Williamson's Many Necessary Existents*

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This note is to show that a well-known point about David Lewis's (1986) modal realism applies to Timothy Williamson's (1998; 2002) theory of necessary existents as well.[1] Each theory, together with certain "recombination" principles, generates individuals too numerous to form a set.

The simplest version of the argument comes from Daniel Nolan (1996).[2] Assume the following recombination principle: *for each cardinal number, ν, it's possible that there exist ν nonsets*. Then given Lewis's modal realism it follows that there can be no set of all (that is, Absolutely All) the nonsets. For suppose for reductio that there were such a set, A; let ν be A's cardinality; and let μ be any cardinal number larger than ν. By the recombination principle, it's possible that there exist μ nonsets; by modal realism, there exists a possible world containing, as parts, μ nonsets; each of these nonsets is a member of A; so A's cardinality cannot have been ν.

On some conceptions of what sets are, Lewis could simply accept this conclusion. But given the iterative conception of set,[3] it seems that there *must* exist a set of all nonsets.[4] According to the iterative conception, sets are "built up" in a series of "stages". At the first stage a set is "formed" whose members are all and only the nonsets. At subsequent stages, sets are formed whose members are sets from earlier stages. The sets, on this conception, are all and only those that are formed at some stage or other. Since a set of all the nonsets is formed at the very first stage, such a set must exist.

My main concern here is not to defend this argument, only to show how an analogous argument against Williamson may be constructed. Still, it's worth noting that the recombination principle on which the argument is premised has a solid intuitive basis. Under the broad sense of 'possible' at issue, there should be no arbitrary limits to what's possible; but any limit to how many nonsets are possible would be arbitrary. It would be strange to say that there could

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[1] Parsons (1994); Linsky and Zalta (1994, 1996) defend somewhat similar views.

[2] The original idea, in somewhat different form, is from Forrest and Armstrong (1984), and is discussed by Lewis (1986, section 2.2).

[3] See Boolos 1971.

[4] See Lewis 1986, p. 104. Nolan rejects this point.

be up to 30,000 electrons, but no more. Or that there could be a countable infinity of electrons, but not an uncountable infinity. Or continuum-many, but not some higher infinite number. And so on. Williamson is of course free to put the brakes on this reasoning in some way. And unlike Lewis, who explicitly endorses (a restricted version of) recombination (1986, section 1.8), Williamson has no particular commitment to recombination. But nor does he have any particular reason to reject it (beyond the present argument); and recombination has general appeal.

Now for the analogous argument that threatens Williamson. Very roughly put: since Williamson accepts the Barcan formula, he thinks that everything that can exist, does exist. So any nonset that can exist, does exist. So if there is no limit to how many nonsets can exist, then there is no limit to how many do in fact exist.

The existence of the analogous argument is hardly surprising since both Lewis and Williamson embrace the crucial assumption that "everything that can exist, does exist". But Nolan's original argument was formulated extensionally, from an "amodal" perspective on Lewis's pluriverse, so to speak. This is indeed the appropriate way to formulate the argument in Lewis's case, given Lewis's reduction of modal facts to extensional facts about the pluriverse; but Williamson's opposition to this reduction is well-known. The analogous argument in Williamson's case must be formulated in terms congenial to Williamson: namely, in a modal language. And it must be done more rigorously than in the preceding paragraph. In fact it's not completely trivial to do this; and as we will see, certain auxiliary modal premises will be required. So I think it's worth working out this argument's details. In what follows I'll do this in two ways.

The first employs an infinitary modal language, specifically, one that allows both arbitrarily long infinitary conjunctions and quantification of arbitrarily many variables.[5] Notation: where X is a (perhaps infinite) set of variables and ϕ is a (perhaps infinitary) formula, let $\sum X \phi$ be the existential quantification of ϕ with respect to the variables in X (thus, $\sum X \phi$ is true iff ϕ is true for some values of the variables in X.) And, where Γ is a (perhaps infinite) set of formulas, let $\bigwedge \Gamma$ be the conjunction of the formulas in Γ (thus, $\bigwedge \Gamma$ is true iff every member of Γ is true.)

In this language we may formulate the recombination principle as the following schema, where 'Sx' means "x is a set" and the schematic variable X may be replaced by any set of variables (of any cardinality):

[5] See Dickmann 1975 on infinitary languages.

Recombination $\Diamond\sum X \bigwedge\{\ulcorner{\sim}Sx\wedge x{\neq}y\urcorner : x,y \text{ distinct variables in } X\}$

An instance of the Recombination schema, where ν is the cardinality of X, says that there could have existed ν distinct nonsets. (Informally, it can be thought of as looking like this:

$$\Diamond\underbrace{\exists x_1\exists x_2\ldots}_{\nu\text{ quantifiers}}({\sim}Sx_1\wedge x_1{\neq}x_2\wedge x_1{\neq}x_3\wedge\cdots\wedge{\sim}Sx_2\wedge x_2{\neq}x_3\wedge x_2{\neq}x_4\wedge\cdots)$$

although this representation is a bit misleading since ν need not be countable.) There is such an instance for each infinite cardinality ν; therefore, the schema's instances collectively have the same upshot as the original recombination principle.

The core of Williamson's theory of necessary existents is his acceptance of the Barcan schema $\forall x\Box\phi\rightarrow\Box\forall x\phi$, or, equivalently, $\Diamond\exists x\phi\rightarrow\exists x\Diamond\phi$. The latter has an infinitary analog, to which, I take it, Williamson is committed:

Infinitary Barcan schema $\Diamond\sum X\phi\rightarrow\sum X\Diamond\phi$

(where X may be replaced by any set of variables and ϕ by any—perhaps infinitary—formula.)

The argument from the infinitary Barcan and Recombination schemas to the conclusion that there can be no set of all nonsets will require some auxiliary assumptions. While they are not strictly required by his theory of necessary existents, Williamson would, I take it, be happy to assume them.[6] First, the logic assumed by the argument will be: S5 propositional modal logic, plus uncontroversial modal predicate logic, plus the converse of the finitary Barcan schema: $\Box\forall x\phi\rightarrow\forall x\Box\phi$. More carefully, since the familiar S5 and modal predicate logic systems are finitary: the argument will employ the obvious infinitary analogs of inferences endorsed by these logics, in addition to the more familiar finitary inferences. Second, the argument will assume the following premises:[7]

[6]See Williamson 1996, and Williamson 1998: sections 3 and 7 on modal logic and p. 266 on the essentiality of sethood. I intend a purely modal reading of 'essentially'. Parsons (1994, 10, and note 5) contemplates the rejection of the essentiality of sethood.

[7]Tim Williamson pointed out that the argument could use the weaker propositional modal system B rather than S5 if the premises were strengthened to: the necessitations of the essentiality of sethood and the necessity of identity ($\Box\Box\forall x(Sx\rightarrow\Box Sx)$ and $\Box\Box\forall x\forall y(x{=}y\rightarrow\Box x{=}y)$). Instead of the S5 theorem schema NE, the argument would use the B theorem schema $\Box\Box(\chi\rightarrow\Box\chi)\rightarrow\Box({\sim}\chi\rightarrow\Box{\sim}\chi)$ in the derivation of the essentiality of nonsethood and the necessity of distinctness.

Essentiality of sethood $\Box\forall x(Sx\rightarrow\Box Sx)$

Necessity of identity $\Box\forall x\forall y(x=y\rightarrow\Box x=y)$

A preliminary step is to move from the essentiality of sethood and the necessity of identity to:

Essentiality of nonsethood $\Box\forall x(\sim Sx\rightarrow\Box\sim Sx)$

Necessity of distinctness $\Box\forall x\forall y(x\neq y\rightarrow\Box x\neq y)$

We do this as follows. Every instance of the following is an S5 theorem:

NE $\Box(\chi\rightarrow\Box\chi)\rightarrow\Box(\sim\chi\rightarrow\Box\sim\chi)$

So in particular, the following two open formulas are theorems:

a) $\Box(Sx\rightarrow\Box Sx)\rightarrow\Box(\sim Sx\rightarrow\Box\sim Sx)$
b) $\Box(x=y\rightarrow\Box x=y)\rightarrow\Box(x\neq y\rightarrow\Box x\neq y)$

But whenever $\Box\phi\rightarrow\Box\psi$ is a theorem, we may move to $\Box\forall x\phi\rightarrow\Box\forall x\psi$ as follows:

1.	$\Box\phi\rightarrow\Box\psi$	
2.	$\forall x(\Box\phi\rightarrow\Box\psi)$	1, universal generalization
3.	$\forall x\Box\phi\rightarrow\forall x\Box\psi$	2, predicate logic
4.	$\forall x\Box\phi\rightarrow\Box\forall x\psi$	3, Barcan schema
5.	$\Box\forall x\phi\rightarrow\Box\forall x\psi$	4, converse Barcan schema

Thus from a) and b) we may move to the essentiality of nonsethood and the necessity of distinctness.

The main argument now runs as follows. Suppose for reductio that there is a set of all nonsets; let ν be its cardinality. Let X be a set of variables with cardinality greater than ν. An instance of Recombination is then:

(1) $\Diamond\sum X\bigwedge\{\ulcorner\sim Sx\wedge x\neq y\urcorner : x,y \text{ distinct variables in } X\}$

By the essentiality of nonsethood and the necessity of distinctness, (1) implies:

(2) $\Diamond\sum X\bigwedge\{\ulcorner\Box\sim Sx\wedge\Box x\neq y\urcorner : x,y \text{ distinct variables in } X\}$

(the reasoning here is the infinitary analog of the inference from $\Box\forall x(\phi_1\rightarrow\psi_1)$ and $\Box\forall x(\phi_2\rightarrow\psi_2)$ and $\Diamond\exists x(\phi_1\wedge\phi_2)$ to $\Diamond\exists x(\psi_1\wedge\psi_2)$, which is valid in any reasonable modal predicate logic.) From (2), via an instance of the infinitary Barcan schema, we obtain:

$$(3)\ \sum X \Diamond \bigwedge\{\ulcorner\Box{\sim}Sx\wedge\Box x\neq y\urcorner : x, y \text{ distinct variables in } X\}$$

By more basic modal predicate logic (the infinitary analog of $\exists x\Diamond(\phi\wedge\psi)\vdash\exists x(\Diamond\phi\wedge\Diamond\psi)$) we get:

$$(4)\ \sum X \bigwedge\{\ulcorner\Diamond\Box{\sim}Sx\wedge\Diamond\Box x\neq y\urcorner : x, y \text{ distinct variables in } X\}$$

And, finally, by the infinitary analog of $\exists x(\Diamond\Box\phi\wedge\Diamond\Box\psi)\vdash\exists x(\phi\wedge\psi)$ (which in turn relies on the S5-valid $\Diamond\Box\chi\vdash\chi$), (4) implies:

$$(5)\ \sum X \bigwedge\{\ulcorner{\sim}Sx\wedge x\neq y\urcorner : x, y \text{ distinct variables in } X\}$$

Since X contains more than ν variables, the truth of (5) implies that there are more than ν nonsets, which contradicts our reductio assumption that there was a set of all nonsets with cardinality ν.

A second way to run the argument dispenses with the infinitary language but requires a further—though quite natural—auxiliary premise about set membership:

∈-rigidity $\Box\forall x\forall y(x\in y\rightarrow\Box x\in y)$

Let L be a first-order modal language, with $\sim$, $\rightarrow$, $\Box$, $\forall$, and $=$ the primitive logical constants, and whose nonlogical vocabulary consists solely of the predicates $\in$ and S, for set membership and sethood, respectively. We first establish the following by induction:[8]

General Rigidity For any formula, ϕ, of L and any assignment, g, to L's variables, $\Box(\phi\rightarrow\Box\phi)$ is true under g

[8] If we were using B propositional modal logic rather than S5 (see note 7), in order to make the inductive proof work we would need to formulate General Rigidity as the claim that $\Box^n(\phi\rightarrow\Box\phi)$ is true for each positive integer n, where $\Box^n$ is $\underbrace{\Box\ldots\Box}_{n\ \Box\text{s}}$. Accordingly, the argument's premises would need to be strengthened to include, for each n, $\Box^n$(the essentiality of sethood) and $\Box^n$(the necessity of distinctness). Thanks again to Tim Williamson.

The base case—that the assertion holds for atomic formulas—follows immediately from the necessity of identity, the essentiality of sethood, and $\in$-rigidity. Assuming next that the assertion holds for ϕ and ψ (the inductive hypothesis, "ih"), it must be shown that $\Box(\sim\phi\rightarrow\Box\sim\phi)$, $\Box((\phi\rightarrow\psi)\rightarrow\Box(\phi\rightarrow\psi))$, $\Box(\Box\phi\rightarrow\Box\Box\phi)$, and $\Box(\forall x\phi\rightarrow\Box\forall x\phi)$ are all true under any assignment g:

- Given ih, $\Box(\phi\rightarrow\Box\phi)$ is true under g; this formula implies $\Box(\sim\phi\rightarrow\Box\sim\phi)$ (by NE); the latter is therefore true under g.[9]

- Given ih, $\Box(\phi\rightarrow\Box\phi)$ and $\Box(\psi\rightarrow\Box\psi)$ are true under g. These two formulas imply $\Box((\phi\rightarrow\psi)\rightarrow\Box(\phi\rightarrow\psi))$:

1.	$\Box(\phi\rightarrow\Box\phi)$	
2.	$\Box(\psi\rightarrow\Box\psi)$	
3.	$\Box(\sim\phi\rightarrow\Box\sim\phi)$	1, NE
4.	$\Box((\sim\phi\vee\psi)\rightarrow(\Box\sim\phi\vee\Box\psi))$	2,3
5.	$\Box((\phi\rightarrow\psi)\rightarrow\Box(\phi\rightarrow\psi))$	4

  The latter is therefore also true under g.

- $\Box(\Box\phi\rightarrow\Box\Box\phi)$ is S5-valid, and so is true under g.[10]

- Let o be any object. By ih we know that $\Box(\phi\rightarrow\Box\phi)$ is true under g_x^o (the assignment just like g except that it assigns o to x). o was arbitrarily chosen, so $\Box(\phi\rightarrow\Box\phi)$ is true under g_x^o for every o. $\forall x\Box(\phi\rightarrow\Box\phi)$ is therefore true under g. But $\forall x\Box(\phi\rightarrow\Box\phi)$ implies $\Box(\forall x\phi\rightarrow\Box\forall x\phi)$:

1.	$\forall x\Box(\phi\rightarrow\Box\phi)$	
2.	$\Box\forall x(\phi\rightarrow\Box\phi)$	1, Barcan schema
3.	$\Box(\forall x\phi\rightarrow\forall x\Box\phi)$	2
4.	$\Box(\forall x\phi\rightarrow\Box\forall x\phi)$	3, Barcan schema

  So, since the former is true under g, the latter is as well.

[9] I assume throughout that consequence in the logic being assumed preserves truth under any variable assignment.

[10] If we were doing this in B (notes 7, 8), what we would need to show here is $\Box^n(\Box\phi\rightarrow\Box\Box\phi)$. But this follows in any normal modal logic from $\Box\Box^n(\phi\rightarrow\Box\phi)$, which the ih would give us.

Next, let the variables A and B range over sets;[11] let NS(α) abbreviate $\forall z(z \in \alpha \rightarrow \sim Sz)$ ("α is a set of (only) nonsets"); and let $\alpha \prec \beta$ mean "the cardinality of α is lower than the cardinality of β" (this can be defined in the language of set theory and hence in L). The recombination principle in the present context can be stated as:

(6) $\forall A \Diamond \exists B(\mathrm{NS}(B) \wedge A \prec B)$

Only assuming $\in$-rigidity is this an appropriate statement of the recombination principle. (6) says that for every actual set, A, no matter what its cardinality, it would have been possible for there to exist a set B of nonsets with a greater cardinality than set A would *then* have had—B is not guaranteed to exceed A's actual cardinality if A might have had different members from what it actually has. Without $\in$-rigidity (or a related assumption, such as the assumption of $\in$-rigidity for pure sets), the recombination principle might not be appropriately stateable in L.

Suppose, then, that (6) is true, and assume for reductio that there is a set of all the nonsets. So, letting ANS(α) abbreviate $\forall z(\sim Sz \rightarrow z \in \alpha)$ ("α is a set containing *all* the nonsets"), the following formula is true:

(7) $\exists A\, \mathrm{ANS}(A)$

Since (6) and (7) imply the following, it too is true:

(8) $\exists A[\mathrm{ANS}(A) \wedge \Diamond \exists B(\mathrm{NS}(B) \wedge A \prec B)]$

Since (8) is true, it is true under some assignment to the variables, g. So, for some o, the following is true under g^o_A:

(9) $\mathrm{ANS}(A) \wedge \Diamond \exists B(\mathrm{NS}(B) \wedge A \prec B)$

By general rigidity, the following is true under g^o_A:

(10) $\Box[\exists B(\mathrm{NS}(B) \wedge A \prec B) \rightarrow \Box \exists B(\mathrm{NS}(B) \wedge A \prec B)]$

(9) and (10) imply $\mathrm{ANS}(A) \wedge \Diamond \Box \exists B(\mathrm{NS}(B) \wedge A \prec B)$ by basic quantified modal logic, which in turn implies (given S5):

[11] Officially: let "$\forall A$" abbreviate "$\forall x(Sx \rightarrow$", let "$\exists B$" abbreviate "$\exists y(Sy \wedge$", etc.

(11) $\text{ANS}(A) \wedge \exists B(\text{NS}(B) \wedge A \prec B)$

So, since (9) and (10) are true under g^o_A, (11) is as well. The following is therefore true under g, and so is true simpliciter:

(12) $\exists A[\text{ANS}(A) \wedge \exists B(\text{NS}(B) \wedge A \prec B)]$

But (12) is a falsehood of set theory: if A contains all nonsets and B contains only nonsets, $B \subseteq A$; but then B cannot have a greater cardinality than A. The reductio assumption is therefore false.

Each version of the argument concludes that there is no set of all the nonsets. Like Lewis, Williamson is free to accept this conclusion, provided he rejects the iterative conception of set. Indeed, he might argue that the iterative conception is in trouble for independent reasons.[12] He might, for instance, argue that there exists a distinct property for each thing, x, even when x is a set (the property of being identical to x, perhaps). Assuming that properties are not sets, it then follows that there cannot be a set of all the nonsets, in apparent violation of the iterative conception. But suppose the iterative conception's first stage is modified to posit only the existence of a set of all those things that are *neither sets nor properties*. Better, suppose that we can identify a category of "iterative" entities, entities that are "iteratively constructed" alongside the sets: properties, propositions, and so on; and let the iterative conception's first stage be modified to posit only the existence of a set of all the *noniterative* things. Provided we can make as much sense out of the iterative construction of properties (propositions, etc.) as we can make of the iterative construction of sets, the modification is not ad hoc, for the idea behind the iterative conception of set was that, when it comes to iterative entities in general, such entities exist only if they show up at some stage in the iterative construction. The argument can then be recast with 'iterative' replacing 'set' throughout, and it looks to retain its strength. In particular, the essentiality of sethood becomes the claim that any iterative entity is necessarily iterative, which looks strong: sets are essentially sets, properties are essentially properties, propositions are essentially propositions, and so on.[13]

[12] Thanks to Tim Williamson here. Also, Stewart Shapiro (2003, section 3) has argued that Williamson's commitments on other fronts (absolutely unrestricted quantification and indefinite extensibility) prohibit the existence of a set of all nonsets.

[13] I should stress that I have not tried to argue in favor of the iterative conception. For all I have said, Williamson could simply reject it, or could insist on some form that does not require the existence of a set of nonsets (or noniterative entities). My aim has simply been to extend the recombination argument to Williamson's theory, not to defend that argument.

Another way out would be to reject the essentiality of sethood. And after all, Williamson's theory of necessary existents leads him to reject other essentialist principles. On his view, although human beings exist necessarily, they are not essentially human since each human being could have lacked spatiotemporal location. Similarly, cats could have been nonspatiotemporal and hence could have failed to be cats, chairs could have been nonspatiotemporal and hence could have failed to be chairs, and so on. So on Williamson's view, for sortal predicates F, Fs are generally not essentially Fs. However, these anti-essentialist claims of Williamson's all derive from a common source: his claim that spatiotemporal objects could have failed to be spatiotemporal. And this common source does not stand in the way of the claim that sets are essentially sets (or the claim that properties are essentially properties, etc.) So nothing in Williamson's view stands in the way of his accepting this piece of orthodoxy.

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