BLINKING FRACTALS AND THEIR QUANTITATIVE ANALYSIS USING INFINITE AND INFINITESIMAL NUMBERS

Yaroslav D. Sergeyev[*]

Dipartimento di Elettronica, Informatica e Sistemistica,
Università della Calabria,
Via P. Bucci, Cubo 42-C,
87030 Rende (CS) – Italy

tel./fax: +39 0984 494855
http://wwwinfo.deis.unical.it/∼yaro
yaro@si.deis.unical.it

Abstract

The paper considers a new type of objects – blinking fractals – that are not covered by traditional theories studying dynamics of self-similarity processes. It is shown that the new approach allows one to give various quantitative characteristics of the newly introduced and traditional fractals using infinite and infinitesimal numbers proposed recently. In this connection, the problem of the mathematical modelling of continuity is discussed in detail. A strong advantage of the introduced computational paradigm consists of its well-marked numerical character and its own instrument – Infinity Computer – able to execute operations with infinite and infinitesimal numbers.

Key Words: Blinking fractals, infinite and infinitesimal numbers, numeral systems, physical continuity, mathematical continuity.

1 Introduction

Fractal objects have been very well studied during last few decades (see, e.g., [8, 18] and references given therein) and have been applied in various fields (see numerous applications given in [4, 7, 8, 10, 18, 27]). However, mathematical analysis of fractals (except, of course, a very well developed theory of fractal dimensions) very often continues to have mainly a qualitative character and tools for a quantitative analysis of fractals at infinity are not very rich yet. Nowadays, the necessity of introduction of such tools becomes very urgent in connection with the appearance of new powerful approaches modelling the spacetime by fractals (see [5, 6, 7, 12, 13, 26] and references given therein).

---

[*]Yaroslav D. Sergeyev works also at the N.I. Lobatchevsky State University, Nizhni Novgorod, Russia (part-time contract) and at the Institute of High Performance Computing and Networking of the National Research Council of Italy (affiliated researcher).
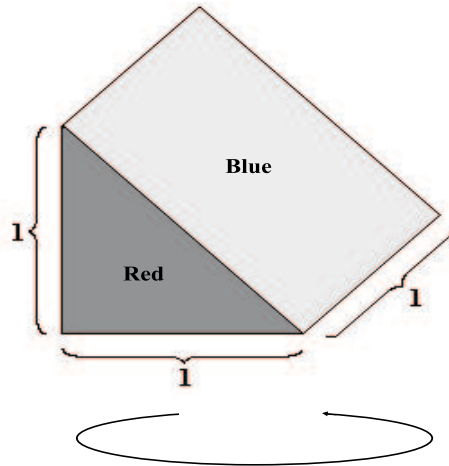
Figure 1: The rotating prism having the triangular face red and the rectangular face blue.

In this paper, we propose to apply a recently developed methodology using explicitly expressible infinite and infinitesimal numbers for two purposes: on the one hand, for a quantitative analysis of traditional and newly introduced blinking fractals; on the other hand, for developing new mathematical tools better describing (in comparison with traditional mathematical instruments developed for this goal) physical notions of continuity and discontinuity.

Let us start by introducing the new class of objects – blinking fractals – that are not covered by traditional theories studying self-similarity processes. Traditional fractals are constructed using the principle of self-similarity that infinitely many times repeats a basic object (some times slightly modified in time). However, there exist processes and objects that evidently are very similar to classical fractals but cannot be covered by the traditional approaches because several self-similarity mechanisms participate in the process of their construction. Before going to a general definition of blinking fractals let us give just three examples of them.

The first example is derived from one of the famous fractal constructions – the coast of Britain – as follows. Suppose that we have made a picture of the coast two times using the same scale of the map: at the moment of the early sunrise and at the moment of late sunset. Then, due to the long shadows present at these moments and directed to the opposite sides we shall have two different pictures. If we suppose, for example, that sunset corresponds to shadows on the left and sunrise to shadows on the right, then we can indicate them as L and R, correspondingly. If now we start to make pictures (starting from sunrise) alternating moments of the photographing from sunrise to sunset and decreasing the scale each time, we shall obtain a series of pictures being very similar to traditional fractals but different because left shadows will alternate right shadows at this sequence as follows: $R, L, R, L, R, L, \ldots$ Thus, there are two fractal mechanisms working in our process. Each of them can be represented by one of subsequences $R, R, R, \ldots$ and $L, L, L, \ldots$ and the traditional analysis does not allow us to say what will be the limit fractal object and will it have L or R type of shadow.

The second example is constructed as follows. Let us take a prism (see Fig. 1)
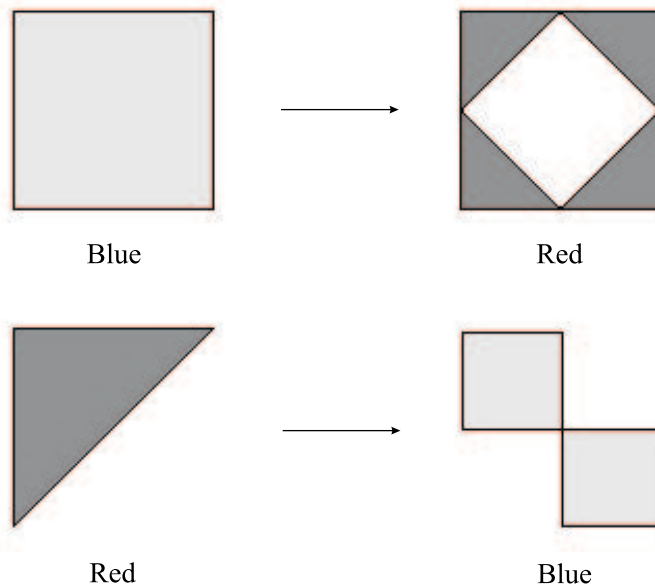
2

Figure 2: We observe that each blue square is transformed in four red triangles and each red triangle is transformed in two blue squares.

that is rotating around its vertical axis and observe it at two different moments. The first is the moment when we see its face being the blue rectangular with sides 1 and $\sqrt{2}$. Since we look exactly at the front of the prism we see the rectangular as the square with the length one on side. The second moment is when we look at the face being the red right isosceles triangle with the legs equal to one. Then we apply to this three-dimensional object the two following self-similarity rules: we substitute each prism by four smaller prisms during the time passing between each even and odd observation and by two smaller prisms during the time passing between each odd and even observation. Thus, at the odd iterations we observe application of the first mechanism shown in the top of Fig. 2. The second mechanism shown in the bottom of this figure is applied during the even iterations. As a result, starting from the blue square one on side at iteration 0 we observe the pictures (see Fig. 3) with alternating blue squares and red triangles. Again, as it was with the above example related to the coast of Britain, we can extract two fractal subsequences being traditional fractals. The mechanism of the first one dealing with blue squares is shown in Fig. 4. The second mechanism dealing with red triangles is presented in Fig. 5. Traditional approaches are not able to say anything about the behavior of this process at infinity. Does there exist a limit object of this process? If it exists, what can we say about its structure? Does it consist of red triangles or blue squares? What is the area of this (again, if it exists) limit object? All these questions remain without answers.

Before we discuss the last example linked, as it was with our first example,

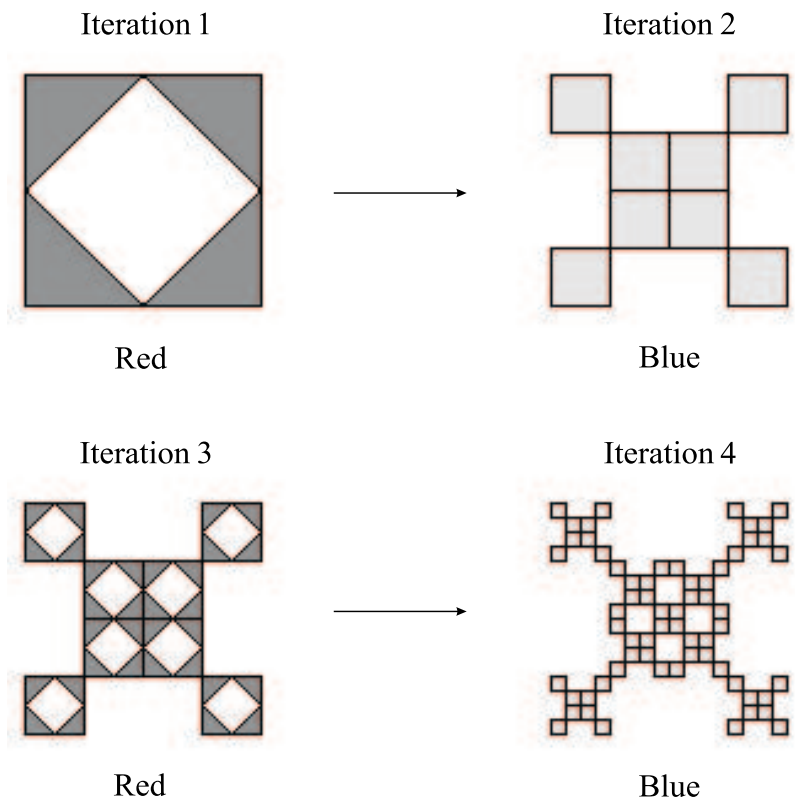| Iteration 1 | Iteration 2 |
Red	Blue
Iteration 3	Iteration 4
Red	Blue

Figure 3: The first four iterations of the process that has started from one blue square and uses two self-similarity mechanisms.

to another famous fractal construction – Cantor's set (see Fig. 6) – let us make a few comments reminding that very often we can give certain numerical answers to questions regarding fractals only if a finite number of steps in the procedure of their construction has been considered. The same questions very often remain without any answer if we consider an infinite number of steps. If a finite number of steps, n, has been done in construction of Cantor's set, then we are able to describe numerically the set being the result of this operation. It will have 2^n intervals having the length $\frac{1}{3^n}$ each. Obviously, the set obtained after $n+1$ iterations will be different and we also are able to measure the lengths of the intervals forming the second set. It will have 2^{n+1} intervals having the length $\frac{1}{3^{n+1}}$ each. The situation changes drastically in the limit because we are not able to distinguish results of n and $n+1$ steps of the construction if n is infinite.

We also are not able to distinguish at infinity the results of the following two processes that both use Cantor's construction but start from different positions. The first one is the usual Cantor's set and it starts from the interval $[0, 1]$, the second starts from the couple of intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. In spite of the fact that for any
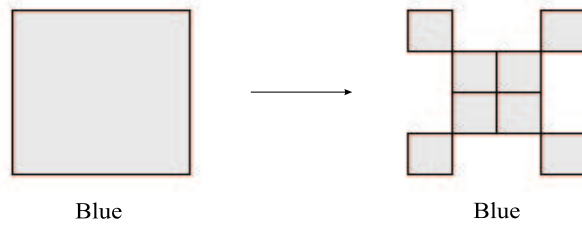
Figure 4: The first traditional fractal mechanism regarding blue squares that can be separated from the process shown in Fig. 3.
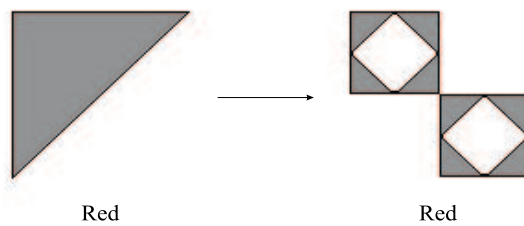

Figure 5: The second traditional fractal mechanism regarding red triangles that can be separated from the process shown in Fig. 3.

given finite number of steps, n, the results of the constructions will be different for these two processes we have no tools to distinguish them at infinity.

Let us now slightly change the process of construction used in Cantor's set to create a new example of a blinking fractal. At each odd iteration we shall maintain Cantor's rule, i.e., we remove the open interval being the middle third part from each of 2^n intervals present in the construction at the n-th iteration, where $n = 2k - 1$. In contrast, if $n = 2k$ from each interval present in the set corresponding to the n-th iteration we remove open intervals being the second and the last fourth parts (see Fig. 7). Again, as it was in the two previous examples, we have two different mechanisms working in this process and we are not able to say anything with respect to the structure of the resulting object at infinity. All the examples considered above have two different fractal mechanisms participating in their construction. Naturally, examples with more than two such mechanisms can be easily given.

To conclude this introduction we give the following general definition of objects that will be studied in this paper together with traditional fractals. Objects constructed using the principle of self-similarity with an infinite cyclic application of *several fractal rules* are called *blinking fractals*.

The rest of the paper is organized as follows. Section 2 introduces the new methodology and Section 3 describes a general framework allowing one to express by a finite number of symbols not only finite but infinite and infinitesimal numbers, too. In Section 4, we show how arithmetical operations with infinite, finite,
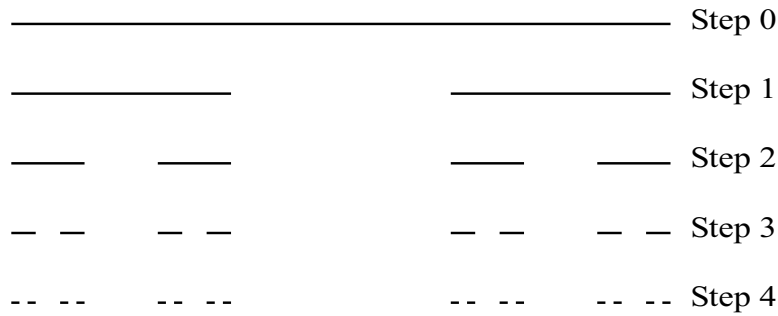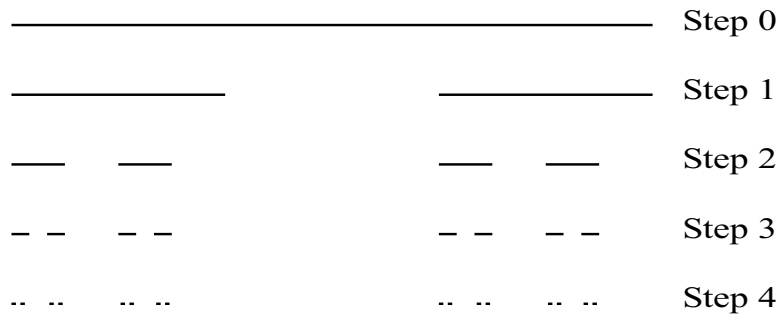
Figure 6: Cantor's set.

Figure 7: At each odd iteration we remove the open interval being the middle third part from each of the intervals present in the construction and at each even iteration from each interval present in the set we remove open intervals being the second and the last fourth parts.

and infinitesimal numbers are executed. Section 5 presents some preliminary results related to infinite sets, divergent series, and limits that will be used in the following sections. Section 6 describes the usage of the infinite and infinitesimal numbers for quantitative analysis of traditional and blinking fractals. Section 7 discusses the notions of continuity and discontinuity in physics and mathematics and introduces new, more physical than the traditional ones, mathematical instruments for describing continuity in mathematics. Finally, Section 8 concludes the paper.

2 Methodological platform

The point of view on infinity accepted nowadays takes its origins from the famous ideas of Georg Cantor who has shown that there exist infinite sets having different number of elements (see [2]). However, it is well known that Cantor's approach leads to some 'paradoxes'. The most famous and simple of them is, probably, Hilbert's paradox of the Grand Hotel. In a normal hotel having a finite number

of rooms no more new guests can be accommodated if it is full. Hilbert's Grand Hotel has an infinite number of rooms (of course, the number of rooms is countable, because the rooms in the Hotel are numbered). If a new guest arrives at the Hotel where every room is occupied, it is, nevertheless, possible to find a room for him. To do so, it is necessary to move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3, etc. In such a way room 1 will be available for the newcomer. Naturally, this paradox is a corollary of Cantor's fundamental result regarding cardinalities of infinite sets.

There exist different ways to generalize traditional arithmetic for finite numbers to the case of infinite and infinitesimal numbers (see [1, 2, 3, 17, 20]). However, arithmetics developed for infinite numbers are quite different with respect to the finite arithmetic we are used to deal with. Very often they leave undetermined many operations where infinite numbers take part (e.g., infinity minus infinity, infinity divided by infinity, sum of infinitely many items, etc.) or use representation of infinite numbers based on infinite sequences of finite numbers.

Usually, when mathematicians deal with infinite objects (sets or processes) it is supposed that human beings are able to execute certain operations infinitely many times. For example, in a fixed numeral system it is possible to write down a numeral with *any* number of digits[1]. However, this supposition is an abstraction (courageously declared by constructivists in [14]) because we live in a finite world and all human beings and/or computers finish operations they have started.

In this paper, we apply a recently developed approach (see [11, 21, 22, 23, 24, 25]) that does not use this abstraction and, therefore, is closer to the world of practical calculus than the traditional approaches. It is important to emphasize that the first simulator of the Infinity Computer able to work with infinite, finite, and infinitesimal numbers introduced in [22, 23, 24, 25] has been already realized (see [11, 21]).

In order to introduce the new methodology, let us consider a study published in *Science* by Peter Gordon (see [9]) where he describes a primitive tribe living in Amazonia - Pirahã - that uses a very simple numeral system for counting: one, two, many. For Pirahã, all quantities bigger than two are just 'many' and such operations as 2+2 and 2+1 give the same result, i.e., 'many'. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, 5, and 6, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that. Moreover, the weakness of their numeral system leads to such results as

$$\text{'many'} + 1 = \text{'many'}, \qquad \text{'many'} + 2 = \text{'many'},$$

which are very familiar to us in the context of views on infinity used in the tradi-

---

[1]We remind that *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '3', 'three', and 'III' are different numerals, but they all represent the same number.

tional calculus

$$\infty + 1 = \infty, \qquad \infty + 2 = \infty.$$

This observation leads us to the following idea: *Probably our difficulty in working with infinity is not connected to the nature of infinity but is a result of inadequate numeral systems used to express numbers.*

We start by introducing three postulates that will fix our methodological positions with respect to infinite and infinitesimal quantities and to mathematics, in general.

Postulate 1. *We postulate existence of infinite and infinitesimal objects but accept that human beings and machines are able to execute only a finite number of operations.*

Thus, we accept that we shall never be able to give a complete description of infinite processes and sets due to our finite capabilities. Particularly, this means that we accept that we are able to write down only a finite number of symbols to express numbers.

The second postulate that will be adopted is due to the following consideration. In natural sciences, researchers use tools to describe the object of their study and the used instrument influences results of observations. When physicists see a black dot in their microscope they cannot say: The object of observation *is* the black dot. They are obliged to say: the lens used in the microscope allows us to see the black dot and it is not possible to say anything more about the nature of the object of observation until we'll not change the instrument - the lens or the microscope itself - by a more precise one.

Due to Postulate 1, the same happens in mathematics studying natural phenomena, numbers, and objects that can be constructed by using numbers. Numeral systems used to express numbers are among the instruments of observations used by mathematicians. Usage of powerful numeral systems gives possibility to obtain more precise results in mathematics in the same way as usage of a good microscope gives a possibility to obtain more precise results in physics. However, the capabilities of the tools will be always limited due to Postulate 1. Thus, following natural sciences, we accept the second postulate.

Postulate 2. *We shall not tell **what are** the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.*

Particularly, this means that from our point of view, axiomatic systems do not define mathematical objects but just determine formal rules for operating with certain numerals reflecting some properties of the studied mathematical objects.

After all, we want to treat infinite and infinitesimal numbers in the same manner as we are used to deal with finite ones, i.e., by applying the philosophical principle of Ancient Greeks 'The part is less than the whole'. This principle, in our opinion, very well reflects organization of the world around us but is not incorporated in many traditional infinity theories where it is true only for finite numbers. Due to this postulate, the traditional point of view on infinity accepting such results as

8

$\infty + 1 = \infty$ should be substituted in such a way that $\infty + 1 > \infty$. Such a substitution has several motivations: one of them can be found in [23], another one has been introduced in connection with the numerals of Pirahã, and now we present one more reason.

Suppose that we are at a point *A* and at another point, *B*, being infinitely far from *A* there is an object. Then, if this object will change its position and will move, let say one meter farther, we shall not be able to register this movement in a quantitative way if we use the traditional rule $\infty + 1 = \infty$ to work with infinity. This rule allows us to say only that the object was infinitely far before the movement and remains to be infinitely far after the movement. In practice, due to this traditional rule, we are forced to negate local movements of objects if they are infinitely far from the observer. In order to avoid similar situations, we introduce the following postulate that, among other things, will allow us to register local movements of objects independently on their location with respect to the origin of the coordinate system.

Postulate 3. *We adopt the principle 'The part is less than the whole' to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).*

Due to this declared applied statement fixed by the three postulates introduced above, such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper because they belong to the theories working with different assumptions[2]. On dependence of the nature of each concrete problem, the user will make a decision which methodology (the traditional one or the new approach presented in this paper) better suits the problem under consideration and will choose the respective mathematical tools. To conclude this section, it is worthwhile to notice that the approach proposed here does not contradict Cantor. In contrast, it evolves his deep ideas regarding existence of different infinite numbers in a more applied way giving them a more quantitative character.

3   Theoretical background

Let us start our consideration by studying situations arising in practice when it is necessary to operate with extremely large quantities (see [22] for a detailed discussion). Imagine that we are in a granary and the owner asks us to count how much grain he has inside it. There are a few possibilities of finding an answer to this question. The first one is to count the grain seed by seed. Of course, nobody can do this because the number of seeds is enormous.

To overcome this difficulty, people take sacks, fill them in with seeds, and count the number of sacks. It is important that nobody counts the number of seeds in a sack. At the end of the counting procedure, we shall have a number of sacks completely filled and some remaining seeds that are not sufficient to complete the

---

[2]As a consequence, the approach used in this paper is different also with respect to the non-standard analysis introduced in [20] and built using Cantor's ideas.

next sack. At this moment it is possible to return to the seeds and to count the number of remaining seeds that have not been put in sacks (or a number of seeds that it is necessary to add to obtain the last completely full sack).

If the granary is huge and it becomes difficult to count the sacks, then trucks or even big train waggons are used. Of course, we suppose that all sacks contain the same number of seeds, all trucks – the same number of sacks, and all waggons – the same number of trucks. At the end of the counting we obtain a result in the following form: the granary contains 16 waggons, 13 trucks, 12 sacks, and 4 seeds of grain. Note, that if we add, for example, one seed to the granary, we can count it and see that the granary has more grain. If we take out one waggon, we again be able to say how much grain has been subtracted.

Thus, in our example it is necessary to count large quantities. They are finite but it is impossible to count them directly using elementary units of measure, u_0, i.e., seeds, because the quantities expressed in these units would be too large. Therefore, people are forced to behave as if the quantities were infinite.

To solve the problem of 'infinite' quantities, new units of measure, u_1, u_2, and u_3, are introduced (units u_1 – sacks, u_2 – trucks, and u_3 – waggons). The new units have the following important peculiarity: it is not known how many units u_i there are in the unit u_{i+1} (we do not count how many seeds are in a sack, we just *complete* the sack). Every unit u_{i+1} is filled in completely by the units u_i. Thus, we know that all the units u_{i+1} contain a certain number K_i of units u_i but this number, K_i, is unknown. Naturally, it is supposed that K_i is the same for all instances of the units. Thus, numbers that it was impossible to express using only initial units of measure are perfectly expressible if new units are introduced. This key idea of counting by introduction of new units of measure will be used in the paper to deal with infinite quantities.

Different numeral systems have been developed by humanity to describe finite numbers. More powerful numeral systems allow us to write down more numerals and, therefore, to express more numbers. In order to have a possibility to write down infinite and infinitesimal numbers by a finite number of symbols, we need at least one new numeral expressing an infinite (or an infinitesimal) number. Then, it is necessary to propose a new numeral system fixing rules for writing down infinite and infinitesimal numerals and to describe arithmetical operations with them.

Note that introduction of a new numeral for expressing infinite and infinitesimal numbers is similar to introduction of the concept of zero and the numeral '0' that in the past have allowed people to develop positional systems being more powerful than numeral systems existing before.

In positional numeral systems fractional numbers are expressed by the record

$$(a_n a_{n-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots a_{-(q-1)} a_{-q})_b \tag{1}$$

where numerals $a_i, -q \leq i \leq n$, are called *digits*, belong to the alphabet $\{0, 1, \ldots, b-1\}$, and the dot is used to separate the fractional part from the integer one. Thus,

the numeral (1) is equal to the sum

$$a_n b^n + a_{n-1} b^{n-1} + \ldots + a_1 b^1 + a_0 b^0 + a_{-1} b^{-1} + \ldots + a_{-(q-1)} b^{-(q-1)} + a_{-q} b^{-q}. \tag{2}$$

In modern computers, the radix $b = 2$ with the alphabet $\{0,1\}$ is mainly used to represent numbers. Numerous ways to represent and to store numbers in computers are described, for example, in [16].

Record (1) uses numerals consisting of one symbol each, i.e., digits $a_i \in \{0,1, \ldots, b-1\}$, to express how many finite units of the type b^i belong to the number (2). Quantities of finite units b^i are counted separately for each exponent i and all symbols in the alphabet $\{0,1,\ldots,b-1\}$ express finite numbers.

A new positional numeral system with infinite radix described in this section evolves the idea of separate count of units with different exponents used in traditional positional systems to the case of infinite and infinitesimal numbers. The infinite radix of the new system is introduced as the number of elements of the set $\mathbb{N}$ of natural numbers expressed by the numeral ① called *grossone*. This mathematical object is introduced by describing its properties postulated by the *Infinite Unit Axiom* consisting of three parts: Infinity, Identity, and Divisibility (we introduce them soon). This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integer numbers are introduced. This means that it is postulated that associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers.

Note that usage of a numeral indicating totality of the elements we deal with is not new in mathematics. It is sufficient to remind the theory of probability where events can be defined in two ways. First, as union of elementary events; second, as a sample space, Ω, of all possible elementary events from where some elementary events have been excluded. Naturally, the second way to define events becomes particularly useful when the sample space consists of infinitely many elementary events.

The Infinite Unit Axiom consists of the following three statements:

Infinity. For any finite natural number n it follows $n < $ ①.

Identity. The following relations link ① to identity elements 0 and 1

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^① = 1, \quad 0^① = 0. \tag{3}$$

Divisibility. For any finite natural number n sets $\mathbb{N}_{k,n}, 1 \le k \le n$, being the nth parts of the set, $\mathbb{N}$, of natural numbers have the same number of elements indicated by the numeral $\frac{①}{n}$ where

$$\mathbb{N}_{k,n} = \{k, k+n, k+2n, k+3n, \ldots\}, \quad 1 \le k \le n, \quad \bigcup_{k=1}^{n} \mathbb{N}_{k,n} = \mathbb{N}. \tag{4}$$

11

Divisibility is based on Postulate 3. Let us illustrate it by three examples. If we take $n = 1$, then $\mathbb{N}_{1,1} = \mathbb{N}$ and Divisibility tells that the set, $\mathbb{N}$, of natural numbers has ① elements. If $n = 2$, we have two sets $\mathbb{N}_{1,2}$ and $\mathbb{N}_{2,2}$ and they have $\frac{①}{2}$ elements each. If $n = 3$, then we have three sets $\mathbb{N}_{1,3}$, $\mathbb{N}_{2,3}$, and $\mathbb{N}_{3,3}$ and they have $\frac{①}{3}$ elements each.

$$
\begin{array}{ll}
① \rightarrow & \mathbb{N} = \quad \{1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad \ldots \} \\
\frac{①}{2} \nearrow & \mathbb{N}_{1,2} = \{1, \qquad 3, \qquad 5, \qquad 7, \quad \ldots \} \\
\phantom{\frac{①}{2}} \searrow & \mathbb{N}_{2,2} = \{ \quad 2, \qquad 4, \qquad 6, \qquad \ldots \} \\
& \mathbb{N}_{1,3} = \{1, \qquad\qquad 4, \qquad\qquad 7, \quad \ldots \} \\
\frac{①}{3} \nearrow & \\
\phantom{\frac{①}{3}} \rightarrow & \mathbb{N}_{2,3} = \{ \quad 2, \qquad\qquad 5, \qquad\qquad \ldots \} \\
\phantom{\frac{①}{3}} \searrow & \\
& \mathbb{N}_{3,3} = \{ \qquad\quad 3, \qquad\qquad 6, \qquad \ldots \}
\end{array}
$$

Before the introduction of the new positional system let us study some properties of grossone. First of all, as was already mentioned above, it is necessary to remind that ① is not either Cantor's $\aleph_0$ or ω that have been introduced in Cantor's theory on the basis of different assumptions. It will be shown hereinafter that grossone unifies both cardinal and ordinal aspects in the same way as finite numerals unify them. Its role in our infinite arithmetic is similar to the role of the number 1 in the finite arithmetic and it will serve us as the basis for construction of other infinite and infinitesimal numbers.

We start by the following important comment: to introduce $\frac{①}{n}$ we do not try to count elements $k, k+n, k+2n, k+3n, \ldots$ In fact, we cannot do this due to the accepted Postulate 1. In contrast, we apply Postulate 3 and state that the number of elements of the nth part of the set, i.e., $\frac{①}{n}$, is n times less than the number of elements of the whole set, i.e., than ①. In terms of our granary example ① can be interpreted as the number of seeds in the sack. Then, if the sack contains ① seeds, its nth part contains $\frac{①}{n}$ seeds. It is worthy to emphasize that, since the numbers $\frac{①}{n}$ have been introduced as numbers of elements of sets $\mathbb{N}_{k,n}$, they are integer.

The introduced numerals $\frac{①}{n}$ and the sets $\mathbb{N}_{k,n}$ allow us immediately to calculate the number of elements of certain infinite sets. For example, due to the introduced axiom, the sets

$$\mathbb{N}_{4,5} = \{4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64, 69, 74, 79, \ldots\}$$

$$\mathbb{N}_{3,11} = \{3, 14, 25, 36, 47, 58, 69, 80, 91, 102, 113, 124, 135, \ldots\},$$

have $\frac{①}{5}$ and $\frac{①}{11}$ elements, correspondingly.

The number of elements of sets being union, intersection, difference, or product of other sets of the type $\mathbb{N}_{k,n}$ is defined in the same way as these operations are defined for finite sets. Thus, we can define the number of elements of sets being

results of these operations with finite sets and infinite sets of the type $\mathbb{N}_{k,n}$. Let us consider three simple examples (a general rule for determining the number of elements of infinite sets having a more complex structure will be given in Section 5). First, we study intersection of the sets $\mathbb{N}_{4,5}$ and $\mathbb{N}_{3,11}$. It follows from the axiom that

$$\mathbb{N}_{4,5} \cap \mathbb{N}_{3,11} = \{14, 69, 124, \ldots\} = \mathbb{N}_{14,55}$$

and, therefore, it has $\frac{①}{55}$ elements. In the second example we consider the set $\mathbb{N}_{4,5} \cup \{2,3,4\}$. Its number of elements is $\frac{①}{5} + 2$ because $4 \in \mathbb{N}_{4,5}$.

It is worthwhile to notice that, as it is for finite sets, operations of union and intersection with finite sets and infinite sets of the type $\mathbb{N}_{k,n}$ enjoy commutative property. Thus, in our example we have

$$\mathbb{N}_{4,5} \cap \mathbb{N}_{3,11} = \mathbb{N}_{3,11} \cap \mathbb{N}_{4,5},$$

$$\mathbb{N}_{4,5} \cup \{2,3,4\} = \{2,3,4\} \cup \mathbb{N}_{4,5}$$

and $\frac{①}{5} + 2 = 2 + \frac{①}{5}$. In the last example we consider the set $\mathbb{N}_{2,5} \cup \{3,5\} \backslash \{2,7,17\}$. It has $\frac{①}{5} - 1$ elements because two elements have been added to and three have been excluded from the set $\mathbb{N}_{2,5} = \{2,7,12,17,\ldots\}$ having $\frac{①}{5}$ elements.

Other results regarding calculating the number of elements of infinite sets can be found in [22]. Particularly, it is shown that the number of elements of the set, $\mathbb{Z}$, of integers is equal to $2①1$ and the number of elements of the set, $\mathbb{Q}$, of different rational numerals is equal to $2①^2 1$. Then, Section 5 shows how to calculate the number of elements of infinite sets defined by formulae.

The new numeral $①$ allows us to write down the set, $\mathbb{N}$, of natural numbers in the form

$$\mathbb{N} = \{1, 2, 3, \quad \ldots \quad ① - 2, \ ① - 1, \ ①\} \tag{5}$$

because *grossone has been introduced as the number of elements of the set of natural numbers* (similarly, the number 3 is the number of elements of the set $\{1, 2, 3\}$). Thus, grossone is the biggest natural number and infinite numbers

$$\ldots \quad ① - 3, \ ① - 2, \ ① - 1 \tag{6}$$

less than grossone are also natural numbers as the numbers $1, 2, 3, \ldots$ They can be viewed both in terms of sets of numbers and in terms of grain. For example, $① - 1$ can be interpreted as the number of elements of the set $\mathbb{N}$ from which a number has been excluded. In terms of our granary example $① - 1$ can be interpreted as a sack minus one seed.

Note that the set (5) is the same set of natural numbers we are used to deal with. Infinite numbers (6) also take part of the usual set, $\mathbb{N}$, of natural numbers[3]. The difficulty to accept existence of infinite natural numbers is in the fact that traditional

---

[3]This point is one of the differences with respect to non-standard analysis (see [19, 20]) where infinite numbers are not included in $\mathbb{N}$.

numeral systems did not allow us to see them. In the same way as Pirahã are not able to see, for instance, numbers 4, 5, and 26 using their weak numeral system, traditional numeral systems did not allow us to see infinite natural numbers that we can see now using the new numeral ①.

We remind also that usage of a numeral indicating the infinite totality of the elements we deal with is not new in mathematics. In the same way as we use ① to indicate the number of all natural numbers, in the probability theory the axiomatic of Kolmogorov uses the symbol Ω to indicate the sample space of all possible elementary events. Then, the events can be described as union of elementary events or as Ω (or its parts) from where some elementary events have been excluded. Analogously, natural numbers can be described as union of finite units or as ① (or its parts) from where some finite units have been excluded.

Now an obvious question arises: Which natural numbers can we express by using the new numeral ①? Suppose that we have a numeral system $\mathcal{S}$ for expressing finite natural numbers and it allows us to express numbers belonging to a set $\mathcal{N}_{\mathcal{S}} \subset \mathbb{N}$. Then, adding ① to this numeral system will allow us to express also infinite natural numbers $\frac{i①}{n} \pm k \leq ①$ where $1 \leq i \leq n$, $k \in \mathcal{N}_{\mathcal{S}}$, $n \in \mathcal{N}_{\mathcal{S}}$ (note that since $\frac{①}{n}$ are integer, $\frac{i①}{n}$ are integer too). Thus, the more powerful system $\mathcal{S}$ is used to express finite numbers, the more infinite numbers can be expressed. This also means that the new numeral system using grossone allows us to express more numbers than traditional numeral systems thanks to the introduced new numerals but, as all numeral systems, it has a limited expressibility.

As an example, let us consider the numeral system, $\mathcal{P}$, of Pirahã able to express only numbers 1 and 2 (the only difference will be in the usage of the numerals '1' and '2' instead of original numerals I and II used by Pirahã). If we add to this system the new numeral ① it becomes possible to express the following numbers

$$\underbrace{1, 2}_{finite}, \quad \ldots \quad \underbrace{\frac{①}{2} - 2, \frac{①}{2} - 1, \frac{①}{2}, \frac{①}{2} + 1, \frac{①}{2} + 2}_{infinite}, \quad \ldots \quad \underbrace{① - 2, ① - 1, ①}_{infinite}.$$

In this record the first two numbers are finite, the remaining eight are infinite, and dots show the natural numbers that are not expressible in this numeral system. This numeral system does not allow us to execute such operation as $2 + 2$ or to add 2 to $\frac{①}{2} + 2$ because their results cannot be expressed in this system but, of course, we do not write that results of these operations are equal, we just say that the results are not expressible in $\mathcal{P}$ and it is necessary to take another, more powerful numeral system.

Note that we have similar crucial limitations working with sets. The numeral system $\mathcal{P}$ allows us to define only the sets $\mathbb{N}_{1,2}$ and $\mathbb{N}_{2,2}$ among all possible sets of the form $\mathbb{N}_{k,n}$ from (4) because we have only two finite numerals, '1' and '2', in $\mathcal{P}$. This numeral system is too weak to define other sets of this type. These limitations have a general deep character and are related to all problems requiring a numerical answer (i.e., an answer expressed only in numerals, without variables). In order to

obtain such an answer, it is necessary to know at least one numeral system able to express numerals required to write down this answer.

The introduction of grossone allows us to obtain the following interesting result: the set $\mathbb{N}$ is not a monoid under addition. In fact, the operation ① $+ 1$ gives us as the result a number grater than ①. Thus, by definition of grossone, ① $+ 1$ does not belong to $\mathbb{N}$ and, therefore, $\mathbb{N}$ is not closed under addition and is not a monoid. This result is a straightforward consequence of the accepted Postulate 3.

This result also means that adding the Infinite Unit Axiom to the axioms of natural numbers defines the set of *extended natural numbers* indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$ as a proper subset

$$\widehat{\mathbb{N}} = \{1, 2, \ldots, ① - 1, ①, ① + 1, \ldots, ①^2 - 1, ①^2, ①^2 + 1, \ldots\}.$$

Again, extended natural numbers grater than grossone can also be interpreted in the terms of sets of numbers. For example, ① $+ 3$ as the number of elements of the set $\mathbb{N} \cup \{a, b, c\}$ where numbers $a, b, c \notin \mathbb{N}$ and ①^2 as the number of elements of the set $\mathbb{N} \times \mathbb{N}$. In terms of our granary example ① $+ 3$ can be interpreted as one sack plus three seeds and ①^2 as a truck if we accept that the numbers K_i from page 10 are such that $K_1 = K_2 = ①$.

Extended natural numbers can be ordered as follows

$$1 < 2 < \ldots < ① - 1 < ① < ① + 1 < \ldots < ①^2 - 1 < ①^2 < ①^2 + 1 < \ldots$$

Let us show, for instance, that ① $< ①^2$. We can write the difference

$$①^2 - ① = ①(① - 1). \tag{7}$$

Due to Infinity property, ① is greater than any finite natural number, therefore, ① > 1 and as a consequence ① $- 1 > 0$. It follows from this inequality and (7) that the number ①$^2 - ①$ is a positive number and, therefore, ①$^2 > ①$.

The set, $\widehat{\mathbb{Z}}$, of *extended integer numbers* can be constructed from the set $\mathbb{Z}$ by a complete analogy and inverse elements with respect to addition are introduced naturally. For example, $6①$ has its inverse with respect to addition equal to $-6①$.

We have already started to write down simple infinite numbers and to execute arithmetical operations with them without concentrating our attention upon this question. Let us consider it systematically.

To express infinite and infinitesimal numbers we shall use records that are similar to (1) and (2) but have some peculiarities. In order to construct a number C in the new numeral positional system with base ① we subdivide C into groups corresponding to powers of ①:

$$C = c_{p_m}①^{p_m} + \ldots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \ldots + c_{p_{-k}}①^{p_{-k}}. \tag{8}$$

Then, the record

$$C = c_{p_m}①^{p_m} \ldots c_{p_1}①^{p_1} c_{p_0}①^{p_0} c_{p_{-1}}①^{p_{-1}} \ldots c_{p_{-k}}①^{p_{-k}} \tag{9}$$

represents the number C, where finite numbers c_i are called *infinite grossdigits* and can be both positive and negative; numbers p_i are called *grosspowers* and can be finite, infinite, and infinitesimal (the introduction of infinitesimal numbers will be given soon). The numbers p_i are such that $p_i > 0, p_0 = 0, p_{-i} < 0$ and

$$p_m > p_{m-1} > \ldots p_2 > p_1 > p_{-1} > p_{-2} > \ldots p_{-(k-1)} > p_{-k}.$$

In the traditional record (1) there exists a convention that a digit a_i shows how many powers b^i are present in the number and the radix b is not written explicitly. In the record (9) we write $①^{p_i}$ explicitly because in the new numeral positional system the number i in general is not equal to the grosspower p_i. This gives possibility to write, for example, such numbers as $7①^{244.5} 3①^{-32}$ where $p_1 = 244.5, p_{-1} = -32$.

Finite numbers in this new numeral system are represented by numerals having only one grosspower equal to zero. In fact, if we have a number C such that $m = k = 0$ in representation (9), then due to (3) we have $C = c_0①^0 = c_0$. Thus, the number C in this case does not contain infinite units and is equal to the grossdigit c_0 which being a conventional finite number can be expressed in the form (1), (2) by any positional system with finite base b (or by another numeral system). It is important to emphasize that the grossdigit c_0 can be integer or fractional and can be expressed by a few symbols in contrast to the traditional record (1) where each digit is integer and is represented by one symbol from the alphabet $\{0, 1, 2, \ldots, b-1\}$. Thus, the grossdigit c_0 shows how many finite units and/or parts of the finite unit, $1 = ①^0$, belong to the number C.

Analogously, in the general case, all grossdigits $c_i, -k \leq i \leq m$, can be integer or fractional and expressed by many symbols. For example, the number $\frac{7}{3}①^4 \frac{84}{19}①^{-3.1}$ has grossdigits $c_4 = \frac{7}{3}$ and $c_{-3.1} = \frac{84}{19}$. All grossdigits show how many corresponding units take part in the number C and it is not important whether this unit is finite or infinite.

Infinite numbers with finite grosspowers in this numeral system are expressed by numerals having at least one grosspower grater than zero. In the following example the left-hand expression presents the way to write down infinite numbers and the right-hand shows how the value of the number is calculated:

$$15①^{14}17.2045①^3 52.1①^{-6} = 15①^{14} + 17.2045①^3 + 52.1①^{-6}.$$

If a grossdigit c_{p_i} is equal to 1 then we write $①^{p_i}$ instead of $1①^{p_i}$. Analogously, if power $①^0$ is the lowest in a number then we often use simply the corresponding grossdigit c_0 without $①^0$, for instance, we write $23①^{14}5$ instead of $23①^{14}5①^0$ or 3 instead of $3①^0$. We also write sometimes $①^1$ simply as $①$.

Numerals with finite grosspowers having only negative grosspowers represent infinitesimal numbers. The simplest number from this group is $①^{-1} = \frac{1}{①}$ being the inverse element with respect to multiplication for $①$:

$$\frac{1}{①} \cdot ① = ① \cdot \frac{1}{①} = 1. \tag{10}$$

Note that all infinitesimals are not equal to zero. Particularly, $\frac{1}{①} > 0$ because $1 > 0$ and $① > 0$. It has a clear interpretation in our granary example. Namely, if we have a sack and it contains $①$ seeds then one sack divided by $①$ is equal to one seed. Vice versa, one seed, i.e., $\frac{1}{①}$, multiplied by the number of seeds in the sack, $①$, gives one sack of seeds.

Inverse elements of more complex numbers including grosspowers of $①$ are defined by a complete analogy. The following two numbers are examples of infinitesimals $3①^{-32}$, $37①^{-2}11①^{-15}$.

The above examples show how we can write down infinite numbers with all grossdigits being finite numbers. Let us see now how we can express a number including infinite grossdigits. The number

$$-14①^2(0.5①+3)①^1(①-4.5)①^{-1} \qquad (11)$$

has $m = 2$, $k = 1$, and the following grossdigits

$$c_2 = -14, \quad c_1 = 0.5①+3, \quad c_{-1} = ①-4.5,$$

where c_2 is finite and c_1, c_{-1} are infinite. The record (11) is correct but not very elegant because the system base $①$ appears in the expressions of grossdigits. In order to overcome this unpleasantness and to introduce a more simple structure of infinite numerals, we rewrite the number (11) in the explicit form (8)

$$-14①^2(0.5①+3)①^1(①-4.5)①^{-1} = -14①^2 + (0.5①+3)①^1 + (①-4.5)①^{-1}.$$

Then we open the parenthesis, collect the items having the same powers of $①$ (taking into account that $①①^{-1} = ①^0$), and finally obtain

$$-14①^2 + (0.5①+3)①^1 + (①-4.5)①^{-1} =$$
$$-14①^2 + 0.5①^2 + 3①^1 + ①①^{-1} - 4.5①^{-1} =$$
$$-13.5①^2 + 3①^1 + ①^0 - 4.5①^{-1} = -13.5①^23①^1①^0(-4.5)①^{-1}. \qquad (12)$$

As can be seen from the record (12), there are no infinite grossdigits in it but negative grossdigits have appeared. Since the record (11) using infinite grossdigits is more cumbersome, we introduce the notion of *finite grossdigit* as a finite number c_i expressed by a finite number of symbols in a numeral system and showing how many infinite units of the type $①^{k_i}$, $-k \le i \le m$, should be added or *subtracted* in order to compose infinite numbers. The record (12) using finite grossdigits is more flexible than the record (11) and will be mainly used hereinafter to express infinite and infinitesimal numbers.

4 Arithmetical operations for infinite, infinitesimal, and finite numbers

Let us now introduce arithmetical operations for infinite, infinitesimal, and finite numbers. The operation of *addition* of two given infinite numbers A and B returns

as the result an infinite number C constructed as follows (the operation of subtraction is a direct consequence of that of addition and is thus omitted). The numbers A, B, and their sum C are represented in the record of the type (12):

$$A = \sum_{i=1}^{K} a_{k_i} ①^{k_i}, \qquad B = \sum_{j=1}^{M} b_{m_j} ①^{m_j}, \qquad C = \sum_{i=1}^{L} c_{l_i} ①^{l_i}. \qquad (13)$$

Then the result C is constructed by including in it all items $a_{k_i} ①^{k_i}$ from A such that $k_i \neq m_j, 1 \leq j \leq M$, and all items $b_{m_j} ①^{m_j}$ from B such that $m_j \neq k_i, 1 \leq i \leq K$. If in A and B there are items such that $k_i = m_j$ for some i and j then this grosspower k_i is included in C with the grossdigit $b_{k_i} + a_{k_i}$, i.e., as $(b_{k_i} + a_{k_i}) ①^{k_i}$. It can be seen from this definition that the introduced operation enjoys the usual properties of commutativity and associativity due to definition of grossdigits and the fact that addition for each grosspower of ① is executed separately.

Let us illustrate the rules by an example (in order to simplify the presentation in all the following examples the radix $b = 10$ is used for writing down grossdigits). We consider two infinite numbers A and B where

$$A = 16.5①^{44.2}(-12)①^{12}17①^{0}1.17①^{-3},$$

$$B = 23①^{14}6.23①^{3}10.1①^{0}(-1.17)①^{-3}11①^{-43}.$$

Their sum C is calculated as follows

$$C = A + B = 16.5①^{44.2} + (-12)①^{12} + 17①^{0} + 1.17①^{-3} +$$

$$23①^{14} + 6.23①^{3} + 10.1①^{0} - 1.17①^{-3} + 11①^{-43} =$$

$$16.5①^{44.2} + 23①^{14} - 12①^{12} + 6.23①^{3} +$$

$$(17 + 10.1)①^{0} + (1.17 - 1.17)①^{-3} + 11①^{-43} =$$

$$16.5①^{44.2} + 23①^{14} - 12①^{12} + 6.23①^{3} + 27.1①^{0} + 11①^{-43} =$$

$$16.5①^{44.2}23①^{14}(-12)①^{12}6.23①^{3}27.1①^{0}11①^{-43}.$$

The operation of *multiplication* of two given infinite numbers A and B from (13) returns as the result the infinite number C constructed as follows.

$$C = \sum_{j=1}^{M} C_j, \quad C_j = b_{m_j} ①^{m_j} \cdot A = \sum_{i=1}^{K} a_{k_i} b_{m_j} ①^{k_i + m_j}, \quad 1 \leq j \leq M. \qquad (14)$$

Similarly to addition, the introduced multiplication is commutative and associative. It is easy to show that the distributive property is also valid for these operations.

Let us illustrate this operation by the following example. We consider two infinite numbers

$$A = ①^{18}(-5)①^{2}(-3)①^{1}0.2, \qquad B = ①^{2}(-1)①^{1}7①^{-3}$$

and calculate the product $C = B \cdot A$. The first partial product C_1 is equal to

$$C_1 = 7①^{-3} \cdot A = 7①^{-3}(①^{18} - 5①^2 - 3①^1 + 0.2) =$$

$$7①^{15} - 35①^{-1} - 21①^{-2} + 1.4①^{-3} = 7①^{15}(-35)①^{-1}(-21)①^{-2}1.4①^{-3}.$$

The other two partial products, C_2 and C_3, are computed analogously:

$$C_2 = -①^1 \cdot A = -①^1(①^{18} - 5①^2 - 3①^1 + 0.2) = -①^{19}5①^33①^2(-0.2)①^1,$$

$$C_3 = ①^2 \cdot A = ①^2(①^{18} - 5①^2 - 3①^1 + 0.2) = ①^{20}(-5)①^4(-3)①^30.2①^2.$$

Finally, by taking into account that grosspowers $①^3$ and $①^2$ belong to both C_2 and C_3 and, therefore, it is necessary to sum up the corresponding grossdigits, the product C is equal (due to its length, the number C is written in two lines) to

$$C = C_1 + C_2 + C_3 = \quad ①^{20}(-1)①^{19}7①^{15}(-5)①^42①^33.2①^2$$
$$(-0.2)①^1(-35)①^{-1}(-21)①^{-2}1.4①^{-3}.$$

In the operation of *division* of a given infinite number C by an infinite number B we obtain an infinite number A and a reminder R that can be also equal to zero, i.e., $C = A \cdot B + R$.

The number A is constructed as follows. The numbers B and C are represented in the form (13). The first grossdigit a_{k_K} and the corresponding maximal exponent k_K are established from the equalities

$$a_{k_K} = c_{l_L}/b_{m_M}, \quad k_K = l_L - m_M. \tag{15}$$

Then the first partial reminder R_1 is calculated as

$$R_1 = C - a_{k_K}①^{k_K} \cdot B. \tag{16}$$

If $R_1 \neq 0$ then the number C is substituted by R_1 and the process is repeated by a complete analogy. The grossdigit $a_{k_{K-i}}$, the corresponding grosspower k_{K-i} and the partial reminder R_{i+1} are computed by formulae (17) and (18) obtained from (15) and (16) as follows: l_L and c_{l_L} are substituted by the highest grosspower n_i and the corresponding grossdigit r_{n_i} of the partial reminder R_i that in its turn substitutes C:

$$a_{k_{K-i}} = r_{n_i}/b_{m_M}, \quad k_{K-i} = n_i - m_M. \tag{17}$$

$$R_{i+1} = R_i - a_{k_{K-i}}①^{k_{K-i}} \cdot B, \quad i \geq 1. \tag{18}$$

The process stops when a partial reminder equal to zero is found (this means that the final reminder $R = 0$) or when a required accuracy of the result is reached.

The operation of division will be illustrated by two examples. In the first example we divide the number $C = -10①^316①^042①^{-3}$ by the number $B = 5①^37$. For these numbers we have

$$l_L = 3, \quad m_M = 3, \quad c_{l_L} = -10, \quad b_{m_M} = 5.$$

It follows immediately from (15) that $a_{k_K}①^{k_K} = -2①^0$. The first partial reminder R_1 is calculated as

$$R_1 = -10①^3 16①^0 42①^{-3} - (-2①^0)\cdot 5①^3 7 =$$

$$-10①^3 16①^0 42①^{-3} + 10①^3 14①^0 = 30①^0 42①^{-3}.$$

By a complete analogy we should construct $a_{k_{K-1}}①^{k_{K-1}}$ by rewriting (15) for R_1. By doing so we obtain equalities

$$30 = a_{k_{K-1}}\cdot 5, \quad 0 = k_{K-1} + 3$$

and, as the result, $a_{k_{K-1}}①^{k_{K-1}} = 6①^{-3}$. The second partial reminder is

$$R_2 = R_1 - 6①^{-3}\cdot 5①^3 7 = 30①^0 42①^{-3} - 30①^0 42①^{-3} = 0.$$

Thus, we can conclude that the reminder $R = R_2 = 0$ and the final result of division is $A = -2①^0 6①^{-3}$.

Let us now substitute the grossdigit 42 by 40 in C and divide this new number $\widetilde{C} = -10①^3 16①^0 40①^{-3}$ by the same number $B = 5①^3 7$. This operation gives us the same result $\widetilde{A}_2 = A = -2①^0 6①^{-3}$ (where subscript 2 indicates that two partial reminders have been obtained) but with the reminder $\widetilde{R} = \widetilde{R}_2 = -2①^{-3}$. Thus, we obtain $\widetilde{C} = B\cdot\widetilde{A}_2 + \widetilde{R}_2$. If we want to continue the procedure of division, we obtain $\widetilde{A}_3 = -2①^0 6①^{-3}(-0.4)①^{-6}$ with the reminder $\widetilde{R}_3 = 0.28①^{-6}$. Naturally, it follows $\widetilde{C} = B\cdot\widetilde{A}_3 + \widetilde{R}_3$. The process continues until a partial reminder $\widetilde{R}_i = 0$ is found or when a required accuracy of the result will be reached.

In all the examples above we have used grosspowers being finite numbers. However, all the arithmetical operations work by a complete analogy also for grosspowers being themselves numbers of the type (9). For example, if

$$X = 16.5①^{44.2①1.17①^{-3}}(-12)①^{12①}1.17①^{-3},$$

$$Y = 23①^{44.2①1.17①^{-3}}(-1.17)①^{-3}11①^{4①^{-23}},$$

then their sum Z is calculated as follows

$$Z = X + Y = 39.5①^{44.2①1.17①^{-3}}(-12)①^{12①}11①^{4①^{-23}}.$$

5 Preliminary results

We start this section by showing how a number of elements of an infinite set can be determined in case when its elements are defined by a formula. We have already discussed in Section 3 how we can do this for sets being results of the usual operations (intersection, union, etc.) with finite sets and infinite sets of the type $\mathbb{N}_{k,n}$. In order to have a possibility to construct more complex infinite sets using these operations and to be able to determine the number of elements of the resulting

sets, let us consider infinite sets having a more general structure than the sets $\mathbb{N}_{k,n}$. Suppose that we have an integer function $f(i) > 0$ strictly increasing on indexes $i = 1, 2, 3, \ldots$ and we wish to know how many elements are there in the set

$$F = \{f(1), f(2), f(3), \ldots\}.$$

In our terminology this question has no any sense because due to Postulate 3 the set F is not defined completely. Let us explain what does this mean.

In the finite case, to define a set it is not sufficient to say that it is finite. It is necessary to indicate explicitly or implicitly its number of elements. For example,

$$F_1 = \{f(i) : 1 \leq i \leq 10\}$$

or

$$F_2 = \{f(i) : i \geq 1, \ f(i) \leq a\}$$

where a is finite.

Now we have mathematical tools to indicate the number of elements for infinite sets, too. Thus, analogously to the finite case and due to Postulate 3, to define a set it is not sufficient to say that the set has infinitely many elements. It is necessary to indicate its number of elements explicitly or implicitly. In the following example, the number of elements of the set

$$F_1 = \left\{f(i) : 1 \leq i \leq \frac{①^2}{2} + 1\right\}$$

is indicated explicitly. It has $\frac{①^2}{2} + 1$ elements. Analogously, the number of elements of the set, $\mathbb{N}$, of natural numbers has been indicated explicitly (see the Infinite Unit Axiom, Divisibility)

$$\mathbb{N} = \{i : 1 \leq i \leq ①\}$$

The number of elements of the set

$$F_2 = \{f(i) : i \geq 1, \ f(i) \leq b\} \tag{19}$$

where b is infinite, is defined implicitly (particularly, if $b = ①$ then the set $F \subseteq \mathbb{N}$ since all its elements are integer, positive, and $f(i) \leq ①$). In both cases, finite and infinite, it is necessary to have numeral systems allowing us to express numbers a and b.

If a set is given in the form (19), then its number of elements J can be determined as

$$J = \max\{i : f(i) \leq b\}.$$

If we are able to determine the inverse function $f^{-1}(x)$ for $f(x)$ then $J = [f^{-1}(b)]$ where $[u]$ is integer part of u.

Let us consider as examples two subsets of $\mathbb{N}$ depending on finite and integer parameters k and n. The first set, F_1, has $f(i) = k + n(i-1)$. Then

$$F_1 = \{f(i) : i \geq 1, \ f(i) \leq ①\} = \mathbb{N}_{k,n}, \qquad 1 \leq k \leq n,$$

where $\mathbb{N}_{k,n}$ is from (4). It has J_1 elements where

$$J_1 = [\frac{①-k}{n} + 1] = [\frac{①-k}{n}] + 1 = \frac{①}{n} - 1 + 1 = \frac{①}{n}.$$

Analogously, the second set

$$F_2 = \{k + ni^3 : i \geq 1, \ k + ni^3 \leq ①\}$$

has $J_2 = [\sqrt[3]{\frac{①-k}{n}}]$ elements.

What can we say now about the number of elements of the sets $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$? Our positional numeral system with the radix ① dose not allow us to say anything because it does not contain numerals able to express such numbers. It is necessary to introduce in a way a more powerful numeral system defining new numerals ②, ③, etc. However, we can work with those subsets of $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{Z}}$ that can be defined by using numerals written down in our positional numeral system with the radix ①.

In order to have a possibility to discuss such important constructions as recursively defined infinite sets we need first to consider infinite sequences from the point of view of the new approach.

We start by proving the following important result: the number of elements of any infinite sequence is less or equal to ①. To demonstrate this we need to recall the definition of the infinite sequence: 'An infinite sequence $\{a_n\}, a_n \in A$ for all $n \in \mathbb{N}$, is a function having as the domain the set of natural numbers, $\mathbb{N}$, and as the codomain a set A'.

We have postulated in the Infinite Unit Axiom that the set $\mathbb{N}$ has ① elements. Thus, due to the sequence definition given above, any sequence having $\mathbb{N}$ as the domain has ① elements.

One of the immediate consequences of the understanding of this result is that any process can have at maximum ① elements. For example, if we consider the set, $\widehat{\mathbb{N}}$, of extended natural numbers then starting from the number 1, it is possible to arrive at maximum to ①

$$\underbrace{1, 2, 3, 4, \ \dots \ ①-2, \ ①-1, ①,}_{①} ①+1, ①+2, ①+3, \dots \qquad (20)$$

Starting from 2 it is possible to arrive at maximum to $①+1$

$$1, \underbrace{2, 3, 4, \ \dots \ ①-2, \ ①-1, ①, ①+1,}_{①} ①+2, ①+3, \dots \qquad (21)$$

Starting from 3 it is possible to to arrive at maximum to $①+2$

$$1,2,\underbrace{3,4, \; \dots \; ①-2, \; ①-1,①,①+1,①+2}_{①},①+3,\dots \qquad (22)$$

Of course, since we have postulated that our possibilities to express numerals are finite, it depends on the chosen numeral system which numbers among ① members of these processes we can observe. It is also very important to notice a deep relation of this observation to the Axiom of Choice. The Infinite Unit Axiom postulates that any process can have at maximum ① elements, thus the process of choice too and, as a consequence, it is not possible to choose more than ① elements from a set. This observation also emphasizes the fact that the parallel computational paradigm is significantly different with respect to the sequential one because p parallel processes can choose $p①$ elements from a set.

Traditionally, the notion of subsequence is introduced as a sequence from which some of its elements have been cancelled. Thus, this definition gives infinite sequences having the number of members less than grossone.

It is appropriate now to define the *complete sequence* as an infinite sequence containing ① elements. For example, the sequence $\{n\}$ of natural numbers is complete, the sequences of even and odd natural numbers are not complete.

Similarly to infinite sets, the Infinite Unit Axiom imposes a more precise description of infinite sequences. To define a sequence $\{a_n\}$ it is not sufficient just to give a formula for a_n. It is necessary also to indicate explicitly how many elements the sequence has. For example, let us consider the following three sequences, $\{a_n\},\{b_n\}$, and $\{c_n\}$:

$$\{a_n\} = \{2, \quad 4, \quad \dots \quad 2(①-1), \quad 2①\};$$

$$\{b_n\} = \{2, \quad 4, \quad \dots \quad 2(\frac{①}{2}-2), \quad 2(\frac{①}{2}-1)\}; \qquad (23)$$

$$\{c_n\} = \{2, \quad 4, \quad \dots \quad 2(\frac{2①}{3}-1), \quad 2(\frac{2①}{3})\}. \qquad (24)$$

They have the same general element equal to $2n$ but are different because they have different number of members. The first sequence has ① elements and is thus complete, the other two sequences are not complete. The second sequence $\{b_n\}$ has $\frac{①}{2}-1$ elements and the third sequence $\{c_n\}$ has $2\frac{①}{3}$ members. Thus, to describe a sequence we should use the record $\{a_n : k\}$ where a_n is, as usual, the general element and k is the number (finite or infinite) of members of the sequence. Note also that among these three sequences only $\{b_n\}$ is a subsequence of the sequence of even natural numbers because its last element has the number $\frac{①}{2}-1 \leq \frac{①}{2}$. Since grossone is the last even natural number, elements of $\{a_n\}$ and $\{c_n\}$ having $n > \frac{①}{2}$ are not natural but extended natural numbers.

In connection to this definition the following natural question arises inevitably. Suppose that we have two sequences, for example, $\{b_n : \frac{①}{2} - 1\}$ and $\{c_n : 2\frac{①}{3}\}$

23

from (23) and (24). Can we create a new sequence, $\{d_n : k\}$, composed from both of them, for instance, as it is shown below

$$b_1, \ b_2, \ \ldots \ b_{\frac{①}{2}-2}, \ b_{\frac{①}{2}-1}, \ c_1, \ c_2, \ \ldots \ c_{2\frac{①}{3}-1}, c_{2\frac{①}{3}}$$

and which will be the value of the number of its elements k?

The answer is 'no' because due to the definition of the infinite sequence, a sequence can be at maximum complete, i.e., it cannot have more than grossone elements. Starting from the element b_1 we can arrive at maximum to the element $c_{\frac{①}{2}+1}$ being the element number ① in the sequence we try to construct

$$\underbrace{b_1, \ \ldots \ b_{\frac{①}{2}-1}, \ c_1, \ \ldots c_{\frac{①}{2}+1}}_{①}, \ \underbrace{c_{\frac{①}{2}+2}, \ldots \ c_{2\frac{①}{3}}}_{\frac{①}{6}-1}. \qquad (25)$$

The remaining members of the sequence $\{c_n : 2\frac{①}{3}\}$ will form the second sequence, $\{g_n : l\}$ having $l = 2\frac{①}{3} - (\frac{①}{2}+1) = \frac{①}{6} - 1$ elements.

Thus, we have formed two sequences, the first of them, $\{d_n : ①\}$, is complete and the second, $\{g_n : \frac{①}{6} - 1\}$, is not, where

$$
\begin{aligned}
d_i &= b_i, \quad 1 \le i \le \tfrac{①}{2} - 1, \\
d_i &= c_j, \quad \tfrac{①}{2} \le i \le ①, \quad 1 \le j \le \tfrac{①}{2} + 1, \\
g_i &= c_j, \quad 1 \le i \le \tfrac{①}{6} - 1, \quad \tfrac{①}{2} + 2 \le j \le 2\tfrac{①}{3}.
\end{aligned}
$$

The given consideration of the infinite sequences allows us to deal with recursively defined sets. Since these sets are constructed sequentially by a process, they can have at maximum ① elements. Again, the number of elements of the set can be defined explicitly or implicitly as it was for the sets with formulae explicitly given to calculate elements of the set.

Let us return to Hilbert's paradox of the Grand Hotel presented in Section 2. In the paradox, the number of the rooms in the Hotel is countable. In our terminology this means that it has ① rooms. When a new guest arrives, Hilbert proposes to move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3, etc. Under the Infinite Unit Axiom this procedure is not possible because the guest from room ① should be moved to room ①+1 and the Hotel has only ① rooms. Thus, when the Hotel is full, no more new guests can be accommodated – the result corresponding perfectly to Postulate 3 and the situation taking place in normal hotels with a finite number of rooms.

Let us give some examples from such an important area as theory of divergent series. We consider two infinite series $S_1 = 1+1+1+\ldots$ and $S_2 = 3+3+3+\ldots$ The traditional analysis gives us a very poor answer that both of them diverge to infinity. Such operations as $S_1 - S_2$ or $\frac{S_1}{S_2}$ are not defined.

In our terminology, we are able to express not only different finite numbers but also different infinite numbers. Thus, the records S_1 and S_2 are not well defined.

It is necessary to indicate explicitly the number of items in the sum and it is not important is it finite or infinite. To calculate the sum it is necessary that the number of items and the result are expressible in the numeral system used for calculations. It is important to notice that even though a sequence cannot have more than ① elements the number of items in a series can be greater than grossone because the process of summing up is not necessary should be executed by a sequential adding items.

Suppose that the series S_1 has k items and S_2 has n items:

$$S_1(k) = \underbrace{1+1+1+\ldots+1}_{k}, \qquad S_2(n) = \underbrace{3+3+3+\ldots+3}_{n}.$$

Then $S_1(k) = k$ and $S_2(n) = 3n$ and by giving numerical values to k and n we obtain numerical values for the sums. If, for instance, $k = n = 5①$ then we obtain $S_1(5①) = 5①$, $S_2(5①) = 15①$ and

$$S_2(5①) - S_1(5①) = 10① > 0.$$

If $k = 5①$ and $n = ①$ we obtain $S_1(5①) = 5①$, $S_2(①) = 3①$ and it follows

$$S_2(①) - S_1(5①) = -2① < 0.$$

If $k = 3①$ and $n = ①$ we obtain $S_1(3①) = 3①$, $S_2(①) = 3①$ and it follows

$$S_2(①) - S_1(3①) = 0.$$

Analogously, the expression $\frac{S_1(k)}{S_2(n)}$ can be calculated.

The infinite and infinitesimal numbers allow us to calculate also arithmetic and geometric series with an infinite number of items. Traditional approaches tell us that if $a_n = a_1 + (n-1)d$ then for a finite n it is possible to use the formula

$$\sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n).$$

Due to Postulate 3, we can use it also for infinite n. For example, the sum of all natural numbers from 1 to ① is calculated as follows

$$1+2+3+\ldots+(①-1)+① = \sum_{i=1}^{①} i = \frac{①}{2}(1+①) = 0.5①^2 0.5①.$$

Let us consider now the geometric series $\sum_{i=0}^{\infty} \frac{1}{q^i}$. Traditional analysis proves that it converges to $\frac{1}{1-q}$ for q such that $-1 < q < 1$. We are able to give a more precise answer for *all* values of q. To do this we should fix the number of items in the sum. If we suppose that it contains n items then

$$Q_n = \sum_{i=0}^{n} q^i = 1+q+q^2+\ldots+q^n. \tag{26}$$

25

By multiplying the left hand and the right hand parts of this equality by q and by subtracting the result from (26) we obtain

$$Q_n - qQ_n = 1 - q^{n+1}$$

and, as a consequence, for all $q \neq 1$ the formula

$$Q_n = \frac{1 - q^{n+1}}{1 - q} \tag{27}$$

holds for finite and infinite n. Thus, the possibility to express infinite and infinitesimal numbers allows us to take into account infinite n too and the value q^{n+1} being infinitesimal for a finite q. Moreover, we can calculate Q_n for $q = 1$ also because in this case we have just

$$Q_n = \underbrace{1 + 1 + 1 + \ldots + 1}_{n+1} = n + 1.$$

As the first example we consider the divergent series

$$1 + 2 + 4 + \ldots = \sum_{i=0}^{\infty} 2^i.$$

To fix it we should decide the number of items, n, at the sum and, for example, for $n = ①^2$ we obtain

$$\sum_{i=0}^{①^2} 2^i = 1 + 2 + 4 + \ldots + 2^{①^2} = \frac{1 - 2^{①^2+1}}{1 - 2} = 2^{①^2+1} - 1.$$

Analogously, for $n = ①^2 + 1$ we obtain

$$1 + 2 + 4 + \ldots + 2^{①^2} + 2^{①^2+1} = 2^{①^2+2} - 1.$$

If we now find the difference between the two sums, we obtain the newly added item $2^{①^2+1}$:

$$2^{①^2+2} - 1 - (2^{①^2+1} - 1) = 2^{①^2+1}(2^1 - 2^0) = 2^{①^2+1}.$$

In the second example we take the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ used in Zeno's Dichotomy paradox. It is known that it converges to one. However, we are able to give a more precise answer. In fact, due to Postulate 3, the formula

$$\sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right) = \frac{1}{2} \cdot \frac{1 - \frac{1}{2}^n}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

can be used directly for infinite n, too. For example, if $n = ①$ then

$$\sum_{i=1}^{①} \frac{1}{2^i} = 1 - \frac{1}{2^①}$$

26

where $\frac{1}{2^①}$ is infinitesimal. Thus, the traditional answer $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ was just a finite approximation to our more precise result using infinitesimals.

Let us consider now the famous divergent series with alternate signs

$$S_3 = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

In literature there exist many approaches giving different answers regarding the value of this series (see [15]). All of them use various notions of average. However, the notions of sum and average are different. In our approach we do not appeal to average and calculate the required sum directly. To do this we should indicate explicitly the number of items, k, in the sum. Then

$$S_3(k) = \underbrace{1 - 1 + 1 - 1 + 1 - 1 + 1 - \ldots}_{k} = \begin{cases} 0, & \text{if } k = 2n, \\ 1, & \text{if } k = 2n + 1, \end{cases}$$

and it is not important is k finite or infinite. For example, $S_3(①) = 0$ because the number $\frac{①}{2}$ being the result of division of ① by 2 has been introduced as the number of elements of a set and, therefore, it is integer. As a consequence, ① is even number. Analogously, $S_3(① - 1) = 1$ because $① - 1$ is odd.

Let us now discuss the limit theory from the point of view of our approach. The concept of limit has been introduced to overcome the difficulties arising when we start to work with the notions of infinite and infinitesimal. In traditional analysis, if a limit $\lim_{x \to a} f(x)$ exists, then it gives us a very poor – just one value – information about the behavior of $f(x)$ when x tends to a. Now we can obtain significantly more rich information because we are able to calculate $f(x)$ directly at any finite, infinite, or infinitesimal point that can be expressed by the new positional system even if the limit does not exist (e.g., we can study divergent processes at various points of infinity).

Thus, limits $\lim_{x \to \infty} f(x)$ equal to infinity can be substituted by precise infinite numerals that are different for different infinite values of x. Naturally, if we speak about limits of sequences, $\lim_{n \to \infty} a(n)$, then $n \in \mathbb{N}$ and, as a consequence, it follows that n should be less or equal to grossone. For instance, the following two limits both give us $+\infty$ as the result

$$\lim_{x \to +\infty} (x^4 + 3x^2) = +\infty,$$

$$\lim_{x \to +\infty} (x^4 + 3x^2 + 1) = +\infty.$$

However, for any finite x it follows that

$$x^4 + 3x^2 < x^4 + 3x^2 + 1.$$

Now this inequality holds for any finite or infinite x. The new positional system with infinite radix allows us to calculate exact values of both expressions, $x^4 + 3x^2$

and $x^4 + 3x^2 + 1$, at any infinite x expressible in this system. For example, if we choose $x = ①$ we obtain numbers $①^4 3①^2$ and $①^4 3①^2 1$. The choice $x = 6①^2$ gives

$$(6①^2)^4 + 3(6①^2)^2 = 1296①^8 108①^4$$

and $1296①^8 108①^4 1$. In both examples both numbers are infinite (in this sense we have the same result with respect to traditional approaches) but our results give a significantly more rich information because we have precise infinite numbers and are able to execute further arithmetical operations with them if it is necessary. For example,

$$1296①^8 108①^4 1 - 1296①^8 108①^4 = 1.$$

It is very important that expressions can be calculated at different infinite points even when limits of these expressions do not exist. For example, the following limit

$$\lim_{n \to +\infty} (-1)^n n$$

does not exist. However, we can calculate expression $(-1)^n n$ at infinite points n: for $n = ①$ it follows $①$ and for $n = ① - 1$ it follows $-① + 1$. Thus, we obtain a very powerful tool for studying divergent processes.

Traditional finite limits in our terminology become just approximations of complete results of expressions having finite and infinitesimal parts to the finite case. For instance, if an expression has a finite limit a and the obtained complete result is expressed in the new positional system as follows

$$a①^0 a_{-1}①^{-1} a_{-2}①^{-2} \ldots$$

then we obtain the traditional limit as the grossdigit corresponding to $①^0$.

Traditional limits with the argument tending to a finite number or zero can be considered analogously. In this case we can calculate the respective expression at any infinitesimal point using the new positional system and to obtain a significantly more reach information again. For example, if x is a fixed finite number then

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = 3x^2. \tag{28}$$

In the new positional system we obtain

$$\frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2. \tag{29}$$

If, for instance, the number $h = ①^{-1}$, the answer is $3x^2①^0 3x①^{-1}①^{-2}$, if $h = ①^{-2}$ we obtain $3x^2①^0 3x①^{-2}①^{-4}$, etc. Thus, the value of limit (28) for a finite x is the finite approximation of the number (29) having finite and infinitesimal parts. If we need only this finite approximation for our eventual further calculations, then we can have it both from (28) and (29). However, if we need an infinitesimal accuracy, only (29) can give it.

Now, when we know how to calculate the number of elements of infinite sets and are able to summing up infinite sums let us apply the developed tools to traditional and blinking fractals.

6 Quantitative analysis of traditional and blinking fractals

We start this section with a few results giving answers to numerical questions stated in Introduction and regarding traditional fractals. Starting from Cantor's set we show how lengths of traditional fractals can be calculated at infinity.

We remind that if a finite number of steps, n, has been executed in Cantor's construction starting from the interval $[0,1]$ then we are able to describe numerically the set being the result of this operation. It will have 2^n intervals having the length $\frac{1}{3^n}$ each. Obviously, the set obtained after $n+1$ iterations will be different and we also are able to measure the lengths of the intervals forming the second set. It will have 2^{n+1} intervals having the length $\frac{1}{3^{n+1}}$ each. The situation changes drastically in the limit because traditional approaches are not able to distinguish results of n and $n+1$ steps of the construction if n is infinite. Now, we can do it using the introduced infinite and infinitesimal numbers.

Since the construction of Cantor's set is a process, it cannot contain more then ① steps (see discussion related to the example (20)-(22)). Thus, if we start the process from the interval $[0,1]$, after ① steps Cantor's set consists of $2^{①}$ intervals and their total length, $L(n)$, is expressed in infinitesimals: $L(①) = \left(\frac{2}{3}\right)^{①}$, i.e., the set has a well defined infinite number of intervals and each of them has the infinitesimal length equal to $3^{-①}$. Analogously, after $①-1$ steps Cantor's set consists of $2^{①-1}$ intervals and their total length is expressed in infinitesimals: $L(①) = \left(\frac{2}{3}\right)^{①-1}$. Thus, the length $L(n)$ for any (finite or infinite) number of steps, n, where $1 \le n \le ①$ and is expressible in the chosen numeral system can be calculated.

It is important to notice here that (again due to the limitation illustrated by the example (20)-(22)) it is not possible to count one by one all the intervals at Cantor's set if their number is superior to ①. For instance, after ① steps it has $2^{①}$ intervals and they cannot be counted one by one because $2^{①} > ①$ and any process (including that of the sequential counting) cannot have more that ① steps.

It becomes possible to study by a complete analogy other classical fractals. For instance, we immediately obtain that the length of the Koch Curve starting from the interval $[0,1]$ after ① steps has the infinite length equal to $\left(\frac{4}{3}\right)^{①}$ because it consists of $4^{①}$ segments having the length $\left(\frac{1}{3}\right)^{①}$ each. In the same way we can calculate the area of the Sierpinski Carpet. If its construction starts from the unit square then after ① steps we obtain the set of squares having the total infinitesimal area equal to $\left(\frac{8}{9}\right)^{①}$ because it consists of $8^{①}$ squares and each of them has area equal to $\left(\frac{1}{9}\right)^{①}$.

Let us consider now two processes that both use Cantor's construction but start from different initial conditions. Traditional approaches do not allow us to distinguish them at infinity in spite of the fact that for any given finite number of steps, n, the results of the constructions are different and can be calculated. Using the new approach we are able to study the processes numerically also at infinity. For

example, if the first process is the usual Cantor's set and it starts from the interval $[0,1]$ and the second one starts from the couple of intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$ then after $\frac{①}{2}$ steps the result of the first process will be the set consisting of $2^{\frac{①}{2}}$ intervals and its length $L(\frac{①}{2}) = (\frac{2}{3})^{\frac{①}{2}}$. The second set after $\frac{①}{2}$ steps will consists of $2^{\frac{①}{2}+1}$ intervals and its length $L(\frac{①}{2}+1) = (\frac{2}{3})^{\frac{①}{2}+1}$.

Let us now consider the following traditional problem: How many points are there at Cantor's set? From our new point of view this formulation is not sufficiently precise. Now, when it becomes possible to distinguish different sets at different iterations we should say: How many points there are at Cantor's set being the result of *n* steps of Cantor's procedure started from the initial set consisting of *k* intervals? In the following without loss of generality we consider the case $k = 1$.

We start our exposition by calculating the number of points of the interval $[0,1)$. To do this we need a definition of the term 'point' and mathematical tools to indicate a point. Since this concept is one of the most fundamental, it is very difficult to find an adequate definition. If we accept (as is usually done in modern mathematics) that the *point x* in an *N*-dimensional space is determined by *N* numerals called *coordinates of the point*

$$(x_1, x_2, \ldots x_{N-1}, x_N) \in \mathbb{T}^N,$$

where $\mathbb{T}^N$ is a set of numerals, then we can indicated the point *x* by its coordinates and are able to execute required calculations.

It is important to emphasize that we have not postulated that $(x_1, x_2, \ldots x_N)$ belongs to the *N*-dimensional set, $\mathbb{R}^N$, of real numbers as it is usually done. Since we can express coordinates only by numerals then different choices of numeral systems lead to various sets of numerals and, as a consequence, to different sets of points we can refer to. The choice of a numeral system will define what is the *point* for us and we'll not be able to work with points coordinates of which are not expressible in the chosen numeral system. Thus, we are able to calculate the number of points if we have decided which numerals will be used to express the coordinates of points.

Therefore, in order to start we should decide which numerals we shall use to express coordinates of the points. Infinitely many variants can be chosen dependent on the precision level we want to obtain. For example, if the numbers $0 \le x < 1$ are expressed in the form $\frac{p-1}{①}, p \in \mathbb{N}$, then the smallest positive number we can distinguish is $\frac{1}{①}$. Therefore, the interval $[0,1)$ contains ① following points

$$0, \quad \frac{1}{①}, \quad \frac{2}{①}, \quad \ldots \quad \frac{①-2}{①}, \quad \frac{①-1}{①}.$$

The interval $[0,1]$ contains one point more than the interval $[0,1)$ and, therefore, in the chosen numeral system $①+1$ points can be distinguished within $[0,1]$.

If we need a higher precision, within each interval

$$[\frac{i-1}{①}, \frac{i}{①}), \quad i \in \mathbb{N},$$

we can distinguish again ① points and the number of points within the interval $[0, 1)$ will become equal to $①^2$.

In some sense, the situation with counting points is similar to the work with a microscope: we decide the level of the precision we need and obtain a result dependent on the chosen level. If we need a more precise or a more rough answer, we change the level of accuracy of our microscope. In general, this situation is typical for natural sciences where it is well known that instruments influence results of observations.

By continuation of analogy with the microscope, we can also decide to change our microscope with a new one. In our terms this means to change one numeral system with another. For example, instead of the numerals considered above we choose a positional numeral system with a finite radix b to express coordinates of the points within the interval $[0, 1)$, i.e.,

$$(.a_{-1}a_{-2}\ldots a_{-(①-1)}a_{-①})_b. \tag{30}$$

Thus, we have ① positions and each of them can be filled in by one of the b digits from the alphabet $\{0, 1, \ldots, b-1\}$. Thus, we have $b^①$ combinations and, as a result, we are able to distinguish $b^①$ points of the form (30) within the interval $[0, 1)$.

It is worthwhile to notice in this occasion that, the traditional point of view on real numbers says that there exist real numbers that can be represented in positional systems by two infinite sequences of digits. For example, in the decimal system the number 1 can be expressed as $1.00000000\ldots$ or as $0.9999999999\ldots$ In contrast, under the Infinite Unit Axiom all the numerals in the positional system with a finite radix represent different numbers. There exists the smallest positive number that can be expressed in the numeral system with the base b. It contains ① digits after the dot

$$(0.\underbrace{000\ldots01}_{①})_b.$$

For instance, in the decimal positional system the numerals

$$0.\underbrace{999\ldots99}_{①}, \quad 1.\underbrace{000\ldots00}_{①}$$

are different and their difference is equal to

$$1.\underbrace{000\ldots00}_{①} - 0.\underbrace{999\ldots9}_{①} = 0.\underbrace{000\ldots01}_{①}.$$

It is obligatory to say in this occasion that the results presented above should be considered as a more precise analysis of the situation discovered by genius of Cantor. He has proved using his famous diagonal argument that the number of elements of the set $\mathbb{N}$ is less than the number of real numbers at the interval $[0, 1)$ *without calculating the latter*. To do this he expressed real numbers in a positional numeral system. We have shown that this number will be different in dependence

on the radix b used in the positional system to express real numbers. However, all of the obtained numbers are superior to the number of elements of the set of natural numbers and, therefore, the diagonal argument maintains its force.

Let us now return back to Cantor's set and to calculations of the number of the points in the set C_n being the result of n steps of Cantor's procedure starting from the interval $[0, 1]$. As it is seen from the analysis made above, we shall be able to do such calculations only if the numeral system chosen to express coordinates of the points will be powerful enough to distinguish the points within the intervals. Moreover, we shall be able to distinguish within C_n no more points than our chosen numeral system will allow us. For instance, if we give to a person from our primitive Pirahã tribe the set C_2 consisting of four intervals, this person operating with his poor numeral system consisting of the numerals *I*, *II*, and 'many' will not be able to say us how many intervals there are in this set and which are coordinates of, for example, their end points. This happens because his system is too poor both for counting the intervals and for expressing coordinates of their end points. His answer will be just 'many' for the number of intervals and he will be able to indicate the coordinate of only one point – 1. However, if we give him the set C_0 his answer will be correct for the intervals – there is one interval – and he will be able to indicate the coordinate of the same point – 1.

Thus, the situation with counting points in Cantor's set again is similar to the work with a microscope: we decide the level of the precision we need and obtain a result dependent on the chosen level. If we need a more precise or a more rough answer, we change the level of accuracy of our microscope. If we need a high precision and need to distinguish many points, we should take a powerful numeral system to express the coordinates. In the case when we need a low precision, a weak numeral system can be taken.

The introduced mathematical tools allow us to give answers to similar questions not only for traditional but for blinking fractals, too. We start by considering the blinking fractal described in Figs. 1– 5. Since the answers depend on the initial conditions, we suppose without loss of generality that the process starts from the blue square one unit of length on side. This means that during any (finite or infinite) even iteration we observe blue squares and during any odd iteration we see red triangles. We shall indicate the set obtained after n iterations by P_n. The area A_n of the set P_n is calculated as follows. For any (finite or infinite) $n = 2k, k \geq 0$, it consists of 2^{3k} squares with the side equal to 2^{-2k}. Thus, the area of P_n is

$$A_{2k} = (2^{-2k})^2 \cdot 2^{3k} = 2^{-k}.$$

For $n = 2k - 1, k \geq 1$, the set P_n consists of 2^{3k-1} right isosceles triangles with the legs equal to 2^{-2k+1}. In this case the area of P_n is calculated as follows

$$A_{2k-1} = 0.5(2^{-2k+1})^2 \cdot 2^{3k-1} = 2^{-k}. \tag{31}$$

For example, for the infinite $n = 0.5①$ the set $P_{0.5①}$ consists of $2^{0.75①}$ blue squares (because the number $0.5①$ is even), their total area is infinitesimal and is equal to

$A_{0.5①} = 2^{-0.25①}$. Analogously, if the number of iterations is $n = 0.5① + 1$ then the set $P_{0.5①+1}$ consists of red triangles and k from (31) is equal to $-0.25① + 1$. The number of triangles is $2^{0.75①+2}$ and their total area is infinitesimal and is equal to $A_{0.5①+1} = 2^{-0.25①+1}$.

Finally, let us consider the blinking fractal from Fig. 7. We shall indicate the set obtained after n iterations by F_n. The length L_n of the set F_n is calculated as follows. For any (finite or infinite) $n = 2k, k \geq 0$, it consists of 2^{2k} intervals and each of them has the length $3^{-k} \cdot 4^{-k}$. Thus,

$$L_{2k} = 2^{2k} \cdot 3^{-k} \cdot 4^{-k} = 3^{-k}.$$

Analogously, for $n = 2k - 1, k \geq 1$, we obtain that F_n consists of 2^{2k-1} intervals and each of them has the length $3^{-k} \cdot 4^{-k+1}$. Thus,

$$L_{2k-1} = 2^{2k-1} \cdot 3^{-k} \cdot 4^{-k+1} = 2 \cdot 3^{-k}.$$

For example, for the infinite odd $n = 0.5① - 1$ the set $F_{0.5①-1}$ consists of $2^{0.5①-1}$ intervals and their total length is infinitesimal and is equal to $L_{0.5①-1} = 2 \cdot 3^{-0.25①}$.

7   Concepts of continuity in physics and mathematics

The goal of this section is to discuss mathematical and physical definitions of continuity and to develop a new, more physical point of view on this notion using the infinite and infinitesimal numbers introduced above. The new point of view is illustrated by a detailed consideration of one of the most fundamental mathematical definitions – function.

In physics, the 'continuity' of an object is relative. For example, if we observe a table by eyes, then we see it continuous. If we use a microscope for our observation, we see that the table is discrete. This means that we decide how to see the object, as a continuous or as a discrete, by the choice of the instrument for observation. A weak instrument – our eyes – is not able to distinguish its internal small separate parts (e.g., molecules) and we see the table as a continuous object. A sufficiently strong microscope allows us to see the separate parts and the table becomes discrete but each small part now is viewed as continuous.

In this connection, fractals become a very useful tool for describing physical objects. Let us return to Figs. 6 and 7 and suppose that we observe two beams consisting of two different materials at Step 0 by eye and we see both of them continuous. Then we take a microscope with a weak lens number 1, look at the microscope and see the pictures corresponding to Step 1 in Figs. 6 and 7, i.e., that the beams are not continuous but consist of two smaller parts that, in their turn, now seem to us to be continuous. Then we proceed by taking a stronger lens number 2, look again at the microscope and see the pictures corresponding to Step 2 in Figs. 6 and 7. First, we see now that the beams consist of four smaller parts and each of them seems to be continuous. Second, we see that their locations are different

(remind, that we have supposed that the beams have been made using different materials). By increasing the force of lenses we can observe pictures viewed at Steps 3, 4, etc. obtaining higher levels of discretization. Thus, continuity in physics is resolution dependent (a strict relation to [12] can be noticed immediately in this occasion) and fractal ideas can serve as a good tool for modelling the physical relative continuity.

In contrast, in the traditional mathematics any mathematical object is either continuous or discrete. For example, the same function cannot be both continuous and discrete. Thus, this contraposition of discrete and continuous in the traditional mathematics does not reflect properly the physical situation that we observe in practice. For fortune, the infinite and infinitesimal numbers introduced in the previous sections give us a possibility to develop a new theory of continuity that is closer to the physical world and better reflects the new discoveries made by physicists (remind, that the foundations of the mathematical analysis have been established centuries ago and, therefore, do not take into account the subsequent revolutionary results in physics, e.g., appearance of quantum physics). We start by introducing a definition of the one-dimensional continuous set of points based on the above consideration and Postulate 2 and establish relations to such a fundamental notion as function using the infinite and infinitesimal numbers.

We remind that traditionally a function $f(x)$ is defined as a binary relation among two sets X and Y (called the *domain* and the *codomain* of the relation) with the additional property that to each element $x \in X$ corresponds exactly one element $f(x) \in Y$. We consider now a function $f(x)$ defined over a one-dimensional interval $[a,b]$. It follows immediately from the previous sections that to define a function $f(x)$ over an interval $[a,b]$ it is not sufficient to give a rule for evaluating $f(x)$ and the values a and b because we are not able to evaluate $f(x)$ at *any* point $x \in [a,b]$ (for example, traditional numeral systems do not allow us to express any irrational number ζ and, therefore, we are not able to evaluate $f(\zeta)$). However, the traditional definition of a function includes in its domain points at which $f(x)$ cannot be evaluated, thus introducing ambiguity.

In order to be precise in the definition of a function, it is necessary to indicate explicitly a numeral system, $\mathcal{S}$, we intend to use to express points from the interval $[a,b]$. Thus, a function $f(x)$ is defined when we know a rule allowing us to obtain $f(x)$ given x and its domain, i.e., the set $[a,b]_{\mathcal{S}}$ of points $x \in [a,b]$ expressible in the chosen numeral system $\mathcal{S}$. We suppose hereinafter that the system $\mathcal{S}$ is used to write down $f(x)$ (of course, the choice of $\mathcal{S}$ determines a class of formulae and/or procedures we are able to express using $\mathcal{S}$) and it allows us to express any number

$$y = f(x), \quad x \in [a,b]_{\mathcal{S}}.$$

The number of points of the domain $[a,b]_{\mathcal{S}}$ can be finite or infinite but the set $[a,b]_{\mathcal{S}}$ is always discrete. This means that for any point $x \in [a,b]_{\mathcal{S}}$ it is possible to determine its closest right and left neighbors, x^+ and x^-, respectively, as follows

$$x^+ = \min\{z : z \in [a,b]_{\mathcal{S}}, \ z > x\}, \quad x^- = \max\{z : z \in [a,b]_{\mathcal{S}}, \ z < x\}. \quad (32)$$
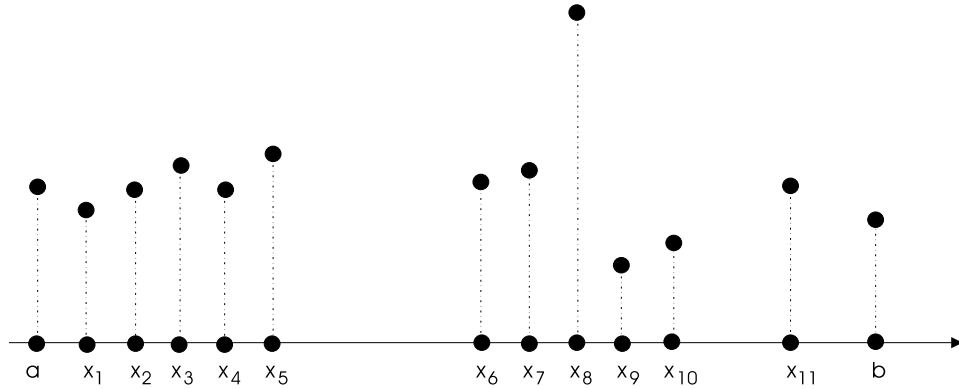
Figure 8: It is not possible to say is this function continuous or discrete until we have not introduced a unit of measure and a numeral system to express distances between the points

Apparently, the obtained discrete construction leads us to the necessity to abandon the nice idea of continuity, which is a very useful notion used in different fields of mathematics. But this is not the case. In contrast, the new approach allows us to introduce a new definition of continuity very well reflecting the physical world.

Let us consider $n+1$ points at a line

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \qquad (33)$$

and suppose that we have a numeral system $\mathcal{S}$ allowing us to calculate their coordinates using a unit of measure μ (for example, meter, inch, etc.) and to construct so the set $X = [a,b]_{\mathcal{S}}$ expressing these points.

The set X is called *continuous in the unit of measure μ* if for any $x \in (a,b)_{\mathcal{S}}$ it follows that the differences $x^+ - x$ and $x - x^-$ from (32) expressed in units μ are equal to infinitesimal numbers. In our numeral system with radix grossone this means that all the differences $x^+ - x$ and $x - x^-$ contain only negative grosspowers. Note that it becomes possible to differentiate types of continuity by taking into account values of grosspowers of infinitesimal numbers (continuity of order $①^{-1}$, continuity of order $①^{-2}$, etc.).

This definition emphasizes the physical principle that there does not exist an absolute continuity: it is relative (see discussion in page 33 and [12]) with respect to the chosen instrument of observation which in our case is represented by the unit of measure μ. Thus, the same set can be viewed as a continuous or not in dependence of the chosen unit of measure.

Example 1. The set of six equidistant points

$$X_1 = \{a, x_1, x_2, x_3, x_4, x_5\} \qquad (34)$$

from Fig. 8 can have the distance d between the points equal to $①^{-1}$ in a unit of

35

measure μ and to be, therefore, continuous in μ. Usage of a new unit of measure $\nu = ①^{-3}\mu$ implies that $d = ①^2$ in ν and the set X_1 is not continuous in ν. □

Note that the introduced definition does not require that all the points from X are equidistant. For instance, if in Fig. 8 for a unit measure μ the largest over the set $[a,b]_S$ distance $x_6 - x_5$ is infinitesimal then the whole set is continuous in μ.

The set X is called *discrete in the unit of measure μ* if for all points $x \in (a,b)_S$ it follows that the differences $x^+ - x$ and $x - x^-$ from (32) expressed in units μ are not infinitesimal numbers. In our numeral system with radix grossone this means that in all the differences $x^+ - x$ and $x - x^-$ negative grosspowers cannot be the largest ones. For instance, the set X_1 from (34) is discrete in the unit of measure ν from Example 1. Of course, it is also possible to consider intermediate cases where sets have continuous and discrete parts (see again discussion in page 33 related to beams from Figs. 6 and 7).

The introduced notions allow us to give the following very simple definition of a function continuous at a point. A function $f(x)$ defined over a set $[a,b]_S$ continuous in a unit of measure μ is called *continuous in the unit of measure μ at a point $x \in (a,b)_S$* if both differences $f(x) - f(x^+)$ and $f(x) - f(x^-)$ are infinitesimal numbers in μ, where x^+ and x^- are from (32). For the continuity at points a, b it is sufficient that one of these differences is infinitesimal. The notions of continuity from the left and from the right in a unit of measure μ at a point are introduced naturally. Similarly, the notions of a function discrete, discrete from the right, and discrete from the left can be defined.

The function $f(x)$ is *continuous in the unit of measure μ over the set $[a,b]_S$* if it is continuous in μ at all points of $[a,b]_S$. Again, it becomes possible to differentiate types of continuity by taking into account values of grosspowers of infinitesimal numbers (continuity of order $①^{-1}$, continuity of order $①^{-2}$, etc.) and to consider functions in such units of measure that they become continuous or discrete over certain subintervals of $[a,b]$. In the further consideration we shall often fix the unit of measure μ and write just 'continuous function' instead of 'continuous function in the unit of measure μ'. Let us give three simple examples illustrating the introduced definitions.

Example 2. We start by showing that the function $f(x) = x^2$ is continuous over the set X_2 defined as the interval $[0,1]$ where numerals $\frac{i}{①}, 0 \le i \le ①$, are used to express its points in units μ. First of all, note that the set X_2 is continuous in μ because its points are equidistant with the distance $d = ①^{-1}$. Since this function is strictly increasing, to show its continuity it is sufficient to check the difference $f(x) - f(x^-)$ at the point $x = 1$. In this case, $x^- = 1 - ①^{-1}$ and we have

$$f(1) - f(1 - ①^{-1}) = 1 - (1 - ①^{-1})^2 = 2①^{-1}(-1)①^{-2}.$$

This number is infinitesimal, thus $f(x) = x^2$ is continuous over the set X_2. □

Example 3. Consider the same function $f(x) = x^2$ over the set X_3 defined as the interval $[① - 1, ①]$ where numerals $① - 1 + \frac{i}{①}, 0 \le i \le ①$, are used to express its

points in units μ. Analogously, the set X_3 is continuous and it is sufficient to check the difference $f(x) - f(x^-)$ at the point $x = ①$ to show continuity of $f(x)$ over this set. In this case,

$$x^- = ① - 1 + \frac{① - 1}{①} = ① - ①^{-1},$$

$$f(x) - f(x^-) = f(①) - f(① - ①^{-1}) = ①^2 - (① - ①^{-1})^2 = 2①^0(-1)①^{-2}.$$

This number is not infinitesimal because it contains the finite part $2①^0$ and, as a consequence, $f(x) = x^2$ is not continuous over the set X_3. $\qquad\square$

Example 4. Consider $f(x) = x^2$ defined over the set X_4 being the interval $[① - 1, ①]$ where numerals $① - 1 + \frac{i}{①^2}, 0 \leq i \leq ①^2$, are used to express its points in units μ. The set X_4 is continuous and we check the difference $f(x) - f(x^-)$ at the point $x = ①$. We have

$$x^- = ① - 1 + \frac{①^2 - 1}{①^2} = ① - ①^{-2},$$

$$f(x) - f(x^-) = f(①) - f(① - ①^{-2}) = ①^2 - (① - ①^{-2})^2 = 2①^{-1}(-1)①^{-4}.$$

Since the obtained result is infinitesimal, $f(x) = x^2$ is continuous over X_4. $\qquad\square$

Let us consider now a function $f(x)$ defined by formulae over a set $X = [a,b]_S$ so that different expressions can be used over different subintervals of $[a,b]$. The term 'formula' hereinafter indicates a single expression used to evaluate $f(x)$.

Example 5. The function $g(x) = 2x^2 - 1, x \in [a,b]_S$, is defined by one formula and function

$$f(x) = \begin{cases} \max\{-10x, 5x^{-1}\}, & x \in [c,0)_S \cup (0,d]_S, \\ 4x, & x = 0, \end{cases} \qquad c < 0, \quad d > 0, \quad (35)$$

is defined by three formulae, $f_1(x), f_2(x)$, and $f_3(x)$ where

$$\begin{aligned} f_1(x) &= -10x, & x \in [c,0)_S, \\ f_2(x) &= 4x, & x = 0, \\ f_3(x) &= 5x^{-1}, & x \in (0,d]_S. \quad \square \end{aligned} \qquad (36)$$

Consider now a function $f(x)$ defined in a neighborhood of a point x as follows

$$f(\xi) = \begin{cases} f_1(\xi), & x - l \leq \xi < x, \\ f_2(\xi), & \xi = x, \\ f_3(\xi), & x < \xi \leq x + r, \end{cases} \qquad (37)$$

where the number l is any number such that the same formula $f_1(\xi)$ is used to define $f(\xi)$ at all points ξ such that $x - l \leq \xi < x$. Analogously, the number r is any number such that the same formula $f_3(\xi)$ is used to define $f(\xi)$ at all points ξ such that $x < \xi \leq x + r$. Of course, as a particular case it is possible that the same formula is used to define $f(\xi)$ over the interval $[x - l, x + r]$, i.e.,

$$f(\xi) = f_1(\xi) = f_2(\xi) = f_3(\xi), \qquad \xi \in [x - l, x + r]. \qquad (38)$$

It is also possible that (38) does not hold but formulae $f_1(\xi)$ and $f_3(\xi)$ are defined at the point x and are such that at this point they return the same value, i.e.,

$$f_1(x) = f_2(x) = f_3(x). \tag{39}$$

If condition (39) holds, we say that function $f(x)$ has *continuous formulae* at the point x. Of course, in the general case, formulae $f_1(\xi), f_2(\xi)$, and $f_3(\xi)$ can be or cannot be defined out of the respective intervals from (37). In cases where condition (39) is not satisfied we say that function $f(x)$ has *discontinuous formulae* at the point x. Definitions of functions having formulae which are continuous or discontinuous from the left and from the right are introduced naturally.

Example 6. Let us study the following function

$$f(x) = \begin{cases} ①^2 + \frac{x^2-1}{x-1}, & x \neq 1, \\ a, & x = 1, \end{cases} \tag{40}$$

at the point $x = 1$. By using designations (37) and the fact that for $x \neq 1$ it follows $\frac{x^2-1}{x-1} = x + 1$ we have

$$f(\xi) = \begin{cases} f_1(\xi) = ①^2 + \xi + 1, & \xi < 1, \\ f_2(\xi) = a, & \xi = 1, \\ f_3(\xi) = ①^2 + \xi + 1, & \xi > 1, \end{cases}$$

Since

$$f_1(1) = f_3(1) = ①^2 + 2, \qquad f_2(1) = a,$$

we obtain that if $a = ①^2 + 2$, then the function (40) has continuous formulae[4] at the point $x = 1$. Analogously, the function (35) has continuous formulae at the point $x = 0$ from the left and discontinuous from the right. □

Thus, functions having continuous formulae at a point can be continuous or discrete at this point in dependence of the chosen unit of measure. Analogously, functions having discontinuous formulae at a point can be continuous or discrete at this point again in dependence of the chosen unit of measure. The notion of continuity of a function depends on the chosen unit of measure and numeral system $\mathcal{S}$ and it can be used for functions defined by formulae, computer procedures, tables, etc. In contrast, the notion of a function having continuous formulae works only for functions defined by formulae and does not depend on units of measure or numeral systems chosen to express its domain. It is related only to properties of formulae.

---

[4]Note, that even if $a = ①^2 + 2 + \varepsilon$, where ε is an infinitesimal number (remind that all infinitesimals are not equal to zero), we are able to establish that the function has discontinuous formulae.

8   Conclusion

In this paper, a new type of objects – blinking fractals – that are not covered by traditional theories studying self-similarity processes have been introduced. They have been studied together with traditional fractals using infinite and infinitesimal numbers proposed recently. It has been shown that notions of length, area, and volume can be extended to the newly introduced and traditional fractals using these new powerful mathematical tools allowing one to study sets obtained after execution of different infinite numbers of steps and from different starting conditions. It has been shown that the new approach allows one to give quantitative characteristics of fractals behavior at infinity.

It has been emphasized that the philosophical triad – researcher, object of investigation, and tools used to observe the object – existing in such natural sciences as physics and chemistry exists in mathematics, too. In natural sciences, the instrument used to observe the object influences results of observations. The same happens in mathematics studying numbers and objects that can be constructed by using numbers. Thus, numeral systems used to express numbers are instruments of observations used by mathematicians. Usage of powerful numeral systems gives a possibility to obtain more precise results in mathematics in the same way as usage of a good microscope gives a possibility to obtain more precise results in physics.

It has been shown in the paper that the new numeral system allowing us to express not only finite but also infinite and infinitesimal numbers gives a lot of new (in comparison with traditional numeral systems able to express only finite numbers) information about behavior of fractal objects at infinity and can be successfully applied for numerical analysis of traditional and blinking fractals. Continuing the analogy with physics we can say that the new numeral system can be compared with a telescope that allows one to see objects situated infinitely far from the observer and in the same time this telescope has a microscope incorporated that gives possibility to see infinitesimal parts of this infinitely far objects.

Moreover, the introduced infinite and infinitesimal numbers have allowed us to propose a new viewpoint on modelling physical continuity in mathematics. The new approach is closer to the physical world than traditional mathematical instruments used for this purpose and gives a possibility to avoid the existing in the traditional mathematics contraposition between the notions *discrete* and *continuous*. It allows us to look at the same object as at a continuous or a discrete in dependence of the instrument that is used for observation in the same way as it happens in physics.

References

[1] V. Benci and M. Di Nasso, *Numerosities of labeled sets: a new way of counting*, Advances in Mathematics **173** (2003), 50–67.

[2] G. Cantor, *Contributions to the founding of the theory of transfinite numbers*, Dover Publications, New York, 1955.

[3] J.H. Conway and R.K. Guy, *The book of numbers*, Springer-Verlag, New York, 1996.

[4] R.L. Devaney, *An introduction to chaotic dynamical systems*, Westview Press Inc., New York, 2003.

[5] M.S. El Naschie, *A review of E-infinity theory*, Chaos, Solitons & Fractals **19** (2004), 209–236.

[6] ———, *A guide to the mathematics of E-infinity Cantorian spacetime theory*, Chaos, Solitons & Fractals **25** (2005), 955–964.

[7] M.S. El Naschie, O.E. Rössler, and I. Prigogine, *Quantum mechanics, diffusion and chaotic fractals*, Pergamon, Oxford, 1995.

[8] K. Falconer, *Fractal geometry: Mathematical foundations and applications*, John Wiley & Sons, Chichester, 1995.

[9] P. Gordon, *Numerical cognition without words: Evidence from Amazonia*, Science **306** (2004), no. 15 October, 496–499.

[10] H.M. Hastings and G. Sugihara, *Fractals: A user's guide for the natural sciences*, Oxford University Press, Oxford, 1994.

[11] http://www.theinfinitycomputer.com.

[12] G. Iovane, *Cantorian spacetime and Hilbert space: Part I – Foundations*, Chaos, Solitons & Fractals **28** (2006), 857–878.

[13] ———, *Cantorian spacetime and Hilbert space: Part II – Relevant consequences*, Chaos, Solitons & Fractals **29** (2006), 1–22.

[14] A.A. Markov Jr. and N.M. Nagorny, *Theory of algorithms*, second ed., FAZIS, Moscow, 1996.

[15] K. Knopp, *Theory and application of infinite series*, Dover Publications, New York, 1990.

[16] D. Knuth, *The art of computer programming*, third ed., Addison-Wesley, Reading, Massachusetts, 1997.

[17] P.A. Loeb and M.P.H. Wolff, *Nonstandard analysis for the working mathematician*, Kluwer Academic Publishers, Dordrecht, 2000.

[18] H.-O. Peitgen, H. Jürgens, and D. Saupe, *Chaos and fractals*, Springer-Verlag, New York, 1992.

[19] A.M. Robert, *Nonstandard analysis*, Dover Publications, New York, 2003.

[20] A. Robinson, *Non-standard analysis*, Princeton Univ. Press, Princeton, 1996.

[21] Ya.D. Sergeyev, *Computer system for storing infinite, infinitesimal, and finite quantities and executing arithmetical operations with them*, patent application, 08.03.04.

[22] ——, *Arithmetic of infinity*, Edizioni Orizzonti Meridionali, CS, 2003.

[23] ——, *A few remarks on philosophical foundations of a new applied approach to Infinity*, Scheria **26-27** (2005), 63–72.

[24] ——, *Mathematical foundations of the Infinity Computer (invited plenary lecture)*, Proc. of the XXI Autumn Meeting of Polish Information Processing Society (2005), 1–8.

[25] ——, *Misuriamo l'infinito*, Periodico di Matematiche **6(2)** (2006), 11–26.

[26] B.G. Sidharth, *The new cosmos*, Chaos, Solitons & Fractals **18** (2003), 197–201.

[27] R.G. Strongin and Ya.D. Sergeyev, *Global optimization and non-convex constraints: Sequential and parallel algorithms*, Kluwer Academic Publishers, Dordrecht, 2000.