Generalizing the Algebra of Physical Quantities

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ABSTRACT

In this paper, I define and study an abstract algebraic structure, the dimensive algebra, which embodies the most general features of the algebra of dimensional physical quantities. I prove some elementary results about dimensive algebras and suggest some directions for future work.

1. INTRODUCTION. Scientists and engineers routinely use physical quantities to represent the measured properties of physical objects. Some mathematicians have studied physical quantities from a more abstract standpoint, with the aim of better understanding the nature and use of those quantities. There are two strictly mathematical disciplines that concern themselves mainly with physical quantities. One is measurement theory, which is a rigorous study of the correspondence between physical quantities and the properties they represent (see, for example, [3]). The other is dimensional analysis, which concerns itself with the algebraic rules governing physical quantities and analyzes those rules to facilitate physical calculations and derivations.

This paper is a study of physical quantities from yet another point of view. In this paper, I investigate some abstract algebraic structures that capture the most general and characteristic features of the algebra of physical quantities. The study of physical quantities from an abstract algebraic standpoint is not a new idea. Earlier authors [1, 4, 5, 6] have used the methods of abstract algebra to rigorize the foundations of dimensional analysis. My aim in this paper is slightly different: I intend to define and study a type of abstract algebraic structure that embodies only the *most general* algebraic properties common to all systems of dimensional quantities. Once these structures have been introduced, one can ask which additional algebraic axioms one must impose on such structures to make certain rules of dimensional analysis hold in them. It will turn out that elementary dimensional analysis, of the kind often taught to beginning physics students, actually requires very little by way of additional constraints on the algebra.

The structures introduced and studied here are called *dimensive algebras*. The main algebraic structures introduced in references [1], [4], [5] and [6] are special cases of dimensive algebras. The present study owes a particular debt to the approach adopted by Whitney, especially in the earlier parts of [5].

The study of dimensive algebras has four potential benefits. First, it can enhance our understanding of the algebra of physical quantities, and in particular of the relationship between physical quantity systems and other algebraic systems. Second, it may have pedagogical benefits, since it shows what assumptions really underlie the most commonly used rules of dimensional analysis. Third, it lets us develop the beginnings of a theory of *generalized* dimensional quantities -- items such as operators, vectors and matrices which have physical dimensions and units, but are not numbers. These objects occur abundantly in physical theory, but are not fully encompassed by existing abstract algebraic treatments of dimensional quantities. Fourth, the study of dimensive algebras may lead to some interesting algebraic problems, quite apart from the original physical motivation of those structures.

The chief aim of this paper is not the proving of theorems, but rather the introduction of potentially interesting concepts. Much of the paper is an exploration of possibilities for algebraic structures modeled on systems of physical quantities. At times I will point out the differences (sometimes striking) between these structures and physical quantity systems. In Section 2, I provide a little more background information relevant to the project of the paper. Beginning with Section 3, I introduce the definitions of dimensive algebra and of several related notions. Although proving theorems is not my principal aim, I do prove or assert some results along the way.

2. DIMENSIONAL QUANTITIES IN SCIENCE. In the physical sciences, as in all disciplines that involve measurement, one often makes use of dimensional quantities -- that is, quantities which can be expressed as a combination of a pure numerical coefficient and a unit of measurement. Examples of such quantities are 1 meter, 20.2 miles, 3 kg, and 5 days. The results of physical measurements typically are quantities of this sort.

Dimensional quantities obey algebraic rules similar to, yet in some respects different from, the rules of algebra for numbers. For example, quantities with units can be multiplied and divided by each other; the result of such an operation is equal to the product or quotient of the coefficients of the two quantities, times the product or quotient of the quantities' units. A quantity with units can be multiplied or divided by a number, and two quantities with units can be added or subtracted provided that the two quantities involved have the same units. However, two quantities with different units cannot be added to or subtracted from one another; the results of such an operation are not defined. This last fact constitutes one of the main differences between the algebra of dimensional quantities and that of pure numbers.

The mathematical literature contains several studies of the algebraic properties of dimensional quantities (see in particular [1, 4, 5, 6]). Many of these studies are concerned with the foundations of dimensional analysis, a mathematical technique used in the sciences to extract information about physical relationships between quantities from information about the quantities' dimensions or units. Dimensional analysis utilizes known algebraic properties of dimensional quantities to discover or verify scientifically useful relationships among physical variables.

3. DIMENSIVE ALGEBRAS. When one compares the set of dimensional quantities to sets of dimensionless numbers (real or natural), one notices a striking difference: among the dimensional quantities, not every pair of quantities can be added. This is perhaps the most striking algebraic feature of dimensional quantities. The addition of such quantities is not a binary operation on the entire set of dimensional quantities. It becomes a binary operation only when restricted to any set of quantities with the same dimensions. On a set containing quantities of different dimensions, addition becomes a partial binary operation (see [3, p. 17]): something like a binary operation, except that not every pair of elements has a sum.

If we want to treat all dimensional quantities together as an algebraic system, we must allow for the possibility of a "binary operation" which is something less than a binary operation on the whole system, but which acts like a binary operation on certain subsets of the system (the sets of quantities of the same dimensions). This generalized notion of a binary operation is captured in the following definitions:

Definition 1. Let S be a set. An *axiation* of S is a pair $\langle X, C\rangle$, where X is a set of subsets of S, $C \subseteq S$, and the following conditions hold: (1) $\cup X = S$; (2) for any A, $B\in X$, $A\cap B = C$; (3) for each $A\in X$, $A\text{-}C \neq \varnothing$. The set C is called the *core* of the axiation.

Definition 2. Let $U = \langle X, C\rangle$ be an axiation of S. A *dimensive addition* on S over U is a map $+\colon \cup_{A\in X} A^2 \to S$ such that for each $A\in X$, if x, $y\in A$ then $+(x,y)\in A$. (We can write $+(x,y)$ as $x+y$.)

An axiation of a set corresponds to the intuitive notion of a subdivision of a set into smaller sets of "similar" elements (in particular, elements having the same physical dimension). If we understand an axiation this way, then a dimensive addition is a partial binary operation which allows us to combine elements of the same kind.

Since the intuitive significance of the core C may not be obvious, a few remarks regarding its intended interpretation are in order. One reason for including C in Definition 1 is to keep the definition of axiation consistent with earlier rigorous treatments of dimensional quantities. Whitney, in particular, has developed a definition of a system of dimensional quantities ("quantity structure") in which a single zero element is shared by all the dimensions represented in the structure [6, p. 235]. (One might think of this as a formalization of the informal scientific practice of jotting down expressions like "$\Delta t = \Delta x = 0$" in place of more formal expressions like "$\Delta t = 0$ sec, $\Delta x = 0$ m.") The core C in Definition 1 allows us to construct axiations in which all axes share the same zero element, by letting C be the singleton of that element. If we do not choose to treat zero quantities in this way, we may set $C = \varnothing$ and distinguish between quantities like 0 m and 0 kg. Another reason for including C is that its presence in the definition helps us draw certain parallels between different algebraic systems (this remark will become clearer later in the paper). However, one can set $C = \varnothing$ throughout most of the rest of the paper if one prefers to do so.

The general restriction against adding quantities with different dimensions is perhaps the most conspicuous feature of the algebra of physical quantities. It is this feature which most clearly differentiates that algebra from the familiar mathematics of numbers. Another conspicuous feature of physical quantity algebra is the fact that the multiplication of two physical quantities always is posible. Multiplication is a binary operation on the set of all dimensional quantities. What is more, this multiplication induces a binary operation on the set of *dimensions* for those quantities. For example, if one multiplies a force by a length, one always gets a quantity with dimensions of energy. (In terms of the familiar notation of dimensional analysis, [force] $\times$ [length] = [energy].) We can express this characteristic of dimensional multiplication in terms of axiations as follows:

Definition 3. Let $U = \langle X, C\rangle$ be an axiation of S. A *dimensive multiplication* on S over U is a binary operation $\bullet$ on S such that for each $A, B \in X$, if $a, b \in A$ and $x, y \in B$ then $a\bullet x$ and $b\bullet y$ belong to the same element of X. Ordinarily we write $a\bullet x$ as ax.

The addition and multiplication of dimensional physical quantities are examples of dimensive addition and multiplication as defined in Definitions 2 and 3.

With the definitions of dimensive addition and multiplication in hand, we can define an algebraic system which has operations of these two sorts.

Definition 4. An *axial algebra* U is a quadruplet $\langle S, \langle X, C\rangle, +, \bullet\rangle$, where S is a set, $\langle X, C\rangle$ is an axiation on S, + is a dimensive addition on S over $\langle X, C\rangle$ (called *addition* in U), and $\bullet$ is a binary operation on S over $\langle X, C\rangle$ (called *multiplication* in U). The sets $A \in X$ are called the *axes* of U. The core C of the axiation is called the *core* of U.

An axial algebra is a system of elements classified into kinds such that quantities of different kinds cannot be added to one another and any two quantities can be multiplied together. Thus, the concept of an axial algebra abstracts out the most general and conspicuous features of the algebra of physical quantities.

Definition 5. A *dimensive algebra* is an axial algebra $\langle S, \langle X, C\rangle, +, \bullet\rangle$ whose multiplication $\bullet$ is a dimensive multiplication on S.

A dimensive algebra is an axial algebra in which the multiplication of elements of any two given kinds always yields an element of the *same* kind. Without this last property, we cannot regard the quantities in an axis as "being of the same dimension" in any reasonable sense.

Throughout the rest of the paper we will be concerned with dimensive algebras and not with general axial algebras. The letter U will denote a dimensive algebra unless otherwise stated; we assume that $U = \langle S, \langle X, C\rangle, +, \bullet\rangle$.

4. A LITTLE DIMENSIONAL ANALYSIS. The definition of a dimensive algebra is extremely general. This definition leaves out some of the simplest algebraic properties of actual physical quantities. For example, we have not yet assumed any associative or distributive laws. More importantly, the definition of a dimensive algebra does not contain anything reminiscent of units. Although the axes of a dimensive algebra are modeled after sets of equidimensional quantities, we have not assumed that all elements of a given axis can be expressed as products of coefficients with the same unit.

Even at this level of generality, we can do a little bit of dimensional analysis in a dimensive algebra. We begin with some definitions and a lemma. (For $A, B \subseteq S$, we use the familiar notation $AB = \{ab \mid a \in A, b \in B\}$.)

Lemma 6. If $A, B \in X$, then there is a unique $D \in X$ such that $AB \subseteq D$.

Proof. This follows from the fact that the product in U is a dimensive multiplication.

Lemma 6 establishes existence and uniqueness for the following definition:

Definition 7. The *dimension groupoid* of U is D(U) = $\langle X, \bullet_D \rangle$, where $\bullet_D$ is a binary operation on X defined as follows: for A, B$\in$X, A$\bullet_D$B is the D$\in$X such that AB$\subseteq$D.

In a dimensive algebra, quantities in the same axis may be regarded as having the same "dimensions," analogous to the dimensions of a physical quantity. The definition of the dimension groupoid formalizes this notion. This groupoid is the set of all possible "dimensions" (in a generalized sense) of quantities in U, together with the binary operation induced by the product in U. We have used the axes themselves as elements of the groupoid; in this we have followed Drobot [1, p. 93] and Whitney [6, pp. 229 and 236], who have identified the set of quantities of given dimension with the dimension itself. However, this way of constructing the groupoid is a convenience rather than a necessity. One could just as well pick out a "dimension symbol" for each axis, analogous to the [L], [T], etc. used in dimensional analysis, and use these symbols to form the groupoid. As long as the assignment of symbols to axes is biunique and preserves products, the resulting groupoid is isomorphic to D(U).

So far, the dimension groupoid D(U) of U does not have any structure other than a binary operation. We can do very little algebra in D(U); we cannot even assume that D(U) is associative, much less that it has an identity or inverses. Of course, all this would change if we imposed further laws on the operations of U, as we would have to do if we were trying to axiomatize dimensional analysis. But in spite of these limitations, we can do a little bit of dimensional analysis in D(U).

Consider, for example, the following simple physics problem:

> A ball of mass m is thrown straight upward in a uniform gravitational field of acceleration g. The ball rises to a height h. Find a formula for the initial kinetic energy E of the ball. (Assume that E depends only on m, g and h.)

One can guess the answer to this problem, up to a dimensionless multiplicative constant, if one knows the following information about the dimensions of the quantities involved:

[m] = [mass]

[g] = [acceleration]

[h] = [length]

[energy] = [force][length]

[force] = [mass][acceleration]

Using the last two relations, one gets [energy] = ([mass][acceleration])[length], from which one arrives at E = (mg)h, which is the correct relationship.

This bit of dimensional reasoning does *not* require any cancellation of dimensions or other operations that might be impossible in a general groupoid of dimensions. Thus, we can carry out dimensional reasoning of this simple kind in a general dimensive algebra, even though we may not be able to carry out any sophisticated dimensional analysis without further constraints on the operations in U.

Many of the dimensional arguments used in introductory physics courses are on the same level of simplicity as the physics problem set forth a few sentences ago, and can be carried through in a dimensive algebra in similar ways. Thus, we can obtain an important, though easy, fragment of dimensional analysis in a general dimensive algebra. We can do this without adding any further constraints such as associative or commutative laws. Note that our solution for the problem of the ball has not been shown to be unique; we have not ruled out the possibility that there are other dimensionally consistent expressions for E in terms of powers of m, g and h. However, introductory physics students do not have to worry about uniqueness after correctly guessing an equation in this way. For some pedagogical purposes, such as demonstrating the plausibility of an equation, the mere *existence* of a dimensional relationship is enough.

For a full rigorous treatment of dimensional analysis in dimensive algebras, we would have to prove an analogue of the well-known pi theorem. Previous authors have proved pi theorem analogues of various sorts for their systems [1, 6, 4]. I will not attempt this for a general dimensive algebra. My point is only that some of the more elementary and useful practices of dimensional analysis are applicable within the context of general dimensive algebras.

5. UNITS AND COEFFICIENTS. One of the most useful properties of dimensional quantities is the possibility of decomposing such quantities into coefficients and units. For every dimensional quantity x, we may write x as a product cu, where u is a unit (a fixed quantity with the same dimensions as x) and c is a coefficient (a dimensionless number). We have not built any decomposition of this sort into the definition of a dimensive algebra. However, we can show that a decomposition of this sort exists, in a very broad sense, for any of a large class of dimensive algebras. The decomposition in this general case may be quite different from the decompositions which we use when working with actual physical quantities. In particular, the coefficients which we get may be far more general than the real numbers. But the chief features of the coefficient-unit decomposition will be discernible.

To begin with, let us define isomorphisms of axial algebras. This is done in the expected way, consistent with universal algebra. An isomorphism of axial algebras is a 1-1 onto map of the underlying sets which preserves axes, cores, and both of the operations.

Definition 8. Let $U = \langle S, \langle X, C\rangle, +, \bullet\rangle$, $V = \langle S', \langle X', C'\rangle, +', \bullet'\rangle$ be axial algebras. An *isomorphism* from U to V is a 1-1 onto map $f:S\rightarrow S'$ such that:

(1) for each $A\in X$, there is exactly one $A'\in X'$ such that $f[A] = A'$;

(2) for each $A'\in X'$, there is exactly one $A\in X$ such that $f[A] = A'$;

(3) $f[C] = C'$;

(4) for each $a, b\in S$, $f(a\bullet b) = f(a)\bullet' f(b)$;

(5) for each $A\in X$ and $a, b\in A$, $f(a+b) = f(a)+' f(b)$.

The following definition also helps.

Definition 9. Let $U = \langle S, \langle X, C\rangle, +, \bullet\rangle$ be an axial algebra. A *section* of U is a map $s{:}X\rightarrow S$ such that for each $A\in X$, $s(A)\in A - C$.

Intuitively, a section corresponds to a system of units for the quantities in the axial algebra.

Given a dimensive algebra $U = \langle S, \langle X, C\rangle, +, \bullet\rangle$, we can construct another dimensive algebra V which is isomorphic to U and in which all quantities are expressed naturally as products of coefficients times units. Here is the procedure:

(1) First, note that each axis A of U is an additive groupoid with $+|_A$ as addition. For each axis A of U, select an additive groupoid A' which is isomorphic to A as an additive groupoid, under isomorphism $f_A{:}A\rightarrow A'$. (*Remark:* Ultimately we will want to regard A' as a set of coefficients for the quantities of dimension A.)

(2) Next, select a section s of U; for each $A\in X$, let $u_A = s(A)$. (*Remark:* This will serve as the "unit" for the quantities of dimension A.)

(3) For each axis A of U, construct a new set $A^* = \{R_A(a) \mid a\in A\}$, where:

$R_A(a) = \{\langle f_A(a), u_A\rangle\}$ for $a\notin C$

$R_A(a) = \{\langle f_A(a), u_A\rangle \mid A\in X\}$ for $a\in C$

The element of A' corresponding to u_A is $f_A(u_A)$; call this $1_{A'}$. (*Warning:* $1_{A'}$ need not be an identity element in A'.) The element of A* corresponding to u_A is $R_A(u_A)$; call this u_A^*. If $a\notin C$, then $u_A^* = \{\langle 1_{A'}, u_A\rangle\}$.

(5) Take the union of all the sets A*; call this union S*. Using the notation $R(a) = [R_A(a)$

where $a \in A$], note that $S^* = \{R(a) \mid a \in S\}$. (The object $R(a)$ is unique even if $a \in C$, since $R_A(a)$ is the same for all A if $a \in C$.) Let $X^* = \{A^* \mid A \in X\}$. Let $C^* = \{R(c) \mid c \in C\}$.

(6) Now we have all the pieces of a new dimensive algebra -- underlying set S*, axis set X*, and core C* -- except for the operations. Define operations +* and •* on S* as follows:

$R(a) +^* R(b) = R(a+b)$
$R(a) \bullet^* R(b) = R(a \bullet b)$

(6) Let $V = \langle S^*, \langle X^*, C^* \rangle, +^*, \bullet^* \rangle$, where $X^* = \{A^* \mid A \in X\}$.

The proof that that V and U are isomorphic should be easy for algebraically inclined readers. Before proving that V is isomorphic to U, let us consider the intuitive interpretation of V in terms of the concept of dimensional quantity from which the concept of a dimensive algebra was abstracted. Each axis A of U has been associated with a set A' of coefficients having the same additive structure as A. The axis A* of V corresponding to A consists essentially of a set of pairs $\langle r, u_A \rangle$, where r is a coefficient for A and u_A is the unit of dimension A. (Actually these "pairs" are singletons of pairs, except for the ones built from the core which are sets of pairs, but these details do not affect the underlying intuitive idea.) If we think of A as a system of quantities with fixed dimensions, then we can think of A* as a representation of the same quantities in terms of pairs of coefficients and units. The unit u_A is itself represented by the pair $\langle 1_{A'}, u_A \rangle$, where $1_{A'}$ is not necessarily an identity element in A' but is simply the coefficient which the quantity u_A gets when expressed in terms of the unit u_A.

This kind of decomposition resembles the customary decomposition into coefficients and units only in its most general outlines. Our new decomposition lacks many important properties of the usual decomposition. In particular, there is no way to multiply a coefficient by a dimensional quantity to obtain a new dimensional quantity. This last limitation seems less unreasonable for general dimensive algebras, since in the general case, the coefficients are *not* quantities within the dimensive algebra. A general dimensive algebra need not contain any elements distinguished as coefficients or as dimensionless quantities. Hence it is not obvious that elements of a dimensive algebra can be multiplied by coefficients; one should not expect to be able to take a product of $a\in A'$ and $b\in A^*$ in the system when a is not an element of the system at all. Later we may find out what we have to assume to overcome this limitation. For now, we simply prove that V and U are isomorphic.

Theorem 10. Let U and V be as in the preceding construction. Let $|C| < 2$. Then the map $R:S\rightarrow S^*$ which takes x to R(x) is an isomorphism of U into V.

Proof. We prove each part of the definition of isomorphism.

(R is 1-1) Case $|C| = 0$: $R(a) = \{\langle f_A(a), u_A\rangle\}$ for the unique A such that $a\in A$. If a and b are in different axes of S, then $u_A \neq u_B$, so $R(a) \neq R(b)$. If a and b are in the same axis but not identical, then since each f_A is 1-1, $R(a) \neq R(b)$. Case $|C| = 1$: If a and b are in C, then $a = b$ and $R(a) = R(b)$. If a and b are not in C, then the argument is similar to the $|C| = 0$ case.

(R is onto) We have observed that $S^* = \{R(a) \mid a\in S\}$.

(1) For each $A\in X$, there is at least one such A', namely A*. To show that this A' is unique, note that $A^* = \{R(a) \mid a\in A\}$.

(2) For each $A^*\in X^*$, we have $f[A] = A^*$. To show that this A is unique, note that any $R(a)\in A^*$ is a set of pairs whose second members are u_A, and that there is a 1-1 onto map between values of A and values of u_A (since u_A is not in C for any choice of A).

(3) By definition of C.

(4) Follows from definition of $\bullet^*$.

(5) Follows from definition of $+^*$.

Note that Theorem 10 does *not* say that every dimensive algebra contains units and coefficients which allow for the customary representation of its elements. Rather, Theorem 10 only says that, given a dimensive algebra U that meets certain conditions, we can find another dimensive algebra which is isomorphic to U and which is made of unit-coefficient pairs. The relationship between U and the new dimensive algebra V may remind one of the relationship between a group and (the image of) its representation. A representation shows how to "express" group elements as matrices; the map R shows how to express elements of a dimensive algebra as coefficient-unit pairs. But just as the representation is not the same as group, the algebra of pairs is not the underlying dimensive algebra by any means.

Now let us return to the question of multiplying coefficients by dimensional quantities. The simplest and most natural way to make this possible is to consider dimensive algebras in which the coefficient set A' for each axis A is itself an axis in the dimensive algebra. This is what happens in the customary algebra of physical quantities, where the sets of coefficients for the quantities are the reals or subsets thereof. One takes the set of coefficients (dimensionless numbers) to be among the physical quantities; then one treats the products and sums of all the quantities on the same footing, letting the sum and product of dimensionless quantities reduce to the usual sum and product of reals when restricted to the real coefficients. In a general dimensive algebra, things can be more complicated. To begin with, there may be different (non-isomorphic) sets of coefficients for different axes. (For example, there might be an axis isomorphic to the reals, for which the reals would form a natural set of coefficients, and there might be another axis isomorphic to some other algebraic structure, say a space of vectors or operators.) Also, the coefficients in a general dimensive algebra need not, in general, be the same as the "dimensionless" quantities in the system. If there is an axis A of U such that for any $a \in A$ and $x \in S$, ax is in the same axis as x, then A may be regarded as an axis of "dimensionless" quantities, and the dimension group for U becomes a groupoid with identity (and a semigroup if the product in U is associative). However, there is no guarantee that such an

A is a suitable set of coefficients for all the axes of U. For example, A might be the set of nonnegative real numbers, and there might be a set of real-valued quantities (positive, zero and negative) in the algebra.

The preceding remarks suggest what kinds of elements might be thought of as "dimensionless" or as coefficients in a general dimensive algebra. In this paper I will not attempt to classify the different possible kinds of dimensionless quantities and coefficients. The following definitions, which are suggested by the preceding remarks, show that there are several possibilities which, in the general case, need not coincide with each other.

Definition 11. Let $A, B \in X$. A is *dimensionless for* B iff $AB \subseteq B$. In this case, we also say that each element of A is dimensionless for B.

Definition 12. Let $A \in X$. A is *globally dimensionless* iff for each $B \in X$, A is dimensionless for B. In this case, we also say that each element of A is globally dimensionless for B.

Definition 13. Let $A, B \in X$. A is a *full coefficient axis for* B iff there is a $u \in B$ such that $Au = B$.

Definition 14. Let $A \in X$. A is a *global full coefficient axis* iff for each $B \in X$, A is a full coefficient axis for B.

Complementary to the concept of a coefficient axis is, of course, the concept of a unit.

Definition 15. Let $A, B \in X$, and let A be a full coefficient axis for B. Let $u \in B$. Then u is a *unit* for B over A iff $Au = B$.

These definitions, like most definitions in this paper, are quite general. The "coefficients" and "dimensionless quantities" that they define need not have the nice properties that we are used to. One of these nice properties is that one can carry out physical calculations by dropping the units and using the coefficients only. Every student of basic physics knows that one can take the equation F = ma and the values m = 2 kg and a = 3 m s^{-1}, and find out the value of F by means of the following reasoning: F = ma = (2 kg)(3 m s^{-1}) --> (2)(3) = 6 --> 6 kg m s^{-1}. It may not be wise to drop the units in this way, since carrying the units through can prevent mistakes, but at least it is possible to drop the units. Even if we carry through the units, we find ourselves behaving as though the coefficients and the units can be multiplied independently of each other: F = ma = (2 kg) • (3 m s^{-1}) --> (2 • 3) • (kg • m s^{-1}) = 6 kg m s^{-1}. Physical mathematics depends upon our ability to treat physical quantities with numerical coefficients as though they were numbers. This is equivalent to our ability to multiply and add dimensional quantities by multiplying and adding units and coefficients separately.

A dimensive algebra in which the coefficients have these vital properties must satisfy the following definition.

Definition 16. A dimensive algebra $U = \langle S, \langle X, C\rangle, +, \bullet\rangle$ is *cosufficient* iff:

(1) for each axis $A \in X$, there is a full coefficient axis $B \in X$ for A;

(2) for each axis $A \in X$, if $B \in X$ is a full coefficient axis for A and u is a unit for A over B, then $(a + b)u = au + bu$ for each $a, b \in B$;

(3) for every $A_1, A_2 \in X$, if

(i) B_1 and B_2 are full coefficient axes for A_1 and A_2 respectively,

and

(ii) u_1 and u_2 are units for A_1 over B_1 and for A_2 over B_2 respectively,

then for each $b_1 \in B_1$ and $b_2 \in B_2$, $(b_1u_1)(b_2u_2) = (b_1b_2)(u_1u_2)$.

A cosufficient dimensive algebra is so called because it has *coefficients* with *sufficient* properties to ensure that coefficients and units can be segregated in the customary way.

Note that a dimensive algebra in which the commutative law for multiplication and the usual distributive laws hold is automatically cosufficient, provided only that each axis has a full coefficient axis.

6. A NOTE ON APPLICATIONS. The treatment of coefficients in this paper allows for dimensive algebras with coefficients of many different kinds. In addition to the usual real-valued coefficients, it allows for the possibility of coefficients that belong to arbitrary groups, rings, fields, etc. Such coefficients are not mere mathematical possibilities; they actually do occur in physics.

A striking example comes from the *operators* used in physics. Many of these clearly have physical dimensions, although they cannot be decomposed into units times numerical coefficients. The familiar differential operator ∇ (del), which plays numerous roles in physics, provides an example of this. ∇ has physical dimensions $[\text{length}]^{-1}$. This implies that it can be assigned units; for example, ∇ has unit m^{-1} in the MKS system. But what is the coefficient that goes with this unit? The coefficient is not a number, but a more general algebraic entity -- specifically, a dimensionless operator. Although ∇ is not a physical quantity in the usual sense, it still is a mathematical object used in physics, and it is manipulated like a physical quantity during quantum mechanical calculations. One can think of it as a generalized dimensional quantity. Certainly it is possible to use dimensional analysis on operator equations; whenever we do so, we tacitly assume that the operators involved are elements of some algebra of dimensional objects.

To apply our familiar concepts of coefficients and units to dimensional objects like ∇, we must allow for coefficients which behave very differently from numbers. For such objects, the multiplication of coefficients is not generally commutative. Dimensive algebras provide an algebraic framework for the treatment of generalized physical quantities. They can accommodate operators and c-numbers on an equal footing.

Operators are not the only examples of dimensional quantities which do not fit into the number-times-unit paradigm of physical quantity. Vector and tensor physical quantities provide further examples. Although such quantities are not the results of measurements in the strict sense (see [5, p. 117] on 3-spaces), they nevertheless are dimensional quantities and can be used in dimensional analysis arguments. (Whitney discusses dimensional vectors and tensors within the context of his theory of "quantity structures" in [6, pp. 246-247].) If one regards these dimensional quantities as coefficients times units, then the coefficients are vectors or tensors, which do not have all the familiar algebraic properties of real numbers. Similar remarks can be made for other multicomponent objects of physical mathematics, such as spinors and matrices.

It is even conceivable that directly measurable scalar quantities may have unexpected algebraic properties in certain cases. Whitney has discussed a case involving the counting of physical objects from a finite set, and has noted that the resulting system of quantities does not satisfy the cancellation laws of the natural numbers [5, p. 117].

In view of these examples, it might be interesting to investigate algebraic structures which reflect, not only the algebraic properties of ordinary measurable physical quantities, but the algebraic properties of all mathematical objects having physical dimensions. Dimensive algebras are structures of this sort.

7. ADDING QUANTITIES WITH DIFFERENT DIMENSIONS (ALMOST). It is a truism of physical quantity algebra that quantities of different dimensions cannot be added.[1] This generalization is reflected in the definition of a dimensive algebra, where addition is restricted to pairs of quantities on the same axis. However, there are some cases in which physical quantities have the same dimensions, but in which the addition of those quantities is formally analogous, in some respects, to addition of quantities with different dimensions.

To see what I mean by this statement, consider vector-valued physical quantities. Ordinarily, such a quantity is regarded as a product of a vector-valued coefficient and a unit. Consider, for example, the position vector **v** of a particle located 2 m along the x-direction and 3 m along the y-direction from the coordinate origin. We can write **v** as (2**i** + 3**j**) m; here m is the unit and the dimensionless vector (2**i** + 3**j**) is the coefficient. Alternatively, we can write **v** as (2**i** m + 3**j** m). In this case, the vector is written as the sum of two quantities, both with the unit m, but with different dimensionless vectors as coefficients. A third approach, formally equivalent to the preceding ones, is to write the vector as (2m)**i** + (3m)**j** , and think of it as a vector with non-real quantities as components. (This last approach is essentially the one Whitney takes in dealing with vector and tensor quantities in [6, pp. 246-247].)

These ways of writing **v** would seem completely natural if the vector 2**i** + 3**j**, or the separate vectors 2**i** and 3**j** (or (2m)**i** and (3m)**j**), were quantities obtained directly from acts of measurement. In real life, however, we do not normally read off quantities like 2**i** + 3**j**, 2**i**, or 3**j** from a meter. Instead, we measure a vector by measuring the components (dimensionless scalars, like 2 and 3) and assembling these components into a vector after the fact. What is more, we do not measure different components of a vector in exactly the same way. If we measure a vector quantity to have the value 2**i** + 3**j**, then the 3 and the 2 are obtained via slightly different procedures (perhaps by laying off a meter stick in two different, perpendicular directions). Thus, the components of the vector do not have exactly the same physical significance; although both represent displacements, they do not represent exactly the same *kind* of displacement. This fact becomes especially prominent when the components are measured in substantially different ways -- for example, when the 3 is measured by an altimeter and the 2 is an east-west distance measured by pacing. The partial vectors 2**i** m and 3**j** m do not have exactly the same intuitive physical meaning. Although both of them are displacements and are expressible in meters, they are not quantities of exactly the same physical sort.

By adding quantities like 2**i**m and 3**j**m, we are combining quantities representing the results of physical measurements of different kinds. Instead of thinking of this sum as a vector coefficient times the unit m (as if it were a length which happens to equal a vector amount of meters!), we might try thinking of it as a combination of two quantities with different dimensions: one with "units" **i**m, representing a "dimension" of "length in the x-direction," and the other with "units" **j**m, representing a "dimension" of "length in the y-direction." Then the vector comes out as a quantity with "mixed dimensions."

Of course, the two component vectors 2**i**m and 3**j**m do not really have different dimensions. But in some respects, they act like they do. They are measured in different ways. (Laying off a meter stick in an east-west direction is a different physical operation from laying off a meter stick in a north-south direction -- albeit only a slightly different operation.) They are represented by mathematical objects which differ in more than just their scalar coefficients. Further, their addition does not lead to a quantity of the same kind as either of the original quantities; the sum 2**i**m + 3**j**m is a quantity of a "new" kind, which differs from both 2**i**m and 3**j**m by something far more than a scalar multiplier.

We have unearthed a potentially interesting fact: that within a class of quantities with the same physical dimensions, there may be various subclasses which are not quite of the same physical kind, and which act, in certain respects, as though they had different dimensions. The quantities in a given subclass are distinguished mainly by the fact that adding them to each other yields a quantity in the same subclass, while adding them to quantities from outside the subclass does not always yield a quantity in the same subclass.

We capture this insight in the following very general definitions.

Definition 17. Let A be an axis in a dimensive algebra U. Let $B \subset A$. Then B is a *hemiaxis* of A iff B is closed under + and, for some $b \in B$ and $c \in A\text{-}B$, $b+c \notin B$.

Definition 18. Let A be an axis in a dimensive algebra U. Let B be a hemiaxis of A. Then B is a *strict hemiaxis* of A iff for all $b \in B$ and $c \in A\text{-}B$, $b+c \notin B$.

Definition 19. Let A be an axis in a dimensive algebra U. Let W be a set of hemiaxes of A. We say that W is a *mixing basis* for A, and that A is a *mixing axis* for W, iff every element a of A can be expressed as a finite sum of elements of hemiaxes in W.

In the case of quantities in a real vector space, the best example of a mixing basis is the set of all one-dimensional subspaces generated by members of a basis of the vector space. These are what one thinks of as axes of the vector space. There is more than a passing analogy between these axes and the axes of a dimensive algebra! In real vector spaces, the zero vector is in all hemiaxes in this mixing basis. The zero quantity in each of the hemiaxes is common to all the hemiaxes; for example, 0**i** m = 0**j** m. Thus, the hemiaxes share something like the core of a dimensive algebra. One reason that I included the core C in the definition of a dimensive algebra is to allow a stronger analogy to be drawn between the structure of a dimensive algebra and that of a mixing axis. Indeed, if one allows dimensive algebras to have a nonempty cores, then a mixing basis sometimes turns out to be a dimensive algebra! (Consider the set of displacements on the axes of a Cartesian coordinate system in Euclidean 3-space, with the cross product as product.) I will not explore this analogy further here, but it might be worth exploring.

We may wish to think of different hemiaxes in an axis as having different "units" in a generalized sense. The following definition captures this generalized concept of "unit."

Definition 20. Let A be an axis in a dimensive algebra U. Let B be a hemiaxis of A. If U contains a coefficient axis C such that B = Cu for some $u \in B$, then we call u a *hemiunit* for B with respect to the coefficients in C.

A mixing axis can occur whenever a dimensive algebra contains an axis of quantities whose coefficients have a set of natural decompositions into components. This includes vector- and tensor-valued quantities, complex- and hypercomplex-valued quantities, and spinor-valued quantities, all of which occur in physics. A quantity of any of these types can be regarded as a multicomponent coefficient times a dimensional unit, or as a sum of two or more quantities, each consisting of a coefficient and a hemiunit. (In the case of a complex-valued quantity with unit u, the hemiunits are u and iu. Thus, for the complex potential (3 + 5i) volts, the hemiunits are volt and i•volt.)

8. CONCLUDING REMARKS. In this paper I have introduced and explored the concept of a dimensive algebra. Dimensive algebras are potentially interesting for at least three reasons. First, they can help us to better understand the foundations of dimensional analysis. Second, they allow us to generalize the algebra of physical quantities to encompass other mathematical objects, such as vectors and operators, which bear physical dimensions but do not fit comfortably within existing treatments of the algebra of dimensional quantities. Third, dimensive algebras may be of interest in themselves as objects of algebraic study. A deeper understanding of the structure of these objects might shed light on other areas of algebra.

These remarks point to two possible directions for further investigation. The first direction is the development of generalizations of dimensional analysis in which the "quantities" are elements of a general dimensive algebra. This project, which might be dubbed "exotic dimensional analysis," could perhaps teach us something about the foundations of ordinary dimensional analysis. Also, this project might help us to better understand other aspects of the mathematics used in physics. One possible problem in this area is the determination of the conditions that a dimensive algebra must satisfy for suitable analogues of the pi theorem to hold in the algebra. The second direction is the study of dimensive algebras from a purely algebraic standpoint. One might study structure and classification for dimensive algebras as one does for groups, rings, fields, etc. Nontrivial problems are far more likely to arise for dimensive algebras which satisfy constraints that connect together addition and multiplication (distributive laws provide simple examples of these constraints). For dimensive algebras without these constraints, the classification problems will tend to collapse into classification problems for semigroups.

The present paper has been devoted mostly to the presentation and motivation of new concepts. Weightier results should be forthcoming once some substantial algebraic problems have been investigated.

FOOTNOTES

1. Ultimately, it is not clear whether the prohibition against such addition is a theoretical necessity or is merely a great convenience. Hoffmann, noting that sums of spatial displacements with forces are not found experimentally and have no name, once remarked that "lack of a name proves no more than that if the resultant exists, it has not hitherto been deemed important enough to warrant a name" [2, p. 13]. Whitney [5, p. 117] has pointed out a simple example in which it seems natural to treat certain discrete count quantities as quantities with mixed units. If quantities with mixed units ever were introduced into science, dimensive algebras with mixing axes would provide a natural framework for their rigorization.

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