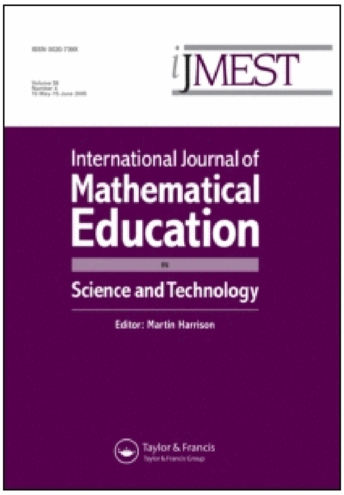


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# Affect, behavioural schemas and the proving process

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# Affect, behavioural schemas and the proving process

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In this largely theoretical article, we discuss the relation between a kind of affect, behavioural schemas and aspects of the proving process. We begin with affect as described in the mathematics education literature, but soon narrow our focus to a particular kind of affect – nonemotional cognitive feelings. We then mention the position of feelings in consciousness because that bears on the kind of data about feelings that students can be expected to be able to report. Next we introduce the idea of *behavioural schemas* as enduring mental structures that link situations to actions, in other words, habits of mind, that appear to drive many mental actions in the proving process. This leads to a discussion of the way feelings can both help cause mental actions and also arise from them. Then we briefly describe a design experiment – a course intended to help advanced undergraduate and beginning graduate mathematics students improve their proving abilities. Finally, drawing on data from the course, along with several interviews, we illustrate how these perspectives on affect and on behavioural schemas appear to explain, and are consistent with, our students' actions.

**Keywords:** tertiary level; design experiment; proof; proving; affect; nonemotional cognitive feelings; behavioural schemas; habits of mind



## 1. Introduction

The proving process plays a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis. In studying such topics, advanced undergraduate and beginning graduate mathematics students not only read and check proofs, but also supplement those proofs with their own subproofs. Their teachers not only explain proofs, but also ask them to construct proofs, especially as a way of assessing students' understandings. As a consequence, we are designing a course to help advanced undergraduate and beginning graduate mathematics students improve their proving abilities. We are also exploring the proving process, attending to whatever contributes to that process, in particular, affect and student actions. In this article, we focus on a particular kind of affect – nonemotional cognitive feelings – and on the implementation of actions via behavioural schemas. We begin with perspectives on affect from the literature.

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## 2. Perspectives on affect

### 2.1. *Beliefs, attitudes, emotions and values*

Affect is often seen as separate from, but related to, cognition. McLeod [1] treated affect as having three main aspects: (1) *beliefs*, for example, the belief that mathematics is based on rules or that teaching is telling; (2) *attitudes*, for example, dislike of geometric proving or enjoyment of problem solving and (3) *emotions*, for example, the joy of, or frustration with, solving nonroutine problems. He described beliefs, attitudes and emotions as of increasing intensity and decreasing stability, with emotions being the most intense and changing the most rapidly.

There are many interesting connections between affect, in the above view, and the teaching, learning and doing of mathematics. For example, while teaching a college algebra course, the first author occasionally gave group quizzes on which students were to solve moderately nonroutine problems. As she walked around answering questions and facilitating the group work, she found that one student was not contributing to her group's solutions, but was instead very quietly repeating to herself, 'I hate this stuff. I hate this stuff'. This student was expressing a negative emotion, and her attention was clearly divided between expressing that emotion and working on the problem at hand. This, no doubt, overburdened her working memory and interfered with trying to solve the problem, and hence interfered with contributing to the group's efforts.

In the above example, it is very likely that the details of the algebra problem and how it was being attempted were, more or less, immaterial to the student. Thus, affect, in this case an emotion, was associated in a rather negative and 'large-grained' way with the doing of mathematics, that is, a way independent of the kind of algebra problem. Much of the research on the relationship between affect and the doing of mathematics seems to have taken a similar large-grained approach [1–5]. However, in this article we take a finer-grained approach, and we begin an analysis of the relationship between specific kinds of affect and specific parts of the proving process.

Here is an example that seems to call for taking a finer-grained view of the proving process, and also for a somewhat expanded perspective on affect. During tutoring by the third author, a student, Sofia, from our Spring 2008 course on proving, had produced what we call the *formal-rhetorical* part of a proof, that is, the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld [6, p. 74]. We call the remaining part of a proof the *problem-centred* part. It is the part that *does* depend on genuine problem solving, intuition and a deeper understanding of the concepts involved [7].

After writing the formal-rhetorical part of the proof, Sofia needed an idea to continue with the problem-centred part of the proof. We inferred that Sofia was aware of needing an idea because she had had experience writing the formal-rhetorical parts of proofs (in particular, in our course) and would have understood that her proof was not complete. As she had done on several previous occasions, Sofia suggested an idea that seemed to us not to be rationally connected to the problem at hand. We have come to call such actions 'unreflective guesses', and more colloquially, 'grasping at straws'. The tutor then needed to take an action in response, as even doing nothing would likely have been interpreted by Sofia as an action or as having meaning. But to act usefully the tutor needed to understand why Sofia made her peculiar suggestion. We believe her suggestion was triggered by some kind of affect, but a kind not among the three major aspects previously discussed. On viewing the video, we found no evidence of an emotional response – she did not in any way appear to be distraught or disturbed. Also, given our informal wide-ranging and substantial discussions in the course and tutoring sessions, we believe any underlying belief or attitude that Sofia had related to this or other similar incidents would have surfaced. But none did. We believe Sofia's suggestion was triggered by a feeling of confusion or by a feeling of not knowing what to do next. Additional examples of students' nonemotional affective responses are given in Section 6.

The above example of Sofia suggests that it would be useful to expand one's view of affect somewhat, and that particular aspects of affect may be linked to specific parts of the proving process. DeBellis and Goldin [2–4] have added a fourth main aspect to affect, namely, *values*. While this addition is a reasonable extension of affect, it also does not account for the kind of behaviour exhibited by Sofia. DeBellis and Goldin further point out that 'affect is not auxiliary to cognition; it is centrally intertwined with it' [5]. They see 'affect as a highly structured system that encodes information, interacting fundamentally – and reciprocally – with cognition' [3, p. 2–249]. We now add a fifth aspect to affect, namely, *feelings*.



### 2.2. *Feelings*

DeBellis and Goldin, as well as Ortony *et al*. [8], see feelings as having an appraisal value that can be either positive or negative. In addition, Clore has considered 'feelings-as-information'. While DeBellis and Goldin [4] refer to 'emotional feelings,' Damasio [9] clearly distinguishes between emotions and feelings, with the former being public and the latter being private (p. 27). 'Emotions play out in the theatre of the body. Feelings play out in the theatre of the mind' (p. 28). Most of the emotions that Damasio considers, such as joy and sorrow, as well as some less intense emotions, are complex collections of chemical and neural responses to a stimulus that may produce bodily changes, such as changes in one's heart rate, temperature and so forth (p. 53).

Because this distinction between feelings and emotions may be somewhat counterintuitive, we paraphrase one of Damasio's salient examples. As doctors were placing tiny electrodes in the mesencephalon of the brain stem of a 65-year old woman suffering from Parkinson's disease, the patient abruptly stopped her ongoing conversation, began to look sad (evidence of emotion) and a *few seconds later* suddenly began to cry. She then said she had no energy to go on living (evidence of a feeling). The doctors quickly removed the offending electrode and the sobbing stopped as abruptly as it had begun, and the sadness vanished from the woman's face.

> The sequence of events in this patient reveals that the emotion sadness came first. The feeling of sadness followed ... Once the stimulation ceased these manifestations waned and then vanished. The emotion disappeared and so did the feeling. ... The importance of this rare neurological incident is apparent. ... As thoughts normally causative of emotions appear in the mind, they cause emotions, which give rise to feelings, which conjure up other thoughts that ... amplify the emotional state. More emotion gives rise to more feeling, and the cycle continues until distraction or reason put an end to it. ... By the time all these sets of phenomena are in full swing ... it is difficult to tell by

introspection which came first. This woman's case helps us see through the conflation [between emotions and feelings]. [9, pp. 67–70]

Thus, emotions and feelings are distinct, with the former being physical and the latter being mental. Each can sometimes elicit the other, and a particular feeling may, or may not, become lastingly associated with a particular emotion.

In discussing feelings, Clore [10] further distinguishes emotional feelings from nonemotional cognitive feelings, such as a feeling of knowing. For example, one might experience a feeling of knowing that one has seen a theorem useful in a current problem, but not be able to bring it to mind at the moment. Such feelings of knowing can guide cognitive actions because they can influence whether one continues a search or aborts it [10, p. 151].

Some nonemotional cognitive feelings, different from a feeling of knowing, are a feeling of familiarity and a feeling of rightness. Mangan [11] distinguishes the two. Of the former, he says that the 'intensity with which we feel familiarity indicates how often a content now in consciousness has been encountered before', and this feeling is different from a feeling of rightness [11, section 1, paragraph 3]. It is rightness, not familiarity that is 'the feeling-of-knowing in implicit cognition' [11, abstract]. Rightness is 'the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge' [11, section 1, paragraph 11].

Some feelings are not only nonemotional, but they also have no sensory component. They are non-sensory experiences – they are, for example, not red or hot. Such feelings do not have a verbal component, but are sometimes described in words. 'The feeling of familiarity is not a color, not an aroma, not a taste, not a sound. It is possible for the feeling of familiarity to merge with, or be absent from, virtually any sensory content found on any sensory dimension' [11, section 1, paragraph 7]. From the point of view of problem solving or proving, an important non-sensory experience is a feeling of rightness [11,12].

In regard to a feeling of rightness, Mangan [11, section 6, paragraph 7] says 'people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction. To explain this capacity, people talk about 'gut feelings', 'just knowing', hunches, intuitions'.

In this article, we will focus on feelings that are often not intense, such as feelings of knowing, of caution, of familiarity, of confusion, of not knowing what to do next, of rightness/appropriateness, of rightness/direction or of rightness/summation. Such feelings are non-sensory experiences that, at any given moment, can pervade one's whole conscious field. However, they may be rather 'vague' and not easily noticed or focused upon, but can influence one's actions [11, section 1, paragraph 4], including actions that are part of the proving process.

Feelings of rightness can give direction, be summative, or suggest appropriateness. Of a feeling of rightness/direction, Mangan says, 'In trying to solve, say, a demanding math problem, [a feeling of] rightness/wrongness gives us a sense of more or less promising directions long before we have the actual solution in hand' [11, section 6, paragraph 3].

A feeling of rightness/summation can integrate and evaluate 'large sets of information necessary for the problem-solving [or theorem proving] processes' [9, p. 177]. For example, often at the end of reading or writing a proof, short-term memory is inadequate to hold sufficient detailed information to allow a conscious rational judgement of whether the proof is correct (i.e. one cannot, all at once, bring to mind a proof, in the way that one can bring to mind the picture of, say, a house). However, something must cause an individual to decide his or her own, or someone else's, proof is correct. We see a feeling of rightness/summation as playing a major role in such decisions. Furthermore, we conjecture that an individual's feelings of rightness/summation about the correctness of proofs can become more accurate through multiple experiences, and that such learning is often tacit.

### 2.3. *Our perspective on affect*

We see affect as having five main aspects: beliefs, values, attitudes, feelings and emotions. Because a person has, or experiences, these aspects of affect, we regard them as part of what one might call the passive mind. In contrast, much of cognition, and in particular the proving process, is part of what one might call the active mind, referring to mental acts, for example, producing inner speech or vision, or bringing something to mind. Beliefs, values and attitudes, like knowledge, are lasting in varying degrees, and can be 'activated', or brought to mind, but they are not themselves continually experienced, that is, they are not usually part of one's consciousness. In contrast, feelings and emotions are experiences, that is, they are part of one's consciousness. We follow Damasio [9] in taking emotions to be physically embodied and feelings to be mental states. Within feelings, some are associated with the senses, for example, a feeling of pain, and some are non-sensory, for example, the feeling that a certain argument is correct. Also, some feelings are associated with emotions, for example, a feeling of joy or sorrow; and some are not, for example, a feeling of familiarity.

Finally, although all of affect can provide information for cognition, certain non-sensory feelings are cognitive in that they are about, or relate to, parts of the cognitive process [13]. For example, the feeling in constructing a proof that one is 'on the right track' is a cognitive feeling. In this article, we focus on nonemotional cognitive feelings and how they might interact with the proving process.



## 3. The difficulty of obtaining information about feelings

Although a person's feelings are not directly observable, they are conscious and potentially reportable. However, not everything experienced is likely to be reported, for example, in 'think aloud' episodes. This is because consciousness has different aspects [11,12], some of which are not easily focused upon in order to formulate a report.

In discussing consciousness, we are referring to phenomenological consciousness, that is, to the subjective experiences that everyone is aware of. For example, this includes experiences arising from the senses (vision, hearing, etc.), one's own speech and other physical actions, as well as the corresponding inner versions of these (inner vision, inner speech, etc.). In addition, consciousness includes more subtle experiences, often without a sensory component, such as the feeling that a proof is correct or that one is 'on the right track'.

At any given moment, a person's conscious field can accommodate a number of experiences but this capacity is limited, and in any one sensory modality, such as hearing, are very limited. Perhaps partly because of this limited capacity, some sensory experiences are focused upon and are of high resolution, while many other sensory experiences are more peripheral, vague and of lower resolution. If new sensory experiences are focused upon, then the earlier focal sensory experiences become more peripheral and vague. In contrast, non-sensory (or 'fringe') experiences can pervade the entire conscious field. Although they can be focused upon, they are often vague and of low resolution [11,12].

In think-aloud problem solving, the reporting process has a more limited capacity than consciousness, and there is no time for reflection, so reports are likely to be restricted mainly to what is being focused upon – probably the experiences of the problem solver's own ongoing actions in the problem-solving process. Except in rare cases, say, of very intense feelings, we expect feelings would only be mentioned in response to direct questions. Hurlburt, in studying consciousness, has developed a 'beeper' technique for obtaining 'snapshot' reports of the contents of a person's consciousness [14], but here again we suspect direct questions would be required to get detailed reports on feelings. Further, 'beeping' a student in the middle of proving a theorem would probably cause the student to lose his or her train of thought. Thus, one often finds oneself in the position of having to infer feelings based upon behaviour. While mental states, or structures, that are not observable should be used cautiously, there is precedent for their having explanatory power, for example, the mental models of Johnson-Laird [15].



## 4. Consciousness, the proving process and behavioural schemas

The psychological process of constructing a proof of a theorem is much more complex than the resulting proof, and no doubt occurs partly outside of consciousness. Below we suggest a theoretical perspective for a considerable part of the proving process.

We see (much of the conscious part of) reasoning in general, and the proving process in particular, as a sequence of mental and physical actions, such as writing or thinking a line in a proof, drawing or visualizing a diagram, reflecting on the results of earlier actions or trying to remember an example. As a person gains experience, much of proof construction appears to be separable into sequences of small parts, consisting of recognizing a situation and taking a mental or physical action. Actions which once may have required a conscious warrant can become automatically linked to triggering situations. From a third-person, or outside, perspective these regularly linked <situation, action> pairs might be regarded as small 'habits of mind' [16]. On the other hand, taking a first-person, inside, or psychological perspective, they are persistent mental structures that we have called *behavioural schemas* [17]. They can be acquired or learned through repeated similar proof constructing experiences, and their automated enactment can reduce the burden on a prover's working memory. Some schemas are beneficial and contribute to constructing correct proofs – they might be regarded as abilities.

### 4.1. *An illustration of a behavioural schema*

Mary, an advanced graduate student in mathematics described to us a <situation, action> pair in proving the theorem that *a compact subset $A$ of $R^n$ is bounded*. Mary and two fellow graduate students assumed $A$ to be unbounded and were able to construct an open cover of $A$ that had no finite subcover. They immediately observed, without further reflection, that this contradicted the compactness of $A$ and that this proved the theorem. Mary, who had never studied formal logic, reported to us that, upon finding the cover had no finite subcover, she immediately knew the theorem had been proved. She did not reflect on the logical structure of what had transpired in an effort to explicitly warrant that the proof was complete. The other two students also did not appear to require reflection or an explicit warrant.

We infer that each student had recognized the situation as similar to one that he or she had experienced many times previously involving a hypothesis, a conclusion assumed false and a resulting contradiction. The mental action was simply deciding the theorem had been proved. The link between the situation and the mental action appeared to be automatic and not to require reflection or a warrant. We see this as due to the students' extensive proof constructing experiences. However, many less experienced students require considerable reflection and often wonder needlessly what should be contradicted. We see such <situation, action> pairs as common and as playing a significant role in the proving process.

### 4.2. *The genesis and enactment of behavioural schemas*

We view behavioural schemas as belonging to a person's knowledge base. The action produced by the enactment of a behavioural schema might be *simple*, as in the above example. It might also be *compound*, such as a procedure consisting of several smaller actions, each produced by the enactment of its own behavioural schema that was 'triggered' by the action of an earlier schema in the procedure. Multi-digit addition of natural numbers is an example of a compound behavioural schema. When viewed in a large grain size, behavioural schemas might also be regarded as habits of mind [16]. Habits of mind are similar to physical habits, and people are similarly often unaware of, or do not remember, them (as habits).

It appears that consciousness plays an essential role in understanding the enactment of behavioural schemas. This is reminiscent of the role consciousness plays in reflection. It is hard to see how reflection, treated as selectively re-presenting past experiences, could be possible without first having had the experiences, that is, without first being conscious of them. We offer the following six-point theoretical sketch of the genesis and enactment of behavioural schemas.

(1) Within very broad contextual considerations, behavioural schemas are immediately available. They do not normally have to be remembered, that is, searched for and brought to mind. This distinguishes them from most conceptual knowledge and episodic and declarative memory, which generally *do* have to be recalled or brought to mind.
(2) Simple behavioural schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action – one just does it.[1] Thus, the enactment of a simple behavioural schema that leads to an error is not under conscious control, and we should not expect that merely understanding the origin of the error would prevent future reoccurrences. Compound behavioural schemas are also largely not under conscious control.
(3) Behavioural schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioural schema as it occurs or immediately after it occurs.

(4) A behavioural schema that would produce a particular action cannot pass that information, outside of consciousness, to be acted on by another behavioural schema. The first action must actually take place and become conscious in order to become information acted on by the second behavioural schema. That is, one cannot 'chain together' behavioural schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action. For example, if the solution to a linear equation would normally require several steps, one cannot give the final answer without being conscious of some of the intermediate steps.
(5) An action due to a behavioural schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person's actions, but such influence is normally not precise enough for doing mathematics.
(6) Behavioural schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing a detrimental behavioural schema requires similar, perhaps longer, practice.



Behavioural schemas, in this view, are instances of procedural knowledge, that is, knowing how to act, as opposed to knowing *that* or *why* something is true. In addition, they have a characteristic that Mason and Spence [19] have called 'knowing-to-act in the moment'. However, we do not mean to suggest here that proof construction, which can require genuine problem solving, intuition, analysis and deep understanding of concepts, is only, or even mainly, procedural in nature. Conceptual knowledge plays a major role in at least two ways. First, each behavioural schema is called into action by a situation that typically has to be interpreted using conceptual understanding. Indeed, over time some kinds of often encountered situations may come themselves to be tacitly treated as concepts, although often without names. For example, theorems requiring the proof of a concluding 'or' statement might tacitly be treated as a concept, that is, as a special category of theorems.[2] Second, the problem-centred part of a proof is likely to call on concepts and relationships between them. These may be brought to mind and combined using behavioural schemas. Thus we suggest behavioural schemas form a kind of network, or matrix, that surrounds and guides the use of conceptual knowledge in proof construction. Elaborations of, and justifications for, this theoretical sketch can be found in [17].

We suggest that many behavioural schemas are learned tacitly, or implicitly [21]. In Section 4.1, we reported Mary's description of a schema in which the situation consisted of an argument that included a hypothesis, a conclusion assumed false and a resulting contradiction, and the action was immediately deciding the argument was a proof. This schema could have been acquired by constructing several similar arguments and providing explicit logical justifications for their correctness. After some time, a student might just start omitting the justification, and might even not be explicitly aware that new knowledge (the schema) had been constructed. This new knowledge (the schema) might also be tacit, or implicit, in the sense that the student might not take explicit note of its application in subsequent arguments. However, if the argument's correctness were challenged, we would expect the student to be able to provide the logical justification.

We conjecture that some behavioural schemas are fully tacit. For example, when working on the problem-centred part of a complicated proof, a behavioural schema might bring to mind some useful insight, even though the prover had forgotten the origins (and existence) of the schema. For the prover, the information would appear to have come 'out of nowhere'.

## 5. The course, and the collection and analysis of data

The setting from which our data are taken is a design experiment [22], consisting of a Modified Moore Method course [23–25], whose sole purpose is to improve the proving skills of beginning graduate and advanced undergraduate mathematics students at an American PhD-granting university. The course is consistent with a constructivist point of view, in that we attempt to help students reflect on, and learn from, their own proof writing experiences. It is also somewhat Vygotskian in that we represent to the students how the mathematics community writes proofs. That is, we see ourselves as instruments in the cultural mediation of community norms and practices. However, we have not used other Vygotskian concepts, such as the zone of proximal development except in individual tutoring sessions.

The students are given self-contained notes consisting of statements of theorems, definitions and requests for examples, but no proofs. The students present their proofs in class, and the proofs are critiqued. Suggestions for improvements in their notation and style of writing are also given. There are no formal lectures, and all comments and conversations are based on students' work. The course carries three credits and lasts one semester. It meets for 1 h and 15 min twice a week, making 30 class meetings per semester. We have now taught four, of a projected eight, iterations of the course. There are two versions of the course, and either or both can be taken for credit. One version covers some basic ideas about sets, functions, real analysis and semigroups. The other version covers sets, functions, some real analysis and topology, but no individual theorems are common to both versions. The specific topics covered are of less importance than giving students opportunities to experience as many different types of proofs as possible.

Data were collected from four iterations of the course beginning Fall 2007. Each semester between four and six students were enrolled in the course. All class meetings were videotaped and field notes were taken by the second author. These videos were viewed and analysed by all three authors the following day in order to determine a hypothetical learning trajectory [26, p. 133] for the next class meeting, and interesting observations were noted so that similar instances could be watched for in future. These planning sessions were also videotaped and notes were taken by the first author. Individual tutoring sessions conducted by the third author were also videotaped. Take-home pretest, take-home and in-class final examinations and student 'scratch work' were collected. All this was done in the spirit of naturalistic inquiry [27] and to inform the future design of the course. Furthermore, retrospective interviews were conducted with Mary and Dr K (Section 6.2.2), after Mary had mentioned to us her experiences with quantification in an earlier real analysis course. All three authors reanalysed the data to confirm, elaborate or adjust our earlier observations regarding nonemotional cognitive feelings and students' actions. We discussed these until we agreed on the interpretations of the illustrative examples provided below (Section 6).

## 6. Feelings and the proving process

### 6.1. *Nonemotional cognitive feelings and the enactment of behavioural schemas*

In constructing a proof, when a person recognizes a situation and enacts a behavioural schema that yields a mental action, a feeling of rightness or appropriateness may soon be experienced. For example, such a feeling can be generated internally by finding a warrant for the action while reflecting on the proof in an effort to establish its correctness. Such a feeling can also be generated by an external authority, such as a teacher certifying that the proof is correct. If the feeling occurs several times when the same behavioural schema is enacted, then the appropriateness of enacting the behavioural schema may be enduringly associated with the situation. We suggest that such an association of a positive feeling with a situation will increase the probability of enacting the schema in future encounters with the situation.



### 6.2. *Edward's and Mary's reactions to considering fixed, but arbitrary elements*

#### 6.2.1. *Edward*

Students are often reluctant to consider an arbitrary, but fixed element in their proofs. We suggest this is because they do not feel it right or appropriate to do so. For example, on the 16th day of the course, just over halfway through the Fall 2007 semester, Edward was proving that the identity function, $f$, on the set of real numbers is continuous. We were using the following definition: A function $f$ is continuous at $a$ means for all $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

In order to write a proof, Edward needed to link 'for all $\varepsilon > 0$' in the definition of continuity (the situation) to writing in the proof something like 'Let $\varepsilon$ be a number greater than 0', meaning $\varepsilon$ is arbitrary, but fixed (the action). Edward was aware of the warrant for this action, namely, that because $\varepsilon$ was arbitrary, anything proved about it must be true for every $\varepsilon$. At least 11 times previously in class, Edward had seen this kind of <situation, action> pair, including its warrant. For example, on the seventh day of the course, Edward had proved $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$ essentially correctly, by choosing an element $x$ in one side and proving $x$ is an element in the other side, and vice versa.

Here is Edward's proof that the identity function, $f$, is continuous on the set of real numbers.

**Proof:** Let $a \in R$ and $\varepsilon > 0$ and $\delta > 0$ where $\delta = \varepsilon/2$. So $\delta < \varepsilon$. For any $\varepsilon > 0$ and any $x \in R$ such that $|x - a| < \delta$. We have that $|f(x) - f(a)| < \delta < \varepsilon$ therefore $f$ is continuous.

In the critique that followed, the teacher said it was inappropriate to write 'For any $\varepsilon > 0$' after writing 'Let $a \in R$ and $\varepsilon > 0$...' because the earlier statement meant $\varepsilon$ represents a particular number. Edward responded, 'It doesn't really matter [meaning the 'For any'] because I had $\varepsilon > 0$ there [meaning at the beginning of his proof].' Our interpretation is that Edward did not really have a feeling of rightness/appropriateness for this <situation, action> pair, even after considerable experience with it. Furthermore, Edward made similar comments several times thereafter.

#### 6.2.2. *Mary*

In a similar situation, Mary, the above mentioned advanced mathematics graduate student (Section 4.1) only slowly developed a feeling of rightness/appropriateness regarding fixed, but arbitrary elements. We conducted interviews with Mary, and her real analysis teacher, Dr K, about events that had taken place two years earlier, when Mary was taking both a pilot version of our proofs course (Section 5) and Dr K's real analysis course. In the homework for Dr K's course, Mary needed to prove many statements that included phrases like 'For all real numbers $x$', where $x$ represented a variable (the situation). In her proofs, Mary needed to write something like 'Let $x$ be a number', where $x$ represented an arbitrary, but fixed number (the action). Dr K often discussed Mary's proofs with her, and in particular, thought she carried out this action based on his authority.

When Mary was interviewed about this <situation, action> pair, she said the following:

Mary: At that point [early in Dr K's real analysis course] my biggest idea was, well he said to 'do it', so I'm going to do it because I want to get full credit. And so I didn't have a real sense of why it worked.
Int: Did you have any feeling ... if it was positive or negative, or extra ...
Mary: Well, I guess I had a feeling of discomfort ...
Int: Did this particular feature [having to fix $x$] keep coming up in proofs?
Mary: ... it comes up a lot and what happened, and I don't remember [exactly] when, is that instead of being rote and kind of uncomfortable, it started to just make sense ... By the end of the semester this was very comfortable for me.

We infer Mary developed both the behavioural schema and the associated feeling of appropriateness only after executing the <situation, action> pair numerous times. In early executions of this <situation, action> pair, Mary carried it out mainly based on Dr K's authority. In addition, after completing each such proof, Mary reported to us that she had attempted to convince herself that considering a fixed, but arbitrary element resulted in a correct proof. Only after repeatedly executing this <situation, action> pair, and convincing herself that the individual proofs were correct, did she develop a feeling of appropriateness.

### 6.3. *Students who focus too soon on the hypothesis*

In a paper reporting on a mid-level undergraduate transition-to-proof course, Moore [28] described students' attempts to prove: *Let $f$ and $g$ be functions on $A$. If $f \circ g$ is one-to-one, then $g$ is one-to-one*. This was a final examination question and 'all but one student [out of 16] incorrectly attempted to begin [the problem-centred part of] the proof with the hypothesis – $f \circ g$ is one-to-one – rather than [starting to prove the conclusion] by assuming $g(x) = g(y)$ for a fixed but arbitrary $x$ and $y$ in $A$.' These students apparently had focused too soon on the hypothesis before examining and unpacking the conclusion.

#### 6.3.1. *Willy*

We have noted a similar, persistent difficulty when students who are unable to construct a complete proof are asked to provide as much of one as they can.

After writing little more than the hypotheses, some students turned immediately to focusing on those hypotheses, after which they could not complete the proof.

For example, on the 26th day of the Spring 2008 course, Willy was asked to prove Theorem 29: *Let X and Y be topological spaces and* $f : X \to Y$ *be a homeomorphism of X onto Y. If X is a Hausdorff space, then so is Y.* Because only 10 min of class time remained, we asked Willy to 'do the set-up', that is, present the formal-rhetorical part of the proof. Willy had indicated that he had not yet proved the theorem. However, early on in the course, he had developed some ability to write the formal-rhetorical parts of simple proofs.

On the left side of the board, Willy wrote:

**Proof:** Let X and $Y$ be topological spaces.
Let $f : X \to Y$ be a homeomorphism of $X$ onto $Y$.
Suppose $X$ is a Hausdorff space.
...
Then $Y$ is a Hausdorff space.

Then, on the right side of the board, he listed:

homeomorphism
one-to-one
onto
continuous ($f$ is open mapping)

and then looked perplexedly back at the left side of the board. Even after two hints to look at the final line of his proof, Willy said, 'And, I was just trying to just think, homeomorphism means one-to-one, onto, ...' After some discussion about the meaning of homeomorphism, the first author said, 'There is no harm in analysing what stuff you might want to use, but there is more to do before you can use any of that stuff', meaning that the conclusion should be examined and unpacked first. Willy did not make further progress that day.

We inferred that Willy was enacting a behavioural schema in which the situation was having written little more than the hypotheses, and the action was focusing on the meaning and potential uses of those hypotheses before examining and unpacking the conclusion. This schema often leads to difficulties, as it did for Willy. We conjectured that Willy and other students, who were reluctant to look at and unpack the conclusion, felt uncomfortable about this, or perhaps felt it more appropriate to begin with the hypotheses and work forward. We wonder whether this might be an expectation built up from proving secondary school geometry theorems, where it often seems that one can prove theorems from the 'top down'.

We also conjectured that, had Willy not been distracted by focusing on the meaning of homeomorphism, he might have written more of the formal-rhetorical part of the proof. That is, he might have filled in the blank space of his proof with something like:

Let $y_1$ and $y_2$ be two elements of $Y$.
...
Thus there are disjoint open sets $U$ and $V$ so that $y_1 \in U$ and $y_2 \in V$.

Writing the formal-rhetorical part of a proof exposes the 'real problem' in this theorem, something Willy might have found tractable. Indeed, by the next class meeting, he had constructed a proof in the way we had expected. Of course, in addition to writing the formal-rhetorical part of the proof, Willy needed to invoke considerable conceptual knowledge about homeomorphisms and the Hausdorff property.

### 6.4. *Sofia's reaction to not having an idea*

In *S*ection 6.2.2, we reported an incident of access to a rare first-person account that illustrated Mary's joint development of a feeling of appropriateness and a behavioural schema. In contrast, in this section, we describe how we used our theoretical perspective about feelings and behavioural schemas to infer that Sophia's progress was blocked by multiple enactments of an 'unreflective guess' schema, and how our intervention appears to have weakened that schema and improved her prospects for later success in graduate school.

#### 6.4.1. *Sofia's 'unreflective guess' schema*

Sofia, the above mentioned first-year graduate student (Section 2.1), was taking the second iteration of our proofs course (Section 5). Our analysis is based on third-person information, that is, observations and video recordings of Sofia's class participation and of her proof constructions during seven individual tutoring sessions with the third author.

Sofia was a diligent student; however, as the course progressed, an unfortunate pattern in her proving attempts emerged. When she did not have an idea for how to proceed, she often produced what one might call an 'unreflective guess' only loosely related to the context at hand, after which she could not make further progress. We do not see the common practice of guessing a direction and then adjusting according to its fruitfulness as inappropriate. However, although we could sometimes speculate on the origins of Sofia's guesses, we could not see how they could reasonably have been helpful in making a proof, nor did she seem to reflect on, or evaluate, them herself.

We inferred that, in such unreflective guessing, Sofia was enacting a behavioural schema that depended on a feeling. She was recognizing a situation, that is, that she had written as much of a proof as she could, and had a feeling of not knowing what to do next. This situation was linked in an automated way to the action of just guessing any approach that usually was only loosely related to the problem at hand without much reflection on its likely usefulness. We also felt that the enactment of an 'unreflective guess' schema was very likely to increase Sofia's feeling of not knowing what to do next, and thus, lead to another unreflective guess, or to abandoning her proof attempt. We judged that without our addressing this unfortunate behavioural schema, Sofia would not make progress on the problem-centred parts of proofs.

We began to suspect Sofia might have a persistent difficulty when she volunteered to present the following 'proof' on the fifth day of class. Only her first and last lines could reasonably be part of a proof.

**Theorem 4:** *For any sets $A$, $B$ and $C$,* if *$A \subseteq B$ then $(A \cap C) \subseteq (B \cap C)$.*

**Proof:** Let $A$, $B$ and $C$ be sets
Suppose $x \notin A$, $x \in B$ and $x \in C$
Then $x \notin (A \cap C)$, but $x \in (B \cap C)$
Therefore $(A \cap C) \subseteq (B \cap C)$ (Sofia did not end her sentences with periods.)

This behaviour persisted for some time. Our intervention consisted of trying, during tutoring sessions, to prevent Sofia from enacting the 'unreflective guess' schema by suggesting substitute actions. These included: draw a figure, look for inferences from the hypotheses, reflect on everything done so far or do something else for a while

### 6.4.2. *Helping Sofia to replace her 'unreflective guess' schema*

Here is an example of how the third author led Sofia towards constructing a beneficial behavioural schema by suggesting substitute actions. This hour and a half tutoring session occurred in the middle of the Spring 2008 course, and was devoted to helping Sofia prove Theorem 20: *Let* $(X, \mathcal{U})$ *be a topological space and* $Y \subseteq X$. *Then* $(Y, \{U \cap Y | U \in \mathcal{U}\})$ *is a topological space* (*called the relative topology on* $Y$).

When Sofia indicated she did not know how to prove the theorem, the tutor, in an attempt to deflect implementation of her 'unreflective guess' schema, suggested that she write the first and last lines of a proof and draw a sketch. With guidance, she then unpacked what was to be proved into four parts (the four defining properties of a topology), and she proved the first one, that is, $Y \in \{U \cap Y | U \in \mathcal{U}\}$. The tutor then tried to encourage her reflection on a key point, saying, 'Now, if I didn't understand that, and I asked you why $Y$ is in there? The thing you said up above [$X \cap Y \in \{U \cap Y | U \in \mathcal{U}\}$] doesn't have any $Y$ in it. You would have to tell me that, well, $Y$ is equal to $X \cap Y$, OK no problem'. Sofia said, 'um hum'.

Then Sofia wrote, '(2) Since [the empty set] $\emptyset \in \mathcal{U}$' and after 30 s said, 'I'm stuck'. It became clear that she did not know, in general, how to show an object is in a set when the defining variable in the set is compound (e.g. $U \cap Y$). The tutor then said to forget the theorem for a bit and turned to the natural numbers. He wrote $\{2n | n \in N\}$ and asked 'Is 6 an element of that set?' Sofia answered yes, and the tutor asked why. Sofia and the tutor then agreed that the answer was yes because $6 = 2 \times 3$ and $3 \in N$. The tutor reiterated that 6 had to be represented in the form $2n$, where $n$ has the appropriate property, that is, $n \in N$. Using this as a model, Sofia was then able to show $\emptyset \in \{U \cap Y | U \in \mathcal{U}\}$. With guidance, but less guidance than before, she also completed the third and fourth parts of the proof.

Here, the tutor's guidance and suggestions not only helped Sofia prove the theorem, they also facilitated the construction of a beneficial behavioural schema in which the situation is needing to show an object is in a set (where the defining variable is compound), and the mental action is forming the intention to represent the object in the proper form and show the defining property of the set is satisfied. Sofia's previous work in the course had suggested she understood the language of sets and could draw appropriate Venn diagrams. However, for her that understanding was not equivalent to knowing how to use her knowledge in constructing a proof of Theorem 20.

As the course ended, our intervention of directing Sofia to do something else, whether it be draw a diagram or review her notes, was beginning to show promise. For example, on the in-class final examination Sofia proved that *if* $f$*,* $g$*, and* $h$ *are functions from a set to itself,* $f$ *is one-to-one, and* $f \circ g = f \circ h$*, then* $g = h$. Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-centred parts of at least a few proofs by the end of the course, and suggests her 'unreflective guess' behavioural schema was weakened.

## 7. Conclusion

In this article, we have described a design experiment to develop a course whose sole purpose is to improve advanced mathematics students' proving skills. The course appears to be useful because many advanced students have difficulty constructing proofs, and students' proofs are used as a major component in assessments of their understanding of content courses, such as abstract algebra or real analysis.

We have discussed the nature of feelings, especially nonemotional cognitive feelings, and treated them as a part of affect. Also, a theoretical framework for analysing student progress with proofs is beginning to emerge. For example, we have suggested it is useful to distinguish between the formal-rhetorical parts of proofs and the problem-centred parts.

We further introduced a theoretical perspective suggesting that much of the proving process depends on procedural knowledge in the form of small habits of mind or behavioural schemas, some of which are beneficial, while others tend to produce difficulties. We pointed out that the way feelings can both arise from, and contribute to, the enactment of behavioural schemas. The example of Mary and Dr K was then provided to show how a feeling of appropriateness can be associated with, and strengthen, a behavioural schema. We also pointed out that Mary's behavioural schema arose from enacting a <situation, action> pair multiple times, based on Dr K's authority.

Our observations of Willy's, and other students', reluctance to examine the conclusion, and their preference for immediately examining the hypotheses, led us to infer that they had a feeling of discomfort or inappropriateness regarding examining the conclusion. That feeling then encouraged the maintenance of a detrimental schema and discouraged the development of a more appropriate one.

Also, our interpretation of Sofia's persistent proving difficulty, as that of an 'unreflective guess' schema, was based on our conjecture that she had a feeling of not knowing what to do next, so she did things or offered suggestions only vaguely related to the theorem at hand. This interpretation led to a beneficial intervention.

We suspect there is much more to be investigated about the way nonemotional cognitive feelings and behavioural schemas relate to each other and to the proving process, and how proving is learned and can be better taught. In addition, the extent of advanced students' proving difficulties, together with the extent of the use of student proofs for assessing their understanding, could be further investigated.



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## Notes

1. This is consistent with Libet and his collaborators' psychological/neurological work suggesting that the immediate cause of (simple physical) actions, such as raising a finger, is an unconscious mental state – rather than the conscious intention to act, as one might expect [18, pp. 329–333].
2. This is similar to Selden *et al.*'s analysis of students' problem-solving processes. In that analysis, 'problem situations' were treated as similar to concepts and were seen as

sometimes linked to 'tentative solution starts' in one's knowledge base [20, p. 146]. Bringing a tentative solution start to mind is a mental action, and consistently pairing a problem situation with the action of bringing to mind a corresponding tentative solution start, can be viewed as a behavioural schema.

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