The Exploratory Role of Idealizations and Limiting Cases in Models

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ABSTRACT

In this article we argue that idealizations and limiting cases in models play an exploratory role in science. Four senses of exploration are presented: exploration of the structure and representational capacities of theory; proof-of-principle demonstrations; potential explanations; and exploring the suitability of target systems. We illustrate our claims through three case studies, including the Aharonov-Bohm effect, the emergence of anyons and fractional quantum statistics, and the Hubbard model of the Mott phase transitions. We end by reflecting on how our case studies and claims compare to accounts of idealization in the philosophy of science literature such as Michael Weisberg's three-fold taxonomy.

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1 Introduction

Idealizations and the use of models, which are by their very nature imperfect or highly fictitious representations of reality, are ubiquitous in science.[1] How is one to make sense of the fact that, in attaining empirical adequacy and giving us knowledge about the world, our best scientific theories invoke falsehoods and distortions of reality? A standard, albeit naïve, response to such a worry has been not to allocate any *substantive* role to idealizations and models. Bluntly put, idealizations and models are used to simplify and abstract away irrelevant details, render computationally tractable various systems of study, or else are taken as auxiliary tools for application of theory. In principle, idealizations and models can be dispensed with.[2]

In contrast, many philosophers of science have attempted to articulate substantive roles for idealizations and models to play in science. The emphasis has largely been on *explanation*.[3] Our goal in this paper is to build upon our own earlier work[4] and join this latter camp. However, whereas previous authors have concentrated on the explanatory role, we wish to fill what we take to be a missing gap in the literature and stress the *exploratory* roles of idealizations and models.[5]

In particular, we will present three case studies that illustrate our claims including the Aharonov-Bohm effect, the emergence of anyons and fractional quantum statistics, and the Hubbard model of the Mott phase transitions (Sections 2-4). Although we do not intend for our list to be exhaustive, we submit that idealizations and models can be exploratory in at least four substantive manners: they may allow for the *exploration of the structure* and representational capacities of theory; feature in *proof-of-principle* demonstrations; generate *potential explanations* of observed (types of) phenomena; and may lead us to assessments of the *suitability of target systems*.[6] Last, we conclude the paper by comparing our case studies with Michael Weisberg's (2007, 2013) recent taxonomy of idealizations and models

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[1] Some examples of idealizations include nonviscous fluid flow, a perfect vacuum, perfectly rational agents, and isolated populations, while examples of (idealized) models include the Ising model, the Hardy-Weinberg equilibrium model, and Schelling's segregation model.

[2] See Norton ([2012]) for a recent defense of the claim that idealizations ought to be dispensed with.

[3] For instance, see Batterman ([2002]) for more on explanatory idealizations, and see Batterman and Rice ([2014]) and Bokulich ([2008]) for more on the explanatory role of idealized and/or imperfect models.

[4] Gelfert (2009), Gelfert (2016) and Shech (2015, 2016, manuscript).

[5] Similar themes have been explored by, among others, Redhead ([1980]), Bailer-Jones ([2002]), Yi ([2002]), Wimsatt ([2007], Ruetsche (2011, p. 337), and Earman ([manuscript]).

[6] This list follows Gelfert's ([2016], pp. 83-94) fourfold distinction of exploratory functions of models. Earman ([manuscript]), Shech ([2015], Section 7]; 2016, Section 5; Manuscript]) discuss exploration of theory structure via idealization.

(Section 5). We argue that his three-fold classificatory scheme is lacking in that it does not make room for the exploratory role of idealizations and models, thereby offering a distorted view of the case studies that we present.[7]

A caveat is in order before beginning. We do not endeavor to define what idealizations and models are. The literature on these questions is vast, and ultimately not much will be at stake for our purposes.[8] Instead, we will appeal to a generic understanding of these notions, on the assumption that, whatever one's preferred account of idealizations and models, the proposal developed in the present paper can be adapted accordingly. This means that at times we will allow ourselves to talk about an 'idealization,' or an idealized system or object, and a 'model' interchangeably since both idealizations and models, insofar as they are used to represent physical phenomena, are *misrepresentations* of sorts.

2 The Aharonov-Bohm Effect

2.1 Case Study: AB Effect

Consider a standard double-slit experiment undertaken with a beam of electrons. Experiments have shown that electrons manifest a behavior consistent with wave interference patterns (see Figure 1). Now add to this configuration an infinitely long and absolutely impenetrable solenoid (in between the double-slit screen and the detector screen) (see Figure 2). If we turn on the solenoid, what type of behavior should we expect to witness? Intuitions may vary on this point, but there is a straightforward sense in which no answer can be given: we cannot ever build an apparatus with an infinitely long and absolutely impenetrable solenoid, so we cannot know what would happen in such a scenario. However, the question can be answered within the context of a theory. For instance, if we take our thought experiment to manifest in a world governed by classical physics, there no reason to think that anything will happen. According to the setup the solenoid is infinitely long so that the magnetic field $\boldsymbol{B}$ produced is wholly confined to a region S_{in} inside the solenoid. The solenoid is also absolutely impenetrable, so that the beam of electrons is completely confined to a region S_{out} outside the solenoid. Since there is no local (physical or causal) interaction between the electrons and the magnetic field, classical physics makes no novel prediction about this particular idealized system.

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[7] We are in debt to John Earman for first bringing this issue to our attention.

[8] For more on idealizations see Weisberg ([2007], [2013]), Ladyman ([2008]), Elliott-Graves and Weisberg ([2014]) and for more models see Morgan and Morrison ([1999]), Frigg and Hartmann ([2012]) and Gelfert ([2016]). See Norton ([2012]) for more on the distinction between idealization and approximation, and see Jones ([2005]) for more on the distinction between idealization and abstraction. Psillos ([2011]) differentiates between the process of idealization/abstraction and the idealized/abstracted system/model that is the product of such a process. Also see Shech ([2014]) for more on misrepresentation.

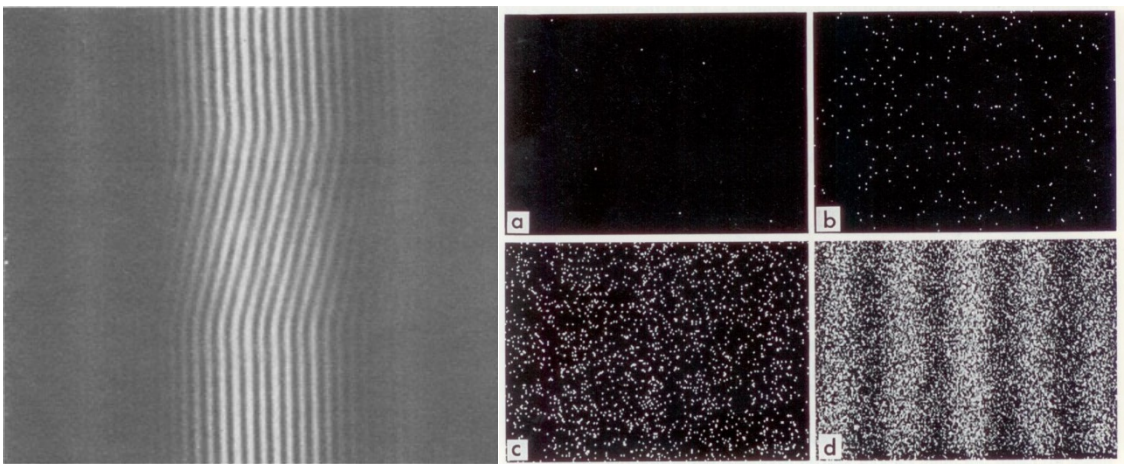

Figure 1. (Left) An example for an interference pattern from a double-slit experiment (from Möllenstedt and Bayh 1962, 304). (Right) Single-electron build-up of (biprism) interference pattern (from Tonomura [1999], p. 15). (a) 8 electrons, (b) 270 electrons, (c) 2000 electrons, and (d) 60,000 electrons.

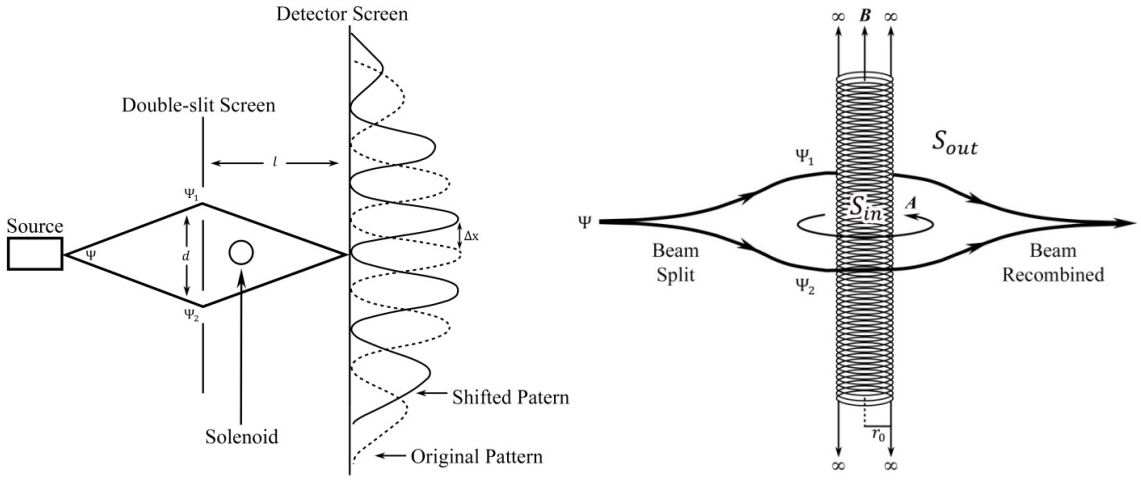

Figure 2. The AB effect. A beam of electrons Ψ is split in a region S_{out}, made to encircle a solenoid (that generates a magnetic field inside the region S_{in}), and then to recombine on a detector screen. The original interference pattern is shifted by an amount Δx.

In start contrast to these classical intuitions, Yakir Aharonov and David J. Bohm ([1959]) showed that quantum mechanics predicts a *shift* in interference pattern, which has become known as the (magnetic) Aharonov-Bohm (AB) effect.[9] In modeling the idealized scenario, they began with the standard Hamiltonian used for a charged particle in electromagnetic fields: $H^I_{AB} = (\boldsymbol{P} - q\boldsymbol{A}/c)^2/2m$, where m and q are the electron mass and charge, respectively, $\boldsymbol{P} = -i\hbar\nabla$ the momentum operator, $\boldsymbol{A}$ the electromagnetic vector potential operator generating the magnetic field such

[9] See Peshking and Tonomura ([1989]) for more on the theory and experimental confirmation of the AB effect.

that $\boldsymbol{B} = \nabla \times \mathbf{A}$, and the electromagnetic scalar potential has been set to zero. Since S_{in} is a region inaccessible to the beam of electrons represented by the quantum state Ψ, H_{AB}^{I} acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3 \setminus S_{in})$ of square-integrable functions defined on a non-simply connected configuration space $\mathbb{R}^3 \setminus S_{in}$, that is, on three dimensional Euclidean space from which the interior of the solenoid has been excised. This means that H_{AB}^{I} is not a self-adjoint operator, and so it does not generate the dynamics of the system. In order to remedy the situation, Aharonov and Bohm ([1959]) chose a unique self-adjoint extension of H_{AB}^{I}, symbolized by $\overline{H}_{AB}^{I} = (\boldsymbol{P} - q\boldsymbol{A}/c)^2/2m$, which is picked out by Dirichlet boundary conditions in which the wavefunction vanishes at the solenoid boundary (i.e., $\Psi = 0$ at the boundary). One can then derive the shift in interference pattern by calculating the relative phase factor $e^{i\theta}$ between the two components of the wave function, Ψ_1 and Ψ_2, as is done in standard textbooks, e.g., Ballentine ([1998], p. 321-325).[10]

2.2 Exploration in the AB Effect

The first sense of exploration that we wish to consider is exploration of the structure of a scientific theory. It is by making use of an idealization, viz., an infinitely long and absolutely impenetrable solenoid, and appealing to the corresponding idealized model $\overline{H}_{AB}^{I}$, that Aharonov and Bohm ([1959]) were able to highlight one of the blatant contrasts between classical and quantum physics: the two theories have a very different structure in that they make vastly different predictions about an idealized model – the model representing the behavior of electrons in the vicinity of a shielded magnetic field. Moreover, exploring the quantum physics of infinite and absolutely impenetrable solenoids is what allowed Aharonov and Bohm ([1959]) to discover a possibly additional manifestation of non-locality in quantum mechanics,[11] since the electrons exhibit a dependency on the magnetic flux while remaining in a region devoid of any such flux.

One may object that it is misleading to talk about the AB effect as an exercise in theoretical exploration via idealization since, in fact, experiments have shown that the AB is a real, physical effect (Tonomura *et al*. [1986]). In reply, we draw an analogy with Shech's ([2013], pp. 1172-1173) distinction between concrete and abstract phase transitions: *concrete phase transitions* are the sharp but continuous changes that arise in various thermodynamic potentials and may be observed in the laboratory. *Abstract phase transitions*, i.e., phase transitions as they are conventionally and theoretically defined, are discontinuous changes governed by a non-analytic partition function that are used to mathematically represent concrete phase transitions (See Figure 3). Abstract phase transitions are defined in idealized

[10] In fact, the calculation is more subtle and complicated than standard textbooks lead one to believe. See Shech ([manuscript]) and Earman ([manuscript], Section 5) for details.

[11] See Healy ([1997], [1999]) and Maudlin ([1998]) for a debate about whether or not the AB effect portrays a type of quantum non-locality comparable with Bell inequalities. In this paper, we shall refrain from making any comment on this issue.

infinite systems through the thermodynamic limit, in which a system's volume and particle number diverge.[12] Similarly, we must make a distinction between two kinds of (magnetic) AB effects. On the one side, the *abstract AB effect* as it is conventionally defined applies only to idealized systems where there is a strictly null intersection between the regions occupied by the electron wavefunction and the magnetic flux. It cannot, in principle, ever manifest in the laboratory, and yet it plays an exploratory role in the senses discussed in this section. On the other side, there is the *concrete AB effect* that has been empirically confirmed and shows that a beam of electrons exhibits a type of quantum dependency on magnetic flux that is unaccounted for by classical physics. Only recent rigorous results in mathematical physics have shown that the abstract AB effect is a good approximation of the concrete one.[13]

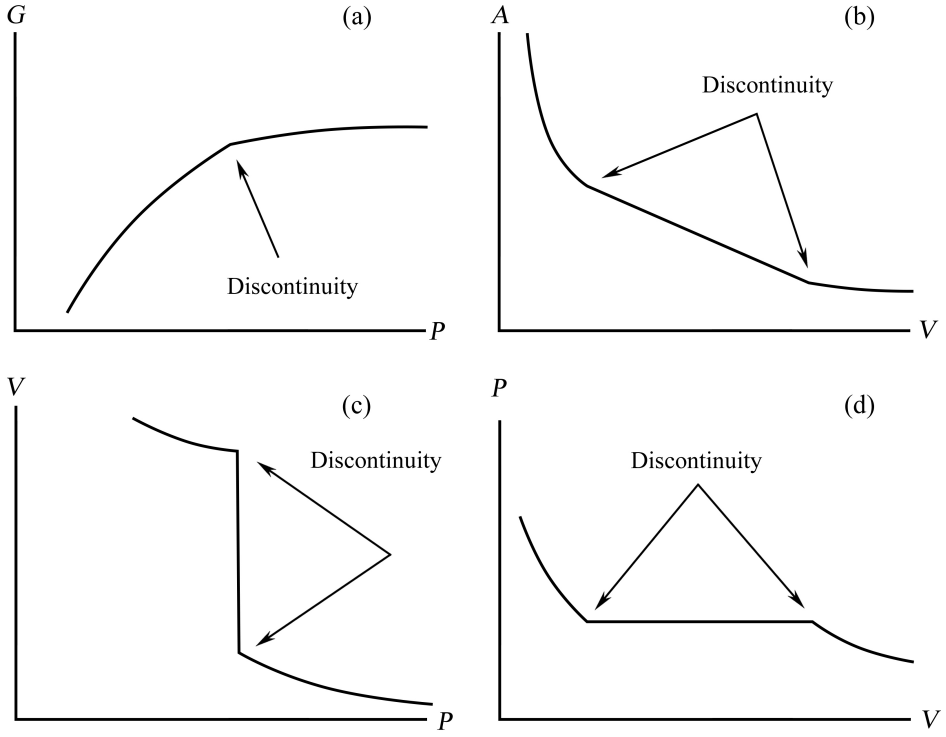

Figure 3. Graphs displaying a first-order phase transition. Graph (a) displays the Gibbs free energy (or Gibbs thermodynamic potential) *G* as a function of the pressure *P*, graph (b) displays the Helmholtz free energy (or Helmholtz thermodynamic potential) *A* as a function of the volume *V*. Graphs (c) and (d) display functional relations between *P* and *V*. Based on Stanley ([1971], p. 31).

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[12] Compare with Kadanoff ([2000], p. 238): 'The existence of a phase transition requires an infinite system. No phase transitions occur in systems with a finite number of degrees of freedom.' See Stanley ([1971]) and Kadanoff ([2000]) for the theoretical treatment of phase transitions.

[13] See Ballesteros and Weder ([2009], [2011]) and de Oliveira and Pereira ([2008], [2010], [2011]) for such results, and see Earman (manuscript) and Shech (2015; manuscript) for a discussion.

A second sense of exploration that may be brought about through the consideration of highly idealized models concerns generating potential explanations, for instance, by envisaging scenarios that, if true, would give rise to the kinds of phenomena that constitute the explanandum.[14] Given the odd nature of the AB effect as a possibly non-local effect, and the fact that the idealization of an infinitely long and absolutely impenetrable solenoid cannot be instantiated in reality, it is not surprising that early claims of experimental verification (e.g., Chambers [1960], Tonomura et al. [1982]) were met with skepticism (e.g., Bocchieri and Loinger [1978]). Attempts to understand and explain the effect and its experimental manifestation took various forms, including potential explanations given within the fiber bundle formalism of electromagnetism.[15] In this context, the electromagnetic fields are represented by the *curvature* of, and the electromagnetic vector potential is represented by a *connection* on, the *principal fiber bundle* appropriate for the formulation of classical electromagnetism (see Table 1). That is to say, a principal bundle where the base space is the spacetime manifold and where the structure group is the group of rotations in the complex plane $U(1)$. The relative phase factor $e^{i\theta}$ which, according to the theory, gives rise to shifted interference pattern that is the AB effect arises as the non-trivial *holonomy* of a closed curve encircling the solenoid.

	Electromagnetic Vector Potential	Magnetic Field Produced by Solenoid	Shift In Interference Pattern (due to a Relative Phase Factor)	Space or Spacetime
Non-Relativistic Quantum Mechanic Formulation	$\boldsymbol{A}$	$\boldsymbol{B}$	$e^{i\theta}$ $= \exp\left(\frac{iq}{\hbar}\oint_C \boldsymbol{A}\cdot d\boldsymbol{r}\right)$	$\mathbb{R}^3$ Or $\mathbb{R}^4$
Fiber Bundle Formulation	Connection	Curvature	Non-trivial Holonomy	Base Space

Table 1. Comparison of terminology between non-relativistic quantum mechanics and the fiber bundle formulation of the AB effect.

[14] See Gelfert ([2016], pp. 87-93).
[15] See Healey ([2007], Ch. 1-2) for an introduction.

It is now possible to generate a potential (although non-actual) explanation of the AB effect. In particular, one may arrive at a non-trivial holonomy by considering a fiber bundle base space that is non-simply connected. The rationale for this explanation is that vanishing electromagnetic fields around the solenoid correspond to a curvature that is zero. Zero curvature means that 'the connection on this bundle is flat everywhere in this region' (Healey [2007], p. 42). Moreover, if 'there is a nontrivial holonomy . . . and if the connection is flat, *the base space* [representing physical space] *must be nonsimply connected*' (Batterman [2003], p. 542; original emphasis). In other words, a non-simply connected base space, which represents physical space (as opposed to the electron configuration space), also allows one to derive a non-trivial holonomy. However, while a derivation based solely on such topological considerations may be considered a potential explanation of the non-trivial holonomy, it is not the actual explanation of the non-trivial holonomy that represents the AB effect. After all, the AB effect is a dynamical effect that depends *on the interaction between* the electron beam and the solenoid (not on holes in physical space or on a particular mathematical formalism). The upshot is that considerations of the highly idealized (abstract AB effect) model, within the fiber bundle formalism, have allowed us to discover a potential explanation of the non-trivial holonomy in terms of a non-simply connected base space.

This concludes our discussion of the AB effect in which we emphasized two senses of exploration: exploring the structure of a theory and generating potential explanations.

3 Anyons and Fractional Quantum Statistics

3.1 Case Study: Anyons

Consider a collection of non-interacting, identical particles in thermal equilibrium. What are the possible ways that such a collection may occupy a set of available discrete energy states? Roughly, quantum and statistical mechanics tell us that there are two such ways, and that the expected number of particles in some specific energy state will depend of the type of particle at hand. Bosons manifest a behavior consistent with Bose-Einstein statistics, while fermions distribute themselves according to Fermi-Dirac statistics. This division into particle types, along with corresponding statistics may be captured by what has become known as the symmetrization/anti-symmetrization postulate: 'The states of a system containing N identical particles are necessarily either all symmetrical or all antisymmetrical with respect to permutation of N particles' (Messiah [1962], p. 595).[16] That is to say, if a collection of N identical particles is represented by the quantum state $\Psi_{(1,2,\ldots,N)}$ and the same collection with, say, particles 1 and 2 permuted is represented by $\Psi_{(2,1,\ldots,N)}$, then the symmetrization/anti-symmetrization postulate tells us that state must be related in the following manner:

[16] See Earman ([2010]) for a discussion.

$$\Psi_{(2,1,\ldots,N)} = e^{i\theta}\Psi_{(1,2,\ldots,N)},$$

where the exchange phase θ can take on a value of $\theta = 0$ for a system of bosons with a corresponding phase factor $e^{i\theta} = +1$ and a symmetric quantum state, or it can take a value $\theta = \pi$ for a system of fermions with a corresponding phase factor of $e^{i\theta} = -1$ and an antisymmetric quantum state.

There are two fundamental frameworks for understanding permutation invariance in quantum mechanics, which ground the symmetrization/anti-symmetrization postulate and its consequences, viz., that there are two basic types of particles and quantum statistics. Following Landsman ([2013]), we will call the first, due to Messiah and Greenberg ([1964]), the *operator* framework, and the second, due to, among others, Laidlaw and DeWitt ([1971]), Leinaas and Myrheim ([1977]), the *configuration space* framework. Landsman ([2013]) has argued that, in dimensions greater than two, both frameworks are equivalent and give equivalent verdicts regarding possible particle types and statistics. However, it turns out that, in two dimensions, according to the configuration space framework the exchange phase can take on *any* value. This allows the framework to represent bosons and fermions, *as well as* other particles known as 'anyons,' which are said to exhibit 'fractional quantum statistics.'[17]

Recall, the manner by which a collection of identical particles occupies energy states will depend on the kind of quantum statistics that such a collection manifests, which in turn depends on the type of particle considered. Particle type is decided by how such a collection behaves under permutation, and such behavior is captured by the value of the exchange phase θ and the corresponding phase factor $e^{i\theta}$. In short, on the configuration space framework, two central theorems (which may be found in Morandi [1992], pp. 119-120) dictate that the phase factor $e^{i\theta}$ is equivalent to the one-dimensional unitary representation γ of the fundamental group π_1 of the configuration space Q of the collection of identical particles, symbolized by $\gamma = e^{i\theta}$.[18] It has been shown by Artin ([1947]), Fadell and Neuwirth

[17] The name is due to Noble laureate Frank Wilczek ([1982]). Note that anyons and fractional statistics have nothing to do with so-called paraparticles and parastatistics (which arise from higher dimensional representations of the permutation group). For more on anyons see Wilczek ([1990]), Khare ([2005]), and references therein.

[18] See Hatcher (2002) for relevant background in algebraic topology. Roughly, the 'one-dimensional unitary representation' will allow us to represent groups with numbers. The 'fundamental group,' also known as the first homotopy group, is a topological invariant that allows one to classify topological spaces according to whether paths or loops in the space can be continuously deformed into each other. For instance, all paths can be continuously deformed into each other, and all loops can be shrunk to a point, in three-dimensional Euclidean space $\mathbb{R}^3$. Such a space is said to be 'simply connected.' The electron configuration space in the context of the

([1962]), and Fox and Neuwirth ([1962]) that the fundamental group for the two-dimensional ($d = 2$) and three-dimensional ($d = 3$) cases are given by:

$$\pi_1 (Q) = B_N \text{ for } d = 2$$
$$\pi_1 (Q) = S_N \text{ for } d = 3$$

where S_N is the permutation group and B_N is the Braid group. In other words, in three dimensions the fundamental group of the configuration space is the (finite and discrete) permutation group S_N which admits of the known one-dimensional unitary representation: $\gamma = \pm 1$ ($+1$ for bosons and -1 for fermions). In two-dimensions, on the other hand, the fundamental group is the (infinite and discrete) braid group B_N with one-dimensional unitary representations giving rise to phase factors of the form: $\gamma_{(\theta)} = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$ so that the exchange phase can take on a continuous range of factors allowing for bosons, fermions, and anyons.

In the remaining part of this subsection, we will be working through the configuration space framework's treatment of the simple two-particle scenario in order to better convey how the appeal to the $d = 2$ idealization brings about the novel mathematical structure needed to represent anyons. Readers uninterested in such technicalities may skip to the next subsection without loss of philosophical continuity.

The fundamental group of the configuration space of the simplest scenario of two particles $N = 2$ in the $d = 2$ and $d = 3$ cases is as follows:

$$\pi_1 \left(\frac{\mathbb{R}^2 \setminus \Delta}{S_2} \right) = \pi_1 (RP_1) = Z \text{ for } d = 2$$
$$\pi_1 \left(\frac{\mathbb{R}^3 \setminus \Delta}{S_2} \right) = \pi_1 (RP_2) = Z_2 \text{ for } d = 3,$$

where S_2 is the permutation group for two particles, Δ an excised set of diagonal points that represent points where particles coincide, Z is the cyclic group of order one, i.e., the infinite group of integers under addition. Z_2 is the cyclic group of order two, i.e., it is the multiplicative group of, say, 1 and -1. RP_1 and RP_2 are the real projective one- and two-dimensional spaces, respectively.

Pictorially, for the $d = 3$ case the configuration space reduces to the real projective space in two dimensions RP_2. This can be visualized as the surface of a three-dimensional sphere with diametrically opposite points identified (see Figure 4) or a hemisphere with opposite points on the equator identified (see Figure 5).

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abstract AB effect, $\mathbb{R}^3 \setminus S_{in}$, is not simply connected because loops encircling the region S_{in} cannot be shrunk to a point.

Consider three scenarios, corresponding to three paths A, B, and C in configuration space including no exchange (Figure 4a), exchange (Figure 4b), and a double exchange (Figure 4c), respectively.

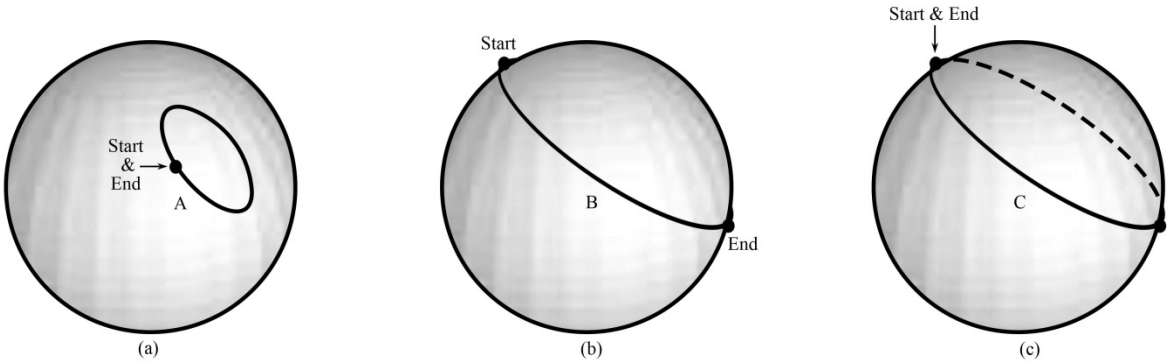

Figure 4. The real projective space in two dimensions RP_2, represented by a sphere with diametrically opposite points identified. Cases (a), (b), and (c), correspond to no exchange, exchange, and double exchange, respectively.

Concentrating on the no exchange case (Figure 4a). We trace a path A in configuration space in which the two particles move and return to their original positions. Path A is a loop in configuration space, with the same fixed start and end points, which can be shrunk to a point. This corresponds to a trivial homotopy class in which the phase factor is trivial.

Moving onto the exchange case (Figure 4b), we start at one end of the configuration space and trace a path B to its diametrically opposite point. This represents an exchange or permutation between the two particles. Notice that since diametrically opposite points are identified (because the particles are identical), this path is actually a closed loop in configuration space. However, since the start and end points of Figure 4b are fixed, the loop cannot be shrunk to point. This corresponds to a non-trivial homotopy class with a non-trivial phase factor.

The double exchange (Figure 4c) case includes tracing a path C in configuration space similar to that of B, but then tracing around the sphere back to the original starting point. Path C is a closed loop in configuration space that can be shrunk to a point, and so it is in the same homotopy class of path A with a corresponding trivial phase factor. Equivalently, we may visualize the paths A, B, C on a hemisphere with opposite points on the equator identified as in Figure 5, where paths A and C can be continuously deformed to a point but path B cannot because of the diametrically opposed fixed start and end point on the equator.

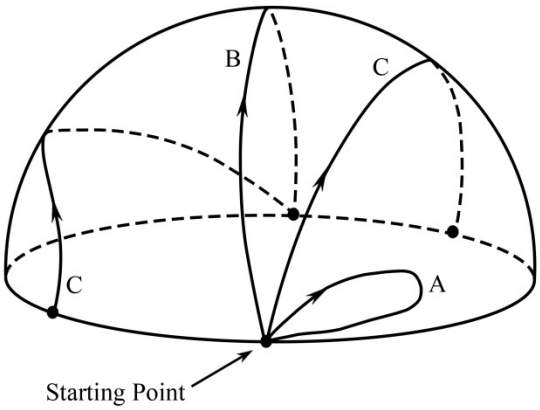

Figure 5. The real projective space in two dimensions RP_2, represented by the northern hemisphere with opposite points on the equator identified.

On the other hand, in the context of the $d = 2$ case, we are dealing with the real projective space in one dimension RP_1. We can visualize this configuration space as a circle with diametrically opposite points identified (see Figure 6). Again, consider three paths A, B, and C in configuration space that correspond to no exchange (Figure 6a), exchange (Figure 6b), and a double exchange (Figure 6c), respectively. Path A traces a closed loop in configuration space (where the particles move but then return to their original positions with no exchange) which can be continuously shrunk to a point and has a corresponding trivial phase factor (as in the $d = 3$ case of figures 4a and 5a). Next, we trace a path B across half the circumference of the circle. Since diametrically opposed points are identified, this represents a particle exchange (Figure 6b). Path B traces a closed loop in configuration space that cannot be continuously shrunk to a point and has a corresponding non-trivial phase factor (as in the $d = 3$ case of figures 4b and 5b).

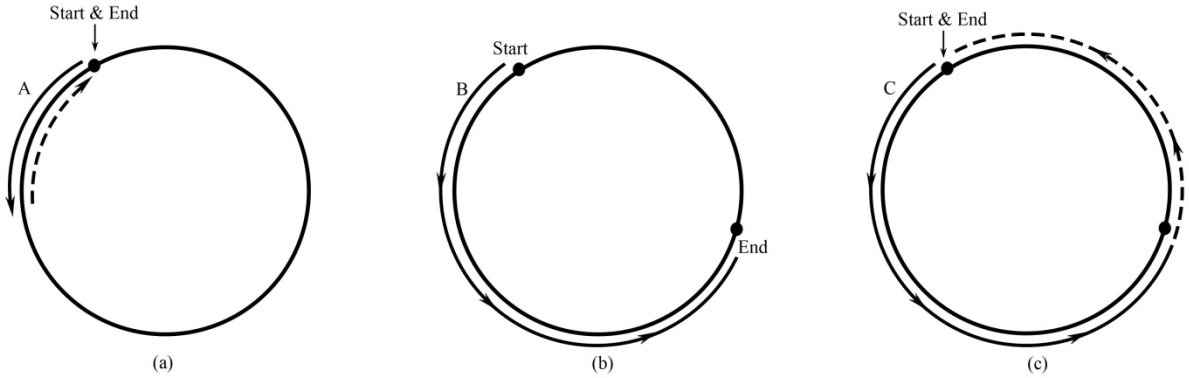

Figure 6. The real projective space in one dimension RP_1, represented by a circle with diametrically opposite points identified. Cases (a), (b), and (c), correspond to no exchange, exchange, and double exchange, respectively.

The main difference between the $d = 3$ and $d = 2$ cases arises when we consider path C (Figure 6c), in which the particles are permuted twice, represented by traversing the entire circular configuration space. Path C is a closed loop in configuration space but, unlike the $d = 3$ case, it cannot be shrunk to a point because the circle itself (so to speak) acts as an obstructive barrier. Moreover, path C cannot even be continuously deformed to overlap with path B. This means that, not only is the phase factor corresponding to the two paths non-trivial, but each path has a different phase factor for each path belongs to a different homotopy class. In fact, for every traversal (in configuration space) of half a circle, we get a closed loop that is in its own homotopy class.[19] In other words, by transitioning from three dimensions to two dimensions, we have transitioned from a doubly connected space to an infinitely connected space, and it is this change in topology that allows for the fractional statistics and the emergence of anyons.

3.2 Exploring Fractional Statistics

The configuration space framework for permutation invariance in quantum mechanics exemplifies, in a very direct way, the manner by which an idealization allows one to explore the structure of a theory and its representational capacities. The transition from $d = 3$ to $d = 2$ is an example of an idealization since two-dimensional systems, strictly speaking, do not exist. However, it is exactly by exploring two-dimensional systems that we discover the full spectrum of the exchange phase, as well as the fact that quantum mechanics has the representational capacities to represent more than just bosons and fermions. The novel mathematical structure that emerges in two-dimensions is the braid group B_N, with its corresponding one-dimensional unitary representation $\gamma_{(\theta)} = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$. In three-dimensions, by contrast, the structure of the permutation group S_N and its one-dimensional unitary representation $\gamma = \pm 1$ is too sparse to represent anyons. Moreover, it is by appealing to the $d = 2$ idealization that we can clearly differentiate between the operator and configuration space frameworks, since it is only in $d = 2$ that the latter differs from the former in its verdict regarding possible particle type with corresponding statistics.

An additional sense of exploration, generating potential explanations, also arises in this context. Currently, the empirical evidence confirming the existence of anyons is inconclusive,[20] but physicists believe that anyons or approximate anyons are likely to manifest in what is known as fractional quantum Hall effect (FQHE)

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[19] If we symbolize this by $\pi_1\,(\,Path)$ we get that $\pi_1\,(Path\;\;A) = 0$ for the trivial homotopy class, but the rest of the paths will be elements of non-trivial homotopy classes: $\pi_1\,(Path\;B) = 1, \pi_1\,(\,Path\;C) = 2$, … and so on, so that we generate all of the integers Z. Negative integers corresponding to traversal of the circular configuration space in the opposite direction.

[20] Recent supposed confirmations include Camino et al. ([2005]), but there is no consensus in the physics community regarding the reality of anyons and fractional statistics.

systems.[21] Insofar as anyons will be found in FQHE systems, which are built to constrain the dynamics of the system to approximately two-dimensions, the configuration space framework allows for one potential explanation of such particles. That is to say, if the idealized two-dimensional systems that allow for anyons and fractional statistics (via the configuration space framework) are good approximations of real FQHE systems, then it is because of the approximate two-dimensional nature of FQHE systems that anyons may emerge in the first place. Ultimately though, since the results derived from the configuration space framework hold for strictly two-dimensional systems and not approximate ones (such as thin layers embedded in three-dimensional space), such an explanation remains only a potential one at this time.

Another example of generating potential explanations can be found in the so-called flux tube model of the anyon introduced in Frank Wilczek's ([1982]) original paper on the subject. The goal is to explain how a composite particle that is neither a boson nor a fermion could come about. Here an anyon is described by spinless particle of charge e in the xy-plane orbiting around a very thin and long solenoid with magnetic flux Φ, set perpendicular to the plane, in the direction of the z-axis (see Figure 7). We are then asked to appeal to further idealizations:

> In the limit where the solenoid becomes extremely narrow and the distance between the solenoid and the charged particle is shrunk to zero, the system may be considered as a single composite object — a charged particle-flux tube composite. Furthermore, for a planar system, there can be no extension in the z-direction. Hence, imagine shrinking the solenoid along the z-directions also to a point. The composite object is now pointlike… (Rao [2001], p. 15)

[21] See Chakraborty et al. ([1995]), Ezawa ([2013]), and references therein for more on both the integer and fractional quantum Hall effects.

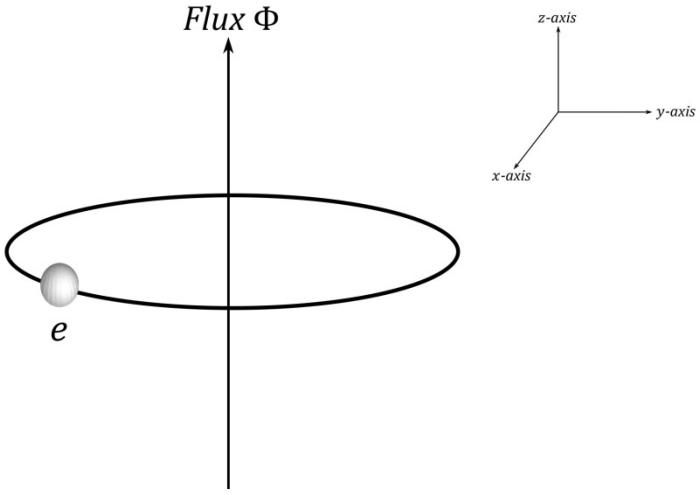

Figure 7. The Flux-tube model of the anyon. A spinless charge *e* particle orbiting around a thin and long solenoid with magnetic flux Φ.

It is difficult to think of this highly idealized model as the actual explanation of how an anyon, thought of as a composite particle, may come about. Nevertheless, the flux tube model of the anyon can be taken as generating a potential explanation – more specifically, a 'how-possibly' explanation, which demonstrates how a particular effect may be brought about *in principle*, without committing itself to the claim that this is how it was *in fact* brought about[22] – and, as such, it may point the way towards more realistic models to be developed in the future.

The third sense of exploration that we wish to discuss concerns proof-of-principle demonstrations, specifically of a kind that establishes the viability of a certain type of approach or methodology for the purpose of generating potential representations of target phenomena. In this sense, our case study shows that it is theoretically fruitful and perhaps empirically viable to consider the physics of systems that are not three-dimensional. This methodology is in part vindicated via the theoretical discover of anyons and has led to the exploration of physics of various two- and one-dimensional systems that may perhaps be manifested approximately in the laboratory. In fact, (non-abelian) anyons, such as bound states of the Mayorana Fermion, are the best candidates from which to build quantum computers (Nayak at al. [2008]), [23] and are regularly discussed in introductory textbooks to the subject (e.g., Pachos [2012]). Also, if the existence of anyons is empirically confirmed, we will know that the configuration space framework is the

[22] For some historical background on the notion of 'how-possibly' explanations, see Gelfert ([2016], p. 92); for a more substantive discussion, which links the distinction between 'how-possibly' and 'how-actually' explanations to different explanatory contexts arising from contrasting ways of framing a problem, see Bokulich ([2014]).
[23] See Mourik et al. ([2012]) for recent experimental results.

correct framework for understanding permutation invariance in quantum mechanics (and vice versa). This shows how an idealization (or idealized model) may establish that a certain type of methodology, namely, a framework for permutation invariance in quantum mechanics, is able to generate a potential representation of anyons (whereas the operator approach cannot).

The last sense of exploration that we wish to discover concerns using idealizations and models to assess the suitability of target systems. We will elaborate on this sense in the following section. Here we only wish to introduce the idea with some background history regarding anyons. In particular, anyons were first discussed in the literature solely as theoretical constructs. Before attempts were made to apply such constructs to FQHE systems, there were different target systems that physicists were hoping fractional statistics could shed light on:

> Surely the most dramatic result of the study of anyon statistics […] has been the demonstration [of] a new mechanism of superfluidity (and, for charged anyons, superconductivity). This superfluidity is quite a robust consequence of fractional quantum statistic at appropriate values of the fraction. […] [It] is tempting to speculate that the anyon mechanism of superconductivity will shed light on the copper oxide high temperature superconductors. Whether or not this speculation works out, the mechanism is of considerable theoretical interest and will undoubtedly play an important role in physics in the future. (Wilczek [1990], p. 325)

In other words, it was first speculated (and hoped) that anyons could shed light on high-temperature superconductivity. When later investigations did not bear fruit in this regard, physicists did not reject anyons altogether as useless idealizations. Instead, a new target system was sought: specifically, FQHE systems. Hence, we see how the application of idealization and the investigation of an idealized model such as the flux tube model of the anyon may lead us to reassessments of the suitability of target systems.

4 The Hubbard Model of the Mott Phase Transition

4.1 Case Study: Hubbard Model

Consider a solid, such as a macroscopic piece of iron, made up of atoms at specific sites in a crystal lattice, and electrons which are either bound to specific atoms, shared between them, or moving about in the solid, as happens in electric conductors where an external electric field can easily bring about an electric current. Whereas in the previous two cases discussed in this paper, the quantum systems were constrained only by geometry, dimensionality, or external fields (as in the Aharonov-Bohm effect), in many-body systems such as crystal domains in a

metal, a vast number of atoms and electrons interact with each other. Of special significance are substances such as iron, nickel, and cobalt, not only because they are ingredients of various technologically important alloys, but also because they exhibit strongly correlated behavior: the Coulomb interaction between the negatively charged electrons in such materials is so strong that the one-particle picture for calculating the electronic band structure, which governs many important physical characteristics, is no longer sufficient to describe them.[24] Such systems exhibit salient many-body effects, which manifest themselves macroscopically, e.g. in ferromagnetic behavior, when a solid exhibits a permanent magnetization even in the absence of an external field. Due to their large number ($\sim 10^{23}$) of interacting particles, strongly correlated electron systems cannot easily be studied on the basis of theoretical 'first principles', but require the use of many-body models.[25]

As a model of strongly correlated electrons in a crystal lattice, the Hubbard model has become one of the most extensively studied models in condensed matter physics. Commonly, electronic states in a solid may be classified into those that are *localized*, i.e., centered around lattice sites and akin to atomic states, and *delocalized* (itinerant) states, i.e., Bloch states filled by electrons in accordance with Fermi statistics. Originally formulated for the study of collective magnetic order in solids, the Hubbard model is now widely employed for the study of various correlation effects in systems with such itinerant electrons. Its uses have proliferated beyond the question of the origins of spontaneous magnetism, and now include the study of metal-insulator transitions, high-temperature superconductivity, lattice gases, organic molecules and nanoparticles. While, at the descriptive level of scientific practice, this proliferation already points towards the exploratory utility of the Hubbard model, the specific character of exploration involved can be made more precise by looking at how features of the model have enabled a reassessment of its initial intended target. This will be done in the second subsection (4.2); first, let us motivate and summarize the Hubbard model.[26]

The Hubbard model was first developed for systems with narrow energy bands, in which the electrons, though delocalized and mobile, are still likely to be found near the lattice sites, i.e., the atoms (ions) that they are associated with. At a general level, the Hamiltonian of any system consisting of electrons and ions can be expressed as the sum of three components

$$H = H_{kin} + H_{ie} + H_{ee}$$

where the first term indicates the purely kinetic energy of the electrons, the second term the interaction between the electrons and the lattice potential, $V(\boldsymbol{r})$, due to the

[24] For relevant background material in solid state/condensed matter physics see standard textbooks such as Ashcroft and Mermin ([1976]).
[25] For a review, see Gelfert ([2015]).
[26] For the canonical derivation due to Hubbard, see his ([1963]).

ions that make up the crystal lattice, and the third term comprises the electron-electron interaction. Assuming the lattice potential $V(\boldsymbol{r})$ to be strong and the mobility of the electrons to be small (though non-negligible), the sum of the *atomic* (single-particle) Hamiltonians at site *i*, $h_{at}^{(i)}$, can still be regarded as an acceptable representation of the system, at least near the lattice sites. (One way to think of this as a starting point for model-building is to regard the atoms as 'too far apart' to yet 'feel' any forces acting between them.) The non-relativistic Schrödinger equation then is $h_{at}^{(i)}\varphi_n(\boldsymbol{r}-\boldsymbol{R}_i) = \varepsilon_n\varphi_n(\boldsymbol{r}-\boldsymbol{R}_i)$, where φ_n is an atomic wave function and *n* signifies the relevant set of quantum numbers. In the case at hand, the overlap between atomic wave functions associated with different lattice sites is assumed to be small, and the wave function will be centered strongly around the respective lattice sites.

The atoms, of course, form a crystal lattice, so electrons will not only feel the atomic Hamiltonian, but also the lattice potential. In the non-interacting parts of the Hamiltonian, $H_0 = H_{kin} + H_{ie} = \sum_{i=1}^{N} h_0^{(i)}$, one therefore needs to include the lattice potential, $h_0^{(i)} = h_{at}^{(i)} + V(\boldsymbol{r})$, which gives rise to the 'non-atomic' single-particle Schrödinger equation:

$$h_0\psi_{n\boldsymbol{k}} = \varepsilon_n(\boldsymbol{k})\psi_{n\boldsymbol{k}}(\boldsymbol{r}).$$

In order to simplify the problem, the wave function $\psi_{n\boldsymbol{k}}$ can be approximated by the atomic wave functions:

$$\psi_{n\boldsymbol{k}}(\boldsymbol{r}) = \frac{1}{\sqrt{N_i}}\sum_{j=1}^{N} e^{i\boldsymbol{k}\boldsymbol{R}_j}\,\varphi_n\big(\boldsymbol{r}-\boldsymbol{R}_j\big).$$

While this approximation introduces some degree of error, the assumption is that the total error will be tolerable, since the approximation is nearly exact near the lattice sites (where the ions render the dynamics of the system almost 'atom-like') and becomes substantial only where the value of φ_n is already very small. It is also worth noting that the formula for $\psi_{n\boldsymbol{k}}$ satisfies Bloch's theorem, according to which, in an idealized system with periodic lattice potential, the wave function should be invariant with respect to translation, except for a phase factor: $\psi_{n\boldsymbol{k}}(\boldsymbol{r}+\boldsymbol{R}^m) = e^{i\boldsymbol{k}\boldsymbol{R}^m}\psi_{n\boldsymbol{k}}(\boldsymbol{r})$.

The (general) Schrödinger equation for the Bloch functions $\psi_{n\boldsymbol{k}}$ can now be evaluated using the approximation, given in the previous formula, by the atomic wave functions. This results in a compact formula for the Bloch energies,

$$\varepsilon_n(\boldsymbol{k}) = \varepsilon_n + \frac{v_n + \frac{1}{\sqrt{N_i}}\sum_{j\neq 0}\gamma_n^{(j)}\, e^{i\boldsymbol{k}\boldsymbol{R}_j}}{1 + \frac{1}{\sqrt{N_i}}\sum_{j\neq 0}\alpha_n^{(j)} e^{i\boldsymbol{k}\boldsymbol{R}_j}}$$

where $v_n = \int d^3r V(\boldsymbol{r})|\varphi_n(\boldsymbol{r})|^2$ reflects the influence of the lattice potential on a single electron, and the *overlap integrals*, $\alpha_n^{(j)} = \int d^3r \varphi_n^*(\boldsymbol{r})\varphi_n(\boldsymbol{r}-\boldsymbol{R}_j)$ and $\gamma_n^{(j)} = \int d^3r \varphi_n^*(\boldsymbol{r})V(\boldsymbol{r})\varphi_n(\boldsymbol{r}-\boldsymbol{R}_j)$, which are a measure of the mutual influence between electrons at different lattice sites, can be assumed to be very small in value for any $\boldsymbol{R}_j \neq 0$, so that higher-order terms can be neglected. Restricting interactions to those between nearest neighbors ('*n.n.*') one thus arrives at

$$\varepsilon_n(\boldsymbol{k}) = T_0^{(n)} + \gamma_n^{(1)} \sum_{n.n.} e^{i\boldsymbol{k}\boldsymbol{R}_{(n.n.)}}$$

which can be translated back into the formalism of creation and annihilation operators, with the non-interacting part of the Hamiltonian reducing to the simple formula $H_0 = \sum_{ij\sigma} T_{ij} a_{i\sigma}^\dagger a_{j\sigma}$ where the T_{ij} are the *hopping integrals* associated with those contributions arising from the movement of a particle at site *j* to another site *i*.

The interacting part of the Hamiltonian can similarly be expressed in terms of creation and annihilation operators:

$$H_{ee} = \frac{1}{2}\sum_{ijkl} v(ij;kl) a_{i\sigma}^\dagger a_{j\sigma'}^\dagger a_{l\sigma'} a_{k\sigma}$$

where the matrix element $v(ij;kl)$ is constructed from atomic wave functions:

$$v(ij;kl) = \frac{e^2}{4\pi\varepsilon_0}\iint d^3r_1 d^3r_2 \frac{\varphi^*(\boldsymbol{r}_1-\boldsymbol{R}_i)\varphi^*(\boldsymbol{r}_2-\boldsymbol{R}_j)\varphi(\boldsymbol{r}_2-\boldsymbol{R}_l)\varphi(\boldsymbol{r}_1-\boldsymbol{R}_k)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}.$$

The matrix element bears a close resemblance to the classical Coulomb potential, but it also takes into account the quantum effects between different particles, as indicated by the 'mixed' integral. Because of the small overlap between atomic wave functions centered on different lattice sites, the intra-atomic matrix element $U = v(ii;ii)$ can be expected to strongly dominate the dynamics of interaction; neglecting, as a final approximation, all other matrix elements, simplifying the

operator combination using the number operator $n_{i\sigma} = a_{i\sigma}^{\dagger} a_{i\sigma}$, and combining the non-interacting and interacting parts of the Hamiltonian, gives the standard *Hubbard Hamiltonian*:

$$H = \sum_{ij\sigma} T_{ij} \hat{a}_{i\sigma}^{\dagger} \hat{a}_{j\sigma} + \frac{1}{2} U \sum_{i\sigma} \hat{n}_{i\sigma} \hat{n}_{i,-\sigma}.$$

The Hubbard model contains only a small number of parameters, most explicitly, the ratio between the Coulomb repulsion and the kinetic energy of the electrons and, less overtly, the filling of the energy band and the geometry of the crystal lattice (which is implicit in the summation range, e.g. by performing the sum over nearest neighbors in a unit cell).

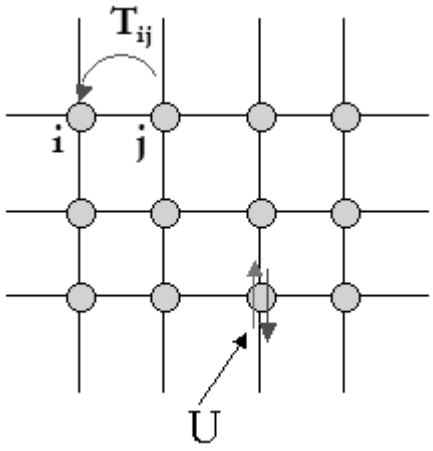

Figure 8. Schematic representation of the processes modeled by the Hubbard Hamiltonian: *U* is the Coulomb repulsion experienced by two electrons (of opposite spin) occupying the same site in a crystal lattice; T_{ij} are the hopping integrals that apply to the movement of an electron from one lattice site, *j*, to another, *i*.

Yet this has proven to be sufficient to render the Hubbard model one of the simplest and most fruitful frameworks for modeling itinerant, strongly correlated electrons in crystal lattices. As a recent *Nature Physics* editorial on 'The Hubbard Model at Half a Century' puts it, while the Hubbard model was initially 'introduced to provide an explanation for the itinerant ferromagnetism of transition metals, such as iron and nickel, […] the past 50 years have seen its relevance go far beyond that original context', and the (anonymous) authors express their confidence that, even after half a century, the model 'should be a stimulus for further explorations' ('The Hubbard Model', [2013], p. 523).

4.2 Exploration in the Hubbard Model

The exploratory utility of the Hubbard model is partly due to its relative simplicity and the ease with which it can be 'customized' to fit single-band and multiband scenarios, and phenomena beyond spontaneous magnetism, such as high-temperature superconductivity and artificial lattices of cold atoms. It has also given rise to other models, such as the *t-J model*, which can be derived from the Hubbard model in the limit of large U via an (operator-level) transformation, in which doubly-occupied electron states are neglected: that is, an electron 'hops' from one lattice site to another (transferring energy t, similar to what happens in the Hubbard model), but only when that destination site is empty. Higher-order processes, in particular, such as two electrons hopping onto the same (previously unoccupied) lattice site, are excluded on this picture.[27] Yet the Hubbard model's simplicity and ease of adaptability alone do not exhaust its exploratory features. In this subsection, we shall discuss a salient example of the fourth type of exploration distinguished earlier, *viz.* exploration via assessing the suitability *of the target system*.

The suggestion that exploration may consist in holding the model and its representational means fixed, while searching for target systems which the model may be able to adequately describe, might at first seem to put the cart before the horse: should we not try to find models that fit our target systems, rather than 'shopping around' for target systems that fit our preferred models? (On this point, see Gelfert 2006, Section 4.5.4.) In response, two points come to mind. First, in the early days of research, before a good theoretical grasp of a phenomenon has been gained, it may be unclear whether the putative phenomenon is indeed the result of a more stable, reproducible set of circumstances. As noted in our earlier discussion of exploration in the Aharonov-Bohm effect (Section 2.2) and of fractional quantum statistics (Section 3.2), it often takes time for theoretical descriptions and experiments to become sufficiently refined to establish that a phenomenon is indeed what physicists call a recurrent 'effect'. In this early stage of research, commitment to putative target systems and phenomena is tentative, and one's choice of target system or phenomenon will be subject to revision. Second, choices of target systems for a given model are not made arbitrarily, but often are the results of *re*assessing a model's target system in the light of new findings. That is, model equations are not deployed arbitrarily to represent a new target; rather, the new target suggests itself on the basis of findings generated in the course of a sustained exploration of the model's utility for describing the initial target. This (fourth) sense of exploration, we submit, can be illustrated in the Hubbard model's evolution from a model for describing itinerant electrons in transition metals (and other conductors) to a potential model for insulators of a particular kind, the so-called *Mott insulators*.

According to A. H. Wilson's metal/non-metal criterion, an insulator (or semiconductor) is characterized by having either completely filled or completely empty energy bands, separated by an energy gap that prevents electrons from

[27] See Spałek ([2007]) for more on the t-J model.

becoming mobile (unless a significant external field is applied).[28] A metal, by contrast, has partially filled bands, such that even the slightest external perturbation is enough to excite an electron into an (infinitesimally higher) empty state in the same energy band. Due to such 'conduction bands', metals are able to conduct electricity, whereas insulators are not. As discussed in the previous section, the Hubbard model, with its emphasis on the ease with which itinerant electrons can 'hop' from one lattice site to the other, was initially intended to model ferromagnetic conductors such as nickel and iron (rather than insulators). Historically, however, it quickly turned out that the classification into metals and non-metals was not as clear-cut as Wilson's criterion had suggested. Thus, as early as the 1930s, it was shown that certain transition-metal oxides were insulators, even though their crystalline structure suggested that they had partially filled bands and should be conductors.

Theoretical physicists, including Rudolf Peierls, Lev Landau, and Nevill Mott, explored the possibility of the breakdown of Wilson's criterion being the result of correlations associated with the repulsive Coulomb interaction between the electrons. Mott, in particular, clarified the metal-insulator criterion in relation to transition-metal monoxides, and the problem became known as the puzzle of 'Mott insulators' (Mott [1949]). Subsequently, in the 1950s, it was realized that all Mott insulators are antiferromagnets and remain insulators even above the Néel temperature, the critical temperature above which antiferromagnetic order is destroyed and a solid turns into a paramagnet. While it was realized, notably by John Slater, that antiferromagnetism led to a 'splitting up' of electron bands, thereby increasing the chances for the emergence of energy gaps characteristic of insulators, numerical simulations indicated that this alone could not explain the behavior of Mott insulators. Exactly why substances that, on the basis of their crystalline structure and electronic characteristics, should be expected to conduct electricity are nonetheless insulators – and why these Mott insulators are antiferromagnets – remained an open question, and a lively debate ensued within theoretical condensed matter physics in the decades to come. This was compounded by the fact that certain types of substances, such as transition metal oxides, displayed a transition from insulating to metallic behavior – a 'Mott transition', as it became known – as the result of certain factors (e.g. slight modifications of its composition known as 'doping').

With the Hubbard model having been developed as a model of ferromagnetic metals, one might wonder how it could possibly shed light on the phenomenon of Mott insulators – which, after all, are *anti*ferromagnetic *insulators* – and their associated phenomena. Yet, in recent years, it has become one of the most extensively studied models for exploring the Mott metal-insulator transition and related phenomena, attesting to the exploratory power of the Hubbard model, which in this case manifests itself in a thoroughgoing reassessment of the range of its intended targets. We are, of course, not suggesting that the Hubbard model is *no*

[28] See Wilson ([1936]) for the original discussion, and Martin ([2004], pp. 40-44) for a more recent textbook presentation.

longer used for its original purposes – it remains the model of choice for many researchers interested in the behavior of ferromagnetic substances and strongly correlated electron systems in model – but, rather, that the range of its target systems has been expanded to include types of targets that could not have been foreseen by, and indeed would likely have seemed outlandish to, its initial proponents.

Why consider the Hubbard model a potential model of Mott insulators in the first place? This latest turn in the Hubbard model's varied career came as something of a surprise to many condensed matter physicists and is due to an exact mapping that obtains in a limiting case of the standard Hubbard model. In particular, it results from the limit of strong interactions, when U is much larger than the other matrix elements (W). Evaluating the model using second-order perturbation theory, the multiple degeneracy of the ground state of the model is lifted and, at half-filling ($n = 1$), only the following effective Hamiltonian remains *as the limiting case of the Hubbard model*:

$$H_{eff} = \sum_{\boldsymbol{R}_1 \neq \boldsymbol{R}_2} \frac{2|t(\boldsymbol{R}_1 - \boldsymbol{R}_2)|^2}{U} \left(\widehat{\boldsymbol{S}}_{\boldsymbol{R}_1} \cdot \widehat{\boldsymbol{S}}_{\boldsymbol{R}_2} - \frac{1}{4} \right)$$

which is rendered a compact formula thanks to the use of spin operators $\widehat{\boldsymbol{S}}_{\boldsymbol{R}_i} = \left(\sum_{\sigma,\sigma'} \hat{a}^{\dagger}_{i\sigma} \boldsymbol{\sigma}_{\sigma,\sigma'} \hat{a}_{i\sigma'} \right)/2$ (with $\boldsymbol{\sigma}$ the vector of Pauli matrices). As it turns out, this limiting case is identical to another much-studied quantum model, the antiferromagnetic *Heisenberg model*

$$H_{Heis} = \sum_{\boldsymbol{R}_1 \neq \boldsymbol{R}_2} J(\boldsymbol{R}_1 - \boldsymbol{R}_2) \left(\widehat{\boldsymbol{S}}_{\boldsymbol{R}_1} \cdot \widehat{\boldsymbol{S}}_{\boldsymbol{R}_2} - \frac{1}{4} \right)$$

with $J(\boldsymbol{R}_1 - \boldsymbol{R}_2) = 2|t(\boldsymbol{R}_1 - \boldsymbol{R}_2)|^2/U$.[29] The Heisenberg model, in turn, has long been known as 'the "standard model" for the description of magnetic insulators' (Gebhard [2000], p. 75).

The situation, then, is this: whereas for $U = 0$, the Hubbard model at half-filling describes a metal, for $U \gg W$ it maps onto the standard model of antiferromagnetic insulators. This behavior, which could not easily be 'read off' from the model's original formulation, and certainly played no role in the type of derivation given in the preceding subsection, provides a stunning example – or so we believe – of the exploratory potential that is afforded by some models in virtue of

[29] For details of the derivation, see various textbooks on quantum magnetism, e.g. Gebhard ([2000]).

their structure and the relations they stand in with other models.[30] For, having convinced themselves that the Hubbard model at half-filling, and for low interaction strength U, describes a metal, and for very large values of U behaves like an antiferromagnetic insulator, researchers 'are confident that the model is indeed capable of describing a Mott transition at a critical interaction strength U_c' somewhere in between.

Once again, exploration is borne out to be a fruitful strategy in model-based research – this time, this time by being open towards the reconsideration of one's target system. This, together with the construction of limiting cases of highly idealized models, led to a widening of the potential domain of applicability of the Hubbard model to include not only ferromagnetic metals, but also Mott insulators such as the transition-metal oxides, in which the phenomenon of Mott insulation was first observed.

5 Conclusion

We have presented three case studies, which illustrate our suggestion that idealizations, in the construction of models or by considering limiting cases, play exploratory roles in science. To end, we first want to consider a general objection to our account, and then wish to reflect on how Michael Weisberg's ([2007], [2013]) recent taxonomy of idealizations and models compares with our case studies.[31]

One of the main worries associated with taking idealizations and models to play a substantive exploratory role in science, is that it may seem that the concept of exploration is so generic as to border on the trivial. For instance, concerning the exploration of theoretical structure, one may raise the following objection: Is it the case that whenever one considers what quantum mechanics says about *any* idealized system, no matter how unrelated to actual scientific investigations, one is 'exploring' the structure of quantum mechanics? If so, haven't we trivialized the notion of exploration? Should there not be any constraints on what counts as a bona fide exploration? Yet, in reply, it is worth emphasizing that the notion of 'exploration' is not a free-for-all: Whether in relation to experimentation, scientific modeling, or, as in our case, idealizations and limiting cases, exploration is marked by recurring strategies, not by haphazard arbitrariness. Indeed, this is why the case studies we have presented are important: They demonstrate not random moves in the investigation of salient scientific questions, but structured approaches that exhibit continuity and stability.

Perhaps, however, the objection is meant to be more specific, targeting not the various forms that exploration in science may take, but the lack of specific criteria of success. Yet, much the same could be said about other basic concepts such

[30] In Gelfert ([2009]), one of us emphasizes the role of mathematically rigorous results and relations in affording such cross-model exploration.

[31] We wish to give John Earman credit for first bringing to our attention (in conversation and correspondence) the idea that Weisberg's ([2007], [2013]) account may well be wanting.

as scientific explanation. For one, we readily accept that there are various notions of scientific explanation, e.g., causal-mechanical, unificationist, deductive-nomological, so that what counts as a successful explanation will greatly depend on which notion is involved and the type of why-question one is looking to answer. Similarly, it is to be expected that there will be various notions of exploration. Second, whether or not an argument proffered merely as a *potential* explanation counts as a good one will not only depend on whether it turns out to be an *actual* explanation, but also on its merits in terms of explanatory power, novelty, simplicity, etc., and historical context (e.g., on whether the explanation does justice to the well accepted science of the time). And just as there are better and worse potential explanations, some exploratory moves in science will be more or less fruitful. As we have seen throughout the brief historical sketches included in our case studies, scientists are often keenly aware of which exploratory moves are further removed from the phenomena, and which stand a realistic chance of leading to the genuine discovery of new phenomena, effects, and explanations.

Finally, we wish to reconsider some of the philosophy of science literature on idealizations in light of the case studies presented here. Michael Weisberg ([2007], [2013]) has recently presented a well-received taxonomy of idealizations. His three-fold classification scheme includes *Galilean* idealizations, *minimalist model* idealizations, and *multiple-model* idealization (see Figure 9). We will consider each notion in turn in light of the case studies that we have discussed. A Galilean idealization is a distortion used to simplify, and render computationally tractable, the treatment of a target system. Can the two-dimensional idealization in the context of anyons and fractional statistics be understood along these lines? We submit that it cannot. By moving from three to two dimensions we saw how a novel type of mathematical structure emerged (namely, the fundamental group for the configuration space became that of the braid group instead of the permutation group), allowing for the representation of anyons and fractional statistics. The goal of such an exercise was not specifically to simplify a target system (although we are happy to grant that simplification may be a partial goal or by-product of such an idealization). In contrast with, for instance, the Ising model, the two-dimensional scenario in the anyon case was *not* implemented because it made the model computationally more tractable.

Next, minimalist model idealizations are distortions used to expose key causal or explanatory factors in the behavior of the target system. In the case of anyons, we grant that the two-dimensional idealization may be used to generate a potential explanation of the phenomenon of fractional statistics. Nevertheless, it is difficult to motivate the idea that the two-dimensional idealization generates the actual explanation of fractional statistics since the results obtained from the configuration space framework hold for systems that are, strictly speaking, exactly two-dimensional. Systems that are approximately two-dimensional cannot manifest anyons and fractional statistics according to the configuration-space framework. Moreover, at no point was there a discussion of any causal factors involved; in this sense, the exploration afforded by the two-dimensional idealization runs deeper

than the search for causes and effects. So, again, a minimalist model idealization does not seem to be the kind of idealization involved in the case of anyons.

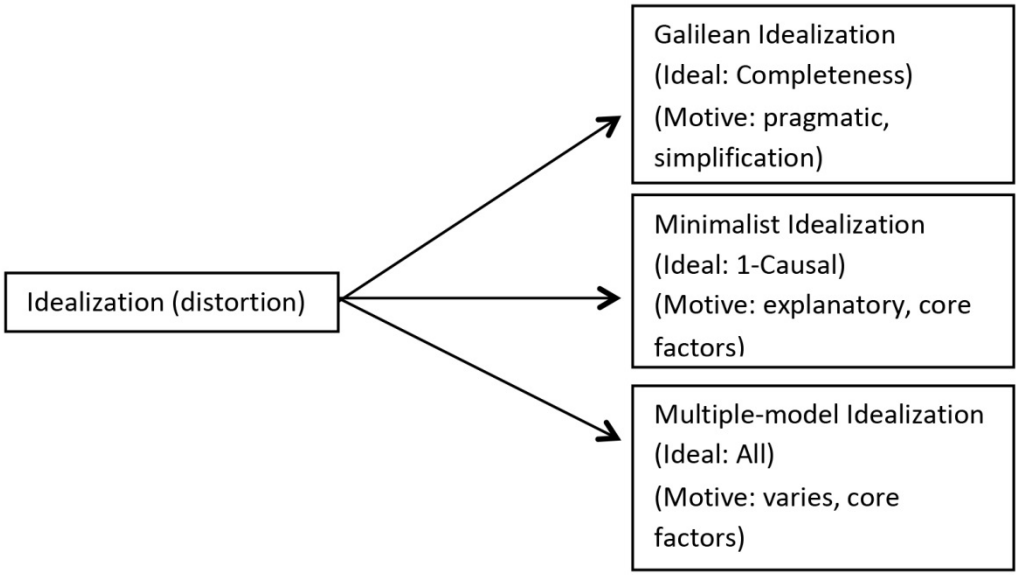

Figure 9. Weisberg's taxonomy of idealizations.

Last, multiple-model idealizations are multiple incomplete models, designed to serve different epistemic/pragmatic goals, which typically trade off against each other. While they 'may retain a single, complex target, [they] construct multiple models for the target' (Weisberg [2013], p. 113). Yet, as we saw in connection with the Hubbard model, exploration may arise from using the *same* model and varying – not arbitrarily, but in a way that is clearly motivated by the course of inquiry – its prospective target system or phenomenon. To be sure, there is a sense in which performing a limiting procedure, for example varying the model's parameters so as to realize the scenario $U \gg W$, gives rise to a sequence of models, one for each set of parameter values. On this understanding, *any* limiting case in scientific modeling would be associated with 'multiple models'. But it seems to us that this is not the sense intended by Weisberg who, rightly, links multiple-model idealization to competing, qualitatively distinct models, whether due to unavoidable tradeoffs or due to the complexity of the target system. In neither sense, however, is the case of the Mott-Hubbard transition adequately characterized by treating it as a multiple-model idealization (which is not to say that, in other circumstances, it cannot be used in this way; the derivation of the t-J model, hinted at in Section 4.2 may be a case in point). Similarly, in the context of anyons and fractional statistics we did not discuss multiple models but only one idealized two-dimensional model or scenario, and there was one only one goal: to show how anyons and fractional statistics may emerge. Indeed, this holds more generally for models aiming at 'how-possibly'

explanations of unexpected phenomena (see footnote 22), where the goal is to demonstrate how a surprising (or perhaps merely speculative) 'effect' *may* possibly arise, while neither requiring that this is how it, in fact, emerges, nor demanding that the model be useful for other purposes.

Thus, we think that Weisberg's ([2007], [2013]) scheme is at the very least incomplete, in that it does not make room for the exploratory role of idealizations and models, thereby offering a distorted view of the types of situations illustrated by our case studies. Moreover, we submit that similar claims can be made about other taxonomies of idealizations and models in the literature, e.g., McMullin ([1985]), Nowak ([1980]), Shaffer ([2012]), but due to space constraints we will leave the details to be worked out in future work. In addition, it seems to us that a deep *understanding* of scientific theory can only be gained by making room for the exploratory role of models and idealizations. If this is the case, we will have found a substantive sense for which idealizations are essential to science, namely, for scientific understanding. Whether or not these further claims can be substantiated given our best accounts of scientific understanding is another issue that we leave for further study. Last, although we have concentrated here on case studies in physics, it would be interesting to see whether the notions of explorations discussed also arise in other sciences as well.

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