Dedekind's Structuralism:
Creating Concepts and Deriving Theorems*

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Abstract

Dedekind's structuralism is a crucial source for the *structuralism of mathematical practice*—with its focus on abstract concepts like groups and fields. It plays an equally central role for the *structuralism of philosophical analysis*—with its focus on particular mathematical objects like natural and real numbers. Tensions between these structuralisms are palpable in Dedekind's work, but are resolved in his essay *Was sind und was sollen die Zahlen?* In a radical shift, Dedekind extends his mathematical approach to "the" natural numbers. He creates the abstract concept of a *simply infinite system*, proves the existence of a "model", insists on the stepwise derivation of theorems, and defines structure-preserving mappings between different systems that fall under the abstract concept. Crucial parts of these considerations were added, however, only to the penultimate manuscript, for example, the very concept of a simply infinite system. The methodological consequences of this radical shift are elucidated by an analysis of Dedekind's metamathematics. Our analysis provides a deeper understanding of the essay and, in addition, illuminates its impact on the evolution of the axiomatic method and of "semantics" before Tarski. This understanding allows us to make connections to contemporary issues in the philosophy of mathematics and science.

Introduction

Dedekind's structuralism has strongly influenced both mathematical practice and philosophical analysis that is concerned with the foundations of mathematics. On the mathematical side, Dedekind was keenly aware of the importance of creating new concepts in mathematics. So he writes in the Preface to his famous essay *Was sind und was sollen die Zahlen?*, ". . . the greatest and most fruitful advances in mathematics and other sciences have invariably been made by the *creation and introduction of new concepts*, rendered necessary by the frequent recurrence of complex phenomena which could be mastered by the old notions only with difficulty" (*WZ*, 792, emphasis added).[1] Dedekind's work on algebraic number theory is characterized by its focus on abstract concepts together with structure-preserving mappings; it impacted modern algebra directly through David Hilbert and Emmy Noether. This development found its expression in van der Waerden's influential monograph *Moderne Algebra*, the first volume of which appeared in 1930 and the second in 1931. More than thirty years later van der Waerden wrote in the Preface to the 1964 edition of Dedekind's *Algebraische Zahlentheorie*:

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*This paper had been in the works for a very long time; it was started in the summer of 2010 and completed in April of 2015. It is most appropriately dedicated to Bill Tait whose perspective on Dedekind influenced us deeply. For WS, Bill's proof theoretic work has been inspiring and his philosophical reflections illuminating; I also treasure the enduring friendship we have formed.

[1] Dedekind's *Was sind und was sollen die Zahlen?* will be referred to as *WZ* throughout this paper, and his earlier essay *Stetigkeit und irrationale Zahlen* will be referred to as *SZ*.

> Evariste Galois and Richard Dedekind are the ones who gave to modern algebra its structure. The supporting skeleton of this structure derives from them.[2]

Moreover, the general methodological aspects that constitute the "supporting skeleton of this structure" have shaped not just modern algebra, but modern mathematics as exemplified in Bourbaki's work.

On the philosophical side, contemporary discussions on structuralism often consider Dedekind's work, and he is taken to represent a variety of different, sometimes conflicting, positions. Dedekind's comments about the free creation of new mathematical objects, as found in his *SZ*, *WZ*, and letters to Weber, feature prominently in such discussions. Some may seek to downplay these remarks, but a number of philosophers, including William Tait (Tait, 1996), Erich Reck (Reck, 2003) and Audrey Yap (Yap, 2009), have argued that Dedekind's comments about creation should not be dismissed. Instead, they maintain that creating (systems of) particular mathematical objects is a crucial feature of Dedekind's structuralism. Notice, then, that Dedekind's focus on the creation of concepts is of central importance for mathematical practice, while his emphasis on the creation of objects has been significant within philosophy. This split in emphasis between concepts, on the one hand, and objects, on the other, is reflected in Dedekind's foundational work. In the penultimate manuscript of *WZ*, Dedekind explicitly created a system of new mathematical objects as *the* abstract natural numbers. This step is no longer taken in the published version. In a *radical shift*, Dedekind introduces instead the concept of a simply infinite system and utilizes metamathematical considerations to ensure that "the definition of the *concept of numbers* given in (73) is completely justified" (*WZ*, 823, emphasis added).

This shift is documented in the Appendix below: it is in the transition from the penultimate version of *WZ* to the publication that the concept of a simply infinite system is introduced for the first time and that the concern with the logical existence of such a system is raised, also for the first time. Taking seriously this radical shift, we interpret Dedekind's new approach as an *extension of his abstract mathematical ways* to "the" natural numbers. We describe his position post-radical-shift in section A, and, in addition, sharpen his views concerning the creation of abstract concepts. Then, in section B, we elaborate the role of structure-preserving mappings between systems falling under such concepts, and explain the character and methodological significance of his metamathematical work. The latter work is primarily, but not exclusively, concerned with categoricity. Relying on a quasi-formal notion of derivation that takes as its starting points the *characteristic conditions* (Merkmale) of abstract concepts, Dedekind proves in *WZ* that all the statements derived from the conditions for the concept of a simply infinite system hold in all particular simply infinite systems. Here we have an incipient model theory (without a Tarskian truth definition) joined with proof theory (without a precise notion of formal derivation).

Equipped with this deeper understanding of Dedekind's axiomatic standpoint, we can compare *WZ* with the earlier essay *SZ*: the treatment of real numbers is already structured in such a way that it can be easily and naturally expanded to parallel the treatment of natural numbers in *WZ*; see section C.1 below. Our interpretation allows us to reemphasize that Hilbert's *axiomatic method*— as formulated and practiced in (Hilbert 1899) and (Hilbert 1900)—is fully in Dedekind's mold. Not surprisingly, as Zermelo was deeply influenced by Hilbert, we can highlight striking parallels between Dedekind's work in *WZ* and Zermelo's formulation of set theory in 1908 as well as its metamathematical investigation in 1930. In the *Concluding Remarks* we point out deep connections with contemporary issues in philosophy of science that are rooted in the co-evolution of mathematics and science in the 19th century, with corresponding changes towards a more scientific philosophy. The underlying intellectual perspective is characteristic of the Göttingen mathematicians: *vide* Gauss,

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[2]Here is the German text: "Evariste Galois und Richard Dedekind sind es, die der modernen Algebra ihre Struktur gegeben haben. Das tragende Skelett dieser Struktur stammt von ihnen." Starting with this remark of van der Waerden's, Mehrtens gives in his (1979) an informative overview of Dedekind's practice of creating mathematical concepts. Van der Waerden's and our perspective on Dedekind's role in the development of modern structural algebra is not shared by Leo Corry in his (2004a).

Dirichlet, and Riemann, with all of whom Dedekind had close relations and in whose tradition Hilbert saw himself.[3]

A. Simply Infinite System: *Concept of Numbers*

In his letter to Keferstein, Dedekind made the following remarkable comment on the genesis of his essay: "... it is a synthesis constructed after protracted labor, based upon a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration." (Dedekind, 1890, 99). The steps of analysis address the questions that his essay *WZ* is supposed to answer:

> What are the mutually independent fundamental properties of the sequence N, that is, those properties that are not derivable from one another but from which all others follow? And how should we divest these properties of their specifically arithmetic character so that they are subsumed under more general concepts and under activities of the understanding, *without* which no thinking is possible at all, but *with* which a foundation is provided for the reliability and completeness of proofs, as well as for the formation of consistent definitions of concepts?[4]

We will trace Dedekind's analytic steps and thus obtain a clearer view of how the fundamental properties of N are "subsumed under more general concepts and under activities of the understanding, without which no thinking is possible". After all, these fundamental properties will be the basis for the systematic, stepwise development of number theory or, what Dedekind calls, its *synthetischer Aufbau*.

A.1. Logical Framework

The questions mentioned in the above quotation clearly express Dedekind's goal of isolating properties that serve to define a higher-order concept of *natural number*. The crucial properties are combined into the concept of a *simply infinite system*. In order to build up to this concept, Dedekind begins the essay by introducing the more general notions of *thing* and *system*. Of the former he writes, "In what follows I understand by *thing* ⟦*Ding*⟧ every object of our thought" (*WZ*, 796), and of the latter he explains, "It very frequently happens that different things, $a, b, c, \ldots$ for some reason can be considered from a common point of view, can be associated in the mind, and we say that they form a system $S \ldots$" (*WZ*, 797). After defining a *part* of a system (a subset, in modern terms) and operations on systems (such as, in modern terms, taking unions and intersections), Dedekind establishes some basic results. For example, he proves the transitivity of the subset relation (#7) and that if $A, B, C, \ldots \subseteq S$ then $\bigcup(A, B, C, \ldots) \subseteq S$ (#10).

Then he introduces a new and extremely important class of mathematical "entities": *mappings*. They are described in Erklärung #21 as follows: "By a mapping ⟦*Abbildung*⟧ φ of a system S we understand a law according to which, to every determinate element s of S, there belongs a determinate thing which is called the image ⟦*Bild*⟧ of s and which is denoted by $\varphi(s) \ldots$" (*WZ*, 799). Dedekind uses the notation $\varphi(S)$ to denote the system of all images $\varphi(s)$ and later writes, "If φ is a ... mapping of a system S, and $\varphi(S)$ part of a system Z, then φ is said to be a mapping of

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[3]This general background is described in detail and with great sensitivity in the first two chapters of Leo Corry's book (2004b).

[4](Dedekind 1890, 99-100). The German text is as follows: "Welches sind die von einander unabhängigen Grundeigenschaften dieser Reihe N, d. h. diejenigen Eigenschaften, welche sich nicht aus einander ableiten lassen, aus denen aber alle anderen folgen? Und wie muß man diese Eigenschaften ihres spezifisch arithmetischen Charakters entkleiden, der Art, daß sie sich allgemeineren Begriffen und solchen Tätigkeiten des Verstandes unterordnen, *ohne* welche überhaupt kein Denken möglich ist, *mit* welchen aber auch die Grundlage gegeben ist für die Sicherheit und Vollständigkeit der Beweise, wie für die Bildung widerspruchsfreier Begriffserklärungen?" (Sinaceur 1974, 272).

S into Z ..." (*WZ*, 802). Of special interest for Dedekind is the case when the system Z is S, and then we have a "... mapping of the system S into *itself* ..." (*WZ*, 802).[5]

Before continuing, we should note that Dedekind distinguishes between function (Funktion) and mapping (Abbildung).[6] This is indicated in Remark #135, where Dedekind refers to applying " ... the definition (set forth in theorem (126)) of a mapping ψ of the number-sequence N (or of the *function $\psi(n)$* determined by it) to the case where the system (there denoted by Ω) in which the image $\psi(N)$ is to be contained is the number series N itself".[7] Dedekind does not explicate the difference between functions and mappings. However, clarifying this distinction will bring out the special and novel character of mappings.

It is clear that *mapping* is a dramatic generalization of *function*, but the precise, distinctive character of function is not as transparent. (The explication we offer here is most closely associated with Dirichlet and Riemann.[8]) Historically we can say, that Dedekind uses the function concept extensively already in his *Habilitationsrede* (1854), whereas he introduces the notion of a mapping only in the manuscript (1872/78). Though *function* is used in the title of (1854), Dedekind discusses at first only *operations*, when treating the seven algebraic functions of elementary arithmetic (addition, subtraction, multiplication, division, exponentiation, taking roots and logarithms); that is also done in (*SZ*, §1, §6) and in Schröder's (1873). The term function is used in the later part of (1854, 434-438), when Dedekind discusses, for example, the trigonometric and elliptic functions. Obviously, that distinction between operations and functions is no longer made in *WZ* as a mapping from N to N "determines" now a function from N to N. So it seems that, in those later considerations, functions are mappings that satisfy two conditions: their domains and co-domains are identical; their domains are particular systems of *numbers*. Dedekind's introduction, in #36, of the notion of "a mapping of S *in itself*" thus satisfies the first of these conditions, but not the last as it does not restrict the domain to particular systems of numbers.

This understanding of Dedekind's distinction between mappings and functions is certainly in accord with his mathematical work (e.g., with Weber in their (1882)). It also reflects the general mathematical understanding of the times, sustained in the second half of the 19th and into the early years of the 20th century. Indeed, the part of mathematics that was called in German "Funktionentheorie" is not a theory of abstract functions (mappings), but rather complex analysis, i.e., the theory of *differentiable* functions of a complex variable. In the 1922 textbook *Funktionentheorie* by Hurwitz and Courant, the term *function* is introduced specifically for calculation procedures leading from a subset Σ of complex numbers to complex numbers as follows: "Now if to each value z may take, i.e. to each number in Σ, is assigned a complex number-value $w = u + iv$ according to a determinate law, we call w a *function* of z."[9]

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[5]Here is the full German text concerning mappings: "Unter einer Abbildung φ eines Systems S wird ein Gesetz verstanden, nach welchem zu jedem bestimmten Element s von S ein bestimmtes Ding gehört, welches das Bild von s heißt und mit $\varphi(s)$ bezeichnet wird; wir sagen auch, daß $\varphi(s)$ dem Element s entspricht, daß $\varphi(s)$ durch die Abbildung φ aus s entsteht oder erzeugt wird, daß s durch die Abbildung φ in $\varphi(s)$ übergeht. Ist nun T irgendein Teil von S, so ist in der Abbildung φ zugleich eine bestimmte Abbildung von T enthalten, welche der Einfachheit wegen wohl mit demselben Zeichen φ bezeichnet werden darf und darin besteht, daß jedem Elemente t des Systems T dasselbe Bild $\varphi(t)$ entspricht, welches t als Element von S besitzt; zugleich soll das System, welches aus allen Bildern $\varphi(t)$ besteht, das Bild von T heißen und mit $\varphi(T)$ bezeichnet werden, wodurch auch die Bedeutung von $\varphi(S)$ erklärt ist. Als ein Beispiel einer Abbildung eines Systems ist schon die Belegung seiner Elemente mit bestimmten Zeichen oder Namen anzusehen."

[6]Ansten Klev offers an alternative interpretation of Dedekind's distinction between function and mapping. See (Klev, 2014).

[7]The fuller German text is: "135. Erklärung. Es liegt nahe, die im Satze 126 dargestellte Definition einer *Abbildung* ψ der Zahlenreihe N oder der durch dieselbe bestimmten *Funktion $\psi(n)$* auf den Fall anzuwenden, wo das dort mit Ω bezeichnete System, in welchem das Bild $\psi(N)$ enthalten sein soll, die Zahlenreihe N selbst ist ...". Jeremy Avigad brought this quotation to our attention and raised the question "What is the difference between mappings and functions for Dedekind?". See (Sieg and Schlimm 2014), where the development of the notion of mapping in Dedekind's work is analyzed, in particular, its structure-preserving variety.

[8]This evolution has received significant attention, but still deserves a deeper treatment. See (Avigad and Morris 2014) and (Sieg and Schlimm 2014), but also the rich secondary literature mentioned in those papers.

[9](Hurwitz and Courant 1922, 17-18). The original German is as follows: "Wenn nun jedem Wert, den z annehmen

The general notion of mapping was absolutely central for Dedekind's foundational thinking; that is obvious from the announcement of the third edition of (Dirichlet 1879), but also from a footnote in that very work. Indeed, Dedekind refers back to these considerations in a note to section 161 of the 1894 fourth edition of Dirichlet's lectures:

> It is stated already in the third edition of the present work (1879, footnote on p. 470) that the entire science of numbers is also based on this intellectual ability to compare a thing a with a thing a', or to let a correspond to a', without which no thinking at all is possible. The development of this thought has meanwhile been published in my essay *Was sind und was sollen die Zahlen?* (Braunschweig, 1888).[10]

Remark #135, which we quoted earlier, is the only place in *WZ* where Dedekind uses the term *function*, when he considers recursively defined mappings from N to N and asserts that they determine functions. When discussing mappings he always refers explicitly to domain and co-domain (or, if not the co-domain, at least the image of the domain) which, moreover, are allowed to be different.[11] The metamathematical work in *WZ* crucially depends on this general concept of mapping: without it he would be unable to form the notion of *similarity* of different systems, which, as we will see, plays an essential role in the essay. A mapping is called *similar*, or in modern terms *injective*, when it maps distinct elements of its domain to distinct elements of its co-domain. This allows Dedekind to introduce the notion of *similarity between systems* in #32: two systems R and S are similar when there is a similar mapping φ from S to R such that $\varphi(S) = R$, i.e., two systems are similar when there is a bijection from one to the other.

Within this set-up, Dedekind introduces two important notions in #37 and #44, namely, that of a *chain* and that of a *chain of a system*. If S is a system, φ a mapping from S to S and K a part of S, then K is a chain if $\varphi(K) \subseteq K$. If S is a system, φ a mapping from S to S and A a part of S, then the commonality (in modern terminology, the intersection) of all of those chains which have A as a part is called the chain of the system A and denoted by A_0. A_0 enjoys a number of properties which, as Dedekind remarks in #48, are sufficient to characterize it completely as *the* "smallest" chain containing A. The characteristic properties that allow the direct proof of this claim are the following facts, formulated as #45, #46, and #47, namely that: (i) $A \subseteq A_0$, (ii) $\varphi(A_0) \subseteq A_0$, and (iii) If $A \subseteq K$ and K is a chain, then $A_0 \subseteq K$. In #64, Dedekind calls a system *infinite* just in case it is similar to a proper part of itself. Then, completing the presentation of the logical framework, Dedekind attempts to establish the existence of an infinite system by a purely logical proof.[12] That sets the stage for the final analytic steps and the subsequent synthetic development

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darf, also jeder Zahl von Σ, nach einem bestimmten Gesetz ein komplexer Zahlenwert $w = u + iv$ zugeordnet ist, so nennen wir w eine *Funktion* von z." Notice that "w ist eine Funktion of z" might best be translated as "w is functionally dependent on z". In his textbook *Funktionentheorie I* (1930), Knopp introduces the concept of "the most general" function in a very similar way, with z and w both ranging over complex values. He adds the notational device $w = f(z)$ and remarks that f stands for "the somehow given calculation procedure ⟦Rechenvorschrift⟧" (Knopp 1930, paragraph 5); the latter allows the determination of w for all z in its domain of variability. We should note, however, that despite Knopp's use of the term "calculation procedure", he does not require the function to be given by an explicit expression. In particular, for domain of variability $\mathfrak{M}$, he claims, "All that is required is that the value w of the function be made to correspond, on the basis of the definition, to each z of $\mathfrak{M}$ in a completely unambiguous manner." (Knopp 1930, paragraph 5)

[10]Here is the German text: "Schon in der dritten Auflage diese Werkes (1879, Anmerkung auf S. 470) ist ausgesprochen, dass auf dieser Fähigkeit des Geistes, ein Ding a mit einem Ding a' zu vergleichen, oder a auf a' zu beziehen, oder dem a ein a' entsprechen zu lassen, ohne welche überhaupt kein Denken möglich ist, auch die gesammte Wissenschaft der Zahlen beruht. Die Durchführung dieses Gedankens ist seitdem veröffentlicht in meiner Schrift *Was sind und was sollen die Zahlen?* (Braunschweig 1888)."(Dirichlet and Dedekind, 1894, footnote on page 456).

[11]Already in #21, when introducing the notion of a mapping φ of a system S, Dedekind describes the image, $\phi(S)$, of S under the mapping. In #25 he is careful to pay attention to the images when composing two mappings. Most importantly, similar observations can be made for §3, where Dedekind defines the similarity of two systems, and for §9, where recursively defined mappings from N to an arbitrary system Ω are introduced.

[12]Though his proof is recognized as highly problematic, given that it appeals to " ... the totality S of all things, which can be objects of my thought ... " (*WZ*, 806), it is nevertheless useful to emphasize why Dedekind tries to

of number theory; the former are described in section A.2, whereas the latter is detailed in A.3.[13]

A.2. Analysis

Dedekind's manuscript (1872/78) has the subtitle *An attempt at analyzing the number concept from the naïve standpoint* (*Versuch einer Analyse des Zahlbegriffs vom naiven Standpuncte aus*). By emphasizing that the attempt is made from the naïve standpoint, Dedekind indicates that the data of ordinary mathematical experience are to be analyzed without philosophical preconceptions. That perspective is also taken many years later in his letter to Keferstein, when Dedekind asserts that the synthetic development is "based on a prior analysis of the sequence of natural numbers as it presents itself, in experience, so to speak, for our consideration". The analysis itself proceeds in steps Dedekind details for Keferstein, as facts (1) through (6). So the analysis from the naïve standpoint allows Dedekind to define the central concept of his investigations, that of a simply infinite system. We quote this definition, in German *Erklärung*, in full:

> 71. Definition [[Erklärung]][14]. A system N is said to be *simply infinite* when there exists a similar mapping φ of N into itself such that N appears as the chain (44) of an element not contained in $\varphi(N)$. We call this element, which we shall denote in what follows by the symbol 1, *the base-element* of N, and say the simply infinite system N is *ordered* [[*geordnet*]] by this mapping φ. If we retain the earlier convenient symbols for images and chains (§4) then the essence of a simply infinite system N consists in the existence of a mapping φ of N and an element 1 which satisfy the following conditions α, β, γ, δ:
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> α. $N' \prec N$.[15]
>
> β. $N = 1_0$.[16]
>
> γ. The element 1 is not contained in N'.
>
> δ. The mapping φ is similar. (*WZ*, 808)

We emphasize most strongly that this is *not* the definition of a particular structure in the sense of modern model theory: the base element and successor operation are quantified out, existentially. I.e., a system of objects N is called *simply infinite* just in case there is an object 1 in N and a function φ from N to N that satisfy the characteristic conditions (α)–(δ).[17] Dedekind thus provides a higher-order, *structural definition* under which particular simply infinite systems fall.

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obtain such a proof: it is to guarantee the internal coherence of the concept of an *infinite system* and, as we will see later, of a *simply infinite system*.

[13]The separation of "analysis" from "synthesis" is a classical methodological approach that goes back to Aristotle. It is described with great sensitivity and acumen in the Introduction to the third book of Lotze's *Logik*, in particular, sections 297–299. This opposition is also formulated in §117 of Kant's *Logik*; there one finds informatively, "Die analytische Methode heißt auch sonst die Methode des *Erfindens*." Beaney's entry "Analysis" for the Stanford Encyclopedia of Philosophy gives an excellent and broad survey, (Beaney 2012).

[14]Dedekind uses the word *Erklärung* frequently, in particular, in *WZ*. It is prominently used to introduce the notion of a *simply infinite system*. (Hilbert also makes regular use of the term in his *Grundlagen der Geometrie* when giving his axiom system for geometry or, rather, when introducing the notion of *Euclidean space*.) It is an open question for us, whether Dedekind's sense of *Erklärung* and *Definition* reflects to a certain extent the distinctions Kant made in his *Kritik der reinen Vernunft* (*Erstes Hauptstück* of the *Transzendentale Methodenlehre*, A709–A739, especially A730). For Kant, the *Erklärung* of a concept splits into *Exposition*, *Explikation*, *Deklaration*, and *Definition*.

[15]N' denotes $\varphi(N)$ under the mapping φ, and '$\prec$' is (close to) Dedekind's symbol for the subsystem relationship, which is today symbolized by '$\subseteq$'.

[16]1_0 is short for $\{1\}_0$, i.e., the chain of the system $\{1\}$.

[17]"Characteristic conditions" (charakteristische Bedingungen) is Dedekind's terminology for the conditions (α)–(δ) at the end of #72.

The systems that fall under the concept are viewed as having their own "canonical language" in terms of which the characteristic conditions are expressed. This idea of a "canonical language" is important to Dedekind's project. The role it plays will be discussed in detail in section B.2, but for now we simply remark that Dedekind was careful to distinguish between language and that which it is used to speak about. Indeed, *WZ* begins: "In order to be able conveniently to speak of things, we designate them by symbols, e.g., by letters, and we venture to speak briefly of the thing a or of a simply, when we mean the thing denoted by a and not at all the letter a itself" (*WZ*, 796–797).[18]

Note that Hilbert's early conception of axiomatics coincides with Dedekind's methodological approach.[19] The axiomatization of geometry in (Hilbert 1899) and of analysis, the theory of real numbers, in (Hilbert 1900a) *are* structural definitions in Dedekind's sense, introduced also under the heading *Erklärung.* The only difference from Dedekind's structural definition lies in the fact that Hilbert calls the characteristic conditions *axioms.* Indeed, Frege in his letter of 27 December 1899 criticized Hilbert's use of the word "axioms" in the *Erklärung* that introduces the geometric principles. In his immediate reply of 29 December 1899 Hilbert writes:

> If you want to call my axioms rather characteristic conditions of the concepts that are given and hence are existent in the 'definitions' ⟦Erklärungen⟧, I would not object to that at all, except perhaps that that conflicts with the custom of mathematicians and physicists; of course I must also be free in giving characteristic conditions.[20]

After some additional remarks, he comes back to this issue, which he considers as the "main issue" (*Hauptsache*), and writes: ". . . the renaming 'characteristic conditions' instead of 'axioms' etc. is surely a formality and moreover a matter of taste—but in any case it is easily achieved."[21]

In a more principled way, Hilbert remarks later on: "Well, it is surely obvious that every theory is only a framework of concepts or a schema of concepts together with their necessary relations to each other, and the basic elements can be thought in arbitrary ways."[22] This reflects Dedekind's methodological perspective perfectly and is also very similar to the modern, algebraic approach to

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[18]Aki Kanamori pointed out the relevance of this remark.

[19]This perspective is now largely shared by José Ferreirós; see (Ferreirós 2007, Epilogue, 458–463). Ansten Klev has subjected the position to a detailed critique, presented in (Klev 2011). Klev maintains that Dedekind and Hilbert, around 1900, "held divergent views on the structure of a deductive science". The two main points of the divergence are concisely formulated in his abstract: "Firstly, . . . Dedekind sees the beginning of a science in concepts, whereas Hilbert sees such beginnings in axioms." However, this conflicts with Hilbert's own words, for example, in the correspondence with Frege that we analyze in our paper below. "Secondly, . . . for Dedekind, the primitive terms of a science are substantive terms whose sense is to be conveyed by elucidation, . . . " As we will see below, especially in section B.2, this is incorrect for Dedekind.

At the root of the difference between Klev's and our understanding is the fact that he compares Dedekind's logical framework for arithmetic with Hilbert's "axiomatic" presentation of Euclidean geometry, not with Hilbert's assumed set theoretic framework for geometry. When examining the "structure of a deductive science" in Dedekind's and Hilbert's work, the *Erklärung* #71 of the concept of a simply infinite system should be set alongside the *Erklärung* of the concept of a Euclidean space that opens Hilbert's *Grundlagen der Geometrie.*

However, Klev's interpretation and our own agree on the following point: Dedekind is concerned with the development of "higher level" concepts. Indeed, as we have already noted, the concept of "simply infinite system" is such a higher order one: individual systems fall under it.

[20]The German text is: "Wollen Sie meine Axiome lieber Merkmale der in den "Erklärungen" gesetzten und dadurch vorhandenen Begriffe nennen, so würde ich dagegen garnichts einzuwenden haben, ausser etwa, dass das der Gewohnheit der Mathematiker und Physiker widerspricht—freilich muss ich auch mit dem Setzen der Merkmale frei schalten können." Quoted in (Frege 1980, 12).

[21]The German text is: "Die Umnennung "Merkmale" statt "Axiome" etc. ist doch nur eine Äusserlichkeit und überdies Geschmackssache—ist aber jedenfalls leicht zu bewerkstelligen." Quoted in (Frege 1980, 12).

[22]Here is the German text: "Ja, es ist doch selbstverständlich eine jede Theorie nur ein Fachwerk von Begriffen oder ein Schema von Begriffen nebst ihren notwendigen Beziehungen zu einander und die Grund-elemente können in beliebiger Weise gedacht werden." Quoted in (Frege 1980, 13). In perhaps the clearest possible way this idea is expressed in (Hilbert 1902, 566). He defines: "*Ein System von Dingen*, welche wir *Punkte und Geraden* nennen und denen wir gewisse gegenseitige Beziehungen auferlegen, wollen wir eine *Geometrie* nennen. *Mit Hilfe der complexen Zahlen* . . . können wir nun leicht eine *Geometrie construiren*, in welcher sämmtliche Axiome *I-IV gelten, V aber nicht.*"

axiomatics which has, of course, its roots in the work of Dedekind and Hilbert. As an example, consider the introduction of the abstract notion of a group. We call any set G equipped with a binary operation $\circledast$ satisfying certain conditions (axioms) a group; but when considering particular groups, we will use the names associated with the mathematical objects involved, instead of 'G' and '$\circledast$'. Before coming back to Dedekind's essay, we mention that Zermelo's 1908 axiomatization of set theory follows the Dedekind-Hilbert approach.[23]

Immediately after having given the definition (Erklärung) of a simply infinite system, Dedekind demonstrates Theorem #72: *every* infinite system contains a simply infinite system as a part. Thus, if the proof of Theorem #66, which claimed that there is an infinite system, had been successful, it would guarantee, together with Theorem #72, the existence of a simply infinite system. What significance does this have? Dedekind answers this question in his letter to Keferstein as follows:

> After the essential character of the simply infinite system, *whose abstract type is the number sequence N*, had been recognized in my analysis (articles 71 and 73), the question arose: does such a system exist at all in the realm of our thoughts? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs (articles 66 and 72 of my essay)[24].

Thus, the aim of Theorems #66 and #72 is to establish the consistency, respectively, of the concept of an *infinite* and a *simply infinite* system. That is to be achieved by "a logical proof of existence", i.e., by specifying within "logic" a particular example of a system falling under the concept of a simply infinite system. We come back a little later to the question, what means are taken for granted as being rooted in "logic"; here we just point out that, with this much completed, Dedekind can define natural numbers in *WZ* as follows:

> 73. Definition ⟦Erklärung⟧. If in the consideration of a simply infinite system N set in order by a mapping φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the ordering mapping φ, then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number series N*. With reference to this liberation of the elements from every other content (abstraction) we are justified in calling the numbers a free creation of the human mind.[25]

There is an important point to note here that corresponds to our claim that, in the final version of *WZ*, Dedekind was primarily concerned with the concept of natural number, rather than identifying natural numbers as particular logical objects. Indeed, Dedekind refers to the *elements of the original simply infinite system* as the natural numbers; for observe that the natural numbers are "these elements", referring back to the elements of the simply infinite system he begins with. Thus, for Dedekind, natural numbers can be taken to be the elements of *any* simply infinite system when

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[23]Cf. (Zermelo 1908, 201). This is discussed in greater detail in C.3.

[24](Dedekind 1890, 101). Here is the German text: "Nachdem in meiner Analyse der wesentliche Charakter des einfach unendlichen Systems, dessen abstrakter Typus die Zahlenreihe N ist, erkannt war (71, 73), fragte es sich: *existiert* überhaupt ein solches System in unserer Gedankenwelt? Ohne den logischen Existenzbeweis würde es immer zweifelhaft bleiben, ob nicht der Begriff eines solchen Systems vielleicht innere Widersprüche enthält. Daher die Notwendigkeit solcher Beweise (66, 72 meiner Schrift)." (Sinaceur 1974, 275)

[25]The German text is as follows: "73. Erklärung. Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung φ geordneten Systems N von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt, in die sie durch die ordnende Abbildung φ zueinander gesetzt sind, so heißen diese Elemente *natürliche Zahlen* oder *Ordinalzahlen* oder auch schlechthin *Zahlen*, und das Grundelement 1 heißt die *Grundzahl der Zahlenreihe N*. In Rücksicht auf diese Befreiung der Elemente von jedem anderen Inhalt (Abstraktion) kann man die Zahlen mit Recht eine freie Schöpfung des menschlichen Geistes nennen."

viewed from a more abstract perspective, neglecting the "special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting mapping φ". That fits in well with his letter to Keferstein, where Dedekind refers to the natural number sequence as the "abstract type" of the notion of a simply infinite system.[26] That is to say, the particular natural number sequence is paradigmatic of all simply infinite systems: it exemplifies exactly and completely the structural properties that each has, as Dedekind comes to prove in #131-134 and which we will consider in detail below. It also fits well with Dedekind's remark in his letter to Weber, written in January of 1888, in which he speaks of "my" ordinal numbers "as the abstract elements of the simply infinite system". (The concepts of abstraction that are used here are discussed in section C.2.)

For now, we draw attention to a number of points that will be important for our later considerations. First of all note that the creation of the concepts "simply infinite system" and "natural numbers" does not end Dedekind's analysis. What is still needed is implicit in what Dedekind describes as the subject-matter of arithmetic in #73: "[T]he relations or laws which are derived entirely from the conditions α, β, γ, δ in (71) and therefore are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements (compare 134), form the first object of the *science of numbers* or *arithmetic*." Note in particular his claim that these laws are *the same* for each simply infinite system, and his reference to Remark #134. The significant foundational issues will be explained in B.2, where we discuss this very remark.

As is clear from the letter to Keferstein, two general and yet highly "practical" questions still need an answer before we can conclude the analysis: (i) the justification of the method of proof by induction, and (ii) the consistent definition of recursive operations for all natural numbers. Attending to (i), Dedekind proves in #80, "... that the form of argument known as complete induction (or the inference from n to $n+1$) is really conclusive ..." (*WZ*, 791). This result is then used to prove Dedekind's recursion theorem: if Ω is a system, θ a mapping of that system in itself (not necessarily similar), and ω a determinate element of Ω, then there exists a unique mapping ψ from N to Ω such that (a) $\psi(1) = \omega$ and (b) $\psi(n') = \theta\psi(n)$, where n' is Dedekind's notation for $\varphi(n)$, the successor of n. This allows the definition of the standard arithmetic operations like addition, multiplication, and exponentiation when $\Omega = N$ and thereby addresses (ii).

This possibility of explicitly defining recursive functions for the number series N is crucial for the development of arithmetic, but the *general* theorem allows Dedekind to prove an important metamathematical result: all simply infinite systems are similar (indeed, isomorphic). In Part B, we will see that this result plays a crucial role in Dedekind's *complete justification* of his concept of numbers. With the consistent definition of recursive functions (from N to N) given by #126 the analysis ends, as Dedekind very clearly emphasizes in his letter to Keferstein. It is followed by the "synthetic development" (*synthetischer Aufbau*) of arithmetic. As to it, Dedekind admits to Keferstein:

> And yet, it [the synthesis] has taken me a great deal of effort! ... The reader of my essay does not have an easy task either; in order to work completely through everything, not only sound common sense is required, but also a strong determination.[27]

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[26]This notion of "abstract type" and, more generally, abstraction is discussed more fully in section C.2. Here we just note that *type* (in German, Typ or Typus) has a number of different meanings. Two seem to be particularly relevant for our investigations. These are, following the Shorter OED: (1) "A class of people or things distinguished by common essential characteristics", and (2) "A person or thing showing the characteristic qualities of a class; a representative specimen; specifically, a person or thing exemplifying the ideal characteristics of a class, the perfect specimen of something". Type in sense (2) is used in biology and refers to a selected individual that serves as the foundation for the scientific description of a taxon, for example, of an animal species. It is in this second sense that "(abstract) type" is to be understood here as well. That is in harmony with the use of "Typus" in philosophy, for example, in Kant's *Kritik der praktischen Vernunft*, A119–A127, and Lotze's *Logik*, #137.

[27]Here is the German text: "Er [der synthetische Aufbau] hat mir doch noch Mühe gemacht! ... Auch der Leser meiner Schrift hat es wahrlich nicht leicht; außer dem gesunden Menschenverstande gehört auch noch ein sehr starker Wille dazu, um Alles vollständig durchzuarbeiten." (Sinaceur 1974, 276)

We now come to consider the question: How is the *synthesis* to be accomplished?

A.3. Synthesis

The *Preface* to the first edition of *WZ* provides an answer to the question we posed at the end of section A.2. The very first sentence articulates what Dedekind considered as the motto for his essay, namely, "In science nothing capable of proof ought to be believed without proof." Lamenting the current state of affairs, he continues: "Though this demand seems reasonable, I cannot regard it as having been met even in the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers" (*WZ*, 790). Dedekind's work is intended to remedy this. He continues in the next paragraph:

> But I know that, in the shadowy forms which I bring before him, many a reader will scarcely recognize his numbers which all his life long have accompanied him as faithful and familiar friends; he will be frightened by the long series of simple inferences corresponding to our step-by-step understanding [[Treppenverstand]], by the matter-of-fact dissection of the chains of reasoning on which the laws of numbers depend ... (*WZ*, 791).

To gain additional insights into this key element of the synthetic development, we explore the central issue: what does Dedekind consider to be a *rigorous proof*? Or, more concretely, how are the "facts" of ordinary arithmetic inferred from the conditions that characterize simply infinite systems? Let us first state very clearly that the facts are not obtained by semantic means. When Dedekind claims in #73 that what we study in arithmetic is what is derived entirely from the conditions α, β, γ, δ, this is to be understood in a quasi-formal manner (noting, as Frege very pointedly did, that Dedekind does not specify rules according to which such derivations are made). This understanding is already suggested by the above passage from the preface in which Dedekind speaks of the "long series of simple inferences corresponding to our step-by-step understanding". Further evidence can be found in numerous places in *WZ* and other of Dedekind's works.[28]

As a first and general observation, we note that Dedekind uses words such as *Ableitung* and *ableiten* and their cognates in *WZ*. These words have the connotation of *being derived* in the formal sense, i.e., as being given by a syntactic proof, rather than as a semantic consequence, for which words such as *Folgerung* and *folgen* are appropriate. More specific evidence can be found by considering Dedekind's descriptions of his work and his opinions about mathematics. Already in *SZ*, Dedekind emphasizes his interest in identifying certain properties, "... that can serve as the basis for *proper*[29] *deductions*" (*SZ*, 771). Moreover, in his letter to Lipschitz, Dedekind explains that, if a mathematical theory has been constructed correctly, then replacing the "terms of art" by other, newly invented terms should not result in the collapse of the theory, see (Sieg and Schlimm 2005, 155). In other words, the proofs of arithmetical theorems should not appeal to intuitions connected with the "terms of art" which fail to constitute part of their definitions, since such "proofs" will crumble after the replacement. Finally, in his 1890 letter to Keferstein, Dedekind rejects one of Keferstein's suggested improvements to his work by asserting: "The danger would immediately arise that from such an

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[28]There is a general philosophical position underlying the insistence on rigorous proof. It is expressed in Dedekind's (1854, 428–429) when he compares a man with "unbounded understanding" with us who are subject "to all the imperfections of [our] mental powers". He claims then that "there would essentially be no more science" for such a man, i.e., "a man for whom the final conclusions, which we obtain through a long chain of inferences, would be immediately evident truths." Dedekind adds, "and this would be so even if he stood in exactly the same relation to the objects of science as we do." Cf. Lotze's remark in *Logik*, section IX of the Introduction to the first book; there Lotze contrasts a mind "that stands in the center of the world and of everything real" with the human mind "that does not stand in this center of things", but rather has its "modest place somewhere in the last ramifications of reality".

[29]Instead of "proper", Ewald uses "valid". The German adjective is *wirklich* and the original German phrase is: "*als Basis für wirkliche Deduktionen*". (Dedekind 1932, 322).

intuition ... we perhaps take as obvious *propositions that must rather be derived quite abstractly from the logical definition of N*."[30]

All of these remarks fit well with a quasi-formal conception of proof. Indeed, to see that this idea has been thoroughly realized, one needs only to follow the development of "elementary set theory" in §1–5 or of arithmetic proper in §11–13 of *WZ*.[31] A quasi-formal conception of proof also underlies Dedekind's argument showing that the laws of arithmetic are *the same* in all simply infinite systems; see subsection B.2. While we wish to emphasize that Dedekind's notion of proof is quasi-formal, we do not wish to imply that he was unconcerned with semantic issues. Indeed, he explores them in a variety of places, including his considerations about consistency of concepts (which, as we have seen, for Dedekind involves exhibiting an appropriate model), categoricity, and independence of characteristic conditions.

Before formulating the central questions for Part B, we want to re-emphasize three points. (1) In *WZ*, Dedekind is concerned with creating an abstract *concept* of number and *not* with identifying (systems of) objects as the numbers. This concern with concepts as opposed to objects is reflected in various places in Dedekind's writings. In earlier manuscripts of *WZ* Dedekind does create abstract mathematical objects. The move from creating objects to creating concepts is the fundamental part of the *radical shift* in Dedekind's position, which we explore in later sections, in particular, section B.3. (2) Dedekind does not have the modern distinction between syntax and semantics, but his notion of inference with respect to arithmetic theorems is nonetheless much more *syntactic* than semantic. It is only *quasi*-formal, however, as the inferential principles that can be appealed to are not explicitly enunciated. This pertains also to Hilbert, whose proofs proceed, around the turn from the 19th to the 20th century, by "finitely many logical steps"; the latter are, as in Dedekind's case, not specified. (3) Much of Dedekind's work in *WZ* is metamathematical. Indeed, it is only in §§11–13 that Dedekind comes to consider the arithmetic operations of addition, multiplication and exponentiation and rigorously proves their basic properties. The crucial and quite novel metamathematical work was missing from the earlier manuscripts of *WZ*, but it is precisely this work that allows Dedekind, as he puts it, to "completely justify" the concept of numbers. We explore next Dedekind's line of argumentation.

B. The Concept of Numbers: *Completely Justified*

In the previous part we have drawn attention to the fact that Dedekind is primarily concerned with the creation of the *concept* of numbers. This understanding is further supported by his letter to Keferstein in which Dedekind responds to Keferstein's objection that he has not given an adequate definition of the number 1. He writes:

> I define the ⟦ordinal⟧ number 1 as the basic element ⟦Grundelement⟧ of the number sequence without any ambiguity in articles 71 and 73, and, just as unambiguously, I arrive at the ⟦cardinal⟧ number 1 in the theorem of article 164 as a consequence of the general definition in article 161. *Nothing further may be added to this at all if the matter is not to be muddled.* (Dedekind 1890, 102, emphasis added)

For Dedekind the number 1 is fully specified as the base element of the number sequence N. How should it be possible to identify it to any further degree, if—as we described already in A.2—the elements of *any* simply infinite system can take on the role of the natural numbers? In subsection

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[30]The fuller German text is: "Dem würde ich mich mit grösster Bestimmtheit widersetzen, weil sofort die Gefahr entsteht, aus einer solchen Anschauung vielleicht unbewusst auch Sätze, als selbstverständlich zu entnehmen, die vielmehr ganz abstrakt aus der logischen Definition von N abgeleitet werden müssen."(Sinaceur 1974, 277)

[31]Whereas in the case of arithmetic the starting points of proofs are the characteristic conditions of the concept of a simply infinite system, in the case of "set theory" Dedekind uses the definition of operations, like union and intersection for systems or composition and inversion for (similar) mappings, as beginnings of proofs.

B.1 we will present a *striking fact* established by Dedekind as Theorem #133: any system Ω that is similar to a simply infinite one is itself simply infinite. The order setting mapping of Dedekind's simply infinite system N (with 1 and φ), for example, can be used (together with the bijection ψ from N to Ω) to fix the base element ω in Ω and an order setting mapping θ on Ω.

In this part we examine Dedekind's metamathematical considerations that *justify* his approach to the natural numbers, as the elements of *any* simply infinite system. Thus, Dedekind must ensure that arithmetic theorems do not depend upon which particular simply infinite system is being considered. He does this by proving a central *representation theorem*, which we describe in B.1; as a consequence one obtains that the concept of a simply infinite system is *categorical*.[32] That is to say, we can map bijectively the elements of one simply infinite system to the elements of any other, whilst respecting the order setting mappings. We will examine in B.2 the consequences of categoricity, including crucially the reason for Dedekind's claim made in #73 that the laws of arithmetic are the same in all simply infinite systems. B.3, finally, is concerned with a very important and quite radical shift away from abstract objects to abstract concepts. (That shift is visible in the penultimate manuscript of *WZ*.)

B.1. A Representation Theorem

Before analyzing Dedekind's metamathematical work in detail, we pause to make two additional observations. The first concerns an example of what falls under the concept of *mapping*. Dedekind considers *naming* the elements of a particular system to constitute a mapping of that system; he writes: "As an example of a mapping of a system we may regard the mere assignment of determinate symbols or names to its elements" (*WZ*, 800). More explicitly, given any system S the following is a mapping of the system: $\chi(s) = \boldsymbol{s}$, where $\boldsymbol{s}$ is a symbol that is used to denote the element s of S. The second observation concerns "the number system N" that is being used in Dedekind's investigations above and below, in particular in the representation theorem. *Any* simply infinite system would do, but in Dedekind's case one may think of the particular system he obtained in #72 from the infinite system whose existence he had "proved" in #66. The reader can replace it by a personally favored modern one, say the von Neumann or Zermelo ordinals, where the distinguished element is the empty set and the successor operation $\varphi(x)$ is given by $x \cup \{x\}$ or $\{x\}$, respectively.

As already indicated, Dedekind's recursion theorem (Theorem #126, stated in the penultimate paragraph of subsection A.2) has important consequences for our purposes. Indeed, it allows us to establish directly the following central result:

> 132. Theorem. All simply infinite systems are similar to the number series N and consequently by (33)[33] also to one another.

Given any simply infinite system Ω (with θ and ω), Dedekind appeals for the proof of #132 to the recursion theorem and obtains a similar mapping ψ from N to Ω satisfying the equations $\psi(1) = \omega$ and $\psi(\varphi(n)) = \theta(\psi(n))$. He then demonstrates that ψ is bijective and, as it maps 1 to ω and commutes with the mappings φ and θ, it is an isomorphism, indeed, a *canonical* one. This establishes the crucial first part of the theorem, which is a *representation theorem*: each simply infinite system is isomorphic to the number series N. To demonstrate that any two simply infinite systems Ω (with θ and ω) and Ω' (with θ' and ω') are isomorphic, Dedekind observes that we have isomorphisms ψ and ψ' from N to Ω and Ω', respectively. But then the composition of $\overline{\psi}$, the inverse of ψ, with ψ' is an isomorphism between Ω and Ω'. This is depicted in Figure 1.

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[32]The emergence of the notion of categoricity is described in (Awodey and Reck 2002a,b). We are applying the term "categorical" here to an abstract notion or structural definition. Awodey and Reck provide a modern perspective on (or rather, framing of) Dedekind's and Hilbert's work concerned with the foundations of number theory and geometry, respectively. Their approach yields insights, but does not highlight the novelty and precision of Dedekind's work and its influence on Hilbert.

[33]Theorem #33 states that if R and S are similar systems, then any system similar to R will be similar to S.
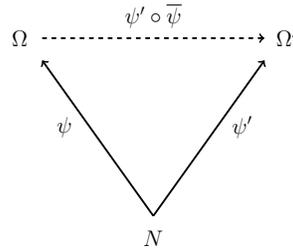
Figure 1

The second part of Theorem #132 consequently formulates the categoricity of the concept of a simply infinite system. It is followed by a striking observation, Theorem #133:

> 133. Theorem. Every system that is similar to a simply infinite system, and therefore by (132), (33) to the number series N, is itself simply infinite.

For any given system Ω whose elements can be bijectively mapped to a simply infinite system Ω', we can obtain a mapping that sets the elements of Ω in order as a simply infinite system. The next two diagrams reflect the idea that is involved in the proof.
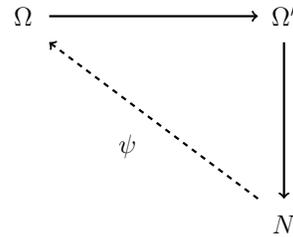

Figure 2

Figure 2 depicts Ω as being similar to Ω'. On account of #132, Ω' is similar to N and thus, as the similarity relation is transitive and symmetric, the system N is similar to the system Ω through the mapping ψ. Figure 3 shows how this fact allows us to construct a mapping that sets the elements of Ω in order.
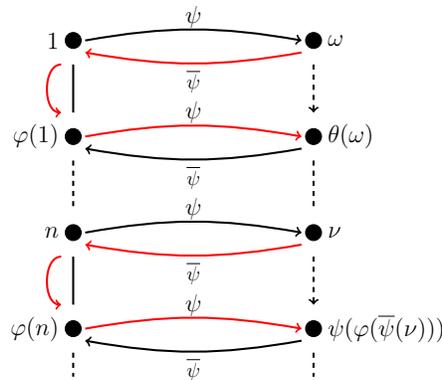

Figure 3

More specifically, with ψ as the similar mapping from N to Ω, we can put $\psi(1) = \omega$ and if $\overline{\psi}$: $\Omega \to N$ denotes, as above, the inverse of ψ, then the mapping $\theta(\nu) = \psi(\varphi(\overline{\psi}(\nu)))$ sets Ω in order, where φ is the successor function on N.

This is then *the striking fact*: the elements of *any* system that can be mapped bijectively to (any simply infinite one and thus to) the natural numbers can be taken to *be* natural numbers. We only have to fix appropriately a base element and an order setting mapping. For example, the system E of even natural numbers is in bijective correspondence with the natural number series. Taking 2 to be the base element and the addition of 2 as the "successor operation", E is a simply infinite system and E's elements can be taken to be the natural numbers. More generally, Dedekind points out at the beginning of Remark #134 that the simply infinite systems form a "class" (*Klasse*) as defined in #32. Considering any simply infinite system T, the systems similar to T are according to Theorems #132 and #133 exactly all the simply infinite systems. Viewing T as the representative of the class, Dedekind remarks "the class is not changed by taking as representative any other system belonging to it" (*WZ*, 802). So it is irrelevant from a "structuralist perspective" which simply infinite system is chosen as a representative.

Before turning in subsection B.2 to Dedekind's unusual and distinctive justification in the next subsection, we recall Dedekind's remarks in *SZ*, §2, and in a letter to Lipschitz that adumbrate the considerations here. In §1 of *SZ*, Dedekind discusses the order relation between the rational numbers whose system R falls under the concept of a field (*Zahlkörper*); he formulates the following laws (*Gesetze*):

(I) If $a > b$ and $b > c$, then $a > c$. We say that b lies between a and c.

(II) If $a \neq c$, then there are infinitely many different numbers between a and c.

(III) If a is a definite number then all numbers of R fall into two classes A_1 and A_2. A_1 contains all numbers a_1 such that $a_1 < a$ and A_2 contains all numbers a_2 such that $a_2 > a$. a can either be added to A_1 or it can be added to A_2. Every element of A_1 is less than every element of A_2.

These laws, Dedekind asserts, remind us of the mutual positional relations (*Lagenbeziehungen*) between the points of the straight line L. In regard to this difference in position (*Lagenverschiedenheit*) analogous laws hold:

(I′) If p is to the right of q and q is to the right of r, then p is to the right of r. We say that q lies between p and r.

(II′) If $p \neq r$, then there are infinitely many different points that lie between p and r.

(III′) If p is a definite point on L then all points that lie in L fall into two classes P_1 and P_2. P_1 contains all points p_1 which lie to the left of p and P_2 contains all numbers p_2 which lie to the right of p. p can either be added to P_1 or to P_2. Every element of P_1 lies to the left of every element of P_2.

The analogy of the laws can be turned into a "real connection" (*wirklicher Zusammenhang*) via a correspondence between the rational numbers a and the associated points p_a on the line that preserves the relationship, namely, if $a < b$ then p_b is to the right of p_a. (We note that this *informal* correspondence was not "mathematically" expressible at the time, as Dedekind introduces general mappings only later.) In sections 4 and 5, Dedekind defines an order relation $\prec$ between cuts and shows that it satisfies the analogous laws. In addition, the system of cuts is continuous or complete. For our considerations here, we just observe that for rational numbers a and b, $a < b$ implies $c_a \prec c_b$, where c_q is the cut associated with the rational number q.

Dedekind emphasizes in section 2 that the fundamental laws for L "are in complete accord" (*entsprechen vollständig*) with those for the rational numbers. The letter to Lip-schitz on which

we reported in A.3 can be read as allowing the extension of the "complete accord" from the basic laws to the theorems that can be inferred, stepwise, from them. In *SZ*, when discussing in section 4 the corresponding order relations for the rational numbers, the points of the geometric line and the real numbers, Dedekind used the phrase *transfer from one area to another* (*Übertragung aus einem Gebiet in ein anderes*). This phrase, as we will see, is being used by Dedekind in *WZ* to describe the same phenomenon between different simply infinite systems. So we have uncovered two pillars on which the accord rests, namely, the structure-preserving correspondence between systems of mathematical objects satisfying basic laws (or falling under a structural definition) and the stepwise proof of theorems from these basic laws (or from the characteristic conditions of the structural definition). We have been discussing the first pillar and have seen shadows of the second; a more thorough treatment of the second is the topic of the next section.

B.2. Transfer By Proofs

Dedekind's remarks in #134 are often taken to be a sketch of the proof that any two structures satisfying the Dedekind-Peano axioms satisfy the same sentences, i.e., are elementarily equivalent. While Dedekind's remark can be understood this way, in this section we will develop an alternative interpretation that is faithful to his metamathematical work. More specifically, instead of interpreting Dedekind's remarks as being about the modern, semantic notion of elementary equivalence, we suggest that he was working with a *syntactically grounded* notion of "proof theoretic equivalence". On this interpretation, two structures are proof theoretically equivalent just in case they fall under the same structural definition and "satisfy" the same set of sentences, namely those that can be deduced step-by-step from the characteristic conditions of the definition. We will now present our analysis of #134. Before proceeding, we want to emphasize that Dedekind was grappling with subtle methodological issues, but without the now familiar modern tools of mathematical logic.

Dedekind, as pointed out in section B.1, begins Remark #134 with the observation that the simply infinite systems form a class. He continues:

> At the same time, with reference to (71), (73) it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N ordered by the mapping φ (and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement φ) possesses perfectly general validity for every other simply infinite system Ω ordered by a mapping θ and its elements ν ... (*WZ*, 823).

How is it that *every* theorem regarding numbers, i.e., the elements of N with mapping φ and distinguished element 1, "possesses perfectly general validity" for any other simply infinite system Ω with mapping θ and distinguished element ω? To answer this question, we follow Dedekind and look in detail at #71 and #73. According to *Erklärung* #71, quoted in full in section A.2, the essence of a simply infinite system consists in the existence of a mapping φ of N and an element 1 satisfying the "characteristic conditions" $\alpha, \beta, \gamma, \delta$. The laws that are exclusively derived from these conditions, Dedekind asserts in #73, constitute the *science of numbers* or *arithmetic*. Laws thus obtained are "therefore in all simply infinite systems always the same, no matter what the accidentally given names of the individual elements might be".[34]

The remarks in #73 aim to explain why the laws obtained from the characteristic conditions for N have general validity for *all* simply infinite systems: one appeals in proofs only to the characteristic conditions and they, together with the statements obtained by stepwise deduction, clearly hold in all such systems. For the understanding of this and the further claim that the laws are "always the

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[34]The full German text is: "Die Beziehungen oder Gesetze, welche ganz allein aus den Bedingungen $\alpha, \beta, \gamma, \delta$ abgeleitet werden und deshalb in allen einfach unendlichen Systemen immer dieselben sind, wie auch die den einzelnen Elementen zufällig gegebenen Namen lauten mögen (vgl. 134), bilden den nächsten Gegenstand der *Wissenschaft von den Zahlen* oder *Arithmetik*."

same, no matter what the accidentally given names of the individual elements might be", we have to move back to #134 and directly continue the above long quotation replacing the ellipsis at its end by the following "conjunct":

> and that the transfer from N to Ω (e.g., also the translation of an arithmetic theorem from one language into another) is effected by the mapping ψ considered in (132) and (133), which transforms every element n of N into an element ν of Ω, i.e., into $\psi(n)$.[35]

The transformation (*Verwandlung*) of the elements of N into elements of Ω underlies the "transfer from N to Ω" and is achieved by the mapping ψ associating $\nu = \psi(n)$ with each n. "This element ν", Dedekind continues, "can be called the nth element of Ω and accordingly the number n is itself the nth number of the number series N." The crucial reason for the transfer, we emphasized already earlier, lies for Dedekind in the fact that ψ is structure-preserving or, to connect this fact most directly to laws, ψ transforms φ into θ and θ into φ; thus, the mappings φ and θ have the "same significance" for the laws in the domains N and Ω. Dedekind expresses that most clearly as follows:

> The same significance which the mapping φ possesses for the laws in the domain N, [in so far as every element n is followed by a determinate element $\varphi(n) = n'$], is found, after the transformation effected by ψ, to belong to the mapping θ for the same laws in the domain Ω, [in so far as the element $\nu = \psi(n)$ arising from the transformation of n is followed by the element $\theta(\nu) = \psi(n')$ arising from the transformation of n']. We are therefore justified in saying that φ is transformed by ψ into θ, which is symbolically expressed by $\theta = \psi\varphi\overline{\psi}$, $\varphi = \overline{\psi}\theta\psi$.[36] (*WZ*, 823).

Before Dedekind moves on to §11 and further applications of the recursion theorem #126, he writes, "These remarks, as I believe, completely justify the definition [[*Erklärung*]] of the concept of numbers given in #73."

To be perfectly clear about the *complete justification* that has been achieved, we have to further clarify the role the canonical isomorphism ψ plays for Dedekind, not only in connecting the elements of N and Ω, but also in translating laws "from one language into another". This is where Dedekind's distinction between an object and its name, and, more particularly, his use of canonical languages, plays a striking role. So let us consider a "language" $\mathfrak{L}$ in which the concept of a simply infinite system can be formulated: X is a simply infinite system if and only if there is a mapping f from X to X and an element d in X, satisfying the characteristic conditions. Consider now an arbitrary such system and instantiate the existential quantifiers (e.g., via there-is elimination of natural deduction), in order to obtain a version of the conditions parameterized with f and d. Let M stand for the conjunction of the characteristic conditions. Replacing the variables f and d by particular symbols $\boldsymbol{\varphi}, \mathbf{1}$, and $\boldsymbol{\theta}, \boldsymbol{\omega}$, denoting $\varphi, 1$ and θ, ω respectively, one obtains "canonical languages" $\mathfrak{L}^N$ and $\mathfrak{L}^\Omega$ for the particular simply infinite systems N (with φ and 1) and Ω (with θ and ω). If A is a statement in $\mathfrak{L}$, then A^N and A^Ω are the statements in these canonical languages obtained from A by the obvious replacements; indeed, there is also a direct translation τ from $\mathfrak{L}^N$ into $\mathfrak{L}^\Omega$ associating A^Ω

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[35] Ewald's translation used "passage" instead of "transfer" for *Übertragung*, and "changes" instead of "transforms" for "verwandelt" The latter holds also for the next long quotation. The full German text is: "und daß die Übertragung von N auf Ω (z.B. auch die Übersetzung eines arithmetischen Satzes aus einer Sprache in eine andere) durch die in 132, 133 betrachtete Abbildung ψ geschieht, welche jedes Element n von N in ein Element ν von Ω, nämlich in $\psi(n)$ verwandelt."

[36] Recall that $\overline{\psi}$ is Dedekind's notation for the inverse of ψ. We put the parallel phrases starting with "insofar as" in parentheses in order to clarify the structure of this long sentence. The German text is: "Dieses Element ν kann man das $n$te Element von Ω nennen und hiernach ist die Zahl n selbst die $n$te Zahl der Zahlenreihe N. Dieselbe Bedeutung, welche die Abbildung φ für die Gesetze im Gebiete N besitzt, insofern jedem Elemente n ein bestimmtes Element $\varphi(n) = n'$ folgt, kommt nach der durch ψ bewirkten Verwandlung der Abbildung θ zu für dieselben Gesetze im Gebiete Ω, insofern dem durch Verwandlung von n entstandenen Elemente $\nu = \psi(n)$ das durch Verwandlung von n' entstandene Element $\theta(\nu) = \psi(n')$ folgt; man kann daher mit Recht sagen, daß φ durch ψ in θ verwandelt wird, was sich symbolisch durch $\theta = \psi\varphi\overline{\psi}$, $\varphi = \overline{\psi}\theta\psi$ ausdrückt."

with A^N. (In a certain way, Pettigrew's Aristotelian perspective in his (2008) joins these approaches by considering the particular symbols φ and $\mathbf{1}$ as parameters.)

Basic for the translation τ is the mapping from (closed) terms in $\mathfrak{L}^N$ to those in $\mathfrak{L}^\Omega$ that satisfies the recursion conditions $\tau(\mathbf{1}) = \boldsymbol{\omega}$ and $\tau(\boldsymbol{\varphi}(\boldsymbol{n})) = \boldsymbol{\theta}(\tau(\boldsymbol{n}))$ as τ is induced by ψ. We can describe τ as a composition of mappings. Let χ_N be the mapping which associates to each element of the number sequence N its name; it maps 1 to $\mathbf{1}$ and $\varphi(n)$ to $\boldsymbol{\varphi}(\boldsymbol{n})$. Similarly let χ_Ω be the mapping which maps the elements of Ω to their names. We take the inverse $\overline{\chi_N}$ of χ_N which has as its domain the names of the natural numbers and observe that $\tau = \chi_\Omega \circ \psi \circ \overline{\chi_N}$. This is illustrated in Figure 4.

Figure 4

This observation tells us that the way in which we name the elements of one simply infinite system and the way in which we name the elements of another simply infinite system are "equivalent up to a translational isomorphism". Given our particular simply infinite system we can use its associated language to talk about the corresponding elements of any simply infinite system, and we can call both n and $\psi(n)$ the nth element of their respective systems and denote them both by $\boldsymbol{n}$. With this set-up we can understand the "transfer from N to Ω" as depicted in Figure 5, where $M \vdash A$ (and the variants) indicate the stepwise deducibility of A from M.
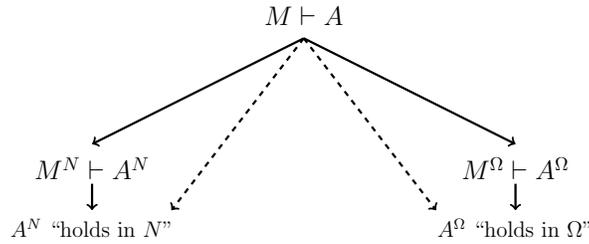

Figure 5

The claim "A^N 'holds in N' " is now simply expressed by the statement in which the names of the form $\boldsymbol{\varphi}(\dots (\boldsymbol{\varphi}(\mathbf{1}))\dots)$ have been replaced by $\varphi(\dots(\varphi(1))\dots)$ and similarly for the claim involving Ω. This seems then to be the context, pedantically reconstructed in more modern terms, that allows the following understanding of the part of #134 we sought to clarify: ν and n are the nth elements of Ω and N, respectively, and ψ preserves the order set by θ and φ for the elements in these systems. The structure-preserving feature (and nothing more) is expressed by the sentence that ends with the observation $\theta = \psi\varphi\overline{\psi}$, $\varphi = \overline{\psi}\theta\psi$. So, the theorems in the parameterized theory M are exactly the statements that "hold" in all simply infinite systems. Through these considerations we can come to understand in what sense the laws obtained by stepwise logical deduction are "always the same" in all simply infinite systems.[37] This analysis can be extended to provide a Dedekindian treatment of

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[37]This understanding of the fundamental "semantic" relationship—a statement holds in a system—is closely related

complete ordered fields. There is, however, a complication because there is no "canonical naming scheme" as for the natural numbers. After all, given any (countable) language, there are always real numbers not named in it. But in a restricted context, e.g., in the proof of a particular theorem, only finitely many real numbers are named and the proof theoretic equivalence can be established as before. Indeed, the issue was addressed by Dedekind in his letter to Weber of January 1888; Dedekind remarked, "Whether the sign language suffices to denote all individuals that have been created does not matter; it [the sign language] is always sufficient to denote the individuals occurring in a (limited) investigation."[38]

We should finally remark that Dedekind's approach in #134 links in closely with the following passage from the Preface to the first edition of *WZ*:

> If we scrutinize closely what is done in counting a set or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. Upon this unique and therefore absolutely indispensable foundation ... the whole science of numbers must, in my opinion, be established.[39]

These considerations emerged already much earlier in the 1870s and are reemphasized in the note to section 161 of (Dirichlet and Dedekind 1894), which we quoted in A.1. Thus it is quite clear that Dedekind has mappings in mind when he refers to "the ability of the mind to relate things to things ... an ability without which no thinking is possible". And as we have seen above, it is the use of various different kinds of mappings that make possible his approach to the natural numbers. Dedekind's comments in Remark #134 indeed justify his attitude towards the natural numbers. For he established (i) that the concept of simply infinite system is *categorical* and (ii) that there is an *isomorphic translation* between any two of the canonical *languages* associated with two simply infinite systems. These results ensure that Dedekind's definition of natural number is not dependent upon the elements of the simply infinite system he starts with. Moreover, they allow Dedekind to explicate the sense in which the theorems proved for one simply infinite system are "the same" as those proved for another: they are proved in a "logically abstract" way from the *Merkmale M* or their instantiation M^N, but leaving "entirely out of consideration the special character of the elements n" of N.

B.3. A Radical Shift

We have argued that Dedekind is in *WZ* primarily concerned with *creating* concepts. However, there are important passages in *SZ* and *WZ* that seem to indicate without any ambiguity that (systems of) objects are created. The considerations in those passages can be supported by remarks Dedekind made in letters to Lipschitz (1876) and to Weber (1878). Consider first a remark in *SZ* directly following Dedekind's proof that there are infinitely many cuts not engendered by rational numbers:

> Thus, whenever we have a cut (A_1, A_2) that is not engendered by a rational number, we *create* a new number, an *irrational* number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that

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to Gödel's perspective in his thesis concerning the completeness of first order logic. See note 53.

[38] Here is the German text: "Ob ferner die Zeichensprache ausreicht, um alle neu zu schaffenden Individuen einzeln zu bezeichnen, fällt nicht ins Gewicht; sie reicht immer dazu aus, um die in irgend einer (begrenzten) Untersuchung auftretenden Individuen zu bezeichnen." (Dedekind 1932, 490).

[39] The German text is as follows: "Verfolgt man genau, was wir bei dem Zählen der Menge oder Anzahl von Dingen tun, so wird man auf die Betrachtung der Fähigkeit des Geistes geführt, Dinge auf Dinge zu beziehen, einem Dinge ein Ding entsprechen zu lassen, oder ein Ding durch ein Ding abzubilden, ohne welche Fähigkeit überhaupt kein Denken möglich ist. Auf dieser einzigen, auch sonst ganz unentbehrlichen Grundlage muß nach meiner Ansicht ... die gesamte Wissenschaft der Zahlen errichtet werden." (Dedekind 1932, 336)

it engenders this cut. From now on, therefore, to every definite cut there corresponds exactly one definite rational or irrational number ... [40]

That view is re-emphasized in the letter of 19 November 1878 to Weber, in which Dedekind points to the fact that the phenomenon of the cut can be used for the introduction of new *numbers*: "so many cuts, so many numbers" (*soviel Schnitte, soviel Zahlen*). He claims that the standard operations of addition, subtraction etc. can be defined for these new numbers. Then he asks rhetorically:

> You also want that students learn how to deal with $\sqrt{2}, \sqrt{3}$ etc.; do you really want that they always view them just as symbols for calculating approximations? Or don't you also think it would be better, if they viewed them as symbols for new numbers that are in complete parity with the old ones?[41]

The introduction of "new numbers that are in complete parity with the old ones" is at the core of *SZ*, and the necessity of such an extension is discussed in two long and deep letters to Lipschitz written on 10 June and 27 July 1876. In them, Dedekind defends his position against Lipschitz's claim that the Euclidean axioms alone, without bringing in the principle of continuity, can serve as the foundation for a complete theory of the real numbers viewed as ratios of magnitudes. At the end of the first letter he asserts:

> ... my theory of irrational numbers creates the perfect pattern of a *continuous* domain which is [on account of its continuity] capable of characterizing every ratio of magnitudes by a determinate individual number that is contained in it [the continuous domain].[42]

In a long footnote to (Dedekind 1877), Dedekind describes how such extensions are to be achieved in a methodologically satisfactory way.[43] However, for a paradigmatic, rigorous execution he refers back to *SZ* and sees at its center "the introduction or the *creation* of new arithmetical elements". He emphasizes that the creation has to be done uniformly in one step and that it has objective boundaries: "The irrational numbers thus defined together with the rational numbers form a continuous domain R without gaps; ... it is impossible to put additional new numbers into the domain R." (Dedekind 1932, 269)

After the publication of *WZ*, in apparent conflict with our understanding of the essay, Dedekind continues to defend this "creative" perspective. At least he seemingly does so in a letter to Weber from 24 January 1888. He responds to Weber's proposal of treating cardinal numbers as classes[44] and writes:

> But if one were to take your route—and I would recommend that it be explored to the very end—then I would advise that by [cardinal] number ⟦Anzahl⟧ one rather not

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[40]The German text is found in (Dedekind 1932, 486): "Jedesmal nun, wenn ein Schnitt (A_1, A_2) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so *erschaffen* wir eine neue, eine *irrationale* Zahl α, welche wir als durch diesen Schnitt (A_1, A_2) vollständig definiert ansehen; wir werden sagen, daß die Zahl α diesem Schnitt entspricht, oder daß sie diesen Schnitt hervorbringt. Es entspricht also von jetzt ab jedem bestimmten Schnitt eine und nur eine bestimmte rationale oder irrationale Zahl ..."

[41]The German text is found in (Dedekind 1932, 486): "Du willst doch auch, daß die Schüler mit $\sqrt{2}, \sqrt{3}$ u.s.w. umgehen lernen; willst Du nun, daß sie darin immer nur Symbole für Näherungsrechnungen sehen? Oder hälst Du es nicht auch für besser, daß sie darin Symbole für neue, mit den alten ganz gleichberechtigte *Zahlen* sehen?"

[42]The German text is found in (Dedekind 1932, 474): " ... durch meine Theorie der irrationalen Zahlen [wird] das vollkommene Muster eines *stetigen* Gebiets erschaffen, welches eben deshalb fähig ist, jedes Grössen-Verhältnis durch ein bestimmtes in ihm enthaltenes Zahl-Individuum zu charakterisieren."

[43]Dedekind connects the general considerations with Kummer's introduction of ideal numbers. See the Note in (Dedekind 1932, 481) based on a report by Bernstein after visiting Dedekind on 6 March 1911; Dedekind related to Bernstein also some informative aspects of his first encounter with Kummer. Note that Hilbert speaks, in the context of extending the field of algebraic numbers to a complete ordered field, of an extension by "ideal" or "irrational" elements; see (Hilbert 1899, 475).

[44]The definition of "class" is given also in *WZ* #32 and subsection B.1 above.

understand the *class* itself (the system of all similar finite systems) but something *new* (corresponding to this class) which the mind *creates*.[45]

How is this remark together with the immediately following observations on the analogous creation of irrational numbers to be understood? The observations seem to paraphrase the problems and their solution from the 1870s. Are they to be viewed as expressing Dedekind's position at this time in 1888? Our answer to the last question is "No". The observations are rather to be understood in the context of the exchange with Weber: *if* one were to follow Weber's proposal, *then* one should do what he, Dedekind, had done for the real numbers in *SZ* and create something new that corresponds to the class. The new mathematical object is presumably, as he put it in *SZ*, *completely defined* by its corresponding class. Appealing to our *Schöpfungskraft* (creative power), Dedekind explains, this path can be taken in a perfectly coherent way. However, when working on the 1887 manuscript, Dedekind had recognized that the creation of an abstract system of abstract new objects is unnecessary, if one focuses on the laws that govern the *relations* between numbers. The manuscript itself ends abruptly with Remark #107 about the "creation of the pure natural numbers":

> *Creation of the pure natural numbers.* It follows from the above that the laws regarding the relations between numbers are completely independent of the choice of that infinite system N, which we have called the number series, as well as independent of the mapping from N, by which N is *ordered* as a simple series.[46]

Here Dedekind formulates the *basic insight* that had prompted a radical revision of section 5 of the manuscript with the introduction of the notion of a *simply infinite system*. That definition is the foundation for the sequence of thoughts we analyzed and that provide the "complete justification" of the concept of numbers in *WZ*. In the letter to Weber, Dedekind's own *new* perspective is indicated only by one short phrase, namely, when "my ordinal numbers" are characterized as "the abstract elements of the ordered simply infinite system".[47]

While revising the 1887 penultimate manuscript, Dedekind not only introduces the notion of a simply infinite system, but eliminates at the same spot an important passage. He had introduced "abstraction" in almost exactly the same way as in *Erklärung* #73 which we quoted in A.2, but included more expansive remarks which are absent from the published version (see the Appendix for more details):

> By this abstraction, the originally given elements n of N are turned into new elements $\mathfrak{n}$, namely into numbers (and N itself is consequently also turned into a new abstract system $\mathfrak{N}$). Thus, one is justified in saying that the numbers owe their existence to an act of free creation of the mind. For our mode of expression, however, *it is more convenient to speak of the numbers as of the original elements of the system N and to disregard the transition from N to $\mathfrak{N}$*, which itself is a similar mapping. Thereby, as one can convince oneself using the theorems regarding definition by recursion ... nothing essential is changed, nor is anything obtained surreptitiously in illegitimate ways. (Emphasis added)[48]

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[45](Dedekind 1932, 489). The German text is: "Will man aber Deinen Weg einschlagen—und ich würde empfehlen, ihn einmal ganz durchzuführen—so möchte ich doch rathen, unter der Zahl (Anzahl, Cardinalzahl) lieber nicht die *Classe* (das System aller ähnlichen endlichen Systeme) selbst zu verstehen, sondern etwas *Neues* (dieser Classe Entsprechendes) was der Geist *erschafft* ..."

[46](Dedekind 1887, 19). The German text is as follows: "*Schöpfung der reinen natürlichen Zahlen.* Aus dem Vorhergehenden ergiebt sich, daß die Gesetze über die Beziehungen zwischen den Zahlen gänzlich unabhängig von der Wahl desjenigen einfach unendlichen Systems N sind, welches wir die Zahlenreihe genannt haben, sowie auch unabhängig von der Abbildung von N, durch welche N als einfache Reihe *geordnet* ist."

[47]The German phrase is: "die abstracten Elemente des geordneten einfach unendlichen Systems."

[48]The German text is as follows: "Da durch diese Abstraction die ursprünglich vorliegenden Elemente n von N (und folglich auch N selbst in ein neues abstraktes System $\mathfrak{N}$) in neue Elemente $\mathfrak{n}$, nämlich in Zahlen umgewandelt

So it seems quite clear that the reflections on proof theoretic equivalence led Dedekind to abandon the creation of the abstract domain $\mathfrak{N}$. What positive role could $\mathfrak{N}$ really play in mathematical practice? After all, due to our intellectual limitations emphasized in (1854, 428-9; cf. Note 28) and the nature of our step-by-step understanding (*Treppenverstand*) described in (*WZ*, 791), we have to give proofs. The fact that the objects in $\mathfrak{N}$ are to have no other properties and relations than those specified by the concept under which they fall is quite directly captured by restrictions on the sequence of thoughts that constitute a *derivation*. Derivations proceed by logic and start either from the characteristic conditions M of the concept or from their instantiations M^N. Thus, in either case, derivations yield a truly "sober dissection of the sequence of thoughts on which the laws of numbers are based".[49] This shift to a schematic, formal approach even for natural numbers is a dramatic change in perspective. It provides, in the end, the complete justification of the *concept* of numbers.

C. Creating by Abstraction

The radical shift in Dedekind's perspective we discussed in B.3 is literally visible on one page of the penultimate manuscript of *WZ*; see our Appendix. In the first half of section C.1 we argue that Dedekind's shift from abstract objects to abstract concepts is the endpoint of a quite natural evolution. It allows us to reinterpret *SZ* from the perspective of that endpoint. We discuss in the second half of C.1 different contemporary perspectives that lead us to the central issues of C.2, the creation of concepts by abstraction. Finally we describe in C.3 how Dedekind's "axiomatic standpoint", as Emmy Noether called his perspective on mathematics, was transformed into structural axiomatics in the hands of Hilbert and Zermelo.

C.1. From Objects to Concepts

Dedekind's motivation for introducing the irrational numbers *corresponding to* or being *completely determined by* cuts is quite straightforward: for the laws real numbers satisfy it is irrelevant that, for example, 1 is an element of the left part of the cut determining $\sqrt{2}$. The *basic insight for natural numbers* described in B.3, and associated with #107 in the penultimate manuscript, was formulated by Dedekind as follows: "[T]he laws regarding the relations between numbers are completely independent of the choice" of the system N, the base element 1, and the mapping φ that orders N as a simple series. That independence, Dedekind realized, does not necessitate the creation of abstract objects, but can be achieved by introducing an appropriate abstract concept and focusing on proofs that use as starting points only characteristic conditions of this abstract concept and proceed otherwise by "logical steps".

   This realization is not far-fetched and did not come out of the blue. There is, first of all, Dedekind's general experience in algebraic number theory and his work with abstract, structural concepts. This general experience is complemented, secondly, by an important aspect in the evolution of Dedekind's thought on "ordinary" numbers. We are alluding to Dedekind's move away from the genetic conception of natural numbers and their generative expansion to integers as well as rational numbers. The genetic conception can be seen in his *Habilitationsrede* of 1854, but also in *SZ*. Right

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sind, so kann man mit Recht sagen, daß die Zahlen ihr Dasein einem freien Schöpfungsacte des Geistes verdanken. Für die Ausdruckweise ist es aber bequemer, von den Zahlen wie von den ursprünglichen Elementen des Systems N zu sprechen, und den Übergang von N zu $\mathfrak{N}$, welcher selbst eine deutliche Abbildung ist, außer Acht zu lassen, wodurch, wie man sich mit Hilfe der Sätze über Definition durch Recursion ... überzeugt, nichts Wesentliches geändert, auch Nichts auf unerlaubte Weise erschlichen wird." (Dedekind 1887, §5)

[49] (Dedekind 1932, 336–7) and (Ewald 1996, 791). The German phrase is: "nüchterne Zergliederung der Gedankenreihen, auf denen die Gesetze der Zahlen beruhen". Dedekind emphasizes the importance of derivations in the very next sentence: "Ich erblicke dagegen in der Möglichkeit, solche Wahrheiten auf andere, einfachere zurückzuführen, mag die Reihe der Schlüsse noch so lang und scheinbar künstlich sein, einen überzeugenden Beweis dafür, daß ihr Besitz oder der Glaube an sie niemals durch innere Anschauung gegeben, sondern immer nur durch eine mehr oder weniger vollständige Wiederholung der einzelnen Schlüsse erworben ist."

after the publication of *SZ*, he began to examine those presuppositions in a critical way. Sieg and Schlimm describe in their (2005, section 3, *Extending Domains*) the increasingly structural approach Dedekind explores for the expansions even before having completed *WZ*. Later on he uses *WZ* as the basis to show, in a perfectly modern way, how to introduce the standard arithmetic operations on (equivalence classes of) pairs of natural numbers and to obtain a system of mathematical objects that falls under the abstract notion of a field; see (Sieg and Schlimm 2005, section 4, *Creating Models*).

This structural approach was extended in *WZ* to the natural numbers themselves; here let us briefly show how it can be used to *reinterpret SZ*. Recalling our discussion at the end of subsection B.1 we observe, in a first step, that *SZ* introduces the structural definition of a "fully continuous domain", which has the system of all cuts of rational numbers (with its induced ordering) as a model.[50] This observation can be broadened, through a second step, to the concept "complete ordered field" by following *SZ* in defining the arithmetic operations on cuts. The final third step, rounding off the metamathematical treatment of complete ordered fields in analogy to that of simply infinite systems, requires Dedekind's general concept of mappings. The latter makes it possible to prove a *representation theorem* ("any complete ordered field is isomorphic to the system of all rational cuts") and then to establish that any two complete ordered fields are isomorphic; i.e., the notion is categorical. The point of these reinterpreting and extrapolating remarks is that the architecture of *SZ* is already refined to such a degree that the analogy to *WZ* is not forced, but rather direct and natural.

The data from *WZ*, i.e, metamathematical results and methodological observations, that we have been appealing to have been used by other commentators in different ways to support particular "structuralist positions". Stewart Shapiro, for example, sees Dedekind as presenting an incipient version of his *ante rem* structuralism (Shapiro 1997), while Charles Parsons considers interpretations of Dedekind as an *eliminative* and *non-eliminative* structuralist; see (Parsons 1990) for the terminology. Parsons highlights, see p. 307, that *Erklärung* #73 and Theorem #132 invite an eliminative structuralist reading, though he does not endorse such an interpretation. More specifically, if we allow $\Gamma(N, 0, S)$ to stand for the characteristic conditions of a simply infinite system, where $N, 0, S$ are variables for the system, base element and successor mapping, respectively, and if we allow $A(N, 0, S)$ to stand for an arithmetic statement, we can consider $A(N, 0, S)$ to be elliptical for:

$$(\forall N)(\forall 0)(\forall S)\,(\Gamma(N, 0, S) \to A(N, 0, S))$$

Note that all occurrences of $N, 0, S$ are "quantified out" and thus removed in a sense that would satisfy an eliminative structuralist.[51]

Erklärung #73 and Theorem #132 are also used to support Tait's non-eliminative interpretation, the crucial aspect of which is "Dedekind abstraction".[52] This abstraction is the process by which, given a particular structure, one introduces a new structure of the same type along with an isomorphism from the original system to the newly introduced one, see (Parsons 1990, 308). On

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[50]Expanding the "order axioms" by the principle of continuity yields the concept of "a fully continuous domain". The main proposition of *SZ* is Theorem *IV* in section 5 stating that the system of all cuts of rational numbers is continuous.

[51]It is important to note that this description of the logical situation is in direct conflict with both Dedekind's and Hilbert's way of articulating their structural definitions of, e.g., simply infinite system and Euclidean space. They give an explicit structural definition applying to systems. This is the reason why Hilbert and Bernays call this way of formulating an axiomatic theory *Existentiale Axiomatik*; they view, as Dedekind did, giving a proof that there is a system falling under the structural definition as a crucial task. That is beautifully and extensively discussed in (Bernays, 1930, 20–21).

[52]Linnebo and Pettigrew discuss Dedekind abstraction in their recent paper "Two Types of Abstraction for Structuralism" (Linnebo and Pettigrew 2014). They argue that this notion of abstraction is unable to satisfy both of two structuralist goals, namely, to ensure that: (i) there is a unique natural number structure, and (ii) the elements of the natural number structure have no "foreign properties". Their arguments are convincing. However, the second of these goals can be achieved by different means for any simply infinite system as we argued in section B above; the first goal is given up by Dedekind.

Tait's interpretation, Dedekind starts with a particular simply infinite system, and, as the concept of a simply infinite system is categorical, he can introduce a new system and corresponding isomorphism. The latter system is then taken to contain *the* natural numbers. However, it is precisely this abstracting move that is removed in the transition from the 1887 manuscript to the published version of *WZ*. The justification for removing it is already indicated in the manuscript. After all, Dedekind claims there that by disregarding the abstraction, " ... as one can convince oneself using the theorems regarding definition by recursion ... nothing essential is changed, nor is anything obtained surreptitiously in illegitimate ways." This highlights Dedekind's "indifference to identification" with respect to natural numbers, using Burgess' apt phrase (Burgess 2011, 8).

Reck and Yap suggest a further, novel interpretation. More particularly, they interpret Dedekind as possessing his own, unique brand of structuralism which they call *logical structuralism.* This is deeply concerned with the creation of mathematical *objects*, in particular natural numbers and their system. Let us sketch the considerations presented in (Reck, 2003) and (Yap, 2009). Their anchor is Dedekind's remark in the very first section of *WZ*, "A thing is completely determined by all that can be affirmed or thought concerning it" (*WZ*, 797); this is apparently Leibniz's principle of the identity of indiscernibles. According to Reck's interpretation, any legitimate attempt to create mathematical objects must consequently ensure that they are completely determined. Reck suggests that, to do just this, Dedekind developed a procedure in which his metamathematics plays a significant role. The procedure, as applied to the natural numbers, aims to establish that they are "completely determined" in the sense that (i) it specifies precisely what statements can be formed about them, and (ii) it ensures, given such a statement ϕ, that exactly one of ϕ or $\neg\phi$ follows *semantically* from Dedekind's "characterizing conditions".[53] Furthermore, each step appeals to purely logical means, hence the name of *logical* structuralism. The procedure consists of the following steps, see (Reck 2003, 395):

(i) Designing a language with which to speak about the natural numbers. This language includes a name for the base element, and a function symbol for the successor function.

(ii) Articulating "the basic definitions and principles" from which all arithmetical truths should follow.

(iii) Establishing that the notion of a simply infinite system is categorical.

Together these steps ensure that the (system of) natural numbers is completely determined: "By (iii), for any sentence ϕ in the language of arithmetic, as specified in (i), either ϕ or $\neg\phi$ follows (semantically) from the basic definitions and principles, as specified in (ii); *tertium non datur*" (Reck 2003, 395). This procedure "creates", according to Reck, the system of natural numbers in the following sense: "It is *identified* as a new system of mathematical objects, one that is neither located in the physical, spatio-temporal world, nor coincides with any of the previously constructed set-theoretic simple infinities ... what has been done is to *determine uniquely* a certain "conceptual possibility", namely a particular simple infinity ... It is that simple infinity whose objects only have arithmetical properties, not any of the additional, "foreign" properties objects in other simple infinities have" (Reck 2003, 400).

Our interpretation is evidently quite different from Reck's. As we have discussed above, Dedekind is concerned with theorems obtained by stepwise derivations, not as semantic consequences. Most

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[53]In *WZ* there is no formal language, nor is there a concept of semantics. Thus, a notion of *semantic consequence*, as is presupposed here, is not available: it emerges only slowly. The development culminates in Gödel's doctoral dissertation (Gödel 1929), but not in the contemporary Tarskian way via a truth-definition. The fundamental "semantic" relation is rather a structure "falling under" or "satisfying" a second-order concept; see (Sieg and Schlimm 2014, note 17). This difference from the modern concept is discussed in (Schiemer and Reck 2013) with respect to Gödel, but also Hilbert and Ackermann whose terminology Gödel followed. However, Schiemer and Reck do not indicate the path from Dedekind's *WZ* through Hilbert's *Grundlagen der Geometrie* to the Hilbert Lectures of 1917/18 and, thence, to Hilbert and Ackermann. Dedekind's grappling with "model theoretic" problems is being analyzed in the present section.

importantly, Dedekind is ultimately creating concepts rather than objects, and "the" natural numbers are in *WZ* explicitly taken as the elements of a previous logical simply infinite system. It is to an understanding of "creation of concepts" that we turn next. This is an intricate, complex subject. We focus on Dedekind and what seem to us the most direct influences on him originating from Kant and Lotze.

C.2. Formation of Concepts (Begriffsbildung)

The emphasis on concepts in Dedekind's reflection on mathematics goes back to his *Habili-tationsrede* of 1854, where he claimed:

> The introduction of such a concept ... is, as it were, a hypothesis which one puts to the inner nature of the science; only in the further development does the science answer it; the greater or lesser *efficacy* of such a concept determines its worth or unworth.[54]

In the Preface to *WZ* he refers back to (1854) and asserts that the most fruitful advances in mathematics as well as in other sciences are made by the "creation and introduction of new concepts". One traditional method of creating new concepts is *abstraction*, and Dedekind appeals to abstraction, when explaining in #73 what natural numbers are, which he had done also in the 1887 manuscript.[55]

But there, as we saw, he introduces new abstract objects that then constitute the abstract system $\mathfrak{N}$. Exactly that step to the system $\mathfrak{N}$ is taken back in the transition from the 1887 draft to *WZ*. And yet, Dedekind refers in his 1888 letter to Weber to "abstract elements" of *the* ordered simply infinite system, and in his 1890 letter to Keferstein he calls the "Zahlenreihe *N*" the "abstract type" of a simply infinite system. Here Dedekind understands "abstract" in the traditional context of conceptual abstraction—no new abstract objects are introduced, but familiar objects fall under more abstract concepts. This "abstraction *from*" is described in Kant's *Logik* (*Elementarlehre* §6, A 145-148): dropping some characteristic conditions or *Merkmale* from a given concept leads to a more abstract one. That's how we read #73 of *WZ*, quoted in full towards the end of section A.2. Dedekind considers there a particular number sequence *N*; he "retains" only the distinguishability of its elements and "takes into account" only their relations to each other that are due to the mapping φ. The elements of *N* are now called numbers, and the abstraction allows one to call *them*, when viewed from this abstracting standpoint, a free creation of the human mind. This is the sense in which we also understand Dedekind's brief remark in the letter to Weber when he describes his ordinal numbers as "the abstract elements of the ordered simply infinite system".[56]

This reading can be much more directly supported, if Dedekind's particular way of defining simply infinite systems is explicitly taken into account: *any* system *N* of objects can be considered to be simply infinite, if there is some element 1 and some mapping φ on *N* that satisfy the notion's characteristic conditions. This definition reflects a form of abstraction that is radically different from traditional abstraction as discussed above following Kant's *Logik*. Lotze introduced it in the 1843 edition of his *Logik* and viewed it as a most significant contribution. It had the clear goal of

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[54](Dedekind 1932, 429) and (Ewald 1996, 756). The German text is as follows: "Die Einführung eines solchen Begriffs ... ist gewissermaßen eine Hypothese, welche man an die innere Natur der Wissenschaft stellt; erst im weitern Verlauf antwortet sie auf dieselbe; die größere oder geringere *Wirksamkeit* eines solchen Begriffs bestimmt seinen Wert oder Unwert." Juliet Floyd argues in her (2013, 1025–1027) that the initial discussion of the *Habilitationschrift* "echoes Kant's account of reflective judgement in his introduction to *The Critique of Judgment*".

[55]Other mathematicians, prominently Cantor, also used "abstraction"; see (Deiser 2010, 60–62) and (Hallett 1984, 119 ff.). Let us mention Ernst Cassirer who in his *Substanzbegriff und Funktionsbegriff* refers to "freie Produktion bestimmter Relationszusammenhänge" (pp. 15-16); he discusses abstraction in general terms on p. 6 and with direct reference to *WZ* on p. 50ff.

[56]This might seem to be a round-about interpretation. We feel, obviously, it is coherent and allows an informed understanding of Dedekind's evolving perspective on mathematics. However, we face here the additional task to explain, how Dedekind could retain the formulation of the manuscript in #73 despite the fact that the context had been changed by the introduction of the concept of a simply infinite system. The issue is also illuminated by the circumstances of the preparation of the final manuscript as described in the Appendix.

reflecting the actual practice of ordinary and scientific thinking. Lotze describes it with pertinent examples in the second edition of his *Logik*, §23, *Die Lehre vom Begriffe*. This abstraction usually does not drop *Merkmale*, but rather replaces some by more general ones and yields in this way "more abstract" concepts.[57] That is indeed done in *WZ*, as we saw in A.2, when the structural definition of a simply infinite system is introduced: any particular system together with its base element and order setting mapping can fall under the more abstract concept. This standpoint clarifies further Dedekind's assertion that the ordered system N is the *abstract type* of simply infinite systems, cf. Note 26; but it also allows us to point to a real difference between the two abstractions, when we join the above considerations with those for deducibility in subsection B.2. In the first case, we accept as principles the interpreted characteristic conditions M^N; in the second case, we work directly with M. Though this is a difference, it is one without consequence as far as the "laws of arithmetic" are concerned; that was established in B.2.

It seems clear then that Lotzean abstraction is at work and leads to the concept of a simply infinite system. However, for it to be applicable, there must be a particular such system from which to abstract. Dedekind has thus to prove the existence of a suitable system; that proof ensures at the same time that the notion has no internal contradictions. For the proof, Dedekind needs objects, a distinguished element, and an order setting mapping that together constitute a simply infinite system. The objects, including his Ego as the base element, and the order setting mapping are obtained in logic. Indeed, he calls his proof of #66 in *WZ*, when writing to Keferstein, a "logical existence proof".[58] The need for such a proof is recognized, it seems, only in the revisions to the penultimate 1887 draft of *WZ*; the emendations are described at great length in (Sieg and Schlimm 2005, section 6.1). That only reemphasizes the dramatically changed perspective underlying the structural definition of a simply infinite system.

It is truly fascinating, given our understanding of *WZ*, that Dedekind extended his structuralist approach from algebraic number theory to the fundamental objects of traditional mathematics, i.e., to number systems beginning with the natural numbers and ending with the real and complex ones. The character of individual numbers is no longer at issue, but rather that of systems whose elements stand in particular relationships. That is made clear in *WZ* last but not least by the systematic metamathematical investigation of simply infinite systems. Dedekind describes in the preface to the first edition of *WZ* (pp. *v–vi*) how, on this basis, a program of "the stepwise extension of the number *concept*" can be carried out, covering negative, fractional, irrational and complex numbers. The extended notions are obtained always by "a reduction to the earlier *concepts*" without mixing in "foreign conceptions" like that of "measurable magnitudes". Indeed, Dedekind claims that the latter magnitudes can be understood in "complete clarity" only through the science of numbers. That is for Dedekind, it seems, the ultimate point of the arithmetization of analysis.

The formation of abstract concepts and the insistence on stepwise argumentation from their characteristic conditions locate Dedekind's structuralism within the logic of his time, in particular, that of Lotze and Schröder. The principles of unrestricted comprehension and extensionality, both used in *WZ* and viewed as logical principles, form the framework for founding number theory—with one crucial addition, the notion of *mapping*. The latter is also part of logic.[59] Dedekind asserted in 1879, long before the completion of *WZ*, "the entire science of numbers is based on this intellectual ability ... without which no thinking at all is possible". The intellectual ability he pointed to is that of establishing correspondences, and that is of course reflected in *WZ* mathematically by mappings. Abstract concepts and structure-preserving mappings are also the crucial tools of Bourbakian math-

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[57]Lotze's abstraction contains the Kantian one as a special case. We have thus isolated two different kinds of conceptual abstraction. Both forms are different from the abstraction used by Frege and Neo-logicists. The latter is, in Kantian terminology, "abstraction *to*". That is made marvelously clear in Chapter 1, *Philosophical Introduction*, of (Fine 2002).

[58]Frege found Dedekind's argument, with the emendation that "thoughts exist independently from our thinking", fully convincing; see the note on pp. 147-8 of (Frege 1969).

[59]Recall our remarks in section A.1. It was only Zermelo who introduced, in his 1908 paper, mappings as set theoretic objects—under the very name of *Abbildungen*; see (Sieg and Schlimm 2014, section 2.2).

ematics concerned with the *principal structures* (structures-mères) of algebra, topology and order. We alluded in the Introduction to this connection with contemporary mathematics; it emerged out of the transformation of the subject that is brilliantly reflected in Dedekind's mathematical and foundational work.[60] In the next section we explore how Dedekind's methodological perspective underlies crucial developments in the early part of the 20$^{\text{th}}$ century, namely, the emergence of structural axiomatics as opposed to its later formal variety; cf. (Sieg 2014). A good starting point is Hilbert's Paris address of 1900 challenging mathematicians with twenty-three problems. The second of these problems is most closely related to the methodological issues we have been discussing.

C.3. Structural Axiomatics

In the Introduction to his Paris talk, Hilbert views the "arithmetic comprehension of the *concept* of the continuum" as one of the most important developments in 19$^{\text{th}}$-century mathematics. At the same time, he wants to refute the view "that only the *concepts* of analysis or even those of arithmetic alone are susceptible to a completely rigorous treatment". On the contrary, he thinks, emphasizing *concepts* throughout:

> ... wherever, from epistemology or in geometry or from scientific theories, mathematical concepts emerge, mathematics has the task to investigate the principles on which these concepts are based and to fix them [the concepts] by a simple and complete system of axioms in such a way that the exactness of the new *concepts* and their use *in deductions* is in no way inferior to the old arithmetic concepts.[61]

For geometry this goal had been pursued and, in Hilbert's view, successfully reached through his *Festschrift* (Hilbert 1899). That work gives, after the addition of the completeness principle, a structuralist foundation for geometries, in particular for Euclidean geometry, in the same way Dedekind's work does for number systems, in particular for the system of real numbers. The emphasis on concepts and their use in deductions in the above quote is noteworthy. Similarly close to Dedekind's view is Hilbert when explaining his views on geometry to Frege:

> If I think of my points as any arbitrary system of things, for example, the system: love, law, chimney sweeps ... and then only assume all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements. *For one merely has to apply an invertible transformation and stipulate that the axioms for the transformed things be the correspondingly same ones.* (Emphasis added.)[62]

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[60]These developments led naturally to category theory. As far as Bourbaki's work is concerned, it began in the early 1930s; see the papers (Bourbaki 1950) and (Dieudonné 1970) that describe the emergence of their work and the influence of Dedekind, Hilbert and Noether. What were the motivating and guiding ideas at that time? Helmut Hasse isolated general methodological features in a talk entitled *Die moderne algebraische Methode* that was given at the annual meeting of the German Association of Mathematicians in September 1929. The methodological features are realized through abstract domains and overarching concepts; they are illustrated by many examples, of course, from algebra. However, Hasse emphasizes that the modern algebraic method permeates all of mathematics: "Everywhere one can apply its principle to find the simplest conceptual foundations for a given theory and to have, in this way, a unifying and systematizing effect" (Hasse, 33). This is supported according to Hasse by a certain philosophical position reflected in the axiomatic standpoint (p. 28).

[61](Hilbert 1900, 49). Here is the German text: " ... wo immer von erkenntnistheoretischer Seite oder in der Geometrie oder aus den Theorien der Naturwissenschaft mathematische Begriffe auftauchen, erwächst der Mathematik die Aufgabe, die diesen Begriffen zu Grunde liegenden Prinzipien zu erforschen und dieselben durch ein einfaches und vollständiges System von Axiomen derart festzulegen, daß die Schärfe der neuen Begriffe und ihre Verwendbarkeit zur Deduktion den alten arithmetischen Begriffen in keiner Hinsicht nachsteht."

[62]The full German text is as follows "Wenn ich unter meinen Punkten irgendwelche Systeme von Dingen, z.B. das System: Liebe, Gesetz, Schornsteinfeger ... denke und dann nur meine sämmtlichen Axiome als Beziehungen zwischen diesen Dingen annehme, so gelten meine Sätze, z.B. der Pythagoras auch von diesen Dingen. Mit andern

This expresses, in a general and informal way, Dedekind's striking insight formulated as #133 (and analyzed in section B.1).

The centrality of *concepts* for modern mathematicians is not special to Dedekind and Hilbert, but seems to be equally important to "classical" constructive mathematicians such as Kronecker: Kronecker's sole extended foundational essay (1887) has the revealing title "On the number concept" (*Über den Zahlbegriff*). However, let us turn to Hilbert's (1900) and his formulation of the second problem that asks one to find a *direct* proof of the consistency of analysis. Hilbert's mathematical formulation and his optimism for finding a direct proof through a suitable modification of the "familiar inference methods in the theory of irrational numbers" are well known; the extended remark he added to his formulation was to characterize the significance of the consistency problem from a different, more philosophical perspective. Here we point out that the consistency of *concepts* is to be established and that their "mathematical existence" is to be guaranteed:

> If one succeeds in proving that the characteristic conditions ⟦*Merkmale*⟧ given to the concept can never lead to a contradiction through a finite number of logical inferences, then I say that the mathematical existence of the concept ... has been proved in this way.[63]

This can be taken as definitional: a concept *exists mathematically* if and only if it is consistent in the sense that no contradiction can be inferred from its characteristic conditions in a finite number of logical steps. If however, as Hilbert clearly does, the mathematical existence of a (second level) concept is taken as the basis for inferring the existence of a system of mathematical objects falling under that concept, then a step is taken that is not justified in general (and no longer maintained in the Hilbert school during the 1920s). The issue is still a topic of wide ranging discussion, frequently in the context of the Frege-Hilbert correspondence.[64] The penetrating reflections of Bernays are articulated in the essay "Mathematische Existenz und Widerspuchsfreiheit" (Bernays 1950) and involve, centrally, a notion of *bezogene Existenz*, i.e., the view that mathematical existence claims are related to a *methodological frame*. A connection to such a broader frame is implicit already in Dedekind's and Hilbert's work.

In the background of Hilbert's considerations concerning arithmetic is *pure logic*, and that was also central for Dedekind.[65] Zermelo sharpened the *logical* framework of Dedekind and Hilbert to a *set theoretic* one and formulated its principles as axioms in his (1908) following Hilbert's structural ways. He considers a *domain* (Bereich) **B** of individuals, "which we call simply *objects* and among which are *sets*". The objects in **B** stand in *fundamental relations* of the form $a \in b$, and *sets* are those objects that have an element; the null or empty set is the only exception. The step from this set-up to the formulation of the characteristic conditions for the domain is taken with the sentence, "The fundamental relations of our domain **B**, now, are subject to the following axioms, or postulates."

Worten: eine jede Theorie kann stets auf unendliche viele Systeme von Grund-elementen angewandt werden. Man braucht ja nur eine umkehrbar eindeutige Transformation anzuwenden und festzusetzen, dass die Axiome für die transformirten Dinge die entsprechend gleichen sein sollen. Thatsächlich wendet man auch diesen Umstand häufig an, z.B. Dualitätsprinzip etc, und ich in meinen Unabhängigkeitsbeweisen." Quoted in (Frege 1980, 14).

[63](Hilbert 1900, 55-56). Here is the German text: "Gelingt es jedoch zu beweisen, daß die dem Begriffe erteilten Merkmale bei Anwendung einer endlichen Anzahl von logischen Schlüssen niemals zu einem Widerspruche führen können, so sage ich, daß damit die mathematische Existenz des Begriffes ... bewiesen worden ist."

[64]See, for example, (Shapiro, 2009). It seems that Hilbert's views around 1900 are complex and sometimes conflicting. An analysis of his lecture notes should provide valuable insights. However, there are a few significant remarks, e.g. (Hilbert 1900a, 1095), (Hilbert 1900b, 1105), and (Hilbert 1905, 137–138). In (Hilbert 1922, 158-9) one finds a slightly modified formulation that is, nevertheless, of real interest. Hilbert describes the axiomatic method as usual and asserts, "The continuum of real numbers is a system of things that are connected to each other by certain relations, so-called axioms. ... [C]onceptually a real number is just a thing of our system." He completes the discussion by adding, "The standpoint just described is altogether logically completely impeccable, and it only remains thereby undecided, whether a system of the required kind *can be thought [ist denkbar]*, i.e., whether the axioms do not lead to a contradiction." Here "Denkbarkeit" not "Existenz" (in some sense) is to be guaranteed by consistency.

[65]Logicist statements are found in Hilbert's early lectures throughout the 1890s; see (Sieg 2013, 83-4).

This classic paper then presents "the axioms of set theory" and develops from them a theory of equivalence that avoids the formal use of cardinal numbers. Dedekind's way of metamathematically investigating simply infinite systems is taken up for *normal domains* (*Normalbereiche*) in (Zermelo 1930) or, to focus on the mathematical work without making a historical connection, Zermelo's analysis can be viewed as being carried out in parallel with Dedekind's. Any domain (of objects) is called *normal* if and only if it satisfies the axioms ZF′, namely, the principles that allow constructing segments of the cumulative hierarchy along suitable ordinals (Separation, Pairing, Power Set, Union and Replacement) together with Extensionality and Foundation.[66] Zermelo's distinction between the abstract concept of a normal domain and concrete, particular normal domains reflects that between the abstract concept of a simply infinite system and concrete, particular instantiations.[67]

Zermelo's remarkable metamathematical investigations of normal domains do not, of course, establish a categoricity result akin to Dedekind's for simply infinite systems. But what they do show is that, given two normal domains N_1 and N_2 (with sets of *urelements* of the same size), either N_1 is isomorphic to an "initial segment" of N_2 or vice versa. It would lead us too far astray to look at the issues Zermelo raises concerning a new axiom that postulates "the existence of an unbounded sequence of boundary numbers" (*die Existenz einer unbegrenzten Folge von Grenzzahlen*), at Gödel's program of ever stronger axioms of infinity, or at Friedman's work on the effect of such axioms on ordinary mathematics. However, set theory can be viewed, and was viewed originally, from a less exalted perspective as a uniform framework for mathematics. In this spirit Zermelo wrote in (1908):

> Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions "number", "order", and "function", taking them in their pristine form, and to develop thereby the logical foundations of all of arithmetic and analysis; thus it constitutes an indispensable component of the science of mathematics. (p. 200)

Hilbert, who had stimulated Zermelo's interest in set theory and encouraged him to publish his investigations, made this connection later in a refined way:

> Set theory encompasses all mathematical theories (such as number theory, analysis, geometry) in the following sense: the relations that hold between objects of one of these mathematical disciplines are represented in a completely corresponding way by relations that obtain [between objects] in a sub-domain of Zermelo's set theory. (Hilbert 1920, 356)

In our way of speaking, the abstract concepts can be instantiated with sets in a segment of the cumulative hierarchy. From work in proof theory we know that, e.g., for the practice of analysis, only a very small part of even the first step in building the cumulative hierarchy is needed, when one takes natural numbers as urelements. So, (fragments of) set theory or other theories of "canonically generated" objects, such as the elements of constructive number classes, can serve as *methodological frames* in the sense of Bernays.[68] Bernays emphasized that his reflections on methodological frames apply not only to mathematics but also to the sciences.

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[66] The axiom of infinity is not viewed as part of *general* set theory, whereas the axiom of choice is being considered as a logical principle on the basis of which one can assume that every set is well-ordered.

[67] It is of interest to note that a very closely related distinction is extracted in (Martin 2010) from Gödel's work on set theory. Martin contrasts two senses of the "concept of set": the instances of the concept of set in "my [i.e., in Martin's] sense" are not sets, but rather concepts of set in "the straightforward sense"; the latter pertain to particular objects considered as sets. That distinction is explicated, most clearly on p. 358 and p. 361, and Martin argues that *his* sense of the concept of set "fits better than the straightforward one with much of what Gödel says about set theory". Relatedly, Parsons' "Platonism and Mathematical Intuition in Kurt Gödel's Thought", in particular sections 4 and 5 (pp. 167–185), is an illuminating discussion of the intricacies of Gödel's thinking on the issue. Feferman presents in his (2009) and (2014) a position he calls *conceptual structuralism*; it is prima facie close to Dedekind's structuralism. However, he does not sustain the distinction that Martin makes for set theory and that we have analyzed in Dedekind, but rather shifts between concepts and conceptions, i.e., abstract concepts and their instantiations.

[68] See (Sieg 2013, Essay III.2) for an indication of the rich material that supports such a *reductive structuralism* in which methodological frames provide, in particular, the basis for consistency proofs, be they semantic or syntactic.

Concluding Remarks

Lotze considered mathematics in his (1989, #18) to be rooted in logic, as did Dedekind, who according to our understanding was deeply influenced by Lotze's views on concept formation, in particular, on abstraction. Those views can be taken as a basis for integrating mathematical with ordinary and scientific knowledge. Let us look at one example in Dedekind, namely, the precise investigation of our *representations* (*Vorstellungen*) *of space*. Already in (*SZ*, 11), he remarks that space, "if it has a real existence", need not be continuous; however, nothing can prevent us from completing it in thought. Coming back to this very idea, he presents in (*WZ*, 793) an analytic model of Euclidean geometry that is nowhere continuous. So, for Dedekind the precise investigation of space, as asserted in (*WZ*, 791), is made possible by relating our representations to the "continuous number-realm"; the latter is obtained in arithmetic, which in turn has a "purely logical development" (*rein logischer Aufbau*).[69]

The idea of considering systems of objects structured in particular ways as corresponding to another realm was taken up by Hilbert already in 1893 for geometry, which he viewed as "the most perfect natural science"; using the notions "*System*" and "*Ding*" so prominent in *WZ*, he formulated the central question for the foundations of geometry as follows:

> What are the necessary and sufficient and mutually independent conditions a system of things has to satisfy, so that to each property of these things a geometric fact corresponds and conversely, thereby making it possible to completely describe and order all geometric facts by means of the above system of things? (Hilbert 1894, 72-3)

Four years later, in the notes for Hilbert's 1898-99 lectures, this remark is almost literally repeated; however, it is now explicitly connected to Hertz's *Prinzipien der Mechanik* and the last part of the above sentence (after "conversely") is replaced by, "thereby having these things provide a complete 'image' of geometric reality".[70]

This broad and open perspective on *theory, model* and *reality*, informed by logical, mathematical and scientific developments, was sustained in Hilbert's work in physics through the 1920s as well as in his comprehensive and ambitious Königsberg lecture of 1930, "Naturerkennen und Logik". It also pervades the reflections of philosophers of science. We are thinking, in particular, of Ernest Nagel and Patrick Suppes.[71] The work of Suppes is perhaps most directly shaped by the structuralist, Bourbakian approach of modern mathematics: the methodological framework is set theory; scientific theories are given as "set theoretic predicates" or, as we would say, by structural definitions; models of a theory are set theoretic structures that satisfy the conditions of the predicate articulating

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In the spectrum of current structuralisms, it combines "relativist" and "predicate" structuralism as characterized in sections 4 and 8 of (Reck and Price 2000). It does so in a principled, but open-ended way, as the domains of "canonically generated" objects can be obtained by a variety of (constructive) operations. In Dedekind's case, the methodological framework is logic in the broad sense of, e.g., Lotze. The latter emphasizes in #18 of his *Logik* that "die Grundbegriffe und Grundsätze des Mathematik ihren systematischen Ort in der Logik haben".

[69] These matters are reflected in Hilbert's early lectures; see (Sieg 2013, 84-5).

[70] This is followed by, "The axioms, as Hertz would say, are images or symbols in our mind, such that consequences of the images are again images of the consequences, i.e., what we derive logically from the images is true again in nature." To see how that is connected to parts of the contemporary discussion in philosophy of science, we recommend reading the "Introduction: the 'picture theory of science'" in van Fraassen's *Scientific Representation*. The general direction of Hilbert's considerations is further expounded in lectures he gave in 1902; see (Hallett and Majer 2004, 540-602). The description of the goals on pp. 540–1 is illuminating: for the construction of the "logisches Fachwerk von Begriffen" for geometry one assumes only the "Gesetze der reinen Logik und Zahlenlehre". This raises immediately the question, "Welche Sätze müssen wir diesem Bereich [der Logik und Zahlenlehre] adjungieren, damit wir die Geometrie aufbauen können?" The main question (Hauptfrage) is then formulated in almost exactly the same way as in (1894).

[71] See Nagel's remarks on the "construction of scientific concepts" (Nagel 1961, 14) and the two chapters of the same book on models (Chapters 5 and 6). As to Suppes, we are referring to his *Representation and Invariance of Scientific Structures*. His earlier expositions of this view, articulated in this book systematically and with a great number of examples, has deeply influenced the emergence of scientific structuralism; see, for example, Chapter 12 of (Suppes 1957).

the theory; structure-preserving mappings are crucial for formulating and proving representation theorems. What is left quite open, in both Dedekind's and Hilbert's remarks, is the correspondence to "reality"; in Suppes, there is a crucial theory of measurement and measurement structures.

Dedekind's and Hilbert's introduction of abstract concepts had a transformative impact on the foundations and practice of mathematics in the 20th century. Our remarks in the last three paragraphs are only intended as an indication of the far-reaching consequences the radical transformation of mathematics in the 19th century had both on scientific practice and on informed philosophical reflections about the scientific enterprise. Conversely, the transformation of mathematics was, undoubtedly, also motivated by the co-evolution of mathematical and scientific developments. We are thinking especially, and very narrowly, of the work of Göttingen mathematicians—Gauss, Dirichlet, and Riemann—that exemplifies this co-evolution. Dedekind was very deeply associated with each of them, and Hilbert stood quite consciously in their tradition. Here is one important locus, in a significantly broader context, of a truly scientific philosophy: free from foundational dogma and open to both intellectual experience and experimentation.

Appendix. Manuscripts of *WZ*

Dedekind started to work intermittently on the foundations for natural numbers right after having completed *SZ*. The result of his work is assembled in a "First Draft" (Dedekind 1872/78); it is published in (Dugac 1976, 293-309), and its mathematical content is described in (Sieg and Schlimm 2005, 140-144). In the Preface to *WZ*, Dedekind remarks on p. iv that this earlier manuscript contains "all essential basic thoughts of my present essay". He mentions as critical points "the sharp distinction between the finite and the infinite", the concept of cardinal, and the justification for the proof principle of induction as well as for the definition principle of recursion. Though Dedekind does treat proof by induction in this manuscript, there are only very brief, almost cryptic remarks concerning definition by recursion.

The second draft was written in June and July of 1887; the third draft was written during the period from August to October 1887. (The dates are in Dedekind's hand on the cover pages of the documents; it should be mentioned that the third draft is still quite different from the published version.) In the second draft, which is incorporated into the third one, a dramatic shift occurs: the concept of a simply infinite system is introduced and major additional changes are made. This happens on page 5 of the manuscript that is fully reproduced below.

The page reproduced below reflects a perfectly standard pattern of writing manuscripts: a sheet is folded in the middle and one writes only on the left half, leaving the right half blank for later additions or corrections. So, the original text of §5 has the heading "*Die Reihe der natürlichen Zahlen*" and begins essentially with the text of #73 of *WZ*. Then, framed by Dedekind in the middle, the creation by abstraction of the elements $\mathfrak{n}$ of $\mathfrak{N}$ is described; we quoted this passage at the very end of B.3. On the right half, the new §5 has the title "*Die einfach unendlichen Systeme (Reihe der natürlichen Zahlen)*". The section begins with the definition (*Erklärung*) of the concept of a simply infinite system, essentially what is presented in #71 of *WZ*. The framed part of this page concerns the creation of $\mathfrak{N}$ and is left out in the published version.

Major changes from the second to the third draft are also found in §4, entitled "*Das Endliche und Unendliche*", where Dedekind formulates a new theorem with number 40*** that states: "*Es giebt unendliche Systeme.*" He added in parentheses that he had made "remarks on the supplementary sheet" (*Bemerkungen auf dem Beiblatt*). Unfortunately, the supplementary sheet seems not to have been preserved. All of these changes were thus made very late: Dedekind dated his Preface to *WZ*, "Harzburg, 5. Oktober 1887"; the final manuscript was delivered to the publishing house *Vieweg und Sohn* on 17 October 1887. That delivery date is mentioned in a letter Dedekind wrote on 30 October of that year to the publisher; here is the relevant passage:

For several reasons it matters a great deal to me that the essay *Was sind und was sollen*

§. 5. Die Reihe der natürlichen Zahlen.

50. Ist S ein unendliches System, giebt es
also eine deutliche Abbildung φ von S, daß
das Bild S' ein echter Theil von S wird,
so wähle ich ein Element von S, welches nicht
in S' enthalten ist, und untersuche die Be-
schaffenheit der Kette dieses Elementes ...
Indem ich hierbei von der ... besonderen
Beschaffenheit der Elemente von N gänzlich
absehe, nur ihre Unterscheidbarkeit festhalte
und lediglich die gegenseitigen Beziehungen
ins Auge fasse, in denen sie zufolge der
vorliegenden Abbildung zu einander stehen,
nenne ich die Elemente von N natürliche
Zahlen oder Ordnungszahlen, und wenn N die
Reihe der natürlichen Zahlen ... [Da ... durch
diese Abstraction ...
Elemente von N (und folglich auch N selbst) zu
... Elemente ..., nämlich ... Zahlen ...
... sind, so kann man mit Recht sagen,
daß die Zahlen ... ... Schöpfung ...,
also das ... verdanken. Für die Aus-
drucksweise ist es aber bequemer, von den
Zahlen wie von den ursprünglichen Elementen
des Systems N zu sprechen und den Übergang
von N zu 𝔑, welcher selbst eine deutliche
Abbildung ist, außer Acht zu lassen, wodurch,
wie man sich mit Hülfe des Satzes über ...
... Abbildung ... ...
... überzeugt, nichts Wesentliches geändert,
auch Nichts auf unerlaubte Weise erschlichen
wird.]

50. Die Eigenschaften von N und φ sind also fol-
gende:

α) φ ist eine deutliche Abbildung von N
β) N', d.h. φ(N) ist Theil von N
γ) N = 1₀, d.h. Nist Kette eines
   Elementes 1 von N.
δ) Das Element 1ist nicht in N' enthalten

Aus δ) folgt, daß das Bild N' ein echter
Theil von N ist, weil aus α), daß jedes
einfach unendliche System N wirklich unendlich :.

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§. 5. Die einfach unendlichen Systeme.
(Reihe der natürlichen Zahlen).

50. Erklärung: Ein System N heißt einfach unendlich
wenn N eine ähnliche Abbildung φ von N
... ... folgende Bedingungen erfüllt
in sich selbst, und zwar von der Art giebt,
daß N als Kette ... ..., in φ(N)
nicht enthaltenen Elementes erscheint.

Also ... diese solche Abbildung φ von
N kann eine Anordnung von N genannt
werden. Die Elemente von N können
(mit Rücksicht auf diese Anordnung) natür-
liche Zahlen heißen.

51. Satz: ... einem unendlichen System ist ein einfach
unendliches System N als Theil enthalten.

... eine einfache Reihe oder Ordnung.

Hier nur ... Note, später ausführlich
als Darstellung der reinen Zahlenreihe zu
behandeln ... neues abstractes System 𝔑)
§. 8 - 107.

Definition durch Recursion (auf Ketten) Ähnlichkeit
aller einfachen Reihen (N)

vielleicht besser in der Folge β, γ, δ, α

die Zahlen?, I submitted as a manuscript on the 17th of this month, be delivered at Christmas; thus, I would like to express my strong desire that printing be started very soon. The setting up [of the manuscript], I believe, does not present any difficulties and, as far as I am concerned, the proofreading will be done by me as quickly as possible.[72]

We mention these details, as it is quite clear that Dedekind made significant, indeed dramatic changes and major additions in a very short period. In particular, the shift in perspective that is reflected in #73 and #134 (with the supporting metamathematical work) was made during that time. So it is perhaps not too surprising that some "tensions" remain between the old and the new perspective. Finally, it seems that Dedekind sent copies of the book to Cantor and Weber. His letter to Weber of 24 January 1888 is responding to remarks on *WZ* that Weber had sent him (and mentions that Cantor had also responded). As to the general reception, there is a report by Hilbert who, from 9 March to 8 April 1888, made his *Rundreise* from Königsberg to other German universities. He arrived in Berlin, his first stop, just as *WZ* had been published; Hilbert recounts that in mathematical circles everyone, young and old, talked about Dedekind's essay, but mostly in an opposing or even hostile sense.[73]

Acknowledgments

References

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[72]Vieweg-Archive, V1D:17. The German text is as follows: "Da mir aus mehreren Gründen sehr daran gelegen sein muß, daß die am 17. d[ieses] M[onats] im Manuskript von mir eingelieferte Schrift "Was sind und was sollen die Zahlen?" zu Weihnachten ausgegeben werden kann, so erlaube ich mir hiermit meinen dringenden Wunsch auszusprechen, daß in diesen Tagen mit dem Druck begonnen werden möge. Der Satz bietet, wie ich glaube, gar keine Schwierigkeiten dar, und soviel an mir liegt, sollen die Korrekturen stets auf das Schnellste von mir besorgt werden". In his next letter of 13 November 1887 to his publisher, Dedekind complains that he still has not received the galleys and emphasizes that he really would like to be able to give two copies of his book as Christmas presents.

[73]That is reported in (Hilbert, 1931, 487), but also in Hilbert's diary from his trip; the latter is part of the Hilbert Nachlass, Cod. MS. 741, 1/5. See also (Dugac 1976, 203).

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