CHAPTER 20: Teaching Proving by Coordinating Aspects of Proofs with Students' Abilities

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In this chapter we introduce concepts for analyzing proofs, and for analyzing undergraduate and beginning graduate mathematics students' proving abilities. We discuss how coordination of these two analyses can be used to improve students' ability to construct proofs.

For this purpose, we need a richer framework for keeping track of students' progress than the everyday one used by mathematicians. We need to know more than that a particular student can, or cannot, prove theorems by induction or contradiction or can, or cannot, prove certain theorems in beginning set theory or analysis. It is more useful to describe a student's work in terms of a finer-grained framework that includes various smaller abilities that contribute to proving and that can be learned in differing ways and at differing periods of a student's development.

Developing a fine-grained framework for analyzing students' abilities is not an especially novel idea. In working with higher primary and secondary students, Gutiérrez and Jaime (1998) developed a fine-grained framework of reasoning processes in order to more accurately and easily assess student van Hiele levels.

For proof construction, there are already a number of abilities suitable for keeping track of students' progress. For example, in comparing undergraduates who had completed a course in abstract algebra with doctoral students in abstract algebra, Weber (2001) found doctoral students more able to use strategic knowledge. For example, when asked whether two specific groups, such as Θ and Z, are isomorphic, undergraduates first looked to see whether the groups had the same cardinality, after which they attempted unsuccessfully to construct an isomorphism between them. The doctoral students considered properties preserved by isomorphism, strategically a better starting point (Weber & Alcock, 2004).

The ability to use strategic information is about the proving processes of individual students. But it is also useful to consider the product of such processes—the proofs themselves—because differing kinds of, and aspects of, proofs can correspond to various abilities needed to produce them. Thus, an analysis of kinds and aspects of proofs should facilitate teaching by coordinating assigned theorems with student abilities. It should also facilitate the assessment of student abilities.

In the first section, we introduce three different global aspects of proofs that we call *structures*, and illustrate them using a real analysis proof that advanced undergraduate or beginning graduate students might construct. In the next section we discuss the coordination of aspects of proofs with student abilities. After that, we make three informal observations. The final section is a conclusion summarizing our main ideas.

Three Structures of Proofs

Mathematics education has often emphasized student learning and activity, rather than the properties of the associated objects, such as the global aspects of proofs that we are calling structures. There are two previous discussions of similar structures. Leron

(1983) suggested presenting proofs (e.g., in a lecture) in a “top down” way, first describing the main ideas before filling in the details. The resulting structure is reminiscent of what we call the hierarchical structure of a proof, except that Leron was not concerned with characterizing the proof itself, but rather with providing a kind of advance organizer for its presentation.

Konior (1993) was also concerned with a hierarchical structure of proofs, but he focused on segmentation, that is, “separation of the whole [mathematical] text into a few parts ... in order to reflect the logical structure of the proof and facilitate the reconstruction of the whole proof.” He discussed delimiters, such as words and paragraph breaks that indicate beginnings and endings of proof segments (e.g., subproofs). While we see this as an interesting approach, an investigation of delimiters is beyond the scope of this chapter.

We introduce three proof structures, that is, aspects of a proof that refer to it as a whole, how it is organized, and how it could have been written. These emerged from our attempts to understand and characterize our students’ proof writing difficulties. For example, such difficulties might arise from a tendency not to focus on producing a subproof, supposing that a proof should be written linearly from the top down as if it were being read, or concentrating on the big picture to the exclusion of the “nuts and bolts” of proving. By attending to these structures a teacher might be able to arrange for the associated student difficulties to arise in a way that allowed them to be overcome.

We discuss (1) a *hierarchical* structure in which we attend to subproofs and subconstructions (such as finding the δ in an $\varepsilon-\delta$ real analysis proof); (2) a *construction path*, that is, a linear path describing one ordering of the steps[1] through which a proof could have been constructed by an idealized prover, one who never erred or followed false leads; and (3) a division of proofs into what we call the *formal-rhetorical* part and the *problem-centered* part. For any given proof these three structures, considered together, can in a qualitative way suggest the proof's complexity. That complexity, in turn, can provide one indication of the proof's difficulty, and knowledge of a proof's difficulty can be an important tool in teaching.

Of course, a particular theorem can often be proved in several ways. However, there are not likely to be very many such ways (except for relatively minor details) for the theorems most advanced undergraduate and beginning graduate students are asked to prove.

The Hierarchy of a Proof and a Possible Construction Path

We analyze a proof of the theorem: $f+g$ *is continuous at a point provided* f *and* g *are*. We build the proof up hierarchically, attending to subproofs and subconstructions. Simultaneously, we build a construction path giving an order in which its sentences, or subsentences, could have been written by an idealized prover. For reference, and to facilitate the subsequent analysis, individual sentences have been numbered in bold brackets (e.g., **[1]**, **[2]**) in their order of appearance in the proof. For example, the sentence labeled **[12, 13, 14, 15]** is divided into four smaller units, one for each statement of equality or inequality.

Proof. **[1]** Let a be a number and f and g be functions continuous at a. **[2]** Let

[1]We speak of steps, or sentences, rather than statements, because proofs can contain sentences that are neither true nor false, but instead give instructions to the reader, such as "Let x be a number."

ε be a number > 0. **[3]** Note that $\varepsilon/2$ > 0. **[4]** Now because f is continuous at a, there is a $\delta_1 > 0$ such that for any x_1, if $|x_1 - a| < \delta_1$, then $|f(x_1) - f(a)| < \varepsilon/2$. **[5]** Also because g is continuous at a, there is a δ_2 > 0 such that for any x_2, if $|x_2 - a| < \delta_2$, then $|g(x_2) - g(a)| < \varepsilon/2$. **[6]** Let $\delta = \min(\delta_1, \delta_2)$. **[7]** Note that $\delta > 0$.

[8] Let x be a number. **[9]** Suppose that $|x - a| < \delta$. **[10]** Then $|x - a| < \delta_1$, so $|f(x) - f(a)| < \varepsilon/2$. **[11]** Also $|x - a| < \delta_2$, so $|g(x) - g(a)| < \varepsilon/2$. **[12, 13, 14, 15]** Now

$$|f(x) + g(x) - (f(a) + g(a))| = |(f(x) - f(a)) + (g(x) - g(a))| \leq$$

$$|(f(x) - f(a))| + |(g(x) - g(a))| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ **[16]** Thus

$|f(x) + g(x) - (f(a) + g(a))| < \varepsilon$. **[17]** Therefore $f + g$ is continuous at a. QED.

The first step of the construction path, appears as **[H1]**, for "hidden." It expresses information needed to construct the proof, but that does not appear in the proof. It is done by rewriting the statement of the theorem in a more formal way, explicitly mentioning variables and quantifiers and using standard logical connectives. This "clarification" yields the statement, **[H1]**: *For all real-valued functions f, all real-valued functions g, and all real numbers a, if f is continuous at a and g is continuous at a, then $f + g$ is continuous at a.*

This version of the statement of the theorem exposes its logical structure. It is essential to understand the logical structure in order to be certain that one's proof proves this theorem, as opposed to some other theorem. The logical structure, independent of the meaning of "function," "+," and "continuous," yields the first and last sentences of the proof. Together they form what we have called a *proof framework*[2] (Selden & Selden, 1995). The resulting partial construction path of our idealized prover is now **[H1]**, **[1]**, **[17]**, **…** and the resulting partial hierarchical structure is shown in Figure 20.1.

<<INSERT FIGURE 20.1>>

Next we clarify line **[17]** by applying a definition of continuous to $f+g$ at the number a. This clarification yields a new statement, **[H2]**, that is needed to construct the proof, but does not appear in it: For every number $\varepsilon > 0$, there is a $\delta > 0$, so that for every number x, if $|x-a| < \delta$ then $|f(x)+g(x)-(f(a)+g(a))| < \varepsilon$. This statement requires a subproof and writing a proof framework for it yields the partial construction path **[H1]**, **[1]**, **[17]**, **[H2]**, **[2]**, **[7]**, **[8]**, **[9]**, **[16]**, **…** and the resulting partial hierarchical structure is shown in Figure 20.2.

<<INSERT FIGURE 20.2>>

Next we add in the construction of δ and finish the proof, yielding the complete hierarchical structure in Figure 20.3.

<<INSERT FIGURE 20.3>>

In Figure 20.3, we first added lines **[3]**–**[6]** of the construction path and then added lines **[10]**–**[15]** of the construction path. The first addition was treated as a new level in the hierarchical structure, but the second was not. This is because lines **[3]**–**[6]** can stand-alone as the construction of δ, but lines **[10]**–**[15]** cannot stand alone as a subproof. Rather, they form a part of **[2]**–**[16]**, the proof of **[H2]**.

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[2] Proof frameworks can be independent of the meanings of certain content words. Selden and Selden (1995, p. 130) illustrated this with a proof framework for a theorem about semigroups that can also serve as a proof framework for a theorem about real intervals. This was accomplished by replacing the words “semigroup” with “set of numbers,” “group” with “interval,” and “a subgroup of” with “a subinterval of” and retaining the usual meaning of the other words.

The final construction path of our idealized prover is now: **[H1]**, **[1]**, **[17]**, **[H2]**, **[2]**, **[7]**, **[8]**, **[9]**, **[16]**, **[3]**, **[4]**, **[5]**, **[6]**, **[10]**, **[11]**, **[12]**, **[13]**, **[14]**, **[15]**.

Although the construction path and the hierarchical structure illustrated above can contribute to observing student abilities, one might also ask: Do undergraduate or graduate students just beginning to construct proofs even tacitly understand such structures? Does this matter? Perhaps a hint at an answer can be obtained by considering the following incident.

During a chance discussion of the above theorem with a mathematics education graduate student, she expressed surprise that constructing proofs might involve the kind of structures illustrated above, rather than progressing linearly from top down. It turned out that she had studied only a little advanced mathematics and much or all of that had been presented "top down" in the traditional definition-theorem-proof style.

Our experience suggests—as this incident does—that at least a tacit understanding of the above proof structures matters and that, early in their exposure to advanced mathematics, some students do not grasp much about them. Such students are likely to have a view of proof construction that we believe will prevent them from succeeding.

The Formal-Rhetorical and Problem-Centered Parts of a Proof

Notice that writing steps **[H1]**, **[1]**, **[17]**, **[H2]**, **[2]**, **[7]**, **[8]**, **[9]**, **[16]** in the construction path of our idealized prover (Figure 20.2) calls on an understanding of the logical structures of the statement of the theorem and of the definition of continuous. This is made clear in the statements **[H1]** and **[H2]**. In addition, one needs to know, and to act on, how parts of the statement of a theorem relate to parts of its proof. We call knowledge of this kind *behavioral knowledge*[3] because a tendency to behave or act is a part of it. It is closely related to "knowing-to-act in the moment" (Mason & Spence, 1999), as well as to the ideas of "concepts-in-action" and "theorems-in-action" (Vergnaud, 1982). For example, if a formal version of a theorem started, "For all real numbers x," then in a direct proof, one might start by "introducing" the variable x, with a statement like "Let x be a real number." For some authors of proofs, this statement might be left implicit instead of being made explicit, especially when x appears in the statement of the theorem. Such an x is usually said to be "arbitrary, but fixed" and does not "vary."

Quite a lot of such behavioral knowledge is required in constructing proofs. While it is not important that a student be able to articulate such behavioral knowledge, it is important that he/she act on it. Although behavioral knowledge neither implies, nor is implied by, the corresponding procedural (knowing how, without necessarily acting) or conceptual knowledge (knowing why), these can also be useful.

For a student with a reasonable repertoire of proof-related behavioral knowledge, writing the steps in Figure 20.2 can be quite straightforward. Such writing can be viewed as invoking a schema. It does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). We call this part of a proof the *formal-rhetorical* part and contrast it with the remainder of the construction path of our idealized prover, **[3]**, **[4]**, **[5]**, **[6]**, and **[10]**, **[11]**, **[12]**, **[13]**, **[14]**, **[15]** (Figure 20.3), that we call the *problem-centered* part. This part of a proof *does* involve problem solving. The steps in the problem-centered part may call

[3] Behavioral knowledge should not be confused with behaviorism, the idea that physically observable, and perhaps measurable, behavior should be the only basis for a scientific treatment of psychology. Taking a behaviorist view would prevent most of the study of the mind, and hence, much current research in mathematics education.

on conceptual knowledge, mathematical intuition, and the ability to bring to mind the "right" resources at the "right time."

Constructing the formal-rhetorical parts of proofs seems to call on a different kind of knowledge than the problem-centered parts do. Helping students acquire these different kinds of knowledge probably involves different kinds, or aspects, of teaching, even though these might be blended in a single course and taught by a single teacher. Such a blending of teaching seems especially appropriate for learning to construct proofs because the two parts of a proof interact. Indeed, there are theorems for which constructing the formal-rhetorical part of a proof can be very helpful in revealing the "real problem" to be solved. Our sample theorem is such a theorem. Since it is about continuous functions, one might expect some kind of visual-spatial intuition about functions might be useful for constructing a proof. However, the formal-rhetorical part (Figure 20.2) reveals that the problem is to find a δ that will yield the inequality $|f(x)+g(x)-(f(a)+g(a))| < \varepsilon$ in step **[16]**. The solution does not arise in any obvious way from intuition. Instead, it involves use of $\varepsilon/2$ in the definition of continuity for both f and g, a choice of δ as the smaller of the resulting two δ's, application of the triangle inequality, and some algebraic rewriting.

In the construction path of the above proof, as in many short proofs, the entire formal-rhetorical part of the proof comes first, followed by the problem-centered part. However, in longer proofs with several subproofs, each subproof may have its own formal-rhetorical part and subsequent problem centered part, and the order of the subproofs within a construction path may vary.

Coordinating Aspects of Proofs with Students' Abilities

Next we discuss some proving abilities that a student might have or that a proof might call for. It is the coordination of these two—what a student might be able to do, or not do, and what a proof might call for—that we suggest may facilitate teaching proving, mainly through students' construction of proofs, rather than in some more teacher-centered way such as lecturing.

Kinds of Proofs

Authors of transition-to-proof textbooks typically distinguish direct proofs, proofs by contradiction, proofs by mathematical induction, proofs by cases, and existence and uniqueness proofs. However, for guiding students' progress, such distinctions are not fine enough. For example, a particular student might be able to construct a number of direct proofs, but not the kind discussed in the first section. This might be due to its complex structure or the number of quantifiers involved. Knowing this would allow a teacher to guide a student's current work and to assign later theorems whose proofs are similarly complex.

A student might be able to construct several proofs by contradiction, but not a proof of: *There is at most one identity element in a semigroup*. The usual proof of this theorem calls for the ability to negate "at most one" as "there are (at least) two," an ability a student might not have.

Formal-Rhetorical Reasoning versus Problem-Centered Reasoning

In the first section, we distinguished between the formal-rhetorical and problem-centered parts of a proof and suggested that writing these two parts calls on different kinds of knowledge, and hence, on different kinds of teaching. We call the abilities to write these two parts, *formal-rhetorical reasoning* and *problem-centered reasoning,* and apply these ideas to the kinds of exploration preceding and surrounding proofs, as well as to writing proofs.

Here is an example of the kind of problem-centered reasoning that can precede a proof. Two students with fairly strong upper-division undergraduate mathematics backgrounds were jointly attempting to prove: *If the number of elements in a set is* n, *then the number of its subsets is* 2^n. They had been advised: *Don't forget to count the empty set,* $\varnothing$, *and the whole set as subsets. For example the subsets of* $\{a,b\}$ *are* $\varnothing$, $\{a\}$, $\{b\}$, *and* $\{a,b\}$. They first considered $\{a,b,c\}$ and wrote $\{c\}$, $\{a,c\}$, $\{b,c\}$, and $\{a,b,c\}$. When asked about this, they indicated that they had "added" c to each of the subsets of $\{a,b\}$ to generate the four additional subsets of $\{a,b,c\}$. Next they considered $\{a,b,c,d\}$ and then seemed puzzled as to how to continue. The teacher intervened, "You can get from 2 to 3, and you can get from 3 to 4. What does that make you think of?" They replied "induction," something they had not thought of. These students could benefit from an opportunity to think independently of using induction on the proof of a subsequent theorem.

Comparing the Difficulty of Proofs

For an individual, a major determiner of the difficulty of constructing proofs seems to be the nature of his/her knowledge base and habits of mind. These are discussed below in connection with problem-centered reasoning. However, in the context of a course in which students prove all, or most, of the theorems, one can often see that one proof is more difficult than another by observing that only a few students can construct it.

This suggests that there are characteristics intrinsic to proofs that make some more difficult than others, at least with respect to rough judgments of difficulty. Certainly, we have found a need to make such judgments especially in teaching Moore Method courses[4], because we prefer to ask a student to prove a theorem that will be challenging, but not so challenging that he/she fails to do so.

To illustrate how rough judgments of the relative difficulties of proofs can sometimes be made independently of specific individuals, we compare our sample proof (Figure 20.3) with the proof (Figure 20.4) of the following theorem: *If f and g are functions from A to A and* $f \circ g$ *is one-to-one, then g is one-to-one.*

<<INSERT FIGURE 20.4>>

The earlier proof is longer, and there are more quantifiers to contend with, but there are other less obvious reasons that the earlier proof might be more difficult. The hierarchical structure of the second proof has only two levels (Figure 20.4) while the earlier one has three levels (Figure 20.3). Also, while the problem-centered part of the second proof consists of just three consecutive steps **[5]**, **[6]**, **[7]**, the problem-centered part of the earlier proof consists of two separate sections, **[3]**, **[4]**, **[5]**, **[6]** and **[10]**, **[11]**, **[12]**, **[13]**, **[14]**, **[15]**, each of which depends on the other. Finally, one might think the earlier proof could be developed from some kind of visual-spatial intuition about continuous functions, but this is unlikely. Hence, the formal rhetorical part of the earlier proof plays a large role, and a kind of technical-algebraic intuition is called far.

[4]In such courses, students are typically given notes containing definitions and statements of theorems, or conjectures, and asked to prove them or to provide counterexamples. The teacher provides the structuring of the notes and critiques the students' efforts. For more information, see Jones (1977) or Mahvier (1999).

Sets and Functions

The language of sets and functions occurs widely in proofs, and undergraduate students are often introduced to it in an abstract way. More important than being able to recall abstract definitions, students need to use them in proofs, that is, be able to carry out appropriate actions effortlessly in order to leave maximum cognitive resources for the problem-centered parts of a proof. In doing this, students need what we are calling behavioral knowledge, and we suspect this is learned more from practice at constructing proofs than from reproducing abstract definitions.

For example, the definition of set equality is usually given as $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, and students are told that this means A and B have the same elements. But in constructing a proof that two sets are equal, it should come to mind easily that this involves showing that an arbitrary element of each set is an element of the other set. That is, normally two subproofs need to be constructed: One beginning *Suppose* $x \in A\ldots$ and ending$\ldots$*Then* $x \in B$; the other beginning *Suppose* $x \in B\ldots$ and ending $\ldots$ *Then* $x \in A$. Logically the definition involving two inclusions and the above "element-chasing" view of set equality are equivalent, but psychologically they appear to be very different. In our experience, naive students often attempt to deduce $A = B$ directly, without considering elements.

Students can have a reasonably good intuitive grasp of the meaning of one-to-one (1-1), but not know how to prove a function f is 1-1. They may even realize that the definition involves an implication, namely, that $f(a) = f(b)$ implies $a = b$, for all a and b in the domain. However, that does not mean they know where to begin a proof (Moore, 1994). It may not be clear to them that they should normally begin the proof by almost automatically writing, *Suppose* $f(a) = f(b)$ ... for arbitrary elements a and b in the domain , and only then attempt to use the hypotheses to arrive at $a = b$.

Logic

Logic does not occur within proofs as often as one might expect. Furthermore, the logic used in proofs is mainly propositional calculus, and there is a tendency, whenever possible, to avoid variables and quantifiers. That is, typically proofs do not contain arguments about all elements x, but instead are about an arbitrary, but fixed, element x. This cannot always be accomplished and our proof about the sum of continuous functions is an exception in that x_1 and x_2 are universally quantified.

Where logic does occur within proofs, it plays an important role. For example, students should have the ability to effortlessly convert *not (p and q)* into *not p or not q*; *if p then q* into *(not p) or q*; and *if p then q* into *if not q then not p*. In addition, they should be able to draw inferences such as *q*, given both *p* and *if p then q* (modus ponens). It seems to us that it would be useful to coordinate a student's abilities to do these various logical activities with proofs that might call on them.

In addition, there are logical and logic-like activities that connect what happens within a proof to the external context. For example, before starting to write a proof of a theorem, a student should be able to unpack its logical structure, making variables and quantifiers explicit, and converting to standard logical connectives, such as if-then. This can be difficult for many students (Selden & Selden, 1995). After that, a student needs a suitable proof framework. For direct proofs, we have illustrated this with our two sample proofs (Figures 20.3 and 20.4). In order to handle proofs by contradiction, students also need to formulate negations of quantified statements. They should be able to negate "for all x, $P(x)$“ almost automatically to get “there is an x such that not $P(x)$.”

Finally, a student must be able to connect a previous theorem, or definition, with a proof in progress. This requires unpacking the previous theorem’s, or definition’s, logical structure, taking an instance thereof, adjusting the symbols to fit those used in the current proof, seeing that the premises are satisfied, and writing the corresponding conclusion into the current proof.

Problem-Centered Reasoning

Abilities in problem-centered reasoning are more difficult to separate out and observe than previously discussed abilities. However, problem-centered reasoning plays a very large role—ultimately a dominant role—in constructing proofs, so it cannot be omitted. As background, Schoenfeld’s (1985) analysis of problem solving should be very helpful, although the time available to solve his problems was less than that usually needed for constructing proofs. This may make a considerable difference. One of the points Schoenfeld makes in regard to control is that students often fail to monitor their work, continuing too long in an unpromising direction. Surely this also happens in student proof construction. However, our experience suggests that another kind of control, persistence, can play a very positive role in proof construction. Strategic knowledge can also be very useful (Weber, 2001; Weber & Alcock, 2004).

A major factor in proof construction is the mathematics a student knows, what Schoenfeld (1985) includes in resources. However, bringing such resources to mind might also be regarded as an ability, because many students cannot do so (Selden, Selden, Hauk, & Mason, 2000). For example, earlier in this section, we discussed two students who appeared to have some knowledge of proof by induction, but neither was able to bring it to mind until the teacher intervened. Bringing to mind appropriate knowledge depends on both the situation, for example, the comments of others, and the interconnected nature of a student's own knowledge.

Finally, intuition has a role in problem-centered reasoning and we suggest it would be useful to consider at least two kinds of intuition that we call *visual-spatial* and *technical-algebraic*. By *visual-spatial* we mean intuition based on pictures or diagrams that can be sketched or visualized. These might be realistic, such a graph, or visually metaphorical, such as a "blob" for an open set in a topological space. Figure 20.5 is an example of a somewhat realistic sketch of the composition of two functions, although one that does not include their Cartesian graphs. Figure 20.5 might also be regarded as a visual metaphor for a similar situation in higher dimensions. In contrast, *technical-algebraic intuition* depends on one's familiarity with the interrelations among definitions and theorems. Earlier in this section, we suggested that part of the difficulty of our proof that the sum of continuous functions is continuous was due to the probable expectation that visual-spatial intuition might be useful, but actually technical-algebraic intuition about manipulating inequalities is called for.

Informal Observations

The Genre of Proof

When students first start constructing their own proofs, they may inquire: What is a proof? They are occasionally told just to write a convincing argument. However, proofs are not just convincing, deductive arguments, they are also texts composed in a special genre[5]. If beginning students are unaware of the need to write in this genre when asked to write "easy" proofs, they may suffer from what might be called the "obviousness obstacle."

For example, Moore (1994, pp. 258-259) reports that when one student was asked on a test to prove: *If* A *and* B *are sets satisfying* $A \cap B = A$, *then* $A \cup B = B$, she drew a Venn diagram with one circle, labeled A, contained in a larger circle, labeled B, and gave an intuitive argument based on her "understanding of set equality, subset, intersection, and union" using informal language, rather than a "proof based only on definitions, axioms, previously proved results, and rules of inference." According to her professor, she had not learned "the language and culture of how we write these things down." When asked what was wrong with her proof, she said "I didn't explain it well enough." Without understanding that there is a genre of proof, such obvious theorems may be very difficult for students.

However, our experience suggests that students who see themselves as learning to write in a special genre will have something positive to do, and hence, be more successful. This seems to be so, even when no detailed description of the genre of proof exists, and students must learn by trial and error or as apprentices.

Convince Yourself

Students are sometimes encouraged to prove theorems by first convincing themselves intuitively, then making their argument more and more precise, eventually arriving at a proof. We suspect that most students would interpret convincing themselves

[5] We are not only suggesting that someone might view proofs in this way, but that it is an empirical fact. That is, with rare exceptions, proofs possess a number of stylistic commonalties not found in other forms of deductive argument. For example, definitions available outside of a proof tend not to be repeated inside it.

intuitively to mean they should visually or spatially manipulate the objects that occur in the statement of the theorem. This is sometimes helpful, but we suggest that for most students there are some theorems that cannot be proved this way, and students who insist on basing their work on refinements of this kind of intuition cannot prove such theorems. Our sample theorem about the sum of continuous functions is such a theorem. For contrast, we now discuss the proof of another theorem that can be obtained by a refinement of visual-spatial intuition.

The theorem is: For all real-valued functions of a real variable, if f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a. At first glance, this theorem seems to be rather like the theorem about the sum of continuous functions. However, one can develop some visual-spatial intuition about it by examining a sketch (Figure 20.5).

<<INSERT FIGURE 20.5>>

Drawing a representation like that in Figure 20.5 might well be within the grasp of an advanced undergraduate student who started with the intuitive view that continuous at a means "points close to a, map close to $f(a)$." This representation not only suggests that the theorem is true; extending the idea of closeness to an $\varepsilon - \delta$ argument also leads to a proof.

Logic

In the United States, most undergraduates, who receive any significant explicit instruction in how to write proofs, are provided it in one mid-level transition-to-proof course. Typically, such courses include logic and a little about topics such as sets, equivalence relations, and functions, as well as some especially accessible mathematics to provide theorems to prove. The logic is often an abstract, symbolic, decontextualized treatment of propositional and predicate calculus, including truth tables and the validity of arguments. It is often taught early in the course, presumably because logic is considered prerequisite to understanding proofs.

Although some parts of logic, such as understanding how to negate a statement, are certainly important for constructing some proofs, we doubt that formal logic often plays a large role in proofs that beginning students typically construct. This somewhat counterintuitive view is supported by an analysis of student-generated proofs from a transition-to-proof course at a large Southwestern university. Although the course was not intended to be the basis of a study, all of the 62 correct student-generated proofs were preserved and later analyzed for a master's thesis (Baker, 2001). These proofs consisted of 926 lines of text whose analysis included noting uses of logic mentioned in the teacher's lecture notes or in the textbook (Velleman, 1994). Logic was used in just 29 lines (and was not mistakenly omitted elsewhere). The remaining lines included advance organizers, assertions of hypotheses or conclusions, statements of whether the proof was direct, by contradiction, or by cases, applications of definitions, and use of subject matter knowledge.

We also examined our sample proof from the first section for uses of the kinds of logic often taught in transition-to-proof courses. In writing this proof, we doubt that a student would call on much formal logic. However, we can find three places where a valid logical argument, together with information from outside the proof, could have been used. These occur in deducing step **[3]** from **[2]**; step **[7]** from parts of **[4]**, **[5]**, and **[6]**; and the premise of **[10]** from **[9]** (and similarly, the premise of **[11]** from **[9]**).

We illustrate how deducing step **[7]** from parts of **[4]**, **[5]**, and **[6]** might be viewed as the result of the valid logical argument: $(P \wedge Q) \rightarrow R$, P, Q, $\therefore$ R First take $(P \wedge Q) \rightarrow R$ to be an instance of a statement from outside the proof, namely, *For all real numbers* a *and* b, *if* $a > 0$ *and* $b > 0$, *then* $\min(a,b) > 0$. If one then interprets P as $\delta_1 > 0$ (from **[4]**) and Q as $\delta_2 > 0$ (from **[5]**), this statement becomes: *If* $\delta_1 > 0$ *and* $\delta_2 > 0$, *then* $\min(\delta_1, \delta_2) > 0$, where R is interpreted as $\min(\delta_1, \delta_2) > 0$. Finally, invoking the valid logical argument $(P \wedge Q) \rightarrow R$, P, Q, $\therefore$ R and identifying δ with $\min(\delta_1, \delta_2)$ (from **[6]**), one gets $\delta > 0$, which is **[7]**.

We believe that few, if any, students would consciously employ such cumbersome arguments. Thus, our sample proof does not call on formal logic. Because many direct proofs call on only a little formal logic, it seems that logic is not prerequisite to understanding proof. Therefore, it should be possible and helpful to teach logic in the context of proofs. Where logic is required in student proofs it plays a central role, so we are not suggesting its teaching be omitted, but rather that the teaching of logic grow out of students' own work with proofs, and thus, take a more practical than formal form.

Instead of **[7]** being the result of a valid logical argument, we see it as the result of an inference-generating schema that accepts both conscious and unconscious inputs, such as parts of **[4]**, **[5]**, and **[6]** and additional information, that is invoked outside of consciousness. Such schemas are not under conscious control so they might best be developed through experiences and reflections on experiences, rather than on abstractions.[6]

[6] The development (i.e., construction) of knowledge through experience and reflections on experience is consistent with a constructivist viewpoint.

Teaching

Teachers of upper-division and graduate mathematics courses, such as real and complex analysis, often ask students to produce proofs as a major part of assessments, presumably because well-written proofs reflect a good understanding of course content. Thus, a student with only modest proving ability is at considerable disadvantage in demonstrating understanding.

How do undergraduate students currently learn to construct proofs—a topic that is mostly part of the implicit curriculum? The only widely taught courses in the U.S. devoted explicitly to teaching undergraduate students how to prove theorems are mid-level transition-to-proof courses (e.g., see Vellman, 1994). Given the above discussion, it seems that students should be able to start proving theorems without prior formal instruction in logic or practice with sets and functions, and develop that knowledge through experience and mentoring.

Students may also learn something of proof construction from lectures on mathematical content, such as real analysis or abstract algebra. However, for teaching proof construction, even well presented lectures may be ineffectual, because a teacher cannot know what features of proofs students are focusing on in class and probably does not know what kinds of proofs students have, or have not, already learned to construct. Consider for example, Dr. T's teaching methods in real analysis, as described by Weber (2004). Dr. T, known for very good teaching, first discussed the formal-rhetorical aspects of proving[7] when presenting theorems about sets and functions. Somewhat later when considering sequential limits, Dr. T concentrated on demonstrating how to manipulate

[7] Weber (2004) refers to Dr. T's teaching at this point in the course as being in a logico-structural style.

absolute value expressions (to find *N*), apparently mistakenly assuming that the students had learned and could supply the required formal-rhetorical parts of such proofs, but they could not.

Homework and tests can also provide opportunities to learn proof construction. However, these too are likely to be ineffectual because often they do not focus directly on teaching how to construct proofs or do not include mentoring students' work. In summary, none of the current ways of teaching proof construction—transition-to-proof courses and homework and tests in content courses—seems adequate. Indeed, it has been our experience that many beginning graduate students at U.S. universities could benefit from a course designed to improve their ability to construct proofs.

What kind of course is likely to be reasonably effective in helping students improve their proof constructing abilities? We suggest that a good way to teach such a course is from a set of notes containing definitions and statements of theorems to prove, with little or no additional explanation. Proofs, as well as examples and nonexamples of definitions, can be provided by the students themselves. At first, students' proof construction might best be done in class, so the teacher can provide adequate mentoring. The teacher should not provide heavy-handed hints, but only enough intervention for students to succeed reasonably often with considerable effort. Every intervention, in a sense, deprives a student of the opportunity to succeed without it. Thus it is probably best if mentoring were not available during all, or even most, of a student's proving of a particular theorem.

Occasionally student proving is likely to require general information, such as how to negate a universally quantified statement or how to prove a function is one-to-one.

When this occurs, a teacher can add such information to a developing set of supplementary notes that students could refer to as needed. The information in such supplementary notes should be especially pertinent as it would have first arisen in context in the form of behavioral knowledge, or the lack of it, and only subsequently developed into conceptual knowledge, rather than the other way round.

In the kind of course we are describing, students seem to do well in small groups. It may be that early on small group discussions alleviate concern over working in an unfamiliar problem-oriented situation. Also, the need to convince one's colleagues of one's ideas may enhance problem-solving control.

In addition, where understanding previous mathematical content (e.g., the definition of uniformly continuous) or bringing it to mind are called for, surely several students have an advantage over one. All of the above suggests that working in a small group raises the probability of a student's successfully proving a theorem. We see raising the probability of success as important because our experience strongly suggests that success breeds success.

There is another way to raise the probability of students successfully proving a theorem that we call "long range priming." We illustrate this by referring to our proof that the sum of continuous functions is continuous. It seems clear this is a difficult proof and depends on students thinking of using minimum to find δ and the triangle inequality to complete the problem-centered part of the proof. The proof might be rendered somewhat less difficult by inserting two earlier theorems in the notes, one requiring minimum and the other requiring the triangle inequality.

The above illustration suggests that course notes should be written as the course progresses, so that the teacher knows which features of proofs students are already familiar with and which kinds of proofs they have successfully constructed thus far. The idea is to provide course notes having "just in time" challenges and information.

There is an additional constraint, the course notes should not be too narrow. Before writing the notes, one might wish to establish priorities on which abilities to include. This might be done in a way that would ensure that the supplemental notes include much of the background material in a transition-to-proof course. All of this suggests the notes may require considerable time to write and raises a question of practicality for most teachers.

Is such a course practical? Can the notes be written in a reasonable length of time and can a reasonable number of students be mentored in class? Our experience from transition-to-proof courses suggests that mentoring would become difficult as class size nears 36, even with groups of four. However, a properly trained student assistant could no doubt be a great help with mentoring. For more advanced undergraduate or graduate students, the writing of notes appears to be manageable for a very small class with three groups of two students each or of three groups of three to four students each.

However, a tool could surely be built that would make writing such notes practical for a larger number of students. It is much easier to select a theorem that requires particular proving abilities than it is to write one "on demand." Thus, what is needed is a reference book, or database, containing branching sequences of definitions, theorems, and proofs, together with an analysis of which previous definitions and theorems, and which abilities, are used in each proof. A teacher could then select a theorem that would "stretch" a particular ability, and working backwards, include any necessary definitions and theorems needed to join the desired theorem to the current notes.

Finally, instead of needing to teach the kind of course we have described, it might be better to integrate the teaching of proving throughout the undergraduate program, and in schools -- as is called for by the *NCTM Standards* (2000). This would require setting aside adequate time for such teaching and providing a good deal of help to school teachers.

Conclusion

This chapter has concerned features of proofs, student abilities called upon by such features, and how coordination of such features and abilities can be used in teaching students to construct proofs. First we introduced three proof structures: a *hierarchical* structure of subproofs and subconstructions; a linear *construction path* giving the order in which an idealized prover could write a proof; and a division of proofs into *formal-rhetorical* and *problem-centered* parts.

We then discussed how several specific features of proofs could be seen as calling on specific student abilities. For example, a proof by induction not only calls on a student's knowledge of induction, but also on the student's ability to bring that knowledge to mind. Attending to the three structures mentioned above can also provide some idea of a proof's complexity. Also, the ability to use the language of sets and functions differs from merely reproducing formal definitions.

Next we discussed three informal ideas: (1) Only arguments written in a particular genre are accepted as proofs. Students who do not understand this seem to have difficulty constructing proofs of statements they see as obvious. (2) Some theorems do not have proofs that are refinements of arguments based on visual-spatial intuition. Such theorems seem to be very difficult for students who habitually depend on such intuition. (3) Logic does not occur so often in student proofs that it must be taught prior to, rather than with, proving. Furthermore, the logic used in student proofs often seems to depend more on a practical, intuitive approach than on an application of formal logic.

Finally we described a method of teaching using mentoring and group work in which instruction is integrated into students' construction of proofs. In such teaching, kinds and features of proofs are more important than specific topics. The key idea is that when students require mentoring on some ability, the teacher tries to assign a later theorem having a proof calling for that ability. We ended with a discussion of the practicality of such teaching.

References

Baker, S. W. (2001). *Proofs and logic: An examination of mathematics bridge course proofs*. Unpublished master's thesis, Tennessee Technological University, Cookeville, Tennessee. USA.

Gutiérrez, A., & Jaime, A. (1998). On the assessment of the van Hiele levels of reasoning. *Focus on Learning Problems in Mathematics*, *20*(*2/3*), 27-46.

James, W. (1890). *The principles of psychology*. New York: Holt.

Jones, F. B. (1977). The Moore Method. *American Mathematical Monthly*, *84*(*4*), 273-278.

Konior, J. (1993) Research into the construction of mathematical texts. *Educational Studies in Mathematics*, *24*(*3*), 251-256.

Leron, U. (1983). Structuring mathematical proofs. *The American Mathematical Monthly*, *90*, 174-184.

Mahavier, W. S. (1999). What is the Moore method?. *PRIMUS*, *9* (December), 339-354.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to-act in the moment. *Educational Studies in Mathematics*, *28*(*1-3*), 135-161.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*(*3*), 249-266.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Selden, A., Selden, J., Hauk, S., & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems? In A. H. Schoenfeld, J. Kaput, & E. Dubinsky, (Eds.), *Research in collegiate mathematics education IV* (pp. 128-153). Providence, RI: American Mathematical Society.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(*2*), 123-151.

Velleman, D. J. (1994). *How to prove it: A structured approach.* Cambridge: Cambridge University Press.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39-59). Hillsdale, NJ: Lawrence Erlbaum Associates.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, *23*(*2*), 115-133.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(*1*), 101-119.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(*3*), 209-234.