Superseparability and its Physical Implications[*]

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Presented at the 16th U. K. and European Meeting
on the Foundations of Physics, Aberdeen, 5–7 July, 2010

Abstract

Since the canonical commutation relations for a finite number of degrees of freedom have many inequivalent irreducible representations, the states of a physical system may span more than one such representation. *Superseparability* is defined to be the case in which no meaning can be attached to a superposition of vectors belonging to inequivalent representations. In this report, which is basically nonmathematical, we trace the origins of superseparability and suggest two experiments that may establish the existence of the phenomenon. We then discuss which-path experiments and interaction-free measurements in the light of superseparability, and conclude with stating some open problems. A mathematical appendix provides the essential mathematical background.

[*]Preliminary version

1 Introduction

In von Neumann's measurement theory, the state vector of a particle can change in two different ways: smoothly, under the Schrödinger equation, or abruptly, by interaction with a measuring apparatus. The second possibility is the infamous 'collapse of the state vector' (demonstrated rigorously, under acceptable physical hypotheses, by Sewell in 2005 [10]).

Actually, the state vector can do worse than collapse; it can escape from its Hilbert space and seek refuge in an altogether different one. This mathematical possibility – arising from the existence of inequivalent irreducible representations of the canonical commutation relations – has been called **superseparability** in [9]. In this talk I shall report on the mathematical phenomenon and explore some of its physical consequences.

The talk itself will be nonmathematical, but it is based on results that have been established rigorously in the quantum mechanics of particles of nonzero mass. A summary of the basic concepts and results is given in the Mathematical appendix of this report.

I shall begin with tracing the mathematical origins of superseparability. I shall then describe an experiment, using the Aharonov-Bohm effect, that should distinguish between superseparability and single-Hilbert-space interpretations. Then I shall attempt to interpret the results of which-path experiments in the light of of superseparability, and consider interaction-free measurements as a variant of which-path experiments. If time allows, I shall conclude with stating some problems that seem inportant to me and have not yet been addressed.

2 Superseparability

The canonical commutation relations (CCR) (we have set $\hbar = 1$)

$$[q_j, p_k] = \mathrm{i}\delta_{jk} I, \;\; [q_j, q_k] = [p_j, p_k] = 0, \quad i, j = 1, \ldots, N \tag{2.1}$$

have infinitely many inequivalent irreducible unitary representations of which the everyday *Schrödinger representation* is a very special one.[1] This fact has been known to mathematicians since 1967 [3], but its relevance to physics was only pointed out by Helmut Reeh in 1988 [6].

The key physical question is whether or not two inequivalent representations of the CCR (2.1) are separated by a superselection rule. (This is the

[1]In this report the term *representation* will always mean an irreducible unitary representation unless the contrary is stated explicitly, and N will be assumed finite. We shall never consider representations that are not unitary.

same as asking whether vectors from two inequivalent representations can be superposed upon each other.) The representations π_1 and π_2 on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ can be combined, trivially, into the direct sum representation $\pi = \pi_1 \oplus \pi_2$ on the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Let $\Pi_{1,2}$ be the projection operators from $\mathfrak{H}$ to $\mathfrak{H}_{1,2}$ and let $\psi_{1,2} \in \mathfrak{H}$ such that $\Pi_1\psi_2 = \Pi_2\psi_1 = 0$. Then, if $(\psi_1, O\psi_2) = 0$ for every observable O on $\mathfrak{H}$, one says that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are separated by a superselection rule. This can only happen if the algebra of observables on $\mathfrak{H}$ has a nontrivial centre. As this requirement is not physically transparent, we shall rephrase it as an independent assumption:

Assumption 2.1 (The superseparability hypothesis) *If π_1 and π_2 are inequivalent representations of the CCR on $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are separated by a superselection rule.*

The superseparability hypothesis merely asserts that there is no *quantum-mechanical observable* that can induce transitions between two inequivalent representations. It does not claim that there are no *classical devices* that can induce such transitions.

3 Origins of superseparability

In quantum mechanics, momentum operators are the generators of translations, i.e.,

$$e^{i\boldsymbol{a}\cdot\boldsymbol{p}} f(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{a}) \tag{3.1}$$

for all $f \in \mathfrak{H}$, where $\mathfrak{H}$ is the Hilbert space of the system. If $\mathfrak{H} = L^2(\mathbb{R}^3, d^3x)$, where d^3x is the Lebesgue measure on $\mathbb{R}^3$, the quantity $f(\boldsymbol{x} + \boldsymbol{a})$ is well-defined for every $\boldsymbol{x}, \boldsymbol{a} \in \mathbb{R}^3$, i.e., the left-hand side of (3.1) is always well-defined. Now excise a point, say the origin O, from $\mathbb{R}^3$. We are then left with $\mathbb{R}^3 \setminus \{O\}$, and for $f \in \mathbb{R}^3 \setminus \{O\}$, the right-hand side of (3.1) is clearly undefined for $\boldsymbol{a} = -\boldsymbol{x}$. It follows that the quantity $\exp(i\boldsymbol{a}\cdot\boldsymbol{p})$ on the left-hand side of (3.1) is not defined. At one level this may be understood as follows: the group of isometries of $\mathbb{R}^3$ is the Euclidean group in three dimensions, but the group of isometries of $\mathbb{R}^3 \setminus \{O\}$ is the group of rotations around O; the translations have disappeared, and that is why the operator $\boldsymbol{p}$ cannot be exponentiated. If one excises another point, say O', the group of isometries of $\mathbb{R}^3 \setminus \{O, O'\}$ will no longer be a continuous group. In some sense, the Hilbert spaces $L^2(\mathbb{R}^3)$, $L^2(\mathbb{R}^3 \setminus \{O\})$ and $L^2(\mathbb{R}^3 \setminus \{O, O'\})$ are very different from each other. The inequivalence of representations of the CCR on these Hilbert spaces is a reflection of this fact.

At the present state of our knowledge, superseparability appears to be an effect of topological origin, the topology being that of X in $L^2(X, \mathrm{d}^3x)$. What is wholly unknown is whether X should be the region defined the apparatus, the entire universe, or something in between. Since superseparability (being a limitation on the superposition principle) is an automatic generator of decoherence, this question is physically nontrivial.

4 Reeh's result

The above examples show that if X and Y are two non-homeomorphic open subsets of $\mathbb{R}^n$, then representations of the CCR on $L^2(X, \mathrm{d}^n x)$ and $L^2(Y, \mathrm{d}^n x)$ may be inequivalent. However, Reeh showed that the matter was not so simple.

Reeh considered representations of the CCR in the presence of an infinite magnetic flux line along the z-axis in $\mathbb{R}^3$ (cf. the Aharonov-Bohm effect). Cylindrical symmetry made the problem into a two-dimensional one. The Hilbert space was $L^2(\mathbb{R}^2 \setminus \{O\})$, and the canonical momenta were modified by the addition of the vector potential ($\boldsymbol{p} \to \boldsymbol{p} + q\boldsymbol{A}$, where q is the charge of the particle), which did not affect the CCR. Reeh showed that the representations corresponding to the trapped fluxes Φ_1 and Φ_2 were inequivalent, except for

$$\Phi_2 = \Phi_1 + n\Phi_0, \tag{4.1}$$

where n is an integer and $\Phi_0 = \pi/e$ is (half of) London's flux quantum; e is the electronic charge. In words, representations for two different fluxes are inequivalent unless the fluxes differ by an integral multiple of Φ_0. A representation with a quantized flux is equivalent to the Schrödinger representation; a quantized flux is topologically active; it can repair holes in the plane.

5 Two experiments

Fig. 1 shows the core of a 'noninterferometer'. It consists of two chambers A and B. One of them, B, contains a flux line of strength Φ (perpendicular to the plane of the paper), but its presence does not influence the vector potential in A; this vector potential is that of a field-free, simply-connected region. That is, the chambers A and B are electromagnetically isolated, and 'leakage' from the entrance and exit slits does not materially affect this electromagnetic isolation. (It may not be easy to meet these conditions in the laboratory.)

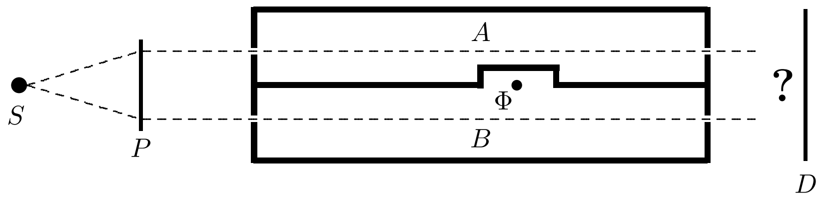


Figure 1: Scheme for a noninterferometer

In the figure, S is a source of charged particles and D the detector. The experiment is a *single-particle interference experiment*, that is, there is only one particle in the noninterferometer at any given time.

The particles are assumed to emerge in a unique (possibly Schrödinger) representation from the source S. The aim of the experiments is to test whether or not the presence of the magnetic flux triggers an escape into another representation. The two experiments are as follows:

1. The flux Φ is variable. The experiment consists of observing changes in the interference pattern at the detector as Φ is varied. If the superseparability assumption holds, interference fringes should be seen only near $\Phi = n\Phi_0$; they should be completely washed out midway between succesive quantized values of the flux, i.e., near $\Phi = (n + 1/2)\Phi_0$.

2. The flux Φ is enclosed in a superconducting tube. Then it is necessarily quantized. The experiment consists of determining whether or not there is an interference pattern at the detector. If the superseparability assumption holds, interference should *always* be observed.

The Aharonov-Bohm fringe shift is not relevant to these experiments. What matters is the existence or nonexistence of interference fringes.

6 Which-path experiments

In a double-slit or (topologically) equivalent interferometer, a classical particle can take one of two paths P_1 and P_2 that are indistinguishable except from the outside. Interference is observed when a quantum object Ω – which may be a photon, electron, neutron, atom or molecule – traverses the interferometer. The symbol Ω stands for a physical entity, which may exist in mathematically distinct states.

The question was asked: is it possible to determine *which path* is taken by Ω without destroying the interference? (This question was much discussed in the 1990s, perhaps earlier; see the references in [1]). Implicitly assuming that Ω is structureless, some authors argued that determination of the path (say P_1) would inject so much momentum (and therefore wavelength) uncertainty into Ω in P_1 as to wash away the interference fringes. Then, in a review article in 1991, Scully, Englert and Walther [8] suggested a way of obtaining which-path information that could defeat the uncertainty principle – provided that Ω had an internal structure that could be exploited for the purpose. Their suggestion was to use rubidium atoms in the long-lived $63p_{3/2}$ Rydberg state as Ω. In one arm of the interferometer the atom would be influenced to emit a photon of about 21 GHz and drop to the $61d_{3/2}$ or $61d_{5/2}$ state. The process will involve no measurable change in the linear momentum of the atom, and should therefore introduce no measurable uncertainty in its de Broglie wavelength. Which-path information will be obtained by detecting the emitted photon.

In 1998, Dürr, Nonn and Rempe performed a refined version of the experiment suggested by Scully et al [1]. In their experiment it was the spin state of the nucleus that was changed, through the emission of a 3 GHz photon. Interference was observed when the spin state of nucleus was left undisturbed, but disappeared as soon as an attempt was made to alter it. Both Scully et al and Dürr et al attribute the disappearance of interference to the correlations between Ω and the which-way detector. In Scully et al's formula, the state of the atom is unchanged, but the state of the which-way detector changes. In Dürr et al's formula, the state of the atom is changed (to an orthogonal one in the same Hilbert space), but the state of the which-way detector remains the same. Both give a vanishing interference term.

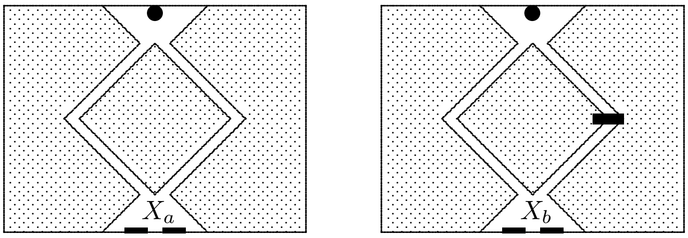
As an alternative to the above, the present author would like to suggest that interaction with a which-way detector changes the representation of the CCR to which Ω initially belongs to an inequivalent one, *which destroys the basic condition under which a particle can interfere with itself.* Note that the effect is strongly local – the representation is changed in one arm of the interferometer, but not in the other.

It should be mentioned that the second noninterferometer experiment of Sec. 5 distinguishes between these two views. When the flux Φ is quantized (and nonzero), the representation is the same as the Schrödinger representation; however, the two paths in chambers A and B are clearly distinguished by the presence of the flux in A.

7 Interaction-free measurements

Consider a two-path photon interferometer with equal path lengths. It is possible to place two photon detectors D_{max} and D_{min} such that D_{max} is at an interference maximum and D_{min} at a minimum. Then only the detector D_{max} will fire. (If there is no interference, each detector has a 50% chance of firing.)

In 1993, Elitzur and Vaidman [2] suggested placing a bomb – which would explode when hit by a single photon – in one of the arms of the interferometer. Fig. 2(a) shows a normal interferometer, and Fig. 2(b) one *with* the Elitzur-Vaidman bomb. The source, in each case, is the black circle at the top. Neither bomb nor detectors (at the bottom) are drawn realistically.



(a) (b)

Figure 2: (a) Normal interferometer; (b) interferometer with bomb

If now n photons are injected into the apparatus 2(b) one after the other and the bomb does not explode, it surely means that all of them have taken the bomb-free arm of the interferometer. As they have passed through the same arm, there is no interference, and therefore both detectors D_{min} and D_{max} will fire; one has detected the presence of the bomb without a single photon intereacting with it! An equivalent experiment has been performed, quite successfully, by Kwiat, Weinfurter and Zeilinger, and called *quantum seeing in the dark* by them (see [4], and the references cited there).

How does one interpret the absence of interference? There are two possibilities:

1. An arm of the interferometer is blocked, which *changes the topology of the apparatus* (see Fig. 2); In the interferometer (a), the space X_a in which the particle can move from source to detector *is not* simply connected, whereas the space X_b in (b) *is.* The results of the two experiments are different because X_a and X_b are topologically distinct. (The Hilbert space $L^2(X_a, \mathrm{d}\mu)$ has an additional structure that the

space $L^2(X_b, \mathrm{d}\mu)$ does not have, but superseparability does not seem to be relevant to these experiments.)

2. An agency (like the magnetic flux in the experiments suggested in Sec. 5), which changes the representation of the particle, is active in one of the arms of the interferometer.

Mere absence of interference cannot distinguish between these two possibilities. One should add the caveat that there is, as yet, no mathematically rigorous theory of zero-mass particles in relativistic physics.

Vaidman's own explanations, based on a many-worlds interpretation of quantum mechanics, may be found in [11].

8 Concluding remarks

In conclusion, I would like to mention a few avenues that seem to me to be worth exploring:

1. The topology of an optical diffraction grating is that of a line segment with a large number of equally spaced holes. Holograms used in electron or atom interferometry are often three-dimensional. The heart of any interferometers is a subspace of $\mathbb{R}^3$ that is not simply connected. Let X_α be a collection of subspaces that are topologically distinct from each other. How do the spaces $\mathfrak{H}_\alpha = L^2(X_\alpha, \mathrm{d}\mu)$ differ from each other? (We are talking here of structures other than the Hilbert space structure of these spaces.) How are the representations π_α of the CCR on the $\mathfrak{H}_\alpha$ related to each other? These are purely mathematical questions which seem to me to be of considerable physical interest.

2. We saw earlier that a quantized magnetic flux can, in effect, repair a (single) hole in the plane. If there exists a physical quantity that has a similar effect in a hologram, it may be possible to devise an experiment that could test superseparability without the need for electromagnetically isolated chambers, as in the experiments suggested in Sec. 5.

3. The wave-particle duality of quantum mechanics is also a local-nonlocal duality. It seems to the present author that Francis Bacon's doctrine of *dissecare naturam* has, at some stage, to come into conflict with the notion of nonlocality. (The Reeh-Schlieder theorem shows that nonlocality cannot be avoided entirely even in relativistic 'local' quantum field theory.) Could part of the confusion in the interpretation of

quantum mechanics be due to a misguided effort to cast an essentially nonlocal theory in a reductionist mould?

4. Can superseparability form the basis for a local theory of decoherence? In particular, does superseparability open up the possibility that a large composite system (a cat?) may not have a quantum state *as a whole*? So far all this is *terra incognita*.

Mathematical appendix

One of the most persistent urban legends in physics is that, for a finite number of degrees of freedom, the canonical commutation relations have only one irreducible unitary representation. This legend arises from a misunderstanding of *von Neumann's uniqueness theorem.* (Physicists need not feel bad; eminent mathematicians have contributed to fostering it).

In 1928, Hermann Weyl (in his book *Gruppentheorie und Quantenmechanik* [12]) replaced the CCR (2.1) by the relations (Einstein's summation convention is not used here)

$$\begin{aligned} A_j(a_j)B(k(b_k) &= B_k(b_k)A_j(a_j)\exp[\mathrm{i}a_jb_k\cdot\delta_{jk}], \\ A_j(a_j)A_j(a'_j) &= A_j(a_j+a'_j), \\ B_k(b_k)B_k(b'_k) &= B_k(b_k+b'_k), \\ A_j(a_j)A_l(a_l) &= A_l(a_l)A_j(a_j), \\ B_k(b_k)B_m(b_m) &= B_m(b_m)B_k(b_k), \end{aligned} \tag{8.1}$$

where $a_j, b_k \in \mathbb{R}$ for all j, k. One may view (8.1) as the defining equations of a Lie *group* (nowadays known as the *Weyl group*). Then (2.1) is the Lie algebra of this group, and $A_j(a_j), B_k(b_k)$ may formally be expressed as

$$\begin{aligned} A_j(a_j) &= \exp[\mathrm{i}a_jq_j], \\ B_k(b_k) &= \exp[\mathrm{i}b_kp_k]. \end{aligned} \tag{8.2}$$

Equations (8.1) are known as the *Weyl form* of the CCR. What von Neumann proved was that the Weyl group (repeat: for a finite number of degrees of freedom) has only one irreducible unitary representation, and he called its generators the *Schrödinger operators.* This representation is known as the *Schrödinger representation.*

A Lie group defines a unique Lie algebra, but the converse is not true. The simplest counterexamples are provided by covering groups of non-simply-connected Lie groups; all covering groups of Lie groups have the same Lie algebra.

The case of Lie algebras of operators on an infinite-dimensional Hilbert space $\mathfrak{H}$ is much more complicated. It is well known that if $[Q,P] = \mathrm{i}I$ then at least one of P, Q is unbounded, i.e., it is not defined everywhere on the Hilbert space $\mathfrak{H}$. The subspace on which it is defined is called its *domain*, and an unbounded operator T is not defined until its domain $\mathfrak{D}(T) \subsetneq \mathfrak{H}$ its specified. (The equalities $[A,B] = 0$ and $[A,B] = \mathrm{i}I$ make sense only on vectors $\varphi \in \mathfrak{D}(A) \cap \mathfrak{D}(B) \subset \mathfrak{H}$; the intersection may well consist of the zero vector alone.) This fact leads to phenomena unheard-of with bounded operators; for example, an unbounded operator need not be exponentiable. In this case there will be no Lie group that 'fathers' the Lie algebra.

The mathematical notion of an unbounded operator may be likened to the zoological notion of a non-elephant; one can only deal with lesser aggregates at a time. We shall restrict ourselves to (unbounded) operators that are defined on dense subspaces of $\mathfrak{H}$ (such operators are called *densely defined*, for short), and have adjoints that are also densely defined. The adjoint of T will be denoted, as usual, by T^*. An operator T_1 is said to be an *extension* of T_0 if $\mathfrak{D}(T_1) \supsetneq \mathfrak{D}(T_0)$ and $T_1\varphi = T_0\varphi$ for all $\varphi \in \mathfrak{D}(T_0)$.

If $\mathfrak{D}A) \subsetneq \mathfrak{D}(A^*)$ and $A\varphi = A^*\varphi$ for all $\varphi \in \mathfrak{D}(A)$, then A is called *symmetric*. If $\mathfrak{D}(A) = \mathfrak{D}(A^*)$ and $A\varphi = A^*\varphi$ for all $\varphi \in \mathfrak{D}(A)$, then A is called *self-adjoint*. A symmetric operator may have no self-adjoint extension, it may have many self-adjoint extensions, or it may have only one. In the last case, it is called *essentially self-adjoint*. The critical difference between symmetric and self-adjoint operators is that self-adjoint operators can be exponentiated, while symmetric operators *cannot*.

We shall denote the unique self-adjoint extension of an essentially self-adjoint operator T by $\bar{T}$.

If A and B are self-adjoint, are defined on a common dense domain $\mathfrak{D}$ and commute on $\mathfrak{D}$, then $\exp(\mathrm{i}aA)$ and $\exp(\mathrm{i}bB)$ $(a, b \in \mathbb{R})$ are defined everywhere, and commute. However, if A and B are merely essentially self-adjoint, defined on $\mathfrak{D}$ and commute on $\mathfrak{D}$, then $\exp(\mathrm{i}a\bar{A})$ and $\exp(\mathrm{i}b\bar{B})$ *do not necessarily commute*. This highly counterintuitive fact was established by Nelson (see [5]).

The physicist, fortified by pleasant experiences with Lie groups, may be forgiven for thinking that the fact unearthed by Nelson is of doubtful physical relevance. But, in 1988, Helmut Reeh showed that that the 'Nelson phenomenon' occurred in the Aharonov-Bohm effect [6]. For brevity, let us call a spinless particle of charge c moving in a plane perpendicular to a trapped magnetic flux – the classical Aharonov-Bohm example – an *AB-particle*. The assumption of cylindrical symmetry makes this into a two-dimensional problem. The formal expressions for the canonical operators of an AB-particle become

$$\mathbf{p} = -\mathrm{i}\frac{\partial}{\partial \mathbf{x}} + c\mathbf{A}, \ \ \mathbf{q} = \mathbf{x}. \tag{8.3}$$

Here boldface symbols denote 2-vectors in the XY-plane, $\mathbf{A}$ is the vector potential (up to a gauge)

$$\mathbf{A} = \frac{\phi}{r}\mathbf{e}$$

(we set have $r = (x^2 + y^2)^{1/2})$), $\mathbf{e}$ being the unit vector at (x, y) tangent to the circle $x^2 + y^2 = r^2$,

$$\mathbf{e} = \left(-\frac{y}{r}, \frac{x}{r}\right)$$

and ϕ is the magnetic flux. We shall set $\alpha = c\phi$.

The problem is to define the formal quantities p_x and p_y in (8.3) as operators on the Hilbert space $L^2(\mathbb{R}^2 \smallsetminus O) = L^2(\mathbb{R}^2)$; excision of a single point, here the origin O, has no effect on an L^2-space. Reeh chose for the common domain of p_x, p_y the space $\mathcal{D}(\mathbb{R}^2 \smallsetminus O)$ of smooth functions of compact support on $\mathbb{R}^2$, excluding the origin, which is dense in $L^2(\mathbb{R}^2)$. Then, since $\operatorname{curl} \mathbf{A} = 0$ away from the origin, it followed that $[p_x, p_y] = 0$. Reeh then proved that the operators p_x and p_y are essentially self-adjoint; they have unique self-adjoint extensions $\bar{p}_x$ and $\bar{p}_y$. Finally, he computed the commutator

$$[\exp{(\mathrm{i}a\bar{p}_x)}, \exp{(\mathrm{i}b\bar{p}_y)}],$$

and found that it vanishes *if and only if* $\alpha = 0, \pm 1, \pm 2, \ldots$ This proved that, for non-integral α, the group generated by the $\bar{p}_x, \bar{p}_y, \bar{q}_x, \bar{q}_y$ is not the Weyl group. If, as is customary, one defines

$$a_{x,y} = \frac{1}{\sqrt{2}}(q_{x,y} + \mathrm{i}p_{x,y}),$$

then (again, for non-integral α) there is no vector in $L^2(\mathbb{R}^2)$ that is annihilated by $a_{x,y}$; the representation is non-Fock.

Finally, we remark that in two dimensions, the quantities P, Q defined by

$$P = -\mathrm{i}\frac{\partial}{\partial x}, \ \ Q = x - i\frac{\partial}{\partial y}$$

formally satisfy the canonical commutation relations. When they are represented as unbounded operators on a Hilbert space, the representation is not equivalent to the Schrödinger representation in the one-dimensional case. This is one of the methods that have been used by Schmüdgen [7] to determine infinitely many inequivalent irreducible representations of the canonical commutation relation $[Q, P] = \mathrm{i}I$.

References

[1] S Dürr, T Nonn and G Rempe (1998). Origin of quantum-mechanical complementarity probed by a ‘which-way’ experiment in an atom interferometer, *Nature* **395**, 33–37.

[2] A Elitzur and L Vaidman (1993). Quantum-mechanical interaction-free measurements, *Found. Phys.* **23**, 987–997.

[3] B Fuglede (1967). On the relation $PQ - QP = \mathrm{i}I$, *Math. Scand.* **29**, 79–88.

[4] P Kwiat, H Weinfurter and A Zeilinger (1996). *Scientific American* **275**, 72–78.

[5] M Reed and B Simon (1972). *Methods of Modern Mathematical Physics*, I. Functional Analysis, New York: Academic Press.

[6] H Reeh (1988). A remark concerning canonical commutation relations, *J. Math. Phys.* **98**, 1535–1536.

[7] K Schmüdgen (1983). On the Heisenberg commutation relation II, *Publ. Res. Inst. Math. Sci. Kyoto* **19**, 601–671.

[8] M O Scully, B-G Englert and H Walther (1991). Quantum optical tests of complementarity, *Nature* **351**, 111–116.

[9] R N Sen (2010). *Causality, Measurement Theory and the Differentiable Structure of Space-Time*, Cambridge University Press, Cambridge. Cambridge Monographs on Mathematical Physics.

[10] G L Sewell (2005). On the mathematical structure of the quantum measurement problem, *Rep. Math. Phys.* **56**, 271–290.

[11] L Vaidman (2001). The paradoxes of the interaction-free measurements, *Z. Naturforsch.* **A 56**, 20–26.

[12] H Weyl (1950?) *The Theory of Groups and Quantum Mechanics*, Dover Press, New York. Translated from the 2nd German edition. The first German edition of *Gruppentheorie und Quantenmechanik* was published in 1928.