Jonathan Shear and Neil Sims

Going outside the system: Gödel and the "I-it" structure of experience

Article (Accepted version) (Refereed)

Original citation:
Shear, Jonathan and Sims, Neil (2018) *Going outside the system: Gödel and the "I-it" structure of experience*. Journal of Consciousness Studies, 25 (5-6). pp. 179-201. ISSN 1355-8250

Going Outside the System: Gödel and the "I-it" Structure of Experience

Jonathan Shear* and Neil Sims

Abstract

It has often been argued that Gödel's first incompleteness theorem has major implications for our understanding of the human mind. Gödel himself hoped that the results of his theorem, combined with Turning's work on computers and phenomenological analysis, would establish that the human mind contains an element totally different from a finite combinatorial mechanism. Decades of attempts to establish this by reasoning about Gödel's theorem and Turing's work are now widely taken to be unsuccessful. The present article, in accord with Gödel's suggestion, adds extended phenomenological analysis to the discussion. It also focuses on the "going outside the system" step central to Gödel's method of proof, rather than on the implications of the theorem itself. Analysis of the "I-it" intentional structure, held by phenomenology to underlie all ordinary experience, yields a simple model that (i) resolves long-standing conceptual problems associated with the "I-it", the most basic structure of phenomenology, (ii) clarifies the "going outside" step crucial to Gödel's method of proof, (iii) avoids conceptual problems associated with this step, (iv) identifies the step as an instance of a natural pre-mathematical operation of ordinary thought, and (v) suggests that the step itself is intrinsically non-algorithmic. Logical analysis of role of this step in Gödel's proof then shows, independently of phenomenological considerations, that anything (human or not) that can prove Gödel's theorem soundly by his method cannot be entirely algorithmic. Further implications for the nature of the mind are then suggested.

*Correspondence to jcs@infionline.net

I. Gödel's First Incompleteness Theorem[1] is one of the most influential theorems of twentieth-century mathematics. It revolutionized fundamental assumptions about the relation of axioms to mathematics and knowledge as a whole. It has also often been taken to indicate that the human mind is capable of operating in mathematically significant ways that no computational machine can match. Gödel himself wanted to establish this, and felt strongly that his theorem, combined with Turing's theoretical work on computers, should yield the result that

> Either there exist infinitely many number theoretic questions which the human mind is unable to answer or the human mind contains an element totally different from a finite combinatorial mechanism (such as a nerve net acting like an electronic computer). I hope to be able to prove on mathematical, philosophical, and psychological grounds that the second alternative holds."[2]

In recent years Roger Penrose has taken up this challenge, arguing that Gödel's and Turing's theorems do in fact imply that the human mind can think in mathematically useful ways that transcend algorithmic[3] computation in general, and those made by mechanical algorithmic com-

---

[1] In what follows the expressions, "Gödel's Theorem" and "Gödel's Incompleteness Theorem," when used without further qualification, will refer to this first theorem.

[2] From one of Gödel's posthumous letters, quoted by Richard Tieszen in his *After Gödel: Platonism and Rationalism in Mathematics and Logic* (Oxford University Press, 2011) p. 183.

[3] The term "algorithmic" will be used throughout this paper in the standard logical sense of

> A precisely described routine procedure that can be applied and systematically followed through to a conclusion. (*Concise Oxford Dictionary of Mathematics*, 4th edition (New York: Oxford U. Press, 2009))

puters in particular.[4] This argument has received considerable attention in recent decades. It clearly resonates with many people's intuition. But it has also been strongly criticized.

These criticisms are often quite technical. But the overall thrust of many of them can be readily understood non-technically. Penrose argues that the "implications of Gödel's theorem… establish that there must be a noncomputational ingredient in human conscious thinking."[5] But the theorem itself is a purely formal abstract mathematical statement. By itself it says nothing at all about human awareness. And abstract theories have to be connected to the empirical world in clear-cut ways before they can properly be taken to imply empirical facts. This requires empirical models, with clear accounts of precisely which phenomena in the empirical world are to be taken as correlates of specific elements of the theory. But as Penrose's critics have often noted, the intended correlates of his theory do not appear to be well specified. Thus in particular, should the "human conscious thinking" in question be taken to be that of *people in general*, or *mathematicians*, or *mistake-free* mathematicians, or those operating in accord *with a finite set of* axioms, or with a finite set of *consistent* axioms, or even with a set of axioms *known to be* consistent? Such progressively qualified conceptions of the kind of "conscious thinking" and/or minds Penrose wants to draw conclusions about have been offered in response to the evolving criticisms. Eventually Penrose felt constrained to suggest that, strictly speaking, the argument actually applies only to the *idealized collective mind* of *mathematicians in general*, and even here only *in principle*.[6] As a result, details of the many criticisms aside, his original thesis, which had appeared to be a statement about real human minds, would now appear to have lost all significant empirical content.

Penrose's attempt to derive a *real* noncomputational ingredient in human conscious thinking (rather than a mere abstract concept of such a thing) from considerations of Gödel's theorem thus appears unsuccessful. And as the above indicates, this is just what ought to be expected from any purely abstract approach to the question, since empirical conclusions require empirical grounds. Abstract considerations are not enough.

Gödel himself appears to have recognized this clearly, as indicated by the last sentence of the passage quoted at the beginning of this paper:

> I hope to be able to prove on mathematical, philosophical, and psychological grounds that the second alternative [i.e. "that the human mind contains an element totally different from a finite combinatorial mechanism"] holds.

This reference to an empirical approach was not a mere afterthought. Gödel was so concerned with formulating this empirical point properly that his posthumous letters contain ten different recastings, with versions such as

---

as explained in more detail in Wikipedia's article, "Algorithm,"

> In mathematics and computer science, an **algorithm** is an unambiguous specification of how to solve a class of problems…an effective method that can be expressed within a finite amount of space and time and in a well-defined formal language for calculating a function. Starting from an initial state and initial input (perhaps empty), the instructions describe a computation that, when executed, proceeds through a finite number of well-defined successive states, eventually producing "output" and terminating at a final ending state. (Algorithm. (2018, February 12). In *Wikipedia, The Free Encyclopedia*. Retrieved 04:12, February 13, 2018, from https://en.wikipedia.org/w/index.php?title=Algorithm&oldid=825280696)

[4] Penrose's arguments are widely regarded as developments of similar approaches taken decades earlier by the philosopher J.R. Lucas. See, for example, "The Lucas-Penrose Argument About Goedel's Theorem," *Internet Encyclopedia of Philosophy*, ("http://www.iep.utm.edu/lp-argue/").

[5] Roger Penrose, "Beyond the Doubting of a Shadow", in *Psyche* 2 (23) 1996, section 1.12

[6] Cp. *ibid*, 6.1

> I conjecture that the second alternative is true and that the transformation of certain aspects of traditional philosophy into an exact science will lead to its proof.

and

> I conjecture that the second alternative is true and perhaps can be verified by a phenomenological investigation of the processes of reasoning.[7]

Gödel's concern for the importance of an empirical[8] approach here, and the tentativeness of his position in the absence of adequate empirical evidence, could not be clearer. It is also consistent with his long-standing interest in phenomenology.

In what follows we will explore some implications of this more empirical approach to reexamine the question of what, if anything, Gödel's theorem and proof might imply about the nature of the human mind.

II. For many centuries it was thought, on the model of Euclid's geometry, that all of mathematics ought be generatable from a finite system of consistent axioms, or, as the idea became revised by the twentieth century, from a system of consistent axioms that, if not finite, can at least be enumerated by some algorithmic procedure. And systematizing all of mathematics in this way was a major hope of mathematicians and scientists in the early decades of the twentieth century. Gödel's Incompleteness Theory however is widely held to have derailed this hope. For it showed that no step-by-step effectively enumerable (i.e., "algorithmic") system of consistent axioms capable of expressing basic arithmetic could demonstrate even all the truths of *arithmetic* (i.e., truths about numbers), much less of mathematics in general. Gödel obtained this result by showing that whatever such consistent axiomatic system S one regards,[9] one can specify an arithmetical statement G that (1) cannot be proven, disproven, or even coherently postulated,[10] within S,[11] but which, at the same time (2) can immediately be recognized ("metamathematically") *from outside* S as both coherent and true. For the straightforward semantical meaning of the mathematical formula, G, is "G cannot be proven in S." So once we know that G

---

[7] Tieszen, *loc. cit.* (note 2)

[8] The term "empirical" will be used throughout this paper, in accord both with empiricists such as Hume and with many contemporary English dictionaries, in the traditional sense of based on, concerned with, or verifiable by observation or experience rather than theory or pure logic" (Oxford Dictionaries, https://en.oxforddictionaries.com/definition/empirical).

[9] His famous proof actually showed this for the system of axioms of Russell and Whitehead's foundational *Principia Mathematica* (at the time referred to as "PM"), and was then quickly generalized.

[10] Cp. Torkel Franzen, *Gödel's Theorem: An Incomplete Guide to its Use and Abuse* (Wellesley, MA: A.K. Peters, 2005) p. 126, quoted at the beginning of Section III below.

[11] The steps leading up to this conclusion are often subtle and difficult. They involve, among other things, generating correspondences between all the symbols and combinations of symbols used in PM, the most widely used mathematical language of the time, and sets of integers specifically constructed in such a way that the logical relations between statements in PM are preserved ("mirrored") in relations between these numerical sets. This allows the logic of mathematical assertions in PM to be recast in terms of assertions about numbers ("arithmetic"). As a result the specially constructed mathematical formula G can specify information about its own non-provability by referring to relationships such as provability between (i) *the number* identified with its own string of symbols and (ii) *the numbers* identified with other strings of symbols. G thus refers *only to numbers* and *their* relationships rather than the strings (including G itself), relationships that are purely numerical and establishable purely syntactically. This, along with the fact that G is concerned with *provability* rather than *truth*, allows G (which states, in effect, that G itself cannot be proven within S) to avoid the semantical problems of self-reference associated with the "liar paradox" ("This sentence is false") that suggested the formulation of G in the first place. Nevertheless, while not self-contradictory, G cannot be coherently postulated within the system S, since *within S*, "G is provable if, and only if, not-G is provable." Ernest Nagel and James R. Newman, *Gödel's Proof* (New York and London: New York University Press, 2001) pp. 93, 99 & footnote 34 p. 99.

is true, we know there is a true statement of arithmetic, G, that S cannot prove. Thus S is *necessarily incomplete*. Gödel then (3) uses this result to show that S is *incompletable*. One might think that this problem of incompleteness could be resolved by adding G as an axiom to S, forming a new axiomatic system S′ that includes this previously excluded mathematical statement. But by the same reasoning above, there will now be a G′ (modeled after G) that is both true and unprovable in S′. Adding G′ to S′ to form a new system S″ would yield a G″ that is both true and unprovable within S″, and so on, *ad infinitum*. As a result S, and more generally, all consistent finitistic axiomatic systems capable of grounding arithmetic, are not only incomplete, but *incompletable*.[12]

Step (1) above, while subtle and difficult, appears to unfold syntactically in terms of the kinds of computational moves a computer can make and check. But step (2), the basis of step (3) and (4), is very different. For this step requires one to (a) *go outside everything the rules of* S *can coherently specify or determine,* in order to (b) evaluate a statement semantically and (c) recognize that it is both true and not expressible within S and its rules. And it is precisely this non-computational step of going outside algorithmic rule-based combinatorial operations, and evaluating something semantically rather than merely syntactically, that leads Penrose to say that

> human minds can reflect on the meanings of the symbols in a formal system, and move from that system to a new system outside of it that has new rules capable of doing useful things the original system cannot do. But a computational system cannot do this, it just has the rules it already follows.[13]

and conclude that the

> implications of Gödel's theorem… establish that there must be a noncomputational ingredient in human conscious thinking"[14]

And it is the step that, as noted earlier, led Gödel to conjecture that

> the human mind contains an element totally different from a finite combinatorial mechanism (such as a nerve net acting like an electronic computer).[15]

III. Using this step of "going outside the system" of Gödel's proof to draw such conclusions, however, raises real problems. For as Torkel Franzen notes, while it is true that

> One cannot prove, or even truly postulate, the Gödel sentence [e.g., G] of the system…in the system, [nevertheless] [t]he image of "going outside the system" is a bit too seductive, in that it suggests there is some generally applicable way of viewing a system "from outside" so as to be able to prove things about it that are not provable in the system. We don't know of any such general method.[16]

A second, more formal problem with attempts to use Gödel's theorem to conclude that the human mind has "an aspect" capable of "going outside" algorithmic systems in a mathematically significant way should also be noted. Gödel's proof *presumes* the mathematician's mind can *always* "go outside" the system in this way, no matter how far along the sequence of systems (S, S′, S″, etc.) one might be. As a result the theorem, at

---

[12] With some additional reasoning Gödel then uses this result to establish that such axiomatic systems can never establish their own consistency (his "Second Incompleteness Theorem"), even when their consistency can be established *from outside* their confines.

[13] Penrose 1994, *Interview with Jane Clark*. Journal of Consciousness Studies, 1, 1:17-24: 19.

[14] Penrose, 1996, *loc. cit*. (note 5)

[15] Tieszen, *loc. cit*. (note 2)

[16] Torkel Franzen, *Gödel's Theorem: An Incomplete Guide to its Use and Abuse* (Wellesley, MA: A.K. Peters, 2005) p. 126.

least as Gödel proved it, can hardly be thought to *establish* the existence of an aspect of mind capable of going outside the systems in this way. For using a theorem to establish something its proof *presumes and relies on* would obviously be circular.

Thus both *what* this "going outside" is and *why* it is *presupposed* by Gödel's method remain unclear, even if the action it describes is something mathematicians regularly both do and take for granted. Given the central role "going outside the system" plays in the discussion, arguments that Gödel's work implies the existence of an algorithm-independent aspect of mind are likely to continue to remain unconvincing unless these problems can be addressed. And addressing them properly cannot rely on abstract mathematical formalisms alone. For insofar as "going outside" is something mathematicians actually *do*, understanding it requires empirical, as well as more formal abstract knowledge.

These critiques are consistent with Gödel's own position that by itself mathematics and logic are inadequate to establish that the human mind can at least sometimes operate independently of algorithmic constraints. For, as we saw above, he explicitly maintained that empirical phenomenological support would be needed, above and beyond such merely formal considerations. Precisely how phenomenology might do this was not clear. But Gödel had high regard for its potential to help us understand foundational questions in mathematics both here and elsewhere, even going so far as to hold that

> Phenomenological investigation of the construction of mathematical objects is of fundamental importance for the foundations of mathematics.[17]

Let us now turn to phenomenology to see some ways it might be able to contribute to our present discussion.

IV. Different schools of phenomenological thought may differ in detail about what phenomenological investigation involves. But there is general agreement that, as the opening sentence of the *Stanford Encyclopedia of Philosophy*'s entry on "Phenomenology" puts it, "Phenomenology is the study of structures of consciousness as experienced from the first-person point of view."[18] And the second sentence asserts, "The central structure of an experience is its intentionality, its being directed toward something."[19] Western philosophers, in other words, generally hold that all experience is "intentional" in structure, characterized by the observer-observed, "I-it" relationship we are all familiar with in ordinary experience. And examination of this fundamental structure of phenomenology will prove most useful for the questions discussed above.

From the perspective of common sense this "I-it" structure might seem indisputable. For what could be more obvious than the fact that experiences are given to us as organized into two relational poles: experiencer and the "object" (sensory perception, image, thought, etc.) experienced. Thus Bertrand Russell could argue shortly after the *Principia Mathematica* was completed that

---

[17] Hao Wang, *A Logical Journey: From Gödel to Philosophy* (USA: Bradford Publishing, MIT Press, 1996) p. 256, para 8.2.8.

[18] Smith, David Woodruff, "Phenomenology", *The Stanford Encyclopedia of Philosophy (Fall 2011 Edition)*, Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2011/entries/phenomenology/>.

[19] *Ibid.*

> It is plain that we are not only *acquainted* with the complex "Self-acquainted-with-A," but we also *know* the proposition "I am acquainted with A."[20]

and add a few pages later that the "I-it" structure of experience is so basic that ultimately

> there are only two words which are strictly proper names of particulars, namely, "I" and "this."[21]

The long-standing problem of locating anything corresponding to the "I" in the perceptual field, however, eventually led Russell to conclude that "[t]he bare subject, if it exists at all, is an inference, … not part of the data" of experience,[22] a mere "logical fiction," "schematically convenient, but not empirically discoverable."[23] Thus, in the final analysis, the term "I" should be excluded from the list of proper names in the strict sense, leaving only "this" referring to whatever happens to be presented in one's perceptual field.[24]

The seeming impossibility of giving clear empirical sense to the "I," generally taken for granted by Western philosophers since Hume and Kant, might well thus seem to render the experiential sense of the "I-it" structure utterly problematic. Nevertheless, despite this conceptual problem, the structure remains obvious to ordinary experience, and basic to Western phenomenology.

V. Eastern phenomenological investigations however suggest that we can address this problem directly. As is well known, Eastern cultures have for many centuries often emphasized the importance of developing and employing systematic meditation procedures to explore inner awareness. The aim of this exploration has generally been to facilitate development of various mental states and abilities, including especially states referred to as "enlightenment," rather than to produce phenomenological knowledge *per se*. But over the centuries a considerable body of phenomenological information has nevertheless been amassed. The nature of one widely reported family of experience in particular will prove useful for our discussion. These experiences are reported in association with many different kinds of meditation procedures[25] designed to allow all ordinary objects of awareness (sense perceptions, thoughts, emotions, etc.) to fade and disappear, while one nevertheless remains awake. Put simply, a major idea underlying these meditation procedures is that removing all the phenomenological objects that ordinarily absorb our waking attention can allow basic, but otherwise "overshadowed" features of consciousness to emerge and become conspicuous, much as removing the film from a projector (while leaving its light on) allows the underlying screen (with its flatness, whiteness, etc.) to emerge into awareness once the film's attention-grabbing action has disappeared.

---

[20] Bertrand Russell, in his famous 1911 article, "Knowledge by Acquaintance and Knowledge by Description," reprinted in his *Mysticism and Logic and Other Essays* (New York: Norton and Co., 1924) pp. 211-2, reproduced at http://www.gutenberg.org/files/25447/25447-h/25447-h.htm .

[21] *Ibid.* p. 224. The term "this" here refers to whatever "it" might be present at the time in one's awareness.

[22] Bertrand Russell, *Our Knowledge of the External World* (London: George Allen & Unwin, 1922) p. 74

[23] Bertrand Russell, "On Propositions: what they are and how they mean," in *Logic and Knowledge, Essays 1901-1950*, p. 305, see also, "On the Nature of Acquaintance," in the same volume, p. 276.

[24] From Russell's footnote, to the passages quoted above from "Knowledge by Acquaintance and Knowledge by Description," added when the article was reprinted in *Mysticism and Logic*. *Ibid*. p. 233.

[25] Compare *The Experience of Meditation: Experts Introduce the Major Traditions*, edited by Jonathan Shear (USA: Paragon House, 2006), p. xvi.

> [Major] traditional meditation procedures…differ with regard to the mental *faculties* they use (attention, feeling, reasoning, visualization, reasoning, memory, bodily awareness, etc.), the *way* these faculties are used (effortlessly, forcefully, actively, passively), and they *objects* they are directed to (thoughts, images,…breath, subtle aspects of the body, love, [concepts of] God).

Various types of experiences are reported to occur as this process of dropping off ordinary contents of awareness in meditation unfolds. The particular experience (or type of experience) we will be concerned with is said to occur just before and/or after all phenomenological objects disappear. Each meditation tradition describes this type of experience in its own particular way. Thus, for example, Maharishi Mahesh Yogi, founder of the Transcendental Meditation program, describes it in the language of Yoga and Vedanta as follows:

> When the object of experience has diminished to the point where it has disappeared, the mind ceases to be the experiencing mind…But during this process of transformation, it first gains the pure state of its own individuality…When the mind gains this state…It holds its individuality in the void—the abstract fullness around it—because there is nothing there for it to experience. It remains undisturbed, awake in itself…a silent wave on a silent ocean…[26]

And the twelfth century Zen (Ch'an) Master Hongzhi Zhengjue offers the following account:

> Take the backward step and directly reach the middle of the circle from where the light [of one's awareness] shines forth…turn within and drop off everything completely…turn your light inwardly to illuminate the self…let everything fall entirely away…[so] body and mind of themselves drop away and the original face [i.e., one's basic nature] is manifested…as expansive as the great emptiness of space. [27]

The language of the two descriptions is different. One is didactic, the other explanatory, and each uses imagery ("light" of awareness, one's "original face," "wave on an ocean") characteristic of its own tradition. Depending on whether terms such as "light" and "illuminate" are taken as metaphoric or phenomenologically literal, the descriptions may or may not appear to refer to precisely the same experience. This will not matter for the purposes of our present discussion however, for what matters here is only that both accounts (along with many others) clearly describe a common *structure*—a bare observing awareness surrounded by phenomenological space devoid of perceptual objects—experienceable by itself when all objects of awareness (other than the space itself and the "light" of awareness that makes it perceptible) disappear in meditation.[28] In time this structure is reported to become clearly recognized as the context within which thoughts, images and other phenomenological objects periodically exist, disappear and reemerge during meditation.[29] Eventually this structure is reported being recognizable as the context of all thought and experience in everyday life as well, clarifying, it would seem, what had formerly been merely subliminal.

The structure itself is very simple. One does not need successful meditation to get a good idea of it. All one has to do is to is imagine floating in space, with no body-awareness, watching the stars all fade out and disappear, leaving (if we ignore the other senses) only a sense of being a localized bare awareness, surrounded by measureless empty space. This simple thought experi-

---

[26] Maharishi Mahesh Yogi, *Maharishi Mahesh Yogi on the Bhagavad-Gita: A New Translation and Commentary, Chapters 1-6* (New York: Penguin Books, 1974) p. 422.

[27] *Cultivating the Empty Field: The Silent Illumination of Zen Master Hongzhi*, edited and translated by Taigen Dan Leighton (Rutland, VT: Tuttle Publishing, 2000) pp. 19-20.

[28] Usually, of course, we do not refer to phenomenological space, much less the "light" of awareness, as phenomenological "objects." But the deeper meditation experience where even these two highly abstract components of experience are absent makes it clear that both are in fact experiential content that can be either present or absent in experience and thus, technically speaking, phenomenological "objects."

[29] Compare Leighton's expression, "the vast luminous space around and beneath…thought nodules." *Loc. cit.*, p. 20.

ment[30] should suggest why, despite their many well-known disputes about what the self "really" is, so many meditation traditions have simply taken it for granted that the experience (or type of experience) we have been discussing displays the basic nature and structure of individual awareness. For it only seems natural to think that what remains when all objects of awareness disappear while one nevertheless remains awake would have to be one's own awareness. And whether or not the above thought-experiment might make it seem intuitively obvious, cross-culturally congruent meditation-related reports clearly indicate that, phenomenologically speaking, *something* does in fact remain when all ordinary phenomenal objects disappear while one remains awake: bare awareness surrounded by empty phenomenological space.

The above sorts of experiential reports raise many methodological and ontological questions, including those of how they might be evaluated objectively, and what their implications might be for various theories about the nature of self. These are important questions, and the literature related to them is growing.[31] Our present concern will not be with these sorts of questions, however, but only with *the structure* of the experiences as described. For this structure suggests a simple model for the "I-it" basic to phenomenology that (a) highlights important features of ordinary experience, (b) avoids problems of the sort Russell raised, and (c) provides a new, clarifying perspective for understanding the "going outside the system" crucial to Gödel's proof.

VI. The structure of the "I-it" model suggested by the above experiences, simply put, is (i) one's bare awareness at the "center" or "here" of one's phenomenological space with (ii) attention oriented outward towards the space and any objects that might appear "there" in it. The "I-pole" of the structure will refer to the locus of one's point of view in this space.[32] Phenomenological objects that may appear in this space will be referred to as "its."

We should note that the "I-it" structure is defined without reference to phenomenological objects or "its" that might be experienced in it. There are several reasons for this. First, the meditation-related reports referred to above specifically assert that the structure can be experienced and identified in the complete absence of all such "its". Secondly, consideration of such unusual experience aside, reflection on ordinary experience[33] makes it clear that the "I-it" relation is identifiable independently of reference to "its." For in ordinary experience one's point of view is

---

[30] Compare Ibn Sina's (Avicenna) famous "Floating Man" thought experiment, offered a thousand years ago to illustrate the self-aware nature of pure individuality independent of all object-oriented awareness. (Cp. "Floating man," (2017, December 15). In *Wikipedia, The Free Encyclopedia*. Retrieved 04:25, February 13, 2018, from https://en.wikipedia.org/w/index.php?title=Floating_man&oldid=815584929.

[31] Much of this research has focused on the deeper, better-known meditation-related experience (often referred to as "pure consciousness" or "pure emptiness") gained when even the abstract, empty phenomenological space with its "I-it" structure has been left behind. See, for example, Jonathan Shear, "Experiential Clarification of the Problem of Self," *J. of Consciousness Studies*, Vol. 4. No. 5/6, 1998; and Shear, "Converging on the Self: Western Philosophy, Eastern Meditation, Scientific Research," in *Interdisciplinary Perspectives on Consciousness and the Self*, edited by S. Menon, A. Sinha and B.V. Sreekantan (Springer, 2014).

[32] The shape of phenomenological space and configuration of the "I-pole" within it can vary depending on such things as state of consciousness and type of mental activity involved. The text above reflects features of the (I –> it) directional arrow common to types of experience (including intellectual work such as mathematical thinking) phenomenology ordinarily studies. We can also note that some meditation-related experiences do not have this structure, but these unusual experiences do not appear relevant to role of the directional arrow in the discussion below.

[33] Expressions such as "ordinary experience" and "phenomenology of ordinary experience" will at times be used below, to allow for the facts that the experience of "pure consciousness"/"pure emptiness" (see note 29 above) is *completely* devoid of all empirical content, including even the "I-it" structure itself, and thus requires phenomenological analysis independent of this structure. This unique experience and associated phenomenology need not be taken into consideration for the purposes of the present discussion, however.

readily recognizable as from "here,"[34] the point of view of one's experience, regardless of what "its" (sense objects, thoughts, etc.) may happen to be present in the experience. This shows that the "here" of one's point of view is recognizable in ordinary experience independently of reference to particular "its," including in particular any "it" or set of "its" that might be supposed to be capable of identifying or defining this phenomenologically central locus.[35]

We should also note that the locus of one's point of view *at* the center of one's phenomenological space is not something that one experiences *in* this space. A little reflection should make it clear why this should be the case. In ordinary experience, attention is always directed *outward from* the locus of one's point of view *to* other places. This remains true, no matter where one directs one's attention. For to use visual imagery, wherever one looks with one's mind's "eye," one remains at one's phenomenological center, the "here" and "now," of one's experience. And since looking is always *from* this center *to* other points, this "center" always remains unseen in itself. In terms of our model this can be expressed by saying that one's point of view, the place from which one perceives "its" and to which they appear, is always *outside the range of* the "its" of the "I-it" structure.[36]

These results are consistent with ordinary phenomenological usage where the "I-it" relation is typically represented by means of a directional arrow (I –> it), to indicate that phenomenal awareness is always (a) directed *from* the perceiving "I" *towards* phenomenological objects and (b) that the "I" (whatever it might be, or not be, in itself) is itself always *unperceived*.

They are also consistent with centuries of philosophical investigations that have generally led Western philosophers since Hume and Kant to conclude that introspection appears completely incapable of locating any internal perceptual object (or set of objects) corresponding to our ordinary notion of the "I" or inner self.

Earlier we saw that Russell, arguing within this tradition, concluded that the inability to identify the "I" in terms of "its" implies that the "I" has to be a mere "logical fiction," "schematically convenient, but not empirically discoverable," and we noted that this conclusion might appear to call into question the empirical significance of the "I-it" structure basic to phenomenology. The inability to identify the "I" in terms of "its," however, presents no difficulty for the "I-it" model discussed here. In the first place this model was defined and identified experientially in terms of the phenomenological locus of *one's point of view*, entirely independently of questions of the empirical significance and/or ontological nature of any "I" supposed to be associated with this

[34] This is of course an *indexical phenomenological* "here," as distinct from a "here" locatable in objective space.

[35] It is also easy to see, purely logically, that any structure capable of being the experiential context of all "it"-oriented experience has to be phenomenologically independent of any particular "it" (or set of "its"). For any structure phenomenologically dependent on experience of any particular "it" (or set of "its") will necessarily be incompatible with all those logically possible experiences where that particular "it" (or set of "its") is absent, and as a result could not be the structure of all possible "it"-oriented experiences (much as, by analogy, glasses of a given color cannot display experiences where that color is absent). This should make it apparent why the "I-it" structure of ordinary experience, the basic structure of phenomenology, should be defined independently of reference to any particular "it" or set of "its." For it is in effect the structure of the space in which such "its" can appear. And above and beyond such logical considerations, it is also, as noted, directly displayed as independent of all "its" in the "empty" meditation-related experience discussed earlier.

[36] The object-oriented space of the "I-it" relation can thus be described as "open" at the "here" of one's phenomenological space (much as a geometrical sphere containing all the points within it except its center is said to be "open" at its center). This appears to hold for all the experiences ordinarily studied by modern phenomenology. Other topologies, to be sure, may appear more appropriate for highly unusual altered states of consciousness, including some of those reported by many meditation traditions. Our concern here however is only with the normal waking state of consciousness in which mathematical work is typically done.

locus. Thus the model (as contrasted with notions of the "I") remains unaffected by whether the "I" itself turns out to be a "logical fiction" as Russell argued. The model also remains unaffected by the putative inability to identify the "I" by means of "its" (or sets of "its"). For as we have seen the "I-it" structure with its "I-pole" is (i) empirically discoverable *and* (ii) independent of any and all phenomenological "its."[37] Indeed, in the meditation experience discussed earlier, the structure with its "I-pole" is empirically discoverable *as* independent of all such "its." This independence is in fact one of its most important features. In short neither the inability to identify the "I" in terms of "its" nor its putative fictitiousness emphasized by Russell present any problems for our "I-it" model at all.[38]

VII. With the above model of the "I-it" structure in hand, let us now return to the problems raised earlier with the "going outside the system" step of Gödel's proof, namely (i) the observation that Gödel's proof, rather than establishing that this "going outside" step can always be done, actually presupposes it, and (ii) Franzen's critique that since we don't actually know of any general way of viewing a system from outside to prove things about it, this crucial step of Gödel's proof remains unexplained.

The application of our "I-it" model to (i) the observation that Gödel's method presupposes rather than establishes that the step of "going outside" the system can always be performed is quite straightforward. For the model implies that it is simply a mistake to think of the mind as *going* outside the system in the first place. This is because if the model is correct, one's point of view, the "eye" (or "I") of the mind, is *always* outside the field of its objects, and thus doesn't need to *go* anywhere to be outside. It is outside *already*. All that is happening here is a shift of attention from (a) things coherently definable within the system S to (b) things coherently definable only outside of the range of S's axioms (namely, S's Gödel statement G and related semantic reasoning), so that one can then (c) begin to draw conclusions about how S and G relate from this new perspective outside the range of S and its axioms.[39] There is nothing strange about this shift, nothing remarkable to be explained. What is shifting from inside the system to outside is not any mathematical entity, it is only one's *attention*. One simply withdraws one's attention and shifts it to something else. And shifting attention in this way is part of how the mind naturally operates. It is so natural that mathematicians (and others) do it all the time, generally without even taking note of this fact.

---

[37] For an extended discussion of relationships between the experience of "pure consciousness"/"pure emptiness" (where even the "I-it" structure is absent) and concepts of "the self" as analyzed by modern Western philosophers such as Descartes, Hume, Kant and Russell, see Jonathan Shear, *The Inner Dimension: Philosophy and the Experience of Consciousness* (Silver Spring MD: Harmonia Books, 2015; New York: Peter Lang Publishing, 1990).

[38] It is also worth noting (although not needed for the analyses of the following sections) that the (i) empirical discoverability and independence of phenomenological "its" that hold for the "I-pole" and "I-it" structure also (ii) hold for the "I" or self we naturally take to experience from the "I-pole." To see why this is so all you have to do is imagine *yourself* having the experience described in the thought-experiment (or, if you have had the relevant meditation experience, simply remember it). For when the "I" or self experiencing at the "I-pole" is *oneself*, it is self-evidentially obvious that *this* "I" or self, namely *yourself*, is not a mere "fiction." It is in other words (i) readily identifiable at this locus, even though this identification, contra Russell, is (ii) in the absence of, and therefore independent of, all perceived "its." (This in fact was exactly what Ibn Sina intended his original thought-experiment to display.) Thus Russell's empirical conclusion, despite its seeming plausibility, is incorrect, and presents no obstacle for the "I-it" model we have been discussing.

[39] It is thus at (c), when the shift of attention has taken place, that the proof shifts to a semantical analysis and concepts such as "truth" that are outside the domain of syntactical structure and content defined by S's axioms.

What draws attention to this shift in this particular case, and makes it especially noteworthy, is the unique logical conclusions it allows us to draw. But the shift of attention itself is a *phenomenological* shift, not a logical one. And in itself it is not about mathematics *per se*. It is a fact about how the mind operates, shifting attention between different "its" within the "I-it" structure, regardless of the logical relationships of the "its," mathematical or otherwise, under consideration. Gödel's tacit presumption that mathematically trained minds can make the "going outside the system" shift basic to his proof is thus an *empirical* rather than a logical presumption. As such, no amount of purely logical reasoning could ever establish it. Nor by itself could such reasoning even explain its bare possibility. But our analysis of the "I-it" structure indicates why this empirical presumption of the ability to shift *attention* away from an axiomatic system, unprovable on purely formal grounds, should hold for appropriately trained normal human minds operating within the most basic structure of ordinary experience.

The application of our model to (ii) Franzen's critique is even simpler, once we separate the two key components of his expression, namely, (a) "generally applicable way of viewing a system from outside," so as (b) "to be able to prove things about it that are not provable in the system." On our analysis (a) presents no problem at all since, phenomenologically speaking, all systems are always *already* viewed "from outside." The only thing that *goes* outside is the mathematician's *attention*. And once one's attention is outside a system, "viewing" things outside it and proving things about it "from outside" (as in (b) above) are no different in principle from viewing things and proving things about it from within it. Thus the only thing remaining to be explained here is how "viewing" systems and proving things about them can ever take place at all, and how attention moves in the first place.[40] These are all of course extremely difficult questions. But they raise no special difficulty for the question at hand, namely, the step of "going outside the system" crucial to Gödel's method.

VIII. As we have seen, the "going outside" step of Gödel's proof is something the mathematician *does*, a phenomenological move, not a logical one. It is, in other words, not a *concept* derived from other concepts, but *a change of perspective in the mathematician's mind*. It occurs in time, and thus, as empirical, has to be *caused* rather than merely *implied*. Awareness of relationships between the concepts one has been thinking about (as contrasted with the mere existence of these relationships) can, of course, prompt this shift of attention away from an axiom system one had been considering. And it is important to note that the move of "going outside" in itself is explicitly just that: going *outside* the constraints of whatever system one may have been considering. In itself the move implies nothing at all about where one's attention might go *to*. Once one's attention has gone outside the constraints of a given system it can in principle go anywhere, including to things irrelevant to the previous line of reasoning and things incompatible

---

[40] It is important to note here that the expression "going outside the system" as used here refers precisely, and only, to shifting attention from (a) the axiomatically constrained contents and relationships of a system that one has been concentrating on to (b) other things, mathematical or not, outside these particular constraints. This move provides a necessary precondition for doing mathematical work that cannot be done within the system. The "going outside" in itself is just that, a shift of attention. In itself it does no other mathematical work, such as proving things, etc. It says nothing about how creatively such work outside the system might be done, or how it might be related to the system from outside. All it does is free it from being confined entirely within the constraints the axioms of a particular system define. And phenomenologically, moving from one axiom system to another is in itself much the same as deciding to play a game according to a different set of rules, as for example when one switches from two to three dimensional chess, or from US football rules to Canadian.

with it. And this is precisely the point of Gödel's "going outside" move—to be able to consider things incompatible with the logical constraints of a system one had been reasoning about.

We should also note here that this kind of "going outside" the axiomatic constraints of systems one had been thinking about is a basic aspect of modern mathematics. Comparing different, often incompatible axiom systems to see what can and cannot be done within each and how they relate as wholes, which obviously involves the "going outside the system" move as described here, has been a mainstay of higher mathematics since the nineteenth century. This has also been a regular feature of secondary school mathematics since the middle of the twentieth century, where students examine the axiomatic differences between Euclidian and non-Euclidian geometry and investigate what can and cannot be done by and within respective axiomatically defined systems—even if the didactic focus is generally on the axiomatic systems themselves, rather than the act of switching back and forth between them. And it is not at all unusual now for mathematicians to intentionally "go outside" an axiom system they have been working within to see, for example, if other axiom systems might produce better results for particular questions. Ordinarily this shifting-of-attention step of a mathematician's research is not part of a proof it helped him/her devise (even if it might be alluded to in accompanying notes). It did not need to be. All that was needed was the deductive results. Gödel's proof was revolutionary in showing that mathematics cannot be completed deductively from finitistic systems alone. But it was also revolutionary both (a) in *relying on* the "going outside" step and (b) *including explicit reference to it* in the proof itself.[41] Here, in effect, Gödel showed that modern mathematics has to include reference to the attentional orientation of the mathematician, which in principle is always free to shift from one object to another within the "I-it" structure of object-oriented awareness, regardless of the nature of the objects being considered. With this, reference to the mathematician's mind is seen to play a vital role in the objective structure of mathematics itself.[42]

IX. Let us now apply the above phenomenological analysis of the "going outside" step to the question of the algorithmicity of human mathematical thinking. It is important to note that the expression "going outside the system" as used here refers to a withdrawal of attention from a system one has been attending to so that one *can* turn to other things. In the case of Gödel's proof, one's attention is immediately redirected to other mathematical objects outside that system's constraints. But the move of withdrawing attention from a system is in itself distinguishable from where it then might happen to go, for once attention has gone outside a system it can go virtually *anywhere.* It can go to another related system, as in the case of Gödel's proof. But it can also go to very different mathematical topics, and outside of mathematics to other topics, and even outside of conscious verbal thinking entirely, as when one becomes absorbed, for example,

---

[41] Cp., for example, the last paragraph of Section I of Gödel's proof:

> From the remark that **[R(q); q]** [the Gödel statement of PM] says about itself that it is not provable it follows at once that **[R(q); q]** is true, for **[R(q); q]** *is* indeed unprovable (being undecidable). Thus, the proposition that is undecidable *in the system **PM*** [Gödel's own emphasis] still was decided by metamathematical considerations [*outside* the system]. The precise analysis of this curious situation leads to surprising results concerning consistency proofs for formal systems, results that will be discussed in more detail in Section 4 (Theorem XI).

English translation by Jean van Heijenoort (vetted and edited by Gödel), in Jean van Heienoort (editor), *From Frege to Gödel: a Source Book in Mathematical Logic, 1879-1931* (Cambridge MA: Harvard U. Press, 1967) p. 599; originally published in *Monatshefte für Mathematik Physik* 38: 173-98, 1931

[42] This result is of course just the kind of thing we might expect if Gödel's claim about the importance of phenomenology for the foundations of mathematics quoted above is correct.

in the sound of a dog's barking. The shift of attention involved in the "going outside the system" move of Gödel's proof, in other words, is simply an instance of a general, pre-mathematical, non-verbal mental ability to shift attention in all sorts of ways within one's awareness. Furthermore, whatever constraints the algorithmic system in question might put on what the "next step" of mathematical thinking should be, human mathematicians can go outside these constraints, whether to the thought of some mathematical object outside the range of the algorithm (as, for example, G in Gödel's proof), or even outside mathematics entirely. And if human mathematical thought can "go outside" *any* arbitrarily chosen algorithmic system, this means it can go outside *every* such system, and this by definition means that it is not entirely algorithmic. In short, the very ability to "go outside" systems in this way identifies human mathematical thought as not completely algorithmic. [43]

X. This result, of course, is precisely what Gödel hoped to prove on the basis of his theorem (and Turing's theoretical work on computers). But as noted earlier, this result, rather than being provable on the basis of Gödel's theorem, is instead already *presumed* by his method of proving the theorem. The phenomenological sections above argue for the reasonableness of this presumption. And the fact that mathematicians for the past nearly ninety years have easily followed and used this step without trouble is further evidence of how commonsensical it is, even if just *why* it was commonsensical was not well understood. The following proof-sketch should now make it clear that the coherence of Gödel's use of the "going outside" step of his proof, and thus the soundness of the proof itself, *requires* this presumption. And this in itself will, as we will see, provide further support, drawn independently of phenomenological considerations, for the conclusion that human mathematical thought is not entirely algorithmic.

Proof-sketch:

Let GT be Gödel's first Incompleteness Theorem.
Let S be an axiom system GT applies to, and Gs be the Gödel statement of S.
Let P be an axiom system supposed capable of proving GT, and Gp be the Gödel statement of P if P has one.
Let GO be the "going outside the system" step of his proof.

1. The first use of GO in the meta-mathematical phase of Gödel's proof presumes GO can apply to *any* arbitrarily chosen system GT applies to.
2. GT applies to all axiom systems that are (i) algorithmic and (ii) capable of expressing basic arithmetic. [Definition of the domain of GT]

[43] It should be noted that the above conclusion that human mathematical thought is not completely algorithmic is a conclusion about relationships between *thoughts* as they exist *within the phenomenological realm*, not a conclusion about relationships between *their presumptive ontological causes*. As such, it does not by itself imply anything about whether brain states thought to cause mathematical thoughts are algorithmic or not. The structure and contents of the phenomenological domain and that of their presumptive physical causes need not be related so simply. Thus, for example, the fact that appearances in a dream (or a Road Runner cartoon) can relate in ways not properly describable by the laws of gravity, does not, obviously, imply that a brain (or a computer) causing these appearances ever operates outside what these laws properly describe. And our current knowledge of how the phenomenological and objective domains are related is insufficient to presume that either algorithmicity or non-algorithmicity in one of these domains implies the same thing in the other. There is at present nothing approaching consensus about how to explain the bare existence of phenomenal consciousness in the physical world. (Cp., Jonathan Shear (editor), *Explaining Consciousness: the Hard Problem* (Cambridge: MIT Press, 1997).)

3. Suppose P is an axiom system capable of proving GT.
4. If P is algorithmic, it fulfills condition (i).
5. P must be capable of expressing basic arithmetic to even express GT, much less prove it. This means that P has to fulfill condition (ii) as well.
6. Therefore any algorithmic system P supposed capable of proving GT fulfills (i) and (ii), and thus has to be one of the systems GT applies to.
7. Thus if P can prove GT via Gödel's method of proof, it must be able to apply GO to every system GT applies to, and thus to itself.
8. But, by definition, no system P can coherently apply GO to itself, since this would involve P's "going outside" itself and making deductions from (a) statements (such as Gp) and (b) semantic terms (such as "truth") that have no coherent significance (i.e., are contradictory and/or undefinable) in P.
9. So if P is algorithmic it cannot use the critical step of GO in Gödel's proof of GT (or even coherently express what this use would involve).
10. Therefore no algorithmic P can prove GT soundly by Gödel's method (or any other method with a comparable use of GO).

This of course means that whatever can coherently apply the "going outside the system" step in Gödel's (or any other comparable) proof, human or not, cannot be modelled as entirely algorithmic. [44] So if (a) all human mathematical thinking is algorithmic, then (b) neither Gödel nor any other human being could have proven Gödel's theorem soundly by his method. The conclusion (b) is of course contrary to the long-standing consensus of mathematicians about the coherence, expression and soundness of Gödel's proof with its critical "going outside the system" step. Their consensus in other words directly supports the negation of premise (a), namely that *not* all human mathematical thinking is algorithmic. The consensus of mathematicians is, of course, a contingent empirical fact, and cannot logically *prove* that human mathematical thinking has a non-algorithmic component. [45] But it provides strong support, independent of and complimentary to the phenomenological considerations implying the same conclusion in the preceding sections.[46]

XI. Gödel's incompleteness theorem has often been thought to indicate that human mathematical thinking must have a non-algorithmic element or ingredient. Gödel himself hoped that

[44] This conclusion follows from the use of the "going outside" step in Gödel's proof, independently of questions of the soundness of the rest of the proof, and of what the theorem itself might imply.

[45] Thus while this evidence has a major logical component, it is in the final analysis empirical and contingent rather than purely logical, even if the consensus of mathematicians and logicians is taken to be the best measure of logical fact we have here.

[46] It is worth noting here that the contrast between this empirical evidence for the non-algorithmicity of aspects of human mathematical thinking and that for the opposing hypothesis that human mathematical thinking is entirely algorithmic is quite striking. For we have at present no real empirical evidence at all to conclude that human mathematical thinking is a product even of a single system that is *logically integrated*, much less one that is *axiomatic*, and even less one that is *entirely algorithmic*. Human mathematical thinking might well, for example, turn out to be a product of coordinated but nevertheless significantly autonomous components that are not entirely integrated logically, axiomatically, or algorithmically. (Compare, by analogy, the relationships between the relatively independent brain regions implicated in vision and language that presumably influence the visual and linguistic aspects of mathematical thinking.) We simply at present do not know enough about the nature of either the mind or the brain to assert the exclusive "all" algorithmic hypothesis with any reasonable empirical confidence.

this conclusion could be established by combining his theoretical work with findings from empirically-oriented disciplines such as phenomenology. The model of the "I-it" structure basic to phenomenology developed in preceding sections allows us to recognize the "going outside the system" step of his proof as a normal shift of attention within this basic structure. And independently of its role in Gödel's proof, phenomenological analysis makes it clear that such shifting of attention is intrinsically non-algorithmic, since (as is obvious) attention can, and often does, move away from any particular thought in this way regardless of the content of the thought, algorithmic or not. Furthermore, as we have just seen, logical analysis of the specific role of the "going outside" step in Gödel's proof directly, and independently of phenomenological considerations, implies that human thought has to be non-algorithmic if anyone has ever applied this step coherently and proven Gödel's theorem by his method. These phenomenological and logical analyses thus together provide strong evidence that human mathematical thinking is not entirely algorithmic, and identify the "I-it" structure, the "I-pole" and movement within it, and the natural ability to shift attention within this structure as the relevant pre-mathematical, non-algorithmic aspects of human thinking implicated in Gödel's work.

Gödel's incompleteness proof thus appears to imply two very different, complimentary kinds of incompleteness with respect to algorithmic systems. On the one hand, moving "outward" from the proof to the theorem and its consequences, it shows that no algorithmic system can generate all the truths of arithmetic, much less mathematics in general. And moving "inward" from the proof to its source, it implies that no such system can model the range of mathematical thought necessary to fully apply (much less create and understand) the proof's crucial "going outside the system" step. In other words, no such system can properly model the "I-it" structure and the full range of its potential content. Gödel, we think, would have been pleased with this result.

Bibliography

Franzen, Torkel, *Gödel's Theorem: An Incomplete Guide to its Use and Abuse* (Wellesley, MA: A.K. Peters, 2005)

Ibn Sina (Avicenna), "Floating man. (2017, December 15). In *Wikipedia, The Free Encyclopedia*. Retrieved 04:25, February 13, 2018, from https://en.wikipedia.org/w/index.php?title=Floating_man&oldid=815584929

Leighton, Taigen Dan (editor and translator), *Cultivating the Empty Field: The Silent Illumination of Zen Master Hongzhi*, (Rutland, VT: Tuttle Publishing, 2000) pp. 19-20.

Nagel, Ernest and Newman, James R, *Gödel's Proof* (New York: New York University Press: 2001)

Penrose, Roger, *Interview with Jane Clark*. Journal of Consciousness Studies, 1:17-24: 1994

-------- "Beyond the Doubting of a Shadow", *Psyche* 2 (23) 1996

Russell, Bertrand, “Knowledge by Acquaintance and Knowledge by Description” (1910), re-printed in *Mysticism and Logic and Other Essays* (New York: Norton and Co., 1924) pp. 211-2.

-------- *Our Knowledge of the External World* (London: George Allen & Unwin, 1922)
"On Propositions: what they are and how they mean," in *Logic and Knowledge, Essays 1901-1950,*

Shear, Jonathan, *The Inner Dimension: Philosophy and the Experience of Consciousness* (New York: Peter Lang Publishing, 1990) and (Silver Spring MD: Harmonia Books, 2015)

--------- “Experiential Clarification of the Problem of Self," *J. of Consciousness Studies*, Vol. 4. No. 5/6, 1998

--------- “Converging on the Self: Western Philosophy, Eastern Meditation, Scientific Research,” in *Interdisciplinary Perspectives on Consciousness and the Self*, edited by S. Menon, A. Sinha and B.V. Sreekantan (Springer, 2014).

Shear, Jonathan (editor) *The Experience of Meditation: Experts Introduce the Major Traditions*, edited by Jonathan Shear (USA: Paragon House, 2006)

Smith, David Woodruff, "Phenomenology", *The Stanford Encyclopedia of Philosophy (Fall 2011 Edition)*, Edward N. Zalta (ed.), http://plato.stanford.edu/archives/fall2011/entries/phenomenology/

Tieszen, Richard, *After Gödel: Platonism and Rationalism in Mathematics and Logic* (Oxford University Press, 2011)

van Heijenoort, Jean, (editor), *From Frege to Gödel: a Source Book in Mathematical Logic, 1879-1931* (Cambridge MA: Harvard U. Press, 1967) p. 599; originally published in *Monatshefte für Mathematik Physik* 38: 173-98, 1931

Wang, Hao, *A Logical Journey: From Gödel to Philosophy* (USA: Bradford Publishing, MIT Press, 1996)

Whitehead, A.N. and Russell, B, *Principia Mathematica* (Cambridge: Cambridge University Press, 1925)

Yogi, Maharishi Mahesh, *Maharishi Mahesh Yogi on the Bhagavad-Gita: A New Translation and Commentary, Chapters 1-6* (New York: Penguin Books, 1974) p. 422.