Journal for the History of Analytical Philosophy
Volume 3, Number 6

Frege on the Foundation of Geometry in Intuition

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I investigate the role of geometric intuition in Frege's early mathematical works and the significance of his view of the role of intuition in geometry to properly understanding the aims of his logicist project. I critically evaluate the interpretations of Mark Wilson, Jamie Tappenden, and Michael Dummett. The final analysis that I provide clarifies the relationship of Frege's restricted logicist project to dominant trends in German mathematical research, in particular to Weierstrassian arithmetization and to the Riemannian conceptual/geometrical tradition at Göttingen. Concurring with Tappenden, I hold that Frege's logicism should not be understood as a continuing a project of reductionist arithmetization. However, Frege does not quite take up the Riemannian banner either. His logicism supports a hierarchical understanding of the structure of mathematical knowledge, according to which arithmetic is applicable to geometry but not vice versa because the former is more general, as revealed by the strictly logical nature of its objects in comparison to the intuitional nature of geometric objects. I suggest, in particular, that Frege intended that foundational work would show the use of geometric intuition in complex analysis, a source of error for Riemann that Weierstrass was proud to have uncovered, to be inessential.

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1. Introduction

Early in his career, Frege made geometrical models of algebra. He was a mathematician, and his motives were plainly mathematical and obviously influenced by the Göttingen mathematics department where he earned his doctorate. Like Cauchy and Gauss, he sought an intuitive basis for understanding the complex numbers, but he was influenced by the neo-Kantian philosopher Lotze as well and took a more philosophically restricted view of geometry than Gauss's protégé Riemann. Frege's early mathematical work displays presuppositions about geometry and geometrical intuition that provide insight into the mathematical and philosophical motivations for Frege's logicist project, and in particular for the restriction of that project to arithmetic.

In this essay I investigate Frege's understanding of the relationship between geometry and arithmetic. Throughout his career, from his early mathematical works on, he maintained that geometry concerns specific objects of intuition. An important feature of Frege's philosophy of mathematics is his conception of arithmetic as more general than geometry; indeed, showing that arithmetic participates in the generality/universality of logic was a primary motivator for Frege's brand of logicism. We will begin with an assessment of Michael Dummett's analysis of Frege's comments on geometry in *Die Grundlagen der Arithmetik*, then will support the overall picture of Frege's view of geometry by investigating Frege's early work and Mark Wilson's commentaries on their significance for understanding his project and by supporting and refining Jamie Tappenden's thesis that Frege's work is best situated in relation to the Riemannian rather than Weierstrassian school.

2. Geometry in the *Grundlagen*

In this section I articulate my view of Frege's mature philosophical position on the foundations of geometry. I begin with a discussion of Lotze, which admittedly draws heavily from the commentary of Roberto Torretti. The discusssion of Lotze sets the stage for an assessment of Michael Dummett's commentary on the sections of *Die Grundlagen der Arithmetik* that deal with geometry. In particular, I want to position Frege as aligned philosophically with Kantian critics of non-Euclidean geometry. Since I will argue that Frege distinguishes geometry from arithmetic in virtue of the former's reliance on intuition, it is important to address statements he makes that might be taken to divorce the objective content of geometry from subjective, sensible intuition.

While Frege's mathematical influences were, as we will discuss subsequently, certainly broadly Riemannian, his philosophical influences included the neo-Kantian Hermann Lotze, from whom Frege took his only philosophy course at Göttingen.[1] Lotze was chair of the Göttingen faculty during the controversy over the philosophical faculty's reception of Riemann's habilitation lecture. The topic was chosen by Gauss himself, and Riemann expounded his foundational conception of manifolds of arbitrary dimension and curvature. Roberto Torretti summarizes Lotze's response and the philosophical point of view that informed it:

> To his mind, the new geometric speculations were "just one big connected mistake" (Lotze, 1879, 234). His criticism of them is set in the context of his metaphysical theory of space. This theory, like Erdmann's, is conceived in terms of the duality of Mind and Things. According to Lotze, space can exist only as space intuition, that is, only insofar as the Mind is aware of it (211). But space is not a mere appearance to which nothing corresponds in reality (*im Reellen*). "Every particular trait of our spatial intuitions corresponds to something that

> is its ground in the world of things." But such ground does not in any way resemble spatial relations. "Not relations, spatial or intelligible, *between* things, but only immediate interactions, which things inflict one another as internal states, are the actual fact whose perception is woven by us into spatial phenomenon" (223) (Torretti, 1978, 286).

Torretti notes that although Lotze's commentary indicates a careless lack of understanding of the Riemannian direction of generalization of geometry, there are nuggets of insight, such as an apparent anticipation of Poincaré's conventionalism. One might add that Lotze also anticipates certain structuralist views in scientific epistemology.[2] One aspect of Lotze's criticism is the merely verbal point that geometry has traditionally been the study of what we call "space", which according to his metaphysics is fixed in the content of intuition. Lotze, in fact, admits of the possibility of non-humans that, in Torretti's characterization, "perceive in a different fashion the same aspects of things which we perceive in space" (Torretti, 1978, 289). But behind this verbal point is a substantive dispute. Riemann, as well, reserved "space" for the space that we perceive, using the generic term "manifold" for "the general notion of multiply extended magnitudes (in which space magnitudes are included)" (Riemann). However, Torretti notes, Riemann was skeptical that the spacial manifold of appearances could by introspection alone be determined as Euclidean while Lotze was without doubt. Doubt concerning the truth of the parallel axiom in the space determined ostensively as the manifold of appearances may be one reason, though only one, why it was natural for Riemann and the mathematicians at Göttingen to adopt a more general conception of geometry; even with respect to the fixed subject matter of perceptual space they considered the geometric structure to be under determined.

Frege appears to have been in agreement with Lotze that the space of intuition is undoubtedly three dimensional and flat, and that the axioms of Euclid characterize self-evident truths concerning geometrical planar objects situated in space, although Frege's view of "intuition" will be up for interpretation and should not be assumed to agree with the strictly subjective, psychological view implied by Torretti's characterization of Lotze's view.[3] That geometry is founded, in some sense, on intuition is the tacit view of his early geometric works and the explicitly stated view in his correspondence with Hilbert and subsequent essays on the foundations of geometry. There is, however, a passage in *Grundlagen* that complicates the interpretation of Frege's view of intuition. In §26 Frege writes:

> Space, according to Kant, belongs to appearance. For other rational beings it might take some form quite different from that in which we know it. Indeed, we cannot even know whether it appears the same to one man as to another; for we cannot, in order to compare them, lay one man's intuition of space beside another's. Yet there is something objective in it all the same; everyone recognizes the same geometrical axioms, even if by his behavior, and must do so if he is to find his way about in the world. What is objective in it is what is subject to laws, what can be conceived and judged, what is expressible in words. What is purely intuitable is not communicable. To make this clear, let us suppose two rational beings such that projective properties and relations are all that they can intuit —the lying of three points on a line, of four points on a plane, and so on; and let what the one intuits as a plane appear to the other as a point, and vice versa, so what for the one is the line joining two points for the other is the line of intersection of two planes, and so on with the one intuition always dual to the other. (Frege, 1980b, §26).

Noting the duality of projective geometry (viz. that swapping the terms "point" and "line" in any theorem yields another theorem), Frege argues that the difference in subjective intuition would be no barrier to communication.

According to Dummett this passage distinguishes Frege from Kant because for Kant both our understanding of the meaning of geometric terms and propositions and our grasp of their truth depend on, presumably subjective, intuition. According to Dummett, that the communicable content of geometry is held by Frege to be independent of intuition shows that the only dependence that geometry has on intuition is epistemic. We could, in principle, grasp the meaning of geometric propositions (as a blind man can form the concept red and aptly use the predicate "red" in communication) without having any intuition whatsoever. In this sense, geometry is objective for Frege. However, Dummett argues, it is because we have the intuitions that we do have that we cannot imagine the axioms of geometry being false:

> We cannot imagine what it would be like for the axioms of geometry to be false, but we can conceive of their falsity, that is, we can think their negations: it follows that their senses are capable of being wholly grasped by conceptual thought in a manner that involves no allusion to our intuitions. It is on the basis of *a priori* intuitions of space that we accept those axioms as true; but the features of those intuitions which the axioms capture are ones which, as being expressible in words, are common to all and could, therefore, be grasped even by a subject whose intuitions differed from ours (Dummett, 1982, 250).

Dummett's interpretation is enticing. It makes sense of how Frege can in the same text endorse the claim that Kant was correct that geometry is synthetic *a priori* and write a passage such as §26, and his interpretation allows for an initial reconciling of §26 with Frege's repeated insistence that geometry is founded in intuition, both prior to *Grundlagen* and after in his correspondence with Hilbert and subsequent writing on the foundations of geometry. Geometry is founded on intuition in a merely epistemic sense, but the objective content (the meaning) of geometric propositions is independent of intuition.

However, upon further reflection this interpretation presents a puzzle. If the contents of geometric propositions, including the axioms, are entirely unrelated to our intuition then how could our intuition provide any assurance of the truth of the axioms? Furthermore, the foundation of geometry in intuition pertains, as Frege stresses in his opposition to Hilbert, to the semantic matter of correct definitions and not merely to the epistemic concerns stressed by Dummett. Indeed, the relationship between definitions and axioms is central to Frege's criticisms of Hilbert and the central matter of this dispute is the relationship of geometry to intuition. Axioms are intuited truths about intuited objects, and proper definitions are nominal because the meanings of the terms contained in the definiens are independently grasped, according to Frege. In an essay "Foundations of Geometry I" (1903) responding to Hilbert's *Grundlagen der Geometrie* Frege expounds:

> In mathematics, what is called a definition is usually the stipulation of the meaning of a word or sign. A definition differs from all other mathematical propositions in that it contains a word or sign which hitherto has had no meaning, but which now acquires one through it. All other mathematical propositions (axiomatic ones and theorems) must contain no proper name, no relation-word, no function-sign whose meaning has not previously been established... [Definitions] are arbitrary and thus differ from all assertoric propositions... No definition extends our knowledge. It is only a means for collecting a manifold content into a brief word or sign, thereby making it easier for us to handle (Frege, 1984, 274).

In the case of geometry, Frege's strictures on definition require a basic grasp of fundamental terms. Dummett's interpretation of §26 leaves it obscure what this fundamental grasp may consist in. Indeed, Dummett's interpretation seems to me to risk pressing Frege into a position on geometry that is not entirely unlike the one adopted by Hilbert, which he vigorously disputes. For, if the meaning of basic terms in geometry is founded on nothing other than public inferential agreement then it would seem that mere assent to the axioms and their consequences may be constitutive of grasping their meaning.

To resolve these puzzles about §26, I propose paying close attention to a potentially important distinction between intuition (*Anschauung*) and pure intuition (*reine Anschauung*). In an earlier passage, criticizing Hankel's proposal to found a theory of the real numbers on an intuition of magnitude Frege writes:

> The expression 'pure intuition of magnitude' gives us pause. If we consider all the different things that are called magnitudes: Numbers, lengths, areas, volumes, angles, curvatures, masses, velocities, forces, illuminations, angles, curvatures, and so forth, we can quite well un-

> derstand how they can all be brought under the single *concept* of magnitude; but the term 'intuition of magnitude,' and still worse 'pure intuition of magnitude', cannot be admitted as appropriate. I cannot even allow an intuition of number in general, not to mention magnitude in general. We are all too ready to invoke inner intuition, whenever we cannot produce any other ground of knowledge. But we have no business, in doing so, to lose sight altogether of the sense of the word 'intuition' (Frege, 1980b, §12).

Frege continues, claiming that Kant himself has both a wide and a narrow sense of intuition. In Kant's *Logik*, Frege notes, an intuition is contrasted with a concept insofar as the former is *individual* while a concept is *general*. Respecting the wide sense of intuition "there is absolutely no mention of any connexion with sensibility." The connection with sensibility arises only in the narrower sense of intuition in *Transcendental Aesthetic*. For Frege, the modifiers "pure" and "inner" are used to indicate the narrow sense of intuition. With this in mind, the passage of §26 can be read as indicating only that geometry does not depend on intuition in the narrow sense.[4]

It is in the broader sense of intuition drawn from Kant's *Logik* that we should understand Frege as holding that geometry, both semantically and epistemologically, depends on intuition. The passage of §26 is resisting both Kant's and Lotze's strictly subjective understanding of geometry as resting on *pure* intuition. As Torretti has noted, Lotze considered the very sort of case that Dummett considers a specifically Fregean, un-Kantian concern:

> Space, as we know it, may be conceived as a special case of the more general concept of an 'order system of empty places'. Nothing prevents us from conceiving several different species of this generic concept, structured by rules other than those that govern space. Other beings might exist, who perceive the same world of things as we do, but under one of these alternative order systems. It is possible that they perceive in a different fashion the same aspects of things which we perceive in space, or that the peculiar structure of their intuition enables them to perceive other aspects of things, which are inaccessible to us. Lotze will not dispute these possibilities. There is, in fact, no way of knowing whether they are fulfilled or not. But Lotze emphatically rejects the contention that other beings, unknown to us, could have a *spatial* intuition different from ours (Torretti, 1978, 287)

Frege disagrees with Lotze on precisely this point. The §26 passage indicates that subjects with differing pure intuitions may nevertheless communicate about the same space. Hence for Frege space is located objectively rather than subjectively. However, Frege agrees with Lotze's position, the core of his criticism of non-Euclidean geometry, that the propositions of geometry, ordinarily understood, are singular propositions incapable of generalization. Importantly for my understanding of Frege's views on geometry, the intuition on which geometry rests is intuition in the *individuating* sense, so that geometry is the objective science of a particular domain. I do not think, however, that Frege anywhere adequately articulates his view of the relationship between pure intuition and intuition in the wider sense. It is tempting to think of the relationship between pure intuition and individuating intuition in terms of the sense and reference, with pure intuitions providing senses that pick out, perhaps in various ways for various subjects, geometrical referents about which we objectively communicate. This suggestion extends beyond the concern of the present essay, which is to articulate Frege's view of the logical structure of our mathematical knowledge.

3. Geometric Representation

I now turn to an investigation of Frege's mathematical works, which will clarify and support the claims of the preceding section. In his dissertation, *On a Geometric Representation of Imaginary Forms in the Plane*, Frege constructs geometric models of functional relations (imaginary forms) between complex numbers. The work begins with an analogy between points at infinity and imaginary forms. The notion of a point at infinity, he says, is technically nonsense since it "would be the end of a distance which had no end."

Frege then notes that we may identify points at infinity with "what is common to all parallels": viz., their direction. The puzzling "points at infinity" are identified with directions. Directions are defined by mentioning only objects and relations that are putatively given in intuition: viz., the relation of parallelism between lines. By calling directions "points", two ways of describing a line are unified. The statement "a line is determined by a point and a direction" becomes a special case of "a line is determined by two points": viz. the case where one point lies "at infinity" (Frege, 1984, 1-3).

Though technically nonsensical, the phrase "point at infinity" is quite expedient. Kepler, in his work on conic sections, assimilates the parabola to the ellipse:

> In the parabola one focus lies within the curve, while the other is represented either outside or within it on its axis at an infinite distance from the first, so far that a line drawn from that blind focus [at either end] to every point of the curve is parallel to the axis[5].

If conic sections are taken as a function of eccentricity (fixing the main focus and the directrix) the second focus of the ellipse approaches infinity as the eccentricity approaches 1. At 1 the section is a parabola. Past eccentricity 1 the section is a hyperbola and the second focus is found on the opposite side of the directrix from the major focus.

Geometric freethinkers, unburdened by tradition, may describe the location of the second focus of the parabola as a limit of the locations of the foci of successive ellipses; the notion of the second focus of the parabola being located "at infinity" is suggested. Indeed, it is inviting to describe the second focus racing ever more swiftly toward infinity as the eccentricity approaches 1, then passing through infinity to the opposite side of the plane, though it is not clear how to associate specific mental images with all the geometric terms of this description. Worried about the status of unintuitive geometrical objects, Frege articulates the voice of Euclidean tradition. As enticing as this description may be, from Frege's point of view in his dissertation, strictly speaking to say that the second focus of a conic section is "at infinity" is just to state prosaically that the conic section lies asymptotically between two parallel lines. One identifies the point at infinity with the shared direction of the asymptotes.[6]

If the direction of a line is the set of lines parallel to it then the sequence of second foci, all of them spatial points, are described by the freethinker as "converging" by motion through space to a non-spatial object.[7] This uncomfortable situation, which may be regarded as a kind of ontological discontinuity, can be avoided in certain geometric representations of infinity. Consider the Riemann sphere as a representation of the extended complex numbers (extended to include a point at infinity). Projected onto the Riemann sphere, a solitary point at infinity is represented by the apex of the sphere. Some sequences that are divergent in the plane project to sequences that converge to the apex of the sphere.

Of course, the Riemann sphere includes, as I have mentioned, only a solitary point representing infinity. Projective geometry on the plane requires a distinct point at infinity in every direction. What I wish to illustrate by the Riemann sphere example, however, is that there are geometric representations, of which Frege was surely aware, in which the, strictly speaking nonsensical, point at infinity is identified with an ordinary point in a higher dimension. This suggests the possibility of identifying points at infinity (and other ideal objects arising in geometry) with geometrical objects rather than logical objects. However, I think that Frege does not embrace this possibility. Frege contends that geometry is founded on intuition, which limits our ascension to higher dimensions because we have no intuition of spaces of greater than three dimensions. This is not to say that, according to Frege, there may not be synthetic, geometrical representations of analytical relations defined in more than three coordinates, but that the representations are intuited geometric objects which cannot be identified as the unique denotations of analytic terms. The Riemann sphere provides such a geometric representation of the extended complex numbers, for example, but the complex numbers are not themselves to be identified with the objects constituting the representation.

Frege's early work sought to obtain intuitive representations of analytic relations on greater than three parameters using strategies of embedding and representation to reduce the analytic dimension number of the relation. His dissertation concerned representation of complex forms, and illustrates this use of intuitive geometry. Pairs of complex numbers are represented as lines between parallel planes labeled "real" and "imaginary", each of which have an intuitive planar geometry. The ordered pair $\{ x + x'i, y + y'i \}$ is represented by $\{\{x, y\}, \{x', y'\}\}$ and may be depicted by a line drawn between parallel planes in which $\{ x, y \}$ and $\{ x', y' \}$ are respectively situated. Frege determines that a complex line $y + y'i = (m + m'i)(x + x'i) + (b + b'i)$ can be regarded as determining a mapping from the real plane containing $\{x, y\}$ to the imaginary plane containing $\{ x', y' \}$. For this to work we should be able to obtain functions f and g such that $x'=f(x, y)$ and $y'=g(x, y)$. By elementary algebra:

$$y+y'i = (m+m'i)(x+x'i)+(b+b'i) =$$
$$y+y'i = (mx-m'x' +b)+(mx' +m'x+b')i$$

So that, separating real and complex parts, gives:

(1) $y = mx-m'x' +b$ and (2) $y' = mx'+m'x+b'$.

Solving for x' in (1) gives:

(3) $x' = \dfrac{mx+b-y}{m'}$

Substituting (3) into (2) gives:

(4) $y' = m(\dfrac{mx+b-y}{m'})+m'x+b'$

By a subtle manipulation of the coordinates for the real and imaginary planes, Frege obtains from the mapping relation a representation of the complex line by pairs of "guide lines" located above and below the real and imaginary planes. The figure shows Frege's illustration of the geometric relation generated by treating complex linear equations as expressing mapping relations between parallel planes with suitably oriented coordinates.

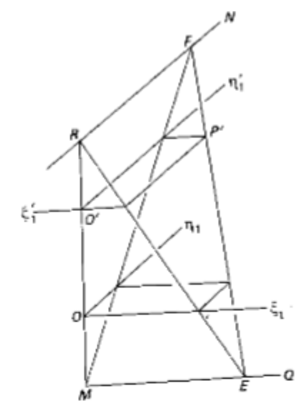

FIGURE 1

This general strategy is repeated for complex figures other than lines as the study of complex lines by their representative guide lines is generalized to a study of complex forms by guide surfaces. To work through the finer points of Frege's approach to providing representations of complex forms is not necessary for the purpose of illustrating Frege's perspective on analytic geometry. Frege is concerned to show that analytic relations are to be regarded as geometrical and intuitable by constructing geometrical representations. In Frege's representation of imaginary forms, non-intuitive, analytic relations between complex numbers are represented in intuition as relations between such objects as guide lines. It is in the context of such representations that the application of intuitive geometrical concepts, like intersection, to analytically defined complex forms is licensed.

What did Frege conceive to be the value of the intuitive representations he constructs? What are they for? Such representations may be regarded as either having an ancillary, heuristic role or as having an essential place in mathematical justification. The tension between these views of representation in intuition is present already in Gauss. Notably, Gauss was, early in his career, guarded about the role that his geometric representation of the complex numbers played in his reasoning. José Ferreirós notes that Gauss' first published proof of the fundamental theorem of algebra deliberately avoided the geometrical considerations which appear in his later proofs. One may speculate that this early reticence to reveal the geometric source of discovery suggests a strictly heuristic role for geometric representations; Gauss's geometric reasoning illustrates a method of discovery, while the earlier proofs provide the logically rigorous justification. As Ferreirós documents, Gauss' stated view was that the use of geometric representations provided intuition and simplicity but did not reveal the "true content" of the theorems proven by their means (Ferreirós, 2007, 54-55):

> I will present the proof in a dressing taken from the geometry of position since in this way it attains its maximum intuitiveness and simplicity. But in essence the true content of the whole argument belongs to a higher domain of the abstract theory of magnitudes, independent of the spatial, the object of which is the combinations, among magnitudes linked by continuity, a domain which until now has been little cultivated, and where we cannot move without language taken from geometrical images (Gauss, 1973, vol 3, 79).

I take Gauss' talk of "true" and "higher" content to be suggesting that the theorems derived express general propositions that hold true in domains other than the specific domain of geometrical representations that may guide the mathematician's reasoning. The quotation does not exactly make the philosophically neat distinction between discovery and justification that one might like to see. While Gauss' comment that the "dressing" of geometrical language does not reveal the higher content of the theorem suggests a merely heuristic role for geometric concepts, the statement that "we cannot move" without this language suggests an essential role for geometric concepts that cannot be eliminated in the context of justification.

Frege, I shall maintain, was motivated by epistemological concerns directly related to the use of geometrical images to study a strictly more general content, and thus took an implicitly critical stance toward Riemann's approach to complex analysis. Frege's project, to be modeled on the logicization of arithmetic, was aiming to show that reasoning about the higher content suggested by Gauss can indeed be undertaken without geometric language. Gauss, of course, would ultimately champion Riemann's efforts to extend the application of specifically geometrical concepts in mathematical reasoning, treating those concepts as essential both to discovery and justification in the context of mathematics conceived as founded in a general theory of manifolds. In the Riemannian approach geometrical concepts were expanded to apply beyond immediate spatial intuition, toward the comprehension of properties of manifolds beyond those available to immediate spacial intuition. I maintain first that Frege was skeptical that geometrical concepts could be expanded because he thought geometrical concepts were tethered to intuition, and second that he held that geometrical intuition had a primarily heuristic role in the investigation of complex functions.

Frege's scattered remarks in his early works on geometric representations do not directly establish whether he sought geometric representations for heuristic value or held geometric concepts to be essential or foundational for higher mathematics such as complex analysis. Concerning his geometric representations, he writes that "these examples may suffice to show how propositions of plane geometry can be translated into our method of representation, and how relations which are quite nonintuitive, or even in conflict with all our intuitions, are made visible by this method in a very simple way," but this merely describes the outcome of the method not the purpose (Frege, 1984, 38). After investigating some basic examples of complex forms represented by his methods, Frege writes:

> These hints may suffice to give a general idea of the nature of guide surfaces and how they can be investigated. It cannot be denied that these figures are usually so complicated that the aim of making imaginary relationships intuitable can be achieved only to a very imperfect degree, at least without models. This desire can be completely fulfilled only in the case of the simplest and most elementary, but for this reason also most important, relationships. However, in the case of these more complicated figures another use comes to the fore: it is possible for us to take the properties of very simple figures of one or two dimensions and by mere translation use them to investigate the nature of far more complicated figures of a higher dimension (Frege, 1984, 45).

We should not expect that the methodological motives for Frege's later logicism are clearly present in his early geometric works, especially not in the dissertation. One may read this passage as supporting a merely heuristic role for intuitive representation by emphasizing the phrase "to investigate the nature" as indicating a context of discovery. However, I think this forces the text into conformity with an interpretation, where in fact there is genuine ambiguity, and it seems to me that Frege also suggests here that by "mere translation" of properties from simple and elementary cases to higher dimensions we may in fact justify our conclusions.

I think Frege later becomes more clearly skeptical about mathematical arguments that involve generalization from intuitive cases. I gather from Frege's approach to mathematical foundations that his later, considered view would have been closer to the conservative view that geometric reasoning can have only a heuristic role in higher analysis. In his early works it is obscure whether he maintained that his geometric representations have heuristic value only. As I will argue a much clearer position is implicit in his foundational works. However, even in the early works Frege's conservative attitude toward geometry is shown the fact that he restricts his representations to geometric objects residing in two and three dimensional Euclidean space. Indeed, such objects seem to be the only objects Frege recognizes as properly geometric objects. Due to the restrictions Frege's conception of intuition place on the terms, concepts, and relations of geometry it is difficult to see how he could have, upon consideration, assigned intuition any role other than as a fruitful and important guide to discovery. By the writing of *Grundlagen* contexts of discovery and of definition or justification are importantly distinguished as Frege states "Never again let us take a description of the origin of an idea for a definition, or an account of the mental and physical conditions on which we became conscious of a proposition for a proof of it" (Frege, 1980b, I-VIe). While this statement occurs in the context of a characteristically Fregean polemic against arithmetic formalism it can equally well be applied as criticism of any view that would mistake an heuristic role of geometry in investigating higher analysis for an indispensably justificatory role or as providing definitions.[8] On the Fregean view, concrete representations of mathematical forms, whether formal arithmetic or intuitive geometric, never exhaust the content of the logical truths in which they partake.

4. Ghost Points and the Context Principle

A different relationship between Frege's ideas about logic and his work in geometry has been suggested by Mark Wilson. In a series of articles and lectures, Mark Wilson has recently discussed Frege's philosophy of mathematics in light of approaches taken to the introduction of ideal terms in geometry by his predecessors von Staudt and Plücker (Wilson, 1992, 2006, 2010). Von Staudt introduced ideal elements in geometry as logical objects, which we may associate with equivalence classes but which von Staudt (and Frege) may have considered as primitive concept correlates rather than classes. (Wilson, 1992) holds that Frege draws from von Staudt's method to frame his logicist program for arithmetic. (Wilson, 2006) suggests that Frege may have considered Plücker's analytic approach to projective geometry as vindicating certain synthetic geometers' views that ideal elements are in fact geometrical objects, but (Wilson, 2010) provides a "relative logicist" reading of Frege's attitudes toward ideal elements which seems to return to von Staudt's synthetic point of view. The discussion to follow is focused as criticism of (Wilson, 2006). In "Ghost World", Plücker's approach is presented by Wilson as vindicating the view of synthetic projective geometers that ideal elements are geometric objects on par with ordinary points, lines, etc. According to Wilson, the homogeneous coordinate system employed by Plücker (described below) provides a crucial context for understanding Frege's famed context principle, one of three guiding methodological principles of *Die Grundlagen der Arithmetik* which instructs us to "never ask for the meaning of a word in isolation, but only in the context of a proposition" (Frege, 1980b, Xe).

Plücker uses homogeneous coordinates as a framework for analytic projective geometry. In Plücker coordinates a point in the plane is coordinatized by an ordered triple $(x_1 : x_2 : x_3)$ (not all 0), with the convention that $(ax_1 : ax_2 : ax_3)$ are coordi nates for the same point as $(x_1 : x_2 : x_3)$. This gives "homogeneous coordinates" for the projective plane $\mathbb{P}_2$. For appropriate homogeneous coordinates, Cartesian planar coordinates may be recovered as $(\frac{x1}{x3}, \frac{x1}{x3})$. When $x3 = 0$, however, a planar coordinate is not defined. We also have coordinates for lines as ordered triples in the following notation $[X_1 : X_2 : X_3]$, with the condition that the point $(x_1 : x_2 : x_3)$ lies on the line $[X_1 : X_2 : X_3]$ when (and only when) $x_1X_1 + x_2X_2 + x_3X_3 = 0$. In the special case that $x_3 \neq 0$, if we set $x = \frac{x1}{x3}$, $y = \frac{x2}{x3}$, $a = X_1$, $b = X_2$, and $c = X_3$ we have that $(x : y : 1)$ lies on $[a : b : c]$ exactly when $ax+by+c = 0$, recovering the familiar algebraic expression for a line. The use of homogeneous coordinates thereby generalizes familiar formulas from the Cartesian to the projective plane, and does so in a way that establishes a clear notational duality between point and line, insofar as each is represented by an ordered triple.[9]

The notation $(x_1 : x_2 : x_3) : [X_1 : X_2 : X_3]$ expresses the incidence of a point upon a line. Such expressions decompose into a variety of truth-valued functions by replacing constants with variables. The incidence relation itself may be treated as a binary truth-valued function taking point and line coordinates as arguments. In general, one obtains unary functions such as $(x_1 : x_2 : \zeta) : [X_1 : X_2 : X_3]$, $(x_1 : x_2 : x_3) : [\zeta' : X_2 : X_3]$ (etc.), binary functions such as $(x_1 : \zeta' : \zeta'') : [X_1 : X_2 : X_3]$, $(x_1 : x_2 : \zeta') : [X_1 : X_2 : \zeta'']$ (etc.), as well as ternary functions and so forth. This situation recalls a discussion in *Begriffsschrift*, in which Frege notes that different functions may be parsed from the ordinary language sentence 'Cato killed Cato', and Frege may indeed have had in mind, there, a generalization from the specifically arithmetic setting of the multiple parsings of incidence expressions in Plücker coordinates as a basis for his general concept-script (Frege, 1952, §9).

Within this broader context, Wilson argues, Frege held that expressions in homogeneous coordinates for points at infinity obtain objective significance. Consider, first, the restriction to functional expressions $(x_1 : x_2 : x_3) : [\zeta' : \zeta'' : \zeta''']$ such that $(x_1 : x_2 : x_3)$ may be represented by the Cartesian point $(\frac{x1}{x3}, \frac{x2}{x3},)$. The course of values of this function is the "pencil" of lines through $(x_1 : x_2 : x_3)$: i.e., the lines intersecting that point. Now, functional expressions of the form $(x_1 : x_2 : 0) : [\zeta' : \zeta'' : \zeta''']$ are just as well behaved. The course of values of such a function is just a range of parallel lines. With $(x_1 : x_2 : 0) : [\zeta' : \zeta'' : \zeta''']$ as well behaved functions, singular expressions of the form $(x_1 : x_2 : 0)$ may now occur as values of functions of the form $(\zeta' : \zeta'' : \zeta''') : [X_1 : X_2 : X_3]$. "Voila!" writes Wilson, "a suitable 'point at infinity,' prized from the woodwork of bland geometrical fact through no other means beyond the twin processes of explicitly defined conceptual enlargement and a Plückerish reorientation of functional activity" (Wilson, 2006, 12). Elaborating on the significance of this method for Frege's later projects, Wilson writes:

> Here, then, is the original purpose of the context principle as I reconstruct it. Introduce by definitional extension a brace of new names and predicative expressions, in a manner such that a *well behaved range of syntactic surrogates* for the desired imaginary and infinitely distant points appear. Then argue philosophically that, since the newly introduced names behave exactly like those for accepted forms of object, these new specimens should qualify, in the same fashion as Frege's point-pairs, as "just as good," ontologically, as regular points. Here our Plückerish capacity to reverse the predicative activity within a complete thought is integral to this convention. But if such tenets are accepted, we require no axiom V to install our new points —or natural numbers —upon the mathematical stage (Wilson, 2006, 13).

Wilson suggests that homogeneous coordinates provide a framework for understanding the introduction of ideal elements in projective geometry according to which such objects retain retain something of a geometrical character, rather than being replaced by logical constructs, such as class of parallel lines. Plücker, as Wilson tells it, thereby vindicates vaguer, quasi-mystical descriptions of "ghost points" given by earlier projective geometers Steiner and Poncelet by providing a rigorous system of coordinates for analytic projective geometry. Furthermore, Wilson speculates that Frege would have hoped that the philosophical account for introducing ideal elements in geometry could be applied to introduce numbers themselves as ideal elements without appealing to Basic Law V.

The case to be made for Wilson's interpretation is largely circumstantial. To be sure, there are passages in *Grundlagen* suggesting that Frege may have held out vague hope for a general theory of concept correlates that are not extensions of concepts, and Wilson's speculation provides an initially plausible account of what that might amount to. However, it is likely that Frege maintained that concept correlates, whether conceived as extensions or not, are always logical objects stripped of any specific characteristics of the domains in which they arise. Furthermore, there are both textual and systematic reasons for holding that Wilson's account must be considered highly speculative.

First, consider the direct textual evidence that Wilson does give. Frege, to be sure, makes use of Plücker coordinates in some of his mathematical work, such as the short work "Lecture on the Geometry of Pairs of Points in the Plane" (Frege, 1984). Wilson characterizes this lecture in terms of the strategy he attributes to Frege (and Plücker) of using homogeneous coordinates to vindicate the introduction of ideal elements in projective geometry as geometric objects:

> Frege employs the trick of reversing the direction of functional activity in a formula (he utilizes the line equation for a degenerate conic) in order to produce suitable coordinates for *pairs of points* regarded as comprising *single fused entities*. That is, Frege invites us to look upon a regular Euclidean plane and "see" it, not as decomposing into solitary points, but instead as fragmenting into a gaggle of point-partnerships bound irrevocably together over long distances (this remotely paired structure is hard to visualize as it constitutes a four dimensional, nonEuclidean geometry) (Wilson, 2006, 10).

I am not convinced that this correctly characterizes Frege's aims in the lecture. Wilson reads Frege as asking us to "see" the familiar and intuitable pairs of points in an unfamiliar and unintuitive way: viz., as a fused entity in four dimensions. I read Frege's objective in this essay differently. He is asking us to understand an unfamiliar, unintuitive four dimensional "geometry", presented to us analytically, by the familiar intuition of pairs of points in three dimensions. Here is how Frege introduces the lecture:

> One of the most far-reaching advances made by analytic geometry in more recent times is that it regards not only points but also other forms (e.g., straight lines, planes, spheres) as elements of space and determines them by means of coordinates. In this way we arrive at geometries of more than three dimensions without leaving the firm ground of intuition. The geometry of straight lines for example is a four dimensional one, and so is the geometry of spheres. But there is a difference between the two, in that a sphere can always be determined unequivocally by four numbers, whereas it would seem that this is not possible in the case of straight lines. we determine a straight line by an equation between six quantities with a quadratic

equation holding between them. We express this peculiarity of the geometry of straight lines by calling it a second order one, whereas the geometry of spheres is of the first order.

The geometry of pairs of points in the plane, with which we will here be concerned, is four-dimensional and of the third order (Frege, 1984, 103).

To settle between Wilson's and my understanding of the lecture we must understand what exactly is meant by "dimension" and "order" in this context. Frege is speaking of ordinary lines and spherical surfaces in three dimensional Euclidean space. These are, respectively, intuitable as one dimensional and two dimensional objects. To call the geometry of these objects four dimensional is to say something about the analytic expressions of which they are representations, not something about the geometric objects themselves.

Consider the "geometry of straight lines". Homogeneous coordinates for points in three dimensional projective space $\mathbb{P}^3$ have the form $(x_1 : x_2 : x_3 : x_4)$. A line is determined by two distinct points $\mathbf{x} = (x_1 : x_2 : x_3 : x_4)$ and $\mathbf{y}=(y_1 : y_2 : y_3 : y_4)$. If there is an a such that for each xi we have $xi = ayi$ then x and y are equivalent in $\mathbb{P}^3$. That is, **x** and **y** must be linearly independent if they are to determine a line. For linear independence, it is sufficient that $\exists i$, j such that $xiyj - xjyi \neq 0$.[10] Define $p_{ij} = x_i y_j - x_j y_i$. Then $p_{ii} = 0$ and $p_{ij} = -p_{ji}$, so the sixteen values for p_{ij} are completely characterized by the six $(p_{12} : p_{13} : p_{14} : p_{34} : p_{42} : p_{23})$. Since p_{ij} cannot all be zero if **x** and **y** are linearly independent, $(p_{12} : p_{13} : p_{14} : p_{34} : p_{42} : p_{23})$ may be regarded as a homogeneous coordinate in $\mathbb{P}^5$. Furthermore, it can be shown that a different choice of initial points **x** and y will give a result differing only by a scalar. This gives an embedding, known as the Plücker embedding, of the one dimensional linear subspaces of $\mathbb{P}^3$ into $\mathbb{P}^5$. It can also be shown that $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$. This is the quadratic expression holding between the six numbers determining a line that is mentioned by Frege. Consequently, the image of the Plücker embedding is a four dimensional surface in $\mathbb{P}^5$. Frege's lecture applies a generalized technique of Plücker embedding.

My understanding of Frege's point of view is that one success of analytic geometry consists in obtaining the higher dimensional analytic geometries from homogeneous coordinates for perfectly intuitable objects in three dimensional space; as I have mentioned, I think that, following Gauss, Frege held these intuitive representations to have heuristic value. In the case of the Plücker embedding, we have seen that a four dimensional "surface" in $\mathbb{P}^5$ can be interpreted as providing an analytic geometry for lines in the three dimensional space that is putatively familiar to intuition. Rather than, as Wilson would seem to have it, the coordinates themselves providing a mode of representation as syntactic surrogates for geometries that are not immediately intuitable, I understand Frege's perspective to be that by our understanding of higher dimensional analytic geometry as operating on homogeneous coordinates for intuitable geometric objects we "arrive at [analytic] geometries of more than three dimensions without leaving the firm ground of [synthetic] intuition."[11] I don't know what sense Wilson, on his interpretation, can make of "without leaving the firm ground of intuition." Although I think that the nature of the "ground" that Frege supposes to be provided by intuition is ambiguous at this point (epistemic? semantic? psychological?), as I read it, *Lecture on the Geometry of Pairs of Points in the Plane* does not provide textual support for Wilson's speculations. Wilson's suggestion, to be clear, is not that, according to Frege, geometry has no foundation in intuition whatsoever, but that geometric contents given in intuition may recarved in various ways. I am arguing that Frege had a much more conservative view, which is skeptical about the introduction of ideal elements. I don't view Frege as having budged from the initial skepticism about, for example, points at infinity expressed in his thesis, according to which a literal point literally at infinity is nonsense. I think that that the grounding of geometry in intuition that Frege has in mind privileges a particular way of carving its contents and that this restricts the sort of sense we can make from uses of terms like "point at infinity." That is, the relation of parallelism is privileged over intersection at infinity. If a surrogate for the troublesome "point at infinity" is to be found, it should be constructed in the manner of von Staudt, and a set of parallel lines is not a geometrical point.

In fact, Jamie Tappenden has provided direct evidence that Frege was enduringly committed to the von Staudt approach. Aiming to show that Frege was up to speed on current mathematics and that his philosophical concerns were continuous with methodological debates of his mathematical colleagues, Tappenden recounts an exchange between Frege and Pasch:

In response to the strictures on definition Frege repeated to everyone, Pasch in 1903 posed a simple problem that he appears to have thought would display the strictures to be unduly harsh: define, if you will, points at infinity (Frege, 1980a, 105). The next letter from Pasch makes it clear that Frege's (lost) reply was what the reader of (Wilson, 1992) would expect: Frege defines points at infinity via the von Staudt method of taking a class of parallel lines, just as he has always thought points at infinity should be defined (Tappenden, 1995, 344).

In (Wilson, 2006) the interpretation of (Wilson, 1992) is retracted in favor of one closer to the interpretation based on Plücker coordinates that we have been discussing. I contend that Frege would not have understood analytic projective geometry in the way that is suggested in (Wilson, 2006). Providing coordinates as "syntactic surrogates" does not secure a *geometric* interpretation, certainly not one that is rooted in the "firm ground of intuition." Indeed, Frege's consistent insistence on "direction" over "point at infinity" is reason to believe that throughout his career he held to the claim made in his thesis that "point at infinity" is, taken literally, nonsense, and the *Grundlagen* strategy, based on Basic Law V, of defining numbers as class-extensions of concepts is modeled on the von Staudt strategy of replacing the synthetic projective geometers' talk of points at infinity with logical objects.[12]

5. Frege and Göttingen

The most important recent work, to my mind, placing Frege in historical and mathematical context has been done by Jamie Tappenden. Quite rightly, Tappenden argues that in order to understand the nature of Frege's foundational project we must reject two competing myths. First, Frege cannot simply be understood as working on a simple extension the Weierstrassian project of arithmetization of analysis, as he was once characterized by Bertrand Russell. Tappenden endorses Kitcher's argument that this foundational project was already complete by the late 19th century (Tappenden, 2006, 113). Yet, Tappenden does not endorse Kitcher's counter-myth that Frege's motivations should be seen as distinctively philosophical, which is to say extra-mathematical. Tappenden argues, in particular, that Frege's work must be situated historically with respect to the methodological divide between Weierstrass and the Berlin school and Riemann and the Göttingen school with respect to foundational issues distinctive to complex analysis. He warns:

> It is important to get the history right, both because it is rich and interesting in its own right and for more specific metaphilosophical reasons... [Wrong history] nourishes an unduly meager conception of the relations of mathematical and philosophical investigation. If we take Frege to be a paradigmatic analytic philosopher, these presumptions can support a quietism about philosophy that sees it as rightly disengaged from mathematical practice (Tappenden, 2006, 113-114).

Tappenden is entirely correct in much of his detailed criticisms of both myth and counter-myth, which I will not recount here. Furthermore, he is right in insisting that Frege's foundational program and his philosophy of mathematics is best understood in relation to the methodological disputes between the Berlin and Göttingen schools. However, I think that Tappenden's view can be refined by more clearly seeing that Frege's response to the research trends at Göttingen was importantly critical, and seeing this reveals the grain of truth in the otherwise over-simplified

Russellian myth. In this section I argue that Frege's views on the relationship between arithmetic and geometry suggest a critical stance toward the Riemannian program to which Tappenden rightly insists we must see him as reacting.

Indeed, Frege's view of the relationship between arithmetic and geometry can be cast in revealing light by comparison with the influential, among Göttingen mathematicians, points of view of Herman Grassmann and Bernhard Riemann. Notably, Hermann Grassmann generalized the technique of Plücker embedding to linear subspaces of arbitrary rank in vector spaces of arbitrary dimension, a critical step in the "far-reaching advances" in analytic geometry described by Frege. Grassmann's work was not widely noticed until his son, then a student at Göttingen, delivered a manuscript to Alfred Clebsch, who along with Hermann Hankel was quite impressed and subsequently influenced by Grassmann's ideas (Tobies, 1996, 119). Clebsch, along with Hankel, brought Grassmann's work to a wider audience. Clebsch was Frege's dissertation advisor at Göttingen, and Tappenden reports that "Frege was apparently thoroughly familiar" with Hankel's work on complex analysis, which drew inspiration from Grassmann's theory of extension (Tappenden, 1995, 327). The welcome reception of Grassmann's ideas at Göttingen was due to the extent to which he anticipated Riemann's general conception of extended manifold embracing geometrical structure in arbitrary dimensions, though Grassmann's studies were restricted to manifolds with zero curvature.

As Tappenden, along with Wilson, has emphasized, Frege may be seen to have been responding to methodological and philosophical considerations arising from mathematicians such as Plücker, Grassmann, and Riemann. For Frege, however, the general theory of extension proposed by Grassmann and Riemann, and popular at Göttingen, is subsumed under arithmetic, losing any specifically geometric character. This is a crucial point in understanding the nature of Frege's logicism and its basis in his view of the relationship between arithmetic/analysis and geometry. This will provide insight into the order of explanation Frege envisions in mathematics, and helps to clarify the methodological and philosophical presuppositions informing his reaction to Hilbert's *Grundlagen der Geometrie*.

Briefly, I will consider Grassmann's ideas in relation to Frege's. Then I will turn to a more extensive discussion of Riemann's influence. Grassmann's 1844 *Die lineal Ausdehnungslehre* received little immediate mathematical attention in part because of the dense philosophical introduction that prefaced the mathematical development of the theory. Grassmann makes a number of subtle philosophical distinctions that, to my satisfaction, have yet to receive complete and adequate interpretation.[13] The distinctions culminate in a division of mathematics into four subdisciplines: (1) Number Theory, which is the study algebraic discrete forms, (2) Combination Theory, which is the study of combinatorial discrete forms, (3) Intensive Magnitude, which is the study of algebraic continuous form, and (4) Extensive Magnitude, which is the study of combinatorial continuous form. Grassmann's new discipline, linear extension theory, is to correspond to the fourth category. Hence the distinction between intensive and extensive magnitude, resting on the continuous generation of combinatorial and algebraic forms respectively, is of fundamental importance for Grassmann's conception of mathematics. Grassman writes: "The intensive magnitude is thus that arising through generation of equals, the extensive magnitude or *extension* that arising through generation of the different" (Grassmann, 1844, 27). Here is Grassmann's characterization of the distinction:

> It is thus somewhat as if the intensive magnitude is number become fluid, the extensive magnitude combination become fluid. The latter is essentially a proceeding of elements mutually apart, retaining them as being mutually parted. With it, the generating element appears as changing, that is as passing through a variety of states, the collection of these various states forming precisely the domain of the extensive magnitude. With the intensive magnitude, its generation produces a series of states equivalent to itself, whose quantity is precisely the in-

tensive magnitude. The best example we can offer for the extensive magnitude is the line segment (displacement), whose elements proceed essentially apart from each other and thus constitute precisely the line as extension; on the other hand, an example of the intensive magnitude is perhaps a point associated with a specific force, since in this case the elements are not removed, but rather are presented only as an intensity, thus forming a specific order of intensification (Grassmann, 1844, 27).

While for Grassmann the distinction between intensive and extensive magnitude was of basic importance, consider the following passage from Frege's "Review of H. Cohen: *Das Prinzip der Infinitesimal Methode und sein Geschichte*" (1885):

> Now the distinction between intensive and extensive magnitude has no sense in pure arithmetic. Nor does it seem to matter anywhere else in the whole of mathematics. The number 3 for example can serve as the measure of a distance in relation to a unit of length; but it can also serve as the measure of an intensive magnitude, e.g., of the intensity of a light measured in units of brightness. The calculation in both cases proceeds according to exactly the same laws. The number 3 is therefore neither an intensive nor an extensive magnitude but rises above this contrast (Frege, 1984, 110).

Note well that Frege's comments here take a clear position against Grassmann on a crucial point. Although it was the mathematical development of linear extension theory which was surely more influential with Clebsch and Hankel, the force of Frege's repudiation of the importance of the distinction suggests to me that he was rejecting a distinction that has previously been presented to him as important and which he wants to make clear is not, at least not for *pure* arithmetic. Also note that the reason he thinks that the distinction is unimportant is that numbers express a generality that "rises above" the contrast. The distinction between extensive and intensive magnitude may be important for certain applications, but Frege would resist Grassmann's attribution of fundamental importance to it; this further illustrates the fact that one of Frege's central motivations was to properly capture the nature of mathematical generality in his foundational program.

The interest that Grassmann held for Clebsch and Hankel stemmed from their association of his approach to linear extension theory with the Riemannian research program. Riemann makes a distinction, which is loosely analogous to the distinction between extensive and intensive magnitude, between manifolds in which measurement is possible and those in which it is not possible. When measurement is possible there is "a means of using one magnitude as the standard for another" (Riemann, §3). Otherwise, only part/whole comparisons are possible. It is inviting to think of manifolds in which measurement is possible as fields of extensive magnitude and those in which measurement is not possible as intensive, but tightening this analogy involves insuperable difficulties that are not worth pursuing presently. In fact, Riemann speaks of both manifolds of measurable and non-measurable quantities as "extended manifolds" (in Clifford's translation), and importantly treats each geometrically, with the more general geometrical properties applying to the non-measurable case and the measurable case determined by the condition that a choice of standard units determine coordinates. With an eye toward applications in function theory, Riemann sets out to understand the structure of manifolds through the application of generalized geometrical concepts. This direction of generalization is pursued by Hilbert and criticized by Frege, and it seems important to me to stress this point when placing Frege in mathematical context. It seems to me that an unarticulated assumption of the epistemology of Riemannian mathematics is that extended manifolds adequately represent fields of both extensive and intensive quantity, including the mixture of the two, and it seems to me that far too little philosophical attention has been given to the interplay between representation and coneptualization in this tradition.

The lasting mathematical importance of Riemann's work consists precisely in this geometric approach to function theory, which is in contrast with Weierstrass' strictly algebraic approach.[14]

Riemann inaugurated both axiomatic/descriptive approaches to characterizing geometric and topological properties and constructive approaches to generating examples of manifolds. Erhard Scholtz summarizes the importance of Riemann's foundational contribution, as follows:

> Thus Riemann presented an outline of a visionary program of a family of geometrical theories, bound together by the manifold concept, diversified by different conceptual and technical levels like topology, differential geometry, complex geometry, algebraic geometry of manifolds, and overarching the whole range from deep inside conceptual ("pure") mathematics to the cognition of physical space and the nature of the constitution and interaction of matter (Scholz, 1999, 27).

In all of these areas, conceptualization of geometric properties, rendering them independent of immediate experience, is indispensable to the Riemannian method. This is not the place to reprise all aspects of Riemannian mathematics and its subsequent development.[15] The point to emphasize is that Frege's foundational works simply do not address the diverse geometric and topological conceptions the germs of which are found in Riemann's work.

Manifolds, for both Grassmann and Riemann, provide a conceptual generalization of the geometric space of empirical intuition. Grassmann had argued that the empirical character of intuitive space is shown by its restriction to three dimensions, but argued that its form could be subsumed in an abstract theory of forms generated by the motion of a point through arbitrarily many laws of evolution:

> The theory of space may again serve as an example. here the collection of elements of a plane are generated from a single element together with two directions when the generating element progresses by arbitrary amounts in the two directions, and the totality of points (elements) so generated are collected together as a single object. The plane is thus the system of second order; in it there is an infinite set of directions dependent on those two original directions. If a third independent direction is added, then by means of this direction, the whole of infinite space (as the system of third order) is produced. In this example one cannot proceed beyond three independent directions (evolutionary laws); but in pure extension theory their number can be infinitely increased (Grassmann, 1844, 29).

Riemann concurred (coincidentally, having no knowledge of Grassmann), and in considering curvature to be unknown regarded our knowledge of empirical space to be incomplete and expanded the domain of manifolds we may consider abstractly (Riemann, intro). Within the Riemannian approach, the theory of manifolds was fundamental and demanded a conceptual generalization of the geometrical and topological properties of our space of intuition. Grassmann and Riemann, then, have this in common: They each viewed classical geometry as an empirical study of a particular determination of a general form. Grassmann's linear extension theory and Riemann's theory of manifolds seek a conceptualization of the properties of empirical space, rendering them general forms applicable to arbitrary domains. Thus for them, as later for Hilbert but not for Frege, "geometry" came to encompass what Kant, dismissed as mere discursive concepts.

Frege, in contrast, considered geometry to be a specific science, restricted by and essentially dependent on intuition, thus incapable of generalization. His account of generality in mathematics was strictly arithmetic. This point is acknowledged by Tappenden, but its importance bears further emphasis:

> In addition to the well-known concern for rigour, Frege also states that geometric interpretations of the complex numbers 'introduce foreign elements' into analysis. The view that 'analysis is infinitely more general than geometry' was a central theme for Frege (as well as Dedekind) and he took the demonstration of this greater generality to be one of his defining objectives (Tappenden, 2006, 124).

If 'geometry' is understood as the study of the space of intuition and 'analysis' is understood as founded on the general theory of manifolds, Frege's statement does not disagree with the Grassmann/Riemann school of thought. However, I think that it should be emphasized that Frege does disagree with the Grassman/Riemann school of thought. In particular, for Frege the generality of analysis derives from its specifically arithmetic character, there is no foundational role for a theory of manifolds, and the heuristic value of geometric intuition in investigating analytic relations is tempered by a skeptical concern that the uncritical use of intuition may lead us to general conclusions which in fact only apply in a restricted class of cases.

Indeed, I think it is likely that, in criticizing proofs that make intuitive leaps and calling for gapless demonstrations, Frege had in mind such uses of intuition in analysis as Riemann's use of the Dirichlet Principle. While Tappenden discusses the Dirichlet Principle to motivate the conclusion that for those swimming in the Göttingen stream the project of factoring logical/analytical content from geometrical intuition was pressing, I would like to go further to suggest that Frege's position was importantly opposed to specific currents in that stream. To be clear, then, I agree with Tappenden that we must see Frege in the Riemannian context (Tappenden, 2006, 133-136), in particular with respect to the general conception of functions, the concern with accounting for applications, and the methodological problems to which we should see Frege as responding, and also that Frege's writings evince a longstanding hostility toward Weierstrass (Tappenden, 2006, 136137). However, Frege's opposition to a foundational, essential, or ineliminable role for geometrical concepts, and not only intuitions, suggests an important departure from the Riemannian program as it developed into modern topology.

The Dirichlet Principle pertains to finding solutions to the equation $\Delta u - f = 0$ when f is given. This is known as Poisson's equation. The operator Δ is called the Laplacian. When applied to the function $u(x)$ it yields another function $\Delta u(x)$. Roughly, the value $\Delta u(p)$ at a point p tells you how much $u(p)$ differs from the average value of $u(x)$ on surrounding points. To illustrate, in one dimension the condition that $\Delta u(x) = 0$, known as Laplace's equation, characterizes the class of lines because the center of any interval is equal to the average of its end points. In the one dimensional case, then, the value $\Delta u(p)$ at a point p measures how different $u(x)$ from a line at p; it measures how curved $u(x)$ is. Beyond the one dimensional case, however, matters are little more interesting because the degree to which the surface deviates from a line in one direction can be canceled by the degree to which it deviates in another direction. Uniquely, the surfaces of harmonic functions satisfy this constraint and are the solutions to Laplace's equation: i.e., harmonic functions are the solutions to Poisson's equation in the case $f = 0$.

The Dirichlet Principle says that in general a solution to Poisson's equation on a bounded domain with a given boundary condition (i.e., given values that $u(x)$ must take on the boundary) can be found by determining a function that minimizes an integral which can be defined using the given function f . As the story goes, Riemann took for granted that the minimum of this integral exists. Weierstrass showed this to be incorrect in general by providing a counter-example.[16] Jeremy Gray has argued out that the story told in these broad strokes obscures important and interesting details, and that, in particular, Riemann himself considered conditions on which the minimum exists and was therefore less naive than is sometimes supposed; nevertheless, criticisms from Riemann's own student Prym and from Weierstrass and the Berlin circle cast serious doubt on the viability of Riemann's approach (Gray, 1994, 50-55).

Having its origin in applications in potential theory, motivation for the Dirichlet Principle can be gained from geometrical intuition. Roughly speaking, it states that a surface exists with minimal area given boundary conditions. One cannot help but think of Dirichlet's Principle in connection with the following passage from the conclusion of *Die Grundlagen der Arithmetik* that sums up the methodological aim of Frege's logicist project to eliminate gaps in reasoning:

> A single step is often really a whole compendium, equivalent to several simple inferences, and into it there can still creep along with these some element from intuition. In proofs as we know them, progress is by jumps, which is why the variety of types of inference in mathematics appears to be so excessively rich; for the bigger the jump, the more diverse are the combinations it can represent of simple inferences with axioms derived from intuition. Often, nevertheless, the correctness of such a transition is immediately self-evident to us, without our ever becoming conscious of the subordinate steps condensed within it; whereupon, since it does not obviously conform to any of the recognized types of logical inference, we are prepared to accept its selfevidence forthwith as intuitive, and the conclusion itself as a synthetic truth —and this even when it obviously holds good of much more than merely what can be intuited (Frege, 1980b, §90, p. 102e-103e).

Sometimes we generalize illicitly, thinking that we are justified in doing so by a kind of abstraction, from results which have been obtained in a manner that restricts their range of application, and in doing so we assert results that may not hold in full generality. It seems to be quite likely that Frege would have thought of Riemann's approach to analysis in connection with this worry, not only Dirichlet's Principle in particular but the entire geometric approach to functions based on the topological study of Riemann surfaces and the manifold as unifying mathematical object. Furthermore, as the quotation more directly states, in making synthetic leaps when analytic results are possible we fail to gain proofs having maximal generality. The variety of proofs of the Fundamental Theorem of Algebra, including Gauss' own production of both algebraic and geometric proofs, may also have been on Frege's mind, but I suspect also that he is thinking again of Riemann's technique, toward which I interpret his general attitude to have been critical if not outright hostile.[17]

There are, then, two concerns associated with proofs that use synthetic inferences where completely analytic ones are possible. The first I call the restriction concern. The worry is that, aware that we have used synthetic means, we will obtain a restricted result when we feel quite confident that a more general one is true. The second concern is the skeptical worry that we will state a general result when our means of proof only warrant a restricted one. I propose to call this the generalization concern, and as I have been suggesting that the application of Dirichlet's Principle by Riemann to general problems in complex analysis seems to be an example that Frege would have in mind that illustrates this concern.

It is noteworthy that the restriction concern is the focus of the concluding remarks of *Die Grundlagen der Arithmetik*, and I will return to this noteworthy point shortly, but first I want to make clear that the generalization concern is not absent from Frege's mind. In an earlier passage that occurs in the context of a discussion that stems from criticisms of the proposal that arithmetic be founded on a theory of aggregation of abstract units, Frege writes:

> Now, a group of points taken together may perhaps fall into some pattern or other like a constellation or may equally arrange themselves somehow or other on a straight line; and a group of identical segments may lie perhaps with their end-points adjacent so as to combine into a single segment or perhaps at a distance from one another. Patterns produced n this way can be completely different while the number of their elements remains the same. So that here once again we should have different distinct fives, sixes, and so forth. Points of time, again, are separated by time intervals, long or short, equal or unequal. All these are relationships which have absolutely nothing to do with number as such. Pervading them all is an element of a special nature mixed in with number, an element which number in its general form leaves far behind. Even a single moment itself has something *sui generis*, which serves to distinguish it from, say, a point of space, and of which there is no trace in the concept of number (Frege, 1980b, §41 53e-54e).

Here, the generality concern seems to me to be more salient. The passage strongly suggests that we should be concerned that conclusions proceeding from properties particular to space or time will be illicitly imported into conclusions about number. Frege's comment here recalls the passage highlighted above from the review of Cohen's work on infinitesimals, in which Frege rejected the relevance of distinguishing kinds of magnitude, and further distinguishes the Fregean program of explaining the application of arithmetic to geometry by demonstrating the generality of arithmetic through its exclusive reduction to logic, which is to be contrasted with the Grassmann/Riemann approach of generalizing geometry itself.

Frege's resistance to these Göttingen currents provides important context for understanding his later resistance to Hilbert's approach to foundations. Frege's understanding of the deductive structure of mathematical knowledge turns fundamentally on his view of the generality of arithmetic, to be demonstrated by its reduction to logic, and, by contrast, the specificity of geometry. Frege's program of factoring the arithmetical content from the geometrical content in mathematics opposed Riemannian mathematics in the following way. By relying on geometric intuition in analysis, for example in Riemann's illicit use of the Dirichlet principle, we proceed from the specific to the general. The error into which we are lead, Frege believes, arises because the specific domain of geometry instantiates many analytic relations, leading us to mistakenly think that those general relations are constituted in geometry. We can sum up Frege's thought neatly (maybe too neatly but informatively nonetheless) in the following way. Weirstrauss' algebraic method gains in rigor while losing an account of content and applications. Riemann's geometric development gains in content and gives a direct account of a restricted class of applications, but loses rigor and generality. Fregean higher analysis, inspired by Gauss and to be modeled on the Fregean development of arithmetic, resolves the difficulties of each approach. This is a more complete picture than that given in Russell's thumbnail placement of Frege in the Weierstrassian context, but it should be noted that Frege's concern for rigor indeed marks a point of alignment with Weierstrass on a fundamental criticism of the Riemannian geometric program. That Frege departs from Weierstrass on other matters, or aligns with Riemann, is no reason to dismiss the point made by Russell as it pertains specifically to placing Frege's work with respect to the historical development of mathematics through emerging standards of rigor. Indeed, Frege holds precisely that arithmetization is a means of obtaining rigor in general results that is absent from those generalizations when they are supported only by geometrical reasoning.

6. Conclusion

I have tried to make it clear that a central motivation for Frege's foundational program was to establish the greater generality of arithmetic and arithmetized analysis by reducing these disciplines, but not geometry, to logic. Hence it is essential to Frege's program that his is a restricted logicism, and this establishes a hierarchical structure of the mathematical discipline. Frege's hierarchical understanding may be contrasted with a more reciprocal relationship, according to which concepts derived from arithmetic representations are applicable to geometric structures and *vice versa*. This reciprocal image of mathematics is present, I think, in the summary article "Algebraic Geometry" by János Kollar in Timothy Gowers' *Princeton Companion to Mathematics*:

> [In] the method of algebraic geometry: a geometric problem is translated into algebra, where it is readily solvable; conversely, we get insight into algebraic problems by using geometry. It is hard to guess the solutions of systems of polynomial equations, but once a corresponding geometric picture is drawn, we start to have a qualitative understanding of them. The precise quantitative answer is then provided by algebra (Kollar, 2008, 363)

In the practice of mathematics, indeed, we find this reciprocity in abundance and applied in diverse circumstance; consider, for example, the use of polynomials to represent knots algebraically and to identify invariants of knots (Lickorish, 2008, 225). An examination of mathematical practice reveals a complicated web of interrelated qualitative representations and abstract concepts to which the Fregean hierarchical image seems inadequate.

However, one ought not conclude that Frege's philosophical and foundational concerns are divorced from the mathematical developments of his time. In his book *Frege: Philosophy of Mathematics*, Michael Dummett concludes that "strictly speaking, [Frege] did not have a philosophy of mathematics: he never enunciated general principles applicable to all branches of mathematics, or to all branches save geometry; he never claimed to have more than a philosophy of arithmetic" (Dummett, 1991, 292). This is correct to a certain extent because Frege is forthright in acknowledging that his view of basic arithmetic is the most worked out part of his program, but it may tend to mislead the reader into thinking that Frege's program was not motivated by a broader image of the subordination of geometry to arithmetic and analysis. Dummett, however, is correct to maintain that Frege's work well articulates many of the basic problems of the philosophy of mathematics, to which I would add the restriction and generalization problems identified in this paper. As these problems arise directly from contemplation of the controversial aspects of Riemannian mathematics, we can follow Tappenden in insisting against Kitcher's assessment that Frege's program was distant from mathematical practice (Kitcher, 1983, 268-269). In my opinion, it's not that Frege's program was distant from mathematical practice, it's just that the hierarchical image that his program embodied was wrong.

The epistemological problems that I have called the restriction problem and the generalization problem, which I have argued animated Frege's research, demand answers: just not, I think, the answers given by Frege. Unfortunately for progress on these problems, philosophers of mathematics have been diverted by other concerns arising from Frege's program and from the era of so-called "foundational crisis." Nevertheless, I am of the view that mathematical and philosophical research has afforded the tools to articulate answers to the restriction and generalization problems, and to thereby gain clarity regarding our other philosophical problems, based on a local conception of the constitution of mathematical objects woven together by relations of structural similarity that have been analyzed by means of logical abstraction. It suffices to conclude this paper with that programmatic suggestion.

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Notes

[1] Much has been made of the influence on Frege of Lotze by Sluga in a series of publications beginning in the late 1970s. Sluga's interpretation was opposed by Michael Dummett. I do not wish to enter into this broader debate, and restrict attention to Lotze's views on non-Euclidean geometry and on intuition and their possible influence on Frege.

[2] Indeed, Lotze was read by Russell and is extensively discussed in Russell's early writings on geometry, which adapted a neo-Kantian framework to tolerance of spaces of constant curvature. The influence of the structuralist strains in Lotze to Russell's later scientific epistemology has not been, to my knowledge, explored.

[3] [3]An anonymous commentator on an earlier version of this paper points out that more original scholarship on Lotze would be nice. In particular, it would be nice to say more about how Lotze responds to concerns about the communicability of subjective content. I agree; though perhaps this task is best left to one who can read German. For my purposes the broad outline provided by Torretti ought to suffice.

[4] To assess whether Frege's understanding of Kant on intuition is correct would require a long excursion into Kant.

[5] Quoted from Davis ”Systems of Conics in Kepler's Work” (Davis, 1975)

[6] This is, of course, to elide the difficulty whether the relation of parallelism, in its Euclidean sense, is directly intuited, which however seems simply to have been assumed by Frege.

[7] I take sets of whatever kind of object to be non-spatial. There is no question of identifying the direction instead with a mereological sum of lines, for this would be the entire plane and equivalent for all directions.

[8] Indeed, Frege's critique of formalism, concerning the adequacy of symbolic representation as a guide to arithmetic truth, suggests an analogous critique of geometric representation as a guide to truths in higher analysis. Moreover, when he criticizes the idea of founding arithmetic on a theory of sets of abstract units, this criticism of the abstract theory of discrete manifolds as a basis for arithmetic, again suggests an analogous critique of the abstract theory of continuous manifolds as a guide to truth in higher analysis.

[9] (Smart, 1998) provides the textbook presentation of analytic projective geometry from which I first learned this material.

[10] I.e., that $\exists M$ such $\det(M) \neq 0$ where M is a 2×2 submatirix of the 4×2 matrix with columns x and y.

[11] I have added “[analytic]” and “[synthetic]” to clarify the sense I make of Frege's point of view.

[12] In another paper (Wilson, 2010), Wilson argues that the method of introducing terms for points at infinity in Plücker coordinates provides an initial model for Frege's project of introducing numbers as logical objects while returning to the thesis of (Wilson, 1992) that the ideal elements introduced as extensions of geometry are to be understood as logical rather than geometrical objects. This strikes me as more plausible, and although, as Wilson notes, the textual evidence is somewhat thin, this interpretation does make sense of some of the vaguely suggestive remarks in *Grundlagen* and the appearance, to some readers, of an abrupt shift of strategy in adopting Basic Law V. Yet, because it is important to characterizing Frege's understanding of the relationship between arithmetic and geometry and the role of logicism in characterizing and explaining this relationship, I think it is important to resist the suggestion in (Wilson, 2006) that merely by introducing suitably well-behaved “syntactic surrogates” can we vindicate talk of unintuitable geometric objects.

[13] Though, see (Cantu, 2011) and (Lewis, 2004), especially Cantu's important contribution to which I would offer only modest criticism.

[14] Bottazzini quotes Weierstrass in correspondence with Schwarz "The more I think about the principles of function theory —and I do so incessantly, the more I am convinced that this must be built on the foundation of algebraic truths" (Bottazzini, 1994, 428).

[15] For this, see (Scholz, 1999).

[16] Ultimately, Hilbert determined the conditions under which the minimum exists and the principle is applicable.

[17] An anonymous referee suggested mentioning the work of Dedekind and Weber toward "logicizing the Riemann-Roch theorem" in connection with this point. Dedekind and Weber proceeded by eliminating appeal to geometric intuitions of continuity and expandability, and I think that, based on what I have said, Frege would have approved of this approach. The alternative approach, which I think Frege did not fully appreciate, would have been to provide a logical analysis liscensing the passage from intuitions to concepts and ideas (see the Kantian epigraph of Hilbert's *Grundlagen der Geometrie*). Dedekind and Weber come near this general standpoint with their algebraic concept of fields. Hilbert's achievement was to extend this perspective to geometry, and I think that this was done in the service of rigorizing, yet more closely following, Riemann's reasoning (as I plan to argue in a subsequent paper) rather than taking an eliminative approach.

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