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WE HOLD THESE TRUTHS TO BE SELF-EVIDENT: BUT WHAT DO WE MEAN BY THAT?

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> Mathematicians at first distrusting the new ideas (Cantor made his first discoveries in 1873), then got used to them; ...
> Waismann (1982, p. 102)

Abstract. At the beginning of Die Grundlagen der Arithmetik (§2) [1884], Frege observes that "it is in the nature of mathematics to prefer proof, where proof is possible". This, of course, is true, but thinkers differ on why it is that mathematicians prefer proof. And what of propositions for which no proof is possible? What of axioms? This talk explores various notions of self-evidence, and the role they play in various foundational systems, notably those of Frege and Zermelo. I argue that both programs are undermined at a crucial point, namely when self-evidence is supported by holistic and even pragmatic considerations.

At the beginning of *Die Grundlagen der Arithmetik* (§2) (1884), Gottlob Frege observes that "it is in the nature of mathematics to prefer proof, where proof is possible", noting that "Euclid gives proofs of many things which anyone would concede him without question". Frege sets himself the task of providing proofs of such basic arithmetic propositions as "every natural number has a successor", the induction principle, and "$1 + 1 = 2$".

Frege's observation was true in Euclid's day, and it remains true now. We still admire the achievements of Euclid, Archimedes, Cauchy, Weierstrass, Bolzano, Dedekind, Frege, and a host of others on providing rigorous proofs of "many things that formerly passed as self-evident", as Frege put it (§1). Many of these are propositions that no one in their right mind would doubt—unless it be on skeptical or nominalist grounds (in which case mathematical proof would not settle the issue).

Nevertheless, thinkers differ widely on *why* it is that we prefer proof, and this question goes to the very heart of mathematics. My topic here is closely related to this. It is a commonplace that one cannot provide a nontrivial or noncircular proof of every known proposition. Frege's observation is that mathematics prefers proof, "*where proof is possible*". What about cases where proof is not possible? What is the epistemic status of the axioms, or basic truths (or inference principles), from which other propositions are derived? If we claim to know the theorems, on the basis of the proofs, then surely we must claim to know the axioms (or inference principles)? How?

§1. Axioms as definitions: They tell us what we are talking about.[1] A relatively recent perspective is that the axioms of a given branch of mathematics serve as an *implicit*

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[1] I do not claim that what is said in this section is definitive, but pursuing the matter further would take us too far afield. For more, see Shapiro (2005). The purpose of this section, and the next two, is to raise and give reason to dismiss certain conceptions of axioms, before we take on the notion of self-evidence.

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definition of the primitive terms of the branch. Hilbert's (1899) *Grundlagen der Geometrie* represents the culmination of a trend toward this orientation toward branches of mathematics. The early pages of the book contain phrases like "the axioms of this group define the idea expressed by the word 'between' . . ." and "the axioms of this group define the notion of congruence or motion". The idea is summed up as follows:

> We think of . . . points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated", "between", "parallel", "congruent", "continuous", etc. The complete and exact description of these relations follows as a consequence of the *axioms of geometry*.

The central aspect of the axiomatization is that *anything at all* can play the role of the undefined primitives of points, lines, planes, etc., so long as the axioms are satisfied. Hilbert was not out to capture the essence of a specific chunk of reality, be it space, the forms of intuition, or anything else. Otto Blumenthal reports that in a discussion in a Berlin train station in 1891, Hilbert said that in an axiomatization of geometry, "one must always be able to say, instead of 'points, straight lines, and planes', 'tables, chairs, and beer mugs'."[2]

Hilbert's protégée Bernays (1967, p. 497) summed up the aims of the new geometry:

> A main feature of Hilbert's axiomatization of geometry is that the axiomatic method is presented and practiced in the spirit of the abstract conception of mathematics that arose at the end of the nineteenth century and which has generally been adopted in modern mathematics. It consists in abstracting from the intuitive meaning of the terms . . . and in understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation . . . for which the axioms are satisfied. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure . . . [On] this conception of axiomatics, . . . it is insisted that in reasoning we should rely only on those properties of a figure that either are explicitly assumed or follow logically from the assumptions and axioms.

Frege had a more traditional orientation toward axioms. In a letter dated December 27, 1899, he tried to correct Hilbert on the nature of definitions and of axioms: definitions give the meanings and fix the denotations of terms, and axioms express truths.[3] These fundamentally different notions should not be confused. Frege remarked that, despite what Hilbert (1899) claims, the book does not provide a definition of, say, "between" since the axiomatization "does not give a characteristic mark by which one could recognize whether the relation Between obtains":

> . . . the meanings of the words "point", "line", "between" are not given, but are assumed to be known in advance . . . [I]t is also left unclear what you call a point. One first thinks of points in the sense of Euclidean geometry, . . . But afterwards you think of a pair of numbers as a point . . .

[2] "Lebensgeschichte" in Hilbert (1935, pp. 388–429); the story is related on p. 403.

[3] The correspondence between Frege and Hilbert is published in Frege (1976) and translated in Frege (1980).

> Here the axioms are made to carry a burden that belongs to definitions ... [B]eside the old meaning of the word "axiom", ... there emerges another meaning but one which I cannot grasp.

According to Frege, axioms, on the other hand,

> ... must not contain a word or sign whose sense and meaning, or whose contribution to the expression of a thought, was not already completely laid down, so that there is no doubt about the sense of the proposition and the thought it expresses. The only question can be whether this thought is true and what its truth rests on. Thus axioms and theorems can never try to lay down the meaning of a sign or word that occurs in them, but it must already be laid down.

In sum, Frege complained that if the terms in the proposed "axioms" do not have a prior meaning, then the sentences cannot be true (or false), and thus they cannot be axioms. If the terms do have meaning beforehand, then the "axioms" cannot be definitions.

The crucial Fregean principle here is that every well-formed sentence in a mathematical theory makes a fixed assertion about a fixed collection of objects and concepts. Each such sentence has a truth-value determined by the referenced objects and concepts. Geometry is about (Euclidean) space; arithmetic is about numbers. Coffa (1991) calls this view "propositionalism", as it holds that mathematical sentences express traditional propositions. Hilbert rejected propositionalism.

Following a suggestion of Geoffrey Hellman's, let us say that for Frege, the axioms and theorems of arithmetic and geometry are *assertory*. For Hilbert, in contrast, axioms and theorems are *algebraic*. Assertory sentences are meant to express propositions with fixed truth-values, while algebraic sentences are schematic, applying to any system of objects that meets the conditions specified by the axioms.[4]

So the algebraist has an answer to our question concerning the epistemic status of axioms: they are *definitions*, and so not asserted as such. Instead, the axioms fix the subject matter, telling us which structure, or structures, we are talking about. So there is no serious question as to how we know these axioms. Strictly speaking, we do not know them—and we do not know the theorems. What we know, it seems, is that the theorems follow from the axioms, or that the theorems are true in any structure described by the axioms. An advocate of the algebraic perspective also has an answer to the question of why the mathematician prefers proof, where proof is possible. By "proof", the algebraist means a derivation in an explicitly formulated axioms system. So proof is a matter of logical hygiene. We need to know the principles that characterize the structure or structures in question, and why the theorem in question follows from these principles.

It is not clear that one can take a Hilbert-style, algebraic perspective for all of mathematics. Consider, for example the following statement:

(*) The statement that any two distinct points line on exactly one line is a logical consequence of the axioms of Euclidean geometry.

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[4] For what it is worth, the ante rem structuralism developed in Shapiro (1997) attempts to bridge the gap between these two perspectives. According to that view, if an implicit definition is satisfiable and categorical, then it characterizes a structure, and the statements in the definition—the axioms—are assertory truths about that structure. So the same sentences can be algebraic in one context, and assertory in another.

Prima facie, (*) is itself assertory, telling us something about the relation of logical consequence. It also seems to be a statement *within* mathematics, to the extent that axiomatizations and logical consequence are mathematical matters. A Hilbertian algebraist says things like "In any Euclidean system *S*, for any two distinct objects *a*, *b* in *S* that are 'points-in-*S*', there is a unique 'line-in-*S*' on which *a* and *b* lie". Presumably, statements like that are made in their capacity as mathematicians.

Can we understand sentences like (*) as algebraic? How? What are the axioms for the theory in which (*) is a theorem? And is there a (*)-like statement for such a meta-theory. We seem to be off on a regress.

Hilbert's *Grundlagen* helped launch the rich field of meta-mathematics. What is the status of *that* putatively mathematical enterprise? Hilbert provided consistency and independence proofs of various sets of axioms by finding interpretations that satisfy them. Typically, he would interpret a batch of axioms with constructions on real numbers. This free reinterpretation of axioms is a main strength of contemporary mathematical logic, and a mainstay of mathematics generally. What is the algebraist to make of the statements of consistency themselves, and the various meta-theoretic proofs that Hilbert provides? What, for example, is the epistemic status of the statement that Euclidean geometry is consistent, and the statement that a given non-Euclidean axiomatization can be interpreted in real analysis? Are such statements algebraic or assertory, or both at once?

This matter is not treated explicitly in Hilbert's *Grundlagen*, and it is hard to be definitive on what his view was, or should have been. But it seems clear that at some level, the meta-theory—the mathematical theory in which the consistency or satisfiability of an axiomatization is established—is *not* to be understood algebraically, not as another theory of whatever satisfies *its* axioms. Rather, the statement that a given theory, such as Euclidean geometry, is consistent is itself assertory. Indeed, Hilbert *asserts* such statements, or at least seems to, and uses them to support his theory. The notion of consistency is a contentful property of theories, and is not to be understood as defined implicitly by the axioms of some meta-theory. For one thing, the meta-theory is not axiomatized in the *Grundlagen*, and so there is no implicit definition of meta-theoretic notions. This, of course, is not decisive, since it would be a routine exercise to axiomatize the meta-theory of Hilbert's *Grundlagen*, at least for a contemporary student of mathematical logic. More importantly, an algebraist who thinks of the axiomatized meta-theory as algebraic would have to worry about *its* consistency. How would we establish that? The ensuing regress is vicious to the epistemological goals of the *Grundlagen*.[5]

If this is correct, and meta-mathematics is assertory, then we can reinstate the question of this paper for it. Within the meta-theory, as anywhere else, mathematics prefers proof where proof is possible. What is the epistemic status of those true meta-theoretic propositions for which no (noncircular or nontrivial) proof is possible? If we claim to know meta-theoretic theorems on the basis of these propositions, then, presumably, we must know the basic propositions themselves. How?

The later Hilbert program (e.g., 1925) provided a framework for answering these very questions. The meta-theory for dealing with consistency was to be finitary arithmetic, focused directly on formal languages themselves. Hilbert (1925) is explicit that this meta-theory is contentful, and thus assertory—not algebraic. To use Coffa's terminology, Hilbert was a propositionalist about his proof theory.

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5 I am indebted to an anonymous referee (for Shapiro, 2005) for this point.

Of course, finitary proof theory, whatever its details, proved to be all but useless for establishing consistency, thanks to Kurt Gödel's incompleteness theorems (but see Detlefsen, 1986). But, as Bernays notes, the powerful algebraic-cum-model-theoretic approach to mathematics continues to thrive, as well it should. And our questions concerning meta-theory remain. In practice, how do we satisfy ourselves that a given characterization—whether it is a traditional axiomatization or a type of category—is coherent, and thus characterizes a structure or a possible system?

On the contemporary scene, questions of coherence, satisfiability, and the like, are referred to set theory, which has become the default meta-mathematics. Prima facie, to play this role of helping to decide questions about how the theories relate to each other, and whether a given theory is consistent or satisfiable, set theory must be assertory, not algebraic. That is, we need to *assert* that a given theory is consistent, satisfiable, or whatever, and it is the language of set theory in which these assertions are negotiated. To play this meta-mathematical role, it seems, set theory is not just a theory of whatever satisfies its axioms. And so our question about the epistemic status of its axioms is still pressing. So the Hilbertian algebraist does not completely avoid the question of this paper, even if she manages to narrow its scope to a single foundational theory.

§2. It is obvious, my dear Watson. Famously, Gödel (1964, p. 484) remarked that the axioms of set theory "force themselves upon us as being true". Although I do not wish to engage exegetical issues here, this at least sounds like a statement of the psychology, or perhaps the phenomenology, of mathematics, or at least of set theory. Gödel notes that the axioms are *obvious*. Write them down, think about what they say, and you will come to believe that they are true.

Note, however, that at least some of the axioms of set theory do not force themselves on us as true on a first or even a second or third reading. The axioms of infinity and replacement, for example, are hardly obvious at first. If these axioms force themselves on us, it is only after we become familiar with the system, and perhaps its intuitive underpinning: the iterative conception of set. We work with the language, derive some basic theorems, develop intuitions, check the theorems against these intuitions, etc. In other words, we immerse ourselves in the practice. Then, perhaps, after a while it all becomes obvious. Then, perhaps, the axioms force themselves on us as being true.

Even after this immersion, there is room for doubt due to the extensive ontological consequences of the axioms. For example, the axioms of set theory entail the existence of a cardinal κ that is a fixed point in the series of alephs: $\aleph_\kappa = \kappa$. As Boolos (1998b) asked, do we really believe in the existence of such a set?[6] To take another example, do we really have to believe in the existence of a decomposition of a unit sphere that, recombined via rigid motions, yields two unit spheres? Does nervousness about these consequences shed some doubt on the axioms, or does it at least shed doubt on the claim that the axioms are obvious?

In any case, we are asking here about the *epistemic* status of the axioms of assertory mathematical theories, not the psychological states of some or even typical practitioners of

[6] The first fixed point in the aleph series might not be all that large. It is consistent with ZFC that this κ is smaller than the continuum. Boolos could have picked another example, perhaps a fixed point in the beth series: a cardinal λ such that $\lambda = \beth_\lambda$. Even this λ is not large by the set-theorist's standards—it is below the first inaccessible—but λ is large enough to boggle the mind of just about anyone else.

the theory. Presumably, the theorems are known on the basis of the axioms (and the logic). How are the axioms *known*? Whether obviousness, or obviousness-after-immersion, has epistemic weight depends on what the correct epistemology for mathematics is. Moreover, if it is immersion in the practice that makes the axioms obvious, then the axioms are not properly foundational. Some of the theorems play a role in how the axioms become obvious—in how they force themselves on us as being true. It is the practice as a whole that generates the psychological state or the phenomenological feel of certainty.

From this perspective, what becomes of Frege's observation that "it is in the nature of mathematics to prefer proof, where proof is possible"? The reason for this, it seems, is to trace logical connections among various propositions within a given mathematical theory. This is part of what it takes to immerse oneself in a practice, which is what it takes for the axioms to force themselves on us as true. The more proofs we have, the stronger the feeling of certainty. But, at this level, it is still merely a feeling.

§3. Maybe axioms are not so evident. Russell's (1993) *Introduction to Mathematical Philosophy*, first published in 1919, opens by describing two opposite directions in which mathematics may be pursued. The more common is toward more complexity: "from integers to fractions, real numbers, complex numbers ... and on to higher mathematics". Russell sometimes calls this "mathematics in the ordinary sense". He calls the other, less familiar direction "mathematical philosophy". It

> ... proceeds, by analysing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterises mathematical philosophy as opposed to ordinary mathematics ... Early Greek geometers, passing from the empirical rules of Egyptian land-surveying to the general propositions by which those rules were found to be justifiable, and thence to Euclid's axioms and postulates, were engaged in mathematical philosophy, according to the above definition; but when once the axioms and postulates had been reached, their deductive employment, as we found it in Euclid, belonged to mathematics in the ordinary sense.

So, according to Russell, axioms and postulates are the results of this "analysis" of mathematics. Although, as above, I do not pursue exegetical issues here, the idea seems to be, or could have been, that the resulting axioms and postulates are known by a sort of abduction, an inference to the best explanation. We believe the axioms *because* we can deduce the more familiar propositions of mathematics from them. The axioms thus organize and explain the more familiar propositions, the ones that are perhaps self-evident, or at least more evident. Russell continues:

> The most obvious and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle. Just as the easiest bodies to see are those that are neither very near nor very far, neither very small nor very great, so the easiest conceptions to grasp are those that are neither very complex nor very simple (using "simple" in

> a *logical* sense). And as we need two sorts of instruments, the telescope and the microscope, for the enlargement of our visual powers, so we need two sorts of instruments for the enlargement of our logical powers, one to take us forward to the higher mathematics, the other to take us backward to the logical foundations of the things we are inclined to take for granted in mathematics. We shall find that by analysing our ordinary mathematical notions we inquire fresh insight, new powers, and the means of reaching whole new mathematical subjects, by adopting fresh lines of advance after our backward journey.

Russell's perspective here seems to be holistic. We know a mature axiomatic mathematical system as a whole, and accept it for its deductive and explanatory power. The system includes its axioms, the derivations of the more certain intermediate theorems, and the derivations of the deeper and less obvious, less self-evident theorems. All of these propositions support each other. In any case, the axioms and postulates of the resulting deductive system may indeed end up as evident, but they are not *self*-evident. The evidence for them comes from some of their consequences, and from the power and coherence of the system as a whole.

So what becomes of Frege's observation from which we began: "it is in the nature of mathematics to prefer proof, where proof is possible"? The above passage suggests that, for Russell, at this phase of his thinking, and for some cases, the idea is that proofs help to systematize mathematics, discovering logical links between various propositions, and they help to forge links between disparate fields. From this perspective, self-evidence is simply not part of the picture or, if it is, only some intermediate propositions have that status.

§4. Epistemic foundation. We turn, at last, to our primary target. The traditional view is that the axioms of a properly formulated, assertory mathematical theory should be *self-evident*, and the proofs should proceed by self-evident steps from such axioms alone. This *Euclidean foundationalist* approach thus puts the theorems on the most solid epistemic foundation possible: they cannot be more secure, or more certain. It is not much of a distortion to describe rationalism as the attempt to extend this Euclidean methodology to all knowledge, or at least all of science.[7]

The typical Euclidean foundationalist is not happy with the perspectives sketched in any of the previous sections. She thinks, first, that the theorems of typical branches of mathematics—arithmetic, real and complex analysis, etc.—are assertory, against the Hilbert-style algebraist. Our Euclidean is also not content with an observation—no doubt true—that typical axioms are obvious to typical practitioners of the field. The Euclidean idea is that proper axioms are *epistemically* secure and not just psychologically certain. We know the axioms, individually, to be truths about their subject matter, and this knowledge does not rely on anything else. The rest of the theory is built on the foundation

[7] The contemporary literature in general epistemology contains a number of internalist foundationalist proposals concerning a priori (and, in some cases, a posteriori) knowledge. Each such program has to have something to say about how propositions at the base of the foundation are known, and this basic knowledge is presumably noninferential. There is no explicit engagement with such proposals here, mostly due to lack of space and my own competence. Present concern is limited to mathematics, and the role of the Euclidean foundational programs there. I will not speculate on the extent to which present considerations bear on foundationalist epistemology generally, even those that rely on a notion of self-evidence.

of the axioms. The Euclidean foundationalist thus also rejects the holistic perspective of the previous section. For the Euclidean, mathematics is to be built, or rebuilt, on a firm epistemological foundation, starting with propositions that are as epistemically secure as possible, known in and of themselves. The theorems are known on the basis of these axioms.

So we repeat our central question. According to the Euclidean, how are the proper axioms known? At this point, "self-evidence" is only a label for the supposed epistemic status enjoyed by certain propositions, the starting point of a Euclidean program. What is this epistemic notion? The quintessential rationalist, René Descartes, said that foundational propositions become known once when one has clear and distinct ideas of the concepts that figure in them. Coffa (1991, p. 10) quipped that despite (or because of) a voluminous literature on this topic, the notion of a "clear and distinct idea" is heterological. It is neither clear nor distinct what it is to be or have a clear and distinct idea of something.

Presumably, one gets a clear and distinct idea, and thus foundational knowledge, of an axiom by fully coming to understand the concepts involved in the axiom. To be sure, one can understand the constituents of a proposition, and even come to know the proposition, in a less than clear and distinct manner. In that case, one does not have the appropriate foundational knowledge of it. Prima facie, no holistic considerations are involved in coming to fully and properly understand a potential axiom: it is not a matter of immersion in the practice. That would undermine the Euclidean program. Rather, we properly reflect on the concepts invoked in the axiom. Once this is accomplished, we know the axiom in the proper manner.

It is not clear how tenable this package is for contemporary mathematical theories. I do not see how one can maintain that the axioms of choice, infinity, powerset, and replacement can be known on the basis of clear and distinct ideas of set-membership, or in terms of the intuitive iterative conception of set (see Boolos, 1989). Even more elementary mathematical theories are suspect. Do we have proper foundational knowledge of the principles that zero exists, that every natural number has a successor, and that the real numbers are complete, in Dedekind's sense? Do we have foundational knowledge that the universe is infinite, and, indeed, uncountable? How did we get such knowledge? Just by reflecting on some concepts?

Our Euclidean foundationalist might argue that the aforementioned principles of set theory and arithmetic are not foundational. Perhaps they can be proved, on the basis of other, more basic primitive propositions, and perhaps we can obtain clear and distinct ideas of those more basic propositions. From this perspective, one of the reasons why "we prefer proof, where proof is possible" is to uncover the foundations of our knowledge. The burden is to provide proper foundational proofs of the basic principles of (assertory) mathematical theories, and then show that the basic principles of such proofs are known, once we have clear and distinct ideas of the relevant concepts. As the saying goes, the would-be foundationalist should put up or shut up.

4.1. Frege: Objective self-evidence. Let us look at (one interpretation of) one foundationalist who did put up, or at least tried to. Frege's views share at least some elements with traditional rationalism, and, as noted above, he made a heroic attempt to provide "proof, where proof is possible", of basic arithmetic principles. In particular, Frege attempted to found arithmetic and analysis on basic, logical principles concerning extensions (Frege, 1884, 1893, 1903). From that perspective, the question of this paper concerns the epistemic property had by those true, known foundational principles for which no (nontrivial) proof is possible.

Frege (1884, §2) tells us *why* it is that it is that mathematicians "prefer proof, where proof is possible":

> The aim of proof is, in fact, not merely to place the truth of the proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is unmoveable, ... there remains the further question, what is it that supports it so securely?

To unpack the metaphor, Frege believed that true or at least knowable propositions have dependence relations to one another. These relations are objective, in the sense that it is not a matter of how some person or other comes to discover or believe a given proposition, or even of how some person or other comes to know the proposition. Rather, it is a matter of what its truth rests upon. Frege (1884, §3) writes:

> ... we are concerned here not with the way in which [the laws of number] are discovered but with the kind of ground on which their proof rests; or in Leibniz's words, "the question here is not one of the history of our discoveries, which is different in different men, but of the connection and natural order of truths, which is always the same" (Frege, 1884, §17; Leibniz, *Nouveaux Essais*, IV, §9)

Frege's account of the notions of analyticity and a priority are formulated in terms of these dependency relations:

> [T]hese distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement. Where there is no justification, the possibility of drawing the distinctions vanishes. When ... a proposition is called a posteriori or analytic in my sense, this is not a judgement about the conditions, psychological, physiological, and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgement about they way in which some other man has come ... to believe it true; rather it is a judgement about the ultimate ground upon which rests the justification for holding it to be true.
>
> This means that the question is removed from the sphere of psychology, and assigned, if the truth concerned is a mathematical one, to the sphere of mathematics. The problem becomes ... that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one ... If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some general science, then the proposition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general ... But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori. (Frege, 1884, §3)

It seems to me that despite the use of terms like "proof" and "justification" here, Frege's relation of dependence is as much metaphysical as it is epistemic—and therein lies at least part of our problem.[8] We have already seen that the dependency relation has nothing to do with how people come to believe propositions. It is also not a matter of *whether* we know, for example, that $7 + 5 = 12$. Skepticism and fictionalism aside, there is no serious question that we do know that. Nor is it a question of *how* we know that $7 + 5 = 12$. That proposition was known long before the foundational work began. Moreover, for most of us, this knowledge need not, and in fact did not, go via the proposed founding definitions. Frege's dependency relationship thus seems to require a distinction between the state of knowing, or the state of being justified, and the *ultimate* or objective *ground* or justification of a proposition. His foundational framework concerns the latter.

To stay in the general realm of epistemology, let us call knowledge that is based on objective grounding relations among the known propositions *proper foundational knowledge*. If one has proper foundational knowledge of a mathematical proposition p, then one has an (or the) explanation of *why* p is true.[9] When we have proper foundational knowledge, and are aware that we do, we then know why the boulder cannot be moved. Crucially for Frege, proper foundational knowledge is needed to determine whether a given proposition is analytic or a priori, since those notions concern the proposition's metaphysical-cum-epistemic pedigree. One of the purposes of his logicism was to demonstrate that arithmetic and analysis are analytic, in his sense of that term. Jeshion (2001, p. 940) calls this the "knowledge-of-sources rationale" for logicism.

Like Bolzano's (1837) ground–consequence relation, Frege's dependency relation is asymmetric: if proposition A depends on proposition B, then B does not depend on A. It follows that the relation is not reflexive: no proposition grounds itself. Presumably, the relation is transitive. Can the dependency relation fail to be well-founded? Can there be infinitely descending dependency chains: A_0 depends on A_1 (plus perhaps other propositions), A_1 depends on A_2 (plus other propositions), A_2 depends on A_3, etc. Let us set aside the issue of whether it is coherent for someone to have proper foundational knowledge of each member of such a series. If there were an infinite dependency chain, the question of this paper need not arise: there will be no proposition in the chain that cannot be "proved" in the sense of being established on the basis of propositions on which it depends. That is, there will be no propositions in the chain for which proof is not possible. In any case, Frege never suggested such a possibility. In what follows, we till take the proposed foundational chains to be finite.

On this metaphysical-cum-epistemic account, then, some true propositions are not grounded on other propositions. They are foundationally secure. Frege's term for such primitive truths is *selbstverständlich*.[10] Jeshion says that there is no proper English

[8] Thanks to Penelope Maddy and Michael Detlefsen here.

[9] From the Euclidean perspective, as articulated here, proper foundational knowledge is of-a-piece with what Kim (1994, p. 51) calls "explanatory knowledge"—knowledge why—as opposed to mere descriptive knowledge—knowledge that. By way of illustration, Kim quotes Aristotle (*Physics*, Chapter 3): "Men do not think they know a thing unless they have grasped the 'why' of it". When put this way, it sounds like proper foundational knowledge is a kind of knowledge, perhaps the best kind. However, Joshua Schechter suggests that what I call "proper foundational knowledge" may not be a special kind of knowledge at all: it is to know and to have an explanation for what one knows. There is no need to settle this matter of classification here.

[10] Some of the material that follows is drawn from the studies in Detlefsen (1988, 1996, pp. 56–70), Burge (1998), and Jeshion (2001, 2004) (see also Jeshion, 2000). For present purposes, I am interested in examining a certain reading of Frege, and less concerned with whether that interpretation is faithful to Frege's own views.

equivalent for this, and so she leaves the term untranslated. I follow that here. To summarize, *selbstverständlich* propositions require no proof and, indeed, no (nontrivial) proof them is possible. All properly foundational proofs must begin with *selbstverständlich* propositions, as axioms. All other known propositions are based on *selbstverständlich* propositions.

This tells us how to formulate the question of this paper: How are *selbstverständlich* truths known or, better, knowable? By definition, *selbstverständlich* propositions are not foundationally known on the basis of anything else. If we are to have proper foundational knowledge of the theorems—if we are to have proper foundational knowledge of anything—we must know the axioms. How?

For Frege, proper axioms have an epistemic property that he calls *einleuchten*, that of being self-evident. Jeshion (2001, p. 953) glosses the property as follows:

(S-E) A proposition *p* is *self-evident* if and only if clearly grasping *p* is [a] sufficient and compelling basis for recognition of *p*'s truth.

As Burge (1998, p. 312) puts it, the epistemic–metaphysical status for proper axioms is "something like *beyond a reasonable doubt by someone who fully understands the relevant propositions*". Let us call this notion *Fregean self-evidence*. The knowledge that comes via Fregean self-evidence cannot rely on any reasoning, or at least not on any deduction. The step from a proper grasp of the proposition to knowledge of it is immediate, and direct.[11]

Jeshion sums up the aims of Frege's logicism as follows:

> **Euclidean rationale:** Frege thought that primitive truths of mathematics have two properties. (i) They are *selbstverständlich*: foundationally secure, yet are not grounded on any other truth, and, as such, do not stand in need of proof. (ii) And they are self-evident: clearly grasping them is a sufficient and compelling basis for recognizing their truth. He also thought that the relations of epistemic justification in science mirrors the natural ordering of truths: in particular, what is self-evident is *selbstverständlich*. Finding many propositions of arithmetic non-self-evident, Frege concluded that they stand in need of proof.

Some of this language is reminiscent of the Cartesian notion of clear and distinct ideas. By grasping the sense of a self-evident proposition, one thereby knows, without doing any more reasoning, that the proposition is true.

With the traditional rationalist, Frege's metaphysical-cum-epistemic picture puts a lot of weight on the power of the unaided human mind to discover truths about the world we find ourselves in. The picture strikes me as containing a large dose of preestablished harmony. According to this reading of Frege, there is a realm of propositions, which can be grasped by human beings. These propositions stand in objective dependency relations to each other, with *selbstverständlich* propositions at the base. These *selbstverständlich* propositions, or

[11] It does not follow that Frege held that self-evident propositions are knowable on the basis of meaning alone (i.e., that they are analytic, in the contemporary sense of that term). Frege followed Kant in holding that the propositions of Euclidean geometry are synthetic. Since knowledge of geometric propositions requires intuition, those propositions are not knowable in the basis of meaning alone. Yet, I presume, Frege took the proper axioms of geometry to be *selbstverständlich* and Fregean self-evident. If all of this is correct (and I realize this is a big "if"), then for Frege, the immediate connection from grasp to knowledge can invoke intuition. What matters is that we go from grasp to knowledge without any deduction.

least the important foundational ones, also happen to be self-evident, in the sense that grasping them is sufficient for knowing them, immediately. And these *selbstverständlich*, Fregean self-evident propositions are sufficient for all mathematical knowledge.

Would that it were so. Even if there is a realm of propositions, distinct from and related to the sentences of natural languages, and even if these propositions enjoy objective dependency relations to each other—a natural order of truths—I see no reason why we should think they are structured in such a pleasing way, a way designed to facilitate proper foundational knowledge of the entire edifice of mathematics by beings just like us. It is natural to hope that the picture is correct. But do we have any reason to think it is?[12]

Fregean self-evidence is not obviousness, not a mere subject feeling of certainty. Reliance on obviousness would smack of the psychologism that Frege so vehemently opposed. There are obvious propositions, such as $2 + 3 = 5$, that are not self-evident. Frege emphasized that to know sums like those, one need not invoke any intuition, Kantian or otherwise, but he insisted that one must *reason* one's way to this knowledge. For present purposes, the more important direction is the reverse: there are, or at least could be, self-evident propositions that, at least at first, are not obvious. Even before he learned of Russell's paradox, Frege conceded that his Basic Law V is not obvious. He wrote that he had "never concealed" from himself Basic Law V's "lack of self-evidence which the others possess, and which must properly be demanded of a law of logic" (Frege, 1903, p. 253).[13] In motivating the system, he wrote:

> If we find everything in order, then we have accurate knowledge of the grounds upon which each individual theorem is based. A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V), which logicians perhaps have not yet expressly enunciated . . . Yet I hold that it is a law of pure logic. (Frege, 1893, vii)

On the reading of Frege under study here, no proposed axiom or basic law can play its assigned foundational role unless it is Fregean self-evident. Yet, Jeshion (2001, p. 960) argues, "Frege believed that what is self-evident does admit of doubt—even rational doubt". Burge (1998, p. 305) writes that Frege "expressed a sophisticated, modern awareness of the fact that what can seem obvious may turn out not even to be true . . . Frege was aware that principles that he put forward as axiomatic—even some that, unlike Axiom V, have endured as basic principles of logic—were not found to be obvious by his peers." Frege's pre-Russell attitude toward Basic Law V looms large in what follows.

In any case, the fact that a proposition is obvious is, presumably, *some* evidence that it is Fregean self-evident. One must start somewhere when setting up a foundational program, and at least some obvious propositions seem to be decent candidates for axioms. Inversely, the fact that a proposition is not obvious is, presumably, some evidence that it is not

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12 As Joshua Schechter noted, this raises a host of familiar issues, many of which are relevant to present concerns; but it would take us too far afield to deal with them here. It is not uncommon to think that the empirical world as structured in a similarly pleasing way, so that beings just like us can discover its most fundamental principles, the scientific laws. One might argue that human cognition evolved, or was designed, just so that what is (Fregean) self-evident correlates with what is metaphysically most fundamental—*selbstverständlich* in the case of mathematics, law-like in the case of empirical science.

13 Frege's term for self-evidence here is "einleuchten". But perhaps the use of that term is not univocal.

Fregean self-evident. But as we have seen, obviousness is neither necessary nor sufficient for Fregean self-evidence. The inference from obvious to Fregean self-evidence is a fallible one, and there are Fregean self-evident propositions that, at least for a time, are not obvious. Burge (1998, p. 327) agrees that Frege "never says or implies that convictions about self-evidence are infallible. I think Frege believed that there is no infallible guarantee that one's commitments on logical or geometric truths are correct".

But how can a Fregean self-evident proposition fail to be obvious? How is it possible for someone to understand a Fregean self-evident proposition and yet not know it? Given the above definition of the term, the answer must be that it is possible to grasp a proposition in a less than perfectly clear manner. This is reminiscent of the rationalist notion of a confused grasp of concepts, a grasp that falls short of clear and distinct understanding. Frege (1914, pp. 216–217) considered the situation at the time, in which different mathematicians gave different analyses of "number". Perhaps they were onto different notions, which happen to have the same name in natural language. Frege gives another take on the situation:

> [I]s not the explanation rather that we have really to do with the same science; that this man *does* attach the same sense to the word "number" as that man, only he doesn't manage to get hold of it properly? Perhaps the sense appears to both through such a haze that when they make to get hold of it, they miss it. One of them makes a grasp to the right perhaps and the other to the left, and so although they mean to get hold of the same thing, they fail to do so. How thick the fog must be for this to be possible!

So how *do* we get proper foundational knowledge of Fregean self-evident propositions. One possibility, I suppose, is to focus one's mind on the concepts and wait for clarity (and distinctness) to come. This does not jibe with the "data". The mathematical giants who provided different "analyses" of, say, the real numbers, can hardly be faulted for failing to concentrate. Their minds are among the finest ever. How did some of them miss the concept, going to the left or the right in the haze?

Moreover, even when clarity *seems* to come, how do we *know* that it has? In general, how do we know, or when are we justified in believing, that the fog has lifted, that now we have a clear grasp of the concepts involved, and thus proper foundational knowledge? Why would Frege think that he, at long last, got it right? Jeshion (2004, p. 967) puts the issue well:

> As he was acutely aware of the possibility of errors resulting from conceptual understanding, Frege regarded reliance on obviousness as insufficient for identifying ... primitive truths. As Frege noted, we are not given concepts 'in their pure form' ([Frege, 1884, vii). Our partial or incorrect understanding of concepts results in mistaken judgements. Such errors are not recognized 'from within' as, perhaps, are mistakes from inattention, sloppiness, or haste in judgement. And they are not remedied merely (!) by exercising control on one's thought, as, perhaps, are the others. The mistakes in question sometimes occur even when exercising tightest control on our intellection.

Apparently, Frege did not say in much detail how we properly come to know the starting points, nor how we know that putative starting points *are* starting points.

4.2. Holistic foundationalism? Here the dialectic takes a most interesting turn. Frege (1914, p. 205) praised the goal of *organizing* mathematical knowledge, presumably in a way that reflects the objective grounding relations:

> The essence of mathematics has to be defined from [a] kernel of truths, and until we have learnt what these primitive truths are, we cannot be clear about the nature of mathematics. If we assume that we have succeeded in discovering these primitive truths, and that mathematics has been developed from them, then it will appear as a system of truths that are connected to one another by logical inference
>
> Euclid had an inclination of this idea of a *system*; but he failed to realize it and it almost seems as if at the present time we were further from this goal than ever. We see mathematicians each pursuing their own work on some fragment of the subject, but these fragments do not fit together into a system; indeed, the idea of a system seems almost to have been lost. And yet the striving for a system is a justified one. We cannot long remain content with the fragmentation that prevails at present. Order can only be created by a system . . .
>
> . . . we must avoid such expressions as 'a moment's reflection shows that' or 'as we can easily see'. We must put the moment's reflection into words so that we can see what inferences it consists of and what premises it makes use of. In mathematics we must never rest content with the fact that something is obvious or that we are convinced of something, but we must seek to obtain a clear insight into the network of inferences that support our conviction. Only in this way can we discover what the primitive truths are.

Burge (1998, p. 328) suggests that at some level, Frege's methodology is holistic, writing that "in arguing for his logic [Frege] made use of methods that were explicitly pragmatic and contextualist . . . " Burge continues:

> In *Basic Laws* (1893) we find Frege recommending to those who are sceptical of his logical system that they get to know it from the inside. He thinks that familiarity with the proofs themselves will engender more confidence in his basic principles . . . In the Introduction to *Basic Laws*, Frege repeatedly appeals to advantages, to simplicity, and to the power of his axioms in producing proofs of widely recognized mathematical principles, as recommendations of his logical axioms. (p. 328)

A bit later, Burge adds that "whatever role self-evidence plays in [Frege's] epistemology seems to be qualified by pragmatic considerations that result from reasoning within and about his system of proofs over time" (p. 335).

Jeshion (2004) also sounds the holistic theme, quite explicitly. According to her reading, we come to know that a given proposition is *selbstverständlich* and Fregean self-evident by examining its role in a carefully worked out scheme of knowledge. She has Frege

> advocating the sane view that what seems obvious may require proof and that obviousness needs supplementation by systematization. To identify a proposition as not needing proof [i.e., *selbstverständlich*] . . . we need

> to systematize our knowledge and see whether the proposition can fulfil the role of an axiom within an ideal Euclidean system of mathematics. It does so by being fruitful, by enabling the derivation of all known mathematical knowledge and by affording means of generating more. It must also satisfy the traditional rationalist goals of *surveyability, simplicity, economy, and unificatory power*. (Jeshion, 2004, p. 969)[14]

As noted, even before he heard from Russell, Frege conceded that Basic Law V is not obvious, and that one can rationally doubt that it is Fregean self-evident (or, indeed, even doubt its truth). In an unpublished note, dated 1906, he wrote that:

> by what right does such a transformation take place, in which concepts correspond to extensions of concepts ... An actual proof can scarcely be furnished. We will have to assume an unprovable law here. Of course it isn't as self-evident [*einleuchtend*] as one would wish for a law of logic. And if it was possible for there to be doubts previously, these doubts have been reinforced by the shock the law has sustained from Russell's paradox. (Frege, 1906, p. 198)

Jeshion provides two "charitable" interpretations of Frege's pre-Russell inclusion of Basic Law V as an axiom. One is that Frege envisioned the possibility that Basic Law V is not really *selbstverständlich*. One day, the principle might be proved from other, even more basic propositions. Presumably, the idea is that one might be able to analyze the concept of "extension" into more primitive terms, and prove Basic Law V from *selbstverständlich*, self-evident propositions about those terms. At the time, Frege did not know how to prove the principle and so, temporarily, he took it as an axiom.

With regard to the theme of this paper, Jeshion's other "charitable" interpretation is more intriguing:

> Frege could have held that the coincidence in sense ... would, in time, come to seem as obvious as the other axioms. Basic Law V seemed true even though it came up short on the obviousness needed for thinking it self-evident ... But, as Frege recognized, that could change ... Frege ... recognized that [the terms] were not fully understood. By proving theorems with the new basic concepts, understanding may well alter enough so that Basic Law V would come to possess the obviousness required for thinking it self-evident. (Jeshion, 2004, pp. 290–291)

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[14] As an aside, of sorts, it seems to me that the goals of surveyability, simplicity, economy, and unificatory power can only fit into a Euclidean program if one accepts an even larger dose of preestablished harmony. For example, why should one think that the propositions that lie at the base of the objective dependency hierarchy happen to have representations short enough to be surveyable by minds like ours? What is so special about the powers of memory and concentration of human mathematicians? Suppose that a given proposition is too complicated for us to understand it in one glance. How can that alone disqualify its foundational status? In any case, these four criteria are not uniquely rationalist; just about everyone lists them as theoretical virtues. The problem is to square these criteria with the alleged goal of the activity in question: how do surveyability, simplicity, economy, and unificatory power bear on what the presumed goal of mathematics and science is?

Jeshion makes an analogy with the $\varepsilon - \delta$ definitions of such notions as convergence, continuity, and differentiability. They were not obvious at first, but in time, it became clear that they are correct, to the extent that the definitions are now obvious. Surely, this clarity emerged through extensive *use* of the definitions in proofs.

This is exactly what was said, two sections ago, when we considered self-evidence as a species of obviousness, as type of psychological certainty. The axioms of set theory, and of most other (assertory) branches of mathematics, do indeed "force themselves on us as true", as Gödel put it, but this forcing does not occur on a first or second reading of them. The axioms only become certain after immersion in the practice. So do the theorems.

On the surface, at least, there is some tension in this picture. Burge (1998, p. 315) writes that "self-evidence must partly be understood in terms of recognizability as true *independently of recognition of other truths*. Sufficient evidence to make believing them rational *is carried in these individual truth themselves*" (my emphasis). This is in line with the above definition of Fregean self-evidence, from Jeshion (2004). But, according to these interpretations, we sometimes get into the position of knowing a proposition independent of other truths only when we see the role of the proposition in a systematization of those other truths. Well, which is it? Are the axioms known individually, in themselves, or are they known holistically, in terms of their role in a successful system?

In sum, Frege elaborated a complex metaphysical-cum-epistemic system, according to which some propositions are objectively grounded on others. When faced with the question of how one comes to know the axioms, the propositions that lie at the base of the edifice, we are told that they are self-evident: one who clearly and fully understands them realizes that they are true, without doing any reasoning. But understanding is sometimes confused, or otherwise less than sufficiently clear. How does one get the proper understanding and thus knowledge of the axioms? How does one get a fully clear grasp of the concepts? At this point, according to Jeshion and Burge, Frege waxes systematic. We get a fully clear grasp of concepts by working with them, by immersing oneself in the practice of deducing the consequences of these axioms, and seeing how everything fits together.

To recall, the Euclidean program under scrutiny here has two key aspects. The first is the framework of objective dependency relations among mathematical propositions; the second is the thesis that the *selbstverständlich* propositions at the base of the framework are self-evident—known individually, in themselves, solely on the basis of clear and distinct understandings of the concepts. I do not claim (here) that the first, foundationalist, aspect of the program is in tension with the holistic elements in Frege's methodology. One very well might argue, or at least claim, that a good systematization of a body of knowledge reveals objective relations among the propositions so systematized. Indeed, one well-discussed view in the philosophy of science is that explanation just is unification (e.g., Kitcher, 1989). All we need to add to this are arguments that the dependency relations under discussion here are explanatory relations, so understood, and that the revealed dependency relations are objective. From a perspective like this, one might also argue that we are justified in accepting certain axioms as true if they play a certain role in a successful systematization. It is a sort of inference to the best explanation. As we saw in §3 above, this, or something like it, underlies Russell's program of mathematical philosophy.

The tension comes with the second aspect of the Euclidean program, the thesis that *selbstverständlich* propositions are self-evident. Recall that, for Burge (1998, p. 312), a self-evident proposition is known "beyond a reasonable doubt by someone who fully understands" it. And according to Jeshion (2001, p. 953), a clear grasp of a Fregean self-evident proposition "is [a] sufficient and compelling basis for recognition of" its truth. How can holistic elements play *any* role in establishing that a proposition is self-evident?

4.3. Ways out. There are at least three ways in which one might resolve this tension. To anticipate my conclusion, I suggest that they all undermine, or at least call into question, the original motivation for the Euclidean program. Things must be left tentative, however, since I do not claim to cover every possible Euclidean program for mathematics, nor every possible way of accommodating holistic elements into it.

According to the above definition, the only prerequisite to coming to know a Fregean self-evident proposition is to fully understand it. Once that is done, one has all she needs to immediately know the proposition, without doing any reasoning. But what of the prerequisite? How does one come to fully understand the meaning of a proposition? Suppose that one adopts a sort of inferentialist account of meaning, or at least of the meaning of much of the terminology in the relevant mathematical propositions.[15] Coming to understand a proposition involves seeing which propositions entail it, and which propositions it entails, perhaps with side premises added. In other words, understanding a proposition consists of understanding its role in the network of inferences.

The first way out of the present dilemma is to supplement the foregoing Euclidean foundationalism with this broad inferentialism. In particular, one must explore the consequences of a proposed Euclidean axiomatization in order to properly understand its axioms. The holistic/systematic elements in the Fregean picture are thus limited to the understanding phase—the one and only prerequisite for knowledge of the Fregean self-evident propositions that lie at the base of the network.

Once again, according to Burge, Fregean self-evident propositions are known "beyond a reasonable doubt by someone who fully understands" them. On our first way out, this full understanding is mediated by systematization. Once a subject grasps the range of consequences of a *selbstverständlich* proposition (or a group of such propositions), she has a "sufficient and compelling basis for recognition of" its (or their) truth, as Jeshion puts it.

Consider a subject, Karl, who has no knowledge of substantial mathematics, say arithmetic or geometry. But assume that Karl knows basic logic: he can draw inferences as well as anybody. Now imagine that Karl is presented with the axioms of a Euclidean system for either geometry or arithmetic. At first, he does not believe these axioms, simply because he does not understand them—he has not pondered the network of inferences needed for understanding. So, at this point, Karl has no reason to think that the axioms are true. He then gets busy and draws an impressive array of consequences of the axioms. Suppose that his results look much like a textbook for the mathematical theory in question. On the inferentialist view, he now fully understands the axioms of the theory, since he is aware of the network of consequences that flow from them. Thus, on the Fregean, foundationalist

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[15] Thanks to Carrie Jenkins here, as well as to other participants of the Philosophy Club at the University of St. Andrews. I should note that in the philosophical literature, "inferentialism" is usually taken to be a view about the understanding (or meaning) of individual *words*, or concepts, such as logical connectives. The view under discussion here is an inferentialism about entire sentences or propositions. To be flippant, the present inferentialism is a combination of a more typical inferentialism with Frege's context principle, that one should never "ask for the meaning of a word in isolation, only in the context of a proposition" (Frege, 1884, Introduction). Slightly less flippantly, the present inferentialism can be thought of as an extension of the context principle to something like this: never ask for the meaning of a word or proposition in isolation, only in the context of a systematization.

view, he has a sufficient and compelling reason to believe these axioms. Karl now *knows* that they are true.

For what it is worth (which is not much), my intuitions say otherwise. If Karl starts off with no understanding of arithmetic or geometry, then, from his perspective, he is drawing inferences from uninterpreted statements, merely doing logic exercises. No matter how long he plays this game, he never gets in position to know the original axioms. So in what sense are the axioms Fregean self-evident? How does Karl know them, after his logic exercise?

In any case, Karl is a fictional character. I suppose it is possible that someone who does not know any mathematics could get that good at drawing inferences from what, for him, are uninterpreted (or at least un-understood) premises, but it does not work that way in practice. Among living humans, the situation is closer to that described by Russell in §3 and/or that attributed to Gödel in §2 above. The axioms become evident—obvious perhaps—once we see that they organize and perhaps explain what *we already know*. In effect, we do not know the theorems *solely* by deriving them from the axioms. In some cases, it is closer to the other way around. Our belief in the axioms is due, in part, to the fact that they organize at least some of the theorems.

My guess is that Frege's own appeal to system is to be understood similarly. Recall (part of) the passage from Frege (1914, p. 205) quoted above:

> Euclid had an inclination of [the] idea of a *system*; but he failed to realize it and it almost seems as if at the present time we were further from this goal than ever. We see mathematicians each pursuing his own work on some fragment of the subject, but these fragments do not fit together into a system; indeed, the idea of a system seems almost to have been lost. And yet the striving for a system is a justified one. We cannot long remain content with the fragmentation that prevails at present. Order can only be created by a system . . .

On Frege's picture, a good systematization reveals the objective dependencies among the propositions. But it seems clear, at least to me, that Frege assumes here that we systematize what we already know. And if we knew at least some of the mathematics before the systematization, then, it seems, we understood it before as well, at least as some level. The epistemic weight that comes to the axioms as a result of this process is due, in the main, to the *prior* understanding of and belief in the relevant theorems. That understanding did not come via an examination of the network of inferences.

To be sure, systematizing a field, and exploring consequences of various propositions, does *improve* understanding of the propositions in the language of the field. Our inferentialist Fregean may even be right that until we have systematized and integrated our mathematics, we do not *fully* understand the propositions. It may be that the prior understanding we had of some of the theorems was partial, foggy, or, to use rationalist terminology, confused. Moreover, systematizing a field can also shore up one's justification of some of the propositions in the field. In a sense, knowledge is improved.

My point here is that the epistemic role of systematization *presupposes* at least a prior (if partial and confused) understanding and knowledge of some of the propositions in the field, and thus it cannot wholly replace this prior understanding and knowledge. Suppose we try to "bracket" our prior knowledge of the relevant theorems, and pretend that we do not believe them, until we have fully grasped (and thus come to know) the axioms, perhaps by exploring their consequences. Then, like our fictional character Karl, we will not come to know anything.

It is, of course, a truism that one can understand a proposition and not know it. Understanding and knowledge are not the same thing. And surely one cannot know a proposition without completely understanding it. Nevertheless, my suggestion here is that knowledge and understanding can be, and typically are, improved together, via systematization. One does not fully understand the axioms (and theorems) of a typical mathematical system unless one already has considerable (albeit partial or confused) knowledge of some of the theorems.

Recall, once more, Jeshion's (2001, p. 953) apt characterization:

(S-E) A proposition *p* is *self-evident* if and only if clearly grasping *p* is [a] sufficient and compelling basis for recognition of *p*'s truth.

On the present way out, Fregean self-evidence sometimes involves considerable knowledge of other propositions, even if this latter knowledge is not fully complete—not sufficiently clear and distinct. In a sense, then, it is not true that we know a Fregean self-evident proposition independent of any other. The best that can be said is that we get *full*—clear and distinct—knowledge of a Fregean self-evident proposition independent of *full* knowledge of any other proposition. And we do not even get that if we cannot fully understand one axiom without understanding the rest of them. Arguably, on the inferentialist model in question, we come to understand the axioms in groups, as we explore their joint consequences.

This is not the place to put forward detailed views on just what it is to understand a proposition, fully or partially, even if I had worked out such views. The Fregean picture we are exploring here has it that once someone comes to fully and properly understand an axiom, she has all she needs to know it. She can then deduce the theorems, and thus get proper foundational knowledge of them as well. I leave it to the reader to determine how much of the original motivation for the Euclidean program remains. Are we really answering the questions that started us down this path, once the concessions to holism, and partial understanding, are made? Why should one remain a Euclidean? What explanatory work does the talk of objective grounding relations, *selbstverständlich*, and self-evident propositions do? Why not just say that we understand and know the whole system, as a whole?

Let us move on, to a second way to accommodate systematic, holistic elements into a Fregean program. Recall that Burge writes that Frege "maintained the views, which he several times expresses in the pre-paradox period, that the basic principles that he proposed are genuine axioms and that axioms are self-evident". Concerning the pre-Russell status of Basic Law V, Burge maintains that since Frege "thought it was an axiom, he must have, at least sometimes, thought that *it* was certain, but because of insufficient analysis or incomplete understanding, *he* was not" (Burge, 1998, pp. 337–338). And this motivates the incursion of holistic elements into the system.

At this point, Burge makes an interpretive proposal that Frege held that only an *ideal mind* can know some self-evident propositions independently of any others. The idea, I take it, is that we fallible, mortal humans invoke systematic considerations to obtain evidence about what an ideal mind can know without reasoning, independently of other propositions.

Burge's distinction, on Frege's behalf, is reminiscent of Leibniz's proposal that God knows every proposition via a clear grasp of the concepts it contains, but we mortals have to know things in a more indirect manner. For the Fregean foundationalist, it seems, the proper epistemic basis of proper axioms is Fregean self-evidence, as above. Understanding them is indeed sufficient for knowing them—ideally. But in practice, humans are not so fortunately situated. We have to rely on holistic or systematic considerations before we are assured of the axioms.

Above, I suggested that the Fregean perspective, as articulated here, requires a fair amount of preestablished harmony. It assumes that each of the crucial *selbstverständlich* propositions that lie at the basis of the dependence hierarchy enjoys the pleasing epistemic property of Fregean self-evidence, of being knowable, individually, solely on the basis of an understanding of the concepts in them. This second way out lessens the preestablished harmony, since the claim, now, is only that the basic propositions are Fregean self-evident for an ideal knower, not for us.

In another sense, however, the amount of preestablished harmony needed here is considerably increased. The second way out presupposes that if we, limited beings find that a set of axioms nicely systematizes a branch of mathematics, then we have reason to think that an ideal being—God—can know each of those very axioms solely on the basis of an understanding of the concepts. Why think this? *Perhaps* there is reason to believe that the axioms in our systematization are objectively *selbstverständlich*, that they do not depend on other propositions. That is the foundationalist element. But what reason is there to think that such axioms are even ideally Fregean self-evident?

A Fregean who takes this way out presumably holds that the inference from being an axiom in a successful systematization to being ideally self-evident is fallible. This is in line with both Jeshion's and Burge's understanding of the Fregean enterprise, as above. Cartesian certainty is not on our agenda anymore, even for the would-be Euclidean foundationalist. The main problem with this second way out is that I see no reason to think that matters of successful systematization are even *relevant* to what is Fregean self-evident, ideally or otherwise. Why think that the fact that a given set of axioms makes for a nice systematization of a body of accepted knowledge (or belief) is *any* evidence that an ideal mind can know each of those axioms on the basis of an understanding of the concepts alone? What is the basis for the even fallible inference from role-in-systematization to Fregean self-evidence?

The idealization invoked here may be the familiar one, invoked throughout mathematics, where we ignore limitations on attention span, memory, lifetime, and the like. Or it may go beyond that, in attributing infinitary capacities to the ideal subjects. I do not challenge the coherence of the idealization here, nor speculate on how it is to be articulated. There is, however, a serious question concerning its relevance to epistemology. At the start of this section, I noted that Frege's notion of objective dependency is as much metaphysical as it is epistemological. We saw that it is not a matter of how ordinary human beings, established mathematicians included, know their axioms. On our second way out, it seems, the same goes for the notion of self-evidence. It is not even a matter of how ordinary human beings *can* come to know axioms. In the crucial cases, it is conceded that *our* methodology, and our epistemology, is holistic. *We* know some axioms, defeasibly, by observing their role in a successful systematization. These axioms are thus not Fregean self-evident *for us*. What I called proper foundational knowledge is simply not always available. To be blunt, what difference does it make if the axioms are knowable some other way to God or to otherwise idealized subjects? Idealized beings can do their own epistemology.

Moving on, the third way out of the present dilemma invokes a distinction familiar to epistemology. Let p be a proper axiom of a Euclidean system. The idea here is that even humans know p independent of any other propositions, on the basis of a clear understanding of p. But sometimes we need holistic or systematic considerations to assure ourselves *that* p is Fregean self-evident, and thus knowable independently of other propositions.

That is, if p is Fregean self-evident, then our knowledge of p is (or can be) unmediated by anything else, but sometimes we only know that p is self-evident, and unmediated by anything else, in a mediated way, by systematic considerations.[16] There is a difference between how we know p and how we know that we know p.

Let Π be any purported epistemological or psychological status that various propositions can enjoy or fail to enjoy. Examples include being self-evident, being knowable via sense perception, being infallible, being obvious, being a priori, being analytic, and being a logical truth. Let p be any proposition that has the status Π. There is, in general, no reason to think that the (true) proposition that p-has-Π itself has the status Π. Sometimes it does, perhaps, but sometimes it does not. We have to consider each case separately, and we will undoubtedly encounter contentious philosophical theses as we do. This is a sensible distinction, and it can be deployed here. It seems to me that this third option gives the Fregean perspective its best shot at accommodating holistic elements.

Notice, first, that this option leaves the foundationalist aspects of the program in place, and it has the advantage of keeping the notion of self-evidence squarely within epistemology. The combination of views makes for a substantial thesis concerning how the crucial propositions are known by at least some ordinary human knowers. The propositions are indeed Fregean self-evident, for us. The holistic considerations come in only at the next level up, so to speak, when we try to work out which propositions have the privileged epistemic status of self-evidence. We sometimes have to go systematic to accomplish that last job.

This way out has a problem shared by the second. Suppose, again, that a proposition p serves as an axiom in a successful systematization of a body of knowledge. An advocate or our third way concludes, perhaps tentatively, that p is Fregean self-evident, that she herself knows p on the basis of understanding alone, independent of any other propositions and without doing any reasoning. Admittedly, she has not contradicted herself by invoking the intense role of deductive reasoning in the systematization. The holism comes in only at the second, more detached level. But, as with the second way out, why should we think that playing a certain role in systematization is even *evidence* for Fregean self-evidence? What is the link between being an axiom in a systematization and being Fregean self-evident, knowable on the basis of understanding alone?

I do not claim to have refuted this perspective, or the two previous ones for that matter. But I do claim that there is a substantial burden to be met, namely of showing the relevance of holistic, systematic considerations to self-evidence, as that notion is presently characterized. Lacking this link, the would-be Fregean may be accused of dignifying a successful systematization with an undeserved metaphysical-cum-epistemic status.

Even if the burden is met, somehow, there is an obvious question of why one should maintain the Euclidean perspective. What philosophical role does the statuses of being *selbstverständlich* and being self-evident play in our overall world view? What light does the program shed on the enterprise of mathematics, or the philosophical subenterprise of figuring out what it is to know mathematics? By relegating holistic considerations to the level of how we know that we know, the Euclidean foundationalist maintains consistency. But it seems to me that even on such a view, the overall methodology remains holistic. Like Russell, and most non-Euclideans, we still justify the system, as a whole, in terms of the role it plays in systematizing our intellectual lives.

To conclude this section, it seems to me that once we go holistic, either at the level of knowledge (for flesh and blood humans) or at the level how we know a proposition's

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[16] Thanks to Richard Heck for pressing this option to me.

status, we compromise the goals and indeed the spirit of the foundationalist program. If, on Jeshion's and Burge's readings of Frege, we are to go holistic at the crucial place, then why not wax holistic from the start? Why do not we claim that we know the whole system—axioms and theorems together—in terms of the role it plays in our overall conceptual scheme, in our form of life, or in the web of belief? The burden on a would-be Euclidean who follows this route is to show that the elements of objective dependence and Fregean self-evidence have some explanatory role to play.

This is not the place, and I am not the author, to explore every possible Euclidean foundationalist account of mathematics. The burden on the Euclidean is to give an account of how we know the axioms, the propositions that lie at the basis of the system. Presumably, they are self-evident. What does this mean?

§5. Zermelo: Self-evidence as unconscious use. Ernst Zermelo's celebrated axiomatization of set theory appeared in Zermelo (1908), which also contains a new proof of the well-ordering theorem, from that axiomatization. The bulk of the article is a section (§2) entitled "Discussion of the objections to the earlier proof" of the well-ordering theorem (i.e., Zermelo, 1904). In the first subsection, "Objections to the principle of choice", Zermelo claims that the axiom of choice is "self-evident". It is interesting that Zermelo maintained that choice is self-evident even after many, probably most, of the leading mathematicians of the day—notably the French analysts Baire, Borel, and Lebesgue—balked at it (see Moore, 1982). Some thought it false, while others expressed doubts, a healthy agnosticism perhaps.

By "self-evident", Zermelo surely did not mean "obvious". One can, of course, hold that a given proposition p is true despite the fact that a large number of respectable thinkers think it false, or are not sure or express doubts. But under these circumstances, one should not maintain that p is obvious. I presume that Zermelo respected his opponents, and was not claiming that they are so dense as to fail to grasp what is obvious. The paper in question is a serious mathematical and philosophical contribution, not an exercise in name-calling.

As far as I know, Zermelo never claimed that the axiom of choice is what I call here "Fregean self-evident". He never intimated that a clear and distinct grasp of the concepts involved in this proposition—the membership relation and the logical terminology—is a sufficient and compelling basis for knowing that the axiom is true. Moreover, he does not claim, or even suggest, that his opponents fail to fully understand the principle.

It is sometimes thought that the issue surrounding the axiom of choice turns on different philosophical conceptions of mathematics and mathematical objects (see, e.g., Maddy, 1990, pp. 121–123; Lavine, 1994, pp. 111–115), and perhaps even the ordinary meaning of the word "choice". If someone holds a constructivist philosophy, maintaining that one must show how to define or construct an object before claiming that it exists, then, presumably, she will doubt the axiom of choice. The principle does not provide a construction or definition of a choice set or function; it only asserts that such a set or function exists. The opposing nonconstructivist, or "combinatorial", view is that sets are arbitrary collections of objects, and that functions are arbitrary correspondences. Sets and functions exist whether or not they can be defined or constructed. Presumably, choice *is* Fregean self-evident, or at least obvious, on the combinatorial view.

Historically, there is some truth to this diagnosis. The French analysts did have constructivist leanings, and Zermelo and some of his supporters, such as Hadamard, did not, explicitly separating out questions of existence from questions of definability and constructibility. Moreover, one can see a trend in the early decades of the twentieth century away from a constructivist perspective on mathematics, along with a trend toward the acceptance of the axiom of choice (see Moore, 1982).

Nevertheless, this is not the entire explanation of the situation on the ground in the early twentieth century, nor is it particularly enlightening in understanding Zermelo's claim of self-evidence. Once again, Zermelo never claimed that choice is *obvious*, as it supposedly is on the combinatorial view. Moreover, at the time, issues of constructivity and definability were not as clearly formulated as they are today, and it is not straightforward to characterize some of the key players as constructivist or combinatorialist, nor is it clear that such a lineup would correspond exactly with stances on choice.

Note also that some versions of choice are true—and obviously true—on a thoroughly constructivist reading. Consider the antecedent of a common instance of the axiom of choice: x is a nonempty set of sets. On a constructive reading, this entails that there is a uniform method for constructing a member of each member of x. For the constructivist, this is what it is for x to *be* a set of nonempty sets. Intuitively, if identity is decidable,[17] this method will produce a choice function for x.

From the other perspective, I would suggest that choice is not quite obvious from a combinatorial point of view. Or at least a reasonable doubt can be raised. For one thing, the axiom has some rather counterintuitive consequences, such as the Tarski–Banach theorem and, for that matter, Zermelo's well-ordering theorem.

In any case, Zermelo's defense of the axiom of choice does not go via the route of obviousness, or even conceptual analysis. He never claims that any (real or ideal) careful thinker will be convinced of its truth just by thinking about what it says. Again, Zermelo does not claim that choice is Fregean self-evident. So what did Zermelo mean by "self-evident"?

In the earlier proof of the well-ordering theorem, Zermelo (1904) claimed that the axiom of choice is a "logical principle", and that it "cannot … be reduced to … still simpler" logical principles. That is, the principle of choice cannot be proved, and so perhaps it is what Frege called *selbstverständlich*. But, Zermelo claimed, "it is applied without hesitation everywhere in mathematical deduction". This, I believe, is the key to understanding his attitude. He cites an example: "the validity of the proposition that the number of parts into which a set decomposes is less than or equal to the number of all of its elements cannot be proved except by associating with each of the parts in question one of its elements".

In the 1908 axiomatization, Zermelo repeats his 1904 assertion that he cannot prove the axiom of choice, and adds that he "therefore cannot compel anyone to accept it apodictically" (§2.a). But he quickly points out that "in mathematics *unprovability* … is in no way equivalent to *nonvalidity*, since, after all not everything can be proved, but every proof in turn presupposes unproved principles". This broaches the theme of the present article. How *do* we know some of these "valid" but unproved principles, especially the ones that are not so obvious? And how do we know which principles those are?

[17] The issue of choice from a constructivist perspective is much more subtle than this. In general, choice fails whenever identity is not decidable. To take one example, the intuitionist accepts the triviality that for each real number x there is a natural number $y > x$. From (one version of) choice, it would follow that there is a function f such that for each real number x, fx is a natural number and $fx > x$. Since f cannot be continuous, the result contradicts Brouwer's theorem that every function on real numbers is continuous. If we assume extensionality, the axiom of choice also fails in intuitionistic set theory (and so Zermelo's well-ordering theorem fails). Thanks to David McCarty and Per Martin-Löf (See Martin-Löf (2009)).

Zermelo goes on to state that "in order to reject such a fundamental principle" as the axiom of choice, "one would have to ascertain that in some particular case it did not hold or to derive contradictory consequences from it; but none of my opponents has made any attempt to do this". Although this seems to merely shift the burden to his opponents, Zermelo goes on to justify his inclusion of a version of choice as an axiom. He observes that "Peano's *Formulaire* (1895) ... rests upon quite a number of unprovable principles". Zermelo claims that Peano arrives "at his own fundamental principles ... by analyzing the modes of inference that in the course of history have come to be recognized as valid and by pointing out that the principles are intuitively evident and necessary for science".

Zermelo then claims that these considerations can "be urged equally well in favor of the disputed principle", namely the axiom of choice. He supports the claim that choice is "necessary for science" by providing a list of seven theorems that seem to rely on it (one of which was the example, noted above, invoked in Zermelo, 1904).

This sounds a holistic theme, which we have encountered already, several times. But what does that have to do with "self-evidence" or "intuitive evidence", which Zermelo cites as a criterion of axiom-acceptance? He continues:

> That this axiom, even though it was never formulated in textbook style, has frequently been used, and successfully at that, in the most diverse fields of mathematics, especially in set theory, by Dedekind, Cantor, F. Bernstein, Schoenflies, J. König, and others is an undisputable fact ... Such an extensive use of a principle can be explained only by its *self-evidence* ... No matter if this self-evidence is to a certain degree subjective—it is surely a necessary source of mathematical principles, even if it is not a tool of mathematical proofs ...

Zermelo thus explicitly invokes an inference to the best explanation here. The truth of the axiom of choice is the explanans, but what is the *explanandum*? What is it that the axiom of choice explains?

Let us call a proposition or rule of inference *Zermelo self-evident* if it is invoked unreflectively, without explicitly citing it, in a wide variety of instances. Besides being vague, this notion is clearly relative. I should perhaps say that p is Zermelo self-evident *for a mathematician* (or a group of mathematicians) *S at time period t* if S applies p unreflectively during t, without explicitly citing it, in a wide variety of instances. A proposition or an inference can become Zermelo self-evident as practices evolve, and a proposition or inference can lose its status as Zermelo self-evident if one stops applying it unconsciously, either by not applying it at all, or by applying it only consciously, explicitly citing instances where it is invoked. In what follows, I will usually drop the relativity, leaving it implicit. In the cases of interest, the "mathematician" is the mathematical community at large, in a certain historical context.

Presumably, logical principles are Zermelo self-evident. Except in logic classes, we apply or-elimination, excluded middle, noncontradiction, and the like, all of the time, without citing them and often without being aware that we are relying on the principle in question. Zermelo claims, then, that the axiom of choice is Zermelo self-evident, and he supports this with a number of examples.

Cantor's famous 1899 letter to Dedekind, reprinted in Van Heijenoort (1967, pp. 113–117), contains an argument for a version of the well-ordering theorem. Cantor shows how to embed ordinals into any given "definite multiplicity" via a transfinite series of choices. Zermelo, who served as editor of Cantor's collected works (1932), added a note criticizing the derivation:

> ... the intuition of time is applied here to a process that goes beyond all intuition, and a fictitious entity is posited of which it is assumed that it could make *successive* arbitrary choices and thereby define a subset ... that ... is precisely *not* definable. Only through the use of the "axiom of choice", which postulates the possibility of a *simultaneous* choice and which Cantor uses unconsciously and instinctively everywhere but does not formulate explicitly anywhere, could [the subset] be defined ... It is precisely doubts of this kind that impelled the editor a few years later to base his own proof of the well-ordering theorem [(Zermelo, 1904)] purely upon the axiom of choice ...

Here we see Zermelo claiming that, for Cantor, the axiom of choice is Zermelo self-evident.[18]

To be sure, what I call Zermelo self-evidence is not exactly what Zermelo meant by "self-evidence" (e.g., in the passage from Zermelo, 1908, §2a, quoted above). He said that the fact that choice is applied widely and unreflectively is *explained* by its self-evidence. To answer a question I posed just above, Zermelo argues that choice is self-evident as an inference to the best explanation of the fact that choice is what I call Zermelo self-evident.

Zermelo's writings contain enough hints and explicit phrases to attribute a platonic ontology to him: the subject matter of set theory is an objectively existing, eternal, abstract realm of sets. It would follow that, in the terminology of §1 above, set theory is assertory. Concerning set theory, at least, Zermelo was not a Hilbert-style algebraist. In an intriguing and insightful article, Taylor (1993) suggests, at least tentatively, that Zermelo also held a kind of platonic epistemology: mathematicians have some access to the platonic realm of sets. In discussing impredicativity, Zermelo (1908, §2b) wrote that "the point of view maintained here [is] that we are dealing with a productive science resting ultimately upon intuition". But, to repeat a theme we encountered above, the access, or "intuition", is sometimes confused, or foggy. Taylor writes:

> Zermelo's historical method for establishing that axioms are intuitively evident suggests that mathematicians must have access to some platonic domain of sets in its entirety. This access will not be direct or immediate ... In this respect, Zermelo is probably not unusual ... More novel is Zermelo's idea that evidence for this access—whatever its nature (and there is little point in speculating on what Zermelo takes the nature of this access to be)—is gathered *empirically* by examination of the work of practicing mathematicians. (Taylor, 1993, p. 557)

The idea, then, is that the unreflective, unconscious use of choice counts as quasi-empirical, a posteriori evidence that mathematicians do "perceive" that it is true. At some perhaps unconscious level, it is indeed intuitively evident that the axiom of choice is true.

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[18] I assume that Zermelo's talk of "instinct" here is only a metaphor. In characterizing the notion of Zermelo self-evidence, I have avoided that term, since I do not want to suggest that the axiom of choice, and other similar principles, are innate.

We, the theorists out to codify and present the mathematical theory, know this—empirically and fallibly—because choice is what I call Zermelo self-evident.

I do not wish to make too much of this proposal for reading Zermelo as endorsing a platonic epistemology, but we will return to it briefly later. Present interest is also somewhat independent of the ontological issues. If someone asserts the axioms of set theory, then, of course, she is committed to holding that these axioms and the theorems that follow from them are true. But we need say little more about set theory's distinctive subject matter—if it has one. What matters, at present, is whether and how we know the axioms, and thus the theorems. In particular, I wish to explore the epistemic status of Zermelo self-evidence.

Zermelo's insight that in the early twentieth century, the axiom of choice was Zermelo self-evident was quite correct. In the end, this proved decisive within the mathematical community. Zermelo (1908) ended the paper with the "hope that in time all of [the] resistance can be overcome through adequate clarification". This is something Frege might have said about Basic Law V, as well. As we saw, this "hope" is part of one of Jeshion's (2004) "charitable" readings of Frege's pre-Russell attitude toward that principle. But in the case of the axiom of choice, the hope was realized. To simplify a complex story, the mathematical community came to realize that the proofs of many major theorems rely on choice principles. Ironically, this included the work of the French analysts, Baire, Borel, and Lebesgue—the major *opponents* of the axiom of choice. Choice was indeed Zermelo self-evident, even for them. This led to an intense effort to prove the theorems without relying on choice, efforts which often (indeed, usually) failed. In many of the cases, it was eventually seen—proved—that the use of choice was necessary, often that the theorem in question implies a choice principle. Over the ensuing decades, the mathematical community at large, with fewer and fewer exceptions, came to accept the axiom of choice—despite its untoward consequences (again, see Moore, 1982). What was done unreflectively is now down explicitly and consciously, without blinking.

Notice that a proposition can be Zermelo self-evident without being "evident" in any intuitive sense. It need not be obvious. Moreover, once a Zermelo self-evident proposition is made explicit, a thinker can balk at it, or even reject it outright, as happened with the axiom of choice. As we have seen, this can happen even if it is *shown* or *made explicit that* the proposition in question is Zermelo self-evident.

There is also an intermediate reaction to a claim of Zermelo self-evidence. One can claim that the arguments rely (or can be made to rely) only on a weak and more defensible version of the principle. For a time, there was a substantial industry of articulating and examining various versions of the axiom of choice. To some extent, the same goes for other disputed principles like the law of excluded middle, impredicative definitions, and the like.

For what it is worth, a similar phenomenon occurs outside of mathematics. In ethical and political considerations, for example, the doctrine of double-effect is, roughly, that an agent is not culpable (or perhaps less culpable) for foreseeable but unintended consequences of his acts. Once the doctrine is made explicit, it is debatable, to say the least. However, it is my experience, at least, that many people apply this principle unreflectively in dispassionate discussions. They speak of collateral damage, and the like. When the "principle" is made explicit, they sometimes withdraw the argument, or else go on to defend the doctrine of double-effect, or perhaps articulate a subtle, nuanced version of it that, they claim, is defensible.

Notice also that a false or even contradictory proposition can be Zermelo self-evident. It is arguable that Frege's Basic Law V, or some other version of an unrestricted comprehension principle, is or was Zermelo self-evident. Frege himself says as much, writing that the indicated "transformation" between concepts and their extensions

> … must be regarded as a law of logic, a law that is invariably employed, *even if tacitly*, whenever discourse is carried on about the extensions of concepts. The whole Leibniz–Boole calculus of logic rests upon it. (Frege, 1893, §9, my emphasis)

In so-called naive set theory, Basic Law V, or comprehension, is applied all the time. So is the principle that every well-ordering has an order type and, at one point in history, the proposition that a subset of a set is smaller than the set, and the proposition that a continuous function is differentiable almost everywhere (or at least somewhere).

Clearly, then, the method of adding an axiom because it is found to be Zermelo self-evident is defeasible. But no one ever said it is not; we are not looking for a safety net. Zermelo (1908, §2a) himself noted that “there are no infallible authorities in mathematics”, echoing a theme we keep encountering. Yet, when developing axiomatic systems, one has to start somewhere, and the fact that a proposition is Zermelo self-evident in an otherwise successful practice is sufficient reason to include it, at least provisionally.[19] Or so it seems to me. As Zermelo (1908, §2a) wrote:

> … so long as … the principle of choice cannot be definitively refuted, no one has the right to prevent the representatives of productive science from continuing to use this “hypothesis”—as one may call it for all I care—and developing its consequences to the greatest extent, especially since any possible contradiction inherent in a given point of view can be discovered only in that way … Banishing fundamental facts or problems from science merely because they cannot be dealt with by means of certain prescribed principles would be like forbidding the further extension of the theory of parallels in geometry because the axiom upon which this theory rests has been shown to be unprovable.

The perspective here seems to be that *any* proposed axioms, whether they be obvious, intuitively evident, or Zermelo self-evident, must pay their dues by playing a role in a systematization of an established and successful practice. The above passage concludes: “… principles must be judged from the point of view of science, and not science from the point of view of principles fixed once and for all”. So we are in a situation much like the one we found above with Jeshion’s and Burge’s Frege, and the concomitant notion of Fregean self-evidence.

The problem here, as in the previous section, is to square these holistic elements with the underlying Euclidean foundationalism. Zermelo employs the traditional vocabulary of “justification” and speaks of what is and what is not “provable”. As Taylor (1993, p. 598) notes, “there seems little doubt that his intention is to … ground mathematics in an epistemic sense”. I presume this invokes something like Fregean objective grounding relations. Taylor continues,

> … when it comes time to defend the axioms, Zermelo adopts a problematic stance … No doubt Zermelo regards some of his axioms as

[19] Of course, the key feature of the proposed axiom is that it figures, or appears to figure, in a successful practice. The fact that the proposed axiom is applied *unreflectively* does not add to its weight. If a proposed axiom were invoked explicitly, then presumably the practitioners could articulate (or did articulate) why it is applied, and it might then fall into some other category, such as being obvious, being Fregean self-evident, or constitutive of the subject matter. Thanks to Joshua Schechter here.

> straightforwardly self-evident. It is clear, however, that he does not regard them all that way, since what he emphasizes is indispensability. We can determine objectively through the examination of mathematical argumentation presented in written texts that a proposition has often been appealed to in the past ... The larger question from the point of view of foundations is this: How are the axioms to ground mathematics if our best evidence for them is that very mathematics? What seems to emerge is a conception of foundations that is not Cartesian at all ...

Just as we saw with Fregean self-evidence, we can wonder what is left of the Euclidean, foundationalist image.

Suppose that a proposition or rule of inference p is Zermelo self-evident for a mathematician, Chris, at time t, and that Chris tacitly invokes p at t. What is *Chris's* (internal) justification for p? She cannot very well be invoking *p because* it is Zermelo self-evident. By definition, Chris applies p unreflectively. If she had a reason she was prepared to cite, p would not be Zermelo self-evident for her.

Suppose that we now convince Chris that p is Zermelo self-evident for her, perhaps by getting her to examine her texts. Can she then cite Zermelo self-evidence to justify *future* uses of the proposition or rule p? That would clearly be circular, and question begging, or else an extreme conservativism. Chris would be saying that she is justified in applying p just because she applied it unreflectively in the past, and got away with it.

Recall that the second of the "ways out" from the previous subsection (§4.3) was to attribute Fregean self-evidence only to an ideal subject, leaving the human mathematician and epistemologist to sometimes fend for himself with systematic or holistic considerations. The third "way out" was to invoke distinctions between how a proposition is known, on one hand, and how it is known that the proposition is known or how we know how it is known on the other. The third proposal was to maintain that proper axioms are indeed Fregean self-evident, but in some cases, we need holistic considerations to know that the axioms are Fregean self-evident. Versions of these two ways out are available here, and they may even be combined. The idea is that the propositions in question—the axiom of choice in particular—are somehow *Fregean* self-evident. They are knowable directly, without doing any derivations, either for us humans or for our idealized counterparts. What I call *Zermelo* self-evidence is defeasible and, indeed, empirical evidence that an otherwise unprovable proposition has the indicated epistemic status.

It seems to me that if Zermelo self-evidence is to have a role in a Euclidean, foundationalist framework, it must be at this "how we know that we know" or perhaps "how we know how we know" level.[20] This sanctions Zermelo's practice of choosing such propositions as axioms, at least tentatively, alongside other, more straightforwardly obvious or self-evident axioms. As with the inclusion of holistic elements in a Fregean system, a Euclidean

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20 Once again, thanks to Richard Heck for pressing this option on me. Joshua Schechter suggested that one might adopt an anti-Euclidean, externalist epistemology here: Zermelo self-evidence does indeed provide justification for a proposition or an inference. On this view, Chris is justified in adopting the proposition or an inference in question—just because it is Zermelo self-evident—although she cannot cite this justification. Her later recognition of its status as Zermelo self-evident provides some assurance at the metalevel (although, ironically, it would then keep the proposition from being Zermelo self-evident, as Chris would then explicitly cite such applications).

foundationalist system is left in place. In invoking Zermelo self-evidence, the Euclidean has not contradicted herself. However, as we found at the end of the previous section, there is still a deep problem of motivating the foundationalist program. What reason is there to think that the fact that a given proposition or inference is applied unreflectively is even defeasible or fallible evidence that it has the proper epistemological status that axioms should have—that of being knowable in themselves, not on the basis of deduction? How is Zermelo self-evidence even *relevant* to the proper epistemic foundational status, even at the "how we know that we know" or "how we know how we know" level?

The platonic epistemology that Taylor attributes to Zermelo (and others at the time) does go some way to answering these questions. The idea, again, is that the mathematician, Chris say, does have some more or less direct grasp of the realm of sets, something analogous to sense perception. But Chris is not always so clear about what she is "perceiving". The epistemic faculty itself is sometimes foggy, and Chris does not have an infallible and direct introspective access to her own mathematical faculty, to make sure it is functioning accurately. Then, it seems, the fact that she applies a given proposition or inference unreflectively does count as some evidence that she "perceives" it directly and thus, that it enjoys the indicated epistemic status. The further fact that Chris's practice, with the invoked proposition or inference, is successful is more defeasible evidence for this epistemological claim on behalf of the proposition.

This inclusion of defeasible and, indeed, empirical elements in an overall platonic epistemology is indeed intriguing. But there is still the burden of justifying the platonic epistemology, and, if one is so inclined, fitting it into a naturalist program. Failing a platonic epistemology, there is much work to do to incorporate the methodology in question into a well-motivated Euclidean foundationalism. By definition, we cannot provide nontrivial proofs of axioms, at least in the system in which they are axioms. But, presumably, we do know these axioms in typical cases. If we have to turn empirical and holistic when asked *how* we know them, then it is fair to question the point of the foundationalist framework. Why not just say that we know the axioms and the theorems alike via their role in systematizing our mathematical theories?

I do not claim to have canvassed every possible notion of self-evidence in play at the foundation of an assertory, Euclidean foundationalist system that underlies a sophisticated mathematical theory, and so I do not take the conclusions here to be definitive. Suppose, however, that we jettison the entire foundationalist enterprise, and opt for a more holistic approach. Although obviousness is, of course, a psychological notion, it has epistemic weight. Adopting axioms because they are obvious is, it seems, a sound policy in striving for reflective equilibrium in mathematics. Obviousness is not infallible; we may have to revise in light of later developments. But we are not looking for infallibility anyway, and it seems evident that in axiomatizing, one should start with what appears obvious, either at first or after immersion in a practice.

But obvious axioms only take us so far. I submit that from the anti-Euclidean, holistic perspective, Zermelo self-evidence also has an important epistemic role to play, and not just at the meta-level of "how we know that we know" or "how we know how we know". That is, Zermelo self-evidence gives one a reason to believe or accept the proposition or inference. Return to our mathematician Chris. If she learns that she has been applying a given proposition or rule of inference in her otherwise successful practice, then she has at least some reason to believe the proposition is true or that the inference is valid. If the empirical analysis leading to the judgment of Zermelo self-evidence is correct, then the proposition or practice in question is indeed part of the successful practice. That, alone, gives us at least *some* reason to believe it.[21] We may be mistaken, and the proposition or inference may have to be retracted later, but for the present, it is good.

It is part of the holistic program that all evidence is tentative, at least in principle. So even if Zermelo self-evidence counts as evidence, it should be supplemented with an exploration of the consequences of the indicated proposition or inference. This is exactly the methodology that Zermelo advocated.

Recall the above passage from Zermelo (1908, §2a):

> ... so long as ... the principle of choice cannot be definitively refuted, no one has the right to prevent the representatives of productive science from continuing to use this "hypothesis"—as one may call it for all I care—and developing its consequences to the greatest extent, especially since any possible contradiction inherent in a given point of view can be discovered only in that way ...

As far as I know, this is the only place where Zermelo invokes the notion of "hypothesis" with respect to the axiom of choice. Notice that he does not call it a hypothesis himself. He is only allowing his opponents to use that word, if they want. All he needs is that his opponents refrain from dismissing his system out of hand.

From the holistic perspective, as from any other, there is of course a difference between a working hypothesis and an established belief (or a known proposition). For the holist, however, the border between those can be fuzzy. A proposition can start life as a working hypothesis, and can later become an established belief or a known fact if it proves fruitful, serving an essential role in a successful system. So Zermelo is not really making any concessions to his opponents by allowing them to call the axiom of choice a hypothesis. The methodology, and the rhetoric, makes eminent sense from an anti-Euclidean, holistic perspective broached above.

Recall the passage from Frege (1884) that got this project underway: "it is in the nature of mathematics to prefer proof, where proof is possible". The holist heartily endorses this. The web of mathematics is supported, and properly pruned and extended, by discovering deductive connections between propositions, both within a given branch of mathematics and between branches. By deriving a hitherto assumed proposition, or a working proposition, from others, we see what is involved in accepting or rejecting it. Putting his Euclidean rhetoric aside, this seems to be what Zermelo urged on his opponents, and what he did himself. The project of systematizing, as endorsed by Frege so enthusiastically, makes sense from the holistic perspective; indeed, systematization is what drives the enterprise.

To conclude, then, what of axioms, those propositions which have no (nontrivial) proof? It should be clear that there is no special problem here for the holist. The axioms and theorems support each other.

§6. Final word. The notion of self-evidence occurs prominently—early and often—in the education of just about every American. The "Declaration of Independence of the Thirteen Colonies" that would later constitute the United States of America reads, "We hold these truths to be self-evident: that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and

[21] As noted, the epistemic bottom line here is the role of the proposition or inference in a successful practice. The fact that the proposition is applied *unreflectively* does not add to this weight.

the pursuit of Happiness”. Let us leave aside the part about the Creator. I have no doubt that all people should indeed be treated as equal in any civilized community, and that the human rights listed are inviolable. But it is not so clear just what the author of this venerable document meant by “self-evident”. The propositions were not, and are not, Zermelo self-evident, applied unreflectively, although it would be a much better world if they were. Perhaps the truths are Fregean self-evident. Perhaps a sufficient grasp of the propositions, and in particular a clear and distinct knowledge of the concept expressed by the word “men” (or “people”, or “human”) is sufficient for one to know, without doing any reasoning, that the propositions are true. Would that it were so, but people who flout human rights can hardly be accused of not understanding the concept “men” (or “people”, or “human”). Like the axiom of choice, the truths in question are indeed obvious now, at least to many of us, but they were hardly obvious then. If they were obvious, the framers would not even have to state them, much less call them self-evident.

To end on a speculative note, perhaps the framers had reflected on the role and nature of society and, in particular, on the place of government, and gotten to the point that the propositions in question were indeed obvious *for them*. They could not see things any other way (their practice as slave-holders notwithstanding). This is a rough analogue of what I called obviousness upon immersion in a practice. To use Gödel’s phrase, the propositions forced themselves on them as true. The framers proposed to codify those insights into the emerging society they were about to found, making them explicit—axiomatic—and they invited others to follow.

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