : j \in J\}$ be a family of partially ordered sets, and
$\mathbf{A} = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in \mathrm{X}$,
$\mu_j \in L_j^X$, $j \in J\}$ and $\mathbf{B} = \{\langle x, (\nu_j(x))_{j\in J}\rangle : x \in X,\ \nu_j \in L_j^X,\ j \in J\}$ be multi-fuzzy sets in a nonempty set $X$. Note that, if the order relation in their membership functions are either product order, dictionary order or reverse dictionary order[16, 21], then;

- $\mathbf{A} = \mathbf{B}$ if and only if $\mu_j(x) = \nu_j(x), \forall x \in X$ and for all $j \in J$
- $\mathbf{A} \sqcup \mathbf{B} = \{\langle x, (\mu_j(x) \vee_j \nu_j(x))_{j\in J}\rangle : x \in X\}$ and
- $\mathbf{A} \sqcap \mathbf{B} = \{\langle x, (\mu_j(x) \wedge_j \nu_j(x))_{j\in J}\rangle : x \in X\}$,

where $\vee_j$ and $\wedge_j$ are the supremum and infimum defined in $L_j$ with partial order relation $\leq_j$. Set inclusion defined as follows:

- In product order, $\mathbf{A} \subset \mathbf{B}$ if and only if $\mu_j(x) < \nu_j(x), \forall x \in X$ and for all $j \in J$.
- In dictionary order, $A \subset B$ if and only if $\mu_1(x) < \nu_1(x)$ or if $\mu_1(x) = \nu_1(x)$ and $\mu_2(x) < \nu_2(x), \forall x \in X$.

**Definition 2.6.** *Let $L$ be a lattice. A mapping $' : L \to L$ is called an order reversing involution [25], if for all $a, b \in L$ :*

1. $a \leq b \Rightarrow b' \leq a'$;
2. $(a')' = a$.

**Definition 2.7.** [23] Let $' : M \to M$ and $' : L \to L$ be order reversing involutions. A mapping $h : M \to L$ is called an order homomorphism, if it satisfies the conditions:

1. $h(0_M) = 0_L$;
2. $h(\vee a_i) = \vee h(a_i)$;
3. $h^{-1}(b') = (h^{-1}(b))'$,

where $h^{-1} : L \to M$ is defined by, for every $b \in L$,
$h^{-1}(b) = \vee\{a \in M : h(a) \leq b\}$.

Generalized Zadeh extension of crisp functions [24] have prime importance in the study of fuzzy mappings. Sabu Sebastian [16, 13]generalized this concept as multi-fuzzy extension of crisp functions and it is useful to map a multi-fuzzy set into another multi-fuzzy set. In the case of a crisp function, there exists infinitely many multi-fuzzy extensions, even though the domain and range of multi-fuzzy extensions are same.

**Definition 2.8.** [16] Let $f : X \to Y$ and $h : \prod M_i \to \prod L_j$ be a functions. The multi-fuzzy extension of $f$ and the inverse of the extension are $f : \prod M_i^X \to \prod L_j^Y$ and $f^{-1} : \prod L_j^Y \to \prod M_i^X$ defined by

$$f(A)(y) = \bigvee_{y=f(x)} h(A(x)),\ A \in \prod M_i^X,\ y \in Y$$

and

$$f^{-1}(B)(x) = h^{-1}(B(f(x))),\ B \in \prod L_j^Y,\ x \in X;$$

where $h^{-1}$ is the upper adjoint [23] of $h$. The function $h : \prod M_i \to \prod L_j$ is called the **bridge function** of the multi-fuzzy extension of $f$.

**Remark 2.9.** In particular, the multi-fuzzy extension of a crisp function $f : X \to Y$ based on the bridge function $h : I^k \to I^n$ can be written as $f : \mathbf{M^kFS}(X) \to \mathbf{M^nFS}(Y)$ and $f^{-1} : \mathbf{M^nFS}(Y) \to \mathbf{M^kFS}(X)$, where

$$f(A)(y) = \sup_{y=f(x)} h(A(x)),\ A \in \mathbf{M^kFS}(X),\ y \in Y$$

and

$$f^{-1}(B)(x) = h^{-1}(B(f(x))), B \in \mathbf{M^nFS}(Y), x \in X.$$

In the following section $\prod M_i = \prod L_j = I^3$.

**Remark 2.10.** There exists infinitely many bridge functions. Lattice homomorphism, order homomorphism, lattice valued fuzzy lattices and strong L-fuzzy lattices are examples of bridge functions.

**Definition 2.11.** [10] A function $t : [0, 1] \times [0, 1] \to [0, 1]$ is a $t$-norm if $\forall a, b, c \in [0, 1]$:**(1)** $t(a, 1) = a$;

**(2)** $t(a, b) = t(b, a)$;

**(3)** $t(a, t(b, c)) = t(t(a, b), c)$;

**(4)** $b \leq c$ implies $t(a, b) \leq t(a, c)$.

Similarly, a $t$-conorm ($s$-norm) is a commutative, associative and non-decreasing mapping $s$ :$[0, 1] \times [0, 1] \to [0, 1]$ that satisfies the boundary condition:

$s(a, 0) = a$, for all $a \in [0, 1]$.

**Definition 2.12.** [9] A function $c : [0, 1] \to [0, 1]$ is called a complement (fuzzy) operation, if it satisfies the following conditions:

**(1)** $c(0) = 1$ and $c(1) = 0$,

**(2)** for all $a, b \in [0, 1]$, if $a \leq b$, then $c(a) \geq c(b)$.

**Definition 2.13.** [9] A $t$-norm $t$ and a $t$-conorm $s$ are dual with respect to a fuzzy complement operation $c$ if and only if

$c(t(a, b)) = s(c(a), c(b))$

and

$c(s(a, b)) = t(c(a), c(b))$,

for all $a, b \in [0, 1]$.

**Definition 2.14.** [9] Let $n$ be an integer greater than or equal to 2. A function $m : [0, 1]^n \to [0, 1]$ is said to be an aggregation operation for fuzzy sets, if it satisfies the following conditions:

1. $m$ is continuous;
2. $m$ is monotonic increasing in all its arguments;
3. $m(0, 0, ..., 0) = 0$;
4. $m(1, 1, ..., 1) = 1$.

## 3 Neutrosophic Sets

In this section, we generalize the definition of neutrosophic sets on [0, 1]. Throughout the following sections $X$is the universe of discourse and $A \in \mathbf{M^3FS}(X)$ means $A$ is a multi-fuzzy sets of dimension 3 with value domain $I^3$, where $I^3 = [0, 1] \times [0, 1] \times [0, 1]$. That is, $A \in (I^3)^X$.

**Definition 3.1.** Let $X$ be a nonempty crisp set and $0 \leq \alpha \leq 3$. A multi-fuzzy set $A \in \mathbf{M^3FS}(X)$is called a neutrosophic set of order $\alpha$, if

$\mathbf{A} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X,$
$0 \leq T_A(x) + I_A(x) + F_A(x) \leq \alpha\}$.

**Definition 3.2.** Let $A$, $B$ be neutrosophic sets in $X$ of order 3 and let $t$, $s$, $m$, $c$ be the $t$-norm, $s$-norm, aggregation operation and complement operation respectively. Then the union, intersection and complement are given by

1. $A \bigcup B = \{\langle x, s(T_A(x), T_B(x)), m(I_A(x), I_B(x)), t(F_A(x), F_B(x))\rangle : x \in X\}$;
2. $A \bigcap B = \{\langle x, t(T_A(x), T_B(x)), m(I_A(x), I_B(x)), s(F_A(x), F_B(x))\rangle : x \in X\}$;
3. $A^c = \{\langle x, c(T_A(x)), c(I_A(x)), c(F_A(x))\rangle : x \in X\}$.

## 4 Extension of crisp functions on neutrosophic set using order homomorphism as bridge function

**Theorem 4.1.** If an order homomorphism $h : I^3 \to I^3$ is the bridge function for the multi-fuzzy extension of a crisp function $f : X \to Y$, then for every $k \in K$ neutrosophic sets $A_k$ in $X$ and $B_k$ in $Y$ of order 3;

1. $A_1 \subseteq A_2$ implies $f(A_1) \subseteq f(A_2)$;
2. $f(\cup A_k) = \cup f(A_k)$;
3. $f(\cap A_k) \subseteq \cap f(A_k)$;
4. $B_1 \subseteq B_2$ implies $f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
5. $f^{-1}(\cup B_k) = \cup f^{-1}(B_k)$;
6. $f^{-1}(\cap B_k) = \cap f^{-1}(B_k)$;
7. $(f^{-1}(B))' = f^{-1}(B')$;
8. $A \subseteq f^{-1}(f(A))$;
9. $f(f^{-1}(B)) \subseteq B$.

*Proof.*

1. $A_1 \subseteq A_2$ implies $A_1(x) \leq A_2(x), \forall x \in X$ and implies $h(A_1(x)) \leq h(A_2(x)), \forall x \in X$.

Hence $\vee\{h(A_1(x)) : x \in X,$ $y = f(x)\} \leq \vee\{h(A_2(x)) : x \in X,$ $y = f(x)\}$ and $f(A_1)(y) \leq f(A_2)(y)$, $\forall y \in Y$. That is, $f(A_1) \subseteq f(A_2)$.

2. For every $y \in Y$,

$$\begin{aligned} f(\cup A_k)(y) &= \vee\{h((\cup A_k)(x)) : x \in X, \ y = f(x)\} \\ &= \vee\{h(\vee A_k(x)) : x \in X, \ y = f(x)\} \\ &= \vee\{\vee_{k\in K} h(A_k(x)) : x \in X, \ y = f(x)\} \\ &= \vee_{k\in K} \vee \{h(A_k(x)) : x \in X, \ y = f(x)\} \\ &= \vee_{k\in K} f(A_k)(y), \end{aligned}$$

thus $f(\cup A_k) = \cup f(A_k)$.

3. For every $y \in Y$,

$$\begin{aligned} f(\cap A_k)(y) &= \vee\{h((\cap A_k)(x)) : x \in X, \ y = f(x)\} \\ &= \vee\{h(\wedge_{k\in K} A_k(x)) : x \in X, \ y = f(x)\} \\ &\leq \vee\{h(A_k(x)) : x \in X, \ y = f(x)\}, \end{aligned}$$

for each $k \in K$. Hence

$f(\cap A_k)(y) \leq \wedge_{k\in K} \vee \{h(A_k(x)) : x \in X,$ $y = f(x)\} = \wedge_{k\in K} f(A_k)(y)$,

thus $f(\cap A_k) \subseteq \cap f(A_k)$.

4. $B_1 \subseteq B_2$ implies $B_1(y) \leq B_2(y), \forall y \in Y$. Hence

$f^{-1}(B_1)(x) = h^{-1}(B_1(f(x))) \leq h^{-1}(B_2(f(x))) =$ $f^{-1}(B_2)(x), \forall x \in X$.

Therefore, $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

5. For every $x \in X$, we have

$$\begin{aligned} f^{-1}(\cup B_k)(x) &= h^{-1}((\cup B_k)(f(x))) = h^{-1}(\sup_{k\in K} B_k(f(x))) \\ &= \sup_{k\in K} h^{-1}(B_k(f(x))) = \sup_{k\in K} f^{-1}(B_k)(x) \\ &= (\cup f^{-1}(B_k))(x). \end{aligned}$$

Hence $f^{-1}(\cup B_k) = \cup f^{-1}(B_k)$.

6. For every $x \in X$, we have

$$\begin{aligned} f^{-1}(\cap B_k)(x) &= h^{-1}((\cap B_k)(f(x))) = h^{-1}(\inf_{k\in K} B_k(f(x))) \\ &= \inf_{k\in K} h^{-1}(B_k(f(x))) = \inf_{k\in K} f^{-1}(B_k)(x) \\ &= (\cap f^{-1}(B_k))(x). \end{aligned}$$

Hence $f^{-1}(\cap B_k) = \cap f^{-1}(B_k)$.

7. For every $x \in X$,

$f^{-1}(B')(x) = h^{-1}(B'(f(x))) = h^{-1}(B(f(x)))' =$ $(f^{-1}(B))'(x)$, since $f^{-1}(B)(x) = h^{-1}(B(f(x)))$.

That is, $f^{-1}(B') = (f^{-1}(B))'$.

8. For every $x_0 \in X$,

$$\begin{aligned} A(x_0) &\leq \vee\{A(x) : x \in X, \ x \in f^{-1}(f(x_0)\} \\ &\leq h^{-1}(h(\vee\{A(x) : x \in X, \ x \in f^{-1}(f(x_0)\})) \\ &= h^{-1}(\vee\{h(A(x)) : x \in X, \ x \in f^{-1}(f(x_0))\}) \\ &= h^{-1}(f(A)(f(x_0))) \\ &= f^{-1}(f(A))(x_0). \end{aligned}$$

9. For every $y \in Y$

$$\begin{aligned} f(f^{-1}(B))(y) &= \sup_{y=f(x)} h(f^{-1}(B)(x)) \\ &= \sup_{y=f(x)} h(h^{-1}(B(f(x)))) \end{aligned}$$

$$= h(h^{-1}(B(y)))$$
$$\leq B(y).$$

Hence $f(f^{-1}(B)) \subseteq B$.

**Proposition 4.2.** If an order homomorphism $h : I^3 \to I^3$ is the bridge function for the extension of a crisp function $f : X \to Y$, then for any $k \in K$ neutrosophic sets $A_k$ in $X$ and $B$ in $Y$:

1. $f(0_X) = 0_Y$;
2. $f(\cup A_k) = \cup f(A_k)$; and
3. $(f^{-1}(B))' = f^{-1}(B')$,

that is, the extension map $f$ is an order homomorphism.

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