Artificial Intelligence and Law **7:** 273–288, 1999.

Fuzzy measurement in the Mishnah and the Talmud

RON A. SHAPIRA[1]
Tel-Aviv University, Faculty of Law, Tel-Aviv, Israel 69978

Abstract. I discuss the attitude of Jewish law sources from the 2nd–5th centuries to the imprecision of measurement. I review a problem that the Talmud refers to, somewhat obscurely, as "impossible reduction". This problem arises when a legal rule specifies an object by referring to a maximized (or minimized) measurement function, e.g., when a rule applies to "the largest part" of a divided whole, or to "the first" incidence that occurs, etc. A problem that is often mentioned is whether there might be hypothetical situations involving more than one maximal (or minimal) value of the relevant measurement and, given such situations, what is the pertinent legal rule. Presumption of simultaneous occurrences or equally measured values are also a source of embarrassment to modern legal systems, in situations exemplified in the paper, where law determines a preference based on measured values. I contend that the Talmudic sources discussing the problem of "impossible reduction" were guided by primitive insights compatible with fuzzy logic presentation of the inevitable uncertainty involved in measurement. I maintain that fuzzy models of data are compatible with a positivistic epistemology, which refuses to assume any precision in the extra-conscious "world" that may not be captured by observation and measurement. I therefore propose this view as the preferred interpretation of the Talmudic notion of "impossible reduction". Attributing a fuzzy world view to the Talmudic authorities is meant not only to increase our understanding of the Talmud but, in so doing, also to demonstrate that fuzzy notions are entrenched in our practical reasoning. If Talmudic sages did indeed conceive the results of measurements in terms of fuzzy numbers, then equality between the results of measurements had to be more complicated than crisp equations. The problem of "impossible reduction" could lie in fuzzy sets with an empty core or whose membership functions were only partly congruent. "Reduction is impossible" may thus be reconstructed as "there is no core to the intersection of two measures". I describe Dirichlet maps for fuzzy measurements of distance as a rough partition of the universe, where for any region A there may be a non-empty set of $\bar{A}-\underline{A}$ (upper approximation minus lower approximation), where the problem of "impossible reduction" applies. This model may easily be combined with probabilistic extention. The possibility of adopting practical decision standards based on α-cuts (and therefore applying interval analysis to fuzzy equations) is discussed in this context. I propose to characterize the uncertainty that was presumably capped by the old sages as "U-uncertainty", defined, for a non-empty fuzzy set A on the set of real numbers, whose α-cuts are intervals of real numbers, as $U(A) = 1/h(A) \int_0^{h(A)} \log[1 + \mu(^{\alpha}A)]d\alpha$, where $h(A)$ is the largest membership value obtained by any element of A and $\mu(^{\alpha}A)$ is the measure of the α-cut of A defined by the Lebesge integral of its characteristic function.

Key words: formalization of legal reasoning, fuzzy equations, imprecision of measurements, Jewish law, simultaneity, U-uncertainty

[1] Tel-Aviv University, Faculty of Law. The author is indebted to Professor Meni Mautner for his valuable comments.

1. Historical Introduction: Observational Conclusion vs. Authoritative Determination of Facts

The formative period of post-dispersion Jewish law, extending from the destruction of the Second Temple (70 AD) to the compilation of the *Babylonian Talmud* (c. 500 AD), was characterized by a progressively increasing awareness of legal reasoning. Whereas the *Mishnah* (c. 200 AD) had been mainly a codification enterprise, focusing on an authoritative restatement of the law, the *Tosephta* (c. 250 AD) and the *Tannaitic Midrashim* (exegetic literature of the 3rd century AD) show a conscious interest in interpretation and argumentation. Both the *Jerusalem Talmud* (c. 400 AD) and the *Babylonian Talmud* were primarily designed with the aim of recording legal debates and deliberations.[2] This increased attention to legal reasoning makes talmudic discourse an important subject of research in the history of legal and legally related ideas.

In this paper, I discuss the attitude of talmudic sources to the imprecision of measurement. A short caveat regarding the subject of the study is required by way of preface. Jewish law is incomprehensible when detached from its religious context.[3] It rests on a deep belief in the superiority of divine decrees over the regularities of nature, and its philosophical assumptions are compatible with a doctrine assuming the infallibility of religious authorities. Even though, unlike Christianity, Judaism has never sustained infallibility as a general approach, it was not considered inconceivable that rabbinical authorities, through delegation from God, might have the power to determine metaphysical truth or to place inanimate objects within their jurisdiction. As described below, this was manifest in regard to the measurement of time: the power of the rabbis to set calendrical time was perceived as a power to command nature.

Although infallibility was philosophically acceptable, Jewish law in late antiquity did acknowledge that rabbinical authorities might err in matters of science or medicine,[4] as shown by two well-known judicial records from the late first century AD, described in the next two paragraphs.

1.1. THE METRECTOMIZED COW

> A story is told about a cow whose womb was removed. Rabbi Tarfon then fed it to dogs. This matter was brought before the sages, who allowed it (for human consumption). Theoderos the physician said: 'No cow or sow ever leave Alexandria without having their womb removed, so that they may not bear offspring'. Said Rabbi Tarfon: 'There goes your donkey, Rabbi Tarfon'.[5]

[2] Menachem Fisch, *To Know Wisdom: Science, Rationality and Torah-Study* (1994), pp. 41–98.

[3] Suzanne Last Stone, "In Pursuit of the Counter-Text: The Turn to the Jewish Legal Model in Contemporary American Legal Theory", 106 *Harv. L. Rev.* 813 (1993).

[4] *See:* Elimelech Westreich, "Medicine and Natural Sciences in Judgments of Rabbinical Courts", 26 *Mishpatim* (Hebrew University Law Review) 425 (1996) [Hebrew].

[5] *Mishnah, Tractate Bekhoroth* [Firstborns], 4:4.

The issue of the discussion concerns the laws of kosher food. A "*treifa*", defined as a moribund animal that is not expected to live longer than a year, is considered non-kosher and fit only for dogs.[6] The Mishnah records a controversy between a majority of the sages and the distinguished Rabbi Tarfon: he ruled that a cow whose womb was removed is a "*treifa*", whereas the majority concluded that it was not. A ruling of this kind carries precedential (or even legislative) value. Rabbi Tarfon, however, was willing to change his ruling when confronted with the testimony of a physician, reporting that metrectomy for cows and pigs exported from Alexandria, probably to preserve an Egyptian monopoly on the breeding of high-quality livestock, was a widespread practice. Having heard the physician's testimony Rabbi Tarfon replied: "There goes your donkey, Rabbi Tarfon", thinking he would have to sell his own donkey in order to compensate the cow owner for the damage he had incurred through his mistake. Subsequently, it emerged that Rabbi Tarfon might enjoy judicial immunity, which is why this passage is often quoted in discussions concerning the immunity of judicial bodies. Yet the interesting point in the present context is that a well-known religious figure such as Rabbi Tarfon did not hesitate to admit that his general rulings might be based on a wrong factual or scientific observation.[7]

1.2. THE CASE OF "THE RED SKIN PATCHES"

> Rabbi Eliezer reported a teaching he had learned from Rabbi Zadok. Once there was a woman who shed red patches of skin. R. Zadok was asked about this and he consulted the sages. The sages summoned the physicians, who said: 'She has an internal wound, which is why she sheds these red patches'.[8]

The problem here was whether the woman should be considered menstruous, and thus to be "set apart" until seven days after the shedding ceased.[9] The sages had no difficulty acknowledging the fact that they were liable to make mistakes, and consequently dependent on expert witnesses regarding matters of crucial importance to their religious decisions.

Eighteen centuries later, the pragmatic assent of religious authorities to scientific expertise may seem almost obvious, but this was not the case in the tannaitic period. Only more than a millennium later did the idea of "laws of nature" emerge in Judeo-Christian philosophy, aiming to account for the alleged universal applicability of global scientific generalizations by associating them with a widely

---

[6] *See*:: *Exodus* 22:30. Note that common English translations of this Biblical verse are inaccurate.

[7] Westreich, *supra* note 4, at p. 418, correctly notes that this early record of talmudic law accepting scientific evidence failed to draw adequate attention.

[8] *Tosephta, Tractate Nidah* [Menstruous woman] 4:3. A later version of the same report is mentioned in the **Babylonian Talmud, Tractate Nidah** [Menstruous woman] 22b. The Talmud adds that the sages finally concluded the case by conducting an experiment: they immersed the "skin patches" in water, and decided that, if they melt, the woman would be declared menstruous.

[9] *Leviticus* 15:19; 18:19.

acknowledged divine law.[10] Admitting that an authorized religious institution depends on physicians to reach general legal conclusions was, therefore, a significant decision. Against this backdrop, however, we may appreciate the absolute power attributed to religious institutions concerning the setting of calendrical time.

1.3. CALENDRICAL INTERCALATION AND THE RECOVERY OF VIRGINITY

The Jewish calendar is lunisolar, namely, the months are reckoned according to the moon and the years according to the sun. A month is the period of time between one conjunction of the moon with the sun and the next one. As the mean synodic month is slightly longer than 29 days and 12 hours whereas the solar year is longer than 365 days and 48 minutes, the solar year exceeds 12 lunar months by about 11 days. The cycle of 12 lunar months must therefore be adjusted to the solar year, so that agricultural festivals may be celebrated at the appropriate season. This adjustment is attained by adding one extra month 7 times in a cycle of 19 years. In the talmudic period, the selection of the 7 intercalated years within every 19 years cycle was decided by the rabbinical courts in the individual years according to climatic and agricultural conditions. The Talmud prescribed a set of climatic signs whose appearance should justify intercalation and, on this basis, rabbinical courts decided which would be the 13 months years.[11] These decisions were of great legal and religious significance. For instance, they had implications for the time set for the performance of obligations by law or contracts, and for the legal capacity of minors.

Prima facie, this function of rabbinical courts might appear as a factual inquiry or a measurement of objective reality, namely, when a set of given factual conditions was met, the court was supposed to declare intercalation. As we know that the rabbis were not considered infallible, we may assume that they acknowledged the possibility of a wrong decision regarding the proclamation of an intercalated year. It appears, however, that the function of rabbinical courts in determining intercalation was perceived differently. The authority to determine time according to data gathered by the rabbis and assessed relative to prescribed talmudic signs was taken as an absolute power to take an authoritative decision that would tame nature to behave according to this determination. This becomes clear in an often-quoted passage from the Jerusalem Talmud, dealing with the virginity of infants.[12]

According to ancient Jewish law, a woman's virginity has several legal implications: the rape of a virgin woman is punished more severely, her own punishment for voluntary "fornication" is harsher, the High Priest is not allowed to marry a "defiled" (non-virgin) woman, etc.. Virginity is an anatomical fact: a woman is

[10] Bas C. van Fraassen, *Laws and Symmetry* (1989) 2–7.

[11] *Babylonian Talmud, Tractate Sanhedrin* [Synod], 11b.

[12] One should not get the impression that ancient Jewish law (or the author of this paper) are obsessed with female organs. The common denominator shared by these recurring examples is completely accidental.

considered to be a virgin when she has a hymen, and talmudic sages knew that some women were born non-virgins. The Talmud dictates, probably as a result of factual observation, that a child younger than three years and one day who was raped may still retain her virginity, because her organs may recover when she grows up. If this happens, a child may legally be considered a virgin after being raped.[13] Therefore, the rapist of a child younger than three years and a day is punished less severely and is not liable to death by stoning. The Jerusalem Talmud deals with a rapist whose victim was thought to be older than three years when the crime took place, but was then retroactively declared younger due to intercalation.

> Rabbi Avunah said: . . . a girl of three years and a day – he who copulates with her shall be stoned to death. If the court decided to intercalate the year and he copulated with her – he shall not be stoned. Rabbi Avin said: . . . a girl of three years and a day – if the court decided to intercalate the year, her virginity returns".[14]

Significantly, the inapplicability of the prescribed capital punishment in this case is not reasoned by specific policy considerations but rather by a factual claim, namely, intercalation may act retroactively on the child's anatomy.[15] The retroactive recovery of the hymen is not presented as a legal fiction or a convenient presumption but as an actual physical occurrence.[16] Relying on this ruling, later authorities prescribed that intercalation may affect menstruation,[17] later to be inherited by Christianity.[21] The frustrating implications of this view for any human attempt to explore the world did not go unnoticed, but it was only since Maimonides[22] and Aquinas[23] that Judaism and Christianity, respectively, drew a clear demarcation line between everyday phenomena and God's original design including perhaps, occasional miraculous interventions. Given occasionalism, and the

---

[13] *Babylonian Talmud, Tractate Ketubboth* [Marriage Contracts], 11b.

[14] *Jerusalem Talmud, Tractate Sanhedrin* [Synod], ch. 1:2; *see also: Jerusalem Talmud, Tractate Nedarim* [Vows], 6:8; Rabbi Avin in: *Jerusalem Talmud, Tractate Ketubboth* [Marriage Contracts] 1:2.

[15] See par. A. *iv.* below.

[16] As has been observed by Ana anonymous referee of this paper, this conclusion might entail such normative absurdities as the retroactive exoneration of a convicted rapist.

[17] *Tofaphoth, Babylonian Talmud, Tractate Nidah* Menstruous Woman], 64b. the length of pregnancy,[18] and the appearance of male puberty signs.[19]

This conclusion was perfectly compatible with the prevalent world view during the talmudic era, which was occasionalism (also known as voluntarism), namely, the view that everything happens because it is directly and individually willed by God. In late antiquity, this was a well-established doctrine in Judaism,[20], 107b: "The Holy One sustains all, from buffalo's horns to nits".

[21] See *Matthew* 10:30: "But the very hairs of your head are all numbered".

[22] *Guide of the perplexed*, 3:17.

[23] *Summa Contra Gentiles*, bk. 2, sec. 4–9.

accepted idea that divine powers may be delegated to humans,[24] there was no reason to assume that the rabbis would be unable to adjust cosmic time.

1.4. "MEASURES SET BY THE SAGES"

A modern secular scholar who studies Jewish law must bear in mind the strong positivistic bent of the Jewish jurisprudential system and the formalistic character of its reasoning. Ancient Jewish law would never justify a legal conclusion by direct application to policy considerations or to an assumed legislative purpose. The verbal content of sanctified texts is usually the exclusive key to their interpretation, and exegeses of these texts are perceived as having a single truth-value.

In the present context, this formalistic trend is expressed in an extremely pedantic approach to the measures set by the law. This approach is summed up in a well-known dictum of Jewish law: "measures set by the sages are as they were set", namely, the slightest deviation from quantitative legal standards would be rejected as a total failure. The Talmud provides several examples of the application of this ruling, and I will consider two of them:[25]

- *Ritual Defilement Through Food*
  Eating certain foods is presumed to cause religious impurity. The law states precisely the quantities of food that might lead to defilement as no less than the volume of one (chicken) egg. The Talmud adds: "one egg minus one sesame seed does not defile".[26]
- *Ritual Immersion*
  Immersion is the standard method of purification. To fulfill this purpose, immersion must take place in a container holding at least 40 *se'ah* of water (about 527 litre). The Talmud emphasizes that "forty *se'ah* minus an eighth of an eighth (namely, the sixty-fourth part) of a *log* (one *log* is 0.549 litre, and the sixty-fourth part is therefore 0.0085 litre) is unfit for immersion".[27]

An alternative approach to measurement, ridiculing the significance of precision, was proposed by Rabbi Yrmiya (early 4th cent. AD), and was firmly rejected.[28] According to the Jewish law of property a young fledgling found near to an aviary is regarded as a possession of the owner of the aviary. The Mishnah set this rule in exact quantitative terms: a fledgling found within fifty cubits from the aviary, and only such an avis, is the possession of the owner of the aviary.[29] Against

[24] See the dispute about the ritual impurity of "the oven of Akhnai" in **Mishnah, tractate Kelim** [Tools] 5:10; *Mishnah, Tractate Eduyot* [Testimonies] 7:7.

[25] *Babylonian Talmud, Tractate Rosh Hashanah* [New Year's Day], 13a.

[26] *Ibid.*

[27] *Ibid.*

[28] M. Silberg, *Principia Talmudica* 45–59 (1961).

[29] *Mishnah, Tractate Bava Batra* [Last Section], chapter B, rule 6.

this background, Rabbi Yrmiya raised the following question: what happens if a fledgling is found "one of its claws within fifty cubits (from the aviary) and the second outside fifty cubits"? The response was harsh: Rabbi Yrmiya was dismissed from college,[30] on the ground of being "troubling the sages",[31] and was allowed back only when he publicly admitted that his query was a "non-question". The reason given for the severe sanction imposed on Rabbi Yrmiya was not that he was overly pedantic or petty. To the contrary, Rabbi Yrmiya was perceived as creating an unnecessary provocation because it was clear from the phrasing of the original rule that it did in fact cover all possibilities, and in particular, that in the borderline case mentioned by Rabbi Yrmiya the fledgling was not to be the property of the owner of the aviary. Rabbi Yrmiya meant to be provocative by reopening this impractical question, which underscored the formalistic nature of the discussion and brought it *ad absurdum*.[32] It was this provocation that was met with tough response.

The positivist and formalistic trend in Jewish law stems from obvious institutional considerations. Rabbis often draw the political justification of their authority from presenting their various functions: textual exegesis, ritual guidance, judicial interpretation, rabbinical decision-making, as simple applications of rules set by God. Thus, even as legislators they generally regard themselves as doing no more than administering God's law through "minor" external modifications, with the purpose of preserving it as He intended.[33] This positivistic approach was intimately connected with the above-mentioned grim perception of precise quantitative rules: when the law prescribes a measure one was not to relax its consequences by questioning the underlying rationale.[34]

1.5. INSTITUTIONAL DIVISION OF NORMATIVE POWER

As noted, talmudic law distinguishes between two categories of factual observation: observations extrinsic to the judicial and legislative functions of rabbinical authorities, which are usually determined by expert witnesses, and observations involving the implementation of rabbinical power to determine physical measures and occurrences. This distinction seems to rest on an institutional division of jurisdiction: physicians' knowledge was appreciated inasmuch as it did not infringe on the role of rabbinical tribunals. The determination of calendrical time was considered to be strictly within the purview of the rabbis: it was their traditional

30 *Babylonian Talmud, Tractate Bava Batra* [Last Section] 23b.

31 *RSI*, *ibid.*

32 Alex Klein, "Rabbi Yrmiya's Questions", 3 *Higayon* 151 (1996) [Hebrew].

For other reflections of the conflict between Rabbi Yrmiya and the majority of sages see: *Babylonian Talmud, Tractate Rosh Hashanah* [New Year's Day] 13a; *Babylonian Talmud, Tractate Sotah* [Faithless Wife] 16b.

33 A. Kirschenbaum, *Equity in Jewish Law*, vol. I, p. 5 (1991).

34 Silberg, *supra* note 28, at p. 52.

province, an important responsibility that rabbinical institutions had discharged for centuries, and a source of great social power. In contrast, general observations concerning the viability of animals or the source of internal bleeding were thought to require a technical expertise that the rabbis did not assume.

This position is not alien to a modern jurist. Modern legal systems are obviously not indifferent to considerations of division of power, particularly in relation to the admissibility of scientific evidence. For example, the rejection of polygraph findings in many modern legal systems, commonly rationalized by their prejudicial effect, is probably also influenced by a feeling that psychologists and technicians threaten a domain traditionally reserved to legal institutions – distinguishing between true statements and lies. Similar imperialistic considerations have probably guided the rabbinical claim of infallibility in regard to the determination of time.

The existence of an established category of factual observations, where rabbinical authorities allege to have privileged access to absolute truth, is crucial for the understanding of the next two sections.

2. The Problem of "Impossible Reduction"

This section reviews a problem that the Talmud refers to, somewhat obscurely, as "impossible reduction". This problem arises when a legal rule specifies an object by referring to a maximized (or minimized) measurement function, e.g., when a rule applies to "the largest part" of a divided whole, or to "the first" incidence that occurs, etc. A problem that is often mentioned is whether there might be hypothetical situations involving more than one maximal (or minimal) value of the relevant measurement and, given such situations, what is the relevant legal rule. For instance, might there be more than one "larger part" of a divided whole or a simultaneous occurrence of more than one incident?

Hypotheticals of simultaneous occurrences are also a source of embarrassment to modern legal systems. For instance, in most criminal legal systems, self defence only applies when the accused reacts against "unlawful" aggression. This formulation probably assumes a temporal ordering: one party to the violent interaction is presumed to be responding to the other party's violence. But what would be the correct legal solution in the case of a strictly symmetrical reciprocal attack? In other words, what happens when two people attack each other simultaneously in an act of self-defence? Any one of them may be justified only if the other acts "unlawfully", but due to the assumed symmetry between them, if one of them is justified then so is the other (and according to the common understanding, a justified action is not "unlawful" in this sense). Moreover, if any of the parties is unjustified (and hence acting unlawfully) there is no reason why his opponent would not enjoy this defence and be justified, but this conclusion also contradicts

the assumption of symmetry. The problem hinges on the possibility of a strictly symmetrical reciprocal attack, which generally requires simultaneity.[35]

The Talmud offers two competing views in this regard: one that acknowledges simultaneous incidents, distances that are equally long, etc., and may thus refer to more than one object as "the first" or "the closest" to a particular place, and another that, in principle, totally denies the possibility of such equations. The latter view is formulated as "reduction is impossible" (namely, the gap between different values can never be completely canceled). The claim that "reduction is impossible" recurs in many talmudic discussions, two of which are described below.[36]

2.1. LEVIRATE

> If brothers dwell together, and one of them die, and have no child, the wife of the dead shall not marry abroad to a stranger: her husband's brother shall go in to her, and take her to him to wife, and perform the duty of a husband's brother to her. And it shall be, that the firstborn which she bears shall succeed in the name of his brother who is dead, that his name be not wiped out in Israel. And if the man like not to take his brother's wife, then let his brother's wife go up to the gate to the elders, and say, My husband's brother refuses to raise up to his brother a name in Israel, he will not perform the duty of a husband's brother. Then the elders of his city shall call him, and speak to him: and he shall stand, and say, I do not wish to take her; then shall his brother's wife approach him in the presence of the elders, and loose his shoe from off his foot, and spit in his face, and shall answer and say, Thus shall it be done to that man that will not build up his brother's house And his name shall be called in Israel, The house of him that had his shoe loosed. (Deuteronomy 25:5–11)

---

[35] *See:* Russell Christopher "Unknowing Justification and the Logical Necessity of the Dadson Principle in Self-Defence", 15 *Oxford J. Legal Stud*. 229 (1995).

[36] Other well-known examples:

(a) On her first delivery, a ewe gives birth to two lambs whose heads come out at the same time. The biblical law states that "the first born male" is to be sacrificed as a priestly offering (*Exodus* 13:12), and a dispute arises as to whether there could be two "firstborns" (**Mishnah, Tractate Bechoroth** [Firstborns] 17:1).

(b) Simultaneous testimonies (*Babylonian Talmud, Tractate Shvu'oth* [Oaths] 32a).

(c) Half-broken partition. A "partition" or fence mark the boundary of a "domain". The precise scope of this domain has important implications concerning property law, the law of defilement, the law of Shabbath, and other legal fields. A rule prescribes that, to determine the scope of the "domains", the broken section of the fence must not be larger than the unimpaired section. The possibility of a partition wherein the broken and unbroken sections are exactly equal is discussed and debated (*Babylonian Talmud, Tractate Eruvin* [Amalgamation of Courts] 5; 15b).

From the 11th century onwards, the rite of removing the brother's shoe, known as *halizah*, has replaced levirate as the standard practice.[37] Either one of these rites – levirate or *halizah* – must be performed for the widow to be allowed to remarry.

A woman whose brother-in-law is prevented from marrying her because of some other reason – for instance, if she is the daughter of her late husband's brother, as a man is not prevented from marrying his niece according to Jewish law – is exempt from these rites, as she is obviously not expected to marry her own father.[38] Similarly, if two brothers marry two sisters and one of the brothers dies, the surviving brother is not allowed to marry his sister-in-law,[39] because she is the sister of his first wife.[40]

The Talmud discusses the following hypothetical: two brothers, who had a third brother, married two sisters. The two brothers died, seemingly at the same time (e.g., at the same accident). Need the two widowed sisters participate in the pertinent rites? According to Rabbi Shimon they do not, because they are both in the same position, and the prohibition against marrying two sisters at the same time prevents the surviving brother from marrying either of them. According to the majority of sages, however, both women are obliged to perform the *halizah* ritual before they marry again, because "reduction is impossible". The two brothers, therefore, could not be said to have passed away strictly at the same time–one of the sisters, we do not know which one, was surely a widow at a moment when she could have, and should have, been levirated, while the other was not. Neither one of the two women, then, will be able to remarry before removing the doubt that has been shed on their marital capacity.[41]

In practical terms, then, the conclusion of the majority opinion implies that the two sisters have to gain the cooperation of their former brother-in-law in order to remarry. This conclusion contrasts with the rabbis' attitude in other contexts where, as noted, they assume almost unlimited power to determine measures and to measure physical objects. As the problem of simultaneous widowhood was clearly within the exclusive jurisdiction of religious authorities, their reluctance in this instance calls for an explanation.

---

[37] *See: RASHI (Rabbi Shlomo Itzhaki), Commentaries on Babylonian Talmud, Tractate Bechoroth* [Firstborns] 13a: a prohibition on performing levirate, unless the man and the widow show that they were exclusively motivated by true religious purposes (11th cent). Cf.: Rabbi Moshe Ben Meimon, *Misneh Torah*, Rules of Levirate and *Halizah*, 1:2.

[38] *Mishnah, Tractate Yevamoth* [Levirate Marriages], 1:1.

[39] According to biblical law a man was allowed to marry more than one woman, and polygamy was abolished only in the 11th century.

[40] *Leviticus* 18:18: "Neither shalt thou take a wife to her sister, to vex her, to uncover her nakedness, beside the other in her life time".

[41] *Babylonian Talmud, Tractate Yevamoth* [Levirate Marriages], 19a.

2.2. BEHEADED HEIFER

> If one be found slain in the land which the Lord thy God gives thee to possess it, lying in the field, and it be not known who has slain him: then thy elders and thy judges shall come out, and they shall measure to the cities which are round about him that is slain and it shall be, that as for the city which is nearest to the slain man, the elders of that city shall take a heifer, which has not been put to work, and which has not drawn in the yoke: and the elders of that city shall bring down the heifer to a rough ravine, which is neither ploughed nor sown, and shall break the heifer's neck there in the ravine: and the priest the sons of Levi shall come near; for them the Lord thy God has chosen to minister to him, and to bless in the name of the Lord; and by their word shall every controversy and every stroke be tried: and all the elders of that city, that are nearest to the slain man, shall wash their hands over the heifer whose neck was broken in the ravine: and they shall answer and say, Our hands have not shed this blood, nor have our eyes seen it (Deuteronomy 21: 1–9).

A preliminary mathematical problem called forth by this passage is drawing the map of Israel by a partition into regions, such that each region includes the points closest to a particular city. In AI literature this is known as constructing the Dirichlet tessellation and it has a standard solution (simpler when the "cities" are just points, a bit more complicated when they have area).[42] The legal problem is more complex than the problem addressed by the standard solution, because the Talmud prescribes that the law of beheaded heifer is applied by a simple measurement of distance only if all the cities in the area are of the same size. Otherwise, the largest city in the area is held responsible for the death of the "slain person".[43] This necessitates an elaborate solution, combining considerations of distance and size.[44] The problem is further muddled due to two special exceptions to the rule, concerning a "slain person" found in vicinity to a major road and a city that does not have a court of its own.

Having solved all these problems, that is, given any map of regions of influence around cities and major roads, what would be the law were measurements to show that two cities are equally "near to the slain man"?

According to one view, there may be a difference between the measurement of physical events (such as a natural death incident) and the measurement of human actions: whereas "reduction is impossible" for the former, it may be possible for the latter.[45] But the distinction between physical events and human actions is not clear

[42] P. J. Green and R. Sibson, "Computing Dirichlet Tessellations in the Plane", 21 *Computer J.* 168 (1978).

[43] *Babylonian Talmud, Tractate Bava Batra* [Last Section] 23b: "majority and vicinity – majority overrides".

[44] See several proposed algorithms in: E. Merzbach, B. Singer "The Nearest Town in the Law of 'Beheaded Heifer' ", 2 **Higayon** 76 (1993) [Hebrew].

[45] *See:* Tosephta, Tractate Eruvin [Amalgamation of Courts] 6a; *Babylonian Talmud, Tractate Bechoroth* [Firstborns] 17a.

in the religious context,[46] because Jewish law usually assumes that human actions are governed by God's decrees, like any other physical event.[47] Indeed, in relation to the problem of "impossible reduction" this distinction is seldom mentioned in the Talmud, and I shall therefore ignore it in the present review.

Concerning the hypothetical of two cities that are "equally near" the slain man, Rabbi Eliezer concluded that both cities should atone for the murder by sacrificing a heifer, because both fall under the biblical definition: "nearest to the slain man".[48] But the majority decision ruled that, as "reduction is impossible", the two cities can fulfill their obligation by sharing one heifer and making a "conditioned sale" transaction, by which they agree to transfer full possession of the heifer to the city that was in fact closer to the slain man.[49] Once again, the majority opinion shows limited confidence in the precision of human measurements, which is hard to account for against the background of the rabbis' assumed power to determine measurements conclusively.

3. Fuzzy World View

The inevitable uncertainty involved in measurement can be dealt with in two ways at least – it can be ascribed to flawed human capabilities or viewed as inherent in the object being measured. Whereas the first approach often employs probability theory, the second typically relies on fuzzy models of data. Prevalent interpretations of probability theory speculate the existence of a crisp world that, in practice or in principle, cannot be measured precisely. Probability theory, however, locates this imprecision in the relation between the data and the observer, while fuzzy logic considers that imprecision is built into the data itself. Fuzzy models of data are thus compatible with a positivistic epistemology, which refuses to assume any precision in the extra-conscious "world" that may not be captured by observation and measurement. I propose this view as the preferred interpretation of the talmudic notion of "impossible reduction". **I believe that attributing a fuzzy world view to the talmudic authorities mentioned above in chapter B will not only increase our understanding of the Talmud but, in so doing, will also demonstrate that fuzzy notions are entrenched in our practical reasoning.**

Although the power of Talmudic sages to determine measurements was considered incontestable even by Nature, the rabbis chose restraint in matters that were clearly within their institutional purview. Given the unlimited trust granted to them by the Jewish legal system, as revealed in their power to set time by processing extrinsic data, there was no institutional reason for the rabbis' reluctance to assume responsibility for the conclusion that events had occurred simultaneously or that a place was equally distant from two cities. This prudence could only rest

46 *See:* Tosephta, Tractate Hullin [Profane Slaughtering] 28b.

47 *Ibid.* See *supra* note 20 and accompanying text (concerning occasionalism).

48 *Babylonian Talmud, Tractate Bechoroth* [Firstborns] 18a.

49 *Ibid.*

on **a conception that ascribes imprecision to the observed objects themselves, a conception that is best formalized through fuzzy logic**.

Crisp logic often seems awkward because of its counter-intuitive rigidity. Probabilistic confidence interval or density functions preserve this awkwardness because they presume ultimate precision in the measured space and, therefore, do not allow final conclusions in the form of "an event x falls around the pre-defined standard m". In contrast, fuzzy sets, and in particular fuzzy numbers,[50] were designed with the aim of amending this shortcoming by capturing the intuitive concept of "numbers that are close to a given number".

If talmudic sages did indeed conceive the results of the measurements they conducted in terms of fuzzy numbers, then equality between the results of two measurements had to be more complicated than crisp equations.[51] Had they been aware of modern formulations of fuzzy set theory, they could have reasoned more clearly the conclusion that "reduction is impossible" in a particular case, by describing the membership functions they had in mind. The problem could lie, perhaps, in fuzzy sets with an empty core (namely, sets where none of the elements has a full membership grade), or whose membership functions were only partly congruent.

In the levirate case, the problem was fuzziness of time. Assume that measurement of time is performed by assigning fuzzy numbers (that represent fuzzy points in time) to events. What would be meant by stating that two events (the death of two persons in this case) were simultaneous, if the results of the (reliable) measurement were fuzzy numbers with an empty core? And how would simultaneity be determined in cases of non-congruent membership functions (even if any of them reaches the maximal degree of membership at some points)?

Suppose, for example, that the following illustration represents the timing of two events, A and B, whose simultaneity has to be decided.

---

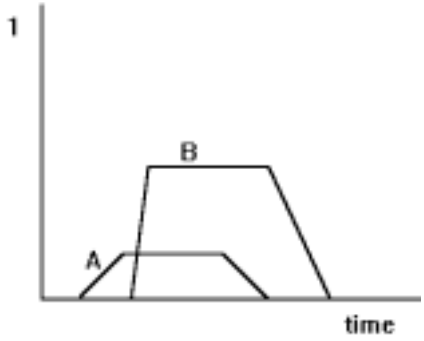
[50] A fuzzy number is a fuzzy set on the set of real numbers which possesses the following properties:

- At least one real number gets the value of 1;
- For every α in (0,1] its α-cut is a closed interval [For α-cuts see *infra*];
- Its support is bounded.

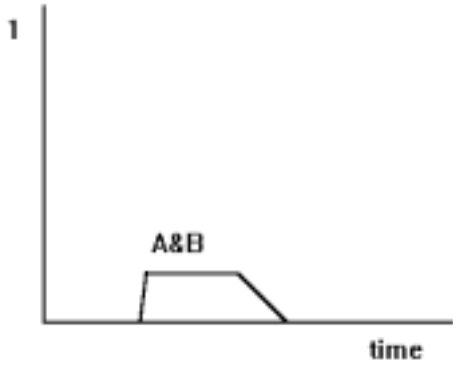
It can easily be demonstrated that, according to these requirements, every fuzzy number can be represented in relation to a non-empty closed interval $[a, b]$, as a composition of three functions: (1) One function that assigns 1 to members of the interval; (2) Another function, from $(-\infty, a)$ to [0, 1], that is monotonic increasing, continuous from the right, and such that its value is 0 for $x \in (-\infty, \omega_1)$; (3) A third function, from (b, ∞) to [0, 1], that is monotonic decreasing, continuous from the right, and such that its value is 0 for $x \in (\omega_2, \infty)$.

See: D. Dubois and H. Prade, "Fuzzy Real Algebra" 2 **Fuzzy Sets and Systems** 327 (1979); D. Dubois, H. Prade, **Possibility Theory** 48–50 (1988).

[51] Notoriously, fuzzy equations behave differently from crisp equations. A standard way of defining arithmetic operations on fuzzy numbers is by operations on their α-cuts (see *infra*) according to the well-established principles of interval analysis. However, according to such a definition we can show, for example, that $B - A$ is not be the solution for $A + x = B$.



The proposition that *A* and *B* has both occurred at a particular point in time is a conjunction of the propositions that *A* has occurred at this point and that *B* has occurred there. Therefore, the function representing their simultaneous presence can easily be understood as the intersection of the two fuzzy numbers. By a standard definition, the membership function of the intersection of ordinary fuzzy sets (type 1 fuzzy sets on the interval [0, 1]) assigns to each element in the universe the minimal membership value the same element gets according to the membership functions of each of the intersecting sets. Therefore, the simultaneous presence of *A* and *B* may be represented by the following illustration.



"Reduction is impossible" is thus reconstructed according to this nodel as "there is no core to the intersection of two measures".

In the beheaded heifer case, it is fuzziness of distance that is at stake. There again, the opinion of the majority of Rabbies may be reconstructed as "there is no core to the intersection of the two measurement results". The corresponding Dirichlet map may be drawn as a rough partition of the universe, in Pawlak's sense of roughness,[52] where for any region *A* there may be a non-empty set of $\overline{A} - \underline{A}$ (upper approximation minus lower approximation), where the problem

[52] Z. Pawlak, "Rough Sets", 17 **Fuzzy Sets and Systems** 99 (1985).

of "impossible reduction" applies. This model may easily be combined with a probabilistic extention.[53]

This fuzzy world view is certainly not inconsistent with the adoption of practical decision standards based on α-cuts.[54] Assuming that the steady opposition to the conclusion that "reduction is impossible" shares the same model, the historical debate can be reconstructed as a debate over the application of α-cuts.[55] According to this interpretation, which seems reasonable, the debate would concern different assessments of uncertainty due to the existence of nonconclusive or borderline situations, as opposed to uncertainty deriving from incomplete information about crisp situations.

Apparently, due to the above mentioned pedantic approach to measures set by law, many rabbis did not consider that decision standards based on α-cuts were applicable to the problem of determining precise simultaneity. But even those rabbis who would have agreed to apply the notion of α-cuts to the problem of "impossible reduction" and, therefore, would also agree to apply interval analysis to fuzzy equations (because any fuzzy number can be uniquely represented by its α-cuts), may not have agreed on the standard of equality. Assuming they did not agree that only equal α-cuts intervals would be considered as an expression of equal fuzzy numbers, they would probably require that the difference between the two measured magnitudes should not exceed a given standard.[56]

Trying to reconstruct the debate in the modern terms of fuzzy logic may involve us in a characterization of the uncertainty that was presumably capped by the old sages. This should be described, in my view, as "U-uncertainty". In order to explain this we must resort once again to the distinction between lack of specific information and fuzzy classification.

Uncertainty resulting from a lack of specific information regarding the object of interest, including the one caused by the finite resolution of measurement instruments, was acknowledged and measured in classical information theory long before the introduction of fuzzy logic. A sensible way to model the uncertainty derived from the coarseness of measurement instruments is to partition the interval of real numbers representing the range of values of the discussed variable into disjoint subintervals, such that values within each subinterval would be considered indistinguishable. The subintervals are usually labeled by real numbers, which may be their respective means or other numbers contained in them, and all the values that fall within the same subinterval are perceived as the same state of the variable,

---

[53] Z. Pawlak, S. K. M. Wong, and W. Ziarko, "Rough Sets: Probabilistic Versus Deterministic Approach", 29 **Int. J. Man-Mach. Stud.** 81 (1988).

[54] The α-cut of a fuzzy set F, denoted ${}^{\alpha}F$, is a crisp set that contains all the elements of the universe whose membership values in F are greater than or equal to the specified value α (which is a member of the range of F).

[55] The opposition may well be reconstructed as claiming that even a core-less intersection may suffice in some cases for the legal purpose.

[56] For the special problematics of such equations see George J. Klir and Bo Yuan, **Fuzzy Sets and Fuzzy Systems: Theory and Application** 114 et seq. (1995).

labeled by the same number. The uncertainty generated by the apparent convergence of values within the same subinterval depends on the size of that interval. According to the standard well-known formula of information theory, the amount of uncertainty associated with the labeling of all values in the interval $[a, b]$ by a single number is $\log_2[1 + b - a]$.

Yet, models based on partitioning of the universe into classes of sharp boundaries (e.g., the partitioning of a measurement range represented by an interval of real numbers into disjoint subintervals) assume that each element (observed magnitude in the same example) fits exactly into one of the classes (subintervals in this example). This is a highly unrealistic assumption as, in practice, measurement errors and observation failures may occur in close proximity to one of the boundaries between classes, thus generating uncertainty regarding the membership of an element in more than one class (or subinterval). In the present context of comparison pre-imposed partition is surely useless, because we are comparing two unknown magnitudes that may fall closer to a predefined boundary between states. In these cases we must resort to genuine multi-valued logic.

Therefore, the amount of uncertainty discussed here cannot be measured simply by the standard formula of information theory but rather by U-uncertainty, which is an extension of that formula. An introduction to the general formulation of U-uncertainty is not required here,[57] and we will confine ourselves to the statement that the U-uncertainty of a set measures the weighted average of the values of uncertainty that have been measured for all its distinct α-cuts.[58] It was the U-uncertainty involved in the comparison between fuzzy states that troubled the rabbis. Due to a high degree of U-uncertainty (and not due to regular uncertainty, that was manageable through authoritative acts of measurement) the rabbis hesitated to commit themselves to a finding of precise simultaneity or equality.

[57] *See*: Klir and Yuan, *ibid.*, at pp. 250–1.

[58] For a non-empty fuzzy set A defined on the set of real numbers, whose α-cuts are intervals of real numbers

$$U(A) = \frac{1}{h(A)} \int_0^{h(A)} \log[1 + \mu({}^{\alpha}A)]\, \mathrm{d}\alpha,$$

where $h(A)$ is the largest membership value obtained by any element of A and $\mu({}^{\alpha}A)$ is the measure of the α-cut of A defined by the Lebesge integral of its characteristic function.