AGPh 2017; 99(1): 102–119 **DE GRUYTER**

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Hume's Fork and Mixed Mathematics

Abstract: Given the sharp distinction that follows from Hume's Fork, the proper epistemic status of propositions of mixed mathematics seems to be a mystery. On the one hand, mathematical propositions concern the relation of ideas. They are intuitive and demonstratively certain. On the other hand, propositions of mixed mathematics, such as in Hume's own example, the law of conservation of momentum, are also matter of fact propositions. They concern causal relations between species of objects, and, in this sense, they are not intuitive or demonstratively certain, but probable or provable. In this article, I argue that the epistemic status of propositions of mixed mathematics is that of matters of fact. I wish to show that their epistemic status is not a mystery. The reason for this is that the propositions of mixed mathematics are dependent on the Uniformity Principle, unlike the propositions of pure mathematics.

DOI 10.1515/agph-2017-0004

1 Introduction

To date, there has been surprisingly little scholarship done on the topic of Hume's mixed mathematics.[1] The only stand-alone article about Hume's mixed mathematics is David Sherry's from 2009, but it does not tackle the specific tension between Hume's Fork (henceforth HF) and mixed mathematics.

HF divides all knowable propositions into relations of ideas and matters of fact. These are two distinctly different kinds of propositions: the former propositions concern only the relations between ideas, whereas the latter concern the relations between species of objects. The former relations are intuitive and demonstrative, whereas the latter are causal. However, the propositions of mixed mathematics concern *both* relations between ideas *and* relations between species of objects. It seems that propositions of mixed mathematics are *both* necessary/certain *and*

1 See Zabeeh 1973, 225, Flew 1961, 62 f., Schmidt 2003, 64 f., and Millican 2003, 133 f.

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non-necessary/fallible. Furthermore, as Peter Millican's research on Hume's logic of induction has shown[2], in propositions of mixed mathematics it is possible to make a demonstrative inference from contingent premises to a contingent conclusion. Since demonstrative inference transfers the truth to the conclusion, it is possible to make *a priori* predictions about the future causal behavior of actual physical objects. This is in stark contrast with Hume's well-known claim that *a priori* demonstration cannot be extended to concern causal relations.

To provide an argument to solve the discrepancy between HF and mixed mathematics, this article is structured into two main sections. In the second section, I introduce HF and the tension it inflicts on mixed mathematics. To do this, it is necessary to understand what Hume means by propositions and relations. For this reason I will trace the first *Enquiry's* fourth section, EHU 4.1 f. (SBN 25 f.), where HF is for the first time made explicit, to *Treatise's* first Book, T 1.3.1 (SBN 69–73), which is the foundation of Hume's theory of relations. This connection has not been previously investigated in Hume scholarship. It is, then, a contribution which illustrates the main difference between propositions concerning the relations of ideas and matters of fact: these propositions concern different kinds of relations. The former relations are intuitive and demonstrative, whereas the latter are causal. In addition, Hume utilizes the principle of contradiction, which separates the two types of propositions in a dichotomous way. Thus there clearly is a tension between HF and his conception of mixed mathematics.

In the third section, I begin to tease out the discrepancy between HF and mixed mathematics. I argue that propositions of mixed mathematics are dependent on the Uniformity Principle (henceforth UP): they presuppose that the future is conformable to the past. This indicates that they are non-necessary, fallible, and *a posteriori* propositions, unlike the propositions of pure mathematics. It is possible to transfer the truth of the premises to the conclusion by necessity and certainty *if* UP is stipulated, but UP itself cannot be deduced. The epistemic status of propositions of mixed mathematics is similar compared to other "common" causal facts of nature that are expressible in qualitative terms. Hume classifies propositions that can be formulated in mathematical terms in the same way as other propositions which describe repetitive causal relations; he labels them as "proofs." This indicates that the propositions of mixed mathematics do not differ from qualitative propositions in any degree of certainty. Rather, as I argue in the conclusion, the appropriate way to understand Hume's propositions of mixed mathematics is that they instantiate epistemic virtues, such as precision, pre-

2 See Millican 2003, 133 f.

dictability, and usefulness. This corroborates Deborah Boyle's observation that "Hume links good causal inference with *virtue*."[3]

Hume's mixed mathematics can be related to a variety of different fields of operations, such as to physics, agriculture, building, and commerce.[4] Yet the only time Hume explicitly mentions mixed mathematics, he refers to the law of conservation of momentum (EHU 4.13; SBN 31 f.) In this article, I wish to take a closer look at Hume's treatment of this law. I will set aside the difficult question of why Hume thinks that certain causal relations can be expressed quantitatively, and others not. I will focus on Hume's conception of mixed mathematics with respect to physics, i. e. with respect to formulations of propositions concerning laws of nature.

2 Hume's Fork as a Dichotomous Distinction between the Two Knowable Propositions

A significant fact about HF is that it divides propositions specifically with respect to relations. The two propositions concern either relations of ideas, or relations between species of objects. Yet in the first *Enquiry* Hume remains silent about what relations actually are. To understand HF and his conception of propositions, it is thus necessary to take a brief sojourn into his doctrine of relations in the *Treatise*, since this is the only source for textual evidence that is available on Hume's position regarding relations.

In the first Book of the *Treatise*, Hume distinguishes between natural and philosophical relations, leaning on the concepts of association and comparison (T 1.1.5.1; SBN 13 f., 1.3.6.16; SBN 94, 1.3.14.31; SBN 170). In natural relations, the mind conceives some relation associatively. In philosophical relations, the mind makes a judgment about relations; they are "subject of comparison" (T 1.1.5.1; SBN 13). In Hume, propositions are either brought to the mind by association, as in natural relations, or they are comparative judgments about philosophical relations.

In the first Book of the *Treatise* (1.3.1; SBN 69–73), Hume classifies philosophical relations with respect to their certainty or probability. According to this fundamental classification, algebraic and arithmetic (but not geometric) relations are

3 Boyle 2012, 158.
4 See Sherry 2009, 57.

certain, whereas causal relations are probable. In the second Book of the *Treatise*, Hume suggests that there are two types of relations that constitute propositions which can be true: "Truth is of two kinds, consisting either in the discovery of the proportions of ideas, consider'd as such, or in the conformity of our ideas of objects to their real existence" (T 2.3.10.2; SBN 449).[5] In the third Book of the *Treatise*, he makes a similar claim, insisting that "truth or falshood consists in an agreement or disagreement either to the *real* relations of ideas, or to *real* existence and matters of fact" (T 3.1.1.9; SBN 458).

Hume's theory of relations in T 1.3.1 (SBN 69–73) is an important background for understanding HF as it appears in the first *Enquiry*. In the fourth section of the first *Enquiry*, the distinction between the two kinds of propositions, that is, between the two kinds of associations or judgments that concern different kinds of relations, is complete:

> All the objects of human reason or enquiry may naturally be divided into two kinds, to wit, *Relations of Ideas*, and *Matters of Fact* (EHU 4.1; SBN 25).

> All reasonings may be divided into two kinds, namely demonstrative reasoning, or that concerning relations of ideas, and moral reasoning, or that concerning matter of fact and existence (EHU 4.18; SBN 35).

The first class of "reasoning," or object "of human reason or enquiry," consists of propositions concerning relations of ideas. As established in T 1.3.1 (SBN 69–73), these are either intuitively certain, or demonstrable by a sequence of intuitions. According to EHU 12.27 (SBN 163), there are three types of propositions that belong to this category: mathematical theorems (algebra and arithmetic),[6] logical

5 It may be noted that Hume allows that there is one exception where a proposition concerns only one idea. In a footnote to the first Book of the *Treatise*, he claims that propositions regarding existence can be formulated only by one idea. In propositions such as "an object *x* exist," "the idea of existence is no distinct idea, which we unite with that of the object." Thus, we can "form a proposition, which contains only one idea" (T 1.3.7.5; SBN 96 f., fn. 20). But as Baxter 2008, 57, indicates, these are mere "trifling" propositions. To say that an object is the same with itself does not bring forth any new information concerning any fact. As Baxter puts it: "In general, a trifling proposition is one in which the proposition as a whole adds nothing to the idea that is the subject."

6 In the *Treatise*, geometry, involved in measurement and diagrammatic reasoning, is for the most part an inexact science, and it depends of the form of physical space. For a detailed argumentation of Hume's different position on geometry in the *Treatise* and the first *Enquiry*, see Batitsky 1998, 10, and Sherry 2009, 65. Still, even in the *Treatise*, Hume is not very consistent on the status of geometry. First, in T 1.3.1.1 (69), he uses geometrical relations as an example of demonstration, but then he goes on to argue against the exactness of geometry at length in (syllogistic) inferences, and definitional truths founded on conventions. Hume does not limit demonstration solely to mathematics,[7] but as this article pursues an adequate understanding of his conception of mixed mathematics, I will limit my study to mathematical propositions only.

In Hume's theory, mathematical propositions express nothing but the relations between ideas of figures, quantities and numbers. As he writes: "[…] the sciences of Geometry, Algebra, and Arithmetic; and in short, every affirmation, which is either intuitively or demonstratively certain […] is a proposition, which expresses a relation between figures, quantities, or numbers" (EHU 4.1; SBN 25). The mind makes a judgment on whether the "component parts" of a mathematical proposition represent ideas that stand in an equal or unequal relation to one another (whether they get "involved" to each other or not, see EHU 12.27; SBN 163). Hume writes in the *Treatise* (1.2.4.21; SBN 46): "equality is a relation, it is not, strictly speaking, a property in the figures themselves, but arises merely from the comparison, which the mind makes betwixt them." The truth or falsity of a mathematical proposition is understood by comparing the relevant ideas of a given mathematical proposition.

In Hume's account, a mathematical theorem is certain because the relation between its ideas is intuition. This relation is invariable, as long as the compared ideas remain the same. Intuition is the basis of mathematical demonstration. Algebraic and arithmetic relations are intuitive. Thus the propositions of these sciences can be demonstrated. Consider the following simple equations:

T 1.3.1.3–1.3.1.6 (SBN 70–2). In the first *Enquiry*, one can also see some tension in the relation between algebra and arithmetic on the one hand and geometry on the other. Geometry is mentioned in EHU 4.1 (SBN 25) as an intuitive and demonstrative science (Hume even goes as far as to claim that "Though there never were a circle or triangle in nature, the truths, demonstrated by Euclid, would for ever retain their certainty and evidence"), but it is not included in EHU 12.27 (SBN 163), or in EHU 12.34 (SBN 165). Furthermore, in the second *Enquiry* (1.5; SBN 171) Hume takes geometry to be a demonstrative science.

7 Owen 1999, 107, implies that Hume limits demonstration to arithmetic and algebra, and drops "talk of syllogisms" altogether. This is a correct reading of the *Treatise*, but it is not the case in the first *Enquiry*. Contrary to Owen's reading, Hume points out in the first *Enquiry* (12.27; SBN 163) that "all those pretended syllogistical reasonings […] may safely, I think, be pronounced" to be "objects of knowledge and demonstration". However, Hume accepts syllogisms only from the viewpoint of his theory of ideas; he is not championing any formally valid deductive modes of inference. Similarly, Hume allows that there are some definitional truths founded on convention: "But to convince us of this proposition, *that where there is no property, there can be no injustice*, it is only necessary to define the terms, and explain injustice to be a violation of property" (EHU 12.27; SBN 163).

$$a + a = 2$$
$$a + a = 2a$$
$$2a = 2a$$
$$a = a$$

$$1 + 1 = 2$$
$$2 = 2$$
$$1 = 1$$

Hume thinks that the former comparisons of quantities and numbers are certain, because "we are possest of a precise standard, by which we can judge of the equality and proportion of numbers." "When two numbers are so combin'd", as in the arithmetic example provided above, "the one has always an unite answering to every unite of the other, we pronounce them equal." So, when two ideas in the mathematical expression are determined to be equal by demonstration, they form a unity that the mind can perceive intuitively. That is why "algebra and arithmetic are the only sciences, in which we can carry on a chain of reasoning to any degree of intricacy, and yet preserve a perfect exactness and certainty" (T 1.3.1.5; SBN 71).

In the *Treatise* 1.3.1.1 (SBN 69), Hume also argues that it can be demonstrated that the sum of the angles of a Euclidian triangle equals two right angles.[8] He writes: "'Tis from the idea of a triangle, that we discover the relation of equality, which its three angles bear to two right ones; and this relation is invariable, as long as our idea remains the same." This proposition can be demonstrated by the following sequence of intuitive, perceivable equalities between figures (Figure 1):[9]

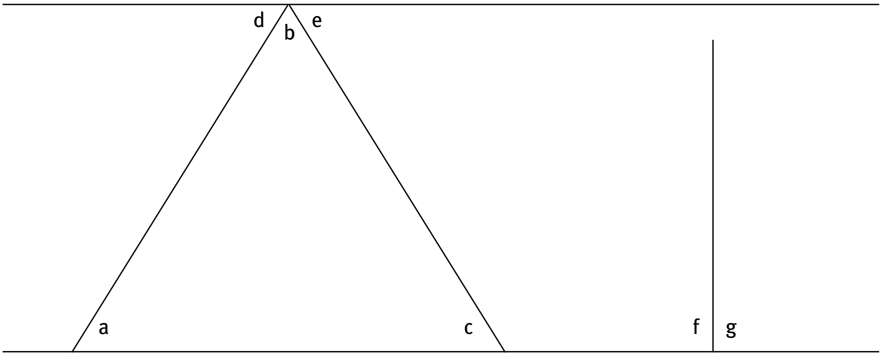

Fig. 1: Demonstration that the sum of the angles of a Euclidian triangle equals two right angles.

8 See footnote 3.

9 A similar geometrical exposition can be found in Sherry 2009, 61 f.

Although Hume's conception of mathematical demonstration ultimately relies on the perception of the equality of ideas through a sequence of intuitions, this argument can also be presented in propositional form as follows:[10]

The sum of the angles of a Euclidian triangle is equal to a straight angle:
$a + b + c = d + b + e.$
The straight angle is equal to two right angles:
$d + b + e = f + g$
Thus the sum of the angles of a Euclidian triangle equal two right angles:
$a + b + c = f + g.$

Hume argues that mathematical theorems are discoverable "by the mere operation of thought" (EHU 4.1; SBN 25). They are *a priori* truths. As Millican explains Hume's position: "What makes a truth a priori is that it can be *justified* without appeal to experience, purely by thinking about the ideas involved."[11] In judging the truth of mathematical propositions, we are relying solely on the mind's capability of comparing ideas to each other:

> Thus as the necessity, which makes two times two equal to four, or three angles of a triangle equal to two right ones, lies only in the act of the understanding, by which we consider and compare these ideas (T 1.3.14.23; SBN 166).

In the quote above, Hume maintains that mathematical truths are necessary. As they are necessary, it must be that we could not somehow conceive them otherwise. In the first *Enquiry*, Hume thinks that the negations of true propositions of mathematics are inconceivable contradictions among ideas.[12] He argues:

> Every proposition, which is not true, is there [in the proper science of mathematics] confused and unintelligible. That the cube root of 64 is equal to the half of 10, is a false proposition, and can never be distinctly conceived (EHU 12.28; SBN 164).

There is still some confusion about the way Hume understands the negations of true mathematical propositions to be inconceivable. The confusion lies in the fact

10 As Owen (1999: 37) notes, this mathematical demonstration is not a formal argument. The argument "counts as a demonstration because we intuitively perceive the relation between the ideas in the chain. Each link in the chain has to be intuitively known."
11 Millican 2007, xxxvi.
12 In the *Treatise*, Hume does not explicitly claim that the denial of mathematical propositions is contradictory. See Steiner 1987, 402.

that in mathematical demonstration, that is, in the *method* and the *process* of proving a conjecture, one does not rely on showing the inconceivability of the negations of propositions. Rather, the inconceivability of the negations of propositions is a *criterion* for something to count as demonstrable. In the *Dialogues* (9.5; KS 189), Hume explains this in the line of Demea: "Nothing is demonstrable, unless the contrary implies a contradiction." Hume's position is that the negations of mathematical theorems (and demonstrable propositions in general) should not involve contradictions. He does not think that this plays any epistemic role in demonstration, that is, proof in the mathematical sense.

Furthermore, "contradiction" in Hume does not mean a logical contradiction, such as: "A is B" and "A is not B" are mutually exclusive. In Hume contradiction means a confusion that cannot be clearly and distinctly conceived by the mind: "'Tis in vain to search for a contradiction in any thing that is distinctly conceiv'd by the mind. Did it imply any contradiction, 'tis impossible it cou'd ever be conceiv'd" (T 1.2.4.11; SBN 43). The same point can be expressed in a propositional way, as Hume writes in the Abstract (18; SBN SBN 652 f.) to the *Treatise*:

> When a demonstration convinces me of any proposition, it not only makes me conceive the proposition, but also makes me sensible, that 'tis impossible to conceive any thing contrary. What is demonstratively false implies a contradiction; and what implies a contradiction cannot be conceived.

Although in the quote above Hume models his position in terms of propositions, it should be emphasized that contradiction is fundamentally inconceivability among ideas. This is because propositions are made out of ideas. In this sense, the negations of mathematical theorems are contradictory in Hume's theory. False mathematical propositions involve at least two confusing and incompatible ideas that do not form a unity, so the mind cannot conceive them clearly and distinctly.

With respect to "all other enquiries of men," they "regard only matter of fact and existence" (EHU 12.28; SBN 163 f.). Factual propositions are not dependent only on the ideas that the mind compares (Owen 1999, 83). By repeated experience, custom, habit, and natural instincts, two species of objects, such as flame and heat, and snow and cold, are related to each other.[13] The relation between these types of objects is not discoverable by intuition, or demonstration, or by *a priori* argumentation (EHU 4.7; SBN 28). We do not acquire factual information

13 Regarding matters of fact propositions in the first *Enquiry*, Hume focuses on causation. The first *Enquiry* does not deal with relations of identity, and space and time, as it is in the first Book of the *Treatise*.

by merely comparing ideas between each other. As Owen points out, "the mere examination of two ideas present in our mind is not enough to tell whether or not they stand in the causal relation."[14]

Our knowledge concerning the relations between species of objects is founded on causality, which is founded on experience (*Abstract* 8; SBN 649, EHU 4.14; SBN 32). Hume is explicit that the source of causal relations is experience: "'tis evident *cause* and *effect* are relations, of which we receive information from experience" (T 1.3.1.1; SBN 69). He ventures

> to affirm, as a general proposition, which admits of no exception, that the knowledge of this relation [of causation] is not, in any instance, attained by reasonings *à priori*; but arises entirely from experience, when we find, that any particular objects are constantly conjoined with each other (EHU 4.6; SBN 27).

In reasoning regarding matters of fact, we proceed "upon the supposition, that the future will be conformable to the past" (EHU 4.19; SBN 36). But "the contrary of every matter of fact is still possible" (EHU 4.2; SBN 25). Hume argues that there is no contradiction in stating that the course of nature could radically change, that some familiar objects could be attended by certain unusual effects. It is distinctly conceivable that there would be snow and frost in July, and heat in January (EHU 4.18; SBN 35); it is distinctly conceivable that unsupported objects would not fall straight to the ground by the force of gravity (EHU 4.9; SBN 29); and it is indeed distinctly conceivable that a struck billiard ball would not continue its motion, to follow Newton's second law, "in the straight line in which that force is impressed" (EHU 4.10; SBN 29 f.).[15] But these are all questions of probability, and no matter of fact is subjected to the principle of contradiction (see EHU 12.28; SBN 168, and DNR 9.5; KS 189).

Hume uses his fork effectively as an epistemological tool by repeatedly contrasting the *a priori* and the empirical in the first *Enquiry* (for example, see 4.6 f.; SBN 27 f., 4.9–11; SBN 30, 4.13; SBN 31 f., 4.18; SBN 35, 12.29; SBN 164). As all knowable propositions fall into two classes which concern two distinct types of relations, Hume insinuates that these types of propositions cannot be legitimately connected with each other. HF is an all-encompassing classification of propositions, and the distinction it implies is a dichotomy: exhaustive and mutually exclusive distinction among propositions of knowledge. Mathematics is confined to the realm of abstract ideas:

14 Owen 1999, 93.

15 See also Newton 1999, 416.

> It seems to me, that the only objects of the abstract sciences or of demonstration are quantity and number, and that all attempts to extend this more perfect species of knowledge beyond these bounds are mere sophistry and illusion (EHU 12.27; SBN 164).

Here Hume claims that the necessity and certainty that is typical to mathematics cannot be *extended* to concern causal relations between real objects. The proper objects of mathematical propositions are quantity, number, and figure. In case of a true mathematical proposition, judgments concerning relations between its component ideas are necessary and certain. Since extending "this more perfect species of knowledge beyond these bounds are mere sophistry and illusion", no causal relation, which is the founding relation in propositions concerning fact or existence (EHU 4.4; SBN 27, 4.14; SBN 32, 4.19; SBN 35), can be known to hold necessarily and certainly.

HF entails a dichotomous classification of propositions, and he claims that mathematics is confined to the realm of abstract ideas. What about Hume's treatment of mixed mathematics in EHU 4.13 (SBN 31)? It seems that reconciling mixed mathematics with HF poses a significant problem. Before rushing into Hume's treatment of mixed mathematics, it is useful to briefly look at the relevant history of this concept.

According to Gary I. Brown's study, the concept of mixed mathematics can be traced to at least as far back as Francis Bacon's 1605 work *Of the Proficience and Advancement of Learnings*.[16] Bacon made an explicit distinction between pure and mixed mathematics in his classification of different kinds of philosophies. The core idea of mixed mathematics is that it took its principles from pure mathematics and applied them to physical reality. As pure mathematics was understood to be absolutely certain, consequently mathematical demonstration about the physical world would also be absolutely certain. This kind of treatment of the application of mathematics was apparent already in Euclid's *Optics*, in Archimedes' *Equilibrium of Planes*, and in Hume's time in the works of Jean le Rond d'Alembert. If this kind of conception of mixed mathematics were correct, HF cannot be right.

Regarding Hume's philosophy, the crux of the problem can be explicated as follows. The propositions of mixed mathematics concern *both* relations between ideas *and* relations between species of objects. The former relations are intuitive and demonstrative, and the latter are causal. Thus propositions of mixed mathematics seem to be *both* necessary/certain *and* non-necessary/fallible. To which class would the propositions of mixed mathematics then belong, according to Hume?

16 Brown 1991, 81 f.

Hume mentions mixed mathematics explicitly only once in all of his works. His own example of mixed mathematics concerns the law of conservation of momentum (EHU 4.13; SBN 32 f.). The law states that in a closed system, the total momentum is a conserved quantity. Momentum, $\vec{P}$, is defined as a product of mass, m, and velocity, $\vec{v}$.[17] The proposition which defines momentum, $\vec{P} = m\vec{v}$, "translation of momentum is a product of the mass and velocity of an object", expresses a relation between ideas. However, as momentum is a conserved quantity, the proposition is informative on how momentum is transferred between real objects, such as between billiard balls in a game of pool.

But how can this be? According to the first Book of the *Treatise* (1.3.1; SBN 69–73), the relations of intuition and demonstration are categorically different kinds of relations compared to the relation of causation. On the one hand, the definitional proposition $\vec{P} = m\vec{v}$ "expresses a relation between these" quantities (EHU 4.1; SBN 26). As such, it is an object of intuition, and demonstration. It can be algebraically manipulated, and the proposition does not refer to anything external. On the other hand, conservation of momentum belongs to "the laws of nature," describing "the operations of bodies," which, "without exception, are known only by experience" (EHU 4.9; SBN 30). Since translation of momentum in a system of bodies is observed to be contiguous between two objects, and there is a temporal sequence between the objects, the mathematical formulation of the law satisfies the conditions that Hume assigns to causal inference (see, section XV of the *Treatise*, "Rules by which to judge of causes and effects", and the *Abstract* (9; SBN 649 f.) of the *Treatise*).

There is another problem in Hume's mixed mathematics. HF in EHU 4.1 f. (SBN 25 f.) divides propositions with respect to contradictions of their negations: the negations of relations of ideas are inconceivable contradictions, whereas the negations of matters of fact are distinctly conceivable. If the rules of algebra are followed, the proposition $\vec{P} = m\vec{v}$ cannot be rendered, without a contradiction, to

17 His own example is literally this: "the moment or force of any body in motion is in the compound ratio or proportion of its solid contents and its velocity" (EHU 4.13; SBN 31).This suggests that Hume conflates

$$\text{moment } \alpha \text{ (solid contents) x (velocity)} \approx \vec{P} = m \times \vec{v}, \text{ and}$$
$$\text{force } \alpha \text{ (solid contents) x (velocity)} \approx \vec{F} = m \times \vec{v},$$

where $\vec{F}$ is the force exerted on the object. As Twardy indicates, this confusion is probably derived from Colin Maclaurin's 1748 textbook *An account of Sir Isaac Newton's philosophical discoveries* (cf. Twardy 2014, 28 f.). Newton's own definition for momentum in the *Principia* is the following: "Quantity of motion is a measure of motion which arises from the velocity and the quantity of matter jointly" (Newton 1999, 404). However, Hume's confusion is not relevant to the problem of mixed mathematics, as I intend to analyze it in this article.

propositions $m = \vec{P}\vec{v}$, or $\vec{v} = \vec{P}m$. In Hume's theory, the negation of $\vec{P} = m\vec{v}$ would be an inconceivable contradiction among the component ideas of this proposition. But it is possible to conceive a situation when a cue ball hits the object ball in the game of pool, the object ball does not continue its motion to the direction of $\vec{P}$ but stays in halt or moves into a direction other than $\vec{P}$. This indicates that referring to the principle of contradiction does not suffice to settle the issue of Hume's mixed mathematics. Rather, as I shall argue next, this requires UP.

3 The Dependency of Mixed Mathematics on the Uniformity Principle

In my interpretation, the reason why Hume does not allow demonstration to be extended to natural events is this: one has to presuppose the uniformity of nature. Instinctively and habitually we, both humans and non-human animals, assume that the future resembles the past. We infer "that the same events will always follow from the same causes" (EHU 9.2; SBN 105). The relevance of UP to Hume's conception of mixed mathematics is also echoed in his own statement: "Every part of mixed mathematics proceeds upon the supposition, that certain laws are established by nature in her operations" (EHU 4.13; SBN 31).

Application of mathematics to factual matters presupposes UP as a non-grounded ground. As Millican 2003, 146 f., formulates Hume's argument: Factual inference to the unobserved is founded on UP and UP is not founded on reason which implies that factual inference to the unobserved is not founded on reason. Hence matter of fact propositions expressed as mathematically formulated laws are justifiable neither by intuitive nor by demonstrative reasoning. We do not know their truth by a mere comparison of the relations of ideas, just by consulting our intellectual faculties. Factual reasoning requires the comparison of how objects are related in the actual world by a customary transition "from the appearance of a cause [...] to the effect" (EHU 7.29; SBN 76 f.). And our knowledge concerning these causal relations, as Hume frequently argues in the first *Enquiry* (4.6; SBN 27, 4.15; SBN 32, 12.29; SBN 164), is not founded on, nor can be justified by, *a priori* reasoning.[18] Consequently, even when matters of fact propositions are formulated mathematically, they are *a posteriori*.

18 And, likewise, neither by experience itself (since this would be circular), nor by direct observation. See the first *Enquiry* (4.19; SBN 35 f.).

One startling objection could still be made to this interpretation. Using a mathematically formulated proposition such as $\vec{P} = m\vec{v}$ *conserved* enables one to deduce a contingent conclusion from contingent premises. To illustrate this, consider the following reformulation of Hume's example about conservation of momentum.

An object A, which is in motion, collides with object B, which is at rest. As a result, the momentum of object A is transferred to the system AB according to the law of conservation of momentum. The contingent premise consist of the initial conditions *i*, the momenta of $\vec{P}_A$ and $\vec{P}_B$. Alike, the final condition *f*, the momentum $\vec{P}_{AB}$ of the system AB, is a contingent conclusion. The initial and final matters of fact are all contingent, since the salient variables, the masses m_A and m_B, and the velocities $(\vec{v}_i)_A$, $(\vec{v}_i)_B$, $(\vec{v}_f)_A$, and $(\vec{v}_f)_B$, are contingent. However, deducing the conclusion from the premise is not a contingent procedure: $\vec{P} = m\vec{v}$ can be, step by step, algebraically manipulated to determine the desired variable. As a mathematical proposition, $\vec{P} = m\vec{v}$ does not refer to anything external. It depends solely on the quantities it is composed of. The situation is like in Figure 2:

contingent premise	**demonstrative inference**	**contingent conclusion**
momenta of A and B	$\vec{P}_i = \vec{P}_f$	momentum of the system
	$(\vec{P}_i)_A + (\vec{P}_i)_B = (\vec{P}_f)_A + (\vec{P}_f)_B$	
$\vec{P}_A$, $\vec{P}_B$	$m_A(\vec{v}_i)_A + m_B(\vec{v}_i)_B = m_A(\vec{v}_f)_A + m_B(\vec{v}_f)_B$	$\vec{P}_{AB}$
	→ deduction of the desired variable	

Fig. 2: Demonstrative inference from a contingent premise to a contingent conclusion.

Given HF, the previous deduction is problematic. As Millican puts it, a demonstrative inference from a contingent premise to a contingent conclusion

> cannot possibly count as 'reasoning concerning matters of fact' as *Hume* understands that phrase, because here the link between premiss and conclusion is deductively certain rather than merely 'probable', is clearly explicable in terms of 'relations of ideas', and hence [...] requires no appeal to experience and no dependence on supposed causal relations. In Hume's terms, therefore, this inference is certainly not an instance of 'reasoning concerning matter of fact'[...][19]

Although the premise of such an argument is contingent, and not in any way necessary, the deduction to the conclusion is necessitated by the given quantities, and hence the process is certain. It can be argued, then, that the demonstration

19 Millican 2003, 119.

in this case brings forward *new information* about some factual matter. Before any experience, that is, before perceiving what happens in the collision of objects A and B, we are able to demonstratively infer what happens to the system AB when A and B are conjoined. But how can this be, when Hume adamantly denies that "there is no demonstration [...] for any conjunction of cause and effect" (*Abstract* 11; SBN 650 f.), that "enquires" regarding "matters of fact and existence [...] are evidently incapable of demonstration" (EHU 12.28; SBN 163 f.), and that "there is an evident absurdity in pretending to demonstrate a matter of fact, or to prove it by any arguments *a priori*" (DNR 9.5; KS 189)?

I think the solution to the former mystery is this: *Given* UP, "*the course of nature continues always uniformly the same",* the premises transfer the truth to the conclusion. As Graciela De Pierris argues, this inference is "licensed by the principle of the uniformity of nature."[20] But in Hume's view, there is no guarantee that "*the future resembles the past*" (See T 1.3.6.4; SBN 89, 1.3.12.9; SBN 134). As UP itself is not provable by intuition or demonstration, it must be that the propositions of mixed mathematics cannot be provable by intuition or demonstration, alone. This also explains a comment Hume makes in the first section to the second *Enquiry*, where he points out that theories about the laws of nature might be refuted, unlike pure, non-applied mathematical theorems: "Propositions in geometry may be proved, systems in physics may be controverted" (EPM 1.5; SBN 171). Propositions concerning laws of nature, although they do express relations between numbers, quantities, and figures, do not share the same necessity and certainty as propositions of pure arithmetic, algebra, and geometry. The former are dependent on UP; the latter are not.

The point why Hume gives a high epistemic status to the laws of nature is that they signify a set of causes and effects which have "hitherto admitted of no exception" (EHU 6.4; SBN 58). In fact, applying mathematics does not guarantee certainty – it is not really mathematics that renders the laws of nature as high-class matters of fact. Rather, it is their regular, unexceptional occurrence. As Hans Reichenbach clarifies Hume's position: "laws of nature are for him statements of an exceptionless repetition – not more."[21] Their epistemic status is similar compared to other "common" causal facts of nature, such as our knowledge of fire having the attribute of burning or water having the attribute of suffocating non-aquatic beings (see EHU 6.4; SBN 57).

The example about burning fire and suffocating water illustrates how Hume understands the epistemic status of propositions of mixed mathematics. In the

20 De Pierris 2006, 305.
21 Reichenbach 1951, 159.

sixth section to the first *Enquiry* (6.4; SBN 57), Hume groups together both mathematically expressible matters of fact propositions (Newton's second law of motion, and the law of universal gravitation), *and* qualitatively expressible matters of fact propositions (fire burns, and water causes drowning to non-aquatic beings). Later, in the tenth section to the first *Enquiry* (10.4; SBN 110), he indicates that the regular, unexceptional occurrence of these kinds of causal relations renders them "proofs." Hume's position can be sketched as follows (see Figure 3):[22]

Hume's classification of "proofs" (EHU 6.4; SBN 57, 10.4; SBN 110)	
Quantitative	qualitative
$\vec{F} \alpha \frac{m_1 m_2}{r^2}$.	Fire burns.
$\vec{F} = \frac{d\vec{P}}{dt}$.	Water causes drowning to non-aquatic beings.

Fig. 3: Examples of propositions which are either quantitative or qualitative, but still instances of an exceptionless repetition.

To Hume "proofs" are high-class, non-necessary propositions about constant conjunction between two species of objects. It is not relevant, according to this classification, if a matter of fact proposition is expressed quantitatively or qualitatively, since, as Hume points out in the fourth section of the first *Enquiry*: "all our reasonings concerning fact are of the same nature" (EHU 4.4; SBN 26, see also *Abstract* of the *Treatise* 10, SBN 650). Mathematical or not, the logic of inductive arguments is the same in both cases: matters of fact propositions presuppose UP as a latent premise.

It should be noted that causal probabilities and proofs do still have a difference in Hume's account. He discusses this difference both in the *Treatise* (1.3.11.2; SBN 124), and in the first *Enquiry* (6, fn. 10). He asserts that proofs "exceed probability," being "entirely free from doubt and uncertainty." They "leave no room for doubt or opposition." Hume thinks that there are empirical proofs which the human reason does not doubt, such as "all humans will eventually die," and that "the sun will rise tomorrow." But these proofs are not absolutely certain, or necessary. The evidence of proofs is higher than probabilities, but it is not as high as in demonstrations. "Proofs" are thus both free of doubt and fallible.

Moreover, when Hume (T 1.3.11.2; SBN 124, EHU 6, fn. 10) introduces the tripartite categorization, which includes demonstration, proof and probability, it

22 Regarding the meaning of "proof" in Hume I follow De Pierris' (2006) interpretation.

would be false to think that this categorization is epistemically fundamental. As I have shown in this article, the fundamental categorization is the dichotomous distinction of HF in the first *Enquiry*, which is grounded in the doctrine of relations in the first Book to the *Treatise* (1.3.1). HF is a distinction of kind, not degree. Although Hume understand proofs and probabilities to clearly differ in the degrees of their evidence (T 1.3.11.2; SBN 124), the proof/probability distinction is not a dichotomous distinction like HF. Hume's fundamental epistemic categorization is between relations of ideas, which are founded on the relation of intuition, and are thus capable of being demonstrated without any appeal to fallible experience, and; matters of fact, which are founded on the relation of causation, and which do require fallible experience.

4 Conclusion: Mixed Mathematics Represents Epistemic Virtues

Hume classifies causal relations that can be expressed in mathematical terms in the same way as causal relations that are expressed in qualitative terms; he labels them as "proofs." As Boyle 2012, 158, points out, it is quite universally accepted in the secondary literature that there is no rational justification for a belief in any causal inference. Even Hume's "proofs" require UP as a latent premise. This principle itself is a customary, habitual, and an instinctive principle, not a principle founded on reason. However, recent scholarship on Hume has emphasized the normative and constructive character of certain causal inferences.[23] Hume talks about "wisdom" and "good" sense, and insists that "a wise man [...] proportions his belief to the evidence" (EHU 10.4; SBN 110). For instance, "proofs" have a specific normative character not shared by mere "probabilities." The former are supported by the whole of past uniform experience, whereas the latter are not. Hume is committed to a normative claim: uniform past experience and repetition are virtues which *should* be appraised by a "wise man."

Since mixed mathematics and qualitative "proofs" are on a par with respect to their certainty, the only difference between these provable causal propositions is that mixed mathematics can be associated with some epistemic virtues. Hume's rhetoric in T 2.3.3.2 (SBN 143) and EHU 4.13 (SBN 31) clearly esteems the application of mathematics. Hume allows that "mixing" mathematics enhances precision, or

23 See, for instance, Millican 1998, 152 f., De Pierris 2001, 351 and 373, Boyle 2012, and Schaffer 2014.

"accuracy of reasoning," and that it "assists experience" in making the discovery and application of laws of nature possible. Mathematics is very useful in mechanical operations: "Mathematics, indeed, are useful in all mechanical operations, and arithmetic in almost every art and profession" (T 2.3.3.2; SBN 413 f.). Hume thinks that it is simply a good thing that mathematics can be used and applied to a variety of different disciplines, such as physics, agriculture, building, and commerce.

Hume's treatment of mixed mathematics, such as in his own example about conservation of momentum, allows that it is possible to make accurate predictions of motions of objects by *a priori* mathematical demonstration. But Hume's logic of induction indicates that predictability is still founded on past experience. The past experience, although it can be brought under a quantitative law, enables one to infer the yet unobserved future, in the exact same way as the past experience enables me to infer that when I will install my finger to a flame I will feel pain and heat.

Hume denies that the necessity and certainty related to abstract mathematical reasoning could be *extended* to concern factual reasoning (EHU 12.27; SBN 164). Matters of fact are founded on causation, which is founded on experience. But Hume does allow that abstract mathematical reasoning can *assist* experience in the discovery and application of laws of nature (EHU 4.13; SBN 31). Thus the appropriate way to understand the epistemic status of propositions of Hume's mixed mathematics is that they neither instantiate necessity, nor do they increase certainty. They are matters of fact that represent epistemic virtues of precision, predictability and usefulness.[24]

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SBN Selby-Bigge, L. A./Nidditch, P. H. (eds.) 1975. *A Treatise of Human Nature*. New York.

EHU Beauchamp, T. L. (ed.) 2000. An *Enquiry concerning Human Understanding*. Oxford.

T Norton, D. F./Norton, M. J. (eds.) 2000. *A Treatise of Human Nature*. Oxford.

24 Concerning this article, I wish to thank the following people for their useful advice and critical feedback: Mikko Yrjönsuuri, Jani Hakkarainen, the anonymous referees of the journal, and the participants of the Nordic Workshop in Early Modern Philosophy held at the University of Turku on June 13–14, 2013. My PhD research has been financially supported by the Finnish Doctoral Programme of Philosophy, funded by the Ministry of Education and Culture together with the Academy of Finland.

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