Florentin Smarandache

(author and editor)

Collected Papers

(on Neutrosophic Theory and Its Applications in Algebra)

Volume IX

Global Knowledge

2022

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Volume IX

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(author and editor)

Collected Papers

(on Neutrosophic Theory and Its Applications in Algebra)

Volume IX

GLOBAL KNOWLEDGE

Publishing House

Miami, United States of America

2022

Publishing:

GLOBAL KNOWLEDGE - Publishing House
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Florida 33131, United States
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ADSUMUS – Scientific and Cultural Society
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ISBN 978-1-59973-733-1

Introductory Note

This ninth volume of *Collected Papers* includes 87 papers comprising 982 pages on Neutrosophic Theory and its applications in Algebra, written between 2014-2022 by the author alone or in collaboration with the following 81 co-authors (alphabetically ordered) from 19 countries: E.O. Adeleke, A.A.A. Agboola, Ahmed B. Al-Nafee, Ahmed Mostafa Khalil, Akbar Rezaei, S.A. Akinleye, Ali Hassan, Mumtaz Ali, Rajab Ali Borzooei , Assia Bakali, Cenap Özel, Victor Christianto, Chunxin Bo, Rakhal Das, Bijan Davvaz, R. Dhavaseelan, B. Elavarasan, Fahad Alsharari, T. Gharibah, Hina Gulzar, Hashem Bordbar, Le Hoang Son, Emmanuel Ilojide, Tèmítópé Gbóláhàn Jaíyéolá, M. Karthika, Ilanthenral Kandasamy, W.B. Vasantha Kandasamy, Huma Khan, Madad Khan, Mohsin Khan, Hee Sik Kim, Seon Jeong Kim, Valeri Kromov, R. M. Latif, Madeleine Al-Tahan, Mehmat Ali Ozturk, Minghao Hu, S. Mirvakili, Mohammad Abobala, Mohammad Hamidi, Mohammed Abdel-Sattar, Mohammed A. Al Shumrani, Mohamed Talea, Muhammad Akram, Muhammad Aslam, Muhammad Aslam Malik, Muhammad Gulistan, Muhammad Shabir, G. Muhiuddin, Memudu Olaposi Olatinwo, Osman Anis, Choonkil Park, M. Parimala, Ping Li, K. Porselvi, D. Preethi, S. Rajareega, N. Rajesh, Udhayakumar Ramalingam, Riad K. Al-Hamido, Yaser Saber, Arsham Borumand Saeid, Saeid Jafari, Said Broumi, A.A. Salama, Ganeshsree Selvachandran, Songtao Shao, Seok-Zun Song, Tahsin Oner, M. Mohseni Takallo, Binod Chandra Tripathy, Tugce Katican, J. Vimala, Xiaohong Zhang, Xiaoyan Mao, Xiaoying Wu, Xingliang Liang, Xin Zhou, Yingcang Ma, Young Bae Jun, Juanjuan Zhang.

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Neutrosophic Closed Set and Neutrosophic Continuous Functions

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A.A. Salama, Florentin Smarandache, Valeri Kromov (2014). Neutrosophic Closed Set and Neutrosophic Continuous Functions. *Neutrosophic Sets and Systems*, 4, 4-8

Abstract
In this paper, we introduce and study the concept of "neutrosophic closed set "and "neutrosophic continuous function". Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Closed Set, Neutrosophic Set; Neutrosophic Topology; Neutrosophic Continuous Function.

1 INTRODUCTION

The idea of "neutrosophic set" was first given by Smarandache [11, 12]. Neutrosophic operations have been investigated by Salama at el. [1-10]. Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts [9, 13]. Here we shall present the neutrosophic crisp version of these concepts. In this paper, we introduce and study the concept of "neutrosophic closed set "and "neutrosophic continuous function".

2 TERMINOLOGIES

We recollect some relevant basic preliminaries, and in particular the work of Smarandache in [11, 12], and Salama at el. [1-10].

2.1 Definition [5]
A neutrosophic topology (NT for short) an a non empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms

(NT_1) $O_N, 1_N \in \tau$,

(NT_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(NT_3) $\cup G_i \in \tau \quad \forall \{G_i : i \in J\} \subseteq \tau$

In this case the pair (X, τ) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in τ is known as neuterosophic open set (NOS for short) in X . The elements of τ are called open neutrosophic sets, A neutrosophic set F is closed if and only if it C (F) is neutrosophic open.

2.1 Definition [5]
The complement of (C (A) for short) of is called a neutrosophic closed set (for short) in A . NOSA NCS X**.**

3 Neutrosophic Closed Set .

3.1 Definition
Let (X, τ) be a neutrosophic topological space. A neutrosophic set A in (X, τ) is said to be neutrosophic closed (in shortly N-closed).
If Ncl (A) ⊆ G whenever A ⊆ G and G is neutrosophic open; the complement of neutrosophic closed set is Neutrosophic open.

3.1 Proposition
If A and B are neutrosophic closed sets then A∪B is Neutrosophic closed set.

3.1 Remark
The intersection of two neutrosophic closed (N-closed for short) sets need not be neutrosophic closed set.

3.1 Example
Let X = {a, b, c} and

A = <(0.5,0.5,0.5) , (0.4,0.5,0.5) , (0.4,0.5,0.5)>

B = <(0.3,0.4,0.4) , (0.7,0.5,0.5) , (0.3,0.4,0.4)>

Then T = { 0_N ,1_N , A, B} is a neutrosophic topology on X. Define the two neutrosophic sets A_1 and A_2 as follows,

A_1 = <(0.5,0.5,0.5),(0.6,0.5,0.5),(0.6,0.5,0.5)>

A_2 = <(0.7,0.6,0.6)(0.3,0.5,0.5),(0.7,0.6,0.6)>

A_1 and A_2 are neutrosophic closed set but $A_1 \cap A_2$ is not a neutrosophic closed set.

3.2 Proposition

Let be a neutrosophic topological space. If B is neutrosophic closed set and B ⊆ A ⊆ Ncl (B), then A is N-closed.

3.4 Proposition

In a (X,τ) neutrosophic topological space (X,T), T=ℑ (the family of all neutrosophic closed sets) iff every neutrosophic subset of (X,T) is a neutrosophic closed set.

Proof.

suppose that every neutrosophic set A of (X,T) is N-closed. Let A∈T, since A ⊆ A and A is N-closed, Ncl (A) ⊆ A. But A ⊆ Ncl (A). Hence, Ncl (A) =A. thus, A ∈ ℑ. Therefore, T ⊆ ℑ. If B ∈ ℑ then 1-B ∈T ⊆ ℑ. and hence B∈T, That is, ℑ ⊆ T. Therefore T=ℑ conversely, suppose that A be a neutrosophic set in (X,T). Let B be a neutrosophic open set in (X,T). such that A ⊆ B. By hypothesis, B is neutrosophic N-closed. By definition of neutrosophic closure, Ncl (A) ⊆ B. Therefore A is N-closed.

3.5 Proposition

Let (X,T) be a neutrosophic topological space. A neutrosophic set A is neutrosophic open iff B ⊆ NInt (A), whenever B is neutrosophic closed and B ⊆ A.

Proof

Let A a neutrosophic open set and B be a N-closed, such that B ⊆ A. Now, B ⊆ A⇒ 1–A ⇒1–B and 1–A is a neutrosophic closed set ⇒ Ncl (1–A) ⊆ 1–B. That is, B=1–(1–B) ⊆ 1–Ncl (1–A). But 1–Ncl (1–A) = Nint (A). Thus, B ⊆ Nint (A). Conversely, suppose that A be a neutrosophic set, such that B ⊆ Nint (A) whenever B is neutrosophic closed and B ⊆ A. Let 1–A ⊆ B ⇒ 1–B ⊆ A. Hence by assumption 1–B ⊆ Nint (A). that is, 1–Nint (A) ⊆ B. But 1–Nint (A) =Ncl (1–A). Hence Ncl(1–A) ⊆ B. That is 1–A is neutrosophic closed set. Therefore, A is neutrosophic open set

3.6 Proposition

If Nint (A) ⊆ B ⊆ A and if A is neutrosophic open set then B is also neutrosophic open set.

4. Neutrosophic Continuous Functions

4.1 Definition

i) If $B=\langle \mu_B,\sigma_B,\nu_B\rangle$ is a NS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is a NS in X defined by $f^{-1}(B)=\left\langle f^{-1}(\mu_B), f^{-1}(\sigma_B), f^{-1}(\nu_\nu)\right\rangle$.

ii) If $A=\langle \mu_A,\sigma_A,\nu_A\rangle$ is a NS in X, then the image of A under f, denoted by $f(A)$, is the a NS in Y defined by $f(A)=\left\langle f(\mu_A), f(\sigma_A), f(\nu_A)^c\right\rangle$.

Here we introduce the properties of images and preimages some of which we shall frequently use in the following sections .

4.1 Corollary

Let A, $\{A_i : i \in J\}$, be NSs in X, and B, $\{B_j : j \in K\}$ NS in Y, and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$,

$B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(b) $A \subseteq f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A)\ )$.

(c) $f^{-1}(f(B)) \subseteq B$ and if f is surjective, then $f^{-1}(f(B)\ ) = B$, .

(d) $f^{-1}(\cup B_i)\ ) = \cup f^{-1}(B_i)$, $f^{-1}(\cap B_i)\ ) = \cap f^{-1}(B_i)$,

(e) $f(\cup A_i) = \cup f(A_i)$; $f(\cap A_i) \subseteq \cap f(A_i)$; and if f is injective, then $f(\cap A_i) = \cap f(A_i)$;

(f) $f^{-1}(!_N) = 1_N$ $f^{-1}(0_N)\ = 0_N$.

(g) $f(0_N) = 0_N$, $f(1_N) = 1_N$ if f is subjective.

Proof

Obvious.

4.2 Definition

Let (X,Γ_1) and (Y,Γ_2) be two NTSs, and let $f : X \to Y$ be a function. Then f is said to be continuous iff the preimage of each NCS in Γ_2 is a NS in Γ_1 .

4.3 Definition

Let (X,Γ_1) and (Y,Γ_2) be two NTSs and let $f : X \to Y$ be a function. Then f is said to be open iff the image of each NS in Γ_1 is a NS in Γ_2 .

4.1 Example

Let (X,Γ_o) and (Y,ψ_o) be two NTSs

(a) If $f : X \to Y$ is continuous in the usual sense, then in this case, f is continuous in the sense of Definition 5.1 too. Here we consider the NTs on X and Y, respectively, as follows : $\Gamma_1 = \left\{\left\langle \mu_G, 0, \mu_G^c \right\rangle : G \in \Gamma_o\right\}$ and

$\Gamma_2 = \left\{\left\langle \mu_H, 0, \mu_H^c \right\rangle : H \in \Psi_o \right\}$,

In this case we have, for each $\left\langle \mu_H, 0, \mu_H^c \right\rangle \in \Gamma_2$, $H \in \Psi_o$,

$f^{-1}\left\langle \mu_H, 0, \mu_H^c \right\rangle = \left\langle f^{-1}(\mu_H), f^{-1}(0), f^{-1}(\mu_H^c) \right\rangle$
$= \left\langle f^{-1}\mu_H, f(0), (f(\mu)^c \right\rangle \in \Gamma_1$.

(b) If $f : X \to Y$ is neutrosophic open in the usual sense, then in this case, f is neutrosophic open in the sense of Definition 3.2.

Now we obtain some characterizations of neutrosophic continuity:

4.1 Proposition

Let $f : (X, \Gamma_1) \to (Y, \Gamma_2)$.

f is neutrosop continuous iff the preimage of each NS (neutrosophic closed set) in Γ_2 is a NS in Γ_2.

4.2 Proposition

The following are equivalent to each other:

(a) $f : (X, \Gamma_1) \to (Y, \Gamma_2)$ is neutrosophic continuous .

(b) $f^{-1}(NInt(B) \subseteq NInt(f^{-1}(B))$ for each CNS B in Y.

(c) $NCl(f^{-1}(B)) \subseteq f^{-1}(NCl(B))$ for each NCB in Y.

4.2 Example

Let (Y, Γ_2) be a NTS and $f : X \to Y$ be a function. In this case $\Gamma_1 = \left\{ f^{-1}(H) : H \in \Gamma_2 \right\}$ is a NT on X. Indeed, it is the coarsest NT on X which makes the function $f : X \to Y$ continuous. One may call it the initial neutrosophic crisp topology with respect to f.

4.4 Definition

Let (X,T) and (Y,S) be two neutrosophic topological space, then

(a) A map f : (X,T) → (Y,S) is called N-continuous (in short N-continuous) if the inverse image of every closed set in (Y,S) is Neutrosophic closedin (X,T).

(b) A map f:(X,T)→ (Y,S) is called neutrosophic-gc irresolute if the inverse image of every Neutrosophic closedset in (Y,S) is Neutrosophic closedin (X,T). Equivalently if the inverse image of every Neutrosophic open set in (Y,S) is Neutrosophic open in (X,T).

(c) A map f:(X,T)→ (Y,S) is said to be strongly neutrosophic continuous if f^{-1}(A) is both neutrosophic open and neutrosophic closed in (X,T) for each neutrosophic set A in (Y,S).

(d) A map f : (X,T) → (Y,S) is said to be perfectly neutrosophic continuous if f^{-1} (A) is both neutrosophic open and neutrosophic closed in (X,T) for each neutrosophic open set A in (Y,S).

(e) A map f:(X,T)→(Y,S) is said to be strongly N-continuous if the inverse image of every Neutrosophic open set in (Y,S) is neutrosophic open in (X,T).

(F) A map f:(X,T)→(Y,S) is said to be perfectly N-continuous if the inverse image of every Neutrosophic open set in (Y,S) is both neutrosophic open and neutrosophic closed in (X,T).

4.3 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces. Let f : (X,T) → (Y,S) be generalized neutrosophic continuous. Then for every neutrosophic set A in X, f(Ncl(A)) ⊆ Ncl(f(A)).

4.4 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces. Let f : (X,T) → (Y,S) be generalized neutrosophic continuous. Then for every neutrosophic set A in Y, Ncl(f^{-1}(A)) ⊆ f^{-1}(Ncl(A)).

4.5 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces. If A is a Neutrosophic closedset in (X,T) and if f : (X,T) → (Y,S) is neutrosophic continuous and neutrosophic-closed then f(A) is Neutrosophic closedin (Y,S).

Proof.

Let G be a neutrosophic-open in (Y,S). If f(A) ⊆ G, then A ⊆ f^{-1}(G) in (X,T). Since A is neutrosophic closedand f^{-1}(G) is neutrosophic open in (X,T), Ncl(A) ⊆ f^{-1}(G), (i.e) f(Ncl(A)⊆G. Now by assumption, f(Ncl(A)) is neutrosophic closed and Ncl(f(A)) ⊆ Ncl(f(Ncl(A))) = f(Ncl(A)) ⊆ G. Hence, f(A) is N-closed.

4.5 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces, If f : (X,T) → (Y,S) is neutrosophic continuous then it is N-continuous.

The converse of proposition 4.5 need not be true. See Example 4.3.

4.3 Example

Let X ={a,b,c} and Y ={a,b,c}. Define neutrosophic sets A and B as follows A = $\langle (0.4,0.4,0.5), (0.2,0.4,0.3), (0.4,0.4,0.5) \rangle$

B = $\langle (0.4,0.5,0.6), (0.3,0.2,0.3), (0.4,0.5,0.6) \rangle$

Then the family T = {0_N, 1_N, A} is a neutrosophic topology on X and S = {0_N, 1_N, B} is a neutrosophic topology on Y. Thus (X,T) and (Y,S) are neutrosophic topological spaces. Define f : (X,T) → (Y,S) as f(a) = b , f(b) = a, f(c) = c. Clearly f is N-continuous. Now f is not neutrosophic continuous, since f^{-1}(B) ∉ T for B ∈ S.

4.4 Example

Let X = {a,b,c}. Define the neutrosophic sets A and B as follows.

A = $\langle (0.4,0.5,0.4), (0.5,0.5,0.5), (0.4,0.5,0.4) \rangle$

$B = \langle (0.7,0.6,0.5), (0.3,0.4,0.5), (0.3,0.4,0.5) \rangle$
and $C = \langle (0.5,0.5,0.5), (0.4,0.5,0.5), (0.5,0.5,0.5) \rangle$
$T = \{0_N, 1_N, A, B\}$
and $S = \{0_N, 1_N, C\}$ are neutrosophic topologies on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Define f: (X,T) → (X,S) as follows f(a) = b, f(b) = b, f(c) = c. Clearly f is N-continuous. Since
$D = \langle (0.6,0.6,0.7), (0.4,0.4,0.3), (0.6,0.6,0.7) \rangle$
is neutrosophic open in (X,S), f^{-1}(D) is not neutrosophic open in (X,T).

4.6 Proposition
Let (X,T) and (Y,S) be any two neutrosophic topological space. If f : (X,T) → (Y,S) is strongly N-continuous then f is neutrosophic continuous.

The converse of Proposition 3.19 is not true. See Example 3.3

4.5 Example
Let X ={a,b,c}. Define the neutrosophic sets A and B as follows.
$A = \langle (0.9,0.9,0.9), (0.1,0.1,0.1), (0.9,0.9,0.9) \rangle$
$B = \langle (0.9,0.9,0.9), (0.1,0.1,0), (0.9,0.1,0.8) \rangle$
and $C = \langle (0.9,0.9,0.9), (0.1,0,0.1), (0.9,0.9,0.9) \rangle$
$T = \{0_N, 1_N, A, B\}$ and $S = \{0_N, 1_N, C\}$ are neutrosophic topologies on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Also define f :(X,T)→ (X,S) as follows f(a) = a, f(b) = c, f(c) = b. Clearly f is neutrosophic continuous. But f is not strongly N-continuous. Since
$D = \langle (0.9,0.9,0.99), (0.05,0,0.01), (0.9,0.9,0.99) \rangle$
Is an Neutrosophic open set in (X,S), f^{-1}(D) is not neutrosophic open in (X,T).

4.7 Proposition
Let (X,T) and (Y, S) be any two neutrosophic topological spaces. If f: (X,T) → (Y,S) is perfectly N-continuous then f is strongly N-continuous.

The converse of Proposition 4.7 is not true. See Example 4.6

4.6 Example
Let X = {a,b,c}. Define the neutrosophic sets A and B as follows.
$A = \langle (0.9,0.9,0.9), (0.1,0.1,0.1), (0.9,0.9,0.9) \rangle$
$B = \langle (0.99,0.99,0.99), (0.01,0,0), (0.99,0.99,0.99) \rangle$
And $C = \langle (0.9,0.9,0.9), (0.1,0.1,0.05), (0.9,0.9,0.9) \rangle$
$T = \{0_N, 1_N, A, B\}$ and $S = \{0_N, 1_N, C\}$ are neutrosophic topologies space on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Also define f : (X,T) → (X,S) as follows f(a) = a, f(b) = f(c) = b. Clearly f is strongly N-continuous. But f is not perfectly N continuous. Since $D = \langle (0.9,0.9,0.9), (0.1,0.1,0), (0.9,0.9,0.9) \rangle$
Is an Neutrosophic open set in (X,S), f^{-1}(D) is neutrosophic open and not neutrosophic closed in (X,T).

4.8 Proposition
Let (X,T) and (Y,S) be any neutrosophic topological spaces. If f: (X,T) → (Y,S) is strongly neutrosophic continuous then f is strongly N-continuous.

The converse of proposition 3.23 is not true. See Example 4.7

4.7 Example
Let X = {a,b,c} and Define the neutrosophic sets A and B as follows.
$A = \langle (0.9,0.9,0.9), (0.1,0.1,0.1), (0.9,0.9,0.9) \rangle$
$B = \langle (0.99,0.99,0.99), (0.01,0,0), (0.99,0.99,0.99) \rangle$
and $C = \langle (0.9,0.9,0.9), (0.1,0.1,0.05), (0.9,0.9,0.9) \rangle$
$T = \{0_N, 1_N, A, B\}$ and $S = \{0_N, 1_N, C\}$ are neutrosophic topologies on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Also define f : (X,T) → (X,S) as follows: f(a) = a, f(b) = f(c) = b. Clearly f is strongly N-continuous. But f is not strongly neutrosophic continuous. Since
$D = \langle (0.9,0.9,0.9), (0.1,0.1,0), (0.9,0.9,0.9) \rangle$
be a neutrosophic set in (X,S), f^{-1}(D) is neutrosophic open and not neutrosophic closed in (X,T).

4.9 Proposition
Let (X,T),(Y,S) and (Z,R) be any three neutrosophic topological spaces. Suppose f : (X,T) → (Y,S), g : (Y,S) → (Z,R) be maps. Assume f is neutrosophic gc-irresolute and g is N-continuous then g o f is N-continuous.

4.10 Proposition
Let (X,T), (Y,S) and (Z,R) be any three neutrosophic topological spaces. Let f : (X,T) → (Y,S), g : (Y,S) → (Z,R) be map, such that f is strongly N-continuous and g is N-continuous. Then the composition g o f is neutrosophic continuous.

4.5 Definition
A neutrosophic topological space (X,T) is said to be neutrosophic $T_{1/2}$ if every Neutrosophic closed set in (X,T) is neutrosophic closed in (X,T).

4.11 Proposition
Let (X,T),(Y,S) and (Z,R) be any neutrosophic topological spaces. Let f : (X,T) → (Y,S) and g : (Y,S) → (Z,R) be mapping and (Y,S) be neutrosophic $T_{1/2}$ if f and g are N-continuous then the composition g o f is N-continuous.

The proposition 4.11 is not valid if (Y,S) is not neutrosophic $T_{1/2}$.

4.8 Example
Let X = {a,b,c}. Define the neutrosophic sets A,B and C as follows.
$A = \langle (0.4,0.4,0.6), (0.4,0.4,0.3) \rangle$
$B = \langle (0.4,0.5,0.6), (0.3,0.4,0.3) \rangle$
and $C = \langle (0.4,0.6,0.5), (0.5,0.3,0.4) \rangle$

Then the family T = $\{0_N, 1_N, A\}$, S = $\{0_N, 1_N, B\}$ and R = $\{0_N, 1_N, C\}$ are neutrosophic topologies on X. Thus (X,T),(X,S) and (X,R) are neutrosophic topological spaces. Also define f : (X,T) $\rightarrow$ (X,S) as f(a) = b, f(b) = a, f(c) = c and g : (X,S) $\rightarrow$ (X,R) as g(a) = b, g(b) = c, g(c) = b. Clearly f and g are N-continuous function. But g o f is not N-continuous. For 1 – C is neutrosophic closed in (X,R). $f^{-1}(g^{-1}(1-C))$ is not N closed in (X,T). g o f is not N-continuous.

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Soft neutrosophic semigroups and their generalization

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Mumtaz Ali, Muhammad Shabir, Munazza Naz, Florentin Smarandache (2014). Soft neutrosophic semigroups and their generalization. *Scientia Magna* 10(1), 93-111

Abstract Soft set theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. In this paper we introduced soft neutrosophic semigroup,soft neutosophic bisemigroup, soft neutrosophic N-semigroup with the discuissionf of some of their characteristics. We also introduced a new type of soft neutrophic semigroup, the so called soft strong neutrosophic semigoup which is of pure neutrosophic character. This notion also foound in all the other corresponding notions of soft neutrosophic thoery. We also given some of their properties of this newly born soft structure related to the strong part of neutrosophic theory.

Keywords Neutrosophic semigroup, neutrosophic bisemigroup, neutrosophic N-semigroup, soft set, soft semigroup, soft neutrosophic semigroup, soft neutrosophic bisemigroup, soft neutrosophic N-semigroup.

§1. Introduction and preliminaries

Florentine Smarandache for the first time introduced the concept of neutrosophy in 1995, which is basically a new branch of philosophy which actually studies the origin, nature, and scope of neutralities. The neutrosophic logic came into being by neutrosophy. In neutrosophic logic each proposition is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Neutrosophic logic is an extension of fuzzy logic. In fact the neutrosophic set is the generalization of classical set, fuzzy conventional set, intuitionistic fuzzy set, and interval valued fuzzy set. Neutrosophic logic is used to overcome the problems of impreciseness, indeterminate, and inconsistencies of date etc. The theory of neutrosophy is so applicable to every field of algebra. W. B. Vasantha Kandasamy and Florentin Smarandache introduced neutrosophic fields, neutrosophic rings,neutrosophic vector spaces,neutrosophic groups,neutrosophic bigroups and neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, and neutrosophic

N-semigroups, neutrosophic loops, nuetrosophic biloops, and neutrosophic N-loops, and so on. Mumtaz ali et al. introduced nuetrosophic LA-semigroups.

Molodtsov introduced the theory of soft set. This mathematical tool is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. This theory has been applied successfully in many fields such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability. Recently soft set theory attained much attention of the researchers since its appearance and the work based on several operations of soft set introduced in [2, 9, 10]. Some properties and algebra may be found in [1] . Feng et al. introduced soft semirings in [5]. By means of level soft sets an adjustable approach to fuzzy soft set can be seen in [6]. Some other concepts together with fuzzy set and rough set were shown in [7, 8].

This paper is about to introduced soft nuetrosophic semigroup, soft neutrosophic group, and soft neutrosophic N-semigroup and the related strong or pure part of neutrosophy with the notions of soft set theory. In the proceeding section, we define soft neutrosophic semigroup, soft neutrosophic strong semigroup, and some of their properties are discussed. In the next section, soft neutrosophic bisemigroup are presented with their strong neutrosophic part. Also in this section some of their characterization have been made. In the last section soft neutrosophic N-semigroup and their corresponding strong theory have been constructed with some of their properties.

§2. Definition and properties

Definition 2.1. Let S be a semigroup, the semigroup generated by S and I i.e. $S \cup I$ denoted by $\langle S \cup I \rangle$ is defined to be a neutrosophic semigroup where I is indeterminacy element and termed as neutrosophic element.

It is interesting to note that all neutrosophic semigroups contain a proper subset which is a semigroup.

Example 2.1. Let $Z = \{$the set of positive and negative integers with zero$\}$, Z is only a semigroup under multiplication. Let $N(S) = \{\langle Z \cup I \rangle\}$ be the neutrosophic semigroup under multiplication. Clearly $Z \subset N(S)$ is a semigroup.

Definition 2.2. Let $N(S)$ be a neutrosophic semigroup. A proper subset P of $N(S)$ is said to be a neutrosophic subsemigroup, if P is a neutrosophic semigroup under the operations of $N(S)$. A neutrosophic semigroup $N(S)$ is said to have a subsemigroup if $N(S)$ has a proper subset which is a semigroup under the operations of $N(S)$.

Theorem 2.1. Let $N(S)$ be a neutrosophic semigroup. Suppose P_1 and P_2 be any two neutrosophic subsemigroups of $N(S)$ then $P_1 \cup P_2$ (i.e. the union) the union of two neutrosophic subsemigroups in general need not be a neutrosophic subsemigroup.

Definition 2.3. A neutrosophic semigroup $N(S)$ which has an element e in $N(S)$ such that $e * s = s * e = s$ for all $s \in N(S)$, is called as a neutrosophic monoid.

Definition 2.4. Let $N(S)$ be a neutrosophic monoid under the binary operation $*$. Suppose e is the identity in $N(S)$, that is $s * e = e * s = s$ for all $s \in N(S)$. We call a proper subset P of $N(S)$ to be a neutrosophic submonoid if

1. P is a neutrosophic semigroup under $*$.

2. $e \in P$, i.e., P is a monoid under $*$.

Definition 2.5. Let $N(S)$ be a neutrosophic semigroup under a binary operation $*$. P be a proper subset of $N(S)$. P is said to be a neutrosophic ideal of $N(S)$ if the following conditions are satisfied.

1. P is a neutrosophic semigroup.

2. For all $p \in P$ and for all $s \in N(S)$ we have $p * s$ and $s * p$ are in P.

Definition 2.6. Let $N(S)$ be a neutrosophic semigroup. P be a neutrosophic ideal of $N(S)$, P is said to be a neutrosophic cyclic ideal or neutrosophic principal ideal if P can be generated by a single element.

Definition 2.7. Let $(BN(S), *, o)$ be a nonempty set with two binary operations $*$ and o. $(BN(S), *, o)$ is said to be a neutrosophic bisemigroup if $BN(S) = P1 \cup P2$ where atleast one of $(P1, *)$ or $(P2, o)$ is a neutrosophic semigroup and other is just a semigroup. $P1$ and $P2$ are proper subsets of $BN(S)$, i.e. $P1 \subsetneq P2$.

If both $(P1, *)$ and $(P2, o)$ in the above definition are neutrosophic semigroups then we call $(BN(S), *, o)$ a strong neutrosophic bisemigroup. All strong neutrosophic bisemigroups are trivially neutrosophic bisemigroups.

Example 2.2. Let $(BN(S), *, o) = \{0, 1, 2, 3, I, 2I, 3I, S(3), *, o\} = (P_1, *) \cup (P_2, o)$ where $(P_1, *) = \{0, 1, 2, 3, I, 2I, 3I, *\}$ and $(P_2, o) = (S(3), o)$. Clearly $(P_1, *)$ is a neutrosophic semigroup under multiplication modulo 4. (P_2, o) is just a semigroup. Thus $(BN(S), *, o)$ is a neutrosophic bisemigroup.

Definition 2.8. Let $(BN(S) = P1 \cup P2; o, *)$ be a neutrosophic bisemigroup. A proper subset $(T, o, *)$ is said to be a neutrosophic subbisemigroup of $BN(S)$ if

1. $T = T1 \cup T2$ where $T1 = P1 \cap T$ and $T2 = P2 \cap T$.

2. At least one of $(T1, o)$ or $(T2, *)$ is a neutrosophic semigroup.

Definition 2.9. Let $(BN(S) = P_1 \cup P_2, o, *)$ be a neutrosophic strong bisemigroup. A proper subset T of $BN(S)$ is called the strong neutrosophic subbisemigroup if $T = T_1 \cup T_2$ with $T_1 = P_1 \cap T$ and $T_2 = P_2 \cap T$ and if both $(T_1, *)$ and (T_2, o) are neutrosophic subsemigroups of $(P_1, *)$ and (P_2, o) respectively. We call $T = T_1 \cup T_2$ to be a neutrosophic strong subbisemigroup, if atleast one of $(T_1, *)$ or (T_2, o) is a semigroup then $T = T_1 \cup T_2$ is only a neutrosophic subsemigroup.

Definition 2.10. Let $(BN(S) = P_1 \cup P_2 *, o)$ be any neutrosophic bisemigroup. Let J be a proper subset of $B(NS)$ such that $J_1 = J \cap P_1$ and $J_2 = J \cap P_2$ are ideals of P_1 and P_2 respectively. Then J is called the neutrosophic bi-ideal of $BN(S)$.

Definition 2.11. Let $(BN(S), *, o)$ be a strong neutrosophic bisemigroup where $BN(S) = P_1 \cup P_2$ with $(P_1, *)$ and (P_2, o) be any two neutrosophic semigroups. Let J be a proper subset of $BN(S)$ where $I = I_1 \cup I_2$ with $I_1 = J \cap P_1$ and $I_2 = J \cap P_2$ are neutrosophic ideals of the neutrosophic semigroups P_1 and P_2 respectively. Then I is called or defined as the strong neutrosophic bi-ideal of $B(N(S))$.

Union of any two neutrosophic bi-ideals in general is not a neutrosophic bi-ideal. This is true of neutrosophic strong bi-ideals.

Definition 2.12. Let $\{S(N), *_1, \ldots, *_N\}$ be a non empty set with N-binary operations defined on it. We call $S(N)$ a neutrosophic N-semigroup (N a positive integer) if the following conditions are satisfied.

1. $S(N) = S_1 \cup \ldots \cup S_N$ where each S_i is a proper subset of $S(N)$ i.e. $S_i \subsetneq S_j$ or $S_j \subsetneq S_i$ if $i \neq j$.

2. $(S_i, *_i)$ is either a neutrosophic semigroup or a semigroup for $i = 1, 2, \ldots, N$.

If all the N-semigroups $(Si, *i)$ are neutrosophic semigroups (i.e. for $i = 1, 2, \ldots, N$) then we call $S(N)$ to be a neutrosophic strong N-semigroup.

Example 2.3. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}$, semigroup under multiplication modulo 12$\}$.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I$, semigroup under multiplication modulo 4$\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle R \cup I \rangle \right\}$, neutrosophic semigroup under matrix multiplication and $S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication.

Definition 2.13. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup. A proper subset $P = \{P_1 \cup P_2 \cup \ldots \cup P_N, *_1, *_2, \ldots, *_N\}$ of $S(N)$ is said to be a neutrosophic Nsubsemigroup if $P_i = P \cap S_i, i = 1, 2, \ldots, N$ are subsemigroups of S_i in which atleast some of the subsemigroups are neutrosophic subsemigroups.

Definition 2.14. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic strong N-semigroup. A proper subset $T = \{T_1 \cup T_2 \cup \ldots \cup T_N, *_1, \ldots, *_N\}$ of $S(N)$ is said to be a neutrosophic strong sub N-semigroup if each $(T_i, *_i)$ is a neutrosophic subsemigroup of $(S_i, *_i)$ for $i = 1, 2, \ldots, N$ where $T_i = T \cap S_i$.

If only a few of the $(Ti, *i)$ in T are just subsemigroups of $(Si, *i)$ (i.e. $(Ti, *i)$ are not neutrosophic subsemigroups then we call T to be a sub N-semigroup of $S(N)$.

Definition 2.15. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup. A proper subset $P = \{P_1 \cup P_2 \cup \ldots \cup P_N, *_1, \ldots, *_N\}$ of $S(N)$ is said to be a neutrosophic N-subsemigroup, if the following conditions are true,

i. P is a neutrosophic sub N-semigroup of $S(N)$.

ii. Each $P_i = P \cap S_i, i = 1, 2, \ldots, N$ is an ideal of S_i.

Then P is called or defined as the neutrosophic N-ideal of the neutrosophic N-semigroup $S(N)$.

Definition 2.16. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic strong N-semigroup. A proper subset $J = \{I_1 \cup I_2 \cup \ldots \cup I_N\}$ where $I_t = J \cap S_t$ for $t = 1, 2, \ldots, N$ is said to be a neutrosophic strong N-ideal of $S(N)$ if the following conditions are satisfied.

1. Each is a neutrosophic subsemigroup of $S_t, t = 1, 2, \ldots, N$ i.e. It is a neutrosophic strong N-subsemigroup of $S(N)$.

2. Each is a two sided ideal of S_t for $t = 1, 2, \ldots, N$.

Similarly one can define neutrosophic strong N-left ideal or neutrosophic strong right ideal of $S(N)$.

A neutrosophic strong N-ideal is one which is both a neutrosophic strong N-left ideal and N-right ideal of $S(N)$.

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subset E$. Molodtsov [12] defined the soft set in the following manner:

Definition 2.17. A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \longrightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-elements of the soft set (F, A), or as the set of e-approximate elements of the soft set.

Example 2.4. Suppose that U is the set of shops. E is the set of parameters and each parameter is a word or senctence. Let E={high rent, normal rent, in good condition, in bad condition}. Let us consider a soft set (F, A) which describes the attractiveness of shops that Mr. Z is taking on rent. Suppose that there are five houses in the universe $U = \{h_1, h_2, h_3, h_4, h_5\}$ under consideration, and that $A = \{e_1, e_2, e_3\}$ be the set of parameters where

e_1 stands for the parameter high rent.

e_2 stands for the parameter normal rent.

e_3 stands for the parameter in good condition.

Suppose that

$F(e_1) = \{h_1, h_4\}$.

$F(e_2) = \{h_2, h_5\}$.

$F(e_3) = \{h_3, h_4, h_5\}$.

The soft set (F, A) is an approximated family $\{F(e_i), i = 1, 2, 3\}$ of subsets of the set U which gives us a collection of approximate description of an object. Thus, we have the soft set (F, A) as a collection of approximations as below:

$(F, A) = \{\text{high rent} = \{h_1, h_4\}, \text{normal rent} = \{h_2, h_5\}, \text{in good condition} = \{h_3, h_4, h_5\}\}$.

Definition 2.18. For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1. $A \subseteq B$.

2. $F(e) \subseteq G(e)$, for all $e \in A$.

This relationship is denoted by $(F, A) \tilde{\subset} (H, B)$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \tilde{\supset} (H, B)$.

Definition 2.19. Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 2.20. (F, A) over U is called an absolute soft set if $F(e) = U$ for all $e \in A$ and we denote it by U.

Definition 2.21. Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by$(F, A) \cap_R (G, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap G(c)$ for all $c \in C = A \cap B$.

Definition 2.22. The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$, $H(e)$ is defined as

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cap_{\varepsilon} (G, B) = (H, C)$.

Definition 2.23. The resticted union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$, $H(e)$ is defined as the soft set $(H, C) = (F, A) \cup_R (G, B)$ where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

Definition 2.24. The extended union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$, $H(e)$ is defined as

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cup_{\varepsilon} (G, B) = (H, C)$.

Definition 2.25. A soft set (F, A) over S is called a soft semigroup over S if $(F, A) \overset{\curlywedge}{\circ} (F, A) \subseteq (F, A)$.

It is easy to see that a soft set (F, A) over S is a soft semigroup if and only if $\phi \neq F(a)$ is a subsemigroup of S.

Definition 2.26. A soft set (F, A) over a semigroup S is called a soft left (right) ideal over S, if $(S, E) \subseteq (F, A), ((F, A) \subseteq (S, E))$.

A soft set over S is a soft ideal if it is both a soft left and a soft right ideal over S.

Proposition 2.1. A soft set (F, A) over S is a soft ideal over S if and only if $\phi \neq F(a)$ is an ideal of S.

Definition 2.27. Let (G, B) be a soft subset of a soft semigroup (F, A) over S, then (G, B) is called a soft subsemigroup (ideal) of (F, A) if $G(b)$ is a subsemigroup (ideal) of $F(b)$ for all $b \in A$.

§3. Soft neutrosophic semigroup

Definition 3.1. Let $N(S)$ be a neutrosophic semigroup and (F, A) be a soft set over $N(S)$. Then (F, A) is called soft neutrosophic semigroup if and only if $F(e)$ is neutrosophic subsemigroup of $N(S)$, for all $e \in A$.

Equivalently (F, A) is a soft neutrosophic semigroup over $N(S)$ if $(F, A) \overset{\curlywedge}{\circ} (F, A) \subseteq (F, A)$, where $\tilde{N}_{(N(S),A)} \neq (F, A) \neq \widetilde{\phi}$.

Example 3.1. Let $N(S) = \langle Z^+ \cup \{0\}^+ \cup \{I\} \rangle$ be a neutrosophic semigroup under $+$. Consider $P = \langle 2Z^+ \cup I \rangle$ and $R = \langle 3Z^+ \cup I \rangle$ are neutrosophic subsemigroup of $N(S)$. Then clearly for all $e \in A$, (F, A) is a soft neutrosophic semigroup over $N(S)$, where $F(x_1) = \{\langle 2Z^+ \cup I \rangle\}$, $F(x_2) = \{\langle 3Z^+ \cup I \rangle\}$.

Theorem 3.1. A soft neutrosophic semigroup over $N(S)$ always contain a soft semigroup over S.

Proof. The proof of this theorem is straight forward.

Theorem 3.2. Let (F, A) and (H, A) be two soft neutrosophic semigroups over $N(S)$. Then their intersection $(F, A) \cap (H, A)$ is again soft neutrosophic semigroup over $N(S)$.

Proof. The proof is staight forward.

Theorem 3.3. Let (F, A) and (H, B) be two soft neutrosophic semigroups over $N(S)$. If $A \cap B = \phi$, then $(F, A) \cup (H, B)$ is a soft neutrosophic semigroup over $N(S)$.

Remark 3.1. The extended union of two soft neutrosophic semigroups (F, A) and (K, B) over $N(S)$ is not a soft neutrosophic semigroup over $N(S)$.

We take the following example for the proof of above remark.

Example 3.2. Let $N(S) = \langle Z^+ \cup I \rangle$ be the neutrosophic semigroup under $+$. Take $P_1 = \{\langle 2Z^+ \cup I \rangle\}$ and $P_2 = \{\langle 3Z^+ \cup I \rangle\}$ to be any two neutrosophic subsemigroups of $N(S)$. Then clearly for all $e \in A$, (F, A) is a soft neutrosophic semigroup over $N(S)$, where $F(x_1) = \{\langle 2Z^+ \cup I \rangle\}, F(x_2) = \{\langle 3Z^+ \cup I \rangle\}$.

Again Let $R_1 = \{\langle 5Z^+ \cup I \rangle\}$ and $R_2 = \{\langle 4Z^+ \cup I \rangle\}$ be another neutrosophic subsemigroups of $N(S)$ and (K, B) is another soft neutrosophic semigroup over $N(S)$, where $K(x_1) = \{\langle 5Z^+ \cup I \rangle\}, K(x_3) = \{\langle 4Z^+ \cup I \rangle\}$.

Let $C = A \cup B$. The extended union $(F, A) \cup_\varepsilon (K, B) = (H, C)$ where $x_1 \in C$, we have $H(x_1) = F(x_1) \cup K(x_1)$ is not neutrosophic subsemigroup as union of two neutrosophic subsemigroup is not neutrosophic subsemigroup.

Proposition 3.1. The extended intersection of two soft neutrosophic semigroups over $N(S)$ is soft neutrosophic semigruop over $N(S)$.

Remark 3.2. The restricted union of two soft neutrosophic semigroups (F, A) and (K, B) over $N(S)$ is not a soft neutrosophic semigroup over $N(S)$.

We can easily check it in above example.

Proposition 3.2. The restricted intersection of two soft neutrosophic semigroups over $N(S)$ is soft neutrosophic semigroup over $N(S)$.

Proposition 3.3. The AND operation of two soft neutrosophic semigroups over $N(S)$ is soft neutrosophic semigroup over $N(S)$.

Proposition 3.4. The OR operation of two soft neutosophic semigroup over $N(S)$ may not be a soft nuetrosophic semigroup over $N(S)$.

Definition 3.2. Let $N(S)$ be a neutrosophic monoid and (F, A) be a soft set over $N(S)$. Then (F, A) is called soft neutrosophic monoid if and only if $F(e)$ is neutrosophic submonoid of $N(S)$, for all $x \in A$.

Example 3.3. Let $N(S) = \langle Z \cup I \rangle$ be a neutrosophic monoid under $+$. Let $P = \langle 2Z \cup I \rangle$ and $Q = \langle 3Z \cup I \rangle$ are neutrosophic submonoids of $N(S)$. Then (F, A) is a soft neutrosophic monoid over $N(S)$, where $F(x_1) = \{\langle 2Z \cup I \rangle\}, F(x_2) = \{\langle 3Z \cup I \rangle\}$.

Theorem 3.4. Every soft neutrosophic monoid over $N(S)$ is a soft neutrosophic semigroup over $N(S)$ but the converse is not true in general.

Proof. The proof is straightforward.

Proposition 3.5. Let (F, A) and (K, B) be two soft neutrosophic monoids over $N(S)$. Then

1. Their extended union $(F,A)\cup_{\varepsilon}(K,B)$ over $N(S)$ is not soft neutrosophic monoid over $N(S)$.

2. Their extended intersection $(F,A)\cap_{\varepsilon}(K,B)$ over $N(S)$ is soft neutrosophic monoid over $N(S)$.

3. Their restricted union $(F,A)\cup_{R}(K,B)$ over $N(S)$ is not soft neutrosophic monoid over $N(S)$.

4. Their restricted intersection $(F,A)\cap_{\varepsilon}(K,B)$ over $N(S)$ is soft neutrosophic monoid over $N(S)$.

Proposition 3.6. Let (F,A) and (H,B) be two soft neutrosophic monoid over $N(S)$. Then

1. Their AND operation $(F,A)\wedge(H,B)$ is soft neutrosophic monoid over $N(S)$.

2. Their OR operation $(F,A)\vee(H,B)$ is not soft neutrosophic monoid over $N(S)$.

Definition 3.3. Let (F,A) be a soft neutrosophic semigroup over $N(S)$, then (F,A) is called Full-soft neutrosophic semigroup over $N(S)$ if $F(x)=N(S)$, for all $x\in A$. We denote it by $N(S)$.

Theorem 3.5. Every Full-soft neutrosophic semigroup over $N(S)$ always contain absolute soft semigroup over S.

Proof. The proof of this theorem is straight forward.

Definition 3.4. Let (F,A) and (H,B) be two soft neutrosophic semigroups over $N(S)$. Then (H,B) is a soft neutrosophic subsemigroup of (F,A), if

1. $B\subset A$.

2. $H(a)$ is neutrosophic subsemigroup of $F(a)$, for all $a\in B$.

Example 3.4. Let $N(S)=\langle Z\cup I\rangle$ be a neutrosophic semigroup under $+$. Then (F,A) is a soft neutrosophic semigroup over $N(S)$, where $F(x_1)=\{\langle 2Z\cup I\rangle\},F(x_2)=\{\langle 3Z\cup I\rangle\},F(x_3)=\{\langle 5Z\cup I\rangle\}$.

Let $B=\{x_1,x_2\}\subset A$. Then (H,B) is soft neutrosophic subsemigroup of (F,A) over $N(S)$, where $H(x_1)=\{\langle 4Z\cup I\rangle\},H(x_2)=\{\langle 6Z\cup I\rangle\}$.

Theorem 3.6. A soft neutrosophic semigroup over $N(S)$ have soft neutrosophic subsemigroups as well as soft subsemigroups over $N(S)$.

Proof. Obvious.

Theorem 3.7. Every soft semigroup over S is always soft neutrosophic subsemigroup of soft neutrosophic semigroup over $N(S)$.

Proof. The proof is obvious.

Theorem 3.8. Let (F,A) be a soft neutrosophic semigroup over $N(S)$ and $\{(H_i,B_i);i\in I\}$ is a non empty family of soft neutrosophic subsemigroups of (F,A) then

1. $\cap_{i\in I}(H_i,B_i)$ is a soft neutrosophic subsemigroup of (F,A).

2. $\wedge_{i\in I}(H_i,B_i)$ is a soft neutrosophic subsemigroup of $\wedge_{i\in I}(F,A)$.

3. $\cup_{i\in I}(H_i,B_i)$ is a soft neutrosophic subsemigroup of (F,A) if $B_i\cap B_j=\phi$, for all $i\neq j$.

Proof. Straightforward.

Definition 3.5. A soft set (F,A) over $N(S)$ is called soft neutrosophic left (right) ideal over $N(S)$ if $N(S)\overset{\curlywedge}{\circ}(F,A)\subseteq(F,A)$, where $\tilde{N}_{(N(S),A)}\neq(F,A)\neq\widetilde{\phi}$ and $N(S)$ is Full-soft neutrosophic semigroup over $N(S)$.

A soft set over $N(S)$ is a soft neutrosophic ideal if it is both a soft neutrosophic left and a soft neutrosophic right ideal over $N(S)$.

Example 3.5. Let $N(S) = \langle Z \cup I \rangle$ be the neutrosophic semigroup under multiplication. Let $P = \langle 2Z \cup I \rangle$ and $Q = \langle 4Z \cup I \rangle$ are neutrosophic ideals of $N(S)$. Then clearly (F, A) is a soft neutrosophic ideal over $N(S)$, where $F(x_1) = \{\langle 2Z \cup I \rangle\}, F(x_2) = \{\langle 4Z \cup I \rangle\}$.

Proposition 3.7. (F, A) is soft neutrosophic ideal if and only if $F(x)$ is a neutrosophic ideal of $N(S)$, for all $x \in A$.

Theorem 3.9. Every soft neutrosophic ideal (F, A) over $N(S)$ is a soft neutrosophic semigroup but the converse is not true.

Proposition 3.8. Let (F, A) and (K, B) be two soft neutrosophic ideals over $N(S)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.
2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.
3. Their restricted union $(F, A) \cup_R (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.
4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.

Proposition 3.9.

1. Let (F, A) and (H, B) be two soft neutrosophic ideal over $N(S)$.
2. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic ideal over $N(S)$.
3. Their OR operation $(F, A) \vee (H, B)$ is soft neutrosophic ideal over $N(S)$.

Theorem 3.10. Let (F, A) and (G, B) be two soft semigroups (ideals) over S and T respectively. Then $(F, A) \times (G, B)$ is also a soft semigroup (ideal) over $S \times T$.

Proof. The proof is straight forward.

Theorem 3.11. Let (F, A) be a soft neutrosophic semigroup over $N(S)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic ideals of (F, A) then

1. $\cap_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of (F, A).
2. $\wedge_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of $\wedge_{i \in I} (F, A)$.
3. $\cup_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of (F, A).
4. $\vee_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of $\vee_{i \in I} (F, A)$.

Definition 3.6. A soft set (F, A) over $N(S)$ is called soft neutrosophic principal ideal or soft neutrosophic cyclic ideal if and only if $F(x)$ is a principal or cyclic neutrosophic ideal of $N(S)$, for all $x \in A$.

Proposition 3.10. Let (F, A) and (K, B) be two soft neutrosophic principal ideals over $N(S)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $N(S)$ is not soft neutrosophic principal ideal over $N(S)$.
2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $N(S)$ is soft neutrosophic principal ideal over $N(S)$.
3. Their restricted union $(F, A) \cup_R (K, B)$ over $N(S)$ is not soft neutrosophic principal ideal over $N(S)$.

4. Their restricted intersection $(F,A) \cap_{\varepsilon} (K,B)$ over $N(S)$ is soft neutrosophic principal ideal over $N(S)$.

Proposition 3.11. Let (F,A) and (H,B) be two soft neutrosophic principal ideals over $N(S)$. Then

1. Their AND operation $(F,A) \wedge (H,B)$ is soft neutrosophic principal ideal over $N(S)$.

2. Their OR operation $(F,A) \vee (H,B)$ is not soft neutrosophic principal ideal over $N(S)$.

§3. Soft neutrosophic bisemigroup

Definition 3.1. Let $\{BN(S), *_1, *_2\}$ be a neutrosophic bisemigroup and let (F,A) be a soft set over $BN(S)$. Then (F,A) is said to be soft neutrosophic bisemigroup over $BN(G)$ if and only if $F(x)$ is neutrosophic subbisemigroup of $BN(G)$ for all $x \in A$.

Example 3.1. Let $BN(S) = \{0,1,2,I,2I,\langle Z \cup I\rangle, \times, +\}$ be a neutosophic bisemigroup. Let $T = \{0,I,2I,\langle 2Z \cup I\rangle, \times, +\}, P = \{0,1,2,\langle 5Z \cup I\rangle, \times, +\}$ and $L = \{0,1,2,Z,\times,+\}$ are neutrosophic subbisemigroup of $BN(S)$. The (F,A) is clearly soft neutrosophic bisemigroup over $BN(S)$, where $F(x_1) = \{0,I,2I,\langle 2Z \cup I\rangle, \times, +\}, F(x_2) = \{0,1,2,\langle 5Z \cup I\rangle, \times, +\}, F(x_3) = \{0,1,2,Z,\times,+\}$.

Theorem 3.1. Let (F,A) and (H,A) be two soft neutrosophic bisemigroup over $BN(S)$. Then their intersection $(F,A) \cap (H,A)$ is again a soft neutrosophic bisemigroup over $BN(S)$.

Proof. Straightforward.

Theorem 3.2. Let (F,A) and (H,B) be two soft neutrosophic bisemigroups over $BN(S)$ such that $A \cap B = \phi$, then their union is soft neutrosophic bisemigroup over $BN(S)$.

Proof. Straightforward.

Proposition 3.1. Let (F,A) and (K,B) be two soft neutrosophic bisemigroups over $BN(S)$. Then

1. Their extended union $(F,A) \cup_{\varepsilon} (K,B)$ over $BN(S)$ is not soft neutrosophic bisemigroup over $BN(S)$.

2. Their extended intersection $(F,A) \cap_{\varepsilon} (K,B)$ over $BN(S)$ is soft neutrosophic bisemigroup over $BN(S)$.

3. Their restricted union $(F,A) \cup_R (K,B)$ over $BN(S)$ is not soft neutrosophic bisemigroup over $BN(S)$.

4. Their restricted intersection $(F,A) \cap_{\varepsilon} (K,B)$ over $BN(S)$ is soft neutrosophic bisemigroup over $BN(S)$.

Proposition 3.2. Let (F,A) and (K,B) be two soft neutrosophic bisemigroups over $BN(S)$. Then

1. Their AND operation $(F,A) \wedge (K,B)$ is soft neutrosophic bisemigroup over $BN(S)$.

2. Their OR operation $(F,A) \vee (K,B)$ is not soft neutrosophic bisemigroup over $BN(S)$.

Definition 3.2. Let (F,A) be a soft neutrosophic bisemigroup over $BN(S)$, then (F,A) is called Full-soft neutrosophic bisemigroup over $BN(S)$ if $F(x) = BN(S)$, for all $x \in A$. We denote it by $BN(S)$.

Definition 3.3. Let (F,A) and (H,B) be two soft neutrosophic bisemigroups over $BN(S)$. Then (H,B) is a soft neutrosophic subbisemigroup of (F,A), if

1. $B \subset A$.

2. $H(x)$ is neutrosophic subbisemigroup of $F(x)$, for all $x \in B$.

Example 3.2. Let $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I\rangle, \times, +\}$ be a neutosophic bisemigroup. Let $T = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}$, $P = \{0, 1, 2, \langle 5Z \cup I\rangle, \times, +\}$ and $L = \{0, 1, 2, Z, \times, +\}$ are neutrosophic subbisemigroup of $BN(S)$. The (F, A) is clearly soft neutrosophic bisemigroup over $BN(S)$, where $F(x_1) = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}$, $F(x_2) = \{0, 1, 2, \langle 5Z \cup I\rangle, \times, +\}$, $F(x_3) = \{0, 1, 2, Z, \times, +\}$.

Then (H, B) is a soft neutrosophic subbisemigroup of (F, A), where $H(x_1) = \{0, I, \langle 4Z \cup I\rangle, \times, +\}$, $H(x_3) = \{0, 1, 4Z, \times, +\}$.

Theorem 3.3. Let (F, A) be a soft neutrosophic bisemigroup over $BN(S)$ and $\{(H_i, B_i); i \in I\}$ be a non-empty family of soft neutrosophic subbisemigroups of (F, A) then

1. $\cap_{i\in I} (H_i, B_i)$ is a soft neutrosophic subbisemigroup of (F, A).

2. $\wedge_{i\in I} (H_i, B_i)$ is a soft neutrosophic subbisemigroup of $\wedge_{i\in I} (F, A)$.

3. $\cup_{i\in I} (H_i, B_i)$ is a soft neutrosophic subbisemigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Theorem 3.4. (F, A) is called soft neutrosophic biideal over $BN(S)$ if $F(x)$ is neutrosophic biideal of $BN(S)$, for all $x \in A$.

Example 3.3. Let $BN(S) = (\{\langle Z \cup I\rangle, 0, 1, 2, I, 2I, +, \times\}(\times$ under multiplication modulo 3)). Let $T = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}$ and $J = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$ are ideals of $BN(S)$. Then (F, A) is soft neutrosophic biideal over $BN(S)$, where $F(x_1) = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}$, $F(x_2) = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$.

Theorem 3.5. Every soft neutrosophic biideal (F, A) over $BS(N)$ is a soft neutrosophic bisemigroup but the converse is not true.

Proposition 3.3. Let (F, A) and (K, B) be two soft neutrosophic biideals over $BN(S)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $BN(S)$ is not soft neutrosophic biideal over $BN(S)$.

2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic biideal over $BN(S)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $BN(S)$ is not soft neutrosophic biideal over $BN(S)$.

4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic biideal over $BN(S)$.

Proposition 3.4. Let (F, A) and (H, B) be two soft neutrosophic biideal over $BN(S)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic biideal over $BN(S)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic biideal over $BN(S)$.

Theorem 3.6.

Let (F, A) be a soft neutrosophic bisemigroup over $BN(S)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic biideals of (F, A) then

1. $\cap_{i\in I} (H_i, B_i)$ is a soft neutrosophic biideal of (F, A).

2. $\wedge_{i\in I} (H_i, B_i)$ is a soft neutrosophic biideal of $\wedge_{i\in I} (F, A)$.

§4. Soft neutrosophic strong bisemigroup

Definition 4.1. Let (F, A) be a soft set over a neutrosophic bisemigroup $BN(S)$. Then (F, A) is said to be soft strong neutrosophic bisemigroup over $BN(G)$ if and only if $F(x)$ is neutrosophic strong subbisemigroup of $BN(G)$ for all $x \in A$.

Example 4.1. Let $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I\rangle, \times, +\}$ be a neutrosophic bisemigroup. Let $T = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}$ and $R = \{0, 1, I, \langle 4Z \cup I\rangle, \times, +\}$ are neutrosophic strong subbisemigroups of $BN(S)$. Then (F, A) is soft neutrosophic strong bisemigroup over $BN(S)$, where $F(x_1) = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}, F(x_2) = \{0, I, 1, \langle 4Z \cup I\rangle, \times, +\}$.

Theorem 4.1. Every soft neutrosophic strong bisemigroup is a soft neutrosophic bisemigroup but the converse is not true.

Proposition 4.1. Let (F, A) and (K, B) be two soft neutrosophic strong bisemigroups over $BN(S)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $BN(S)$ is not soft neutrosophic strong bisemigroup over $BN(S)$.
2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $BN(S)$ is soft neutrosophic stong bisemigroup over $BN(S)$.
3. Their restricted union $(F, A) \cup_{R} (K, B)$ over $BN(S)$ is not soft neutrosophic stong bisemigroup over $BN(S)$.
4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $BN(S)$ is soft neutrosophic strong bisemigroup over $BN(S)$.

Proposition 4.2. Let (F, A) and (K, B) be two soft neutrosophic strong bisemigroups over $BN(S)$. Then

1. Their AND operation $(F, A) \wedge (K, B)$ is soft neutrosophic strong bisemigroup over $BN(S)$.
2. Their OR operation $(F, A) \vee (K, B)$ is not soft neutrosophic strong bisemigroup over $BN(S)$.

Definition 4.2. Let (F, A) and (H, B) be two soft neutrosophic strong bisemigroups over $BN(S)$. Then (H, B) is a soft neutrosophic strong subbisemigroup of (F, A), if

1. $B \subset A$.
2. $H(x)$ is neutrosophic strong subbisemigroup of $F(x)$, for all $x \in B$.

Example 4.2. Let $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I\rangle, \times, +\}$ be a neutrosophic bisemigroup. Let $T = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}$ and $R = \{0, 1, I, \langle 4Z \cup I\rangle, \times, +\}$ are neutrosophic strong subbisemigroups of $BN(S)$. Then (F, A) is soft neutrosophic strong bisemigroup over $BN(S)$, where $F(x_1) = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}, F(x_2) = \{0, I, \langle 4Z \cup I\rangle, \times, +\}$.

Then (H, B) is a soft neutrosophic strong subbisemigroup of (F, A), where $H(x_1) = \{0, I, \langle 4Z \cup I\rangle, \times, +\}$.

Theorem 4.2. Let (F, A) be a soft neutrosophic strong bisemigroup over $BN(S)$ and $\{(H_i, B_i); i \in I\}$ be a non empty family of soft neutrosophic strong subbisemigroups of (F, A) then

1. $\cap_{i \in I}(H_i, B_i)$ is a soft neutrosophic strong subbisemigroup of (F, A).
2. $\wedge_{i \in I}(H_i, B_i)$ is a soft neutrosophic strong subbisemigroup of $\wedge_{i \in I}(F, A)$.

3. $\cup_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong subbisemigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Definition 4.3. (F, A) over $BN(S)$ is called soft neutrosophic strong biideal if $F(x)$ is neutosophic strong biideal of $BN(S)$, for all $x \in A$.

Example 4.3. Let $BN(S) = (\{\langle Z \cup I\rangle, 0, 1, 2, I, 2I\}, +, \times(\times$ under multiplication modulo 3)). Let $T = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}$ and $J = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$ are neutrosophic strong ideals of $BN(S)$. Then (F, A) is soft neutrosophic strong biideal over $BN(S)$, where $F(x_1) = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}, F(x_2) = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$.

Theorem 4.3. Every soft neutrosophic strong biideal (F, A) over $BS(N)$ is a soft neutrosophic bisemigroup but the converse is not true.

Theorem 4.4. Every soft neutrosophic strong biideal (F, A) over $BS(N)$ is a soft neutrosophic strong bisemigroup but the converse is not true.

Proposition 4.3. Let (F, A) and (K, B) be two soft neutrosophic strong biideals over $BN(S)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $BN(S)$ is not soft neutrosophic strong biideal over $BN(S)$.

2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic strong biideal over $BN(S)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $BN(S)$ is not soft neutrosophic strong biideal over $BN(S)$.

4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic stong biideal over $BN(S)$.

Proposition 4.4. Let (F, A) and (H, B) be two soft neutrosophic strong biideal over $BN(S)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic strong biideal over $BN(S)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic strong biideal over $BN(S)$.

Theorem 4.5. Let (F, A) be a soft neutrosophic strong bisemigroup over $BN(S)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic strong biideals of (F, A) then

1. $\cap_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong biideal of (F, A).

2. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong biideal of $\wedge_{i\in I}(F, A)$.

§5. Soft neutrosophic N-semigroup

Definition 5.1. Let $\{S(N), *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup and (F, A) be a soft set over $\{S(N), *_1, \ldots, *_N\}$. Then (F, A) is termed as soft neutrosophic N-semigroup if and only if $F(x)$ is neutrosophic sub N-semigroup, for all $x \in A$.

Example 5.1. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}$, semigroup under multiplication modulo 12$\}$.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I$, semigroup under multiplication modulo 4$\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle R \cup I \rangle \right\}$, neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication. Let $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$ is a neutosophic sub 4-semigroup of $S(4)$, where $T_1 = \{0,2,4,6,8,10\} \subseteq Z_{12}$, $T_2 = \{0, I, 2I, 3I\} \subset S_2$, $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Q \cup I \rangle \right\} \subset S_3$, $T4 = \{\langle 5Z \cup I \rangle\} \subset S_4$, the neutrosophic semigroup under multiplication. Also let $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$ be another neutrosophic sub 4-semigroup of $S(4)$, where $P_1 = \{0,6\} \subseteq Z_{12}$, $P_2 = \{0,1,I\} \subset S_2$, $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Z \cup I \rangle \right\} \subset S_3$, $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$. Then (F, A) is soft neutrosophic 4-semigroup over $S(4)$, where

$$F(x_1) = \{0,2,4,6,8,10\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\},$$

$$F(x_2) = \{0,6\} \cup \{0,1,I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}.$$

Theorem 5.1. Let (F, A) and (H, A) be two soft neutrosophic N-semigroup over $S(N)$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic N-semigroup over $S(N)$.

Proof. Straightforward.

Theorem 5.2. Let (F, A) and (H, B) be two soft neutrosophic N-semigroups over $S(N)$ such that $A \cap B = \phi$, then their union is soft neutrosophic N-semigroup over $S(N)$.

Proof. Straightforward.

Proposition 5.1. Let (F, A) and (K, B) be two soft neutrosophic N-semigroups over $S(N)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $S(N)$ is not soft neutrosophic N-semigroup over $S(N)$.
2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $S(N)$ is soft neutrosophic N-semigroup over $S(N)$.
3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic N-semigroup over $S(N)$.
4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $S(N)$ is soft neutrosophic N-semigroup over $S(N)$.

Proposition 5.2. Let (F, A) and (K, B) be two soft neutrosophic N-semigroups over $S(N)$. Then

1. Their AND operation $(F, A) \wedge (K, B)$ is soft neutrosophic N-semigroup over $S(N)$.
2. Their OR operation $(F, A) \vee (K, B)$ is not soft neutrosophic N-semigroup over $S(N)$.

Definition 5.2. Let (F, A) be a soft neutrosophic N-semigroup over $S(N)$, then (F, A) is called Full-soft neutrosophic N-semigroup over $S(N)$ if $F(x) = S(N)$, for all $x \in A$. We denote it by $S(N)$.

Definition 5.3. Let (F,A) and (H,B) be two soft neutrosophic N-semigroups over $S(N)$. Then (H,B) is a soft neutrosophic sub N-semigroup of (F,A), if

1. $B \subset A$.
2. $H(x)$ is neutrosophic sub N-semigroup of $F(x)$, for all $x \in B$.

Example 5.2. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}$, semigroup under multiplication modulo 12$\}$.

$S_2 = \{0,1,2,3,I,2I,3I$, semigroup under multiplication modulo 4$\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle R \cup I \rangle \right\}$, neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication. Let $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$ is a neutosophic sub 4-semigroup of $S(4)$, where $T_1 = \{0,2,4,6,8,10\} \subseteq Z_{12}$, $T_2 = \{0,I,2I,3I\} \subset S_2$, $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Q \cup I \rangle \right\} \subset S_3$, $T_4 = \{\langle 5Z \cup I \rangle\} \subset S_4$, the neutrosophic semigroup under multiplication. Also let $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$ be another neutrosophic sub 4-semigroup of $S(4)$, where $P_1 = \{0,6\} \subseteq Z_{12}$, $P_2 = \{0,1,I\} \subset S_2$, $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Z \cup I \rangle \right\} \subset S_3$, $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$. Also let $R = \{R_1 \cup R_2 \cup R_3 \cup R_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic sub 4-semigroup os $S(4)$ where $R_1 = \{0,3,6,9\}, R_2 = \{0,I,2I\}, R_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle 2Z \cup I \rangle \right\}, R_4 = \{\langle 3Z \cup I \rangle\}$. Then (F,A) is soft neutrosophic 4-semigroup over $S(4)$, where

$$
\begin{aligned}
F(x_1) &= \{0,2,4,6,8,10\} \cup \{0,I,2I,3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\}, \\
F(x_2) &= \{0,6\} \cup \{0,1,I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}, \\
F(x_3) &= \{0,3,6,9\} \cup \{0,I,2I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle 2Z \cup I \rangle \right\} \cup \{\langle 3Z \cup I \rangle\}.
\end{aligned}
$$

Clearly (H,B) is a soft neutrosophic sub N-semigroup of (F,A), where

$$
\begin{aligned}
H(x_1) &= \{0,4,8\} \cup \{0,I,2I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle Z \cup I \rangle \right\} \cup \{\langle 10Z \cup I \rangle\}, \\
H(x_3) &= \{0,6\} \cup \{0,I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \langle 4Z \cup I \rangle \right\} \cup \{\langle 6Z \cup I \rangle\}.
\end{aligned}
$$

Theorem 5.3. Let (F,A) be a soft neutrosophic N-semigroup over $S(N)$ and $\{(H_i,B_i); i \in I\}$ is a non empty family of soft neutrosophic sub N-semigroups of (F,A) then

1. $\cap_{i\in I}(H_i, B_i)$ is a soft neutrosophic sub N-semigroup of (F, A).

2. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic sub N-semigroup of $\wedge_{i\in I}(F, A)$.

3. $\cup_{i\in I}(H_i, B_i)$ is a soft neutrosophic sub N-semigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Definition 5.4. (F, A) over $S(N)$ is called soft neutrosophic N-ideal if $F(x)$ is neutosophic N-ideal of $S(N)$, for all $x \in A$.

Theorem 5.4. Every soft neutrosophic N-ideal (F, A) over $S(N)$ is a soft neutrosophic N-semigroup but the converse is not true.

Proposition 5.3. Let (F, A) and (K, B) be two soft neutrosophic N-ideals over $S(N)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $S(N)$ is not soft neutrosophic N-ideal over $S(N)$.

2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic N-ideal over $S(N)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic N-ideal over $S(N)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic N-ideal over $S(N)$.

Proposition 5.4. Let (F, A) and (H, B) be two soft neutrosophic N-ideal over $S(N)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic N-ideal over $S(N)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic N-ideal over $S(N)$.

Theorem 5.5. Let (F, A) be a soft neutrosophic N-semigroup over $S(N)$ and $\{(H_i, B_i);\ i \in I\}$ is a non empty family of soft neutrosophic N-ideals of (F, A) then

1. $\cap_{i\in I}(H_i, B_i)$ is a soft neutrosophic N-ideal of (F, A).

2. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic N-ideal of $\wedge_{i\in I}(F, A)$.

§6. Soft neutrosophic strong N-semigroup

Definition 6.1. Let $\{S(N), *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup and (F, A) be a soft set over $\{S(N), *_1, \ldots, *_N\}$. Then (F, A) is called soft neutrosophic strong N-semigroup if and only if $F(x)$ is neutrosophic strong sub N-semigroup, for all $x \in A$.

Example 6.1. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \langle Z_6 \cup I\rangle$, a neutrosophic semigroup.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I,$ semigroup under multiplication modulo $4\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle R \cup I\rangle \right\}$, neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication. Let $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$ is a neutosophic strong sub 4-semigroup of $S(4)$, where $T_1 = \{0, 3, 3I\} \subseteq \langle Z_6 \cup I \rangle$, $T_2 = \{0, I, 2I, 3I\} \subset S_2$, $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \subset S_3$, $T_4 = \{\langle 5Z \cup I \rangle\} \subset S_4$, the neutrosophic semigroup under multiplication. Also let $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$ be another neutrosophic strong sub 4-semigroup of $S(4)$, where $P_1 = \{0, 2I, 4I\} \subseteq \langle Z_6 \cup I \rangle$, $P_2 = \{0, 1, I\} \subset S_2$, $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \subset S_3$, $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$. Then (F, A) is soft neutrosophic strong 4-semigroup over $S(4)$, whereThen (F, A) is soft neutrosophic 4-semigroup over $S(4)$, where

$$\begin{aligned} F(x_1) &= \{0, 3, 3I\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\}, \\ F(x_2) &= \{0, 2I, 4I\} \cup \{0, 1, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}. \end{aligned}$$

Theorem 6.1. Every soft neutrosophic strong N-semigroup is trivially a soft neutrosophic N-semigroup but the converse is not true.

Proposition 6.1. Let (F, A) and (K, B) be two soft neutrosophic strong N-semigroups over $S(N)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $S(N)$ is not soft neutrosophic strong N-semigroup over $S(N)$.
2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic strong N-semigroup over $S(N)$.
3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic strong N-semigroup over $S(N)$.
4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic strong N-semigroup over $S(N)$.

Proposition 6.2. Let (F, A) and (K, B) be two soft neutrosophic strong N-semigroups over $S(N)$. Then

1. Their AND operation $(F, A) \wedge (K, B)$ is soft neutrosophic strong N-semigroup over $S(N)$.
2. Their OR operation $(F, A) \vee (K, B)$ is not soft neutrosophic strong N-semigroup over $S(N)$.

Definition 6.2. Let (F, A) and (H, B) be two soft neutrosophic strong N-semigroups over $S(N)$. Then (H, B) is a soft neutrosophic strong sub N-semigroup of (F, A), if

1. $B \subset A$.
2. $H(x)$ is neutrosophic strong sub N-semigroup of $F(x)$, for all $x \in B$.

Theorem 6.2.

1. Let (F, A) be a soft neutrosophic strong N-semigroup over $S(N)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic stong sub N-semigroups of (F, A) then
2. $\cap_{i \in I} (H_i, B_i)$ is a soft neutrosophic strong sub N-semigroup of (F, A).

3. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong sub N-semigroup of $\wedge_{i\in I}(F, A)$.

4. $\cup_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong sub N-semigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Definition 6.3. (F, A) over $S(N)$ is called soft neutrosophic strong N-ideal if $F(x)$ is neutosophic strong N-ideal of $S(N)$, for all $x \in A$.

Theorem 6.3. Every soft neutrosophic strong N-ideal (F, A) over $S(N)$ is a soft neutrosophic strong N-semigroup but the converse is not true.

Theorem 6.4. Every soft neutrosophic strong N-ideal (F, A) over $S(N)$ is a soft neutrosophic N-semigroup but the converse is not true.

Proposition 6.3. Let (F, A) and (K, B) be two soft neutrosophic strong N-ideals over $S(N)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $S(N)$ is not soft neutrosophic strong N-ideal over $S(N)$. 2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic strong N-ideal over $S(N)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic strong N-ideal over $S(N)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic strong N-ideal over $S(N)$.

Proposition 6.4. Let (F, A) and (H, B) be two soft neutrosophic strong N-ideal over $S(N)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic strong N-ideal over $S(N)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic strong N-ideal over $S(N)$.

Theorem 6.5. Let (F, A) be a soft neutrosophic strong N-semigroup over $S(N)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic strong N-ideals of (F, A) then

1. $\cap_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong N-ideal of (F, A).

2. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong N-ideal of $\wedge_{i\in I}(F, A)$.

Conclusion

This paper is an extension of neutrosphic semigroup to soft semigroup. We also extend neutrosophic bisemigroup, neutrosophic N-semigroup to soft neutrosophic bisemigroup, and soft neutrosophic N-semigroup. Their related properties and results are explained with many illustrative examples, the notions related with strong part of neutrosophy also established within soft semigroup.

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(T, I, F)-Neutrosophic Structures

Florentin Smarandache

Florentin Smarandache (2015). (T, I, F)-Neutrosophic Structures. Proceedings of the Annual Symposium of the Institute of Solid Mechanics and Session of the Commission of Acoustics, SISOM 2015 Bucharest 21-22 May; *Acta Electrotechnica* 57(1-2); N*eutrosophic Sets and Systems* 8, 3-10

Abstract. In this paper we introduce for the first time a new type of structures, called (T, I, F)-Neutrosophic Structures, presented from a neutrosophic logic perspective, and we show particular cases of such structures in geometry and in algebra. In any field of knowledge, each structure is composed from two parts: a space, and a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms (laws), has some indeterminacy, that structure is a (T, I, F)-Neutrosophic Structure. The (T, I, F)-Neutrosophic Structures [based on the components T=truth, I=indeterminacy, F=falsehood] are different from the Neutrosophic Algebraic Structures [based on neutrosophic numbers of the form a+bI, where I=indeterminacy and In = I], that we rename as Neutrosophic I-Algebraic Structures (meaning algebraic structures based on indeterminacy "I" only). But we can combine both and obtain the (T, I, F)-Neutrosophic I-Algebraic Structures, i.e. algebraic structures based on neutrosophic numbers of the form a+bI, but also having indeterminacy related to the structure space (elements which only partially belong to the space, or elements we know nothing if they belong to the space or not) or indeterminacy related to at least one axiom (or law) acting on the structure space. Then we extend them to Refined (T, I, F)-Neutrosophic Refined I-Algebraic Structures.

Keywords: Neurosophy, algebraic structures, neutrosophic sets, neutrosophic logics.

1. Neutrosophic Algebraic Structures [or Neutrosophic I-Algebraic Structures].

A previous type of neutrosophic structures was introduced in algebra by W.B. Vasantha Kandasamy and Florentin Smarandache [1-56], since 2003, and it was called Neutrosophic Algebraic Structures. Later on, more researchers joined the neutrosophic research, such as: Mumtaz Ali, A. A. Salama, Muhammad Shabir, K. Ilanthenral, Meena Kandasamy, H. Wang, Y.-Q. Zhang, R. Sunderraman, Andrew Schumann, Salah Osman, D. Rabounski, V. Christianto, Jiang Zhengjie, Tudor Paroiu, Stefan Vladutescu, Mirela Teodorescu, Daniela Gifu, Alina Tenescu, Fu Yuhua, Francisco Gallego Lupiañez, etc.

The neutrosophic algebraic structures are algebraic structures based on sets of neutrosophic numbers of the form N = a + bI, where a, b are real (or complex) numbers, and a is called the determinate part on N and b is called the indeterminate part of N, while I = indeterminacy,

$$\text{with } mI + nI = (m + n)I,\ 0\cdot I = 0,\ I^n = I \text{ for integer } n \geq 1, \text{ and } I / I = \text{undefined}.$$

When a, b are real numbers, then a + bI is called a neutrosophic real number. While if a, b are complex numbers, then a + bI is called a neutrosophic complex number.

We may say "indeterminacy" for "I" from a+bI, and "degree of indeterminacy" for "I" from (T, I, F) in order to distinguish them.

The neutrosophic algebraic structures studied by Vasantha-Smarandache in the period 2003-2015 are: neutrosophic groupoid, neutrosophic semigroup, neutrosophic group, neutrosophic ring, neutrosophic field, neutrosophic vector space, neutrosophic linear algebras etc., which later (between 2006-2011) were generalized by the same researchers to neutrosophic bi-algebraic structures, and more general to neutrosophic N-algebraic structures.

Afterwards, the neutrosophic structures were further extended to neutrosophic soft algebraic structures by Florentin Smarandache, Mumtaz Ali, Muhammad Shabir, and Munazza Naz in 2013-2014.

In 2015 Smarandache refined the indeterminacy I into different types of indeterminacies (depending on the problem to solve) such as $I_1, I_2, \ldots, I_p$ with integer $p \geq 1$, and obtained the refined neutrosophic numbers of the form $N_p = a+b_1I_1+b_2I_2+\ldots+b_pI_p$ where a, $b_1, b_2, \ldots, b_p$ are real or complex numbers, and a is called the determinate part of N_p, while for each $k \in \{1, 2, \ldots, p\}$ I_k is called the k-th indeterminate part of N_p, and for each $k \in \{1, 2, \ldots, p\}$, and similarly

$mI_k + nI_k = (m + n)I_k$, $0\cdot I_k = 0$, $I_k^n = I_k$ for integer $n \geq 1$, and

I_k / I_k = undefined.

The relationships and operations between I_j and I_k, for $j \neq k$, depend on each particular problem we need to solve.

Then consequently Smarandache [2015] extended the neutrosophic algebraic structures to Refined Neutrosophic Algebraic Structures [or Refined Neutrosophic I-Algebraic Structures], which are algebraic structures based on the sets of the refined neutrosophic numbers $a+b_1I_1+b_2I_2+\ldots+b_pI_p$.

2. (T, I, F)-Neutrosophic Structures.

We now introduce for the first time another type of neutrosophic structures. These structures, in any field of knowledge, are considered from a neutrosophic logic point of view, i.e. from the truth-indeterminacy-falsehood (T, I, F) values. In neutrosophic logic every proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F), where T, I, F are standard or non-standard subsets of the non-standard unit interval $]^-0, 1^+[$. In technical applications T, I, and F are only standard subsets of the standard unit interval [0, 1] with:

$$^-0 \leq \sup(T) + \sup(I) + \sup(F) \leq 3^+$$

where sup(Z) means superior of the subset Z.

In general, each structure is composed from: a space, endowed with a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms, has some indeterminacy, we consider it as a (T, I, F)-Neutrosophic Structure.

Indeterminacy with respect to the space is referred to some elements that partially belong [i.e. with a neutrosophic value (T, I. F)] to the space, or their appurtenance to the space is unknown.

An axiom (or law) which deals with indeterminacy is called neutrosophic axiom (or law).

We introduce these new structures because in the world we do not always know exactly or completely the space we work in; and because the axioms (or laws) are not always well defined on this space, or may have indeterminacies when applying them.

3. Refined (T, I, F)-Neutrosophic Structures [or (T_j, I_k, F_l)-Neutrosophic Structures]

In 2013 Smarandache [76] refined the neutrosophic components (T, I, F) into

$$(T_1, T_2, \ldots, T_m;\ I_1, I_2, \ldots, I_p;\ F_1, F_2, \ldots, F_r),$$

where m, p, r are integers ≥ 1.

Consequently, we now [2015] extend the (T, I, F)-Neutrosophic Structures to $(T_1, T_2, \ldots, T_m;\ I_1, I_2, \ldots, I_p;\ F_1, F_2, \ldots, F_r)$-Neutrosophic Structures, that we called Refined (T, I, F)-Neutrosophic Structures [or (T_j, I_k, F_l)-Neutrosophic Structures]. These are structures whose elements have a refined neutrosophic value of the form $(T_1, T_2, \ldots, T_m;\ I_1, I_2, \ldots, I_p;\ F_1, F_2, \ldots, F_r)$ or the space has some indeterminacy of this form

4. (T, I, F)-Neutrosophic I-Algebraic Structures.

The (T, I, F)-Neutrosophic Structures [based on the components T=truth, I=indeterminacy, F=falsehood] are different from the Neutrosophic Algebraic Structures [based on neutrosophic numbers of the form a+bI]. We may rename the last ones as Neutrosophic I-Algebraic Structures (meaning: algebraic structures based on indeterminacy "I" only).

But we can combine both of them and obtain a (T, I, F)-Neutrosophic I-Algebraic Structures, i.e. algebraic structures based on neutrosophic numbers of the form a+bI, but also have indeterminacy related to the structure space (elements which only partially belong to the space, or elements we know nothing if they belong to the space or not) or indeterminacy related to at least an axiom (or law) acting on the structure space.

Even more, we can generalize them to Refined (T, I, F)-Neutrosophic Refined I-Algebraic Structures, or (T_j, I_k, F_l)-Neutrosophic I_s-Algebraic Structures.

5. Example of Refined I-Neutrosophic Algebraic Structure

Let the indeterminacy I be split into I_1 = contradiction (i.e. truth and falsehood simultaneously), I_2 = ignorance (i.e. truth or falsehood), and I_3 = unknown, and the corresponding 3-refined neutrosophic numbers of the form $a+b_1I_1+b_2I_2+b_3I_3$.

The (G, *) be a groupoid. Then the 3-refined I-neutrosophic groupoid is generated by I_1, I_2, I_3 and G under * and it is denoted by

$$N_3(G) = \{(G \cup I_1 \cup I_2 \cup I_3), *\}$$

$$= \{ a+b_1I_1+b_2I_2+b_3I_3 \,/\, a, b_1, b_2, b_3 \in G \}.$$

6. Example of Refined (T, I, F)-Neutrosophic Structure

Let (T, I, F) be split as (T_1, T_2; I_1, I_2; F_1, F_2, F_3). Let H = ({h_1, h_2, h_3}, #) be a groupoid, where h_1, h_2, and h_3 are real numbers. Since the elements h_1, h_2, h_3 only partially belong to H in a refined way, we define a refined (T, I, F)-neutrosophic groupoid { or refined (2; 2; 3)-neutrosophic groupoid, since T was split into 2 parts, I into 2 parts, and F into 3 parts } as
H = {h_1(0.1, 0.1; 0.3, 0.0; 0.2, 0.4, 0.1), h_2(0.0, 0.1; 0.2, 0.1; 0.2, 0.0, 0.1), h_3(0.1, 0.0; 0.3, 0.2; 0.1, 0.4, 0.0)}.

7. Examples of (T, I, F)-Neutrosophic I-Algebraic Structures.

1. Indeterminate Space (due to Unknown Element). And Neutrosophic Number included. Let B = {2+5I, -I, -4, b(0, 0.9, 0)} a neutrosophic set, which contain two neutrosophic numbers, 2+5I and -I, and we know about the element b that its appurtenance to the neutrosophic set is 90% indeterminate.
2. Indeterminate Space (due to Partially Known Element). And Neutrosophic Number included. Let C = {-7, 0, 2+I(0.5, 0.4, 0.1), 11(0.9, 0, 0) }, which contains a neutrosophic number 2+I, and this neutrosophic number is actually only partially in C; also, the element 11 is also partially in C.
3. Indeterminacy Axiom (Law). Let D = [0+0I, 1+1I] = {c+dI, where c, d ϵ [0, 1]}. One defines the binary law # in the following way:
$$\# : D\times D \rightarrow D$$
$x \# y = (x_1 + x_2I) \# (y_1 + y_2I) = [(x_1 + x_2)/y_1] + y_2I$, but this neutrosophic law is undefined (indeterminate) when $y_1 = 0$.
4. Little Known or Completely Unknown Axiom (Law). Let us reconsider the same neutrosophic set D as above. But, about the binary neutrosophic law Θ that D is endowed with, we only know that it associates the neutrosophic numbers 1+I and 0.2+0.3I with the neutrosophic number 0.5+0.4I, i.e. (1+I)Θ(0.2+0.3I) = 0.5+0.4I.

There are many cases in our world when we barely know some axioms (laws).

8. Examples of Refined (T, I, F)-Neutrosophic Refined I-Algebraic Structures.

We combine the ideas from Examples 5 and 6 and we construct the following example. Let's consider, from Example 5, the groupoid (G, *), where G is a subset of positive real numbers, and its extension to a 3-refined I-neutrosophic groupoid, which was generated by I_1, I_2, I_3 and G under the law * that was denoted by
$N_3(G) = \{ a+b_1I_1+b_2I_2+b_3I_3 / a, b_1, b_2, b_3 \in G \}$.

We then endow each element from $N_3(G)$ with some (2; 2; 3)-refined degrees of membership/indeterminacy/nonmembership, as in Example 6, of the form (T_1, T_2; I_1, I_2; F_1, F_2, F_3), and we obtain a
$N_3(G)_{(2;2;3)} = \{ a+b_1I_1+b_2I_2+b_3I_3(T_1, T_2; I_1, I_2; F_1, F_2, F_3) / a, b_1, b_2, b_3 \in G \}$,
where

$$T_1 = \frac{a}{a+b_1+b_2+b_3},\ T_2 = \frac{0.5a}{a+b_1+b_2+b_3}$$

$$I_1 = \frac{b_1}{a+b_1+b_2+b_3}, I_2 = \frac{b_2}{a+b_1+b_2+b_3};$$

$$F_1 = \frac{0.1b_3}{a+b_1+b_2+b_3}, F_2 = \frac{0.2b_1}{a+b_1+b_2+b_3},$$
$$F_3 = \frac{b_2+b_3}{a+b_1+b_2+b_3}.$$

Therefore, $N_3(G)_{(2;2;3)}$ is a refined (2; 2; 3)-neutrosophic groupoid and a 3-refined I-neutrosophic groupoid.

9. Neutrosophic Geometric Examples.

a) Indeterminate Space.
We might not know if a point P belongs or not to a space S [we write P(0, 1, 0), meaning that P's indeterminacy is 1, or completely unknown, with respect to S].
Or we might know that a point Q only partially belongs to the space S and partially does not belong to the space S [for example Q(.3, 0.4, 0.5), which means that with respect to S, Q's membership is 0.3, Q's indeterminacy is 0.4, and Q's nonmembership is 0.5].
Such situations occur when the space has vague or unknown frontiers, or the space contains ambiguous (not well defined) regions.

b) Indeterminate Axiom.
Also, an axiom (α) might not be well defined on the space S, i.e. for some elements of the space the axiom (α) may be valid, for other elements of the space the axiom (α) may be indeterminate (meaning neither valid, nor invalid), while for the remaining elements the axiom (α) may be invalid.
As a concrete example, let's say that the neutrosophic values of the axiom (α) are (0.6, 0.1, 0.2) = (degree of validity, degree of indeterminacy, degree of invalidity).

10. (T, I, F)-Neutrosophic Geometry as a Particular Case of (T, I, F)-Neutrosophic Structures.

As a particular case of (T, I, F)-neutrosophic structures in geometry, one considers a (T, I, F)-Neutrosophic Geometry as a geometry which is defined either on a space with some indeterminacy (i.e. a portion of the space is not known, or is vague, confused, unclear, imprecise), or at least one of its axioms has some indeterminacy (i.e. one does not know if the axiom is verified or not in the given space).
This is a generalization of the Smarandache Geometry (SG) [57-75], where an axiom is validated and invalidated in the same space, or only invalidated, but in multiple ways. Yet the SG has no degree of indeterminacy related to the space or related to the axiom.
A simple Example of a SG is the following – that unites Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries altogether, in the same space, considering the Fifth Postulate of Euclid: in one region of the SG space the postulate is validated (only one parallel trough a point to a given line), in a second region of SG the postulate is invalidated (no parallel through a point to a given line – elliptical geometry), and in a third region of SG the postulate is invalidated but in a different way (many parallels through a point to a given line – hyperbolic geometry). This simple example shows a hybrid geometry which is partially Euclidean, partially Non-Euclidean Elliptic, and partially Non-Euclidean Hyperbolic. Therefore, the fifth postulate (axiom) of Euclid is true for some regions, and false for others, but it is not indeterminate for any region (i.e. not knowing how many parallels can be drawn through a point to a given line).
We can extend this hybrid geometry adding a new space region where one does not know if there are or there are not parallels through some given points to the given lines (i.e. the Indeterminate component) and we form a more complex (T, I, F)-Neutrosophic Geometry.

12. Neutrosophic Algebraic Examples.

1) Indeterminate Space (due to Unknown Element).
Let the set (space) be NH = {4, 6, 7, 9, a}, where the set NH has an unknown element "a", therefore the whole space has some degree of indeterminacy. Neutrosophically, we write a(0, 1, 0), which means the element a is 100% unknown.
2) Indeterminate Space (due to Partially Known Element).
Given the set M = {3, 4, 9(0.7, 0.1, 0.3)}, we have two elements 3 and 4 which surely belong to M, and one writes them neutrosophically as 3(1, 0, 0) and 4(1, 0, 0), while the third element 9 belongs only partially (70%) to M, its appurtenance to M is indeterminate (10%), and does not belong to M (in a percentage of 30%).
Suppose M is endowed with a neutrosophic law* defined in the following way:
$x_1(t_1, i_1, f_1)* x_2(t_2, i_2, f_2) = \max\{x_1, x_2\}(\min\{t_1, t_2\}, \max\{i_1, i_2\}, \max\{f_1, f_2\})$,
which is a neutrosophic commutative semigroup with unit element 3(1, 0 ,0).
Clearly, if x, y ϵ M, then x*y ϵ M. Hence the neutrosophic law * is well defined.
Since max and min operators are commutative and associative, then * is also commutative and associative.
If x ϵ M, then x*x = x.
Below, examples of applying this neutrosophic law *:
3*9(0.7, 0.1, 0.3) = 3(1, 0, 0)*9(0.7, 0.1, 0.3) = max{3, 9}(min{1, 0.7}, max{0, 0.1}, max{0, 0.3}) = 9(0.7, 0.1, 0.3).
3*4 = 3(1, 0, 0)*4(1, 0, 0) = max{3, 4}(min{1, 1}, max{0, 0}, max{0, 0}) = 4(1, 0, 0).
2) Indeterminate Law (Operation).
For example, let the set (space) be NG = ({0, 1, 2}, /), where "/" means division.
NG is a (T, I, F)-neutrosophic groupoid, because the operation "/" (division) is partially defined and undefined (indeterminate). Let's see:
2/1 = 1, which belongs to NG;
1/2 = 0.5, which does not belongs to NG;
1/0 = undefined (indeterminate).
So the law defined on the set NG has the properties that:

- applying this law to some elements, the results are in NG [well defined law];
- applying this law to other elements, the results are not in NG [not well defined law];
- applying this law to again other elements, the results are undefined [indeterminate law].

We can construct many such algebraic structures where at least one axiom has such behavior (such indeterminacy in principal).

12. Websites at UNM for Neutrosophic Algebraic Structures and respectively Neutrosophic Geometries:

http://fs.gallup.unm.edu/neutrosophy.htm
and
http://fs.gallup.unm.edu/geometries.htm respectively.

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IV. Refined Neutrosophics

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Neutrosophic filters in BE-algebras

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Akbar Rezaei, Arsham Borumand Saeid, Florentin Smarandache (2015).
Neutrosophic filters in BE-algebras. *Ratio Mathematica* 29, 65-79

Abstract

In this paper, we introduce the notion of (implicative) neutrosophic filters in BE-algebras. The relation between implicative neutrosophic filters and neutrosophic filters is investigated and we show that in self distributive BE-algebras these notions are equivalent.

Keywords: BE-algebra, neutrosophic set, (implicative) neutrosophic filter.

1 Introduction

Neutrosophic set theory was introduced by Smarandache in 1998 ([10]). Neutrosophic sets are a new mathematical tool for dealing with uncertainties which are free from many difficulties that have troubled the usual theoretical approaches. Research works on neutrosophic set theory for many applications such as infor-mation fussion, probability theory, control theory, decision making, measurement theory, etc. Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures ([3, 4, 5]). Since then many researchers worked in this area and lots of literatures had been produced about the theory of neutrosophic set. In the neutrosophic set one can have elements which have paraconsistent information (sum of components > 1), others incomplete information (sum of components < 1), others consistent information (in the case when the sum of components =1) and others interval-valued components (with no restriction on their superior or inferior sums).

H.S. Kim and Y.H. Kim introduced the notion of a BE-algebra as a generaliza-tion of a dual BCK-algebra ([6]). B.L. Meng give a procedure which generated a filter by a subset in a transitive BE-algebra ([7]). A. Walendziak introduced the no-tion of a normal filter in BE-algebras and showed that there is a bijection between congruence relations and filters in commutative BE-algebras ([11]). A. Borumand Saeid and et al. defined some types of filters in BE-algebras and showed the re-lationship between them ([1]). A. Rezaei and et al. discussed on the relationship between BE-algebras and Hilbert algebras ([9]). Recently, A. Rezaei and et al. introduced the notion of hesitant fuzzy (implicative) filters and get some results on BE-algebras ([8]).

In this paper, we introduce the notion of (implicative) neutrosophic filters and study it in details. In fact, we show that in self distributive BE-algebras concepts of implicative neutrosophic filter and neutrosophic filter are equivalent.

2 Preliminaries

In this section, we cite the fundamental definitions that will be used in the sequel:

Definition 2 .1. [6] By a BE-algebra we shall mean an algebra $\mathfrak{X}= (X; *, 1)$ of type $(2, 0)$ satisfying the Aollowing axioms:

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

From now on, $\mathfrak{X}$ is a BE-algebra, unless otherwise is stated. We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$. A BE-algebra $\mathfrak{X}$ is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$. A BE-algebra $\mathfrak{X}$ is said to be commutative if satisfies:

$$(x * y) * y = (y * x) * x, \text{ for all } x, y \in X.$$

Proposition 2.1. [11] If $\mathfrak{X}$ is a commutative BE-algebra, then for all $x, y \in X$,

$$x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

We note that "$\leq$" is reflexive by (BE1). If $\mathfrak{X}$ is self distributive then relation "$\leq$" is a transitive ordered set on X, because if $x \leq y$ and $y \leq z$, then

$$x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1.$$

Hence $x \leq z$. If $\mathfrak{X}$ is commutative then by Proposition 2.1, relation "$\leq$" is anti-symmetric. Hence if $\mathfrak{X}$ is a commutative self distributive BE-algebra, then relation "$\leq$" is a partial ordered set on $\mathfrak{X}$.

Proposition 2.2. [6] In a BE-algebra $\mathfrak{X}$, the following hold:

(i) $x * (y * x) = 1$,

(ii) $y * ((y * x) * x) = 1$, for all $x, y \in X$.

A subset F of X is called a filter of $\mathfrak{X}$ if it satisfies: (F1) $1 \in F$, (F2) $x \in F$ and $x * y \in F$ imply $y \in F$. Define

$$A(x, y) = \{z \in X : x * (y * z) = 1\},$$

which is called an upper set of x and y. It is easy to see that $1, x, y \in A(x, y)$, for any $x, y \in X$. Every upper set $A(x, y)$ need not be a filter of $\mathfrak{X}$ in general.

Definition 2.2. [1] A non-empty subset F of X is called an implicative filter if satisfies the following conditions:

(IF1) $1 \in F$,

(IF2) $x * (y * z) \in F$ and $x * y \in F$ imply that $x * z \in F$, for all $x, y, z \in X$.

If we replace x of the condition (IF2) by the element 1, then it can be easily observed that every implicative filter is a filter. However, every filter is not an implicative filter as shown in the following example.

Example 2.1. Let $X = \{1, a, b\}$ be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	a	1

Then $F = \{1, a\}$ is a filter of X, but it is not an implicative filter, since
$1 * (a * b) = 1 * a = a \in F$ and $1 * a = a \in F$ but $1 * b = b \notin F$.

Definition 2 .3. [10] Let X be a set. A neutrosophic subset A of X is a triple (T_A, I_A, F_A) where $T_A : X \to [0, 1]$ is the membership function, $I_A : X \to [0, 1]$ is the indeterminacy function and $F_A : X \to [0, 1]$ is the nonmembership function. Here for each $x \in X$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are all standard real numbers in $[0, 1]$.

We note that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, for all $x \in X$. The set of neutrosophic subset of X is denoted by NS(X).

Definition 2 .4. [10] Let A and B be two neutrosophic sets on X . Define $A \leq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$, for all $x \in X$.

Definition 2 .5. Let $\mathfrak{X}_1 = (X_1; *, 1)$ and $\mathfrak{X}_2 = (X_2; \circ, 1')$ be two BE-algebras. Then a mapping $f : X_1 \to X_2$ is called a homomorphism if, for all $x_1, x_2 \in X_1$ $f(x_1 * x_2) = f(x_1) \circ f(x_2)$. It is clear that if $f : X_1 \to X_2$ is a homomorphism, then $f(1) = 1'$.

3 Neutrosophic Filters

Definition 3 .1. A neutrosophic set A of $\mathfrak{X}$ is called a *neutrosophic filter if satisfies the following conditions:*

(NF1) $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$,

(NF2) $\min\{T_A(x * y), T_A(x)\} \leq T_A(y)$, $\min\{I_A(x * y), I_A(x)\} \geq I_A(y)$ and $\min\{F_A(x * y), F_A(x)\} \geq F_A(y)$, *for all* $x, y \in X$.

The set of neutrosophic filter of $\mathfrak{X}$ is denoted by NF($\mathfrak{X}$).

Example 3.1. In Example 2.1, put $T_A(1) = 0.9$, $T_A(a) = T_A(b) = 0.5$, $I_A(1) = 0.2$, $I_A(a) = I_A(b) = 0.35$ and $F_A(1) = 0.1$, $F_A(a) = F_A(b) = 0$. Then $A = (T_A, I_A, F_A)$ is a neutrosophic filter.

Proposition 3.1. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then*

(i) if $x \leq y$, *then* $T_A(x) \leq T_A(y)$, $I_A(x) \geq I_A(y)$ and $F_A(x) \geq F_A(y)$,

(ii) $T_A(x) \leq T_A(y * x)$, $I_A(x) \geq I_A(y * x)$ and $F_A(x) \geq F_A(y * x)$,

(iii) $\min\{T_A(x), T_A(y)\} \leq T_A(x * y)$, $\min\{I_A(x), I_A(y)\} \geq I_A(x * y)$ and $\min\{F_A(x), F_A(y)\} \geq F_A(x * y)$,

(iv) $T_A(x) \leq T_A((x*y)*y)$, $I_A(x) \geq I_A((x*y)*y)$ and $F_A(x) \geq F_A((x*y)*y)$,

(v) $\min\{T_A(x), T_A(y)\} \leq T_A((x * (y * z)) * z)$,
$\min\{I_A(x), I_A(y)\} \geq I_A((x * (y * z)) * z)$ and
$\min\{F_A(x), F_A(y)\} \geq F_A((x * (y * z)) * z)$,

(vi) *if* $\min\{T_A(y), T_A((x * y) * z)\} \leq T_A(z * x)$, *then* T_A *is order reversing and* I_A, F_A *are order* (i.e. if $x \leq y$, then $T_A(y) \leq T_A(x)$, $I_A(y) \geq I_A(x)$ and $F_A(y) \geq F_A(x)$)

(vii) *if* $z \in A(x, y)$, *then* $\min\{T_A(x), T_A(y)\} \leq T_A(z)$,
$\min\{I_A(x), I_A(y)\} \geq I_A(z)$ and $\min\{F_A(x), F_A(y)\} \geq F_A(z)$

(viii) *if* $\prod_{i=1}^{n} a_i * x = 1$, *then* $\bigwedge_{i=1}^{n} T_A(a_i) \leq T_A(x)$, $\bigwedge_{i=1}^{n} I_A(a_i) \geq I_A(x)$ and $\bigwedge_{i=1}^{n} F_A(a_i) \geq F_A(x)$ *where* $\prod_{i=1}^{n} a_i * x = a_n * (a_{n-1} * (\dots (a_1 * x) \dots))$.

Proof. (i). Let $x \leq y$. Then $x * y = 1$ and so

$$T_A(x) = \min\{T_A(x), T_A(1)\} = \min\{T_A(x), T_A(x * y)\} \leq T_A(y),$$

$$I_A(x) = \min\{I_A(x), I_A(1)\} = \min\{I_A(x), I_A(x * y)\} \geq I_A(y),$$

$$F_A(x) = \min\{F_A(x), F_A(1)\} = \min\{F_A(x), F_A(x * y)\} \geq F_A(y).$$

(ii). Since $x \leq y * x$, by using (i) the proof is clear.
(iii). By using (ii) we have

$$\min\{T_A(x), T_A(y)\} \leq T_A(y) \leq T_A(x * y),$$

$$\min\{I_A(x), I_A(y)\} \geq I_A(y) \geq I_A(x * y),$$

$$\min\{F_A(x), F_A(y)\} \geq F_A(y) \geq F_A(x * y).$$

(iv). It follows from Definition 3.1,

$$\begin{aligned} T_A(x) &= \min\{T_A(x), T_A(1)\} \\ &= \min\{T_A(x), T_A((x * y) * (x * y))\} \\ &= \min\{T_A(x), T_A(x * ((x * y) * y))\} \\ &\leq T_A((x * y) * y). \end{aligned}$$

Also, we have

$$\begin{aligned} I_A(x) &= \min\{I_A(x), I_A(1)\} \\ &= \min\{I_A(x), I_A((x*y)*(x*y))\} \\ &= \min\{I_A(x), I_A(x*((x*y)*y))\} \\ &\geq I_A((x*y)*y) \end{aligned}$$

and

$$\begin{aligned} F_A(x) &= \min\{F_A(x), F_A(1)\} \\ &= \min\{F_A(x), F_A((x*y)*(x*y))\} \\ &= \min\{F_A(x), F_A(x*((x*y)*y))\} \\ &\geq F_A((x*y)*y). \end{aligned}$$

(v). From (iv) we have

$$\begin{aligned} \min\{T_A(x), T_A(y)\} &\leq \min\{T_A(x), T_A((y*(x*z))*(x*z))\} \\ &= \min\{T_A(x), T_A((x*(y*z))*(x*z))\} \\ &= \min\{T_A(x), T_A(x*(x*(y*z))*z))\} \\ &\leq T_A((x*(y*z))*z)), \end{aligned}$$

$$\begin{aligned} \min\{I_A(x), I_A(y)\} &\geq \min\{I_A(x), I_A((y*(x*z))*(x*z))\} \\ &= \min\{I_A(x), I_A((x*(y*z))*(x*z))\} \\ &= \min\{I_A(x), I_A(x*(x*(y*z))*z))\} \\ &\geq I_A((x*(y*z))*z)) \end{aligned}$$

and

$$\begin{aligned} \min\{F_A(x), F_A(y)\} &\geq \min\{F_A(x), F_A((y*(x*z))*(x*z))\} \\ &= \min\{F_A(x), F_A((x*(y*z))*(x*z))\} \\ &= \min\{F_A(x), F_A(x*(x*(y*z))*z))\} \\ &\geq F_A((x*(y*z))*z)). \end{aligned}$$

(vi). Let $x \leq y$, that is, $x * y = 1$.

$T_A(y) = \min\{T_A(y), T_A(1*1)\} = \min\{T_A(y), T_A((x*y)*1)\} \leq T_A(1*x) = T_A(x)$,

$I_A(y) = \min\{I_A(y), I_A(1*1)\} = \min\{I_A(y), I_A((x*y)*1)\} \geq I_A(1*x) = I_A(x)$,

$$F_A(y) = \min\{F_A(y), F_A(1 * 1)\} = \min\{F_A(y), F_A((x * y) * 1)\} \geq F_A(1 * x) = F_A(x).$$

(vii). Let $z \in A(x, y)$. Then $x * (y * z) = 1$. Hence

$$\begin{aligned} \min\{T_A(x), T_A(y)\} &= \min\{T_A(x), T_A(y), T_A(1)\} \\ &= \min\{T_A(x), T_A(y), T_A(x * (y * z))\} \\ &\leq \min\{T_A(y), T_A(y * z)\} \\ &\leq T_A(z). \end{aligned}$$

Also, we have

$$\begin{aligned} \min\{I_A(x), I_A(y)\} &= \min\{I_A(x), I_A(y), I_A(1)\} \\ &= \min\{I_A(x), I_A(y), I_A(x * (y * z))\} \\ &\geq \min\{I_A(y), I_A(y * z)\} \\ &\geq I_A(z), \end{aligned}$$

and

$$\begin{aligned} \min\{F_A(x), F_A(y)\} &= \min\{F_A(x), F_A(y), F_A(1)\} \\ &= \min\{F_A(x), F_A(y), F_A(x * (y * z))\} \\ &\geq \min\{F_A(y), F_A(y * z)\} \\ &\geq F_A(z). \end{aligned}$$

(viii). The proof is by induction on n. By (vii) it is true for $n = 1, 2$. Assume that it satisfies for $n = k$, that is,

$\prod_{i=1}^{k} a_i * x = 1 \Rightarrow \bigwedge_{i=1}^{k} T_A(a_i) \leq T_A(x), \bigwedge_{i=1}^{k} I_A(a_i) \geq I_A(x)$ and $\bigwedge_{i=1}^{k} F_A(a_i) \geq F_A(x)$ for all $a_1, \ldots, a_k, x \in X$.

Suppose that $\prod_{i=1}^{k+1} a_i * x = 1$, for all $a_1, \ldots, a_k, a_{k+1}, x \in X$. Then

$\bigwedge_{i=2}^{k+1} T_A(a_i) \leq T_A(a_1 * x)$, $\bigwedge_{i=2}^{k+1} I_A(a_i) \geq I_A(a_1 * x)$, and $\bigwedge_{i=2}^{k+1} F_A(a_i) \geq F_A(a_1 * x)$.

Since A is a neutrosophic filter of $\mathfrak{X}$, we have

$$\bigwedge_{i=1}^{k+1} T_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} T_A(a_i)), T_A(a_1)\} \leq \min\{T_A(a_1 * x), T_A(a_1)\} \leq T_A(x),$$

$$\bigwedge_{i=1}^{k+1} I_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} I_A(a_i)), I_A(a_1)\} \geq \min\{I_A(a_1 * x), I_A(a_1)\} \geq I_A(x)$$

and

$$\bigwedge_{i=1}^{k+1} F_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} F_A(a_i)), F_A(a_1)\} \geq \min\{F_A(a_1 * x), F_A(a_1)\} \geq F_A(x).$$

□

Theorem 3.1. *If $\{A_i\}_{i\in I}$ is a family of neutrosophic filters in $\mathfrak{X}$, then $\bigcap\limits_{i\in I} A_i$ is too.*

Theorem 3.2. *Let $A \in \mathrm{NF}(\mathfrak{X})$. Then the sets*

(i) $X_{T_A} = \{x \in X : T_A(x) = T_A(1)\}$,

(ii) $X_{I_A} = \{x \in X : I_A(x) = I_A(1)\}$,

(iii) $X_{F_A} = \{x \in X : F_A(x) = F_A(1)\}$,

are filters of $\mathfrak{X}$.

Proof. (i). Obviously, $1 \in X_{h_A}$. Let $x, x * y \in X_{T_A}$. Then $T_A(x) = T_A(x * y) = T_A(1)$. Now, by (NF1) and (NF2), we have

$$T_A(1) = \min\{T_A(x), T_A(x * y)\} \leq T_A(y) \leq T_A(1).$$

Hence $T_A(y) = T_A(1)$. Therefore, $y \in X_{T_A}$.

The proofs of (ii) and (iii) are similar to (i).□

Definition 3.2. A neutrosophic set A of $\mathfrak{X}$ is called an implicative neutrosophic filter of $\mathfrak{X}$ if satisfies the following conditions:

(INF1) $T_A(1) \geq T_A(x)$,

(INF2) $T_A(x * z) \geq \min\{T_A(x * (y * z)), T_A(x * y)\}$,
$I_A(x * z) \leq \min\{I_A(x * (y * z)), I_A(x * y)\}$ and
$F_A(x * z) \leq \min\{F_A(x * (y * z)), F_A(x * y)\}$, for all $x, y, z \in X$.

The set of implicative neutrosophic filter of $\mathfrak{X}$ is denoted by INF($\mathfrak{X}$). If we replace x of the condition (INF2) by the element 1, then it can be easily observed that every implicative neutrosophic filter is a neutrosophic filter. However, every neutrosophic filter is not an implicative neutrosophic filter as shown in the following example.

Example 3.2. Let $X = \{1, a, b, c, d\}$ be a BE-algebra with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then $\mathfrak{X} = (X; *, 1)$ is a BE-algebra. Define a neutrosophic set A on $\mathfrak{X}$ as follows:

$$T_A(x) = \begin{cases} 0.85 & \text{if } x = 1, a \\ 0.12 & otherwise \end{cases}$$

and $I_A(x) = F_A(x) = 0.5$, for all $x \in X$.

Then clearly $A = (T_A, I_A, F_A)$ is a neutrosophic filter of $\mathfrak{X}$, but it is not an implicative neutrosophic filter of $\mathfrak{X}$, since

$$T_A(b * c) \ngeq \min\{T_A(b * (d * c)), T_A(b * d)\}.$$

Theorem 3.3. *Let $\mathfrak{X}$ be a self distributive BE-algebra. Then every neutrosophic filter is an implicative neutrosophic filter.*

Proof. Let $A \in \mathrm{NF}(\mathfrak{X})$ and $x \in X$. Obvious that $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$. By self distributivity and (NF2), we have

$$\min\{T_A(x*(y*z)), T_A(x*y)\} = \min\{T_A((x*y)*(x*z)), T_A(x*y)\} \leq T_A(x*z),$$

$$\min\{I_A(x*(y*z)), I_A(x*y)\} = \min\{I_A((x*y)*(x*z)), I_A(x*y)\} \geq I_A(x*z)$$

and

$$\min\{F_A(x*(y*z)), F_A(x*y)\} = \min\{F_A((x*y)*(x*z)), F_A(x*y)\} \geq F_A(x*z).$$

Therefore $A \in \mathrm{INF}(\mathfrak{X})$.□

Let $t \in [0, 1]$. For a neutrosophic filter A of $\mathfrak{X}$, t-level subset which denoted by $U(A; t)$ is defined as follows:

$$U(A; t) := \{x \in A : t \leq T_A(x), I_A(x) \leq t \text{ and } F_A(x) \leq t\}$$

and strong t-level subset which denoted by $U(A; t)_>$ as

$$U(A; t)_> := \{x \in A : t < T_A(x), I_A(x) < t \text{ and } F_A(x) < t\}.$$

Theorem 3.4. *Let $A \in \mathrm{NS}(\mathfrak{X})$. The following are equivalent:*

(i) $A \in \mathrm{NF}(\mathfrak{X})$,

(ii) $(\forall t \in [0, 1])$ $U(A; t) \neq \emptyset$ *imply* $U(A; t)$ *is a filter of* $\mathfrak{X}$.

Proof. (i)⇒(ii). Let $x, y \in X$ be such that $x, x * y \in U(A; t)$, for any $t \in [0, 1]$. Then $t \leq T_A(x)$ and $t \leq T_A(x*y)$. Hence $t \leq \min\{T_A(x), T_A(x*y)\} \leq T_A(y)$. Also, $I_A(x) \leq t$ and $I_A(x * y) \leq t$ and so $t \geq \min\{I_A(x), I_A(x * y)\} \geq I_A(y)$. By a similar argument we have $t \geq \min\{F_A(x), F_A(x * y)\} \geq F_A(y)$. Therefore, $y \in U(A; t)$.

(ii)⇒(i). Let $U(A; t)$ be a filter of $\mathfrak{X}$, for any $t \in [0, 1]$ with $U(A; t) \neq \emptyset$. Put $T_A(x) = I_A(x) = F_A(x) = t$, for any $x \in X$. Then $x \in U(A; t)$. Since $U(A; t)$ is a filter of $\mathfrak{X}$, we have $1 \in U(A; t)$ and so $T_A(x) = t \leq T_A(1)$. Now, for any $x, y \in X$, let $T_A(x * y) = I_A(x * y) = F_A(x * y) = t_{x*y}$ and $T_A(x) = I_A(x) = F_A(x) = t_x$. Put $t = \min\{t_{x*y}, t_x\}$. Then $x, x * y \in U(A; t)$, so $y \in U(A; t)$. Hence $t \leq T_A(y)$, $t \geq I_A(y)$, $t \geq F_A(y)$ and so

$$\min\{T_A(x * y), T_A(x)\} = \min\{t_{x*y}, t_x\} = t \leq T_A(y),$$

$$\min\{I_A(x * y), I_A(x)\} = \min\{t_{x*y}, t_x\} = t \geq I_A(y),$$

and

$$\min\{F_A(x * y), F_A(x)\} = \min\{t_{x*y}, t_x\} = t \geq F_A(y).$$

Therefore, $A \in \mathrm{NF}(\mathfrak{X})$.□

Theorem 3.5. *Let $A \in \mathrm{NF}(\mathfrak{X})$. Then we have*

$$(\forall a, b \in X)\ (\forall t \in [0, 1])\ (a, b \in U(A; t) \Rightarrow A(a, b) \subseteq U(A; t)).$$

Proof. Assume that $A \in \mathrm{NF}(\mathfrak{X})$. Let $a,\, b \in X$ be such that $a,\, b \in U(A;\, t)$. Then $t \leq T_A(a)$ and $t \leq T_A(b)$. Let $c \in A(a,\, b)$. Hence $a * (b * c) = 1$. Now, by Proposition 3.1(v) and (BE3), we have

$$t \leq \min\{T_A(a), T_A(b)\} \leq T_A((a * (b * c) * c)) = T_A(1 * c) = T_A(c),$$

$$t \geq \min\{I_A(a), I_A(b)\} \geq I_A((a * (b * c) * c)) = I_A(1 * c) = I_A(c)$$

and

$$t \geq \min\{F_A(a), F_A(b)\} \geq F_A((a * (b * c) * c)) = F_A(1 * c) = F_A(c).$$

Then $c \in U(A; t)$. Therefore, $A(a, b) \subseteq U(A; t)$).□

Corolary 3.1. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then*

$$(\forall t \in [0,1])\ (U(A;t) \neq \emptyset \ \Rightarrow\ U(A;t) = \bigcup_{a,b \in U(A;t)} A(a,b)).$$

Proof. It is sufficient prove that $U(A;t) \subseteq \bigcup_{a,b \in U(A;t)} A(a,b)$. For this, assume that $x \in U(A;t)$. Since $x * (1 * x) = 1$, we have $x \in A(x,1)$. Hence

$$U(A;t) \subseteq A(x,1) \subseteq \bigcup_{x \in U(A;t)} A(x,1) \subseteq \bigcup_{x,y \in U(A;t)} A(x,y).$$

□

Theorem 3.6. *Let* $\mathfrak{X}$ *be a self distributive BE-algebra and* $A \in \mathrm{NF}(\mathfrak{X})$. *Then the following conditions are equivalent:*

(i) $A \in \mathrm{INF}(\mathfrak{X})$,

(ii) $T_A(y * (y * x)) \leq T_A(y * x)$, $I_A(y * (y * x)) \geq I_A(y * x)$ *and* $F_A(y * (y * x)) \geq F_A(y * x)$,

(iii) $\min\{T_A((z * (y * (y * x))), T_A(z)\} \leq T_A(y * x)$,
$\min\{I_A((z * (y * (y * x))), I_A(z)\} \geq I_A(y * x)$ *and*
$\min\{F_A((z * (y * (y * x))), F_A(z)\} \geq F_A(y * x)$.

Proof. (i)⇒(ii). Let $A \in \mathrm{NF}(\mathfrak{X})$. By (INF1) and (BE1) we have

$$\begin{aligned} T_A(y*(y*x)) &= \min\{T_A(y*(y*x)), T_A(1)\} \\ &= \min\{T_A(y*(y*x)), T_A(y*y)\} \\ &\leq T_A(y*x), \end{aligned}$$

$$\begin{aligned} I_A(y*(y*x)) &= \min\{I_A(y*(y*x)), I_A(1)\} \\ &= \min\{I_A(y*(y*x)), I_A(y*y)\} \\ &\geq I_A(y*x) \end{aligned}$$

and

$$\begin{aligned} F_A(y*(y*x)) &= \min\{F_A(y*(y*x)), F_A(1)\} \\ &= \min\{F_A(y*(y*x)), F_A(y*y)\} \\ &\geq F_A(y*x). \end{aligned}$$

(ii)⇒(iii). Let A be a neutrosophic filter of $\mathfrak{X}$ satisfying the condition (ii). By using (NF2) and (ii) we have

$$\begin{aligned} \min\{T_A(z*(y*(y*x))), T_A(z)\} &\leq T_A(y*(y*x)) \\ &\leq T_A(y*x), \end{aligned}$$

$$\begin{aligned} \min\{I_A(z*(y*(y*x))), I_A(z)\} &\geq I_A(y*(y*x)) \\ &\geq I_A(y*x) \end{aligned}$$

and

$$\begin{aligned} \min\{F_A(z*(y*(y*x))), F_A(z)\} &\geq F_A(y*(y*x)) \\ &\geq F_A(y*x). \end{aligned}$$

(iii)⇒(i). Since

$$x*(y*z) = y*(x*z) \leq (x*y)*(x*(x*z)),$$

we have $T_A(x*(y*z)) \leq T_A((x*y)*(x*(x*z)))$,
$I_A(x*(y*z)) \geq I_A((x*y)*(x*(x*z)))$ and
$F_A(x*(y*z)) \geq F_A((x*y)*(x*(x*z)))$, by Proposition 3.1(i). Thus

$$\begin{aligned} \min\{T_A(x*(y*z)), T_A(x*y)\} &\leq \min\{T_A((x*y)*(x*(x*z))), T_A(x*y)\} \\ &\leq T_A(x*z). \end{aligned}$$

$$\begin{aligned}\min\{I_A(x*(y*z)), I_A(x*y)\} &\geq \min\{I_A((x*y)*(x*(x*z))), I_A(x*y)\}\\ &\geq I_A(x*z)\end{aligned}$$

and

$$\begin{aligned}\min\{F_A(x*(y*z)), F_A(x*y)\} &\geq \min\{F_A((x*y)*(x*(x*z))), F_A(x*\\ y)\} &\geq F_A(x*z).\end{aligned}$$

Therefore, $A \in \mathrm{INF}(\mathfrak{X})$. Let $f : X \to Y$ be a homomorphism of BE-algebras

and $A \in \mathrm{NS}(\mathfrak{X})$.
Define tree maps T_{A^f}: $X \to [0,1]$ such that $T_{A^f}(x) = T_A(f(x))$,
$I_{A^f} : X \to [0,1]$ such that $I_{A^f}(x) = I_A(f(x))$ and $F_{A^f} : X \to [0,1]$ such that $F_{A^f}(x) = F_A(f(x))$, for all $x \in X$. Then T_{A^f}, I_{A^f} and F_{A^f} are well-define and $A^f = (T_{A^f}, I_{A^f}, F_{A^f}) \in \mathrm{NS}(\mathfrak{X})$.□

Theorem 3.7. *Let $f : X \to Y$ be an onto homomorphism of BE-algebras and $A \in \mathrm{NS}(\mathfrak{Y})$. Then $A \in \mathrm{NF}(\mathfrak{Y})$ (resp. $A \in \mathrm{INF}(\mathfrak{Y})$) if and only if $A^f \in \mathrm{NF}(\mathfrak{X})$ (resp. $A^f \in \mathrm{INF}(\mathfrak{X})$).*

Proof. Assume that $A \in \mathrm{NF}(\mathfrak{Y})$. For any $x \in X$, we have

$$T_{A^f}(x) = T_A(f(x)) \leq T_A(1_Y) = T_A(f(1_X)) = T_{A^f}(1_X),$$

$$I_{A^f}(x) = I_A(f(x)) \geq I_A(1_Y) = I_A(f(1_X)) = I_{A^f}(1_X)$$

and

$$F_{A^f}(x) = F_A(f(x)) \geq F_A(1_Y) = F_A(f(1_X)) = F_{A^f}(1_X).$$

Hence (NF1) is valid. Now, let $x, y \in X$. By (NF1) we have

$$\begin{aligned}\min\{T_{A^f}(x*y), T_{A^f}(x)\} &= \min\{T_A(f(x*y)), T_A(f(x))\}\\ &= \min\{T_A(f(x)*f(y)), T_A(f(x))\}\\ &\leq T_A(f(y))\\ &= T_{A^f}(y)\end{aligned}$$

Also,

$$\begin{aligned}\min\{I_{A^f}(x*y), I_{A^f}(x)\} &= \min\{I_A(f(x*y)), I_A(f(x))\}\\ &= \min\{I_A(f(x)*f(y)), I_A(f(x))\}\\ &\geq I_A(f(y))\\ &= I_{A^f}(y).\end{aligned}$$

By a similar argument we have $\min\{F_{A^f}(x*y), F_{A^f}(x)\} \geq F_{A^f}(y)$. Therefore, $A^f \in \mathrm{NF}(\mathfrak{X})$.

Conversely, Assume that $A^f \in \mathrm{NF}(\mathfrak{X})$. Let $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Then

$$T_A(y) = T_A(f(x)) = T_{A^f}(x) \leq T_{A^f}(1_X) = T_A(f(1_X)) = T_A(1_Y),$$

$$I_A(y) = I_A(f(x)) = I_{A^f}(x) \geq I_{A^f}(1_X) = I_A(f(1_X)) = I_A(1_Y)$$

and

$$F_A(y) = F_A(f(x)) = F_{A^f}(x) \geq F_{A^f}(1_X) = F_A(f(1_X)) = F_A(1_Y),$$

Now, let $x, y \in Y$. Then there exists $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Hence we have

$$\begin{aligned}
\min\{T_A(x*y), T_A(x)\} &= \min\{T_A(f(a)*f(b)), T_A(f(a))\} \\
&= \min\{T_A(f(a*b)), T_A(f(a))\} \\
&= \min\{T_{A^f}(a*b), T_{A^f}(a)\} \\
&\leq T_{A^f}(b) \\
&= T_A(f(b)) \\
&= T_A(y).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\min\{I_A(x*y), I_A(x)\} &= \min\{I_A(f(a)*f(b)), I_A(f(a))\} \\
&= \min\{I_A(f(a*b)), I_A(f(a))\} \\
&= \min\{I_{A^f}(a*b), I_{A^f}(a)\} \\
&\geq I_{A^f}(b) \\
&= I_A(f(b)) \\
&= I_A(y).
\end{aligned}$$

By a similar argument we have $\min\{F_A(x*y), F_A(x)\} \geq F_A(y)$.
Therefore, $A \in \mathrm{NF}(\mathfrak{Y})$.□

4 Conclusion

F. Smarandache as an extension of intuitionistic fuzzy logic introduced the concept of neutrosophic logic and then several researchers have studied of some neutrosophic algebraic structures. In this paper, we applied the theory of neutrosophic sets to BE-algebras and introduced the notions of (implicative) neutrosophic filters and many related properties are investigated.

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Isomorphism of Bipolar Single Valued Neutrosophic Hypergraphs

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Muhammad Aslam Malik, Ali Hassan, Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache (2016). Isomorphism of Bipolar Single Valued Neutrosophic Hypergraphs. *Critical Review* XIII, 79-102

Abstract

In this paper, we introduce the homomorphism, the weak isomorphism, the co-weak isomorphism, and the isomorphism of the bipolar single valued neutrosophic hypergraphs. The properties of order, size and degree of vertices are discussed. The equivalence relation of the isomorphism of the bipolar single valued neutrosophic hypergraphs and the weak isomorphism of bipolar single valued neutrosophic hypergraphs, together with their partial order relation, is also verified.

Keywords

homomorphism, weak-isomorphism, co-weak-isomorphism, isomorphism, bipolar single valued neutrosophic hypergraphs.

1 Introduction

The neutrosophic set - proposed by Smarandache [8] as a generalization of the fuzzy set [14], intuitionistic fuzzy set [12], interval valued fuzzy set [11] and interval-valued intuitionistic fuzzy set [13] theories - is a mathematical tool created to deal with incomplete, indeterminate and inconsistent information in the real world. The characteristics of the neutrosophic set are the truth-membership function (t), the indeterminacy-membership function (i), and the falsity membership function (f), which take values within the real standard or non-standard unit interval $]^-0 , 1^+[$.

A subclass of the neutrosophic set, the single-valued neutrosophic set (SVNS), was intoduced by Wang et al. [9]. The same authors [10] also introduced a generalization of the single valued neutrosophic set, namely the interval valued neutrosophic set (IVNS), in which the three membership functions are independent, and their values belong to the unit interval [0, 1]. The IVNS is more precise and flexible than the single valued neutrosophic set.

More works on single valued neutrosophic sets, interval valued neutrosophic sets and their applications can be found on *http://fs.gallup.unm.edu/NSS/*.

In this paper, we extend the isomorphism of the bipolar single valued neutrosophic hypergraphs, and introduce some of their relevant properties.

1 Preliminaries

Definition 2.1

A hypergraph is an ordered pair H = (X, E), where:

(1) $X = \{x_1, x_2, \dots, x_n\}$ is a finite set of vertices.

(2) $E = \{E_1, E_2, \dots, E_m\}$ is a family of subsets of X.

(3) E_j are non-void for j = 1, 2, 3, ..., m, and $\cup_j (E_j) = X$.

The set X is called 'set of vertices', and E is denominated as the 'set of edges' (or 'hyper-edges').

Definition 2.2

A fuzzy hypergraph H = (X, E) is a pair, where X is a finite set and E is a finite family of non-trivial fuzzy subsets of X, such that $X = \cup_j Supp(E_j)$, $j = 1, 2, 3, \dots, m$.

Remark 2.3

The collection $E = \{E_1, E_2, E_3, \dots, E_m\}$ is a collection of edge set of H.

Definition 2.4

A fuzzy hypergraph with underlying set X is of the form H = (X, E, R), where $E = \{E_1, E_2, E_3, \dots, E_m\}$ is the collection of fuzzy subsets of X, that is $E_j : X \to [0, 1]$, j = 1, 2, 3, ..., m, and $R : E \to [0, 1]$ is the fuzzy relation of the fuzzy subsets E_j, such that:

$$R(x_1, x_2, \dots, x_r) \le \min(E_j(x_1), \dots, E_j(x_r)), \qquad (1)$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Definition 2.5

Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by its truth membership function T_A(x), its indeterminacy membership function I_A(x), and its falsity membership function F_A(x). For each point, x ∈ X; T_A(x), I_A(x), F_A(x) ∈ [0, 1].

Definition 2.6

A single valued neutrosophic hypergraph is an ordered pair $H = (X, E)$, where:

(1) $X = \{x_1, x_2, \dots, x_n\}$ is a finite set of vertices.

(2) $E = \{E_1, E_2, \dots, E_m\}$ is a family of SVNSs of X.

(3)$E_j \neq 0 = (0, 0, 0)$ for $j = 1, 2, 3, \dots, m$, and $\bigcup_j Supp(E_j) = X$.

The set X is called set of vertices and E is the set of SVN-edges (or SVN-hyperedges).

Proposition 2.7

The single valued neutrosophic hypergraph is the generalization of fuzzy hypergraphs and intuitionistic fuzzy hypergraphs.

Note that a given SVNHG$H = (X, E, R)$, with underlying set X, where $E = \{E_1, E_2, \dots, E_m\}$, is the collection of the non-empty family of SVN subsets of X, and R is the SVN relation of the SVN subsets E_j, such that:

$$R_T(x_1, x_2, \dots, x_r) \leq \min([T_{E_j}(x_1)], \dots, [T_{E_j}(x_r)]), \quad (2)$$

$$R_I(x_1, x_2, \dots, x_r) \geq \max([I_{E_j}(x_1)], \dots, [I_{E_j}(x_r)]), \quad (3)$$

$$R_F(x_1, x_2, \dots, x_r) \geq \max([F_{E_j}(x_1)], \dots, [F_{E_j}(x_r)]), \quad (4)$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Definition 2.8

Let X be a space of points (objects) with generic elements in X denoted by x.

A bipolar single valued neutrosophic set A (BSVNS A) is characterized by the positive truth membership function $PT_A(x)$, the positive indeterminacy membership function $PI_A(x)$, the positive falsity membership function $PF_A(x)$, the negative truth membership function $NT_A(x)$, the negative indeterminacy membership function $NI_A(x)$, and the negative falsity membership function $NF_A(x)$.

For each point x ∈X; $PT_A(x)$, $PI_A(x)$,$PF_A(x)$ ∈ [0, 1], and $NT_A(x)$, $NI_A(x)$, $NF_A(x)$ ∈ [-1, 0].

Definition 2.9

A bipolar single valued neutrosophic hypergraph is an ordered pair $H = (X, E)$, where:

(1) $X = \{x_1, x_2, \dots, x_n\}$ is a finite set of vertices.

(2) $E = \{E_1, E_2, \dots, E_m\}$ is a family of BSVNSs of X.

(3) $E_j \neq 0 = ([0, 0], [0, 0], [0, 0])$ for $j = 1, 2, 3, \dots, m$, and $\bigcup_j Supp(E_j) = X$.

The set X is called the 'set of vertices' and E is called the 'set of BSVN-edges' (or 'IVN-hyper-edges'). Note that a given BSVNHG$H = (X, E, R)$, with underlying set X, where $E = \{E_1, E_2, \dots, E_m\}$ is the collection of non-empty family of BSVN subsets of X, and R is the BSVN relation of BSVN subsets E_j such that:

$$R_{PT}(x_1, x_2, \dots, x_r) \leq \min([PT_{E_j}(x_1)], \dots, [PT_{E_j}(x_r)]), \quad (5)$$

$$R_{PI}(x_1, x_2, \dots, x_r) \geq \max([PI_{E_j}(x_1)], \dots, [PI_{E_j}(x_r)]), \quad (6)$$

$$R_{PF}(x_1, x_2, \dots, x_r) \geq \max([PF_{E_j}(x_1)], \dots, [PF_{E_j}(x_r)]), \quad (7)$$

$$R_{NT}(x_1, x_2, \dots, x_r) \geq \max([NT_{E_j}(x_1)], \dots, [NT_{E_j}(x_r)]), \quad (8)$$

$$R_{NI}(x_1, x_2, \dots, x_r) \leq \min([NI_{E_j}(x_1)], \dots, [NI_{E_j}(x_r)]), \quad (9)$$

$$R_{NF}(x_1, x_2, \dots, x_r) \leq \min([NF_{E_j}(x_1)], \dots, [NF_{E_j}(x_r)]), \quad (10)$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Proposition 2.10

The bipolar single valued neutrosophic hypergraph is the generalization of the fuzzy hypergraph, intuitionistic fuzzy hypergraph, bipolar fuzzy hypergraph and intuitionistic fuzzy hypergraph.

Example 2.11

Consider the BSVNHG $H = (X, E, R)$, with underlying set $X = \{a, b, c\}$, where $E = \{A, B\}$, and R defined in *Tables* below:

H	A	B
a	(0.2, 0.3, 0.9, -0.2, -0.2, -0.3)	(0.5, 0.2, 0.7, -0.4, -0.2, -0.3)
b	(0.5, 0.5, 0.5, -0.4, -0.3, -0.3)	(0.1, 0.6, 0.4, -0.9, -0.3,-0.4)
c	(0.8, 0.8, 0.3, -0.9, -0.2, -0.3)	(0.5, 0.9, 0.8, -0.1, -0.2, -0.3)

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.2	0.8	0.9	-0.1	-0.4	-0.5
B	0.1	0.9	0.8	-0.1	-0.5	-0.6

By routine calculations, $H = (X, E, R)$ is BSVNHG.

3 Isomorphism of BSVNHGs

Definition 3.1

A homomorphism $f: H \to K$ between two BSVNHGs $H = (X, E, R)$ and $K = (Y, F, S)$ is a mapping $f: X \to Y$ which satisfies the conditions:

$$\min[PT_{E_j}(x)] \leq \min[PT_{F_j}(f(x))], \quad (11)$$

$$\max[PI_{E_j}(x)] \geq \max[PI_{F_j}(f(x))], \quad (12)$$

$$\max[PF_{E_j}(x)] \geq \max[PF_{F_j}(f(x))], \quad (13)$$

$$\max[NT_{E_j}(x)] \geq \max[NT_{F_j}(f(x))], \quad (14)$$

$$\min[NI_{E_j}(x)] \leq \min[NI_{F_j}(f(x))], \quad (15)$$

$$\min[NF_{E_j}(x)] \leq \min[NF_{F_j}(f(x))], \quad (16)$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) \leq S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \quad (17)$$

$$R_{PI}(x_1, x_2, \dots, x_r) \geq S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \quad (18)$$

$$R_{PF}(x_1, x_2, \dots, x_r) \geq S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \quad (19)$$

$$R_{NT}(x_1, x_2, \dots, x_r) \geq S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \quad (20)$$

$$R_{NI}(x_1, x_2, \dots, x_r) \leq S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \quad (21)$$

$$R_{NF}(x_1, x_2, \dots, x_r) \leq S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \quad (22)$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Example 3.2

Consider the two BSVNHGs $H = (X, E, R)$ and $K = (Y, F, S)$ with underlying sets $X = \{a, b, c\}$ and $Y = \{x, y, z\}$, where $E = \{A, B\}$, $F = \{C, D\}$, R and S, which are defined in *Tables* given below:

H	A	B
a	(0.2, 0.3, 0.9,-0.2,-0.2,-0.3)	(0.5, 0.2, 0.7,-0.4,-0.2,-0.3)
b	(0.5, 0.5, 0.5,-0.4,-0.3,-0.3)	(0.1, 0.6, 0.4,-0.9,-0.3,-0.4)
c	(0.8, 0.8, 0.3,-0.9,-0.2,-0.3)	(0.5, 0.9, 0.8,-0.1,-0.2,-0.3)

K	C	D
x	(0.3, 0.2, 0.2,-0.9,-0.2,-0.3)	(0.2, 0.1, 0.3,-0.6,-0.1,-0.2)
y	(0.2, 0.4, 0.2,-0.4,-0.2,-0.3)	(0.3, 0.2, 0.1,-0.7,-0.2,-0.1)
z	(0.5, 0.8, 0.2,-0.2,-0.1,-0.3)	(0.9, 0.7, 0.1,-0.2,-0.1,-0.3)

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.2	0.8	0.9	-0.1	-0.4	-0.5
B	0.1	0.9	0.8	-0.1	-0.5	-0.6

S	S_{PT}	S_{PI}	S_{PF}	S_{NT}	S_{NI}	S_{NF}
C	0.2	0.8	0.3	-0.1	-0.2	-0.3
D	0.1	0.7	0.3	-0.1	-0.2	-0.3

and $f: X \rightarrow Y$ defined by: $f(a)=x$, $f(b)=y$ and $f(c)=z$. Then, by routine calculations, $f: H \rightarrow K$ is a homomorphism between H and K.

Definition 3.3

A weak isomorphism $f: H \rightarrow K$ between two BSVNHGs $H = (X, E, R)$ and $K = (Y, F, S)$ is a bijective mapping $f: X \rightarrow Y$ which satisfies f is homomorphism, such that:

$$\min[PT_{E_j}(x)] \leq \min[PT_{F_j}(f(x))], \tag{23}$$

$$\max[PI_{E_j}(x)] \geq \max[PI_{F_j}(f(x))], \tag{24}$$

$$\max[PF_{E_j}(x)] \geq \max[PF_{F_j}(f(x))], \tag{25}$$

$$\max[NT_{E_j}(x)] \geq \max[NT_{F_j}(f(x))], \tag{26}$$

$$\min[NI_{E_j}(x)] \leq \min[NI_{F_j}(f(x))], \tag{27}$$

$$\min[NF_{E_j}(x)] \leq \min[NF_{F_j}(f(x))], \tag{28}$$

for all $x \in X$.

Note

The weak isomorphism between two BSVNHGs preserves the weights of vertices.

Example 3.4

Consider the two BSVNHGs $H = (X, E, R)$ and $K = (Y, F, S)$ with underlying sets $X = \{a, b, c\}$ and $Y = \{x, y, z\}$, where $E = \{A, B\}$, $F = \{C, D\}$, R and S, which are defined by *Tables* given below, and $f: X \rightarrow Y$ defined by: $f(a)=x$, $f(b)=y$ and $f(c)=z$. Then, by routine calculations, $f: H \rightarrow K$ is a weak isomorphism between H and K.

H	A	B
a	(0.2, 0.3, 0.9,-0.2,-0.2,-0.3)	(0.5, 0.2, 0.7,-0.4,-0.2,-0.3)
b	(0.5, 0.5, 0.5,-0.4,-0.3,-0.3)	(0.1, 0.6, 0.4,-0.9,-0.3,-0.4)
c	(0.8, 0.8, 0.3,-0.9,-0.2,-0.3)	(0.5, 0.9, 0.8,-0.1,-0.2,-0.3)

K	C	D
x	(0.2, 0.3, 0.2,-0.9,-0.2,-0.3)	(0.2, 0.1, 0.8,-0.6,-0.1,-0.4)
y	(0.2, 0.4, 0.2,-0.4,-0.3,-0.3)	(0.1, 0.6, 0.5,-0.6,-0.2,-0.3)
z	(0.5, 0.8, 0.9,-0.2,-0.2,-0.3)	(0.9, 0.9, 0.1,-0.1,-0.3,-0.3)

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.2	0.8	0.9	-0.1	-0.4	-0.3
B	0.1	0.9	0.9	-0.1	-0.3	-0.5

S	S_{PT}	S_{PI}	S_{PF}	S_{NT}	S_{NI}	S_{NF}
C	0.2	0.8	0.9	-0.1	-0.3	-0.2
D	0.1	0.9	0.8	-0.1	-0.3	-0.4

Definition 3.5

A co-weak isomorphism *f: H → K* between two BSVNHGs *H = (X, E, R)* and *K = (Y, F, S)* is a bijective mapping *f: X → Y* which satisfies *f* is homomorphism, such that:

$$R_{PT}(x_1, x_2, \dots, x_r) = S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{29}$$

$$R_{PI}(x_1, x_2, \dots, x_r) = S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{30}$$

$$R_{PF}(x_1, x_2, \dots, x_r) = S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{31}$$

$$R_{NT}(x_1, x_2, \dots, x_r) = S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{32}$$

$$R_{NI}(x_1, x_2, \dots, x_r) = S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{33}$$

$$R_{NF}(x_1, x_2, \dots, x_r) = S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{34}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of *X*.

Note

The co-weak isomorphism between two BSVNHGs preserves the weights of edges.

Example 3.6

Consider the two BSVNHGs *H = (X, E, R)* and *K = (Y, F, S)* with underlying sets *X = {a, b, c}* and *Y = {x, y, z}*, where *E = {A, B}*, *F = {C, D}*, *R* and *S,* which are defined in *Tables* given below, and *f : X → Y* defined by: *f(a)=x* , *f(b)=y* and *f(c)=z*. Then, by routine calculations, *f: H → K* is a co-weak isomorphism between *H* and *K*.

H	A	B
a	(0.2, 0.3, 0.9,-0.4,-0.2,-0.3)	(0.5, 0.2, 0.7,-0.1,-0.2,-0.3)
b	(0.5, 0.5, 0.5,-0.4,-0.2,-0.3)	(0.1, 0.6, 0.4,-0.4,-0.2,-0.3)
c	(0.8, 0.8, 0.3,-0.1,-0.2,-0.3)	(0.5, 0.9, 0.8,-0.4,-0.2,-0.3)

K	C	D
x	(0.3, 0.2, 0.2,-0.9,-0.2,-0.3)	(0.2, 0.1, 0.3,-0.4,-0.2,-0.3)
y	(0.2, 0.4, 0.2,-0.4,-0.2,-0.3)	(0.3, 0.2,0.1,-0.9,-0.2,-0.3)
z	(0.5, 0.8, 0.2,-0.1,-0.2,-0.3)	(0.9, 0.7, 0.1,-0.1,-0.2,-0.3)

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.2	0.8	0.9	-0.1	-0.2	-0.3
B	0.1	0.9	0.8	-0.1	-0.2	-0.3

S	S_{PT}	S_{PI}	S_{PF}	S_{NT}	S_{NI}	S_{NF}
C	0.2	0.8	0.9	-0.1	-0.2	-0.3
D	0.1	0.9	0.8	-0.1	-0.2	-0.3

Definition 3.7

An isomorphism $f : H \to K$ between two BSVNHGs $H = (X, E, R)$ and $K = (Y, F, S)$ is a bijective mapping $f : X \to Y$ which satisfies the conditions:

$$\min[PT_{E_j}(x)] = \min[PT_{F_j}(f(x))], \tag{35}$$

$$\max[PI_{E_j}(x)] = \max[PI_{F_j}(f(x))], \tag{36}$$

$$\max[PF_{E_j}(x)] = \max[PF_{F_j}(f(x))], \tag{37}$$

$$\max[NT_{E_j}(x)] = \max[NT_{F_j}(f(x))], \tag{38}$$

$$\min[NI_{E_j}(x)] = \min[NI_{F_j}(f(x))], \tag{39}$$

$$\min[NF_{E_j}(x)] = \min[NF_{F_j}(f(x))], \tag{40}$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) = S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{41}$$

$$R_{PI}(x_1, x_2, \dots, x_r) = S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{42}$$

$$R_{PF}(x_1, x_2, \dots, x_r) = S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{43}$$

$$R_{NT}(x_1, x_2, \dots, x_r) = S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{44}$$

$$R_{NI}(x_1, x_2, \dots, x_r) = S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{45}$$

$$R_{NF}(x_1, x_2, \dots, x_r) = S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{46}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Note

The isomorphism between two BSVNHGs preserves the both weights of vertices and weights of edges.

Example 3.8

Consider the two BSVNHGs $H = (X, E, R)$ and $K = (Y, F, S)$ with underlying sets $X = \{a, b, c\}$ and $Y = \{x, y, z\}$, where $E = \{A, B\}$, $F = \{C, D\}$, R and S, which are defined by *Tables* given below:

H	A	B
a	(0.2, 0.3, 0.7,-0.2,-0.2,-0.3)	(0.5, 0.2, 0.7,-0.6,-0.6,-0.6)
b	(0.5, 0.5, 0.5,-0.4,-0.3,-0.3)	(0.1, 0.6, 0.4,-0.1,-0.2,-0.7)
c	(0.8, 0.8, 0.3,-0.9,-0.2,-0.4)	(0.5, 0.9, 0.8,-0.7,-0.2,-0.3)

K	C	D
x	(0.2, 0.3, 0.2,-0.2,-0.2,-0.4)	(0.2, 0.1, 0.8,-0.3,-0.2,-0.3)
y	(0.2, 0.4, 0.2,-0.6,-0.2,-0.3)	(0.1, 0.6, 0.5,-0.1,-0.2,-0.7)
z	(0.5, 0.8, 0.7,-0.4,-0.3,-0.3)	(0.9, 0.9, 0.1,-0.9,-0.6,-0.3)

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.2	0.8	0.9	-0.1	-0.3	-0.4
B	0.0	0.9	0.8	-0.1	-0.7	-0.8

S	S_{PT}	S_{PI}	S_{PF}	S_{NT}	S_{NI}	S_{NF}
C	0.2	0.8	0.9	-0.1	-0.3	-0.4
D	0.0	0.9	0.8	-0.1	-0.7	-0.8

and $f : X \rightarrow Y$ defined by: $f(a)=x$, $f(b)=y$ and $f(c)=z$. Then, by routine calculations, $f: H \rightarrow K$ is an isomorphism between H and K.

Definition 3.9

Let $H = (X, E, R)$ be a BSVNHG, then the order of H is denoted and defined by as follows:

$$O(H) = (\sum min\left(PT_{E_j}(x)\right), \sum max\left(PI_{E_j}(x)\right), \sum max\left(PF_{E_j}(x)\right), \sum max\left(NT_{E_j}(x)\right), \sum min\left(NI_{E_j}(x)\right), \sum min\left(NF_{E_j}(x)\right)) \quad (47)$$

The size of H is denoted and defined by:

$$S(H) = (\sum R_{PT}(E_j), \sum R_{PI}(E_j), \sum R_{PF}(E_j), \sum R_{NT}(E_j), \sum R_{NI}(E_j), \sum R_{NF}(E_j)) \quad (48)$$

Theorem 3.10

Let $H = (X, E, R)$ and $K = (Y, F, S)$ be two BSVNHGs such that H is isomorphic to K, then:

(1) O(H) = O(K),

(2) S(H) = S(K).

Proof

Let *f: H → K* be an isomorphism between two BSVNHGs *H* and *K* with underlying sets *X* and *Y* respectively; then, by definition:

$$\min[PT_{E_j}(x)] = \min[PT_{F_j}(f(x))], \quad (49)$$

$$\max[PI_{E_j}(x)] = \max[PI_{F_j}(f(x))], \quad (50)$$

$$\max[PF_{E_j}(x)] = \max[PF_{F_j}(f(x))], \quad (51)$$

$$\max[NT_{E_j}(x)] = \max[NT_{F_j}(f(x))], \quad (52)$$

$$\min[NI_{E_j}(x)] = \min[NI_{F_j}(f(x))], \quad (53)$$

$$\min[NF_{E_j}(x)] = \min[NF_{F_j}(f(x))], \quad (54)$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \ldots, x_r) = S_{PT}(f(x_1), f(x_2), \ldots, f(x_r)), \quad (55)$$

$$R_{PI}(x_1, x_2, \ldots, x_r) = S_{PI}(f(x_1), f(x_2), \ldots, f(x_r)), \quad (56)$$

$$R_{PF}(x_1, x_2, \ldots, x_r) = S_{PF}(f(x_1), f(x_2), \ldots, f(x_r)), \quad (57)$$

$$R_{NT}(x_1, x_2, \ldots, x_r) = S_{NT}(f(x_1), f(x_2), \ldots, f(x_r)), \quad (58)$$

$$R_{NI}(x_1, x_2, \ldots, x_r) = S_{NI}(f(x_1), f(x_2), \ldots, f(x_r)), \quad (59)$$

$$R_{NF}(x_1, x_2, \ldots, x_r) = S_{NF}(f(x_1), f(x_2), \ldots, f(x_r)), \quad (60)$$

for all $\{x_1, x_2, \ldots, x_r\}$ subsets of *X*.

Consider:

$$O_{PT}(H) = \sum \min PT_{E_j}(x) = \sum \min PT_{F_j}(f(x)) = O_{PT}(K) \quad (61)$$

$$O_{NT}(H) = \sum \max NT_{E_j}(x) = \sum \max NT_{F_j}(f(x)) = O_{NT}(K) \quad (62)$$

Similarly, $O_{PI}(H) = O_{PI}(K)$ and $O_{PF}(H) = O_{PF}(K)$, $O_{NI}(H) = O_{NI}(K)$ and $O_{NF}(H) = O_{NF}(K)$**,** hence *O(H) = O(K).*

Next:

$$S_{PT}(H) = \sum R_{PT}(x_1, x_2, \ldots, x_r)$$

$$= \sum S_{PT}(f(x_1), f(x_2), \ldots, f(x_r)) = S_{PT}(K). \quad (63)$$

Similarly,

$$S_{NT}(H) = \sum R_{NT}(x_1, x_2, \ldots, x_r)$$

$$= \sum S_{NT}(f(x_1), f(x_2), \ldots, f(x_r)) = S_{NT}(K). \quad (64)$$

and $S_{PI}(H) = S_{PI}(K)$, $S_{PF}(H) = S_{PF}(K)$, $S_{NI}(H) = S_{NI}(K)$, $S_{NF}(H) = S_{NF}(K)$, hence $S(H) = S(K)$.

Remark 3.11

The converse of the above theorem need not to be true in general.

Example 3.12

Consider the two BSVNHGs *H = (X, E, R)* and *K = (Y, F, S)* with underlying sets *X = {a, b, c, d}* and *Y = {w, x, y, z}*, where *E = {A, B}*, *F = {C, D}*, *R* and *S* are defined in *Tables* given below:

H	A	B
a	(0.2, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.14, 0.5, 0.3, -0.1, -0.2, -0.3)
b	(0.0,0.0,0.0, 0.0, 0.0, 0.0)	(0.2, 0.5, 0.3, -0.4, -0.2, -0.3)
c	(0.33, 0.5, 0.3, -0.4, -0.2, -0.3)	(0.16, 0.5, 0.3, -0.1, -0.2, -0.3)
d	(0.5, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.0,0.0,0.0, 0.0, 0.0, 0.0)

K	C	D
w	(0.14, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.2, 0.5, 0.33, -0.4, -0.2, -0.3)
x	(0.16, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.33,0.5, 0.33, -0.1, -0.2, -0.3)
y	(0.25, 0.5, 0.3, -0.1,-0.2, -0.3)	(0.2, 0.5, 0.33, -0.1, -0.2, -0.3)
z	(0.5, 0.5, 0.3, -0.4, -0.2, -0.3)	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.2	0.5	0.3	-0.1	-0.2	-0.3
B	0.14	0.5	0.3	-0.1	-0.2	-0.3

S	S_{PT}	S_{PI}	S_{PF}	S_{NT}	S_{NI}	S_{NF}
C	0.14	0.5	0.3	-0.1	-0.2	-0.3
D	0.2	0.5	0.3	-0.1	-0.2	-0.3

where *f* is defined by: *f(a)=w, f(b)=x, f(c)=y, f(d)=z.*

Here, *O(H) = (1.0,2.0, 1.2,-0.7,-0.8,-1.2) = O(K)* and *S(H)=(0.34, 1.0, 0.9, -0.2, -0.4, -0.9)=S(K)*, but, by routine calculations, *H* is not an isomorphism to *K*.

Corollary 3.13

The weak isomorphism between any two BSVNHGs *H* and *K* preserves the orders.

Remark 3.14

The converse of the above corollary need not to be true in general.

Example 3.15

Consider the two BSVNHGs *H = (X, E, R)* and *K = (Y, F, S)* with underlying sets *X = {a, b, c, d}* and *Y = {w, x, y, z}*, where *E = {A, B}*, *F = {C, D}*, *R* and *S* are defined in *Tables* given below**,** where *f* is defined by: *f(a)=w, f(b)=x, f(c)=y, f(d)=z:*

H	A	B
a	(0.2, 0.5, 0.3,-0.1,-0.2,-0.3)	(0.14, 0.5, 0.3,-0.4,-0.2,-0.3)
b	(0.0,0.0,0.0,0.0,0.0,0.0)	(0.2, 0.5, 0.3,-0.1,-0.2,-0.3)
c	(0.33, 0.5, 0.3,-0.4,-0.2,-0.3)	(0.16, 0.5, 0.3,-0.1,-0.2,-0.3)
d	(0.5, 0.5, 0.3,-0.1,-0.2,-0.3)	(0.0,0.0,0.0,0.0,0.0,0.0)

K	C	D
w	(0.14, 0.5, 0.3,-0.1,-0.2,-0.3)	(0.16, 0.5, 0.3,-0.1,-0.2,-0.3)
x	(0.0, 0.0, 0.0,0.0,0.0,0.0)	(0.16, 0.5, 0.3,-0.1,-0.2,-0.3)
y	(0.25, 0.5, 0.3,-0.1,-0.2,-0.3)	(0.2, 0.5, 0.3,-0.4,-0.2,-0.3)
z	(0.5, 0.5, 0.3,-0.1,-0.2,-0.3)	(0.0, 0.0, 0.0,0.0,0.0,0.0)

Here, $O(H)= (1.0, 2.0, 1.2, -0.4, -0.8, -1.2) = O(K)$, but, by routine calculations, H is not a weak isomorphism to K.

Corollary 3.16

The co-weak isomorphism between any two BSVNHGs H and K preserves sizes.

Remark 3.17

The converse of the above corollary need not to be true in general.

Example 3.18

Consider the two BSVNHGs $H = (X, E, R)$ and $K = (Y, F, S)$ with underlying sets $X = \{a, b, c, d\}$ and $Y = \{w, x, y, z\}$, where $E = \{A, B\}$, $F = \{C, D\}$,R and S are defined in *Tables* given below,

H	A	B
a	(0.2, 0.5, 0.3,-0.1,-0.2,-0.3)	(0.14, 0.5, 0.3, -0.1,-0.2,-0.3)
b	(0.0,0.0,0.0,0.0,0.0,0.0)	(0.16, 0.5, 0.3, -0.1,-0.2,-0.3)
c	(0.3, 0.5, 0.3, -0.1,-0.2,-0.3)	(0.2, 0.5, 0.3,-0.4,-0.2,-0.3)
d	(0.5, 0.5, 0.3, -0.1,-0.2,-0.3)	(0.0,0.0,0.0,0.0,0.0,0.0)

K	C	D
w	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.2, 0.5, 0.3, -0.1,-0.2,-0.3)
x	(0.14,0.5,0.3, -0.1,-0.2,-0.3)	(0.25, 0.5, 0.3, -0.1,-0.2,-0.3)
y	(0.5, 0.5, 0.3, -0.1,-0.2,-0.3)	(0.2, 0.5, 0.3,-0.4,-0.2,-0.3)
z	(0.3, 0.5, 0.3, -0.1,-0.2,-0.3)	(0.0,0.0,0.0,0.0,0.0,0.0)

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.2	0.5	0.3	-0.1	-0.2	-0.3
B	0.14	0.5	0.3	-0.1	-0.2	-0.3

S	S_{PT}	S_{PI}	S_{PF}	S_{NT}	S_{NI}	S_{NF}
C	0.14	0.5	0.3	-0.1	-0.2	-0.3
D	0.2	0.5	0.3	-0.1	-0.2	-0.3

where *f* is defined by: *f(a)=w, f(b)=x, f(c)=y, f(d)=z.*

Here *S(H)= (0.34, 1.0, 0.6, -0.2, -0.4, -0.6) = S(K)*, but, by routine calculations, *H* is not a co-weak isomorphism to *K*.

Definition 3.19

Let *H* = *(X, E, R)* be a BSVNHG, then the degree of vertex x_i, which is denoted and defined by:

$$\deg(x_i) = (deg_{PT}(x_i), deg_{PI}(x_i), deg_{PF}(x_i), deg_{NT}(x_i), deg_{NI}(x_i), deg_{NF}(x_i) \tag{65}$$

where:

$$deg_{PT}(x_i) = \sum R_{PT}(x_1, x_2, \dots, x_r), \tag{66}$$

$$deg_{PI}(x_i) = \sum R_{PI}(x_1, x_2, \dots, x_r), \tag{67}$$

$$deg_{PF}(x_i) = \sum R_{PF}(x_1, x_2, \dots, x_r), \tag{68}$$

$$deg_{NT}(x_i) = \sum R_{NT}(x_1, x_2, \dots, x_r), \tag{69}$$

$$deg_{NI}(x_i) = \sum R_{NI}(x_1, x_2, \dots, x_r), \tag{70}$$

$$deg_{NF}(x_i) = \sum R_{NF}(x_1, x_2, \dots, x_r), \tag{71}$$

for $x_i \neq x_r$.

Theorem 3.20

If *H* and *K* be two isomorphic BSVNHGs, then the degree of their vertices are preserved.

Proof

Let *f: H → K* be an isomorphism between two BSVNHGs *H* and *K* with underlying sets *X* and *Y* respectively, then, by definition, we have:

$$\min[PT_{E_j}(x)] = \min[PT_{F_j}(f(x))], \tag{72}$$

$$\max[PI_{E_j}(x)] = \max[PI_{F_j}(f(x))], \tag{73}$$

$$\max[PF_{E_j}(x)] = \max[PF_{F_j}(f(x))], \tag{74}$$

$$\max[NT_{E_j}(x)] = \max[NT_{F_j}(f(x))], \tag{75}$$

$$\min[NI_{E_j}(x)] = \min[NI_{F_j}(f(x))], \tag{76}$$

$$\min[NF_{E_j}(x)] = \min[NF_{F_j}(f(x))], \tag{77}$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) = S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{78}$$

$$R_{PI}(x_1, x_2, \dots, x_r) = S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{79}$$

$$R_{PF}(x_1, x_2, \dots, x_r) = S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{80}$$

$$R_{NT}(x_1, x_2, \dots, x_r) = S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{81}$$

$$R_{NI}(x_1, x_2, \dots, x_r) = S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{82}$$

$$R_{NF}(x_1, x_2, \dots, x_r) = S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{83}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Consider:

$$deg_{PT}(x_i) = \sum R_{PT}(x_1, x_2, \dots, x_r)$$
$$= \sum S_{PT}(f(x_1), f(x_2), \dots, f(x_r))$$
$$= deg_{PT}(f(x_i)), \tag{84}$$

and similarly:

$$deg_{NT}(x_i) = deg_{NT}(f(x_i)), \tag{85}$$

$$deg_{PI}(x_i) = deg_{PI}(f(x_i)), deg_{PF}(x_i) = deg_{PF}(f(x_i)) \tag{86}$$

$$deg_{NI}(x_i) = deg_{NI}(f(x_i)), deg_{NF}(x_i) = deg_{NF}(f(x_i)) \tag{87}$$

Hence:

$$deg(x_i) = deg(f(x_i)). \tag{88}$$

Remark 3.21

The converse of the above theorem may not be true in general.

Example 3.22

Consider the two BSVNHGs *H = (X, E, R)* and *K = (Y, F, S)* with underlying sets *X = {a, b}* and *Y = {x, y}*, where *E = {A, B}*, *F = {C, D}*, *R* and *S* are defined by *Tables* given below:

H	A	B
a	(0.5, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.3, 0.5, 0.3, -0.1, -0.2, -0.3)
b	(0.25, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.2, 0.5, 0.3, -0.1, -0.2, -0.3)

K	C	D
x	(0.3, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.5,0.5,0.3, -0.1, -0.2, -0.3)
y	(0.2, 0.5, 0.3, -0.1, -0.2, -0.3)	(0.25, 0.5, 0.3, -0.1, -0.2, -0.3)

S	S_{PT}	S_{PI}	S_{PF}	S_{NT}	S_{NI}	S_{NF}
C	0.2	0.5	0.3	-0.1	-0.2	-0.3
D	0.25	0.5	0.3	-0.1	-0.2	-0.3

R	R_{PT}	R_{PI}	R_{PF}	R_{NT}	R_{NI}	R_{NF}
A	0.25	0.5	0.3	-0.1	-0.2	-0.3
B	0.2	0.5	0.3	-0.1	-0.2	-0.3

where *f* is defined by: *f(a)=x, f(b)=y,* here *deg(a) = (0.8, 1.0, 0.6, -0.2, -0.4, -0.6) = deg(x)* and *deg(b) = (0.45, 1.0, 0.6, -0.2, -0.4, -0.6) = deg(y).*

But *H* is not isomorphic to *K*, i.e. *H* is neither weak isomorphic, nor co-weak isomorphic to *K*.

Theorem 3.23

The isomorphism between BSVNHGs is an equivalence relation.

Proof

Let *H = (X, E, R), K = (Y, F, S)* and *M = (Z, G, W)* be BSVNHGs with underlying sets X, Y and *Z*, respectively:

Reflexive

Consider the map (identity map) $f : X \to X$ defined as follows: $f(x) = x$ for all $x \in X$, since the identity map is always bijective and satisfies the conditions:

$$\min[PT_{E_j}(x)] = \min[PT_{E_j}(f(x))], \tag{89}$$

$$\max[PI_{E_j}(x)] = \max[PI_{E_j}(f(x))], \tag{90}$$

$$\max[PF_{E_j}(x)] = \max[PF_{E_j}(f(x))], \tag{91}$$

$$\max[NT_{E_j}(x)] = \max\left[NT_{E_j}(f(x))\right], \tag{92}$$

$$\min[NI_{E_j}(x)] = \min[NI_{E_j}(f(x))], \tag{93}$$

$$\min[NF_{E_j}(x)] = \min[NF_{E_j}(f(x))], \tag{94}$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) = R_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{95}$$

$$R_{PI}(x_1, x_2, \dots, x_r) = R_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{96}$$

$$R_{PF}(x_1, x_2, \dots, x_r) = R_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{97}$$

$$R_{NT}(x_1, x_2, \dots, x_r) = R_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{98}$$

$$R_{NI}(x_1, x_2, \dots, x_r) = R_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{99}$$

$$R_{NF}(x_1, x_2, \dots, x_r) = R_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{100}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of *X*.

Hence *f* is an isomorphism of BSVNHG *H* to itself.

Symmetric

Let *f: X → Y* be an isomorphism of *H* and *K*, then *f* is a bijective mapping defined as *f(x) = y* for all $x \in X$.

Then, by definition:

$$\min[PT_{E_j}(x)] = \min[PT_{F_j}(f(x))], \quad (101)$$

$$\max[PI_{E_j}(x)] = \max[PI_{F_j}(f(x))], \quad (102)$$

$$\max[PF_{E_j}(x)] = \max[PF_{F_j}(f(x))], \quad (103)$$

$$\max[NT_{E_j}(x)] = \max[NT_{F_j}(f(x))], \quad (104)$$

$$\min[NI_{E_j}(x)] = \min[NI_{F_j}(f(x))], \quad (105)$$

$$\min[NF_{E_j}(x)] = \min[NF_{F_j}(f(x))], \quad (106)$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) = S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \quad (107)$$

$$R_{PI}(x_1, x_2, \dots, x_r) = S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \quad (108)$$

$$R_{PF}(x_1, x_2, \dots, x_r) = S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \quad (109)$$

$$R_{NT}(x_1, x_2, \dots, x_r) = S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \quad (101)$$

$$R_{NI}(x_1, x_2, \dots, x_r) = S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \quad (111)$$

$$R_{NF}(x_1, x_2, \dots, x_r) = S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \quad (112)$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of *X*.

Since *f* is bijective, then we have:

$$f^{-1}(y) = x \text{ for all } y \in Y.$$

Thus, we get:

$$\min[PT_{E_j}(f^{-1}(y))] = \min[PT_{F_j}(y)], \quad (113)$$

$$\max[PI_{E_j}(f^{-1}(y))] = \max[PI_{F_j}(y)], \quad (114)$$

$$\max[PF_{E_j}(f^{-1}(y))] = \max[PF_{F_j}(y)], \quad (115)$$

$$\max[NT_{E_j}(f^{-1}(y))] = \max[NT_{F_j}(y)], \quad (116)$$

$$\min[NI_{E_j}(f^{-1}(y))] = \min[NI_{F_j}(y)], \quad (117)$$

$$\min[NF_{E_j}(f^{-1}(y))] = \min[NF_{F_j}(y)], \quad (118)$$

for all $x \in X$.

$$R_{PT}(f^{-1}(y_1), f^{-1}(y_2), \dots, f^{-1}(y_r)) = S_{PT}(y_1, y_2, \dots, y_r), \quad (119)$$

$$R_{PI}(f^{-1}(y_1), f^{-1}(y_2), \dots, f^{-1}(y_r)) = S_{PI}(y_1, y_2, \dots, y_r), \quad (120)$$

$$R_{PF}(f^{-1}(y_1), f^{-1}(y_2), \dots, f^{-1}(y_r)) = S_{PF}(y_1, y_2, \dots, y_r), \quad (121)$$

$$R_{NT}\left(f^{-1}(y_1), f^{-1}(y_2), \dots, f^{-1}(y_r)\right) = S_{NT}(y_1, y_2, \dots, y_r), \tag{122}$$
$$R_{NI}\left(f^{-1}(y_1), f^{-1}(y_2), \dots, f^{-1}(y_r)\right) = S_{NI}(y_1, y_2, \dots, y_r), \tag{123}$$
$$R_{NF}\left(f^{-1}(y_1), f^{-1}(y_2), \dots, f^{-1}(y_r)\right) = S_{NF}(y_1, y_2, \dots, y_r), \tag{124}$$

for all $\{y_1, y_2, \dots, y_r\}$ subsets of *Y*.

Hence, we have a bijective map $f^{-1} : Y \rightarrow X$ which is an isomorphism from *K* to *H*.

Transitive

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two isomorphism of BSVNHGs of *H* onto *K* and *K* onto *M*, respectively. Then $g \ o \ f$ is bijective mapping from *X* to *Z*, where $g \ o \ f$ is defined as $(g \ o \ f)(x) = g\big(f(x)\big)$ for all $x \in X$.

Since f is an isomorphism, then by definition $f(x) = y$ for all $x \in X$, which satisfies the conditions:

$$\min[PT_{E_j}(x)] = \min[PT_{F_j}\big(f(x)\big)], \tag{125}$$
$$\max[PI_{E_j}(x)] = \max[PI_{F_j}\big(f(x)\big)], \tag{126}$$
$$\max[PF_{E_j}(x)] = \max[PF_{F_j}\big(f(x)\big)], \tag{127}$$
$$\max[NT_{E_j}(x)] = \max[NT_{F_j}\big(f(x)\big)], \tag{128}$$
$$\min[NI_{E_j}(x)] = \min\left[NI_{F_j}\big(f(x)\big)\right], \tag{129}$$
$$\min[NF_{E_j}(x)] = \min[NF_{F_j}\big(f(x)\big)], \tag{130}$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) = S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{131}$$
$$R_{PI}(x_1, x_2, \dots, x_r) = S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{132}$$
$$R_{PF}(x_1, x_2, \dots, x_r) = S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{133}$$
$$R_{NT}(x_1, x_2, \dots, x_r) = S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{134}$$
$$R_{NI}(x_1, x_2, \dots, x_r) = S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{135}$$
$$R_{NF}(x_1, x_2, \dots, x_r) = S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{136}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of *X*.

Since $g : Y \rightarrow Z$ is an isomorphism, then by definition $g(y) = z$ for all $y \in Y$ satisfying the conditions:

$$\min[PT_{F_j}(y)] = \min[PT_{G_j}\big(g(y)\big)], \tag{137}$$
$$\max[PI_{F_j}(y)] = \max[PI_{G_j}\big(g(y)\big)], \tag{138}$$

$$\max[PF_{F_j}(y)] = \max[PF_{G_j}(g(y))], \tag{139}$$

$$\max[NT_{F_j}(y)] = \max\left[NT_{G_j}(g(y))\right], \tag{140}$$

$$\min[NI_{F_j}(y)] = \min[NI_{G_j}(g(y))], \tag{141}$$

$$\min[NF_{F_j}(y)] = \min\left[NF_{G_j}(g(y))\right], \tag{142}$$

for all $x \in X$.

$$S_{PT}(y_1, y_2, \dots, y_r) = W_{PT}(g(y_1), g(y_2), \dots, g(y_r)), \tag{143}$$

$$S_{PI}(y_1, y_2, \dots, y_r) = W_{PI}(g(y_1), g(y_2), \dots, g(y_r)), \tag{144}$$

$$S_{PF}(y_1, y_2, \dots, y_r) = W_{PF}(g(y_1), g(y_2), \dots, g(y_r)), \tag{145}$$

$$S_{NT}(y_1, y_2, \dots, y_r) = W_{NT}(g(y_1), g(y_2), \dots, g(y_r)), \tag{146}$$

$$S_{NI}(y_1, y_2, \dots, y_r) = W_{NI}(g(y_1), g(y_2), \dots, g(y_r)), \tag{147}$$

$$S_{NF}(y_1, y_2, \dots, y_r) = W_{NF}(g(y_1), g(y_2), \dots, g(y_r)), \tag{148}$$

for all $\{y_1, y_2, \dots, y_r\}$ subsets of *Y*.

Thus, from above equations we conclude that:

$$\min[PT_{E_j}(x)] = \min[PT_{G_j}(g(f(x)))], \tag{149}$$

$$\max[PI_{E_j}(x)] = \max[PI_{G_j}(g(f(x)))], \tag{150}$$

$$\max[PF_{E_j}(x)] = \max[PF_{G_j}(g(f(x)))], \tag{151}$$

$$\max[NT_{E_j}(x)] = \max\left[NT_{G_j}\left(g(f(x))\right)\right], \tag{152}$$

$$\min[NI_{E_j}(x)] = \min[NI_{G_j}(g(f(x)))], \tag{153}$$

$$\min[NF_{E_j}(x)] = \min[NF_{G_j}(g(f(x)))], \tag{154}$$

for all $x \in X$.

$$R_{PT}(x_1, \dots, x_r) = W_{PT}(g(f(x_1)), \dots, g(f(x_r))), \tag{155}$$

$$R_{PI}(x_1, \dots, x_r) = W_{PI}(g(f(x_1)), \dots, g(f(x_r))), \tag{156}$$

$$R_{PF}(x_1, \dots, x_r) = W_{PF}(g(f(x_1)), \dots, g(f(x_r))), \tag{157}$$

$$R_{NT}(x_1, \dots, x_r) = W_{NT}(g(f(x_1)), \dots, g(f(x_r))), \tag{158}$$

$$R_{NI}(x_1, \dots, x_r) = W_{NI}(g(f(x_1)), \dots, g(f(x_r))), \tag{159}$$

$$R_{NF}(x_1, \dots, x_r) = W_{NF}(g(f(x_1)), \dots, g(f(x_r))), \tag{160}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of *X*.

Therefore $g\ o\ f$ is an isomorphism between *H* and *M*.

Hence, the isomorphism between BSVNHGs is an equivalence relation.

Theorem 3.24

The weak isomorphism between BSVNHGs satisfies the partial order relation.

Proof

Let $H = (X, E, R)$, $K = (Y, F, S)$ and $M = (Z, G, W)$ be BSVNHGs with underlying sets X, Y and Z, respectively:

Reflexive

Consider the map (identity map) $f : X \to X$ defined as follows: $f(x)=x$ for all $x \in X$, since the identity map is always bijective and satisfies the conditions:

$$\min[PT_{E_j}(x)] = \min[PT_{E_j}(f(x))], \tag{161}$$

$$\max[PI_{E_j}(x)] = \max[PI_{E_j}(f(x))], \tag{162}$$

$$\max[PF_{E_j}(x)] = \max[PF_{E_j}(f(x))], \tag{163}$$

$$\max[NT_{E_j}(x)] = \max[NT_{E_j}(f(x))], \tag{164}$$

$$\min[NI_{E_j}(x)] = \min[NI_{E_j}(f(x))], \tag{165}$$

$$\min[NF_{E_j}(x)] = \min[NF_{E_j}(f(x))], \tag{166}$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) \le R_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{167}$$

$$R_{PI}(x_1, x_2, \dots, x_r) \ge R_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{168}$$

$$R_{PF}(x_1, x_2, \dots, x_r) \ge R_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{169}$$

$$R_{NT}(x_1, x_2, \dots, x_r) \ge R_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{170}$$

$$R_{NI}(x_1, x_2, \dots, x_r) \le R_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{171}$$

$$R_{NF}(x_1, x_2, \dots, x_r) \le R_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{172}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Hence, f is a weak isomorphism of BSVNHG H to itself.

Anti-symmetric

Let f be a weak isomorphism between H onto K, and g be a weak isomorphic between K and H, that is $f: X \to Y$ is a bijective map defined by: $f(x) = y$ for all $x \in X$ satisfying the conditions:

$$\min[PT_{E_j}(x)] = \min[PT_{F_j}(f(x))], \tag{173}$$

$$\max[PI_{E_j}(x)] = \max[PI_{F_j}(f(x))], \tag{174}$$

$$\max[PF_{E_j}(x)] = \max[PF_{F_j}(f(x))], \tag{175}$$

$$\max[NT_{E_j}(x)] = \max[NT_{F_j}(f(x))], \tag{176}$$
$$\min[NI_{E_j}(x)] = \min[NI_{F_j}(f(x))], \tag{177}$$
$$\min[NF_{E_j}(x)] = \min[NF_{F_j}(f(x))], \tag{178}$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) = S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{179}$$
$$R_{PI}(x_1, x_2, \dots, x_r) = S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{180}$$
$$R_{PF}(x_1, x_2, \dots, x_r) = S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{181}$$
$$R_{NT}(x_1, x_2, \dots, x_r) = S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{182}$$
$$R_{NI}(x_1, x_2, \dots, x_r) = S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{183}$$
$$R_{NF}(x_1, x_2, \dots, x_r) = S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{184}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Since g is also bijective map $g(y) = x$ for all $y \in Y$ satisfying the conditions:

$$\min[PT_{F_j}(y)] = \min[PT_{E_j}(g(y))], \tag{185}$$
$$\max[PI_{F_j}(y)] = \max[PI_{E_j}(g(y))], \tag{186}$$
$$\max[PF_{F_j}(y)] = \max[PF_{E_j}(g(y))], \tag{187}$$
$$\max[NT_{F_j}(y)] = \max[NT_{E_j}(g(y))], \tag{188}$$
$$\min[NI_{F_j}(y)] = \min[NI_{E_j}(g(y))], \tag{189}$$
$$\min[NF_{F_j}(y)] = \min[NF_{E_j}(g(y))], \tag{190}$$

for all $y \in Y$.

$$R_{PT}(y, y_2, \dots, y_r) \le S_{PT}(g(y_1), g(y_2), \dots, g(y_r)), \tag{191}$$
$$R_{PI}(y_1, y_2, \dots, y_r) \ge S_{PI}(f(y_1), f(y_2), \dots, f(y_r)), \tag{192}$$
$$R_{PF}(y_1, y_2, \dots, y_r) \ge S_{PF}(f(y_1), f(y_2), \dots, f(y_r)), \tag{193}$$
$$R_{NT}(y, y_2, \dots, y_r) \ge S_{NT}(g(y_1), g(y_2), \dots, g(y_r)), \tag{194}$$
$$R_{NI}(y_1, y_2, \dots, y_r) \le S_{NI}(f(y_1), f(y_2), \dots, f(y_r)), \tag{195}$$
$$R_{NF}(y_1, y_2, \dots, y_r) \le S_{NF}(f(y_1), f(y_2), \dots, f(y_r)), \tag{196}$$

for all $\{y_1, y_2, \dots, y_r\}$ subsets of Y.

The above inequalities hold for finite sets *X* and *Y* only whenever *H* and *K* have same number of edges and corresponding edge have same weights, hence *H* is identical to *K*.

Transitive

Let $f: X \to Y$ and $g: Y \to Z$ be two weak isomorphism of BSVNHGs of *H* onto *K* and *K* onto *M*, respectively. Then $g\ o\ f$ is bijective mapping from *X* to *Z*, where $g\ o\ f$ is defined as $(g\ o\ f)(x) = g(f(x))$ for all $x \in X$.

Since f is a weak isomorphism, then by definition $f(x) = y$ for all $x \in X$ which satisfies the conditions:

$$\min[PT_{E_j}(x)] = \min[PT_{F_j}(f(x))], \tag{197}$$

$$\max[PI_{E_j}(x)] = \max[PI_{F_j}(f(x))], \tag{198}$$

$$\max[PF_{E_j}(x)] = \max[PF_{F_j}(f(x))], \tag{199}$$

$$\max[NT_{E_j}(x)] = \max[NT_{F_j}(f(x))], \tag{200}$$

$$\min[NI_{E_j}(x)] = \min[NI_{F_j}(f(x))], \tag{201}$$

$$\min[NF_{E_j}(x)] = \min[NF_{F_j}(f(x))], \tag{202}$$

for all $x \in X$.

$$R_{PT}(x_1, x_2, \dots, x_r) \leq S_{PT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{203}$$

$$R_{PI}(x_1, x_2, \dots, x_r) \geq S_{PI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{204}$$

$$R_{PF}(x_1, x_2, \dots, x_r) \geq S_{PF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{205}$$

$$R_{NT}(x_1, x_2, \dots, x_r) \geq S_{NT}(f(x_1), f(x_2), \dots, f(x_r)), \tag{206}$$

$$R_{NI}(x_1, x_2, \dots, x_r) \leq S_{NI}(f(x_1), f(x_2), \dots, f(x_r)), \tag{207}$$

$$R_{NF}(x_1, x_2, \dots, x_r) \leq S_{NF}(f(x_1), f(x_2), \dots, f(x_r)), \tag{208}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Since $g : Y \to Z$ is a weak isomorphism, then by definition $g(y) = z$ for all $y \in Y$, satisfying the conditions:

$$\min[PT_{F_j}(y)] = \min[PT_{G_j}(g(y))], \tag{209}$$

$$\max[PI_{F_j}(y)] = \max[PI_{G_j}(g(y))], \tag{210}$$

$$\max[PF_{F_j}(y)] = \max[PF_{G_j}(g(y))], \tag{211}$$

$$\max[NT_{F_j}(y)] = \max[NT_{G_j}(g(y))], \tag{212}$$

$$\min[NI_{F_j}(y)] = \min[NI_{G_j}(g(y))], \tag{213}$$

$$\min[NF_{F_j}(y)] = \min[NF_{G_j}(g(y))], \tag{214}$$

for all $x \in X$.

$$S_{PT}(y_1, y_2, \dots, y_r) \leq W_{PT}(g(y_1), g(y_2), \dots, g(y_r)), \tag{215}$$

$$S_{PI}(y_1, y_2, \dots, y_r) \geq W_{PI}(g(y_1), g(y_2), \dots, g(y_r)), \tag{216}$$

$$S_{PF}(y_1, y_2, \dots, y_r) \geq W_{PF}(g(y_1), g(y_2), \dots, g(y_r)), \tag{217}$$

$$S_{NT}(y_1, y_2, \dots, y_r) \geq W_{NT}(g(y_1), g(y_2), \dots, g(y_r)), \tag{218}$$

$$S_{NI}(y_1, y_2, \dots, y_r) \leq W_{NI}(g(y_1), g(y_2), \dots, g(y_r)), \tag{219}$$

$$S_{NF}(y_1, y_2, \dots, y_r) \leq W_{NF}(g(y_1), g(y_2), \dots, g(y_r)), \tag{220}$$

for all $\{y_1, y_2, \dots, y_r\}$ subsets of Y.

Thus, from above equations, we conclude that:

$$\min[PT_{E_j}(x)] = \min[PT_{G_j}(g(f(x)))], \tag{221}$$

$$\max[PI_{E_j}(x)] = \max[PI_{G_j}(g(f(x)))], \tag{222}$$

$$\max[PF_{E_j}(x)] = \max[PF_{G_j}(g(f(x)))], \tag{223}$$

$$\max[NT_{E_j}(x)] = \max[NT_{G_j}(g(f(x)))], \tag{224}$$

$$\min[NI_{E_j}(x)] = \min[NI_{G_j}(g(f(x)))], \tag{225}$$

$$\min[NF_{E_j}(x)] = \min[NF_{G_j}(g(f(x)))], \tag{226}$$

for all $x \in X$.

$$R_{PT}(x_1, \dots, x_r) \leq W_{PT}(g(f(x_1)), \dots, g(f(x_r))), \tag{227}$$

$$R_{PI}(x_1, \dots, x_r) \geq W_{PI}(g(f(x_1)), \dots, g(f(x_r))), \tag{228}$$

$$R_{PF}(x_1, \dots, x_r) \geq W_{PF}(g(f(x_1)), \dots, g(f(x_r))), \tag{229}$$

$$R_{NT}(x_1, \dots, x_r) \geq W_{NT}(g(f(x_1)), \dots, g(f(x_r))), \tag{230}$$

$$R_{NI}(x_1, \dots, x_r) \leq W_{NI}(g(f(x_1)), \dots, g(f(x_r))), \tag{231}$$

$$R_{NF}(x_1, \dots, x_r) \leq W_{NF}(g(f(x_1)), \dots, g(f(x_r))), \tag{232}$$

for all $\{x_1, x_2, \dots, x_r\}$ subsets of X.

Therefore $g \, o \, f$ is a weak isomorphism between H and M.

Hence, the weak isomorphism between BSVNHGs is a partial order relation.

4 Conclusion

The bipolar single valued neutrosophic hypergraph can be applied in various areas of engineering and computer science. In this paper, the isomorphism between BSVNHGs is proved to be an equivalence relation and the weak isomorphism is proved to be a partial order relation. Similarly, it can be proved that co-weak isomorphism in BSVNHGs is a partial order relation.

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Regular Bipolar Single Valued Neutrosophic Hypergraphs

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Muhammad Aslam Malik, Ali Hassan, Said Broumi, F. Smarandache (2016). Regular Bipolar Single Valued Neutrosophic Hypergraphs. *Neutrosophic Sets and Systems,* 13, 84-89

Abstract. In this paper, we define the regular and totally regular bipolar single valued neutrosophic hypergraphs, and discuss the order and size along with properties of regular and totally regular bipolar single valued neutrosophic hypergraphs. We extend work on completeness of bipolar single valued neutrosophic hypergraphs.

Keywords: bipolar single valued neutrosophic hypergraphs, regular bipolar single valued neutrosophic hypergraphs and totally regular bipolar single valued neutrosophic hyper graphs.

1 Introduction

The notion of neutrosophic sets (NSs) was proposed by Smarandache [8] as a generalization of the fuzzy sets [14], intuitionistic fuzzy sets [12], interval valued fuzzy set [11] and interval-valued intuitionistic fuzzy sets [13] theories. The neutrosophic set is a powerful mathematical tool for dealing with incomplete, indeterminate and inconsistent information in real world. The neutrosophic sets are characterized by a truth-membership function (*t*), an indeterminacy-membership function (*i*) and a falsity membership function (*f*) independently, which are within the real standard or nonstandard unit interval $]^{-}0\ ,\ 1^{+}[$. In order to conveniently use NS in real life applications, Wang et al. [9] introduced the concept of the single-valued neutrosophic set (SVNS), a subclass of the neutrosophic sets. The same authors [10] introduced the concept of the interval valued neutrosophic set (IVNS), which is more precise and flexible than the single valued neutrosophic set. The IVNS is a generalization of the single valued neutrosophic set, in which the three membership functions are independent and their value belong to the unit interval [0, 1]. More works on single valued neutrosophic sets, interval valued neutrosophic sets and their applications can be found on *http://fs.gallup.unm.edu/NSS/.*

Hypergraph is a graph in which an edge can connect more than two vertices, hypergraphs can be applied to analyse architecture structures and to represent system partitions, Mordesen J.N and P.S Nasir gave the definitions for fuzzy hypergraphs. Parvathy. R and M. G. Karunambigai's paper introduced the concepts of Intuitionistic fuzzy hypergraphs and analyse its components, Nagoor Gani. A and Sajith Begum. S defined degree, order and size in intuitionistic fuzzy graphs and extend the properties. Nagoor Gani. A and Latha. R introduced irregular fuzzy graphs and discussed some of its properties.

Regular intuitionistic fuzzy hypergraphs and totally regular intuitionistic fuzzy hypergraphs are introduced by Pradeepa. I and Vimala. S in [0]. In this paper we extend regularity and totally regularity on bipolar single valued neutrosophic hypergraphs.

2 Preliminaries

In this section we discuss the basic concept on neutrosophic set and neutrosophic hyper graphs.

Definition 2.1 Let X be the space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth membership function $T_A(x)$, indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$. For each point x $\in$X; $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$.

Definition 2.2 Let X be a space of points (objects) with generic elements in X denoted by x. A bipolar single valued neutrosophic set A (BSVNS A) is characterized by positive truth membership function $PT_A(x)$, positive indeterminacy membership function $PI_A(x)$ and a positive falsity membership function $PF_A(x)$ and negative truth membership function $NT_A(x)$, negative indeterminacy membership function $NI_A(x)$ and a negative falsity membership function $NF_A(x)$.

For each point $x \in$ X; $PT_A(x)$, $PI_A(x)$, $PF_A(x) \in [0, 1]$ and $NT_A(x)$, $NI_A(x)$, $NF_A(x) \in [-1, 0]$.

Definition 2.3 Let A be a BSVNS on X then support of A is denoted and defined by

$Supp(A) = \{x : x \in X, PT_A(x) > 0, PI_A(x) > 0, PF_A(x) > 0, NT_A(x) < 0, NI_A(x) < 0, NF_A(x) < 0\}$.

Definition 2.4 A hyper graph is an ordered pair $H = (X, E)$, where

(1) $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of vertices.

(2) $E = \{E_1, E_2, \ldots, E_m\}$ *be* a family of subsets of

X. (3) E_j for $j= 1,2,3,\ldots,m$ and $\bigcup_j(E_j)= X$.

The set X is called set of vertices and E is the set of edges (or hyper edges).

Definition 2.5 A bipolar single valued neutrosophic hypergraph is an ordered pair $H = (X, E)$, where

(1) $X = \{x_1, x_2, \ldots, x_m\}$ be a finite set of vertices.

(2) $E = \{E_1, E_2, \ldots, E_m\}$ be a family of BSVNSs of X.

(3) $E_j \neq O = (0, 0, 0)$ for $j= 1,2,3,\ldots,m$ and $\bigcup_j Supp(E_j)= X$.

The set X is called set of vertices and E is the set of BSVN-edges (or BSVN-hyper edges).

Proposition 2.6 The bipolar single valued neutrosophic hyper graph is the generalization of fuzzy hyper graphs, intuitionistic fuzzy hyper graphs, bipolar fuzzy hyper graphs and single valued neutrosophic hypergraphs.

3 Regular and totally regular BSVNHGs

Definition 3.1 The open neighbourhood of a vertex x in bipolar single valued neutrosophic hypergraphs (BSVNHGs) is the set of adjacent vertices of x, excluding that vertex and is denoted by $N(x)$.

Definition 3.2 The closed neighbourhood of a vertex x in bipolar single valued neutrosophic hypergraphs (BSVNHGs) is the set of adjacent vertices of x, including that vertex and is denoted by $N[x]$.

Example 3.3 Consider a bipolar single valued neutrosophic hypergraphs $H = (X, E)$ where, $X = \{a, b, c, d, e\}$ and $E = \{P, Q, R, S\}$, which is defined by

P = {(a, 0.1, 0.2, 0.3, -0.4, -0.6 -0.8), (b, 0.4, 0.5, 0.6, -0.4, -0.6 -0.8)}

Q = {(c, 0.1, 0.2, 0.3, -0.4, -0.4 -0.9), (d, 0.4, .5, 0.6, -0.3, -0.5 -0.6), (e, 0.7, 0.8, 0.9, -0.7, -0.9, -0.2)}

R = {(b, 0.1, 0.2, 0.3, -0.2, -0.5, -0.8), (c, 0.4, 0.5, 0.6, -0.9, -0.7 -0.4)}

S = {(a, 0.1, 0.2, 0.3, -0.7, -0.6, -0.9), (d, 0.9, 0.7, 0.6, -0.4, -0.7, -0.9)}

Then the open neighbourhood of a vertex a is the b and d, and closed neighbourhood of a vertex b is b, a and c.

Definition 3.4 Let $H = (X, E)$ be a BSVNHG, the open neighbourhood degree of a vertex x, which is denoted and defined by

$deg(x) = (deg_{PT}(x), deg_{PI}(x), deg_{PF}(x), deg_{NT}(x), deg_{NI}(x), deg_{NF}(x))$

where

$$deg_{PT}(x) = \sum_{x \in N(x)} PT_E(x)$$

$$deg_{PI}(x) = \sum_{x \in N(x)} PI_E(x)$$

$$deg_{PF}(x) = \sum_{x \in N(x)} PF_E(x)$$

$$deg_{NT}(x) = \sum_{x \in N(x)} NT_E(x)$$

$$deg_{NI}(x) = \sum_{x \in N(x)} NI_E(x)$$

$$deg_{NF}(x) = \sum_{x \in N(x)} NF_E(x)$$

Example 3.5 Consider a bipolar single valued neutrosophic hypergraphs $H = (X, E)$ where, $X = \{a, b, c, d, e\}$ and $E = \{P, Q, R, S\}$, which are defined by

P = {(a, .1, .2, .3, -0.1, -0.2, -0.3), (b, .4, .5, .6, -0.1, -0.2, -0.3)}

Q = {(c, .1, .2, .3, -0.1, -0.2, -0.3), (d, .4, .5, .6, -0.1, -0.2, -0.3), (e, .7, .8, .9, -0.1, -0.2, -0.3)}

R = {(b, .1, .2, .3, -0.1, -0.2, -0.3), (c, .4, .5, .6, -0.1, -0.2, -0.3)}

S = {(a, .1, .2, .3, -0.1, -0.2, -0.3), (d, .4, .5, .6, -0.1, -0.2, -0.3)}

Then the open neighbourhood of a vertex a contain b and d and therefore open neighbourhood degree of a vertex a is (.8, 1, 1.2, -0.2, -0.4, -0.6).

Definition 3.6 Let $H = (X, E)$ be a BSVNHG, the closed neighbourhood degree of a vertex x is denoted and defined by

$deg[x] = (deg_{PT}[x], deg_{PI}[x], deg_{PF}[x], deg_{NT}[x], deg_{NI}[x], deg_{NF}[x])$

which are defined by

$$deg_{PT}[x] = deg_{PT}(x) + PT_E(x)$$
$$deg_{PI}[x] = deg_{PI}(x) + PI_E(x)$$
$$deg_{PF}[x] = deg_{PF}(x) + PF_E(x)$$
$$deg_{NT}[x] = deg_{NT}(x) + NT_E(x)$$
$$deg_{NI}[x] = deg_{NI}(x) + NI_E(x)$$
$$deg_{NF}[x] = deg_{NF}(x) + NF_E(x)$$

Example 3.7 Consider a bipolar single valued neutrosophic hypergraphs $H = (X, E)$ where, $X = \{a, b, c, d, e\}$ and $E = \{P, Q, R, S\}$, which is defined by

P = {(a, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (b, 0.4, 0.5, 0.6, -0.1, -0.2, -0.3)}

Q = {(c, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (d, 0.4, 0.5, 0.6, -0.1, -0.2, -0.3), (e, 0.7, 0.8, 0.9, -0.1, -0.2, -0.3)}

R = {(b, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (c, 0.4, 0.5, 0.6, -0.1, -0.2, -0.3)}

S = {(a, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (d, 0.4, 0.5, 0.6, -0.1, -0.2, -0.3)}

The closed neighbourhood of a vertex a contain a, b and d, hence the closed neighbourhood degree of a vertex $\underline{a}$ is (0.9, .1.2, 1.5, -0.3, -0.6, -0.9).

Definition 3.8 Let $H = (X, E)$ be a BSVNHG, then H is said to be an n-regular BSVNHG if all the vertices have the same open neighbourhood degree $n = (n_1, n_2, n_3, n_4, n_5, n_6)$

Definition 3.9 Let $H = (X, E)$ be a BSVNHG, then H is said to be m-totally regular BSVNHG if all the vertices have the same closed neighbourhood degree $m = (m_1, m_2, m_3, m_4, m_5, m_6)$.

Proposition 3.10 A regular BSVNHG is the generalization of regular fuzzy hypergraphs, regular intuitionistic fuzzy hypergraphs, regular bipolar fuzzy hypergraphs and regular single valued neutrosophic hypergraphs.

Proposition 3.11 A totally regular BSVNHG is the generali-zation of totally regular fuzzy hypergraphs, totally regular intuitionistic fuzzy hypergraphs, totally regular bipolar fuzzy hypergraphs and totally regular single valued neu-trosophic hypergraphs.

Example 3.12 Consider a bipolar single valued neutrosophic hypergraphs $H = (X, E)$ where, $X = \{a, b, c, d\}$ and $E = \{P, Q, R, S\}$ which is defined by

P = {(a, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (b, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

Q = {(b, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (c, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

R = {(c, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (d, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

S = {(d, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (a, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

Here the open neighbourhood degree of every vertex is *(1.6, 0.4, 0.6, -0.2, -0.4, -0.6)* hence H is regular BSVNHG and closed neighbourhood degree of every vertex is *(2.4, 0.6, 0.9, -0.3, -0.6, -0.9)*, Hence H is both regular and totally regular BSVNHG.

Theorem 3.13 Let $H = (X, E)$ be a BSVNHG which is both regular and totally regular BSVNHG then E is constant.

Proof: Suppose H is an n-regular and m-totally regular BSVNHG. Then $deg(x) = n = (n_1, n_2, n_3, n_4, n_5, n_6)$ and $deg[x] = m = (m_1, m_2, m_3, m_4, m_5, m_6)$ $\forall x \in E_i$. Consider $deg[x] = m$. Hence by definition, $deg(x) + E_i(x) = m$ this implies $E_i(x) = m - n$ for all $x \in E_i$. Hence E is constant.

Remark 3.14 The converse of above theorem need not to be true in general.

Example 3.15 Consider a bipolar single valued neutrosophic hypergraphs $H = (X, E)$ where, $X = \{a, b, c, d\}$ and $E = \{P, Q, R, S\}$, which is defined by

P = {(a, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (b, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

Q = {(b, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (d, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

R = {(c, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (d, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

S = {(d, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3), (a, 0.8, 0.2, 0.3, -0.1, -0.2, -0.3)}

Here E is constant but *deg(a) = (1.6, 0.4, 0.6, -0.2, -0.4, -0.6)* and *deg(d) = (2.4, 0.6, 0.9, -0.3, -0.6, -0.9)* i.e deg(a) and deg(d) are not equals hence H is not regular BSVNHG. Next *deg[a] = (2.4, 0.6, 0.9, -0.3, -0.6, -0.9)* and *deg[d]= (3.2, 0.8, 1.2, -.4, -0.8, -1.2)*, hence deg[a] and deg[d] are not equals hence H is not totally regular BSVNHG, Thus that H is neither regular and nor totally regular BSVNHG.

Theorem 3.16 Let $H = (X, E)$ be a BSVNHG then E is constant on X if and only if following are equivalent,

(1) H is regular BSVNHG.

(2) H is totally regular BSVNHG.

Proof: Suppose $H = (X, E)$ be a BSVNHG and E is constant in H, that is $E_i(x) = c = (c, c, c, c, c, c)$ $\forall x \in E$.

Suppose H is n-regular BSVNHG, then $deg(x) = n = (n_1, n_2, n_3, n_4, n_5, n_6)$ $\forall x \in E_i$, consider $deg[x] = deg(x) + E_i(x) = n + c$ $\forall x \in E_i$, hence H is totally regular BSVNHG.

Next suppose that H is m-totally regular BSVNHG, then $deg[x] = m = (m_1, m_2, m_3, m_4, m_5, m_6)$ for all $x \in E_i$, that is $deg(x) + E_i(x) = m$ $\forall x \in E_i$, this implies that $deg(x) = m - c$ $\forall x \in E_i$. Thus H is regular BSVNHG, thus *(1)* and *(2)* are equivalent.

Conversely: Assume that *(1)* and *(2)* are equivalent. That is H is regular BSVNHG if and only if H is totally regular BSVNHG. Suppose contrary E is not constant, that is E_i(x) and E_i(y) not equals for some x and y in X. Let $H = (X, E)$ be n-regular BSVNHG, then $deg(x) = n = (n_1, n_2, n_3, n_4, n_5, n_6)$ for all $x \in E_i$. Consider

$$deg[x] = deg(x) + E_i(x) = n + E_i(x)$$

$$deg[y] = deg(y) + E_i(y) = n + E_i(y)$$

Since E_i(x) and E_i(y) are not equals for some x and y in X. Hence $deg[x]$ and $deg[y]$ are not equals, thus H is not totally regular BSVNHG, which contradict to our assumption.

Next let H be totally regular BSVNHG, then $deg[x] = deg[y]$, that is $deg(x) + E_i(x) = deg(y) + E_i(y)$ and $deg(x) - deg(y) = E_i(y) - E_i(x)$, since RHS of last equation is nonzero, hence LHS of above equation is also nonzero, thus $deg(x)$ and $deg(y)$ are not equals, so H is not regular BSVNHG, which is again contradict to our assumption, thus our supposition was wrong, hence E must be constant, this completes the proof.

Definition 3.17 Let $H = (X, E)$ be a regular BSVNHG, then the order of BSVNHG H is denoted and defined by

$O(H) = (p, q, r, s, t, u)$, where $p = \sum_{x \in X} PT_{E_i}(x)$, $q = \sum_{x \in X} PI_{E_i}(x)$, $r = \sum_{x \in X} PF_{E_i}(x)$, $s = \sum_{x \in X} NT_{E_i}(x)$, $t = \sum_{x \in X} NI_{E_i}(x)$, $u = \sum_{x \in X} NF_{E_i}(x)$. For every $x \in X$ and size of regular BSVNHG is denoted and defined by $S(H) = \sum_{i=1}^{n}(S_{E_i})$, where $S(E_i) = (a, b, c, d, e, f)$ which is defined by

$$a = \sum_{x \in E_i} PT_{E_i}(x)$$

$$b = \sum_{x \in E_i} PI_{E_i}(x)$$

$$c = \sum_{x \in E_i} PF_{E_i}(x)$$

$$d = \sum_{x \in E_i} NT_{E_i}(x)$$

$$e = \sum_{x \in E_i} NI_{E_i}(x)$$

$$f = \sum_{x \in E_i} NF_{E_i}(x)$$

Example 3.18 Consider a bipolar single valued neutrosophic hypergraphs $H = (X, E)$ where, $X = \{a, b, c, d\}$ and

$E = \{P, Q, R, S\}$, which is defined by

$P = \{(a, .8, .2, .3, -.1, -.2, -.3), (b, .8, .2, .3, -.1, -.2, -.3)\}$

$Q = \{(b, .8, .2, .3, -.1, -.2, -.3), (c, .8, .2, .3, -.1, -.2, -.3)\}$

$R = \{(c, .8, .2, .3, -.1, -.2, -.3), (d, .8, .2, .3, -.1, -.2, -.3)\}$

$S = \{(d, .8, .2, .3, -.1, -.2, -.3), (a, .8, .2, .3, -.1, -.2, -.3)\}$

Here order and size of H are given (3.2, .8, 1.2, -.4, -.8, -1.2) and (6.4, 1.6, 2.4, -.8, -1.6, -2.4) respectively.

Proposition 3.19 The size of an n-regular BSVNHG $H = (H, E)$ is $nk/2$, where $|X| = k$.

Proposition 3.20 If $H = (X, E)$ be m-totally regular BSVNHG then $2S(H) + O(H) = mk$, where $|X| = k$.

Corollary 3.21 Let $H = (X, E)$ be a n-regular and m-totally regular BSVNHG then $O(H) = k(m - n)$, where $|X| = k$.

Proposition 3.22 The dual of n-regular and m-totally regular BSVNHG $H = (X, E)$ is again an n-regular and m-totally regular BSVNHG.

Definition 3.23 A bipolar single valued neutrosophic hypergraph (BSVNHG) is said to be complete BSVNHG if for every x in X, $N(x) = \{x: x \text{ in } X-\{x\}\}$, that is $N(x)$ contains all remaining vertices of X except x.

Example 3.24 Consider a bipolar single valued neutrosophic hypergraphs $H = (X, E)$, where $X = \{a, b, c, d\}$ and $E = \{P, Q, R\}$, which is defined by

$P = \{(a, 0.4, 0.6, 0.3, -0.5, -0.2, -0.3), (c, 0.8, 0.2, 0.3, -0.1, -0.8, -0.3)\}$

$Q = \{(a, 0.8, 0.8, 0.3, -0.1, -0.6, -0.3), (b, 0.8, 0.2, 0.1, -0.1, -0.2, -0.3), (d, 0.8, 0.2, 0.1, -0.1, -0.9, -0.3)\}$

$R = \{(c, 0.4, 0.9, 0.9, -0.1, -0.2, -0.3), (d, 0.7, 0.2, 0.1, -0.5, -0.9, -0.3), (b,$

0.4, 0.2, 0.1, -0.8, -0.4, -0.2)}. Here *N(a) = {b, c, d}* , *N(b) = {a, c, d}*, *N(c) = {a, b, d}*, *N(d) = {a, b, c}* hence *H* is complete BSVNHG.

Remark 3.25 In a complete BSVNHG *H = (X, E),* the cardi-nality of *N(x)* is same for every vertex.

Theorem 3.26 Every complete BSVNHG *H = (X, E)* is both regular and totally regular if *E* is constant in *H*.

Proof: Let *H = (X, E)* be complete BSVNHG, suppose *E* is constant in *H*, so that $E_i(x) = c = (c_1, c_2, c_3, c_4, c_5, c_6)$ $\forall x \in E_i$, since BSVNHG is complete, then by definition for every vertex *x* in *X*, *N(x) = {x: x in X-{x}}*, the open neighbourhood degree of every vertex is same. That is $deg(x) = n = (n_1, n_2, n_3, n_4, n_5, n_6)$ $\forall x \in E_i$. Hence complete BSVNHG is regular BSVNHG. Also, $deg[x] = deg(x) + E_i(x) = n + c$ $\forall x \in E_i$. Hence *H* is totally regular BSVNHG.

Remark 3.27 Every complete BSVNHG is totally regular even if *E* is not constant.

Definition 3.28 A BSVNHG is said to be *k*-uniform if all the hyper edges have same cardinality.

Example 3.29 Consider a bipolar single valued neutrosophic hypergraphs *H = (X, E),* where *X = {a, b, c, d}* and *E = {P, Q, R},* which is defined by

P = {(a, 0.8, 0.4, 0.2,-0.4, -0.6, -0.2), (b, 0.7, 0.5, 0.3, -0.7, -0.1, -0.2)}

Q = {(b, 0.9, 0.4, 0.8, -0.3, -0.2, -0.9), (c, 0.8, 0.4, 0.2, -0.4, -0.3, -0.7)}

R = {(c, 0.8, 0.6, 0.4, -0.3, -0.7, -0.2), (d, 0.8, 0.9, 0.5, -0.4, -0.8, -0.9)}

4 Conclusion

Theoretical concepts of graphs and hypergraphs are utilized by computer science applications. Single valued neutrosophic hypergraphs are more flexible than fuzzy hypergraphs and intuitionistic fuzzy hypergraphs. The concepts of single valued neutrosophic hypergraphs can be applied in various areas of engineering and computer science. In this paper, we defined the regular and totally regular bipolar single valued neutrosophic hyper graphs. We plan to extend our research work to irregular and totally irregular on bipolar single valued neutrosophic hyper graphs.

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Neutrosophic Soluble Groups, Neutrosophic Nilpotent Groups and Their Properties

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Mumtaz Ali, Florentin Smarandache (2016). Neutrosophic Soluble Groups, Neutrosophic Nilpotent Groups and Their Properties. Annual Symposium of the Institute of Solid Mechanics, SISOM 2015, Robotics and Mechatronics. Special Session and Work Shop on VIPRO Platform and RABOR Rescue Robots, Romanian Academy, Bucharest, 21-22 May 2015; *Acta Electrotehnica* 57(1/2), 153-159

Abstract

The theory of soluble groups and nilpotent groups is old and hence a generalized on. In this paper, we introduced neutrosophic soluble groups and neutrosophic nilpotent groups which have some kind of indeterminacy. These notions are generalized to the classic notions of soluble groups and nilpotent groups. We also derive some new type of series which derived some new notions of soluble groups and nilpotent groups. They are mixed neutrosophic soluble groups and mixed neutrosophic nilpotent groups as well as strong neutrosophic soluble groups and strong neutrosophic nilpotent groups.

Key words: Soluble group, nilpotent group, neutrosophic group, neutrosophic soluble group, neutrosophic nilpotent group.

1. Introduction

Smarandache [15] in 1980 introduced neutrosophy which is a branch of philosophy that studies the origin and scope of neutralities and their interaction with ideational spectra. The concept of neutrosophic set and logic came into being due to neutrosophy, where each proposition is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Neutrosophic sets are the generalization to all other traditional theories of logics. This mathematical framework is used to handle problems with uncertaint, imprecise, indeterminate, incomplete and inconsistent etc. Kandasamy and Smarandache apply the concept of indeterminacy factor in algebraic structures by inserting the indeterminate element I in the algebraic notions with respect to the opeartaion *. This phenomenon generates the corresponding neutrosophic algebraic notion. They called that indeterminacy element I, a neutrosophic element which is unknown in some sense. This approach a relatively large structure which contain the old classic alegebraic structure. In this way, they studied several neutrosophic algebraic structures in [9,10,11,12]. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on. Mumtaz et al.[1] introduced neutrosophic left almost semigroup in short neutrosophic LA-semigroup and their generalization [2]. Further, Mumtaz et al. studied neutrosophic LA-semigroup rings and their generalization.

Groups [5,7] are the most rich algebraic structures in the theory of algebra.They shared common features to all the algebraic structures. Soluble groups [13,14] are important notions in the theory of groups as they are studied on the basis of some kind of series structures of the subgroups of the group. A soluble group is constructed by using abelian groups through the extension. A nilpotent group [13] is one whose which has finite length of central series. Thus a nilplotent group is also a soluble group. It is a special type of soluble group because every soluble group has a abelian series. A huge amount of literature on soluble groups and nilpotent groups can be found in [6,8,16,17,18].

In this paper, we introduced neutrosophic soluble groups and neutrosophic nilpotent groups and investigate some of their propertied. The organization of this paper is as follows: In section 1, we give a brief introduction of neutrosophic algebraic structures in terms of I and soluble groups and nilpotent groups. In the next section 2, some basic concept have been studied which we have used in the rest of the paper. In section 3, we introduced neutrosophic soluble groups and investigate some of their basic properties. In section 4, the notions of neutrosophic nilpotent groups are introduced and studied their basic properties. Conclusion is placed in section 5.

2. Fundamental Concepts

Definition 2.1: Let $(G,*)$ be a group . Then the neutrosophic group is generated by G and I under $*$ denoted by $N(G)=\{\langle G\cup I\rangle,*\}$. The identity element is represented by e and $\{e\}$ represents the trivial subgroup of G. I is called the indeterminate element with the property $I^2=I$. For an integer n , $n+I$ and nI are neutrosophic elements and $0.I=0$. I^{-1} , the inverse of I is not defined and hence does not exist.

Definition 2.2: Let $N(G)$ be a neutrosophic group and H be a neutrosophic subgroup of $N(G)$. Then H is a neutrosophic normal subgroup of $N(G)$ if $xH=Hx$ for all $x\in N(G)$.

Definition 2.3: Let $N(G)$ be a neutrosophic group. Then center of $N(G)$ is denoted by $C(N(G))$ and defined as $C(N(G))=\{x\in N(G): ax=xa$ for all $a\in N(G)\}$.

Definition 2.4: Let G be a group and $H_1,H_2,...,H_n$ be the subgroups of G. Then

$$\{e\}=H_0\leq H_1\leq H_2\leq ...\leq H_{n-1}\leq H_n=G$$

is called subgroup series of G .

Definition 2.5: Let G be a group and e be the identity element. Then

$$\{e\}=H_0\lhd H_1\lhd H_2\lhd ...\lhd H_{n-1}\lhd H_n=G$$

is called subnormal series. That is H_j is normal subgroup of H_{j+1} for all j .

Definition 2.6: Let

$$\{e\}=H_0\lhd H_1\lhd H_2\lhd ...\lhd H_{n-1}\lhd H_n=G$$

be a subnormal series of G. If each H_j is normal in G for all j, then this subnormal series is called normal series.

Definition 2.7: A normal series

$$\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_{n-1} \lhd H_n = G$$

is called an abelian series if the factor group H_{j+1}/H_j is an abelian group.

Definition 2.8: A group G is called a soluble group if G has an abelian series.

Definition 2.9: Let G be a soluble group. Then length of the shortest abelian series of G is called derived length.

Definition 2.10: Let G be a group. The series

$$\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_{n-1} \lhd H_n = G$$

is called central series if $H_{j+1}/H_j \subseteq Z\left(G/H_j\right)$ for all j.

Definition 2.11: A group G is called a nilpotent group if G has a central series.

3. Neutrosophic Soluble Groups

Definition 3.1: Let $N(G) = \langle G \cup I \rangle$ be a neutrosophic group and let $H_1, H_2, \ldots, H_n$ be the neutrosophic subgroups of $N(G)$. Then a neutrosophic subgroup series is a chain of neutrosophic subgroups such that

$$\{e\} = H_0 \leq H_1 \leq H_2 \leq \ldots \leq H_{n-1} \leq H_n = N(G).$$

Example 3.2: Let $N(G) = \langle \mathbb{Z} \cup I \rangle$ be a neutrosophic group of integers. Then the following are the neutrosophic subgroups series of the group $N(G)$. Here the identity element is 0 and {0} is the trivial subgroup of Z.

$$\{0\} \leq 4\mathbb{Z} \leq 2\mathbb{Z} \leq \langle 2\mathbb{Z} \cup I \rangle \leq \langle \mathbb{Z} \cup I \rangle,$$

$$\{0\} \leq \langle 4\mathbb{Z} \cup I \rangle \leq \langle 2\mathbb{Z} \cup I \rangle \leq \langle \mathbb{Z} \cup I \rangle,$$

$$\{0\} \leq 4\mathbb{Z} \leq 2\mathbb{Z} \leq \mathbb{Z} \leq \langle \mathbb{Z} \cup I \rangle.$$

Definition 3.3: Let $\{e\} = H_0 \leq H_1 \leq H_2 \leq \ldots \leq H_{n-1} \leq H_n = N(G)$ be a neutrosophic subgroup series of the neutrosophic group $N(G)$. Then this series of subgroups is called a strong neutrosophic subgroup series if each H_j is a neutrosophic subgroup of $N(G)$ for all j.

Example 3.4: Let $N(G) = \langle \mathbb{Z} \cup I \rangle$ be a neutrosophic group. Then the following neutrosophic subgroup series of $N(G)$ is a strong neutrosophic subgroup series:

$$\{0\} \leq \langle 4\mathbb{Z} \cup I \rangle \leq \langle 2\mathbb{Z} \cup I \rangle \leq \langle \mathbb{Z} \cup I \rangle.$$

Theorem 3.5: Every strong neutrosophic subgroup series is trivially a neutrosophic subgroup series but the converse is not true in general.

Definition 3.6: If some H_j's are neutrosophic subgroups and some H_k's are just subgroups of $N(G)$. Then that neutrosophic subgroups series is called mixed neutrosophic subgroup series.

Example 3.7: Let $N(G) = \langle \mathbb{Z} \cup I \rangle$ be a neutrosophic group. Then the following neutrosophic subgroup series of $N(G)$ is a mixed neutrosophic subgroup series:

$$\{0\} \leq 4\mathbb{Z} \leq 2\mathbb{Z} \leq \langle 2\mathbb{Z} \cup I \rangle \leq \langle \mathbb{Z} \cup I \rangle.$$

Theorem 3.8: Every mixed neutrosophic subgroup series is trivially a neutrosophic subgroup series but the converse is not true in general.

Definition 3.9: If H_j's in $\{e\} = H_0 \leq H_1 \leq H_2 \leq \ldots \leq H_{n-1} \leq H_n = N(G)$ are only subgroups of the neutrosophic group $N(G)$, then that series is termed as subgroup series of the neutrosophic group $N(G)$.

Example 3.10: Let $N(G) = \langle \mathbb{Z} \cup I \rangle$ be a neutrosophic group. Then the following neutrosophic subgroup series of $N(G)$ is just a subgroup series:

$$\{0\} \leq 4\mathbb{Z} \leq 2\mathbb{Z} \leq \mathbb{Z} \leq \langle \mathbb{Z} \cup I \rangle.$$

Theorem 3.11: A neutrosophic group $N(G)$ has all three type of neutrosophic subgroups series.

Theorem 3.12: Every subgroup series of the group G is also a subgroup series of the neutrosophic group $N(G)$.

Proof: Since G is always contained in $N(G)$. This directly followed the proof.

Definition 3.13: Let $\{e\} = H_0 \leq H_1 \leq H_2 \leq \ldots \leq H_{n-1} \leq H_n = N(G)$ be a neutrosophic subgroup series of the neutrosophic group $N(G)$. If

$$\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_{n-1} \lhd H_n = N(G) \ldots\ldots\ldots\ldots\ldots (1)$$

That is each H_j is normal in H_{j+1}. Then (1) is called a neutrosophic subnormal series of the neutrosophic group $N(G)$.

Example 3.14: Let $N(G)=\langle A_4 \cup I\rangle$ be a neutrosophic group, where A_4 is the alternating subgroup of the permutation group S_4. Then the following is the neutrosophic subnormal series of the group $N(G)$.

$$\{e\} \lhd C_2 \lhd V_4 \lhd \langle V_4 \cup I\rangle \lhd \langle A_4 \cup I\rangle.$$

Definition 3.15: A neutrosophic subnormal series is called strong neutrosophic subnormal series if all H_j's are neutrosophic normal subgroups in (1) for all j.

Example 3.16: Let $N(G)=\langle \mathbb{Z} \cup I\rangle$ be a neutrosophic group of integers. Then the following is a strong neutrosophic subnormal series of $N(G)$.

$$\{0\} \lhd \langle 4\mathbb{Z} \cup I\rangle \lhd \langle 2\mathbb{Z} \cup I\rangle \lhd \langle \mathbb{Z} \cup I\rangle.$$

Theorem 3.17: Every strong neutrosophic subnormal series is trivially a neutrosophic subnormal series but the converse is not true in general.

Definition 3.18: A neutrosophic subnormal series is called mixed neutrosophic subnormal series if some H_j's are neutrosophic normal subgroups in (1) while some H_k's are just normal subgroups in (1) for some j and k.

Example 3.19: Let $N(G)=\langle \mathbb{Z} \cup I\rangle$ be a neutrosophic group of integers. Then the following is a mixed neutrosophic subnormal series of $N(G)$.

$$\{0\} \lhd 4\mathbb{Z} \lhd 2\mathbb{Z} \lhd \langle 2\mathbb{Z} \cup I\rangle \lhd \langle \mathbb{Z} \cup I\rangle.$$

Theorem 3.20: Every mixed neutrosophic subnormal series is trivially a neutrosophic subnormal series but the converse is not true in general.

Definition 3.21: A neutrosophic subnormal series is called subnormal series if all H_j's are only normal subgroups in (1) for all j.

Theorem 3.22: Every subnormal series of the group G is also a subnormal series of the neutrosophic group $N(G)$.

Definition 3.23: If H_j are all normal neutrosophic subgroups in $N(G)$. Then the neutrosophic subnormal series (1) is called neutrosophic normal series.

Theorem 3.24: Every neutrosophic normal series is a neutrosophic subnormal series but the converse is not true.

For the converse, see the following Example.

Example 3.25: Let $N(G)=\langle A_4 \cup I\rangle$ be a neutrosophic group, where A_4 is the alternating subgroup of the permutation group S_4. Then the following are the neutrosophic subnormal series of the group $N(G)$.

$$\{e\} \lhd C_2 \lhd V_4 \lhd \langle V_4 \cup I \rangle \lhd \langle A_4 \cup I \rangle .$$

This series is not neutrosophic normal series as C_2 (cyclic group of order 2) is not normal in V_4 (Klein four group).

Similarly we can define strong neutrosophic normal series, mixed neutrosophic normal series and normal series respectively on the same lines of the neutrosophic group $N(G)$.

Definition 3.26: The neutrosophic normal series

$$\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_{n-1} \lhd H_n = N(G) \ldots\ldots\ldots (2)$$

is called neutrosophic abelian series if the factor group H_{j+1}/H_j are all abelian for all j.

Example 3.27: Let $N(G) = \langle S_3 \cup I \rangle$ be a neutrosophic group, where S_3 is the permutation group. Then the following is the neutrosophic abelian series of the group $N(G)$.

$$\{e\} \lhd A_3 \lhd \langle A_3 \cup I \rangle \lhd \langle S_3 \cup I \rangle .$$

We explain it as following:

Since $\langle S_3 \cup I \rangle / \langle A_3 \cup I \rangle \simeq \mathbb{Z}_2$ and $\mathbb{Z}_2$ is cyclic which is abelian. Thus $\langle S_3 \cup I \rangle / \langle A_3 \cup I \rangle$ is an abelian neutrosophic group.

Also,

$\langle A_3 \cup I \rangle / A_3 \simeq \mathbb{Z}_2$ and this is factor group is also cyclic and every cyclic group is abelian. Hence $\langle A_3 \cup I \rangle / A_3$ is also ablian group. Finally,

$A_3 / I \simeq \mathbb{Z}_3$ which is again abelian group. Therefore the series is a neutrosophic abelian series of the group $N(G)$.

Thus on the same lines, we can define strong neutrosophic abelian series, mixed neutrosophic abelian series and abelian series of the neutrosophic group $N(G)$.

Definition 3.28: A neutrosophic group $N(G)$ is called neutrosophic soluble group if $N(G)$ has a neutrosophic abelian series.

Example 3.29: Let $N(G) = \langle S_3 \cup I \rangle$ be a neutrosophic group, where S_3 is the permutation group. Then the following is the neutrosophic abelian series of the group $N(G)$,

$$\{e\} \lhd A_3 \lhd \langle A_3 \cup I \rangle \lhd \langle S_3 \cup I \rangle .$$

Then clearly $N(G)$ is a neutrosophic soluble group.

Theorem 3.30: Every abelian series of a group G is also an abelian series of the neutrosophic group $N(G)$.

Theorem 3.31: If a group G is a soluble group, then the neutrosophic group $N(G)$ is also soluble neutrosophic group.

Theorem 3.32: If the neutrosophic group $N(G)$ is an abelian neutrosophic group, then $N(G)$ is a neutrosophic soluble group.

Theorem 3.33: If $N(G)=\mathrm{C}(N(G))$, then $N(G)$ is a neutrosophic soluble group.

Proof: Suppose the $N(G)=\mathrm{C}(N(G))$. Then it follows that $N(G)$ is a neutrosophic abelian group. Hence by above Theorem 3.35, $N(G)$ is a neutrosophic soluble group.

Theorem 3.34: If the neutrosophic group $N(G)$ is a cyclic neutrosophic group, then $N(G)$ is a neutrosophic soluble group.

Definition 3.35: A neutrosophic group $N(G)$ is called strong neutrosophic soluble group if $N(G)$ has a strong neutrosophic abelian series.

Theorem 3.36: Every strong neutrosophic soluble group $N(G)$ is trivially a neutrosophic soluble group but the converse is not true.

Definition 3.37: A neutrosophic group $N(G)$ is called mixed neutrosophic soluble group if $N(G)$ has a mixed neutrosophic abelian series.

Theorem 3.38: Every mixed neutrosophic soluble group $N(G)$ is trivially a neutrosophic soluble group but the converse is not true.

Definition 3.39: A neutrosophic group $N(G)$ is called soluble group if $N(G)$ has an abelian series.

Definition 3.40: Let $N(G)$ be a neutrosophic soluble group. Then length of the shortest neutrosophic abelian series of $N(G)$ is called derived length.

Example 3.41: Let $N(G)=\langle \mathbb{Z}\cup I\rangle$ be a neutrosophic soluble group. The following is a neutrosophic abelian series of the group $N(G)$.

$$\{0\}\lhd 4\mathbb{Z}\lhd 2\mathbb{Z}\lhd\langle 2\mathbb{Z}\cup I\rangle\lhd\langle \mathbb{Z}\cup I\rangle.$$

Then $N(G)$ has derived length 4.

Remark 3.42: Neutrosophic group of derive length zero is trivial neutrosophic group.

Proposition 3.43: Every neutrosophic subgroup of a neutrosophic soluble group is soluble.

Proposition 3.44: Quotient neutrosophic group of a neutrosophic soluble group is soluble.

4. Neutrosophic Nilpotent Groups

Definition 4.1: Let $N(G)$ be a neutrosophic group. The series

$$\{e\}=H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft H_{n-1} \triangleleft H_n = N(G) \ldots\ldots\ldots\ldots\ldots (3)$$

is called neutrosophic central series if $H_{j+1}/H_j \subseteq C\left(N(G)/H_j\right)$ for all j.

Definition 4.2: A neutrosophic group $N(G)$ is called a neutrosophic nilpotent group if $N(G)$ has a neutrosophic central series.

Theorem 4.3: Every neutrosophic central series is a neutrosophic abelian series.

Theorem 4.4: If $N(G) = \mathrm{C}(N(G))$, then $N(G)$ is a neutrosophic nilpotent group.

Theorem 4.5: Every neutrosophic nilpotent group $N(G)$ is a neutrosophic soluble group.

Theorem 4.6: All neutrosophic abelian groups are neutrosophic nilpotent groups.

Theorem 4.7: All neutrosophic cyclic groups are neutrosophic nilpotent groups.

Theorem 4.8: The direct product of two neutrosophic nilpotent groups is nilpotent.

Definition 4.9: Let $N(G)$ be a neutrosophic group. Then the neutrosophic central series (3) is called strong neutrosophic central series if all H_j 's are neutrosophic normal subgroups for all j.

Theorem 4.10: Every strong neutrosophic central series is trivially a neutrosophic central series but the converse is not true in general.

Theorem 4.11: Every strong neutrosophic central series is a strong neutrosophic abelian series.

Definition 4.12: A neutrosophic group $N(G)$ is called strong neutrosophic nilpotent group if $N(G)$ has a strong neutrosophic central series.

Theorem 4.13: Every strong neutrosophic nilpotent group is trivially a neutrosophic nilpotent group.

Theorem 4.14: Every strong neutrosophic nilpotent group is also a strong neutrosophic soluble group.

Definition 4.15: Let $N(G)$ be a neutrosophic group. Then the neutrosophic central series (3) is called mixed neutrosophic central series if some H_j 's are neutrosophic normal subgroups while some H_k 's are just normal subgroups for j, k.

Theorem 4.16: Every mixed neutrosophic central series is trivially a neutrosophic central series but the converse is not true in general.

Theorem 4.17: Every mixed neutrosophic central series is a mixed neutrosophic abelian series.

Definition 4.18: A neutrosophic group $N(G)$ is called mixed neutrosophic nilpotent group if $N(G)$ has a mixed neutrosophic central series.

Theorem 4.19: Every mixed neutrosophic nilpotent group is trivially a neutrosophic nilpotent group.

Theorem 4.20: Every mixed neutrosophic nilpotent group is also a mixed neutrosophic soluble group.

Definition 4.21: Let $N(G)$ be a neutrosophic group. Then the neutrosophic central series (3) is called central series if all H_j 's are only normal subgroups for all j.

Theorem 4.22: Every central series is an abelian series.

Definition 4.23: A neutrosophic group $N(G)$ is called nilpotent group if $N(G)$ has a central series.

Theorem 4.24: Every nilpotent group is also a soluble group.

Theorem 4.25: If G is nilpotent group, then $N(G)$ is also a neutrosophic nilpotent group.

5. Conclusion

In this paper, we initiated the study of neutrosophic soluble groups and neutrosophic nilpotent groups which are the generalization of soluble groups and nilpotent groups. We also investigate their properties. Strong neutrosophic soluble and strong neutrosophic nilpotent groups are introduced which are completely new in their nature and properties. We also study the notions of mixed neutrosophic soluble groups and mixed neutrosophic nilpotent groups. These notions are studied on the basis of their serieses. In future, a lot of study can be carried out on neutrosophic nilpotent groups and neutrosophic soluble groups and their related properties.

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Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets

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Florentin Smarandache (2016). Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets. *Journal of Mathematics and Informatics* 5, 63-67

Abstract. We have defined *Neutrosophic Over-/Under-/Off-Set and Logic* for the first time in 1995 and published in 2007. During 1995-2016 we presented them to various national and international conferences and seminars. These new notions are totally different from other sets/logics/probabilities.

We extended the neutrosophic set respectively to *Neutrosophic Overset* {when some neutrosophic component is > 1}, to *Neutrosophic Underset* {when some neutrosophic component is < 0}, and to *Neutrosophic Offset* {when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and other neutrosophic component < 0}.

This is no surprise since our real-world has numerous examples and applications of over-/under-/off-neutrosophic components.

Keywords. neutrosophic overset, neutrosophic underset, neutrosophic offset, neutrosophic over logic, neutrosophic under logic, neutrosophic off logic, neutrosophic over probability, neutrosophic under probability, neutrosophic off probability, over membership (membership degree > 1), under membership (membership degree < 0), off membership (membership degree off the interval [0, 1]).

1. Introduction

In the classical set and logic theories, in the fuzzy set and logic, and in intuitionistic fuzzy set and logic, the degree of membership and degree of non-membership have to belong to, or be included in, the interval [0, 1]. Similarly, in the classical probability and in imprecise probability the probability of an event has to belong to, or respectively be included in, the interval [0, 1].

Yet, we have observed and presented to many conferences and seminars around the globe {see [12]-[33]} and published {see [1]-[8]} that in our real world there are many cases when the degree of membership is greater than 1. The set, which has elements whose membership is over 1, we called it *Overset*.

Even worst, we observed elements whose membership with respect to a set is under 0, and we called it *Underset*.

In general, a set that has elements whose membership is above 1 and elements whose membership is below 0, we called it *Offset* (i.e. there are elements whose memberships are off (over and under) the interval [0, 1]).

2. Example of over membership and under membership

In a given company a full-time employer works 40 hours per week. Let's consider the last week period.

Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was $30/40 = 0.75 < 1$.

John worked full-time, 40 hours, so he had the membership degree $40/40 = 1$, with respect to this company.

But George worked overtime 5 hours, so his membership degree was $(40+5)/40 = 45/40 = 1.125 > 1$. Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was $0/40 = 0$.

Yet, Richard, who was also hired as a full-time, not only didn't come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was $- 20/40 = - 0.50 < 0$.

Consequently, we need to make distinction between employees who produce damage, and those who produce profit, or produce neither damage no profit to the company.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to take them into consideration.

Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively *Neutrosophic Over-/Unde-r/Off-Logic, -Measure, -Probability, -Statistics* etc. [Smarandache, 2007].

3. Definition of single-valued neutrosophic overset

Let U be a universe of discourse and the neutrosophic set $A_1 \subset U$.

Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_1:

$T(x), I(x), F(x) : U \to [0, \Omega]$

where $0 < 1 < \Omega$, and Ω is called overlimit.

A Single-Valued Neutrosophic Overset A_1 is defined as:

$A_1 = \{(x, <T(x), I(x), F(x)>), x \in U\}$, such that there exists at least one element in A_1 that has at least one neutrosophic component that is > 1, and no element has neutrosophic components that are < 0.

For **example**: $A_1 = \{(x_1, <1.3, 0.5, 0.1>), (x_2, <0.2, 1.1, 0.2>)\}$, since $T(x_1) = 1.3 > 1$, $I(x_2) = 1.1 > 0$, and no neutrosophic component is < 0.

Also $O_2 = \{(a, <0.3, -0.1, 1.1>)\}$, since $I(a) = -0.1 < 0$ and $F(a) = 1.1 > 1$.

4. Definition of single-valued neutrosophic underset

Let U be a universe of discourse and the neutrosophic set $A_2 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_2:

$T(x), I(x), F(x) : U \rightarrow [\Psi, 1]$

where $\Psi < 0 < 1$, and Ψ is called underlimit.
A Single-Valued NeutrosophicUndersetA₂is defined as:
$A_2 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exists at least one element in A_2 that has at least one neutrosophic component that is < 0, and no element has neutrosophic components that are > 1.
For **example**: $A_2 = \{(x_1, <-0.4, 0.5, 0.3>), (x_2, <0.2, 0.5, -0.2>)\}$, since $T(x_1) = -0.4 < 0$, $F(x_2) = -0.2 < 0$, and no neutrosophic component is > 1.

5. Definition of single-valued neutrosophic offset

Let U be a universe of discourse and the neutrosophic set $A_3 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the set A_3:

$T(x), I(x), F(x) : U \rightarrow [\Psi, \Omega]$

where $\Psi < 0 < 1 < \Omega$, and Ψ is called under limit, while Ω is called overlimit.
A Single-Valued Neutrosophic Offset A_3 is defined as:
$A_3 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exist some elements in A_3 that have at least one neutrosophic component that is > 1, and at least another neutrosophic component that is < 0.
For **examples**: $A_3 = \{(x_1, <1.2, 0.4, 0.1>), (x_2, <0.2, 0.3, -0.7>)\}$, since $T(x_1) = 1.2 > 1$ and $F(x_2) = -0.7 < 0$.
Also, $B_3 = \{(a, <0.3, -0.1, 1.1>)\}$, since $I(a) = -0.1 < 0$ and $F(a) = 1.1 > 1$.

6. Neutrosophic overset / underset / offset operators

Let U be a universe of discourse and $A = \{(x, <T_A(x), I_A(x), F_A(x)>), x \in U\}$ and and $B = \{(x, <T_B(x), I_B(x), F_B(x)>), x \in U\}$ be two single-valued neutrosophic oversets / undersets / offsets.

$T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x): U \rightarrow [\Psi, \Omega]$

where $\Psi \leq 0 < 1 \leq \Omega$, and Ψ is called underlimit, while Ω is called overlimit.
We take the inequality sign ≤ instead of < on both extremes above, in order to comprise all three cases: overset {when $\Psi = 0$, and $1 < \Omega$}, underset {when $\Psi < 0$, and $1 = \Omega$}, and offset{when $\Psi < 0$, and $1 < \Omega$}.

Neutrosophic Overset / Underset / Offset Union.
Then $A \cup B = \{(x, <\max\{T_A(x), T_B(x)\}, \min\{I_A(x), I_B(x)\}, \min\{F_A(x), F_B(x)\}>), x \in U\}$

Neutrosophic Overset / Underset / Offset Intersection.
Then $A \cap B = \{(x, <\min\{T_A(x), T_B(x)\}, \max\{I_A(x), I_B(x)\}, \max\{F_A(x), F_B(x)\}>), x \in U\}$

Neutrosophic Overset / Underset / Offset Complement.
The complement of the neutrosophic set A is
$C(A) = \{(x, <F_A(x), \Psi + \Omega - I_A(x), T_A(x)>), x \in U\}$.

7. Conclusion

The membership degrees over 1 (over membership), or below 0 (undermembership) are part of our real world, sotheydeserve more study in the future. The neutrosophic over set / under set / off set together with neutrosophic over logic / under logic / off logic and especially neutrosophic over probability / under probability / and off probability have many applications in technology, social science, economics and so on that the readers may be interested in exploring.

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Interval-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets

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Florentin Smarandache (2016). Interval-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets. *International Journal of Science and Engineering Investigations* 5, 54, Paper ID: 55416-01, 4 p.

Abstract-We have proposed since 1995 the existence of degrees of membership of an element with respect to a neutrosophic set to also be partially or totally above 1 (over-membership), and partially or totally below 0 (under-membership) in order to better describe our world problems [published in 2007].

Keywords-*interval neutrosophic overset, interval neutrosophic underset, interval neutrosophic offset, interval neutrosophic overlogic, interval neutrosophic underlogic, interval neutrosophic offlogic, interval neutrosophic overprobability, interval neutrosophic underprobability, interval neutrosophic offprobability, interval overmembership (interval membership degree partially or totally above 1), interval undermembership (interval membership degree partially or totally below 0), interval offmembership (interval membership degree off the interval [0, 1]).*

I. INTRODUCTION

"Neutrosophic" means based on three components *T (truth-membership), I (indeterminacy)*, and *F (falsehood-non-membership)*. And "over" means above 1, "under" means below 0, while "offset" means behind/beside the set on both sides of the interval [0, 1], over and under, more and less, supra and below, out of, off the set. Similarly, for "offlogic", "offmeasure", "offprobability", "offstatistics" etc..

It is like a pot with boiling liquid, on a gas stove, when the liquid swells up and leaks out of pot. The pot (the interval [0, 1]) can no longer contain all liquid (i.e., all neutrosophic truth/indeterminate/falsehood values), and therefore some of them fall out of the pot (i.e., one gets neutrosophic truth/indeterminate/falsehood values which are > 1), or the pot cracks on the bottom and the liquid pours down (i.e., one gets neutrosophic truth/indeterminate/falsehood values which are < 0).

Mathematically, they mean getting values off the interval [0, 1].

The American aphorism "think outside the box" has a perfect resonance to the neutrosophic offset, where the box is the interval [0, 1], yet values outside of this interval are permitted.

II. EXAMPLE OF MEMBERSHIP ABOVE 1 AND MEMBERSHIP BELOW 0

Let's consider a spy agency $S = \{S_1, S_2, \ldots, S_{1000}\}$ of a country Atara against its enemy country Batara. Each agent S_j, $j \in \{1, 2, \ldots, 1000\}$, was required last week to accomplish 5 missions, which represent the full-time contribution/membership.

Last week agent S_{27} has successfully accomplished his 5 missions, so his membership was $T(S_{27}) = 5/5 = 1 = 100\%$ (full-time membership).

Agent S_{32} has accomplished only 3 missions, so his membership is $T(S_{32}) = 3/5 = 0.6 = 60\%$ (part-time membership).

Agent S_{41} was absent, without pay, due to his health problems; thus $T(S_{41}) = 0/5 = 0 = 0\%$ (null-membership).

Agent A_{53} has successfully accomplished his 5 required missions, plus an extra mission of another agent that was absent due to sickness, therefore $T(S_{53}) = (5+1)/5 = 6/5 = 1.2 > 1$ (therefore, he has membership above 1, called over-membership).

Yet, agent S_{75} is a double-agent, and he leaks highly confidential information about country Atara to the enemy country Batara, while simultaneously providing misleading information to the country Atara about the enemy country Batara. Therefore A_{75} is a negative agent with respect to his country Atara, since he produces damage to Atara, he was estimated to having intentionally done wrongly all his 5 missions, in addition of compromising a mission of another agent of country Atara, thus his membership $T(S_{75}) = -(5+1)/5 = -6/5 = -1.2 < 0$ (therefore, he has a membership below 0, called under-membership).

III. DEFINITION OF INTERVAL-VALUED NEUTROSOPHIC OVERSET

Let U be a universe of discourse and the neutrosophic set $A_1 \subset U$. Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and non-membership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_1:

$$T(x), I(x), F(x) : U \rightarrow P([0, \Omega]),$$

where $0 < 1 < \Omega$, and Ω is called over limit,

T(x), I(x), F(x) $\subseteq [0, \Omega]$, and P($[0, \Omega]$) is the set of all subsets of $[0, \Omega]$.

An Interval-Valued Neutrosophic Overset A_1 is defined as:

A_1 = {(x, <T(x), I(x), F(x)>), x ∈ U},

such that there exists at least one element in A_1 that has at least one neutrosophic component that is partially or totally above 1, and no element has neutrosophic components that is partially or totally below 0.

For *example*: A_1 = {(x_1, <(1, 1.4], 0.1, 0.2>), (x_2, <0.2, [0.9, 1.1], 0.2>)}, since T(x_1) = (1, 1.4] is totally above 1, I(x_2) = [0.9, 1.1] is partially above 1, and no neutrosophic component is partially or totally below 0.

IV. Definition of Interval-Valued Neutrosophic Underset

Let U be a universe of discourse and the neutrosophic set $A_2 \subset$ U. Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element x ∈ U, with respect to the neutrosophic set A_2:

T(x), I(x), F(x) : U → $[\Psi, 1]$,

where $\Psi < 0 < 1$, and Ψ is called underlimit,

T(x), I(x), F(x) $\subseteq [\Psi, 1]$, and P($[\Psi, 1]$) is the set of all subsets of $[\Psi, 1]$.

An Interval-Valued Neutrosophic Underset A_2 is defined as:

A_2 = {(x, <T(x), I(x), F(x)>), x ∈ U},

Such that there exists at least one element in A_2 that has at least one neutrosophic component that is partially or totally below 0, and no element has neutrosophic components that are partially or totally above 1.

For *example*: A_2 = {(x_1, <(-0.5,-0.4), 0.6, 0.3>), (x_2, <0.2, 0.5, [-0.2, 0.2]>)}, since T(x_1) = (-0.5, -0.4) is totally below 0, F(x_2) = [-0.2, 0.2] is partially below 0, and no neutrosophic component is partially or totally above 1.

V. Definition of Interval-Valued Neutrosophic offset

Let U be a universe of discourse and the neutrosophic set $A_3 \subset$ U. Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element x ∈ U, with respect to the set A_3:

T(x), I(x), F(x) : U → P($[\Psi, \Omega]$),

where $\Psi < 0 < 1 < \Omega$, and Ψ is called underlimit, while Ω is called overlimit,

T(x), I(x), F(x) $\subseteq [\Psi, \Omega]$, and P($[\Psi, \Omega]$) is the set of all subsets of $[\Psi, \Omega]$.

An Interval-Valued Neutrosophic Offset A_3 is defined as:

A_3 = {(x, <T(x), I(x), F(x)>), x ∈ U},

such that there exist some elements in A_3 that have at least one neutrosophic component that is partially or totally above 1, and at least another neutrosophic component that is partially or totally below 0.

For *examples*: A_3 = {(x_1, <[1.1, 1.2], 0.4, 0.1>), (x_2, <0.2, 0.3, (-0.7, -0.3)>)}, since T(x_1) = [1.1, 1.2] that is totally above 1, and F(x_2) = (-0.7, -0.3) that is totally below 0.

Also B_3 = {(a, <0.3, [-0.1, 0.1], [1.05, 1.10]>)}, since I(a) = [- 0.1, 0.1] that is partially below 0, and F(a) = [1.05, 1.10] that is totally above 1.

VI. Interval-Valued Neutrosophic Overset / Underset / Offset Operators

Let U be a universe of discourse and A = {(x, <T_A(x), I_A(x), F_A(x)>), x ∈ U} and B = {(x, <T_B(x), I_B(x), F_B(x)>), x ∈ U} be two interval-valued neutrosophic oversets / undersets / offsets.

T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x): U →P($[\Psi, \Omega]$),

where P($[\Psi, \Omega]$) means the set of all subsets of $[\Psi, \Omega]$,

and T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x) $\subseteq [\Psi, \Omega]$,

with $\Psi \le 0 < 1 \le \Omega$, and Ψ is called underlimit, while Ω is called overlimit.

We take the inequality sign ≤ instead of < on both extremes above, in order to comprise all three cases: overset {when Ψ = 0, and $1 < \Omega$}, underset {when $\Psi < 0$, and $1 = \Omega$}, and offset {when $\Psi < 0$, and $1 < \Omega$}.

A. Interval-Valued Neutrosophic Overset / Underset / Offset Union

Then A∪B =
{(x, <[max{inf(T_A(x)), inf(T_B(x))}, max{sup(T_A(x)), sup(T_B(x)}],
[min{inf(I_A(x)), inf(I_B(x))}, min{sup(I_A(x)), sup(I_B(x)}],
[min{inf(F_A(x)), inf(F_B(x))}, min{sup(F_A(x)), sup(F_B(x)}]>, x ∈ U}.

B. Interval-Valued Neutrosophic Overset / Underset / Offset Intersection

Then $A \cap B =$

$\{(x, <[\min\{\inf(T_A(x)), \inf(T_B(x))\}, \min\{\sup(T_A(x)), \sup(T_B(x)\}],$

$[\max\{\inf(I_A(x)), \inf(I_B(x))\}, \max\{\sup(I_A(x)), \sup(I_B(x)\}],$

$[\max\{\inf(F_A(x)), \inf(F_B(x))\}, \max\{\sup(F_A(x)), \sup(F_B(x)\}]>, x \in U\}.$

C. Interval-Valued Neutrosophic Overset / Underset / Offset Complement

The complement of the neutrosophic set A is

$C(A) = \{(x, <F_A(x),$

$[\Omega + \Psi - \sup\{I_A(x)\}, \Omega + \Psi - \inf\{I_A(x)\}], T_A(x)>), x \in U\}.$

VII. CONCLUSION

After designing the neutrosophic operators for single-valued neutrosophic overset/underset/offset, we extended them to interval-valued neutrosophic overset/underset/offset operators. We also presented another example of membership above 1 and membership below 0.

Of course, in many real world problems the neutrosophic union, neutrosophic intersection, and neutrosophic complement for interval-valued neutrosophic overset/underset/offset can be used. Future research will be focused on practical applications.

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Subtraction and Division of Neutrosophic Numbers

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Florentin Smarandache (2016). Subtraction and Division of Neutrosophic Numbers. *Critical Review* XIII, 103-110

Abstract

In this paper, we define the subtraction and the division of neutrosophic single-valued numbers. The restrictions for these operations are presented for neutrosophic single-valued numbers and neutrosophic single-valued overnumbers / undernumbers / offnumbers. Afterwards, several numeral examples are presented.

Keywords

neutrosophic calculus, neutrosophic numbers, neutrosophic summation, neutrosophic multiplication, neutrosophic scalar multiplication, neutrosophic power, neutrosophic subtraction, neutrosophic division.

1 Introduction

Let $A = (t_1, i_1, f_1)$ and $B = (t_2, i_2, f_2)$ be two single-valued neutrosophic numbers, where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, and $0 \leq t_1, i_1, f_1 \leq 3$ and $0 \leq t_2, i_2, f_2 \leq 3$.

The following operational relations have been defined and mostly used in the neutrosophic scientific literature:

1.1 Neutrosophic Summation

$$A \oplus B = (t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2) \tag{1}$$

1.2 Neutrosophic Multiplication

$$A \otimes B = (t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2) \tag{2}$$

1.3 Neutrosophic Scalar Multiplication

$$\lambda A = (1 - (1 - t_1)^{\lambda}, i_1^{\lambda}, f_1^{\lambda}), \tag{3}$$

where $\lambda \in \mathbb{R}$, and $\lambda > 0$.

1.4 Neutrosophic Power

$$A^{\lambda} = (t_1^{\lambda}, 1-(1-i_1)^{\lambda}, 1-(1-f_1)^{\lambda}), \quad (4)$$

where $\lambda \in \mathbb{R}$, and $\lambda > 0$.

2 Remarks

Actually, the neutrosophic scalar multiplication is an extension of neutrosophic summation; in the last, one has $\lambda = 2$.

Similarly, the neutrosophic power is an extension of neutrosophic multiplication; in the last, one has $\lambda = 2$.

Neutrosophic summation of numbers is equivalent to neutrosophic union of sets, and neutrosophic multiplication of numbers is equivalent to neutrosophic intersection of sets.

That's why, both the neutrosophic summation and neutrosophic multiplication (and implicitly their extensions neutrosophic scalar multiplication and neutrosophic power) can be defined in many ways, i.e. equivalently to their neutrosophic union operators and respectively neutrosophic intersection operators.

In general:

$$A \oplus B = (t_1 \vee t_2, i_1 \wedge i_2, f_1 \wedge f_2), \quad (5)$$

or

$$A \oplus B = (t_1 \vee t_2, i_1 \vee i_2, f_1 \vee f_2), \quad (6)$$

and analogously:

$$A \otimes B = (t_1 \wedge t_2, i_1 \vee i_2, f_1 \vee f_2) \quad (7)$$

or

$$A \otimes B = (t_1 \wedge t_2, i_1 \wedge i_2, f_1 \vee f_2), \quad (8)$$

where "$\vee$" is the fuzzy OR (fuzzy union) operator, defined, for $\alpha, \beta \in [0, 1]$, in three different ways, as:

$$\alpha \overset{1}{\vee} \beta = \alpha + \beta - \alpha\beta, \quad (9)$$

or

$$\alpha \overset{2}{\vee} \beta = max\{\alpha, \beta\}, \quad (10)$$

or

$$\alpha \overset{3}{\vee} \beta = min\{x + y, 1\}, \quad (11)$$

etc.

While "∧" is the fuzzy AND (fuzzy intersection) operator, defined, for $\alpha, \beta \in [0, 1]$, in three different ways, as:

$$\alpha \underset{1}{\wedge} \beta = \alpha\beta, \tag{12}$$

or

$$\alpha \underset{2}{\wedge} \beta = min\{\alpha, \beta\}, \tag{13}$$

or

$$\alpha \underset{3}{\wedge} \beta = max\{x + y - 1, 0\}, \tag{14}$$

etc.

Into the definitions of $A \oplus B$ and $A \otimes B$ it's better if one associates $\overset{1}{\vee}$ with $\underset{1}{\wedge}$, since $\overset{1}{\vee}$ is opposed to $\underset{1}{\wedge}$, and $\overset{2}{\vee}$ with $\underset{2}{\wedge}$, and $\overset{3}{\vee}$ with $\underset{3}{\wedge}$, for the same reason. But other associations can also be considered.

For examples:

$$A \oplus B = (t_1 + t_2 - t_1t_2, i_1 + i_2 - i_1i_2, f_1f_2), \tag{15}$$

or

$$A \oplus B = (max\{t_1, t_2\}, min\{i_1, i_2\}, min\{f_1, f_2\}), \tag{16}$$

or

$$A \oplus B = (max\{t_1, t_2\}, max\{i_1, i_2\}, min\{f_1, f_2\}), \tag{17}$$

or

$$A \oplus B = (min\{t_1 + t_2, 1\}, max\{i_1 + i_2 - 1, 0\}, max\{f_1 + f_2 - 1, 0\}). \tag{18}$$

where we have associated $\overset{1}{\vee}$ with $\underset{1}{\wedge}$, and $\overset{2}{\vee}$ with $\underset{2}{\wedge}$, and $\overset{3}{\vee}$ with $\underset{3}{\wedge}$.

Let's associate them in different ways:

$$A \oplus B = (t_1 + t_2 - t_1t_2, min\{i_1, i_2\}, min\{f_1, f_2\}), \tag{19}$$

where $\overset{1}{\vee}$ was associated with $\underset{2}{\wedge}$ and $\underset{3}{\wedge}$; or:

$$A \oplus B = (max\{t_1, t_2\}, i_1, i_2, max\{f_1 + f_2 - 1, 0\}), \tag{20}$$

where $\overset{2}{\vee}$ was associated with $\underset{1}{\wedge}$ and $\underset{3}{\wedge}$; and so on.

Similar examples can be constructed for $A \otimes B$.

3 Neutrosophic Subtraction

We define now, for the first time, the subtraction of neutrosophic number:

$$A \ominus B = (t_1, i_1, f_1) \ominus (t_2, i_2, f_2) = \left(\frac{t_1 - t_2}{1 - t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) = C, \tag{21}$$

for all $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, with the restrictions that: $t_2 \neq 1$, $i_2 \neq 0$, and $f_2 \neq 0$.

So, the neutrosophic subtraction only partially works, i.e. when $t_2 \neq 1$, $i_2 \neq 0$, and $f_2 \neq 0$.

The restriction that

$$\left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) \in ([0, 1], [0, 1], [0, 1]) \tag{22}$$

is set when the classical case when the neutrosophic number components t, i, f are in the interval $[0, 1]$.

But, for the general case, when dealing with neutrosophic overset / underset /offset [1], or the neutrosophic number components are in the interval $[\Psi, \Omega]$, where Ψ is called *underlimit* and Ω is called *overlimit*, with $\Psi \leq 0 < 1 \leq \Omega$, i.e. one has *neutrosophic overnumbers / undernumbers / offnumbers*, then the restriction (22) becomes:

$$\left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) \in ([\Psi, \Omega], [\Psi, \Omega], [\Psi, \Omega]). \tag{23}$$

3.1 Proof

The formula for the subtraction was obtained from the attempt to be consistent with the neutrosophic addition.

One considers the most used neutrosophic addition:

$$(a_1, b_1, c_1) \oplus (a_2, b_2, c_2) = (a_1 + a_2 - a_1 a_2, b_1 b_2, c_1 c_2), \tag{24}$$

We consider the $\ominus$ neutrosophic operation the opposite of the $\oplus$ neutrosophic operation, as in the set of real numbers the classical subtraction $-$ is the opposite of the classical addition $+$.

Therefore, let's consider:

$$(t_1, i_1, f_1) \ominus (t_2, i_2, f_2) = (x, y, z), \tag{25}$$

$$\oplus\, (t_2, i_2, f_2) \qquad\qquad \oplus\, (t_2, i_2, f_2)$$

where $x, y, z \in \mathbb{R}$.

We neutrosophically add $\oplus\, (t_2, i_2, f_2)$ on both sides of the equation. We get:

$$(t_1, i_1, f_1) = (x, y, z) \oplus (t_2, i_2, f_2) = (x + t_2 - x t_2, y i_2, z f_2). \tag{26}$$

Or,

$$\left\{\begin{aligned} t_1 &= x + t_2 - x t_2, \text{whence } x = \frac{t_1-t_1}{1-t_2}\ ; \\ i_1 &= y i_2, \text{whence } y = \frac{i_1}{i_2}\ ; \\ f_1 &= z f_2, \text{whence } z = \frac{f_1}{f_2}\ . \end{aligned}\right. \tag{27}$$

3.2 Checking the Subtraction

With $A = (t_1, i_1, f_1)$, $B = (t_2, i_2, f_2)$, and $C = \left(\frac{t_1 - t_2}{1 - t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right)$,
where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, and $t_2 \neq 1$, $i_2 \neq 0$, and $f_2 \neq 0$, we have:

$$A \ominus B = C. \tag{28}$$

Then:

$$B \oplus C = (t_2, i_2, f_2) \oplus \left(\frac{t_1 - t_2}{1 - t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) = \left(t_2 + \frac{t_1 - t_2}{1 - t_2} - t_2 \cdot \frac{t_1 - t_2}{1 - t_2}, i_2, \frac{i_1}{i_2}, f_2, \frac{f_1}{f_2}\right) = \left(\frac{t_2 - t_2^2 + t_1 - t_2 - t_1 t_2 + t_2}{1 - t_2}, i_1, f_1\right) = \left(\frac{t_1(1 - t_2)}{1 - t_2}, i_1, f_1\right) = (t_1, i_1, f_1). \tag{29}$$

$$A \ominus C = (t_1, i_1, f_1) \ominus \left(\frac{t_1 - t_2}{1 - t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) = \left(\frac{t_1 - \frac{t_1 - t_2}{1 - t_2}}{1 - \frac{t_1 - t_2}{1 - t_2}}, \frac{i_1}{\frac{i_1}{i_2}}, \frac{f_1}{\frac{f_1}{f_2}}\right) = \left(\frac{\frac{t_1 - t_1 t_2 - t_1 + t_2}{1 - t_2}}{\frac{1 - t_2 - t_1 + t_2}{1 - t_2}}, i_2, f_2\right) = \left(\frac{-t_1 t_2 + t_2}{1 - t_2}, i_2, f_2\right) = \left(\frac{t_2(-t_1 + 1)}{1 - t_2}, i_2, f_2\right) = (t_2, i_2, f_2). \tag{30}$$

4 Division of Neutrosophic Numbers

We define for the first time the division of neutrosophic numbers:

$$A \oslash B = (t_1, i_1, f_1) \oslash (t_2, i_2, f_2) = \left(\frac{t_1}{t_2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2}\right) = D, \tag{31}$$

where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, with the restriction that $t_2 \neq 0$, $i_2 \neq 1$, and $f_2 \neq 1$.

Similarly, the division of neutrosophic numbers only partially works, i.e. when $t_2 \neq 0$, $i_2 \neq 1$, and $f_2 \neq 1$.

In the same way, the restriction that

$$\left(\frac{t_1}{t_2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2}\right) \in ([0, 1], [0, 1], [0, 1]) \tag{32}$$

is set when the traditional case occurs, when the neutrosophic number components t, i, f are in the interval [0, 1].

But, for the case when dealing with neutrosophic overset / underset /offset [1], when the neutrosophic number components are in the interval $[\Psi, \Omega]$, where Ψ is called *underlimit* and Ω is called *overlimit*, with $\Psi \leq 0 < 1 \leq \Omega$, i.e. one has *neutrosophic overnumbers / undernumbers / offnumbers*, then the restriction (31) becomes:

$$\left(\frac{t_1}{t_2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2}\right) \in ([\Psi, \Omega], [\Psi, \Omega], [\Psi, \Omega]). \quad (33)$$

4.1 Proof

In the same way, the formula for division $\oslash$ of neutrosophic numbers was obtained from the attempt to be consistent with the neutrosophic multiplication.

We consider the $\oslash$ neutrosophic operation the opposite of the $\otimes$ neutrosophic operation, as in the set of real numbers the classical division $\div$ is the opposite of the classical multiplication $\times$.

One considers the most used neutrosophic multiplication:

$$(a_1, b_1, c_1) \otimes (a_2, b_2, c_2)$$
$$= (a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2), \quad (34)$$

Thus, let's consider:

$$\underset{\otimes(t_2, i_2, f_2)}{(t_1, i_1, f_1) \oslash (t_2, i_2, f_2)} = \underset{\otimes(t_2, i_2, f_2)}{(x, y, z)}, \quad (35)$$

where $x, y, z \in \mathbb{R}$.

We neutrosophically multiply $\otimes$ both sides by (t_2, i_2, f_2). We get

$$(t_1, i_1, f_1) = (x, y, z) \otimes (t_2, i_2, f_2)$$
$$= (x t_2, y + i_2 - y i_2, z + f_2 - z f_2). \quad (36)$$

Or,

$$\begin{cases} t_1 = x t_2, \text{whence } x = \frac{t_1}{t_2};: \\ i_1 = y + i_2 - y i_2, \text{whence } y = \frac{i_1 - i_2}{1 - i_2}; \\ f_1 = z + f_2 - z f_2, \text{whence } z = \frac{f_1 - f_2}{1 - f_2}. \end{cases} \quad (37)$$

4.2 Checking the Division

With $A = (t_1, i_1, f_1)$, $B = (t_2, i_2, f_2)$, and $D = \left(\frac{t_1}{t_2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2}\right)$,

where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, and $t_2 \neq 0$, $i_2 \neq 1$, and $f_2 \neq 1$, one has:

$$A * B = D. \quad (38)$$

Then:

$$\frac{B}{D} = (t_2, i_2, f_2) \times \left(\frac{t_1}{t_2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2}\right) = \left(t_2 \cdot \frac{t_1}{t_2}, i_2 + \frac{i_1 - i_2}{1 - i_2} - i_2 \cdot \frac{i_1 - i_2}{1 - i_2}, f_2 + \frac{f_1 - f_2}{1 - f_2} - f_2 \cdot \frac{f_1 - f_2}{1 - f_2}\right) =$$

$$\left(t_1, \frac{i_2-i_2^2+i_1-i_2-i_1i_2+i_2^2}{1-i_2}, \frac{f_2-f_2^2+f_1-f_2-f_1f_2+f_2^2}{1-f_2}\right) =$$
$$\left(t_1, \frac{i_1(1-i_2)}{1-i_2}, \frac{f_1(1-f_2)}{1-f_2}\right) = (t_1, i_1, f_1) = A. \quad (39)$$

Also:

$$\frac{A}{D} = \frac{(t_1, i_1, f_1)}{\left(\frac{t_1}{t_2}, \frac{i_1-i_2}{1-i_2}, \frac{f_1-f_2}{1-f_2}\right)} = \left(\frac{t_1}{\frac{t_1}{t_2}}, \frac{i_1-\frac{i_1-i_2}{1-i_2}}{1-\frac{i_1-i_2}{1-i_2}}, \frac{f_1-\frac{f_1-f_2}{1-f_2}}{1-\frac{f_1-f_2}{1-f_2}}\right) =$$
$$\left(t_2, \frac{\frac{i_1-i_1i_2-i_1+i_2}{1-i_2}}{\frac{1-i_2-i_1+i_2}{1-i_2}}, \frac{\frac{f_1-f_1f_2-f_1+f_2}{1-f_2}}{\frac{1-f_2-f_1+f_2}{1-f_2}}\right) = \left(t_2, \frac{\frac{i_2(-i_1+1)}{1-i_2}}{\frac{1-i_1}{1-i_2}}, \frac{\frac{f_2(-f_1+1)}{1-f_2}}{\frac{1-f_1}{1-f_2}}\right) =$$
$$\left(t_2, \frac{i_2(1-i_1)}{1-i_1}, \frac{f_2(1-f_1)}{1-f_1}\right) = (t_2, i_2, f_2) = B. \quad (40)$$

5 Conclusion

We have obtained the formula for the subtraction of neutrosophic numbers $\ominus$ going backwords from the formula of addition of neutrosophic numbers $\oplus$.

Similarly, we have defined the formula for division of neutrosophic numbers $\oslash$ and we obtained it backwords from the neutrosophic multiplication $\otimes$.

We also have taken into account the case when one deals with classical neutrosophic numbers (i.e. the neutrosophic components t, i, f belong to [0, 1]) as well as the general case when t, i, f belong to $[\Psi, \Omega]$, where the underlimit $\Psi \leq 0$ and the overlimit $\Omega \geq 1$.

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On Neutrosophic Quadruple Algebraic Structures

S.A. Akinleye, F. Smarandache, A.A.A. Agboola

S.A. Akinleye, Florentin Smarandache, A.A.A. Agboola (2016). On Neutrosophic Quadruple Algebraic Structures. *Neutrosophic Sets and Systems* 12, 122-126

Abstract. In this paper we present the concept of neutrosophic quadruple algebraic structures. Specially, we study neutrosophic quadruple rings and we present their elementary properties.

Keywords: Neutrosophy, neutrosophic quadruple number, neutrosophic quadruple semigroup, neutrosophic quadruple group, neutrosophic quadruple ring, neutrosophic quadruple ideal, neutrosophic quadruple homomorphism.

1 Introduction

The concept of neutrosophic quadruple numbers was introduced by Florentin Smarandache [3]. It was shown in [3] how arithmetic operations of addition, subtraction, multiplication and scalar multiplication could be performed on the set of neutrosophic quadruple numbers. In this paper, we studied neutrosophic sets of quadruple numbers together with binary operations of addition and multiplication and the resulting algebraic structures with their elementary properties are presented. Specially, we studied neutrosophic quadruple rings and we presented their basic properties.

Definition 1.1 [3]

A neutrosophic quadruple number is a number of the form (a, bT, cI, dF), where T, I, F have their usual neutrosophic logic meanings and $a, b, c, d \in \mathbb{R}$ or $\mathbb{C}$. The set NQ defined by

$$NQ = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\} \quad (1)$$

is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number (a, bT, cI, dF), representing any entity which may be a number, an idea, an object, etc., a is called the known part and (bT, cI, dF) is called the unknown part.

Definition 1.2

Let

$$a = (a_1, a_2T, a_3I, a_4F),$$
$$b = (b_1, b_2T, b_3I, b_4F) \in NQ.$$

We define the following:

$$a + b = (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F) \quad (2)$$

$$a - b = (a_1 - b_1, (a_2 - b_2)T, (a_3 - b_3)I, (a_4 - b_4)F). \quad (3)$$

Definition 1.3

Let

$$a = (a_1, a_2T, a_3I, a_4F) \in NQ$$

and let α be any scalar which may be real or complex, the scalar product $\alpha.a$ is defined by

$$\alpha.a = \alpha.(a_1, a_2T, a_3I, a_4F) = (\alpha a_1, \alpha a_2T, \alpha a_3I, \alpha a_4F) \quad (4)$$

If $\alpha = 0$, then we have $0.a = (0, 0, 0, 0)$ and for any non-zero scalars m and n and $b = (b_1, b_2\text{T}, b_3\text{I}, b_4\text{F})$, we have:

$$(m + n)a = ma + na,$$
$$m(a + b) = ma + mb,$$
$$mn(a) = m(na),$$
$$-a = (-a_1, -a_2T, -a_3I, -a_4F).$$

Definition 1.4 [3] [Absorbance Law]

Let X be a set endowed with a total order $x < y$, named "x prevailed by y" or "x less strong than y" or "x less preferred than y". $x \leq y$ is considered as "x prevailed by or equal to y" or "x less strong than or equal to y" or "x less preferred than or equal to y".

For any elements $x, y \in X$, with $x \leq y$, absorbance law is defined as

$$x \cdot y = y \cdot x = \text{absorb}(x, y) = \max\{x, y\} = y \quad (5)$$

which means that the bigger element absorbs the smaller element (*the big fish eats the small fish*). It is clear from (5) that

$$x \cdot x = x^2 = absorb(x, x) = max\{x, x\} = x \quad (6)$$

and

$$x_1 \cdot x_2 \cdots x_n = max\{x_1, x_2, \cdots, x_n\}. \quad (7)$$

Analogously, if $x > y$, we say that "x prevails to y" or "x is stronger than y" or "x is preferred to y". Also, if $x \geq y$, we say that "x prevails or is equal to y" or "x is stronger than or equal to y" or "x is preferred or equal to y".

Definition 1.5

Consider the set $\{T, I, F\}$. Suppose in an optimistic way we consider the prevalence order $T > I > F$. Then we have:

$$TI = IT = \max\{T, I\} = T, \quad (8)$$
$$TF = FT = \max\{T, F\} = T, \quad (9)$$
$$IF = FI = \max\{I, F\} = I, \quad (10)$$
$$TT = T^2 = T, \quad (11)$$
$$II = I^2 = I, \quad (12)$$
$$FF = F^2 = F. \quad (13)$$

Analogously, suppose in a pessimistic way we consider the prevalence order $T < I < F$. Then we have:

$$TI = IT = max\{T, I\} = I, \quad (14)$$
$$TF = FT = max\{T, F\} = F, \quad (15)$$
$$IF = FI = max\{I, F\} = F, \quad (16)$$
$$TT = T^2 = T, \quad (17)$$
$$II = I^2 = I, \quad (18)$$
$$FF = F^2 = F. \quad (19)$$

Definition 1.6

Let

$a = (a_1, a_2T, a_3I, a_4F)$,
$b = (b_1, b_2T, b_3I, b_4F) \in NQ$.

Then (20)

$$a.b = (a_1, a_2T, a_3I, a_4F).(b_1, b_2T, b_3I, b_4F) = (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, (a_1b_4 + a_2b_4, a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F).$$

2 Main Results

All neutrosophic quadruple numbers to be considered in this section will be real neutrosophic quadruple numbers i.e $a, b, c, d \in \mathbb{R}$ for any neutrosophic quadruple number $(a, bT, cI, dF) \in NQ$.

Theorem 2.1

$(NQ, +)$ is an abelian group.

Proof.

Suppose that

$a = (a_1, a_2T, a_3I, a_4F)$,
$b = (b_1, b_2T, b_3I,$
$c = (c_1, c_2T, c_3I, c_4F \in NQ$

are arbitrary.

It can easily be shown that

$$a + b = b + a \cdot a + (b + c) = (a + b) + c \cdot a + (0,0,0,0) = (0,0,0,0) = a$$

and

$$a + (-a) = -a + a = (0,0,0,0).$$

Thus, $0 = (0,0,0,0)$ is the additive identity element in $(NQ, +)$ and for any $a \in NQ, -a$ is the additive inverse. Hence, $(NQ, +)$ is an abelian group.

Theorem 2.2

$(NQ, .)$ is a commutative monoid.

Proof.

Let

$a = (a_1, a_2T, a_3I, a_4F)$,
$b = (b_1, b_2T, b_3I,$
$c = (c_1, c_2T, c_3I, c_4F$

be arbitrary elements in NQ. It can easily be shown that

$$ab = ba \cdot a(bc) = (ab)c \cdot a \cdot (1, 0, 0, 0) = a.$$

Thus, $e = (1, 0, 0, 0)$ is the multiplicative identity element in $(NQ, .)$. Hence, $(NQ, .)$ is a commutative monoid.

Theorem 2.3

$(NQ, .)$ is not a group.

Proof.

Let

$$x = (a, bT, cI, dF)$$

be any arbitrary element in NQ.

Since we cannot find any element $y = (p, qT, rI, sF) \in NQ$ such that $xy = yx = e = (1, 0, 0, 0)$, it follows that $x - 1$ does not exist in NQ for any given $a, b, c, d \in \mathbb{R}$ and consequently, $(NQ, .)$ cannot be a group.

Example 1.

Let $X = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{Z}_n\}$. Then $(X, +)$ is an abelian group.

Example 2.

Let

$$(M_{2\times2}, .) = \left\{ \begin{bmatrix} (a, bT, cI, dF) & (e, fT, gI, hF) \\ (i, jT, kI, lF) & (m, nT, pI, qF) \end{bmatrix} : \atop \text{a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, q} \in \mathbb{R} \right\}$$

Then $(M_{2\times2}, .)$ is a non-commutative monoid.

Theorem 2.4

$(NQ, +, .)$ is a commutative ring.

Proof.

It is clear that $(NQ,+)$ is an abelian group and $(NQ,.)$ is a semigroup. To complete the proof, suppose that

$a = (a_1, a_2T, a_3I, a_4F)$,
$b = (b_1, b_2T, b_3I,$
$c = (c_1, c_2T, c_3I, c_4F \in NQ$

are arbitrary. It can easily be shown that $a(b+c) = ab + ac, (b + c)a = ba + ca$ and $ab = ba$. Hence, $(NQ,+,.)$ is a commutative ring.

From now on, the ring $(NQ,+,.)$ will be called neutrosophic quadruple ring and it will be denoted by NQR. The zero element of NQR will be denoted by (0, 0, 0, 0) and the unity of NQR will be denoted by (1, 0, 0, 0).

Example 3.

(i) Let X be as defined in EXAMPLE 1. Then $(X,+,.)$ is a commutative neutrosophic quadruple ring called a neutrosophic quadruple ring of integers modulo n.

It should be noted that $NQR(\mathbb{Z}_n)$ has 4^n elements and for $NQR(\mathbb{Z}_2)$we have

$NQR(\mathbb{Z}_2) =$
$= \{(0,0,0,0), (1,0,0,0), (0,T,0,0), (0,0,I,0), (0,0,0,F),$
$(0,T,I,F), (0,0,I,F), (0,T,I,0), (0,T,0,F), (1,T,0,0),$
$(1,0,I,0), (1,0,0,F), (1,T,0,F), (1,0,I,F), (1,T,I,0),$
$(1,T,I,F)\}$.

(ii) Let $M_{2\times2}$ be as defined in EXAMPLE 2. Then $(M_{2\times2},.)$ is a non-commutative neutrosophic quadruple ring.

Definition 2.5

Let NQR be a neutrosophic quadruple ring.

(i) An element $a \in NQR$ is called idempotent if $a^2 = a$.

(ii) An element $a \in NQR$ is called nilpotent if there $exists\ n \in Z^+$ such that $a^n = 0$.

Example 4.

(i) In $NQR(\mathbb{Z}_2)$, $(1,T,I,F)$ and $(1,T,I,0)$ are idempotent elements.

(ii) In $NQR(\mathbb{Z}_4)$, $(2,2T,2I,2F)$ is a nilpotent element.

Definition 2.6

Let NQR be a neutrosophic quadruple ring.

NQR is called a neutrosophic quadruple integral domain if for $x, y \in NQR$, $xy = 0$ implies that $x = 0$ or $y = 0$.

Example 5.

$NQR(\mathbb{Z})$ the neutrosophic quadruple ring of integers is a neutrosophic quadruple integral domain.

Definition 2.7

Let NQR be a neutrosophic quadruple ring.

An element $x \in NQR$ is called a zero divisor if there exists a nonzero element $y \in NQR$ such that $xy = 0$. For example in $NQR(\mathbb{Z}_2)$, $(0,0,I,F)$ and $(0,T,I,0)$ are zero divisors even though $\mathbb{Z}_2$ has no zero divisors.

This is one of the distinct features that characterize neutrosophic quadruple rings.

Definition 2.8

Let NQR be a neutrosophic quadruple ring and let NQS be a nonempty subset of NQR. Then NQS is called a neutrosophic quadruple subring of NQR if $(NQS,+,.)$ is itself a neutrosophic quadruple ring. For example, $NQR(n\mathbb{Z})$ is a neutrosophic quadruple subring of $NQR(\mathbb{Z})$ for $n = 1, 2, 3, \cdots$.

Theorem 2.9

Let NQS be a nonempty subset of a neutrosophic quadruple ring NQR. Then NQS is a neutrosophic quadruple subring if and only if for all $x, y \in NQS$, the following conditions hold:

(i) $x - y \in NQS$

and

(ii) $xy \in NQS$.

Proof.

Same as the classical case and so omitted.

Definition 2.10

Let NQR be a neutrosophic quadruple ring.

Then the set

$Z(NQR) = \{x \in NQR: xy = yx \ \forall \ y \in NQR\}$

is called the centre of NQR.

Theorem 2.11

Let NQR be a neutrosophic quadruple ring.

Then $Z(NQR)$ is a neutrosophic quadruple subring of NQR.

Proof.

Same as the classical case and so omitted.

Theorem 2.12

Let NQR be a neutrosophic quadruple ring and let NQS_j be families of neutrosophic quadruple subrings of NQR. Then

$$\bigcap_{j=1}^{} n\, NQS_j$$

is a neutrosophic quadruple subring of NQR.

Definition 2.13

Let NQR be a neutrosophic quadruple ring.

If there exists a positive integer n such that $nx = 0$ for each $x \in NQR$, then the smallest such positive integer is called the characteristic of NQR. If no such positive integer exists, then NQR is said to have characteristic zero. For example, $NQR(\mathbb{Z})$ has characteristic zero and $NQR(\mathbb{Z}_n)$ has characteristic n.

Definition 2.14

Let NQJ be a nonempty subset of a neutrosophic quadruple ring NQR. NQJ is called a neutrosophic quadruple ideal of NQR if for all $x, y \in NQJ$, $r \in NQR$, the following conditions hold:

(i) $x - y \in NQJ$.

(ii) $xr \in NQJ$ and $rx \in NQJ$.

Example 6.

(i) $NQR(3\mathbb{Z})$ is a neutrosophic quadruple ideal of $NQR(\mathbb{Z})$.

(ii) Let

$NQJ = \{(0,0,0,0), (2,0,0,0), (0,2T, 2I, 2F), (2,2T, 2I, 2F)\}$

be a subset of $NQR(\mathbb{Z}_4)$. Then NQJ is a neutrosophic quadruple ideal.

Theorem 2.15

Let NQJ and NQS be neutrosophic quadruple ideals of NQR and let

$\{NQJ_j\}_{j=1}^{n}$

be a family of neutrosophic quadruple ideals of NQR. Then:

(i) $NQJ + NQJ = NQJ$.

(ii) $x + NQJ = NQJ$ for all $x \in NQJ$.

(iii)

$\bigcap_{j=1} n\, NQS_j$

is a neutrosophic quadruple ideal of NQR.

(iv) $NQJ + NQS$ is a neutrosophic quadruple ideal of NQR.

Definition 2.16

Let NQJ be a neutrosophic quadruple ideal of NQR. The set

$NQR/NQJ = \{x + NQJ : x \in NQR\}$

is called a neutrosophic quadruple quotient ring.

If $x + NQJ$ and $y + NQJ$ are two arbitrary elements of NQR/NQJ and if $\oplus$ and $\odot$ are two binary operations on NQR/NQJ defined by:

$(x + NQJ) \oplus (y + 4\,NQJ) = (x + y) + NQJ,$
$(x + NQJ) \odot (y + NQJ) = (xy) + NQJ,$

it can be shown that $\oplus$ and $\odot$ are well defined and that (NQR/NQJ, $\oplus$, $\odot$) is a neutrosophic quadruple ring.

Example 7.

Consider the neutrosophic quadruple ring $NQR(\mathbb{Z})$ and its neutrosophic quadruple ideal $NQR(2\mathbb{Z})$. Then

$$\frac{NQR(\mathbb{Z})}{NQR(2\mathbb{Z})} = \{NQR(2\mathbb{Z}), (1,0,0,0) + NQR(2\mathbb{Z}), (0, T, 0,0) + NQR(2\mathbb{Z}), (0,0, I, 0) + NQR(2\mathbb{Z}), (0,0,0, F) + NQR(2\mathbb{Z}), (0, T, I, F) + NQR(2\mathbb{Z}), (0,0, I, F) + NQR(2\mathbb{Z}), (0, T, I, 0) + NQR(2\mathbb{Z}), (0, T, 0, F) + NQR(2\mathbb{Z}), (1, T, 0,0) + NQR(2\mathbb{Z}), (1,0, I, 0) + NQR(2\mathbb{Z}), (1,0,0, F) + NQR(2\mathbb{Z}), (1, T, 0, F) + NQR(2\mathbb{Z}), (1,0, I, F) + NQR(2\mathbb{Z}), (1, T, I, 0) + NQR(2\mathbb{Z}), (1, T, I, F) + NQR(2\mathbb{Z})\}.$$

which is clearly a neutrosophic quadruple ring.

Definition 2.17

Let NQR and NQS be two neutrosophic quadruple rings and let $\varphi : NQR \to NQS$ be a mapping defined for all $x, y \in NQR$ as follows:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$.
(ii) $\varphi(xy) = \varphi(x)\varphi(y)$.
(iii) $\varphi(T) = T, \varphi(I) = I$ and $\varphi(F) = F$.
(iv) $\varphi(1,0,0,0) = (1,0,0,0)$.

Then φ is called a neutrosophic quadruple homomorphism. Neutrosophic quadruple monomorphism, endomorphism, isomorphism, and other morphisms can be defined in the usual way.

Definition 2.18

Let $\varphi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism.

(i) The image of φ denoted by $Im\varphi$ is defined by the set $Im\varphi = \{y \in NQS : y = \varphi(x)$, for some $x \in NQR\}$.

(ii) The kernel of φ denoted by $Ker\varphi$ is defined by the set $Ker\varphi = \{x \in NQR : \varphi(x) = (0,0,0,0)\}$.

Theorem 2.19

Let $\varphi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism. Then:

(i) $Im\varphi$ is a neutrosophic quadruple subring of NQS.
(ii) $Ker\varphi$ is not a neutrosophic quadruple ideal of NQR.

Proof.

(i) Clear.

(ii) Since T, I, F cannot have image (0,0,0,0) under φ, it follows that the elements $(0, T, 0,0), (0,0, I, 0), (0,0,0, F)$ cannot be in the $Ker\varphi$. Hence, $Ker\varphi$ cannot be a neutrosophic quadruple ideal of NQR.

Example 8.

Consider the projection map

$\varphi : NQR(\mathbb{Z}_2) \times NQR(\mathbb{Z}_2) \rightarrow NQR(\mathbb{Z}_2)$

defined by $\varphi(x,y) = x$ for all $x, y \in NQR(\mathbb{Z}_2)$.

It is clear that φ is a neutrosophic quadruple homomorphism and its kernel is given as

$$Ker\varphi = \{((0,0,0,0),(0,0,0,0)),((0,0,0,0),(1,0,0,0)), \big((0,0,0,0),(0,T,0,0)\big),\big((0,0,0,0),(0,0,I,0)\big), \big((0,0,0,0),(0,0,0,F)\big),\big((0,0,0,0),(0,T,I,F)\big), \big((0,0,0,0),(0,0,I,F)\big),\big((0,0,0,0),(0,T,I,0)\big), \big((0,0,0,0),(0,T,0,F)\big),\big((0,0,0,0),(1,T,0,0)\big), \big((0,0,0,0),(1,0,I,0)\big),\big((0,0,0,0),(1,0,0,F)\big), \big((0,0,0,0),(1,T,0,F)\big),\big((0,0,0,0),(1,0,I,F)\big), ((0,0,0,0),(1,T,I,0)),((0,0,0,0),(1,T,I,F))\}.$$

Theorem 2.20

Let φ: NQR(Z) → NQR(Z)/NQR(nZ) be a mapping defined by φ(x) = x + NQR(nZ) for all x ∈ NQR(Z) and n = 1, 2, 3, … . Then φ is not a neutrosophic quadruple ring homomorphism.

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Neutrosophic quadruple algebraic hyperstructures

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A.A.A. Agboola, B. Davvaz, Florentin Smarandache (2017). Neutrosophic quadruple algebraic hyperstructures. Annals of Fuzzy Mathematics and Informatics 14(1), 29-42

ABSTRACT. The objective of this paper is to develop neutrosophic quadruple algebraic hyperstructures. Specifically, we develop neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings and we present elementary properties which characterize them.

Keywords: Neutrosophy, Neutrosophic quadruple number, Neutrosophic quadruple semihypergroup, Neutrosophic quadruple canonical hypergroup, Neutrosophic quadruple hyperrring.

1. Introduction

The concept of neutrosophic quadruple numbers was introduced by Florentin Smarandache [18]. It was shown in [18] how arithmetic operations of addition, subtraction, multiplication and scalar multiplication could be performed on the set of neutrosophic quadruple numbers. In [1], Akinleye et.al. introduced the notion of neutrosophic quadruple algebraic structures. Neutrosophic quadruple rings were studied and their basic properties were presented. In the present paper, two hyperoperations $\hat{+}$ and $\hat{\times}$ are defined on the neutrosophic set NQ of quadruple numbers to develop new algebraic hyperstructures which we call neutrosophic quadruple algebraic hyperstructures. Specifically, it is shown that $(NQ, \hat{\times})$ is a neutrosophic quadruple semihypergroup, $(NQ, \hat{+})$ is a neutrosophic quadruple canonical hypergroup and $(NQ, \hat{+}, \hat{\times})$ is a neutrosophic quadruple hyperrring and their basic properties are presented.

Definition 1.1 ([18])**.** A neutrosophic quadruple number is a number of the form (a, bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and $a, b, c, d \in \mathbb{R}$ or $\mathbb{C}$. The set NQ defined by

$$NQ = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\} \tag{1.1}$$

is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number (a, bT, cI, dF) representing any entity which may be a number, an idea, an object, etc, a is called the known part and (bT, cI, dF) is called the unknown part.

Definition 1.2. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. We define the following:

$$a + b = (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F), \tag{1.2}$$
$$a - b = (a_1 - b_1, (a_2 - b_2)T, (a_3 - b_3)I, (a_4 - b_4)F). \tag{1.3}$$

Definition 1.3. Let $a = (a_1, a_2T, a_3I, a_4F) \in NQ$ and let α be any scalar which may be real or complex, the scalar product $\alpha.a$ is defined by

$$\alpha.a = \alpha.(a_1, a_2T, a_3I, a_4F) = (\alpha a_1, \alpha a_2T, \alpha a_3I, \alpha a_4F). \tag{1.4}$$

If $\alpha = 0$, then we have $0.a = (0, 0, 0, 0)$ and for any non-zero scalars m and n and $b = (b_1, b_2T, b_3I, b_4F)$, we have:

$$\begin{aligned} (m + n)a &= ma + na, \\ m(a + b) &= ma + mb, \\ mn(a) &= m(na), \\ -a &= (-a_1, -a_2T, -a_3I, -a_4F). \end{aligned}$$

Definition 1.4 ([18])**.** [Absorbance Law] Let X be a set endowed with a total order $x < y$, named " x prevailed by y" or "x less stronger than y" or "x less preferred than y". $x \leq y$ is considered as "x prevailed by or equal to y" or "x less stronger than or equal to y" or "x less preferred than or equal to y".

For any elements $x, y \in X$, with $x \leq y$, absorbance law is defined as

$$x.y = y.x = \text{absorb}(x, y) = \max\{x, y\} = y \tag{1.5}$$

which means that the bigger element absorbs the smaller element (the big fish eats the small fish). It is clear from (1.5) that

$$x.x = x^2 = \text{absorb}(x, x) = \max\{x, x\} = x \quad \text{and} \tag{1.6}$$
$$x_1.x_2 \cdots x_n = \max\{x_1, x_2, \cdots, x_n\}. \tag{1.7}$$

Analogously, if $x > y$, we say that "x prevails to y" or "x is stronger than y" or "x is preferred to y". Also, if $x \geq y$, we say that "x prevails or is equal to y" or "x is stronger than or equal to y" or "x is preferred or equal to y".

Definition 1.5. Consider the set $\{T, I, F\}$. Suppose in an optimistic way we consider the prevalence order $T > I > F$. Then we have:

$$TI = IT = \max\{T, I\} = T, \tag{1.8}$$
$$TF = FT = \max\{T, F\} = T, \tag{1.9}$$
$$IF = FI = \max\{I, F\} = I, \tag{1.10}$$
$$TT = T^2 = T, \tag{1.11}$$
$$II = I^2 = I, \tag{1.12}$$
$$FF = F^2 = F. \tag{1.13}$$

Analogously, suppose in a pessimistic way we consider the prevalence order $T < I < F$. Then we have:

$$TI = IT = \max\{T, I\} = I, \tag{1.14}$$
$$TF = FT = \max\{T, F\} = F, \tag{1.15}$$
$$IF = FI = \max\{I, F\} = F, \tag{1.16}$$
$$TT = T^2 = T, \tag{1.17}$$
$$II = I^2 = I, \tag{1.18}$$
$$FF = F^2 = F. \tag{1.19}$$

Except otherwise stated, we will consider only the prevalence order $T < I < F$ in this paper.

Definition 1.6. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. Then

$$\begin{aligned} a.b &= (a_1, a_2T, a_3I, a_4F).(b_1, b_2T, b_3I, b_4F) \\ &= (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, \\ &\quad (a_1b_4 + a_2b_4, a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F). \end{aligned} \tag{1.20}$$

Theorem 1.7 ([1])**.** *$(NQ, +)$ is an abelian group.*

Theorem 1.8 ([1])**.** *$(NQ, .)$ is a commutative monoid.*

Theorem 1.9 ([1])**.** *$(NQ, .)$ is not a group.*

Theorem 1.10 ([1])**.** *$(NQ, +, .)$ is a commutative ring.*

Definition 1.11. Let NQR be a neutrosophic quadruple ring and let NQS be a nonempty subset of NQR. Then NQS is called a neutrosophic quadruple subring of NQR, if $(NQS, +, .)$ is itself a neutrosophic quadruple ring. For example, $NQR(n\mathbb{Z})$ is a neutrosophic quadruple subring of $NQR(\mathbb{Z})$ for $n = 1, 2, 3, \cdots$.

Definition 1.12. Let NQJ be a nonempty subset of a neutrosophic quadruple ring NQR. NQJ is called a neutrosophic quadruple ideal of NQR, if for all $x, y \in NQJ, r \in NQR$, the following conditions hold:

(i) $x - y \in NQJ$,
(ii) $xr \in NQJ$ and $rx \in NQJ$.

Definition 1.13 ([1])**.** Let NQR and NQS be two neutrosophic quadruple rings and let $\phi : NQR \to NQS$ be a mapping defined for all $x, y \in NQR$ as follows:

(i) $\phi(x + y) = \phi(x) + \phi(y)$,
(ii) $\phi(xy) = \phi(x)\phi(y)$,
(iii) $\phi(T) = T$, $\phi(I) = I$ and $\phi(F) = F$,
(iv) $\phi(1, 0, 0, 0) = (1, 0, 0, 0)$.

Then ϕ is called a neutrosophic quadruple homomorphism. Neutrosophic quadruple monomorphism, endomorphism, isomorphism, and other morphisms can be defined in the usual way.

Definition 1.14. Let $\phi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism.

(i) The image of ϕ denoted by $Im\phi$ is defined by the set

$$Im\phi = \{y \in NQS : y = \phi(x), \text{for some } x \in NQR\}.$$

(ii) The kernel of ϕ denoted by $Ker\phi$ is defined by the set

$$Ker\phi = \{x \in NQR : \phi(x) = (0,0,0,0)\}.$$

Theorem 1.15 ([1]). *Let $\phi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism. Then:*

(1) *$Im\phi$ is a neutrosophic quadruple subring of NQS,*
(2) *$Ker\phi$ is not a neutrosophic quadruple ideal of NQR.*

Theorem 1.16 ([1]). *Let $\phi : NQR(\mathbb{Z}) \to NQR(\mathbb{Z})/NQR(n\mathbb{Z})$ be a mapping defined by $\phi(x) = x + NQR(n\mathbb{Z})$ for all $x \in NQR(\mathbb{Z})$ and $n = 1, 2, 3, \ldots$. Then ϕ is not a neutrosophic quadruple ring homomorphism.*

Definition 1.17. Let H be a non-empty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:
(i) $x + y = y + x$, for all $x, y \in H$,
(ii) $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
(iii) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
(iv) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$.

A nonempty subset A of H is called a subcanonical hypergroup, if A is a canonical hypergroup under the same hyperaddition as that of H that is, for every $a, b \in A$, $a - b \in A$. If in addition $a + A - a \subseteq A$ for all $a \in H$, A is said to be normal.

Definition 1.18. A hyperring is a tripple $(R, +, .)$ satisfying the following axioms:
(i) $(R, +)$ is a canonical hypergroup,
(ii) $(R, .)$ is a semihypergroup such that $x.0 = 0.x = 0$ for all $x \in R$, that is, 0 is a bilaterally absorbing element,
(iii) for all $x, y, z \in R$,

$$x.(y + z) = x.y + x.z \text{ and } (x + y).z = x.z + y.z.$$

That is, the hyperoperation . is distributive over the hyperoperation $+$.

Definition 1.19. Let $(R, +, .)$ be a hyperring and let A be a nonempty subset of R. A is said to be a subhyperring of R if $(A, +, .)$ is itself a hyperring.

Definition 1.20. Let A be a subhyperring of a hyperring R. Then

(i) A is called a left hyperideal of R if $r.a \subseteq A$ for all $r \in R, a \in A$,
(ii) A is called a right hyperideal of R if $a.r \subseteq A$ for all $r \in R, a \in A$,
(iii) A is called a hyperideal of R if A is both left and right hyperideal of R.

Definition 1.21. Let A be a hyperideal of a hyperring R. A is said to be normal in R, if $r + A - r \subseteq A$, for all $r \in R$.

For full details about hypergroups, canonical hypergroups, hyperrings, neutrosophic canonical hypergroups and neutrosophic hyperrings, the reader should see [3, 14]

2. Development of neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings

In this section, we develop two neutrosophic hyperquadruple algebraic hyperstructures namely neutrosophic quadruple canonical hypergroup and neutrosophic quadruple hyperring . In what follows, all neutrosophic quadruple numbers will be real neutrosophic quadruple numbers i.e $a, b, c, d \in \mathbb{R}$ for any neutrosophic quadruple number $(a, bT, cI, dF) \in NQ$.

Definition 2.1. Let $+$ and $.$ be hyperoperations on $\mathbb{R}$ that is $x+y \subseteq \mathbb{R}, x.y \subseteq \mathbb{R}$ for all $x, y \in \mathbb{R}$. Let $\hat{+}$ and $\hat{\times}$ be hyperoperations on NQ. For $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F) \in NQ$ with $x_i, y_i \in \mathbb{R}, i = 1, 2, 3, 4$, define:

$$x\hat{+}y = \{(a, bT, cI, dF) : a \in x_1 + y_1, b \in x_2 + y_2, c \in x_3 + y_3, d \in x_4 + y_4\}, \tag{2.1}$$

$$\begin{aligned} x\hat{\times}y = \{&(a, bT, cI, dF) : a \in x_1.y_1, b \in (x_1.y_2) \cup (x_2.y_1) \cup (x_2.y_2), c \in (x_1.y_3) \\ &\cup (x_2.y_3) \cup (x_3.y_1) \cup (x_3.y_2) \cup (x_3.y_3), d \in (x_1.y_4) \cup (x_2.y_4) \\ &\cup (x_3.y_4) \cup (x_4.y_1) \cup (x_4.y_2) \cup (x_4.y_3) \cup (x_4.y_4)\}. \end{aligned} \tag{2.2}$$

Theorem 2.2. *$(NQ, \hat{+})$ is a canonical hypergroup.*

Proof. Let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$.

(i) To show that $x\hat{+}y = y\hat{+}x$, let

$$x\hat{+}y = \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1 + y_1, a_2 \in x_2 + y_2, a_3 \in x_3 + y_3, a_4 \in x_4 + y_4\},$$
$$y\hat{+}x = \{b = (b_1, b_2T, b_3I, b_4F) : b_1 \in y_1 + x_1, b_2 \in y_2 + x_2, b_3 \in y_3 + b_3, b_4 \in y_4 + x_4\}.$$

Since $a_i, b_i \in \mathbb{R}, i = 1, 2, 3, 4$, it follows that $x\hat{+}y = y\hat{+}x$.

(ii) To show that that $x\hat{+}(y\hat{+}z) = (x\hat{+}y)\hat{+}z$, let

$$y\hat{+}z = \{w = (w_1, w_2T, w_3I, w_4F) : w_1 \in y_1 + z_1, w_2 \in y_2 + z_2, w_3 \in y_3 + z_3, w_4 \in y_4 + z_4\}.$$ Now,

$$\begin{aligned} x\hat{+}(y\hat{+}z) &= x\hat{+}w \\ &= \{p = (p_1, p_2T, p_3I, p_4F) : p_1 \in x_1 + w_1, p_2 \in x_2 + w_2, p_3 \in x_3 + w_3, \\ &\quad p_4 \in x_4 + w_4\} \\ &= \{p = (p_1, p_2T, p_3I, p_4F) : p_1 \in x_1 + (y_1 + z_1), p_2 \in x_2 + (y_2 + z_2), \\ &\quad p_3 \in x_3 + (y_3 + z_3), p_4 \in x_4 + (y_4 + z_4)\}. \end{aligned}$$

Also, let $x\hat{+}y = \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in x_1 + y_1, u_2 \in x_2 + y_2, u_3 \in x_3 + y_3, u_4 \in x_4 + y_4\}$ so that

$$\begin{aligned} (x\hat{+}y)\hat{+}z &= u\hat{+}z \\ &= \{q = (q_1, q_2T, q_3I, q_4F) : q_1 \in u_1 + z_1, q_2 \in u_2 + z_2, q_3 \in u_3 + z_3, \\ &\quad q_4 \in u_4 + z_4\} \\ &= \{q = (q_1, q_2T, q_3I, q_4F) : q_1 \in (x_1 + y_1) + z_1, q_2 \in (x_2 + y_2) + z_2, \\ &\quad q_3 \in (x_3 + y_3) + z_3, q_4 \in (x_4 + y_4) + z_4\}. \end{aligned}$$

Since $u_i, p_i, q_i, w_i, x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$, it follows that $x\hat{+}(y\hat{+}z) = (x\hat{+}y)\hat{+}z$.

(iii) To show that $0 = (0, 0, 0, 0) \in NQ$ is a neutral element, consider

$$\begin{aligned} x\hat{+}(0,0,0,0) &= \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1 + 0, a_2 \in x_2 + 0, a_3 \in x_3 + 0, \\ &\quad a_4 \in x_4 + 0\} \\ &= \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in \{x_1\}, a_2 \in \{x_2\}, a_3 \in \{x_3\}, \\ &\quad a_4 \in \{x_4\}\} \\ &= \{x\}. \end{aligned}$$

Similarly, it can be shown that $(0, 0, 0, 0)\hat{+}x = \{x\}$. Hence $0 = (0, 0, 0, 0) \in NQ$ is a neutral element.

(iv) To show that that for every $x \in NQ$, there exists a unique element $\hat{-}x \in NQ$ such that $0 \in x\hat{+}(\hat{-}x) \cap (\hat{-}x)\hat{+}x$, consider

$$\begin{aligned} x\hat{+}(\hat{-}x) \cap (\hat{-}x)\hat{+}x &= \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1 - x_1, a_2 \in x_2 - x_2, \\ &\quad a_3 \in x_3 - x_3, a_4 \in x_4 - x_4\} \cap \{b = (b_1, b_2T, b_3I, b_4F) : \\ &\quad b_1 \in -x_1 + x_1, b_2 \in -x_2 + x_2, b_3 \in -x_3 + x_3, b_4 \in -x_4 + x_4\} \\ &= \{(0, 0, 0, 0)\}. \end{aligned}$$

This shows that for every $x \in NQ$, there exists a unique element $\hat{-}x \in NQ$ such that $0 \in x\hat{+}(\hat{-}x) \cap (\hat{-}x)\hat{+}x$.

(v) Since for all $x, y, z \in NQ$ with $x_i, y_1, z_i \in \mathbb{R}, i = 1, 2, 3, 4$, it follows that $z \in x\hat{+}y$ implies $y \in \hat{-}x\hat{+}z$ and $x \in z\hat{+}(\hat{-}y)$. Hence, $(NQ, \hat{+})$ is a canonical hypergroup. □

Lemma 2.3. *Let $(NQ, \hat{+})$ be a neutrosophic quadruple canonical hypergroup. Then*

(1) $\hat{-}(\hat{-}x) = x$ *for all* $x \in NQ$,

(2) $0 = (0, 0, 0, 0)$ *is the unique element such that for every* $x \in NQ$, *there is an element* $\hat{-}x \in NQ$ *such that* $0 \in x\hat{+}(\hat{-}x)$,

(3) $\hat{-}0 = 0$,

(4) $\hat{-}(x\hat{+}y) = \hat{-}x\hat{-}y$ *for all* $x, y \in NQ$.

Example 2.4. Let $NQ = \{0, x, y\}$ be a neutrosophic quadruple set and let $\hat{+}$ be a hyperoperation on NQ defined in the table below.

$\hat{+}$	0	x	y
0	0	x	y
x	x	$\{0, x, y\}$	y
y	y	y	$\{0, y\}$

Then $(NQ, \hat{+})$ is a neutrosophic quadruple canonical hypergroup.

Theorem 2.5. $(NQ, \hat{\times})$ *is a semihypergroup.*

Proof. Let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$.

(i)

$$\begin{aligned} x\hat{\times}y &= \{a=(a_1,a_2T,a_3I,a_4F): a_1\in x_1y_1, a_2\in x_1y_2\cup x_2y_1\cup x_2y_2, a_3\in x_1y_3 \\ &\quad \cup x_2y_3\cup x_3y_1\cup x_3y_2\cup x_3y_3, a_4\in x_1y_4\cup x_2y_4 \\ &\quad \cup x_3y_4\cup x_4y_1\cup x_4y_2\cup x_4y_3\cup x_4y_4\} \\ &\subseteq NQ. \end{aligned}$$

(ii) To show that $x\hat{\times}(y\hat{\times}z)=(x\hat{\times}y)\hat{\times}z$, let

$$\begin{aligned} y\hat{\times}z &= \{w=(w_1,w_2T,w_3I,w_4F): w_1\in y_1z_1, w_2\in y_1z_2\cup y_2z_1\cup y_2z_2, \\ &\quad w_3\in y_1z_3\cup y_2z_3\cup y_3z_1\cup y_3z_2\cup y_3z_3, w_4\in y_1z_4)\cup y_2z_4 \\ &\quad \cup y_3z_4\cup y_4z_1\cup y_4z_2\cup y_4z_3\cup y_4z_4\} \end{aligned} \tag{2.3}$$

so that

$$\begin{aligned} x\hat{\times}(y\hat{\times}z) &= x\hat{\times}w \\ &= \{p=(p_1,p_2T,p_3I,p_4F): p_1\in x_1w_1, p_2\in x_1w_2\cup x_2w_1\cup x_2w_2, \\ &\quad p_3\in x_1w_3\cup x_2w_3\cup x_3w_1\cup x_3w_2\cup x_3y_3, p_4\in x_1w_4\cup x_2w_4 \\ &\quad \cup x_3w_4\cup x_4w_1\cup x_4w_2\cup x_4w_3\cup x_4w_4\}. \end{aligned} \tag{2.4}$$

Also, let

$$\begin{aligned} x\hat{\times}y &= \{u=(u_1,u_2T,u_3I,u_4F): u_1\in x_1y_1, u_2\in x_1y_2\cup x_2y_1\cup x_2y_2, u_3\in x_1y_3 \\ &\quad \cup x_2y_3\cup x_3y_1\cup x_3y_2\cup x_3y_3, u_4\in x_1y_4\cup x_2y_4 \\ &\quad \cup x_3y_4\cup x_4y_1\cup x_4y_2\cup x_4y_3\cup x_4y_4\} \end{aligned} \tag{2.5}$$

so that

$$\begin{aligned} (x\hat{\times}y)\hat{\times}z &= u\hat{\times}z \\ &= \{q=(q_1,q_2T,q_3I,q_4F): q_1\in u_1z_1, q_2\in u_1z_2\cup u_2z_1\cup u_2z_2, \\ &\quad q_3\in u_1z_3\cup u_2z_3\cup u_3z_1\cup u_3z_2\cup u_3z_3, q_4\in u_1z_4\cup u_2z_4 \\ &\quad \cup u_3z_4\cup u_4z_1\cup u_4z_2\cup u_4z_3\cup u_4z_4\}. \end{aligned} \tag{2.6}$$

Substituting w_i of (2.3) in (2.4) and also substituting u_i of (2.5) in (2.6), where $i=1,2,3,4$ and since $p_i,q_i,u_i,w_i,x_i,z_i\in\mathbb{R}$, it follows that $x\hat{\times}(y\hat{\times}z)=(x\hat{\times}y)\hat{\times}z$. Consequently, $(NQ,\hat{\times})$ is a semihypergroup which we call neutrosophic quadruple semihypergroup. □

Remark 2.6. $(NQ,\hat{\times})$ is not a hypergroup.

Definition 2.7. Let $(NQ,\hat{+})$ be a neutrosophic quadruple canonical hypergroup. For any subset NH of NQ, we define

$$\hat{-}NH=\{\hat{-}x : x\in NH\}.$$

A nonempty subset NH of NQ is called a neutrosophic quadruple subcanonical hypergroup, if the following conditions hold:

(i) $0=(0,0,0,0)\in NH$,

(ii) $x\hat{-}y\subseteq NH$ for all $x,y\in NH$.

A neutrosophic quadruple subcanonical hypergroup NH of a netrosophic quadruple canonical hypergroup NQ is said to be normal, if $x\hat{+}NH\hat{-}x\subseteq NH$ for all $x\in NQ$.

Definition 2.8. Let $(NQ, \hat{+})$ be a neutrosophic quadruple canonical hypergroup. For $x_i \in NQ$ with $i = 1, 2, 3 \ldots, n \in \mathbb{N}$, the heart of NQ denoted by NQ_ω is defined by

$$NQ_\omega = \bigcup \sum_{i=1}^{n} (x_i \hat{-} x_i) .$$

In Example 2.4, $NQ_\omega = NQ$.

Definition 2.9. Let $(NQ_1, \hat{+})$ and $(NQ_2, \hat{+}')$ be two neutrosophic quadruple canonical hypergroups. A mapping $\phi : NQ_1 \to NQ_2$ is called a neutrosophic quadruple strong homomorphism, if the following conditions hold:

(i) $\phi(x \hat{+} y) = \phi(x) \hat{+}' \phi(y)$ for all $x, y \in NQ_1$,
(ii) $\phi(T) = T$,
(iii) $\phi(I) = I$,
(iv) $\phi(F) = F$,
(v) $\phi(0) = 0$.

If in addition ϕ is a bijection, then ϕ is called a neutrosophic quadruple strong isomorphism and we write $NQ_1 \cong NQ_2$.

Definition 2.10. Let $\phi : NQ_1 \to NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then the set $\{x \in NQ_1 : \phi(x) = 0\}$ is called the kernel of ϕ and it is denoted by $Ker\phi$. Also, the set $\{\phi(x) : x \in NQ_1\}$ is called the image of ϕ and it is denoted by $Im\phi$.

Theorem 2.11. *$(NQ, \hat{+}, \hat{\times})$ is a hyperring.*

Proof. That $(NQ, \hat{+})$ is a canonical hypergroup follows from Theorem 2.2. Also, that $(NQ, \hat{\times})$ is a semihypergroup follows from Theorem 2.4.

Next, let $x = (x_1, x_2T, x_3I, x_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$. Then

$$\begin{aligned} x \hat{\times} 0 &= \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in x_1.0, u_2 \in x_1.0 \cup x_2.0 \cup x_2.0, u_3 \in x_1.0 \\ &\quad \cup x_2.0 \cup x_3.0 \cup x_3.0 \cup x_3.0, u_4 \in x_1.0 \cup x_2.0 \cup x_3.0 \cup x_4.0 \cup x_4.0 \\ &\quad \cup x_4.0 \cup x_4.0\} \\ &= \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in \{0\}, u_2 \in \{0\}, u_3 \in \{0\}, u_4 \in \{0\}\} \\ &= \{0\}. \end{aligned}$$

Similarly, it can be shown that $0 \hat{\times} x = \{0\}$. Since x is arbitrary, it follows that $x \hat{\times} 0 = 0 \hat{\times} x = \{0\}$, for all $x \in NQ$. Hence, $0 = (0, 0, 0, 0)$ is a bilaterally absorbing element.

To complete the proof, we have to show that $x \hat{\times} (y \hat{+} z) = (x \hat{\times} y) \hat{+} (x \hat{\times} z)$, for all $x, y, z \in NQ$. To this end, let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ$ be arbitrary with $x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4$. Let

$$\begin{aligned} y \hat{+} z &= \{w = (w_1, w_2T, w_3I, w_4F) : w_1 \in y_1 + z_1, w_2 \in y_2 + z_2, w_3 \in y_3 + z_3, \\ &\quad w_4 \in y_4 + z_4\} \end{aligned} \tag{2.7}$$

so that

$$\begin{aligned} x\hat{\times}(y\hat{+}z) &= x\hat{\times}w \\ &= \{p=(p_1,p_2T,p_3I,p_4F): p_1\in x_1w_1, p_2\in x_1w_2\cup x_2w_1\cup x_2w_2, \\ &\quad p_3\in x_1w_3\cup x_2w_3\cup x_3w_1\cup x_3w_2\cup x_3y_3, p_4\in x_1w_4\cup x_2w_4 \\ &\quad \cup x_3w_4\cup x_4w_1\cup x_4w_2\cup x_4w_3\cup x_4w_4\}. \end{aligned} \tag{2.8}$$

Substituting $w_i, i=1,2,3,4$ of (2.7) in (2.8), we obtain the following:

$$p_1\in x_1(y_1+z_1), \tag{2.9}$$

$$p_2\in x_1(y_2+z_2)\cup x_2(y_1+z_1)\cup x_2(y_2+z_2), \tag{2.10}$$

$$p_3\in x_1(y_3+z_3)\cup x_2(y_3+z_3)\cup x_3(y_1+z_1)\cup x_3(y_2+z_2)\cup x_3(y_3+z_3), \tag{2.11}$$

$$\begin{aligned} &p_4\in x_1(y_4+z_4)\cup x_2(y_4+z_4)\cup x_3(y_4+z_4)\cup x_4(y_1+z_1)\cup x_4(y_2+z_2), \\ &\cup x_4(y_3+z_3)\cup x_4(y_4+z_4). \end{aligned} \tag{2.12}$$

Also, let

$$\begin{aligned} x\hat{\times}y &= \{u=(u_1,u_2T,u_3I,u_4F): u_1\in x_1y_1, u_2\in x_1y_2\cup x_2y_1\cup x_2y_2, \\ &\quad u_3\in x_1y_3\cup x_2y_3\cup x_3y_1\cup x_3y_2\cup x_3y_3, u_4\in x_1y_4\cup x_2y_4 \\ &\quad \cup x_3y_4\cup x_4y_1\cup x_4y_2\cup x_4y_3\cup x_4y_4\} \end{aligned} \tag{2.13}$$

$$\begin{aligned} x\hat{\times}z &= \{v=(v_1,v_2T,v_3I,v_4F): v_1\in x_1z_1, v_2\in x_1z_2\cup x_2z_1\cup x_2z_2, \\ &\quad v_3\in x_1z_3\cup x_2z_3\cup x_3z_1\cup x_3z_2\cup x_3z_3, v_4\in x_1z_4\cup x_2z_4 \\ &\quad \cup x_3z_4\cup x_4z_1\cup x_4z_2\cup x_4z_3\cup x_4z_4\} \end{aligned} \tag{2.14}$$

so that

$$\begin{aligned} (x\hat{\times}y)\hat{+}(x\hat{\times}z) &= u\hat{+}v \\ &= \{q=(q_1,q_2T,q_3I,q_4F): q_1\in u_1+v_1, q_2\in u_2+v_2, \\ &\quad q_3\in u_3+v_3, q_4\in u_4+v_4\}. \end{aligned} \tag{2.15}$$

Substituting u_i of (2.13) and v_i of (2.14) in (2.15), we obtain the following:

$$q_1\in u_1+v_1\subseteq x_1y_1+x_1z_1\subseteq x_1(y_1+z_1), \tag{2.16}$$

$$\begin{aligned} &q_2\in u_2+v_2\subseteq (x_1y_2\cup x_2y_1\cup x_2y_2) \\ &+(x_1z_2\cup x_2z_1\cup x_2(z_2) \\ &\subseteq x_1(y_2+z_2)\cup x_2(y_1+z_1)\cup x_2(y_2+z_2), \end{aligned} \tag{2.17}$$

$$\begin{aligned} &q_3\in u_3+v_3\subseteq (x_1y_3\cup x_2y_3\cup x_3y_1)\cup x_3y_2\cup x_3y_3) \\ &+(x_1z_3\cup x_2z_3\cup x_3z_1)\cup x_3z_2\cup x_3z_3) \\ &\subseteq x_1(y_3+z_3)\cup x_2(y_3+z_3)\cup x_3(y_1+z_1)\cup x_3(y_2+z_2)\cup x_3(y_3+z_3). \end{aligned} \tag{2.18}$$

$$\begin{aligned} &q_4\in u_4+v_4\subseteq (x_1y_4\cup x_2y_4\cup x_3y_4)\cup x_4y_1\cup x_4y_2)\cup x_4y_3\cup x_4y_4) \\ &+(x_1z_4\cup x_2z_4\cup x_3z_4)\cup x_4z_1\cup x_4z_2)\cup x_4z_3\cup x_4z_4) \\ &\subseteq x_1(y_4+z_4)\cup x_2(y_4+z_4)\cup x_3(y_4+z_4)\cup x_4(y_1+z_1)\cup x_4(y_2+z_2) \\ &\cup x_4(y_3+z_3)\cup x_4(y_4+z_4). \end{aligned} \tag{2.19}$$

Comparing (2.9), (2.10), (2.11) and (2.12) respectively with (2.16), (2.17), (2.18) and (2.19), we obtain $p_i=q_i$, $i=1,2,3,4$. Hence, $x\hat{\times}(y\hat{+}z)=(x\hat{\times}y)\hat{+}(x\hat{\times}z)$, for all

$x, y, z \in NQ$. Thus, $(NQ, \hat{+}, \hat{\times})$ is a hyperring which we call neutrosophic quadruple hyperring. $\square$

Theorem 2.12. *$(NQ, \hat{+}, \circ)$ is a Krasner hyperring where $\circ$ is an ordinary multiplicative binary operation on NQ.*

Definition 2.13. Let $(NQ, \hat{+}, \hat{\times})$ be a neutrosophic quadruple hyperring. A nonempty subset NJ of NQ is called a neutrosophic quadruple subhyperring of NQ, if $(NJ, \hat{+}, \hat{\times})$ is itself a neutrosophic quadruple hyperring.

NJ is called a neutrosophic quadruple hyperideal if the following conditions hold:

(i) $(NJ, \hat{+})$ is a neutrosophic quadruple subcanonical hypergroup.

(ii) For all $x \in NJ$ and $r \in NQ$, $x\hat{\times}r, r\hat{\times}x \subseteq NJ$.

A neutrosophic quadruple hyperideal NJ of NQ is said to be normal in NQ, if $x\hat{+}NJ\hat{-}x \subseteq NJ$, for all $x \in NQ$.

Definition 2.14. Let $(NQ_1, \hat{+}, \hat{\times})$ and $(NQ_2, \hat{+}', \hat{\times}')$ be two neutrosophic quadruple hyperrings. A mapping $\phi : NQ_1 \to NQ_2$ is called a neutrosophic quadruple strong homomorphism, if the following conditions hold:

(i) $\phi(x\hat{+}y) = \phi(x)\hat{+}'\phi(y)$, for all $x, y \in NQ_1$,

(ii) $\phi(x\hat{\times}y) = \phi(x)\hat{\times}'\phi(y)$, for all $x, y \in NQ_1$,

(iii) $\phi(T) = T$,

(iv) $\phi(I) = I$,

(v) $\phi(F) = F$,

(vi) $\phi(0) = 0$.

If in addition ϕ is a bijection, then ϕ is called a neutrosophic quadruple strong isomorphism and we write $NQ_1 \cong NQ_2$.

Definition 2.15. Let $\phi : NQ_1 \to NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple hyperrings. Then the set $\{x \in NQ_1 : \phi(x) = 0\}$ is called the kernel of ϕ and it is denoted by $Ker\phi$. Also, the set $\{\phi(x) : x \in NQ_1\}$ is called the image of ϕ and it is denoted by $Im\phi$.

Example 2.16. Let $(NQ, \hat{+}, \hat{\times})$ be a neutrosophic quadruple hyperring and let NX be the set of all strong endomorphisms of NQ. If $\oplus$ and $\odot$ are hyperoperations defined for all $\phi, \psi \in NX$ and for all $x \in NQ$ as

$$\begin{aligned} \phi \oplus &= \{\nu(x) : \nu(x) \in \phi(x)\hat{+}\psi(x)\}, \\ \phi \odot &= \{\nu(x) : \nu(x) \in \phi(x)\hat{\times}\psi(x)\}, \end{aligned}$$

then $(NX, \oplus, \odot)$ is a neutrosophic quadruple hyperring.

3. Characterization of neutrosophic quadruple canonical hypergroups and neutrosophic hyperrings

In this section, we present elementary properties which characterize neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings.

Theorem 3.1. *Let NG and NH be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$. Then*

(1) $NG \cap NH$ is a neutrosophic quadruple subcanonical hypergroup of NQ,

(2) $NG \times NH$ *is a neutrosophic quadruple subcanonical hypergroup of* NQ.

Theorem 3.2. *Let* NH *be a neutrosophic quadruple subcanonical hypergroup of a neutrosophic quadruple canonical hypergroup* $(NQ, \hat{+})$. *Then*

(1) $NH \hat{+} NH = NH$,
(2) $x \hat{+} NH = NH$, *for all* $x \in NH$.

Theorem 3.3. *Let* $(NQ, \hat{+})$ *be a neutrosophic quadruple canonical hypergroup.* NQ_ω, *the heart of* NQ *is a normal neutrosophic quadruple subcanonical hypergroup of* NQ.

Theorem 3.4. *Let* NG *and* NH *be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup* $(NQ, \hat{+})$.

(1) *If* $NG \subseteq NH$ *and* NG *is normal, then* NG *is normal.*
(2) *If* NG *is normal, then* $NG \hat{+} NH$ *is normal.*

Definition 3.5. Let NG and NH be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$. The set $NG \hat{+} NH$ is defined by

$$NG \hat{+} NH = \{x \hat{+} y : x \in NG, y \in NH\}. \tag{3.1}$$

It is obvious that $NG \hat{+} NH$ is a neutrosophic quadruple subcanonical hypergroup of $(NQ, \hat{+})$.

If $x \in NH$, the set $x \hat{+} NH$ is defined by

$$x \hat{+} NH = \{x \hat{+} y : y \in NH\}. \tag{3.2}$$

If x and y are any two elements of NH and τ is a relation on NH defined by $x \tau y$ if $x \in y \hat{+} NH$, it can be shown that τ is an equivalence relation on NH and the equivalence class of any element $x \in NH$ determined by τ is denoted by $[x]$.

Lemma 3.6. *For any* $x \in NH$, *we have*

(1) $[x] = x \hat{+} NH$,
(2) $[\hat{-}x] = \hat{-}[x]$.

Proof. (1)

$$\begin{aligned} [x] &= \{y \in NH : x \tau y\} \\ &= \{y \in NH : y \in x \hat{+} NH\} \\ &= x \hat{+} NH. \end{aligned}$$

(2) Obvious. □

Definition 3.7. Let NQ/NH be the collection of all equivalence classes of $x \in NH$ determined by τ. For $[x], [y] \in NQ/NH$, we define the set $[x] \hat{\oplus} [y]$ as

$$[x] \hat{\oplus} [y] = \{[z] : z \in x \hat{+} y\}. \tag{3.3}$$

Theorem 3.8. $(NQ/NH, \hat{\oplus})$ *is a neutrosophic quadruple canonical hypergroup.*

Proof. Same as the classical case and so omitted. □

Theorem 3.9. *Let $(NQ, \hat{+})$ be a neutrosophic quadruple canonical hypergroup and let NH be a normal neutrosophic quadruple subcanonical hypergroup of NQ. Then, for any $x, y \in NH$, the following are equivalent:*

(1) $x \in y\hat{+}NH$,
(2) $y\hat{-}x \subseteq NH$,
(3) $(y\hat{-}x) \cap NH \neq \varnothing$

Proof. Same as the classical case and so omitted. □

Theorem 3.10. *Let $\phi : NQ_1 \to NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then*

(1) *$Ker\phi$ is not a neutrosophic quadruple subcanonical hypergroup of NQ_1,*
(2) *$Im\phi$ is a neutrosophic quadruple subcanonical hypergroup of NQ_2.*

Proof. (1) Since it is not possible to have $\phi((0, T, 0, 0)) = \phi((0, 0, 0, 0))$, $\phi((0, 0, I, 0)) = \phi((0, 0, 0, 0))$ and $\phi((0, 0, 0, F)) = \phi((0, 0, 0, 0))$, it follows that $(0, T, 0, 0), (0, 0, I, 0)$ and $(0, 0, 0, F)$ cannot be in the kernel of ϕ. Consequently, $Ker\phi$ cannot be a neutrosophic quadruple subcanonical hypergroup of NQ_1.

(2) Obvious. □

Remark 3.11. If $\phi : NQ_1 \to NQ_2$ is a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups, then $Ker\phi$ is a subcanonical hypergroup of NQ_1.

Theorem 3.12. *Let $\phi : NQ_1 \to NQ_2$ be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then*

(1) *$NQ_1/Ker\phi$ is not a neutrosophic quadruple canonical hypergroup,*
(2) *$NQ_1/Ker\phi$ is a canonical hypergroup.*

Theorem 3.13. *Let NH be a neutrosophic quadruple subcanonical hypergroup of the neutrosophic quadruple canonical hypergroup $(NQ, \hat{+})$. Then the mapping $\phi : NQ \to NQ/NH$ defined by $\phi(x) = x\hat{+}NH$ is not a neutrosophic quadruple strong homomorphism.*

Remark 3.14. Isomorphism theorems do not hold in the class of neutrosophic quadruple canonical hypergroups.

Lemma 3.15. *Let NJ be a neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. Then*

(1) $\hat{-}NJ = NJ$,
(2) $x\hat{+}NJ = NJ$, *for all* $x \in NJ$,
(3) $x\hat{\times}NJ = NJ$, *for all* $x \in NJ$.

Theorem 3.16. *Let NJ and NK be neutrosophic quadruple hyperideals of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. Then*

(1) *$NJ \cap NK$ is a neutrosophic quadruple hyperideal of NQ,*
(2) *$NJ \times NK$ is a neutrosophic quadruple hyperideal of NQ,*
(3) *$NJ\hat{+}NK$ is a neutrosophic quadruple hyperideal of NQ.*

Theorem 3.17. *Let NJ be a normal neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. Then*

(1) $(x\hat{+}NJ)\hat{+}(y\hat{+}NJ) = (x\hat{+}y)\hat{+}NJ$, *for all* $x, y \in NJ$,
(2) $(x\hat{+}NJ)\hat{\times}(y\hat{+}NJ) = (x\hat{\times}y)\hat{+}NJ$, *for all* $x, y \in NJ$,
(3) $x\hat{+}NJ = y\hat{+}NJ$, *for all* $y \in x\hat{+}NJ$.

Theorem 3.18. *Let* NJ *and* NK *be neutrosophic quadruple hyperideals of a neutrosophic quadruple hyperring* $(NQ, \hat{+}, \hat{\times})$ *such that* NJ *is normal in* NQ. *Then*
(1) $NJ \cap NK$ *is normal in* NJ,
(2) $NJ\hat{+}NK$ *is normal in* NQ,
(3) NJ *is normal in* $NJ\hat{+}NK$.

Let NJ be a neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring $(NQ, \hat{+}, \hat{\times})$. For all $x \in NQ$, the set NQ/NJ is defined as

$$NQ/NJ = \{x\hat{+}NJ : x \in NQ\}. \tag{3.4}$$

For $[x], [y] \in NQ/NJ$, we define the hyperoperations $\hat{\oplus}$ and $\hat{\otimes}$ on NQ/NJ as follows:

$$[x]\hat{\oplus}[y] = \{[z] : z \in x\hat{+}y\}, \tag{3.5}$$

$$[x]\hat{\otimes}[y] = \{[z] : z \in x\hat{\times}y\}. \tag{3.6}$$

It can easily be shown that $(NQ/NH, \hat{\oplus}, \hat{\otimes})$ is a neutrosophic quadruple hyperring.

Theorem 3.19. *Let* $\phi : NQ \to NR$ *be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple hyperrings and let* NJ *be a neutrosophic quadruple hyperideal of* NQ. *Then*
(1) $Ker\phi$ *is not a neutrosophic quadruple hyperideal of* NQ,
(2) $Im\phi$ *is a neutrosophic quadruple hyperideal of* NR,
(3) $NQ/Ker\phi$ *is not a neutrosophic quadruple hyperring*,
(4) $NQ/Im\phi$ *is a neutrosophic quadruple hyperring*,
(5) *The mapping* $\psi : NQ \to NQ/NJ$ *defined by* $\psi(x) = x\hat{+}NJ$, *for all* $x \in NQ$ *is not a neutrosophic quadruple strong homomorphism.*

Remark 3.20. The classical isomorphism theorems of hyperrings do not hold in neutrosophic quadruple hyperrings.

4. Conclusion

We have developed neutrosophic quadruple algebraic hyperstrutures in this paper. In particular, we have developed new neutrosophic algebraic hyperstructures namely neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings. We have presented elementary properties which characterize the new neutrosophic algebraic hyperstructures.

Acknowledgements. The authors thank all the anonymous reviewers for useful observations and critical comments which have improved the quality of the paper.

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Neutrosophic $\mathcal{N}$-Structures Applied to BCK/BCI-Algebras

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Young Bae Jun, Florentin Smarandache, Hashem Bordbar (2017). Neutrosophic N-Structures Applied to BCK/BCI-Algebras. *Information* 8, 128; DOI: 10.3390/info8040128

Abstract: Neutrosophic $\mathcal{N}$-structures with applications in BCK/BCI-algebras is discussed. The notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra are introduced, and several related properties are investigated. Characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are considered, and relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are stated. Conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal are provided.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-subalgebra; (closed) neutrosophic $\mathcal{N}$-ideal

1. Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (MV-algebras). Additionally, Iséki introduced the notion of a BCI-algebra, which is a generalization of a BCK-algebra (see [2]).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far, most of the generalizations of the crisp set have been conducted on the unit interval $[0,1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool. To attain such an object, Jun et al. [3] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Zadeh [4] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [5] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components:

(t, i, f) = (truth, indeterminacy, falsehood)

For more details, refer to the following site:

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In this paper, we discuss a neutrosophic $\mathcal{N}$-structure with an application to BCK/BCI-algebras. We introduce the notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra, and investigate related properties. We consider characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We discuss relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal.

2. Preliminaries

We let $K(\tau)$ be the class of all algebras with type $\tau = (2,0)$. A *BCI-algebra* refers to a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

(I) $((x*y)*(x*z))*(z*y) = \theta$,
(II) $(x*(x*y))*y = \theta$,
(III) $x*x = \theta$,
(IV) $x*y = y*x = \theta \Rightarrow x = y$.

for all $x, y, z \in X$. If a BCI-algebra X satisfies $\theta * x = \theta$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\ (x \preceq y \Rightarrow x*y = \theta)$$

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x*\theta = x) \tag{1}$$
$$(\forall x, y, z \in X)\ ((x*y)*z = (x*z)*y) \tag{2}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y \in S$ for all $x, y \in S$.

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following:

(I1) $0 \in I$,
(I2) $(\forall x, y \in X)(x*y \in I,\ y \in I \Rightarrow x \in I)$.

We refer the reader to the books [6,7] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

We denote by $\mathcal{F}(X, [-1,0])$ the collection of functions from a set X to $[-1,0]$. We say that an element of $\mathcal{F}(X, [-1,0])$ is a *negative-valued function* from X to $[-1,0]$ (briefly, $\mathcal{N}$*-function* on X). An $\mathcal{N}$*-structure* refers to an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X (see [3]). In what follows, we let X denote the nonempty universe of discourse unless otherwise specified.

A *neutrosophic* $\mathcal{N}$*-structure* over X (see [8]) is defined to be the structure:

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{3}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X, which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

We note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$$

3. Application in BCK/BCI-Algebras

In this section, we take a BCK/BCI-algebra X as the universe of discourse unless otherwise specified.

Definition 1. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-subalgebra of X if the following condition is valid:*

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\} \end{array} \right) \tag{4}$$

Example 1. *Consider a BCK-algebra $X = \{\theta, a, b, c\}$ with the following Cayley table.*

$*$	$\boldsymbol{\theta}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
θ	θ	θ	θ	θ
a	a	θ	θ	a
b	b	a	θ	b
c	c	c	c	θ

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.7,-0.2,-0.6)}, \frac{a}{(-0.5,-0.3,-0.4)}, \frac{b}{(-0.5,-0.3,-0.4)}, \frac{c}{(-0.3,-0.8,-0.5)} \right\}$$

over X is a neutrosophic $\mathcal{N}$-subalgebra of X.

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$\begin{aligned} T_N^{\alpha} &:= \{x \in X \mid T_N(x) \leq \alpha\} \\ I_N^{\beta} &:= \{x \in X \mid I_N(x) \geq \beta\} \\ F_N^{\gamma} &:= \{x \in X \mid F_N(x) \leq \gamma\} \end{aligned}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the (α, β, γ)-*level set* of $X_{\mathbf{N}}$. Note that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$$

Theorem 1. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is a subalgebra of X.*

Proof. Let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$ and $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$. If $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (4) that

$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha$,
$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta$, and
$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma$.

Hence, $x * y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, and therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subalgebra of X. □

Theorem 2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are subalgebras of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Proof. Assume that there exist $a, b \in X$ such that $T_N(a * b) > \bigvee\{T_N(a), T_N(b)\}$. Then $T_N(a * b) > t_\alpha \geq \bigvee\{T_N(a), T_N(b)\}$ for some $t_\alpha \in [-1, 0)$. Hence $a, b \in T_N^{t_\alpha}$ but $a * b \notin T_N^{t_\alpha}$, which is a contradiction. Thus

$$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\}$$

for all $x, y \in X$. If $I_N(a * b) < \bigwedge\{I_N(a), I_N(b)\}$ for some $a, b \in X$, then

$$I_N(a * b) < t_\beta < \bigwedge\{I_N(a), I_N(b)\}$$

where $t_\beta := \frac{1}{2}\{I_N(a * b) + \bigwedge\{I_N(a), I_N(b)\}\}$. Thus $a, b \in I_N^{t_\beta}$ and $a * b \notin I_N^{t_\beta}$, which is a contradiction. Therefore

$$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\}$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(a * b) > t_\gamma \geq \bigvee\{F_N(a), F_N(b)\}$$

Then $a, b \in F_N^{t_\gamma}$ and $a * b \notin F_N^{t_\gamma}$, which is a contradiction. Hence

$$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

Because $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3. *If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic $\mathcal{N}$-subalgebras of X, then $(\{X_{N_i} \mid i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Proposition 1. *If a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is a neutrosophic $\mathcal{N}$-subalgebra of X, then $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 4. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_N(a_n) = -1$, $\lim_{n\to\infty} I_N(a_n) = 0$ and $\lim_{n\to\infty} F_N(a_n) = -1$, then $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$.*

Proof. By Proposition 1, we have $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Hence $T_N(\theta) \leq T_N(a_n)$, $I_N(a_n) \leq I_N(\theta)$ and $F_N(\theta) \leq F_N(a_n)$ for every positive integer n. It follows that

$$
\begin{aligned}
-1 &\leq T_N(\theta) \leq \lim_{n\to\infty} T_N(a_n) = -1 \\
0 &\geq I_N(\theta) \geq \lim_{n\to\infty} I_N(a_n) = 0 \\
-1 &\leq F_N(\theta) \leq \lim_{n\to\infty} F_N(a_n) = -1
\end{aligned}
$$

Hence $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$. □

Proposition 2. *If every neutrosophic $\mathcal{N}$-subalgebra $X_{\mathbf{N}}$ of X satisfies:*

$$
T_N(x*y) \leq T_N(y), I_N(x*y) \geq I_N(y), F_N(x*y) \leq F_N(y) \tag{5}
$$

for all $x, y \in X$, then $X_{\mathbf{N}}$ is constant.

Proof. Using Equations (1) and (5), we have $T_N(x) = T_N(x*\theta) \leq T_N(\theta)$, $I_N(x) = I_N(x*\theta) \geq I_N(\theta)$ and $F_N(x) = F_N(x*\theta) \leq F_N(\theta)$ for all $x \in X$. It follows from Proposition 1 that $T_N(x) = T_N(\theta)$, $I_N(x) = I_N(\theta)$ and $F_N(x) = F_N(\theta)$ for all $x \in X$. Therefore $X_{\mathbf{N}}$ is constant. □

Definition 2. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-ideal of X if the following assertion is valid:*

$$
(\forall x, y \in X) \begin{pmatrix} T_N(\theta) \leq T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} \\ I_N(\theta) \geq I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} \\ F_N(\theta) \leq F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} \end{pmatrix} \tag{6}
$$

Example 2. *The neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X in Example 1 is a neutrosophic $\mathcal{N}$-ideal of X.*

Example 3. *Consider a BCI-algebra $X := Y \times \mathbb{Z}$ where $(Y, *, \theta)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see [6]). Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X given by*

$$
X_{\mathbf{N}} = \left\{ \tfrac{x}{(\alpha,0,\gamma)} \mid x \in Y \times (\mathbb{N} \cup \{0\}) \right\} \cup \left\{ \tfrac{x}{(0,\beta,0)} \mid x \notin Y \times (\mathbb{N} \cup \{0\}) \right\}
$$

where $\alpha, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.

Proposition 3. *Every neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X satisfies the following assertions:*

$$
(x, y \in X)\ (x \preceq y \ \Rightarrow\ T_N(x) \leq T_N(y), I_N(x) \geq I_N(y), F_N(x) \leq F_N(y)) \tag{7}
$$

Proof. Let $x, y \in X$ be such that $x \preceq y$. Then $x * y = \theta$, and so

$$
\begin{aligned}
&T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} = \bigvee\{T_N(\theta), T_N(y)\} = T_N(y) \\
&I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} = \bigwedge\{I_N(\theta), I_N(y)\} = I_N(y) \\
&F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} = \bigvee\{F_N(\theta), F_N(y)\} = F_N(y)
\end{aligned}
$$

This completes the proof. □

Proposition 4. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then*

(1) $T_N(x*y) \leq T_N((x*y)*y) \ \Leftrightarrow\ T_N((x*z)*(y*z)) \leq T_N((x*y)*z)$
(2) $I_N(x*y) \geq I_N((x*y)*y) \ \Leftrightarrow\ I_N((x*z)*(y*z)) \geq I_N((x*y)*z)$
(3) $F_N(x*y) \leq F_N((x*y)*y) \ \Leftrightarrow\ F_N((x*z)*(y*z)) \leq F_N((x*y)*z)$

for all $x, y, z \in X$.

Proof. Note that

$$((x*(y*z))*z)*z \preceq (x*y)*z \tag{8}$$

for all $x, y, z \in X$. Assume that $T_N(x*y) \leq T_N((x*y)*y)$, $I_N(x*y) \geq I_N((x*y)*y)$ and $F_N(x*y) \leq F_N((x*y)*y)$ for all $x, y \in X$. It follows from Equation (2) and Proposition 3 that

$$\begin{aligned} T_N((x*z)*(y*z)) &= T_N((x*(y*z))*z) \\ &\leq T_N(((x*(y*z))*z)*z) \\ &\leq T_N((x*y)*z) \end{aligned}$$

$$\begin{aligned} I_N((x*z)*(y*z)) &= I_N((x*(y*z))*z) \\ &\geq I_N(((x*(y*z))*z)*z) \\ &\geq I_N((x*y)*z) \end{aligned}$$

and

$$\begin{aligned} F_N((x*z)*(y*z)) &= F_N((x*(y*z))*z) \\ &\leq F_N(((x*(y*z))*z)*z) \\ &\leq F_N((x*y)*z) \end{aligned}$$

for all $x, y \in X$.

Conversely, suppose

$$\begin{aligned} T_N((x*z)*(y*z)) &\leq T_N((x*y)*z) \\ I_N((x*z)*(y*z)) &\geq I_N((x*y)*z) \\ F_N((x*z)*(y*z)) &\leq F_N((x*y)*z) \end{aligned} \tag{9}$$

for all $x, y, z \in X$. If we substitute z for y in Equation (9), then

$$\begin{aligned} T_N(x*z) &= T_N((x*z)*\theta) = T_N((x*z)*(z*z)) \leq T_N((x*z)*z) \\ I_N(x*z) &= I_N((x*z)*\theta) = I_N((x*z)*(z*z)) \geq I_N((x*z)*z) \\ F_N(x*z) &= F_N((x*z)*\theta) = F_N((x*z)*(z*z)) \leq F_N((x*z)*z) \end{aligned}$$

for all $x, z \in X$ by using (III) and Equation (1). □

Theorem 5. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is an ideal of X.*

Proof. Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Clearly, $\theta \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Let $x, y \in X$ be such that $x*y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$ and $y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x*y) \leq \alpha$, $I_N(x*y) \geq \beta$, $F_N(x*y) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (6) that

$$\begin{aligned} T_N(x) &\leq \bigvee\{T_N(x*y), T_N(y)\} \leq \alpha \\ I_N(x) &\geq \bigwedge\{I_N(x*y), I_N(y)\} \geq \beta \\ F_N(x) &\leq \bigvee\{F_N(x*y), F_N(y)\} \leq \gamma \end{aligned}$$

so that $x \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is an ideal of X. □

Theorem 6. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Proof. If there exist $a, b, c \in X$ such that $T_N(\theta) > T_N(a)$, $I_N(\theta) < I_N(b)$ and $F_N(\theta) > F_N(c)$, respectively, then $T_N(\theta) > a_t \geq T_N(a)$, $I_N(\theta) < b_i \leq I_N(b)$ and $F_N(\theta) > c_f \geq F_N(c)$ for some $a_t, c_f \in [-1, 0)$ and $b_i \in (-1, 0]$. Then $\theta \notin T_N^{a_t}$, $\theta \notin I_N^{b_i}$ and $\theta \notin F_N^{c_f}$. This is a contradiction. Hence, $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_N(a_t) > \bigvee\{T_N(a_t * b_t), T_N(b_t)\}$, $I_N(a_i) < \bigwedge\{I_N(a_i * b_i), I_N(b_i)\}$ and $F_N(a_f) > \bigvee\{F_N(a_f * b_f), F_N(b_f)\}$. Then there exist $s_t, s_f \in [-1, 0)$ and $s_i \in (-1, 0]$ such that

$$
\begin{aligned}
&T_N(a_t) > s_t \geq \bigvee\{T_N(a_t * b_t), T_N(b_t)\} \\
&I_N(a_i) < s_i \leq \bigwedge\{I_N(a_i * b_i), I_N(b_i)\} \\
&F_N(a_f) > s_f \geq \bigvee\{F_N(a_f * b_f), F_N(b_f)\}
\end{aligned}
$$

It follows that $a_t * b_t \in T_N^{s_t}$, $b_t \in T_N^{s_t}$, $a_i * b_i \in I_N^{s_i}$, $b_i \in I_N^{s_i}$, $a_f * b_f \in F_N^{s_f}$ and $b_f \in F_N^{s_f}$. However, $a_t \notin T_N^{s_t}$, $a_i \notin I_N^{s_i}$ and $a_f \notin F_N^{s_f}$. This is a contradiction, and so

$$
\begin{aligned}
&T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \\
&I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \\
&F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\}
\end{aligned}
$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X. □

Proposition 5. *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X, we have*

$$
(\forall x, y, z \in X)\left(x * y \preceq z \ \Rightarrow \begin{cases} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{cases} \right) \tag{10}
$$

Proof. Let $x, y, z \in X$ be such that $x * y \preceq z$. Then $(x * y) * z = \theta$, and so

$$
\begin{aligned}
&T_N(x * y) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} = \bigvee\{T_N(\theta), T_N(z)\} = T_N(z) \\
&I_N(x * y) \geq \bigwedge\{I_N((x * y) * z), I_N(z)\} = \bigwedge\{I_N(\theta), I_N(z)\} = I_N(z) \\
&F_N(x * y) \leq \bigvee\{F_N((x * y) * z), F_N(z)\} = \bigvee\{F_N(\theta), F_N(z)\} = F_N(z)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \leq \bigvee\{T_N(y), T_N(z)\} \\
&I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \geq \bigwedge\{I_N(y), I_N(z)\} \\
&F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \leq \bigvee\{F_N(y), F_N(z)\}
\end{aligned}
$$

This completes the proof. □

Theorem 7. *In a BCK-algebra, every neutrosophic $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-subalgebra.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of a BCK-algebra X. For any $x, y \in X$, we have

$$\begin{aligned} T_N(x*y) &\leq \bigvee\{T_N((x*y)*x), T_N(x)\} = \bigvee\{T_N((x*x)*y), T_N(x)\} \\ &= \bigvee\{T_N(\theta*y), T_N(x)\} = \bigvee\{T_N(\theta), T_N(x)\} \\ &\leq \bigvee\{T_N(x), T_N(y)\} \end{aligned}$$

$$\begin{aligned} I_N(x*y) &\geq \bigwedge\{I_N((x*y)*x), I_N(x)\} = \bigwedge\{I_N((x*x)*y), I_N(x)\} \\ &= \bigwedge\{I_N(\theta*y), I_N(x)\} = \bigwedge\{I_N(\theta), I_N(x)\} \\ &\geq \bigwedge\{I_N(y), I_N(x)\} \end{aligned}$$

and

$$\begin{aligned} F_N(x*y) &\leq \bigvee\{F_N((x*y)*x), F_N(x)\} = \bigvee\{F_N((x*x)*y), F_N(x)\} \\ &= \bigvee\{F_N(\theta*y), F_N(x)\} = \bigvee\{F_N(\theta), F_N(x)\} \\ &\leq \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of a BCK-algebra X. □

The converse of Theorem 7 may not be true in general, as seen in the following example.

Example 4. *Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table.*

$*$	$\boldsymbol{\theta}$	**1**	**2**	**3**	**4**
θ	θ	θ	θ	θ	θ
1	1	θ	θ	θ	θ
2	2	1	θ	1	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X, which is given as follows:

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.8, 0, -1)}, \frac{1}{(-0.8, -0.2, -0.9)}, \right. \\ \left. \frac{2}{(-0.2, -0.6, -0.5)}, \frac{3}{(-0.7, -0.4, -0.7)}, \frac{4}{(-0.4, -0.8, -0.3)} \right\}$$

Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, but it is not a neutrosophic $\mathcal{N}$-ideal of X as $T_N(2) = -0.2 > -0.7 = \bigvee\{T_N(2*3), T_N(3)\}$, $I_N(4) = -0.8 < -0.4 = \bigwedge\{I_N(4*3), I_N(3)\}$, or $F_N(4) = -0.3 > -0.7 = \bigvee\{F_N(4*3), F_N(3)\}$.

Theorem 7 is not valid in a BCI-algebra; that is, if X is a BCI-algebra, then there is a neutrosophic $\mathcal{N}$-ideal that is not a neutrosophic $\mathcal{N}$-subalgebra, as seen in the following example.

Example 5. *Consider the neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X in Example 3. If we take $x := (\theta, 0)$ and $y := (\theta, 1)$ in $Y \times (\mathbb{N} \cup \{0\})$, then $x * y = (\theta, 0) * (\theta, 1) = (\theta, -1) \notin Y \times (\mathbb{N} \cup \{0\})$. Hence*

$$\begin{aligned} T_N(x*y) &= 0 > \alpha = \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) &= \beta < 0 = \bigwedge\{I_N(x), I_N(y)\} \text{ or} \\ F_N(x*y) &= 0 > \gamma = \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

Therefore $X_{\mathbf{N}}$ is not a neutrosophic $\mathcal{N}$-subalgebra of X.

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$X_{\mathbf{N}}^{\omega_t} := \{x \in X \mid T_N(x) \leq T_N(\omega_t)\}$$
$$X_{\mathbf{N}}^{\omega_i} := \{x \in X \mid I_N(x) \geq I_N(\omega_i)\}$$
$$X_{\mathbf{N}}^{\omega_f} := \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\}$$

Clearly, $\omega_t \in X_{\mathbf{N}}^{\omega_t}$, $\omega_i \in X_{\mathbf{N}}^{\omega_i}$ and $\omega_f \in X_{\mathbf{N}}^{\omega_f}$.

Theorem 8. *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X.*

Proof. Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t}$, $\theta \in X_{\mathbf{N}}^{\omega_i}$ and $\theta \in X_{\mathbf{N}}^{\omega_f}$. Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$T_N(x*y) \leq T_N(\omega_t),\ T_N(y) \leq T_N(\omega_t)$$
$$I_N(x*y) \geq I_N(\omega_i),\ I_N(y) \geq I_N(\omega_i)$$
$$F_N(x*y) \leq F_N(\omega_f),\ F_N(y) \leq F_N(\omega_f)$$

It follows from Equation (6) that

$$T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} \leq T_N(\omega_t)$$
$$I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} \geq I_N(\omega_i)$$
$$F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} \leq F_N(\omega_f)$$

Hence $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X. □

Theorem 9. *Let $\omega_t, \omega_i, \omega_f \in X$ and let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X. Then*

(1) *If $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X, then the following assertion is valid:*

$$(\forall x, y, z \in X) \left(\begin{array}{l} T_N(x) \geq \bigvee\{T_N(y*z), T_N(z)\} \ \Rightarrow\ T_N(x) \geq T_N(y) \\ I_N(x) \leq \bigwedge\{I_N(y*z), I_N(z)\} \ \Rightarrow\ I_N(x) \leq I_N(y) \\ F_N(x) \geq \bigvee\{F_N(y*z), F_N(z)\} \ \Rightarrow\ F_N(x) \geq F_N(y) \end{array} \right) \tag{11}$$

(2) *If $X_{\mathbf{N}}$ satisfies Equation (11) and*

$$(\forall x \in X)\ (T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x), F_N(\theta) \leq F_N(x)) \tag{12}$$

then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X for all $\omega_t \in \mathrm{Im}(T_N)$, $\omega_i \in \mathrm{Im}(I_N)$ and $\omega_f \in \mathrm{Im}(F_N)$.

Proof. (1) Assume that $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X for $\omega_t, \omega_i, \omega_f \in X$. Let $x, y, z \in X$ be such that $T_N(x) \geq \bigvee\{T_N(y*z), T_N(z)\}$, $I_N(x) \leq \bigwedge\{I_N(y*z), I_N(z)\}$ and $F_N(x) \geq \bigvee\{F_N(y*z), F_N(z)\}$. Then $y*z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, where $\omega_t = \omega_i = \omega_f = x$. It follows from (I2) that $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ for $\omega_t = \omega_i = \omega_f = x$. Hence $T_N(y) \leq T_N(\omega_t) = T_N(x)$, $I_N(y) \geq I_N(\omega_i) = I_N(x)$ and $F_N(y) \leq F_N(\omega_f) = F_N(x)$.

(2) Let $\omega_t \in \mathrm{Im}(T_N)$, $\omega_i \in \mathrm{Im}(I_N)$ and $\omega_f \in \mathrm{Im}(F_N)$ and suppose that $X_{\mathbf{N}}$ satisfies Equations (11) and (12). Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ by Equation (12). Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$\begin{aligned} &T_N(x*y) \leq T_N(\omega_t),\ T_N(y) \leq T_N(\omega_t) \\ &I_N(x*y) \geq I_N(\omega_i),\ I_N(y) \geq I_N(\omega_i) \\ &F_N(x*y) \leq F_N(\omega_f),\ F_N(y) \leq F_N(\omega_f) \end{aligned}$$

which implies that $\bigvee\{T_N(x*y), T_N(y)\} \leq T_N(\omega_t)$, $\bigwedge\{I_N(x*y), I_N(y)\} \geq I_N(\omega_i)$, and $\bigvee\{F_N(x*y), F_N(y)\} \leq F_N(\omega_f)$. It follows from Equation (11) that $T_N(\omega_t) \geq T_N(x)$, $I_N(\omega_i) \leq I_N(x)$ and $F_N(\omega_f) \geq F_N(x)$. Thus, $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X. □

Definition 3. *A neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X is said to be closed if it is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Example 6. *Consider a BCI-algebra $X = \{\theta, 1, a, b, c\}$ with the following Cayley table.*

$*$	θ	1	a	b	c
θ	θ	θ	a	b	c
1	1	θ	a	b	c
a	a	a	θ	c	b
b	b	b	c	θ	a
c	c	c	b	a	θ

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X which is given as follows:

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.9,-0.3,-0.8)}, \frac{1}{(-0.7,-0.4,-0.7)}, \frac{a}{(-0.6,-0.8,-0.3)}, \frac{b}{(-0.2,-0.6,-0.3)}, \frac{c}{(-0.2,-0.8,-0.5)} \right\}$$

Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X.

Theorem 10. *Let X be a BCI-algebra, For any $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [-1, 0)$ and $\beta_1, \beta_2 \in (-1, 0]$ with $\alpha_1 < \alpha_2$, $\gamma_1 < \gamma_2$ and $\beta_1 > \beta_2$, let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X given as follows:*

$$T_N : X \to [-1, 0],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in X_+ \\ \alpha_2 & \text{otherwise} \end{cases}$$

$$I_N : X \to [-1, 0],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in X_+ \\ \beta_2 & \text{otherwise} \end{cases}$$

$$F_N : X \to [-1, 0],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in X_+ \\ \gamma_2 & \text{otherwise} \end{cases}$$

where $X_+ = \{x \in X \mid \theta \preceq x\}$. Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X.

Proof. Because $\theta \in X_+$, we have $T_N(\theta) = \alpha_1 \leq T_N(x)$, $I_N(\theta) = \beta_1 \geq I_N(x)$ and $F_N(\theta) = \gamma_1 \leq F_N(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then

$$\begin{aligned} &T_N(x) = \alpha_1 \leq \bigvee\{T_N(x*y), T_N(y)\} \\ &I_N(x) = \beta_1 \geq \bigwedge\{I_N(x*y), I_N(y)\} \\ &F_N(x) = \gamma_1 \leq \bigvee\{F_N(x*y), F_N(y)\} \end{aligned}$$

Suppose that $x \notin X_+$. If $x * y \in X_+$ then $y \notin X_+$, and if $y \in X_+$ then $x * y \notin X_+$. In either case, we have

$$\begin{aligned} T_N(x) &= \alpha_2 = \bigvee\{T_N(x*y), T_N(y)\} \\ I_N(x) &= \beta_2 = \bigwedge\{I_N(x*y), I_N(y)\} \\ F_N(x) &= \gamma_2 = \bigvee\{F_N(x*y), F_N(y)\} \end{aligned}$$

For any $x, y \in X$, if any one of x and y does not belong to X_+, then

$$\begin{aligned} T_N(x*y) &\le \alpha_2 = \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) &\ge \beta_2 = \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x*y) &\le \gamma_2 = \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

If $x, y \in X_+$, then $x * y \in X_+$. Hence

$$\begin{aligned} T_N(x*y) &= \alpha_1 = \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) &= \beta_1 = \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x*y) &= \gamma_1 = \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

Therefore $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X. □

Proposition 6. *Every closed neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of a BCI-algebra X satisfies the following condition:*

$$(\forall x \in X)\ (T_N(\theta * x) \le T_N(x),\ I_N(\theta * x) \ge I_N(x),\ F_N(\theta * x) \le F_N(x)) \tag{13}$$

Proof. Straightforward. □

We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be closed.

Theorem 11. *Let X be a BCI-algebra. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X that satisfies the condition of Equation (13), then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and hence is a closed neutrosophic $\mathcal{N}$-ideal of X.*

Proof. Note that $(x * y) * x \preceq \theta * y$ for all $x, y \in X$. Using Equations (10) and (13), we have

$$\begin{aligned} T_N(x*y) &\le \bigvee\{T_N(x), T_N(\theta*y)\} \le \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) &\ge \bigwedge\{I_N(x), I_N(\theta*y)\} \ge \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x*y) &\le \bigvee\{F_N(x), F_N(\theta*y)\} \le \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and is therefore a closed neutrosophic $\mathcal{N}$-ideal of X. □

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Neutrosophic Commutative $\mathcal{N}$-Ideals in BCK-Algebras

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Seok-Zun Song, Florentin Smarandache, Young Bae Jun (2017). Neutrosophic Commutative N-Ideals in BCK-Algebras. *Information* 8, 130; DOI: 10.3390/info8040130

Abstract: The notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras is introduced, and several properties are investigated. Relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal are discussed. Characterizations of a neutrosophic commutative $\mathcal{N}$-ideal are considered.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-ideal; neutrosophic commutative $\mathcal{N}$-ideal

1. Introduction

As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set.

Smarandache proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutrosophy" [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy"/"intuitionistic fuzzy" logic/set and "neutrosophic" logic/set, i.e., the *included middle* component (Lupasco–Nicolescu's logic in philosophy), i.e., the neutral/indeterminate/unknown part (besides the "truth"/"membership" and "falsehood"/"non-membership" components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components

$$(\mathrm{t, i, f}) = (\mathrm{truth, indeterminacy, falsehood}).$$

For more details, refer to the site http://fs.gallup.unm.edu/FlorentinSmarandache.htm.

Jun et al. [2] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. Khan et al. [3] introduced the notion of neutrosophic $\mathcal{N}$-structure and applied it to a semigroup. Jun et al. [4] applied the notion of neutrosophic $\mathcal{N}$-structure to BCK/BCI-algebras. They introduced the notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra, and investigated related properties. They also considered characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal, and discussed relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. They provided conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal. BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [5], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (= MV-algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra (see [6]).

In this paper, we introduce the notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras, and investigate several properties. We consider relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal. We discuss characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

2. Preliminaries

By a BCI-algebra, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x*y)*(x*z))*(z*y) = 0,$
(II) $(x*(x*y))*y = 0,$
(III) $x*x = 0,$
(IV) $x*y = y*x = 0 \ \Rightarrow\ x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra.

We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\ (x \preceq y \ \Rightarrow\ x * y = 0).$$

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x * 0 = x), \tag{1}$$
$$(\forall x, y, z \in X)\ ((x*y)*z = (x*z)*y). \tag{2}$$

A BCK-algebra X is said to be *commutative* if it satisfies the following equality:

$$(\forall x, y \in X)\ (x*(x*y) = y*(y*x)). \tag{3}$$

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0 \in I, \tag{4}$$
$$(\forall x, y \in X)\ (x*y \in I,\ y \in I \ \Rightarrow\ x \in I). \tag{5}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X if it satisfies (4) and

$$(\forall x, y, z \in X)\ ((x*y)*z \in I,\ z \in I \ \Rightarrow\ x*(y*(y*x)) \in I). \tag{6}$$

Lemma 1. *An ideal I is commutative if and only if the following assertion is valid.*

$$(\forall x, y \in X)\ (x*y \in I \ \Rightarrow\ x*(y*(y*x)) \in I). \tag{7}$$

We refer the reader to the books [7,8] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$*-structure*, we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X (see [2]). A *neutrosophic* $\mathcal{N}$*-structure* over a nonempty universe of discourse X (see [3]) is defined to be the structure

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{8}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

Note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X) \left(-3 \le T_N(x) + I_N(x) + F_N(x) \le 0 \right).$$

3. Neutrosophic Commutative $\mathcal{N}$-Ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 1 ([4]). *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-ideal of X if the following assertion is valid.*

$$(\forall x, y \in X) \begin{pmatrix} T_N(0) \le T_N(x) \le \bigvee\{T_N(x * y), T_N(y)\} \\ I_N(0) \ge I_N(x) \ge \bigwedge\{I_N(x * y), I_N(y)\} \\ F_N(0) \le F_N(x) \le \bigvee\{F_N(x * y), F_N(y)\} \end{pmatrix}. \tag{9}$$

Definition 2. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic commutative $\mathcal{N}$-ideal of X if the following assertions are valid.*

$$(\forall x \in X) \left(T_N(0) \le T_N(x),\ I_N(0) \ge I_N(x),\ F_N(0) \le F_N(x) \right), \tag{10}$$

$$(\forall x, y, z \in X) \begin{pmatrix} T_N(x * (y * (y * x))) \le \bigvee\{T_N((x * y) * z), T_N(z)\} \\ I_N(x * (y * (y * x))) \ge \bigwedge\{I_N((x * y) * z), I_N(z)\} \\ F_N(x * (y * (y * x))) \le \bigvee\{F_N((x * y) * z), F_N(z)\} \end{pmatrix}. \tag{11}$$

Example 1. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1.*

Table 1. Cayley table for the binary operation "*".

*	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.8,-0.2,-0.9)}, \frac{1}{(-0.3,-0.9,-0.5)}, \frac{2}{(-0.7,-0.7,-0.4)}, \frac{3}{(-0.3,-0.6,-0.7)}, \frac{4}{(-0.5,-0.3,-0.1)} \right\}$$

over X is a neutrosophic commutative $\mathcal{N}$-ideal of X.

Theorem 1. *Every neutrosophic commutative $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-ideal.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic commutative $\mathcal{N}$-ideal of X. For every $x, z \in X$, we have

$$T_N(x) = T_N(x * (0 * (0 * x))) \leq \bigvee\{T_N((x * 0) * z), T_N(z)\} = \bigvee\{T_N(x * z), T_N(z)\},$$
$$I_N(x) = I_N(x * (0 * (0 * x))) \geq \bigwedge\{I_N((x * 0) * z), I_N(z)\} = \bigwedge\{I_N(x * z), I_N(z)\},$$
$$F_N(x) = F_N(x * (0 * (0 * x))) \leq \bigvee\{F_N((x * 0) * z), F_N(z)\} = \bigvee\{F_N(x * z), F_N(z)\}$$

by putting $y = 0$ in (11) and using (1). Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

The converse of Theorem 1 is not true in general as seen in the following example.

Example 2. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 2.*

Table 2. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.8, -0.1, -0.7)}, \frac{1}{(-0.7, -0.6, -0.6)}, \frac{2}{(-0.6, -0.2, -0.4)}, \frac{3}{(-0.3, -0.8, -0.4)}, \frac{4}{(-0.3, -0.8, -0.4)} \right\}$$

over X is a neutrosophic $\mathcal{N}$-ideal of X. But it is not a neutrosophic commutative $\mathcal{N}$-ideal of X since $F_N(2 * (3 * (3 * 2)) = F_N(2) = -0.4 \nleq -0.7 = \bigvee\{F_N((2 * 3) * 0), F_N(0)\}$.

We consider characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

Theorem 2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x * (y * (y * x))) \leq T_N(x * y), \\ I_N(x * (y * (y * x))) \geq I_N(x * y), \\ F_N(x * (y * (y * x))) \leq F_N(x * y) \end{array} \right). \tag{12}$$

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. The assertion (12) is by taking $z = 0$ in (11) and using (1) and (10).

Conversely, suppose that a neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X satisfies the condition (12). Then,

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x * y) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} \\ I_N(x * y) \geq \bigwedge\{I_N((x * y) * z), I_N(z)\} \\ F_N(x * y) \leq \bigvee\{F_N((x * y) * z), F_N(z)\} \end{array} \right). \tag{13}$$

It follows that the condition (11) is induced by (12) and (13). Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

Lemma 2 ([4]). *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X, we have*

$$(\forall x, y, z \in X) \left(x * y \preceq z \Rightarrow \left\{ \begin{array}{l} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{array} \right. \right). \tag{14}$$

Theorem 3. *In a commutative BCK-algebra, every neutrosophic $\mathcal{N}$-ideal is a neutrosophic commutative $\mathcal{N}$-ideal.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of a commutative BCK-algebra X. For any $x, y, z \in X$, we have

$$\begin{aligned}
&((x * (y * (y * x))) * ((x * y) * z)) * z \\
&= ((x * (y * (y * x))) * z) * ((x * y) * z) \\
&\preceq (x * (y * (y * x))) * (x * y) \\
&= (x * (x * y)) * (y * (y * x)) = 0,
\end{aligned}$$

that is, $(x * (y * (y * x))) * ((x * y) * z) \preceq z$. It follows from Lemma 2 that

$$\begin{aligned}
&T_N(x * (y * (y * x))) \leq \bigvee\{T_N((x * y) * z),\ T_N(z)\}, \\
&I_N(x * (y * (y * x))) \geq \bigwedge\{I_N((x * y) * z),\ I_N(z)\}, \\
&F_N(x * (y * (y * x))) \leq \bigvee\{F_N((x * y) * z),\ F_N(z)\}.
\end{aligned}$$

Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets.

$$\begin{aligned}
T_N^{\alpha} &:= \{x \in X \mid T_N(x) \leq \alpha\}, \\
I_N^{\beta} &:= \{x \in X \mid I_N(x) \geq \beta\}, \\
F_N^{\gamma} &:= \{x \in X \mid F_N(x) \leq \gamma\}.
\end{aligned}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the (α, β, γ)*-level set* of $X_{\mathbf{N}}$. It is clear that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}.$$

Theorem 4. *If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ whenever they are nonempty.*

We call T_N^{α}, I_N^{β} and F_N^{γ} *level commutative ideals* of $X_{\mathbf{N}}$.

Proof. Assume that T_N^{α}, I_N^{β} and F_N^{γ} are nonempty for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, $x \in T_N^{\alpha}$, $y \in I_N^{\beta}$ and $z \in F_N^{\gamma}$ for some $x, y, z \in X$. Thus, $T_N(0) \leq T_N(x) \leq \alpha$, $I_N(0) \geq I_N(y) \geq \beta$, and $F_N(0) \leq F_N(z) \leq \gamma$, that is, $0 \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. Let $(x * y) * z \in T_N^{\alpha}$ and $z \in T_N^{\alpha}$. Then, $T_N((x * y) * z) \leq \alpha$ and $T_N(z) \leq \alpha$, which imply that

$$T_N(x * (y * (y * x))) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} \leq \alpha,$$

that is, $x * (y * (y * x)) \in T_N^{\alpha}$. If $(a * b) * c \in I_N^{\beta}$ and $c \in I_N^{\beta}$, then $I_N((a * b) * c) \geq \beta$ and $I_N(c) \geq \beta$. Thus

$$I_N(a * (b * (b * c))) \geq \bigwedge\{I_N((a * b) * c), I_N(c)\} \geq \beta,$$

and so $a * (b * (b * c)) \in I_N^{\beta}$. Finally, suppose that $(u * v) * w \in F_N^{\gamma}$ and $w \in F_N^{\gamma}$. Then, $F_N((u * v) * w) \leq \gamma$ and $F_N(w) \leq \gamma$. Thus,

$$F_N(u * (v * (v * w))) \leq \bigvee \{F_N((u * v) * w), F_N(w)\} \leq \gamma,$$

that is, $u * (v * (v * w)) \in F_N^{\gamma}$. Therefore, T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of X. □

Corollary 1. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is a commutative ideal of X.*

Proof. Straightforward. □

Lemma 3 ([4]). *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Theorem 5. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X.*

Proof. If T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of X, then they are ideals of X. Hence, $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X by Lemma 3. Let $x, y \in X$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ such that $T_N(x * y) = \alpha$, $I_N(x * y) = \beta$ and $F_N(x * y) = \gamma$. Then, $x * y \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. Since $T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$ is a commutative ideal of X, it follows from Lemma 1 that $x * (y * (y * x)) \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

Hence

$$\begin{aligned} T_N(x * (y * (y * x))) &\leq \alpha = T_N(x * y), \\ I_N(x * (y * (y * x))) &\geq \beta = I_N(x * y), \\ F_N(x * (y * (y * x))) &\leq \gamma = F_N(x * y). \end{aligned}$$

Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X by Theorem 2. □

Theorem 6. *Let $f : X \to X$ be an injective mapping. Given a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X, the following are equivalent.*

(1) *$X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, satisfying the following condition.*

$$(\forall x \in X) \begin{pmatrix} T_N(f(x)) = T_N(x) \\ I_N(f(x)) = I_N(x) \\ F_N(f(x)) = F_N(x) \end{pmatrix}. \tag{15}$$

(2) *T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of $X_{\mathbf{N}}$, satisfying the following condition.*

$$f(T_N^{\alpha}) = T_N^{\alpha}, f(I_N^{\beta}) = I_N^{\beta}, f(F_N^{\gamma}) = F_N^{\gamma}. \tag{16}$$

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic commutative $\mathcal{N}$-ideal of X, satisfying the condition (15). Then, T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of $X_{\mathbf{N}}$ by Theorem 4. Let $\alpha \in \mathrm{Im}(T_N)$, $\beta \in \mathrm{Im}(I_N)$, $\gamma \in \mathrm{Im}(F_N)$ and $x \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. Then $T_N(f(x)) = T_N(x) \leq \alpha$, $I_N(f(x)) = I_N(x) \geq \beta$ and $F_N(f(x)) = F_N(x) \leq \gamma$. Thus, $f(x) \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$, which shows that $f(T_N^{\alpha}) \subseteq T_N^{\alpha}$, $f(I_N^{\beta}) \subseteq I_N^{\beta}$ and $f(F_N^{\gamma}) \subseteq F_N^{\gamma}$. Let $y \in X$ be such that $f(y) = x$. Then, $T_N(y) = T_N(f(y)) = T_N(x) \leq \alpha$, $I_N(y) = I_N(f(y)) = I_N(x) \geq \beta$

and $F_N(y) = F_N(f(y)) = F_N(x) \leq \gamma$, which imply that $y \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. Thus, $x = f(y) \in f(T_N^{\alpha}) \cap f(I_N^{\beta}) \cap f(F_N^{\gamma})$, and so $T_N^{\alpha} \subseteq f(T_N^{\alpha})$, $I_N^{\beta} \subseteq f(I_N^{\beta})$ and $F_N^{\gamma} \subseteq f(F_N^{\gamma})$. Therefore (16) is valid.

Conversely, assume that T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of $X_{\mathbf{N}}$, satisfying the condition (16). Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X by Theorem 5. Let $x, y, z \in X$ be such that $T_N(x) = \alpha$, $I_N(y) = \beta$ and $F_N(z) = \gamma$. Note that

$$
\begin{aligned}
&T_N(x) = \alpha \Longleftrightarrow x \in T_N^{\alpha} \text{ and } x \notin T_N^{\tilde{\alpha}} \text{ for all } \alpha > \tilde{\alpha}, \\
&I_N(y) = \beta \Longleftrightarrow y \in I_N^{\beta} \text{ and } y \notin I_N^{\tilde{\beta}} \text{ for all } \beta < \tilde{\beta}, \\
&F_N(z) = \gamma \Longleftrightarrow z \in F_N^{\gamma} \text{ and } z \notin F_N^{\tilde{\gamma}} \text{ for all } \gamma > \tilde{\gamma}.
\end{aligned}
$$

It follows from (16) that $f(x) \in T_N^{\alpha}$, $f(y) \in I_N^{\beta}$ and $f(z) \in F_N^{\gamma}$. Hence, $T_N(f(x)) \leq \alpha$, $I_N(f(y)) \geq \beta$ and $F_N(f(z)) \leq \gamma$. Let $\tilde{\alpha} = T_N(f(x))$, $\tilde{\beta} = I_N(f(y))$ and $\tilde{\gamma} = F_N(f(z))$. If $\alpha > \tilde{\alpha}$, then $f(x) \in T_N^{\tilde{\alpha}} = f\left(T_N^{\tilde{\alpha}}\right)$, and thus $x \in T_N^{\tilde{\alpha}}$ since f is one to one. This is a contradiction. Hence, $T_N(f(x)) = \alpha = T_N(x)$. If $\beta < \tilde{\beta}$, then $f(y) \in I_N^{\tilde{\beta}} = f\left(I_N^{\tilde{\beta}}\right)$ which implies from the injectivity of f that $y \in I_N^{\tilde{\beta}}$, a contradiction. Hence, $I_N(f(x)) = \beta = I_N(x)$. If $\gamma > \tilde{\gamma}$, then $f(z) \in F_N^{\tilde{\gamma}} = f\left(F_N^{\tilde{\gamma}}\right)$. Since f is one to one, we have $z \in F_N^{\tilde{\gamma}}$ which is a contradiction. Thus, $F_N(f(x)) = \gamma = F_N(x)$. This completes the proof. □

For any elements ω_t, ω_i, $\omega_f \in X$, we consider sets:

$$
\begin{aligned}
X_{\mathbf{N}}^{\omega_t} &:= \{x \in X \mid T_N(x) \leq T_N(\omega_t)\}, \\
X_{\mathbf{N}}^{\omega_i} &:= \{x \in X \mid I_N(x) \geq I_N(\omega_i)\}, \\
X_{\mathbf{N}}^{\omega_f} &:= \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\}.
\end{aligned}
$$

Obviously, $\omega_t \in X_{\mathbf{N}}^{\omega_t}$, $\omega_i \in X_{\mathbf{N}}^{\omega_i}$ and $\omega_f \in X_{\mathbf{N}}^{\omega_f}$.

Lemma 4 ([4])**.** *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X.*

Theorem 7. *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are commutative ideals of X.*

Proof. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then it is a neutrosophic $\mathcal{N}$-ideal of X and so $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X by Lemma 4. Let $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ for any $x, y \in X$. Then, $T_N(x * y) \leq T_N(\omega_t)$, $I_N(x * y) \geq T_N(\omega_i)$ and $F_N(x * y) \leq F_N(\omega_f)$. It follows from Theorem 2 that

$$
\begin{aligned}
T_N(x * (y * (y * x))) &\leq T_N(x * y) \leq T_N(\omega_t), \\
I_N(x * (y * (y * x))) &\geq I_N(x * y) \geq I_N(\omega_i), \\
F_N(x * (y * (y * x))) &\leq F_N(x * y) \leq F_N(\omega_f).
\end{aligned}
$$

Hence, $x * (y * (y * x)) \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are commutative ideals of X by Lemma 1. □

Theorem 8. *Any commutative ideal of X can be realized as level commutative ideals of some neutrosophic commutative $\mathcal{N}$-ideal of X.*

Proof. Let A be a commutative ideal of X and let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X in which

$$T_N : X \to [-1,0],\ x \mapsto \begin{cases} \alpha & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$
$$I_N : X \to [-1,0],\ x \mapsto \begin{cases} \beta & \text{if } x \in A, \\ -1 & \text{otherwise,} \end{cases}$$
$$F_N : X \to [-1,0],\ x \mapsto \begin{cases} \gamma & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha, \gamma \in [-1,0)$ and $\beta \in (-1,0]$. Division into the following cases will verify that $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X.

If $(x*y)*z \in A$ and $z \in A$, then $x*(y*(y*x) \in A$. Thus,

$$T_N((x*y)*z) = T_N(z) = T_N(x*(y*(y*x))) = \alpha,$$
$$I_N((x*y)*z) = I_N(z) = I_N(x*(y*(y*x))) = \beta,$$
$$F_N((x*y)*z) = F_N(z) = F_N(x*(y*(y*x))) = \gamma,$$

and so (11) is clearly verified.

If $(x*y)*z \notin A$ and $z \notin A$, then $T_N((x*y)*z) = T_N(z) = 0$, $I_N((x*y)*z) = I_N(z) = -1$ and $F_N((x*y)*z) = F_N(z) = 0$. Hence

$$T_N(x*(y*(y*x))) \leq \bigvee\{T_N((x*y)*z),\ T_N(z)\},$$
$$I_N(x*(y*(y*x))) \geq \bigwedge\{I_N((x*y)*z),\ I_N(z)\},$$
$$F_N(x*(y*(y*x))) \leq \bigvee\{F_N((x*y)*z),\ F_N(z)\}.$$

If $(x*y)*z \in A$ and $z \notin A$, then $T_N((x*y)*z) = \alpha$, $T_N(z) = 0$, $I_N((x*y)*z) = \beta$, $I_N(z) = -1$, $F_N((x*y)*z) = \gamma$ and $F_N(z) = 0$. Therefore,

$$T_N(x*(y*(y*x))) \leq \bigvee\{T_N((x*y)*z),\ T_N(z)\},$$
$$I_N(x*(y*(y*x))) \geq \bigwedge\{I_N((x*y)*z),\ I_N(z)\},$$
$$F_N(x*(y*(y*x))) \leq \bigvee\{F_N((x*y)*z),\ F_N(z)\}.$$

Similarly, if $(x*y)*z \notin A$ and $z \in A$, then (11) is verified. Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. Obviously, $T_N^{\alpha} = A$, $I_N^{\beta} = A$ and $F_N^{\gamma} = A$. This completes the proof. □

4. Conclusions

In order to deal with the negative meaning of information, Jun et al. [2] have introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. The concept of neutrosophic set (NS) has been developed by Smarandache in [9,10] as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. In this article, we have introduced the notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras, and investigated several properties. We have considered relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal. We have discussed characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

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Compact Open Topology and Evaluation Map via Neutrosophic Sets

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R. Dhavaseelan, Saeid Jafari, Florentin Smarandache (2017). Compact Open Topology and Evaluation Map via Neutrosophic Sets. *Neutrosophic Sets and Systems* 16, 35-38

Abstract: The concept of neutrosophic locally compact and neutrosophic compact open topology are introduced and some interesting propositions are discussed.

Keywords: neutrosophic locally Compact Hausdorff space; neutrosophic product topology; neutrosophic compact open topology; neutrosophic homeomorphism; neutrosophic evaluation map; Exponential map.

1 Introduction and Preliminaries

In 1965, Zadeh [19] introduced the useful notion of a fuzzy set and Chang [6] three years later offered the notion of fuzzy topological space. Since then, several authors have generalized numerous concepts of general topology to the fuzzy setting. The concept of intuitionistic fuzzy set was introduced and studied by Atanassov [1] and subsequently some important research papers published by him and his colleagues [2,3,4]. The concept of fuzzy compact open topology was introduced by S.Dang and A . Behera[9]. The concepts of intuitionistic evaluation maps by R.Dhavaseelan et al[9]. After the introduction of the concepts of neutrosophy and neutrosophic set by F. Smarandache [[11], [12]], the concepts of neutrosophic crisp set and neutrosophic crisp topological spaces were introduced by A. A. Salama and S. A. Alblowi[10].

In this paper the notion of neutrosophic compact open topology is introduced. Some interesting properties are discussed. Moreover, neutrosophic local compactness and neutrosophic product topology are developed. We have also utilized the notion of fuzzy locally compactness due to Wong[17], Christoph [8] and fuzzy product topology due to Wong [18].

Throughout this paper neutrosophic topological spaces (X,T),(Y,S) and (Z,R) will be replaced by X,Y and Z respectively.

Definition 1.1. Let T,I,F be real standard or non standard subsets of $]0^-,1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
$sup_I = i_{sup}, inf_I = i_{inf}$
$sup_F = f_{sup}, inf_F = f_{inf}$
$n-sup = t_{sup} + i_{sup} + f_{sup}$
$n-inf = t_{inf}+i_{inf}+f_{inf}$. T,I,F are neutrosophic components.

Definition 1.2. Let X be a nonempty fixed set. A neutrosophic set [briefly NS] A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

Remark 1.1. (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-,1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

We introduce the neutrosophic sets 0_N and 1_N in X as follows:

Definition 1.3. $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.4. [8] A neutrosophic topology (NT) on a nonempty set X consists of a family T of neutrosophic sets in X which satisfies the following:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X,T) or simply X is called a neutrosophic topological space (NTS) and each neutrosophic set in T is called a neutrosophic open set (NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (NCS) in X.

Definition 1.5. [8] Let A be a neutrosophic subset of a neutrosophic topological space X. The neutrosophic interior and neutrosophic closure of A are denoted and defined by
$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and

$G \subseteq A\}$;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$.

2 Neutrosophic Locally Compact and Neutrosophic Compact Open Topology

Definition 2 .1. Let X be a nonempty set and $x \in X$ a fixed element in X. If $r, t \in I_0 = (0, 1]$ and $s \in I_1 = [0, 1)$ are fixed real numbers such that $0 < r + t + s < 3$, then $x_{r,t,s} = \langle x, r, t, s\rangle$ is called a neutrosophic point (in short NP) in X, where r denotes the degree of membership of $x_{r,t,s}$, t denotes the degree of indeterminacy and s denotes the degree of nonmembership of $x_{r,t,s}$ and $x \in X$ the support of $x_{r,t,s}$.

The neutrosophic point $x_{r,t,s}$ is contained in the neutrosophic $A(x_{r,t,s} \in A)$ if and only if $r < \mu_A(x), t < \sigma_A(x), s > \gamma_A(x)$.

Definition 2.2. A neutrosophic set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ in a neutrosophic topological space (X, T) is said to be a neutrosophic neighbourhood of a neotrosophic point $x_{r,t,s}, x \in X$, if there exists a neutrosophic open set $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ with $x_{r,t,s} \subseteq B \subseteq A$.

Definition 2.3. *Let X and Y be neutrosophic topological spaces.A mapping $f : X \to Y$ is said to be a neutrosophic homeomorphism if f is bijective, neutrosophic continuous and neutrosophic open.*

Definition 2.4. *An neutrosophic topological space (X, T) is called a neutrosophic Hausdorff space or T_2-space if for any pair of distinct neutrosophic points(i.e., neutrosophic points with distinct supports) $x_{r,t,s}$ and $y_{u,v,w}$,there exist neutrosophic open sets U and V such that $x_{r,t,s} \in U$,$y_{u,v,w} \in V$ and $U \wedge V = 0_N$*

Definition 2.5. *An neutrosophic topological space (X, T) is said to be neutrosophic locally compact if and only if for every neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set $U \in T$ such that $x_{r,t,s} \in U$ and U is neutrosophic compact,i.e., each neutrosophic open cover of U has a finite subcover.*

Definition 2.6. *Let $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle$ and $B = \langle y, \mu_B(y), \sigma_B(y), \gamma_B(y)\rangle$ be neutrosophic sets of X and Y respectively.The product of two neutrosophic sets A and B in a neutrosophic topological space X is defined as $(A \times B)(x, y) = \langle (x, y), min(\mu_A(x), \mu_B(y)), min(\sigma_A(x), \sigma_B(y)), max(\gamma_A(x), \gamma_B(y))\rangle$ for all $(x, y) \in X \times Y$.*

Definition 2.7. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by: $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))\ \forall(x_1, x_2) \in X_1 \times X_2$.*

Lemma 2.1. *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be functions and U, V are neutrosophic sets of Y_1, Y_2, respectively, then $(f_1 \times f_2)^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V)\ \forall\, U \times V \in Y_1 \times Y_2$*

Definition 2.8. *A mapping $f : X \to Y$ is neutrosophic continuous iff for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic neighbourhood B of $f(x_{r,t,s})$ in Y,there is a neutrosophic neighbourhood A of $x_{r,t,s}$ in X such that $f(A) \subseteq B$.*

Definition 2.9. *A mapping $f : X \to Y$ is said to be neutrosophic homeomorphism if f is bijective ,neutrosophic continuous and neutrosophic open.*

Definition 2.10. *A neutrosophic topological space X is called a neutrosophic Hausdorff space or T_2 space if for any distinct neutrosophic points $x_{r,t,s}$ and $y_{u,v,w}$,there exists neutrosophic open sets G_1 and G_2, such that $x_{r,t,s} \in G_1$,$y_{u,v,w} \in G_2$ and $G_1 \cap G_2 = 0_\sim$*

Definition 2.11. *A neutrosophic topological space X is said to be a neutrosophic locally compact iff for any neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set $U \in T$ such that $x_{r,t,s} \in U$ and U is neutrosophic compact that is, each neutrosophic open cover of U has a finite subcover.*

Proposition 2.1. In a neutrosophic Hausdorff topological space X, the following conditions are equivalent.

(a) X is a neutrosophic locally compact

(b) for each neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set G in X such that $x_{r,t,s} \in G$ and $Ncl(G)$ is neutrosophic compact

Proof. $(a) \Rightarrow (b)$ By hypothesis for each neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set G which is neutrosophic compact.Since X is neutrosophic Hausdorff (neutrosophic compact subspace of neutrosophic Hausdorff space is neutrosophic closed), G is neutrosophic closed, thus $G = Ncl(G)$. Hence $x_{r,t,s} \in G$ and $Ncl(G)$ is neutrosophic compact.
$(b) \Rightarrow (a)$ Proof is simple. □

Proposition 2.2. Let X be a neutrosophic Hausdorff topological space.Then X is neutrosophic locally compact at a neutrosophic point $x_{r,t,s}$ in X iff for every neutrosophic open set G containing $x_{r,t,s}$ there exists a neutrosophic open set V such that $x_{r,t,s} \in V$, $Ncl(V)$ is neutrosophic compact and $Ncl(V) \subseteq G$.

Proof. Suppose that X is neutrosophic locally compact at a neutrosophic point $x_{r,t,s}$. By Definition 2.11, there exists a neutrosophic open set G such that $x_{r,t,s} \in G$ and G is neutrosophic compact. Since X is a neutrosophic Hausdorff space,(neutrosophic compact subspace of neutrosophic Hausdorff space is neutrosophic closed), G is neutrosophic closed. Thus $G = Ncl(G)$. Consider a neutrosophic point $x_{r,t,s} \in \overline{G}$. Since X is neutrosophic Hausdorff space, by Definition 2.10, there exist neutrosophic open sets C and D such that $x_{r,t,s} \in C$, $y_{u,v,w} \in D$ and $C \cap D = 0_\sim$. Let $V = C \cap G$. Hence $V \subseteq G$ implies $Ncl(V) \subseteq Ncl(G) = G$. Since $Ncl(V)$ is neutrosophic closed and G is neutrosophic compact, (every neutrosophic closed subset of a neutrosophic compact space is neutrosophic compact) it follows that $Ncl(V)$ is neutrosophic compact. Thus $x_{r,t,s} \in Ncl(V) \subseteq G$ and $Ncl(G)$ is neutrosophic compact.

The converse follows from Proposition 2.1(b). □

Definition 2.12. *Let X and Y be two neutrosophic topological spaces.The function $T : X \times Y \to Y \times X$ defined by $T(x, y) = (y, x)$ for each $(x, y) \in X \times Y$ is called a switching map.*

Proposition 2.3. The switching map $T : X \times Y \to Y \times X$ defined as above is neutrosophic continuous.

We now introduce the concept of a neutrosophic compact open topology in the set of all neutrosophic continuous functions from a neutrosophic topological space X to a neutrosophic topological space Y.

Definition 2.13. *Let X and Y be two neutrosophic topological spaces and let $Y^X = \{f : X \to Y$ such that f is neutrosophic continuous$\}$. We give this class Y^X a topology called the neutrosophic compact open topology as follows: Let $\mathcal{K} = \{K \in I^X :$ K is neutrosophic compact on $X\}$ and $\mathcal{V} = \{V \in I^Y : V$ is neutrosophic open in $Y\}$.For any $K \in \mathcal{K}$ and $V \in \mathcal{V}$,let $S_{K,V} = \{f \in Y^X : f(K) \subseteq V\}$.*

The collection of all such $\{S_{K,V} : K \in \mathcal{K}, V \in \mathcal{V}\}$ is a neutrosophic subbase to generate a neutrosophic topology on the class Y^X. The class Y^X with this topology is called a neutrosophic compact open topological space.

3 Neutrosophic Evaluation Map and Exponential Map

We now consider the neutrosophic product topological space $Y^X \times X$ and define a neutrosophic continuous map from $Y^X \times X$ into Y.

Definition 3.1. *The mapping $e : Y^X \times X \to Y$ defined by $e(f, x_{r,t,s}) = f(x_{r,t,s})$ for each neutrosophic point $x_{r,t,s} \in X$ and $f \in Y^X$ is called the neutrosophic evaluation map.*

Definition 3.2. *Let X,Y,Z be neutrosophic topological spaces and $f : Z \times X \to Y$ be any function. Then the induced map $\widehat{f} : X \to Y^Z$ is defined by $(\widehat{f}(x_{r,t,s}))(z_{t,u,v}) = f(z_{t,u,v}, x_{r,t,s})$ for neutrosophic point $x_{r,t,s} \in X$ and $z_{t,u,v} \in Z$.*
Conversely, given a function $\widehat{f} : X \to Y^Z$, a corresponding function f can also be defined by the same rule.

Proposition 3.1. Let X be a neutrosophic locally compact Hausdorff space. Then the neutrosophic evaluation map $e : Y^X \times X \to Y$ is neutrosophic continuous.

Proof. Consider $(f, x_{r,t,s}) \in Y^X \times X$,where $f \in Y^X$ and $x_{r,t,s} \in X$.Let V be a neutrosophic open set containing $f(x_{r,t,s}) = e(f, x_{r,t,s})$ in Y. Since X is neutrosophic locally compact and f is neutrosophic continuous, by Proposition 2.2, there exists a neutrosophic open set U in X such that $x_{r,t,s} \in Ncl(U)$ is neutrosophic compact and $f(Ncl(U)) \subseteq V$.

Consider the neutrosophic open set $S_{Ncl(U),V} \times U$ in $Y^X \times X$. Clearly $(f, x_{r,t,s}) \in S_{Ncl(U),V} \times U$.Let $(g, x_{t,u}) \in S_{Ncl(U),V} \times U$ be arbitrary. Thus $g(Ncl(U)) \subseteq V$. Since $x_{t,u} \in U$,we have $g(x_{t,u}) \in V$ and $e(g, x_{t,u}) = g(x_{t,u}) \in V$.Thus $e(S_{Ncl(U),V} \times U) \subseteq V$.Hence e is neutrosophic continuous. □

Proposition 3.2. Let X and Y be two neutrosophic topological spaces with Y being neutrosophic compact. Let $x_{r,t,s}$ be any neutrosophic point in X and N be a neutrosophic open set in the neutrosophic product space $X \times Y$ containing $\{x_{r,t,s}\} \times Y$. Then there exists some neutrosophic neighbourhood W of $x_{r,t,s}$ in X such that $\{x_{r,t,s}\} \times Y \subseteq W \times Y \subseteq N$.

Proposition 3.3. Let Z be a neutrosophic locally compact Hausdorff space and X, Y be arbitrary neutrosophic topological spaces. Then a map $f : Z \times X \to Y$ is neutrosophic continuous iff $\widehat{f} : X \to Y^Z$ is neutrosophic continuous,where $\widehat{f}$ is defined by the rule $(\widehat{f}(x_{r,t,s}))(z_{t,u,v}) = f(z_{t,u,v}, x_{r,t,s})$.

Proposition 3.4. Let X and Z be a neutrosophic locally compact Hausdorff spaces. Then for any neutrosophic topological space Y,the function $E : Y^{Z\times X} \to (Y^Z)^X$ defined by $E(f) = \widehat{f}$(that is $E(f)(x_{r,t,s})(z_{t,u,v}) = f(z_{t,u,v}, x_{r,t,s}) = (\widehat{f}(x_{r,t,s})(z_{t,u,v})))$ for all $f : Z \times X \to Y$ is a neutrosophic homeomorphism.

Proof. (a) Clearly E is onto.

(b) For E to be injective, let $E(f) = E(g)$ for $f, g : Z \times X \to Y$. Thus $\widehat{f} = \widehat{g}$, where $\widehat{f}$ and $\widehat{g}$ are the induced map of f and g, respectively. Now for any neutrosophic point $x_{r,t,s}$ in X and any neutrosophic point $z_{t,u,v}$ in Z, $f(z_{t,u,v}, x_{r,t,s}) = (\widehat{f}(x_{r,t,s})(z_{t,u,v})) = (\widehat{g}(x_{r,t,s})(z_{t,u,v})) = g(z_{t,u,v}, x_{r,t,s})$. Thus $f = g$.

(c) For proving the neutrosophic continuity of E, consider any neutrosophic subbasis neighbourhood V of $\widehat{f}$ in $(Y^Z)^X$, i.e V is of the form $S_{K,W}$ where K is a neutrosophic compact subset of X and W is neutrosophic open in Y^Z. Without loss of generality, we may assume that $W = S_{L,U}$, where L is a neutrosophic compact subset of Z and U is a neutrosophic open set in Y. Then $\widehat{f}(K) \subseteq S_{L,U} = W$ and this implies that $\widehat{f}(K)(L) \subseteq U$. Thus for any neutrosophic point $x_{r,t,s}$ in K and for every neutrosophic point $z_{t,u,v}$ in L, we have $(\widehat{f}(x_{r,t,s}))(z_{t,u,v}) \in U$, that is $f(z_{t,u,v}, x_{r,t,s}) \in U$ and therefore $f(L \times K) \subseteq U$. Now since L is a neutrosophic compact in Z and K is a neutrosophic compact in X, $L \times K$ is also a neutrosophic compact in $Z \times X$[7] and since U is a neutrosophic open set in Y, we conclude that $f \in S_{L\times K,U} \subseteq Y^{Z\times X}$. We assert that $E(S_{L\times K,U}) \subseteq S_{K,W}$. Let $g \in S_{L\times K,U}$ be arbitrary. Thus $g(L \times K) \subseteq U$, i.e $g(z_{t,u,v}, x_{r,t,s}) = (\widehat{g}(x_{r,t,s}))(z_{t,u,v}) \in U$ for all neutrosophic points $z_{t,u,v} \in L \subseteq Z$ and for every neutrosophic point $x_{r,t,s} \in L \subseteq X$. So $(\widehat{g}(x_{r,t,s}))(L) \subseteq U$ for every neutrosophic point $x_{r,t,s} \in K \subseteq X$, that is $(\widehat{g}(x_{r,t,s})) \in S_{L,U} = W$ for every neutrosophic points $x_{r,t,s} \in K \subseteq X$, that is $\widehat{g}(x_{r,t,s}) \in S_{L,U} = W$ for every neutrosophic point $x_{r,t,s} \in K \subseteq U$. Hence we have $\widehat{g}(K) \subseteq W$, that is $\widehat{g} = E(g) \in S_{K,W}$ for any $g \in S_{L\times K,U}$.

Thus $E(S_{L\times K,U}) \subseteq S_{K,W}$. This proves that E is a neutrosophic continuous.

(d) For proving the neutrosophic continuity of E^{-1},we consider the following neutrosophic evaluation maps: $e_1 : (Y^Z)^X \times X \to Y^Z$ defined by $e_1(\widehat{f}, x_{r,t,s}) = \widehat{f}(x_{r,t,s})$ where $\widehat{f} \in (Y^Z)^X$ and $x_{r,t,s}$ is any neutrosophic point in X and $e_2 : Y^Z \times Z \to Y$ defined by $e_2(g, z_{t,u,v}) = g(z_{t,u,v})$, where $g \in Y^Z$ and $z_{t,u,v}$ is a neutrosophic point in Z. Let denote the composition of the following neutrosophic continuous functions $\psi : (Z\times X)\times (Y^Z)^X \xrightarrow{T} (Y^Z)^X \times (Z\times X) \xrightarrow{i\times t} (Y^Z)^X \times (X\times Z) \xrightarrow{=} ((Y^Z)^X \times X)\times Z \xrightarrow{e_1\times i_Z} (Y^Z)\times Z \xrightarrow{e_2} Y$, where i, i_Z denote the neutrosophic identity maps on $(Y^Z)^X$ and Z, respectively and T, t denote the switching maps. Thus $: (Z\times X)\times (Y^Z)^X \to Y$, that is $\in Y^{(Z\times X)\times (Y^Z)^X}$. We consider the map $\widetilde{E} : Y^{(Z\times X)\times (Y^Z)^X} \to (Y^{(Z\times X)})^{(Y^Z)^X}$ (as defined in the statement of the Proposition 3.4 in fact it is E). So $\widetilde{E}(\psi) : (Y^Z)^X \to Y^{(Z\times X)}$. Now for any neutrosophic points $z_{t,u,v} \in Z, x_{r,t,s} \in X$ and $f \in Y^{(Z\times X)}$, again we have that $(\widetilde{E}(\psi)\circ E)(f)(z_{t,u,v}, x_{r,t,s}) = f(z_{t,u,v}, x_{r,t,s})$;hence $\widetilde{E}(\psi)\circ E$=identity. Similarly for any $\widehat{g} \in (Y^Z)^X$ and neutrosophic points $x_{r,t,s} \in X, z_{t,u,v} \in Z$, so we have that $(E\circ \widetilde{E}(\psi))(\widehat{g})(x_{r,t,s}, z_{t,u,v}) = (\widehat{g}(x_{r,t,s}))(z_{t,u,v})$;hence $E\circ\widetilde{E}(\psi)$=identity. Thus E is a neutrosophic homeomorphism.

$\square$

Definition 3.3. *The map E in Proposition 3.4 is called the exponential map.*

As easy consequence of Proposition 3.4 is as follows.

Proposition 3.5. Let X, Y, Z be neutrosophic locally compact Hausdorff spaces. Then the map $N : Y^X \times Z^Y \to Z^X$ defined by $N(f, g) = g\circ f$ is neutrosophic continuous.

Proof. Consider the following compositions: $X\times Y^X\times Z^Y \xrightarrow{T} Y^X\times Z^Y\times X \xrightarrow{t\times i_X} Z^Y\times Y^X\times X \xrightarrow{=} Z^Y\times (Y^X\times X) \xrightarrow{i\times e_2} Z^Y\times Y \xrightarrow{e_2} Z$, where T, t denote the switching maps, i_X, i denote the neutrosophic identity functions on X and Z^Y, respectively and e_2 denotes the neutrosophic evaluation maps. Let $\varphi = e_2\circ (i\times e_2)\circ (t\times i_X)\circ T$. By proposition 3.4, we have an exponential map $E : Z^{X\times Y^X\times Z^Y} \to (Z^X)^{Y^X\times Z^Y}$. Since $\varphi \in Z^{X\times Y^X\times Z^Y}$, $E(\varphi) \in (Z^X)^{Y^X\times Z^Y}$. Let $N = E(\varphi)$ that is, $N : Y^X\times Z^Y \to Z^X$ is neutrosophic continuous. For $f \in Y^X, g\in Z^Y$ and for any neutrosophic point $x_{r,t,s} \in X$,it easy to see that $N(f,g)(x_{r,t,s}) = g(f(x_{r,t,s}))$. $\square$

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On Neutrosophic Semi-Supra Open Set and Neutrosophic Semi-Supra Continuous Functions

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R. Dhavaseelan, M. Parimala, S. Jafari, Florentin Smarandache (2017). On Neutrosophic Semi-Supra Open Set and Neutrosophic Semi-Supra Continuous Functions. *Neutrosophic Sets and Systems* 16, 39-43

Abstract: In this paper, we introduce and investigate a new class of sets and functions between topological space called neutrosophic semi-supra open set and neutrosophic semi-supra open continuous functions respectively.

Keywords: Supra topological spaces; neutrosophic supra-topological spaces; neutrosophic semi-supra open set.

1 Introduction and Preliminaries

Intuitionistic fuzzy set is defined by Atanassov [2] as a generalization of the concept of fuzzy set given by Zadesh [14]. Using the notation of intuitionistic fuzzy sets, Coker [3] introduced the notion of an intuitionistic fuzzy topological space. The supra topological spaces and studied s-continuous functions and s^*-continuous functions were introduced by A. S. Mashhour [6] in 1993. In 1987, M. E. Abd El-Monsef et al. [1] introduced the fuzzy supra topological spaces and studied fuzzy supra continuous functions and obtained some properties and characterizations. In 1996, Keun Min [13] introduced fuzzy s-continuous, fuzzy s-open and fuzzy s-closed maps and established a number of characterizations. In 2008, R. Devi et al. [4] introduced the concept of supra α-open set, and in 1983, A. S. Mashhour et al. introduced the notion of supra-semi open set, supra semi-continuous functions and studied some of the basic properties for this class of functions. In 1999, Necla Turan [11] introduced the concept of intuitionistic fuzzy supra topological space. The concept of intuitionistic fuzzy semi-supra open set was introduced by Parimala and Indirani [7]. After the introduction of the concepts of neutrosophy and a neutrosophic se by F. Smarandache [[9], [10]], A. A. Salama and S. A. Alblowi[8] introduced the concepts of neutrosophic crisp set and neutrosophic topological spaces.

The purpose of this paper is to introduce and investigate a new class of sets and functions between topological space called neutrosophic semi-supra open set and neutrosophic semi-supra open continuous functions, respectively.

Definition 1.1. Let T, I, F be real standard or non standard subsets of $]0^-, 1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
$sup_I = i_{sup}, inf_I = i_{inf}$
$sup_F = f_{sup}, inf_F = f_{inf}$
$n-sup = t_{sup} + i_{sup} + f_{sup}$
$n-inf = t_{inf} + i_{inf} + f_{inf}$. T, I, F are neutrosophic components.

Definition 1.2. Let X be a nonempty fixed set. A neutrosophic set [briefly NS] A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

Remark 1.1. (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.3. Let X be a nonempty set and the neutrosophic sets A and B in the form
$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$; [Complement of A]

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$;

(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$;

(f) $[\,]A = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle : x \in X\}$;

(g) $\langle\rangle A = \{\langle x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.4. Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$;

(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we must introduce the neutrosophic sets 0_N and 1_N in X as follows:

Definition 1.5. $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.6. [5] A neutrosophic topology (NT) on a nonempty set X is a family T of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X, T) or simply X is called a neutrosophic topological space (NTS) and each neutrosophic set in T is called a neutrosophic open set (NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (NCS) in X.

Definition 1.7. [5] Let A be a neutrosophic set in a neutrosophic topological space X. Then

$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and $G \subseteq A\}$ is called the neutrosophic interior of A;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$ is called the neutrosophic closure of A.

Definition 1.8. Let X be a nonempty set. If r, t, s be real standard or non standard subsets of $]0^-, 1^+[$, then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point(in short NP)in X given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,t,s}$, where r denotes the degree of membership value ,t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

Now we shall define the image and preimage of neutrosophic sets. Let X and Y be two nonempty sets and $f : X \to Y$ be a function.

Definition 1.9. [5]

(a) If $B = \{\langle y, \mu_B(y), \sigma_B(y), \gamma_B(y)\rangle : y \in Y\}$ is a neutrosophic set in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the neutrosophic set in X defined by
$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\sigma_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\}$.

(b) If $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ is a neutrosophic set in X, then the image of A under f, denoted by $f(A)$, is the neutrosophic set in Y defined by
$f(A) = \{\langle y, f(\mu_A)(y), f(\sigma_A)(y), (1 - f(1 - \gamma_A))(y)\rangle : y \in Y\}$. where

$$f(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(\sigma_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \sigma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$(1 - f(1 - \gamma_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \gamma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

For the sake of simplicity, let us use the symbol $f_-(\gamma_A)$ for $1 - f(1 - \gamma_A)$.

Corollary 1.1. [5] Let A , $A_i (i \in J)$ be neutrosophic sets in X, B, $B_i (i \in K)$ be neutrosophic sets in Y and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,

(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(c) $A \subseteq f^{-1}(f(A))$ { If f is injective,then $A = f^{-1}(f(A))$ } ,

(d) $f(f^{-1}(B)) \subseteq B$ { If f is surjective,then $f(f^{-1}(B)) = B$ },

(e) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,

(f) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,

(g) $f(\bigcup A_i) = \bigcup f(A_i)$,

(h) $f(\bigcap A_i) \subseteq \bigcap f(A_i)$ { If f is injective,then $f(\bigcap A_i) = \bigcap f(A_i)$},

(i) $f^{-1}(1_N) = 1_N$,

(j) $f^{-1}(0_N) = 0_N$,

(k) $f(1_N) = 1_N$, if f is surjective

(l) $f(0_N) = 0_N$,

(m) $\overline{f(A)} \subseteq f(\overline{A})$, if f is surjective,

(n) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

2 Main Results

Definition 2.1. A neutrosophic set A in a neutrosophic topological space (X, T) is called

1) a neutrosophic semiopen set (NSOS) if $A \subseteq Ncl(Nint(A))$.

2) a neutrosophic α open set $(N\alpha OS)$ if $A \subseteq Nint(Ncl(Nint(A)))$.

3) a neutrosophic preopen set (NPOS) if $A \subseteq Nint(Ncl(A))$.

4) a neutrosophic regular open set (NROS) if $A = Nint(Ncl(A))$.

5) a neutrosophic semipre open or β open set $(N\beta OS)$ if $A \subseteq Ncl(Nint(Ncl(A)))$.

A neutrosophic set A is called a neutrosophic semiclosed set, neutrosophic α closed set, neutrosophic preclosed set, neutrosophic regular closed set and neutrosophic β closed set, respectively (NSCS, NαCS, NPCS, NRCS and NβCS, resp), if the complement of A is a neutrosophic semiopen set, neutrosophic α-open set, neutrosophic preopen set, neutrosophic regular open set, and neutrosophic β-open set, respectively.

Definition 2.2. Let (X, T) ba a neutrosophic topological space. A neutrosophic set A is called a neutrosophic semi-supra open set (briefly NSSOS) if $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. The complement of a neutrosophic semi-supra open set is called a neutrosophic semi-supra closed set.

Proposition 2.1. Every neutrosophic supra open set is neutrosophic semi-supra open set.

Proof. Let A be a neutrosophic supra open set in (X, T). Since $A \subseteq s\text{-}Ncl(A)$, we get $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. Then $s\text{-}Nint(A) \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. Hence $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. □

The converse of Proposition 2.1., need not be true as shown in Example 2.1.

Example 2.1. Let $X = \{a, b\}$. Define the neutrosophic sets A, B and C in X as follows:
$A = \langle x, (\frac{a}{0.2}, \frac{b}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6})\rangle$, $B = \langle x, (\frac{a}{0.6}, \frac{b}{0.2}), (\frac{a}{0.6}, \frac{b}{0.2}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$
and $C = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle$. Then the families $T = \{0_N, 1_N, A, B, A \cup B\}$ is neutrosophic topology on X. Thus, (X, T) is a neutrosophic topological space. Then C is called neutrosophic semi-supra open but not neutrosophic supra open set.

Proposition 2.2. Every neutrosophic α-supra open is neutrosophic semi-supra open

Proof. Let A be a neutrosophic α-supra open in (X, T), then $A \subseteq s\text{-}Nint(s\text{-}Ncl(s\text{-}Nint(A)))$. It is obvious that $s\text{-}Nint(s\text{-}Ncl(s\text{-}Nint(A))) \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. Hence $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$.

The converse of Proposition 2.2., need not be true as shown in Example 2.2. □

Example 2.2. Let $X = \{a, b\}$. Define the neutrosophic sets A, B and C in X as follows:
$A = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.3})\rangle$, $B = \langle x, (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.6}, \frac{b}{0.5})\rangle$
and $C = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle$. Then the families $T = \{0_N, 1_N, A, B, A \cup B\}$ is neutrosophic topology on X.Thus, (X, T) is a neutrosophic topological space. Then C is called neutrosophic semi-supra open but not neutrosophic α-supra open set.

Proposition 2.3. Every neutrosophic regular supra open set is neutrosophic semi-supra open set

Proof. Let A be a neutrosophic regular supra open set in (X, T). Then $A \subseteq (s\text{-}Ncl(A))$. Hence $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$.

The converse of Proposition 2.3., need not be true as shown in Example 2.3. □

Example 2.3. Let $X = \{a, b\}$. Define the neutrosophic sets A, B and C in X as follows:
$A = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.3})\rangle$, $B = \langle x, (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.6}, \frac{b}{0.5})\rangle$
and $C = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle$. Then the families $T = \{0_N, 1_N, A, B, A \cup B\}$ is neutrosophic topology on X. Thus, (X, T) is a neutrosophic topological space. Then C is neutrosophic semi-supra open but not neutrosophic regular-supra open set.

Definition 2.3. The neutrosophic semi-supra closure of a set A is denoted by $semi\text{-}s\text{-}Ncl(A) = \bigcup\{$ G :G is aneutrosophic semi-supra open set in X and $G \subseteq A\}$ and the neutrosophic semi-supra interior of a set A is denoted by $semi\text{-}s\text{-}Nint(A) = \bigcap\{$G :G is a neutrosophic semi-supra closed set in X and $G \supseteq A\}$.

Remark 2.1. It is clear that $semi\text{-}s\text{-}Nint(A)$ is a neutrosophic semi-supra open set and $semi\text{-}s\text{-}Ncl(A)$ is a neutrosophic semi-supra closed set.

Proposition 2.4. i) $\overline{semi - s - Nint(A)}$ = semi s-Ncl $(\overline{A})$

ii) $\overline{semi - s - Ncl(A)}$ = semi s-int $(\overline{A})$

iii) if $A \subseteq B$ then $semi\text{-}s\text{-}Ncl(A) \subseteq semi\text{-}s\text{-}Ncl(B)$ and $semi\text{-}s\text{-}Nint(A) \subseteq semi\text{-}s\text{-}Nint(B)$

Proof. It is obvious. □

Proposition 2.5. (i) The intersection of a neutrosophic supra open set and a neutrosophic semi-supra open set is a neutrosophic semi- supra open set.

(ii) The intersection of a neutrosophic semi-supra open set and aneutrosophic pre-supra open set is a neutrosophic pre-supra open set.

Proof. It is obvious. □

Definition 2.4. Let (X, T) and (Y, S) be two neutrosophic semi-supra open sets and R be a associated supra topology with T. A map $f : (X, T) \to (Y, S)$ is called neutrosophic semi- supra continuous map if the inverse image of each neutrosophic open set in Y is a neutrosophic semi- supra open in X.

Proposition 2.6. Every neutrosophic supra continuous map is neutrosophic semi-supra continuous map.

Proof. Let $f : (X, T) \to (Y, S)$ be a neutrosophic supra continuous map and A is a neutrosophic open set in Y. Then $f^{-1}(A)$ is a neutrosophic open set in X. Since R is associated with T. Then $T \subseteq R$. Therefore $f^{-1}(A)$ is a neutrosophic supra open set in X which is a neutrosophic supra open set in X. Hence f is aneutrosophic semi-supra continuous map. □

Remark 2.2. Every neutrosophic semi-supra continuous map need not be neutrosophic supra continuous map.

Proposition 2.7. Let (X, T) and (Y, S) be two neutrosophic topological spaces and R be a associated neutrosophic supra topology with T. Let f be a map from X into Y. Then the following are equivalent.

i) f is a neutrosophic semi-supra continuous map.

ii) The inverse image of a neutrosophic closed sets in Y is a neutrosophic semi closed set in X.

iii) Semi-s-Ncl($f^{-1}(A)$) $\subseteq$ $f^{-1}(Ncl(A))$ for every neutrosophic set A in Y.

iv) $f(semi\text{-}s\text{-}Ncl(A)) \subseteq Ncl(f(A))$ for every neutrosophic set A in X.

v) $f^{-1}(Nint(B)) \subseteq semi\text{-}s\text{-}Nint(f^{-1}(B))$ for every neutrosophic set B in Y.

Proof. $(i) \Rightarrow (ii)$: Let A be a neutrosophic closed set in Y. Then $\overline{A}$ is neutrosophic open in Y, Thus $f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$ is neutrosophic semi-open in X. It follows that $f^{-1}(A)$ is a neutrosophic semi-s closed set of X.
$(ii) \Rightarrow (iii)$: Let A be any subset of X. Since $Ncl(A)$ is neutrosophic closed in Y then it follows that $f^{-1}(Ncl(A))$ is neutrosophic semi-s closed in X. Therefore, $f^{-1}(Ncl(A)) = semi\text{-}s\text{-}Ncl(f^{-1}(Ncl(A)) \supseteq semi\text{-}s\text{-}Ncl(f^{-1}(A))$
$(iii) \Rightarrow (iv)$: Let A be any subset of X. By (iii) we obtain $f^{-1}(Ncl(f((A))) \supseteq semi\text{-}s\text{-}Ncl(f^{-1}(f(A))) \supseteq semi\text{-}s\text{-}Ncl(A)$ and hence $f(semi\text{-}s\text{-}Ncl(A)) \subseteq Ncl(f(A))$.
$(iv) \Rightarrow (v)$: Let $f(semi\text{-}s\text{-}Ncl(A)) \subseteq f(Ncl(A)$ for every neutrosophic set A in X. Then $semi\text{-}s\text{-}Ncl(A)) \subseteq f^{-1}(Ncl(f(A))$. $\overline{semi - s - Ncl(A)} \supseteq \overline{f^{-1}(Ncl(f(A)))}$ and $semi\text{-}s\text{-}Nint(\overline{A}) \supseteq f^{-1}(Nint(\overline{f(A)}))$. Then $semi\text{-}s\text{-}Nint(f^{-1}(B)) \supseteq f^{-1}(Nint(B))$. Therefore $f^{-1}(Nint(B)) \subseteq s\text{-}Nint(f^{-1}(B))$ for every B in Y.
$(v) \Rightarrow (i)$: Let A be a neutrosophic open set in Y. Therefore $f^{-1}(Nint(A)) \subseteq semi\text{-}s\text{-}Nint(f^{-1}(A))$, hence $f^{-1}(A) \subseteq semi\text{-}s\text{-}Nint(f^{-1}(A))$. But we know that $semi\text{-}s\text{-}Nint(f^{-1}(A)) \subseteq f^{-1}(A)$, then $f^{-1}(A) = semi\text{-}s\text{-}Nint(f^{-1}(A))$. Therefore $f^{-1}(A)$ is a neutrosophic semi-s-open set. □

Proposition 2.8. If a map $f : (X, T) \to (Y, S)$ is a neutrosophic semi-s-continuous and $g : (Y, S) \to (Z, R)$ is neutrosophic continuous, Then $g \circ f$ is neutrosophic semi-s-continuous.

Proof. Obvious. □

Proposition 2.9. Let a map $f : (X, T) \to (Y, S)$ be a neutrosophic semi-supra continuous map, then one of the following holds

i) $f^{-1}(semi\text{-}s\text{-}Nint(A)) \subseteq Nint(f^{-1}(A))$ for every neutrosophic set A in Y.

ii) $Ncl(f^{-1}(A)) \subseteq f^{-1}(semi\text{-}s\text{-}Ncl(A))$ for every neutrosophic set A in Y.

iii) $f(Ncl(B)) \subseteq semi\text{-}s\text{-}Ncl(f(B))$ for every neutrosophic set B in X.

Proof. Let A be any neutrosophic open set of Y, then condition (i) is satisfied, then $f^{-1}(semi\text{-}s\text{-}Nint(A)) \subseteq Nint(f^{-1}(A))$. We get, $f^{-1}(A) \subseteq Nint(f^{-1}(A))$. Therefore $f^{-1}(A)$ is a neutrosophic supra open set. Every neutrosophic supra open set is a neutrosophic semi supra open set. Hence f is a neutrosophic semi-s-continuous function. If condition (ii) is satisfied, then we can easily prove that f is a neutrosophic semi -s continuous function if condition (iii) is satisfied, and A is any neutrosophic open set of Y, then $f^{-1}(A)$ is a set in X and $f(Ncl(f^{-1}(A)) \subseteq semi\text{-}s\text{-}Ncl(f(f^{-1}(A)))$. This implies $f(Ncl(f^{-1}(A))) \subseteq semi\text{-}s\text{-}Ncl(A)$. This is nothing but condition (ii). Hence f is a neutrosophic semi-s-continuous function. □

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Neutrosophic Regular Filters and Fuzzy Regular Filters in Pseudo-BCI Algebras

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Xiaohong Zhang, Yingcang Ma, Florentin Smarandache (2017). Neutrosophic Regular Filters and Fuzzy Regular Filters in Pseudo-BCI Algebras. *Neutrosophic Sets and Systems* 17, 10-15

Abstract. Neutrosophic set is a new mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Pseudo-BCI algebra is a kind of non-classical logic algebra in close connection with various non-commutative fuzzy logics. Recently, we applied neutrosophic set theory to pseudo-BCI algebras. In this paper, we study neutrosophic filters in pseudo-BCI algebras. The concepts of neutrosophic regular filter, neutrosophic closed filter and fuzzy regular filter in pseudo-BCI algebras are introduced, and some basic properties are discussed. Moreover, the relationships among neutrosophic regular filter, fuzzy filters and anti-grouped neutrosophic filters are presented, and the results are proved: a neutrosophic filter (fuzzy filter) is a neutrosophic regular filter (fuzzy regular filter), if and only if it is both a neutrosophic closed filter (fuzzy closed filter) and an anti-grouped neutrosophic filter (fuzzy anti-grouped filter).

Keywords: Neutrosophic set, Pseudo-BCI algebra, Neutrosophic Filter, Neutrosophic Regular Filter, Fuzzy Regular Filter.

1 Introduction

In 1998, Florentin Smarandache introduced the concept of a neutrosophic set from a philosophical point of view (see [16, 17, 18]). The neutrosophic set is a powerful general formal framework that generalizes the concept of fuzzy set and intuitionistic fuzzy set. In this paper we work with special neutrosophic sets, they are called single valued neutrosophic set (see [21]). The neutrosophic set theory is applied to many scientific fields (see [18, 19, 20]), and also applied to algebraic structures (see [1, 2, 15, 19]), it is similar to the applications of fuzzy set (soft set, rough set) theory in algebraic structures (see [11, 14, and 23]).

In 2008, W. A. Dudek and Y. B. Jun [3] introduced the notion of pseudo-BCI algebra as a generalization of BCI algebra, it is also as a generalization of pseudo-BCK algebra (which is close connection with various non-commutative fuzzy logic formal systems, see [4, 24, 26, 27, 28, and 32]). For non-classical logic algebra systems, the theory of filters (ideals) plays an important role (see [9, 12, 13, 25, and 30]). In [7], the notion of pseudo-BCI filter (ideal) of pseudo-BCI algebras is introduced. In 2009, some special pseudo-BCI filters (ideals) are discussed in [10]. Since then, some articles related filters of pseudo-BCI algebras are published (see [29, 31, 33, and 34]).

Recently, we applied neutrosophic set theory to pseudo -BCI algebras in [35]**.** This paper we further study on the applications of neutrosophic sets to pseudo-BCI algebras. We introduce the new concepts of neutrosophic regular filter, neutrosophic closed filter and fuzzy regular filter in pseudo-BCI algebras, and investigate their basic properties and present relationships among neutrosophic regular filters, anti-grouped neutrosophic filter and fuzzy filters.

Note that, the notion of pseudo-BCI algebra in this paper is a dual of the original definition in [3], so the notion of filter is a dual of (pseudo-BCI) ideal in [7, 10].

2 Some basic concepts and properties

2.1 On neutrosophic sets

Definition 2.1[17, 18, 19] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. That is, $T_A(x): X\to\ ]^-0, 1^+[$, $I_A(x): X\to\ ]^-0, 1^+[$, and $F_A(x): X\to\ ]^-0, 1^+[$. Thus, there is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, so $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

Definition 2.2[21] Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$. Then, a simple valued neutrosophic set A can be denoted by

$$A=\{\langle x , T_A(x), I_A(x), F_A(x) \rangle \mid x\in X\},$$

where $T_A(x)$, $I_A(x)$, $F_A(x)\in[0, 1]$ for each point x in X. Therefore, the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.3[21] The complement of a simple valued neutrosophic set A is denoted by A^c and is defined as ($\forall x\in X$)

$$T_{A^c}(x) = F_A(x),\ I_{A^c}(x) = 1 - I_A(x),\ F_{A^c}(x) = T_A(x).$$

Then

$$A^c=\{\langle x , F_A(x), 1-I_A(x), T_A(x)\rangle \mid x\in X\}.$$

Definition 2.4[21] A simple valued neutrosophic set A is contained in the other simple valued neutrosophic set B, denote $A\subseteq B$, if and only if $T_A(x)\le T_B(x)$, $I_A(x) \le I_B(x)$, $F_A(x)\ge F_B(x)$ for any x in X.

Definition 2.5[21] Two simple valued neutrosophic sets A and B are equal, written as $A = B$, if and only if $A\subseteq B$ and $B\subseteq A$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.6[21] The union of two neutrosophic sets A and B is a neutrosophic set C, written as $C=A\cup B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x)=\max(T_A(x), T_B(x)),\ I_C(x)=\max(I_A(x), I_B(x)),$$
$$F_C(x)=\min(F_A(x), F_B(x)),\ \forall x\in X.$$

Definition 2.7[21] The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C=A\cap B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x)= \min(T_A(x), T_B(x)),\ I_C(x)=\min(I_A(x), I_B(x)),$$
$$F_C(x)=\max(F_A(x), F_B(x)),\ \forall x\in X.$$

Definition 2.8[20] Let A be a neutrosophic set in X and $\alpha, \beta, \gamma\in[0, 1]$ with $0\le\alpha+\beta+\gamma \le 3$ and (α, β, γ)-level set of A denoted by $A^{(\alpha, \beta, \gamma)}$ is defined as:

$$A^{(\alpha, \beta, \gamma)}=\{ x\in X \mid T_A(x)\ge\alpha, I_A(x)\ge\beta, F_A(x)\le\gamma\}.$$

2.2 On pseudo-BCI algebras

Definition 2.9[3] A pseudo-BCI algebra is a structure $(X; \le, \to, \rightsquigarrow, 1)$, where "$\le$" is a binary relation on X, "$\to$" and "$\rightsquigarrow$" are binary operations on X and "1" is an element of X, verifying the axioms: for all $x, y, z\in X$,

(1) $y\to z\le(z\to x)\rightsquigarrow(y\to x)$, $y\rightsquigarrow z\le(z\rightsquigarrow x)\to(y\rightsquigarrow x)$;
(2) $x\le(x\to y)\rightsquigarrow y$, $x\le(x\rightsquigarrow y)\to y$;
(3) $x\le x$;
(4) $x\le y$, $y\le x \Rightarrow x=y$;
(5) $x\le y \Leftrightarrow x\to y =1 \Leftrightarrow x\rightsquigarrow y =1$.

If $(X; \le, \to, \rightsquigarrow, 1)$ is a pseudo-BCI algebra satisfying $x\to y = x\rightsquigarrow y$ for all $x, y\in X$, then $(X; \to, 1)$ is a BCI-algebra.

Proposition 2.1[3, 7, 10] Let $(X; \le, \to, \rightsquigarrow, 1)$ be a pseudo-BCI algebra, then X satisfy the following properties ($\forall x, y, z\in X$):

(1) $1\le x \Rightarrow x=1$;
(2) $x\le y \Rightarrow y\to z\le x\to z$, $y\rightsquigarrow z\le x\rightsquigarrow z$;
(3) $x\le y$, $y\le z \Rightarrow x\le z$;
(4) $x\rightsquigarrow(y\to z)=y\to(x\rightsquigarrow z)$;
(5) $x\le y\to z \Leftrightarrow y\le x\rightsquigarrow z$;
(6) $x\to y\le(z\to x)\to(z\to y)$, $x\rightsquigarrow y\le(z\rightsquigarrow x)\rightsquigarrow(z\rightsquigarrow y)$;
(7) $x\le y \Rightarrow z\to x\le z\to y$, $z\rightsquigarrow x\le z\rightsquigarrow y$;
(8) $1\to x=x$, $1\rightsquigarrow x=x$;
(9) $((y\to x)\rightsquigarrow x)\to x=y\to x$, $((y\rightsquigarrow x)\to x)\rightsquigarrow x=y\rightsquigarrow x$;
(10) $x\to y\le(y\to x)\rightsquigarrow 1$, $x\rightsquigarrow y \le(y\rightsquigarrow x)\to 1$;
(11) $(x\to y)\to 1=(x\to 1)\rightsquigarrow(y\rightsquigarrow 1)$,
$(x\rightsquigarrow y)\rightsquigarrow 1=(x\rightsquigarrow 1)\to(y\to 1)$;
(12) $x\to 1=x\rightsquigarrow 1$.

Definition 2.10[7] A nonempty subset F of pseudo-BCI algebra X is called a pseudo-BCI filter (briefly, filter) of X if it satisfies:

(F1) $1\in F$;
(F2) $x\in F$, $x\to y\in F \Rightarrow y\in F$;
(F3) $x\in F$, $x\rightsquigarrow y\in F \Rightarrow y\in F$.

Definition 2.11[29] A pseudo-BCI algebra X is said to be anti-grouped pseudo-BCI algebra if it satisfies the following identity:

(G1) $\forall x, y, z\in X$, $(x\to y)\to(x\to z)= y\to z$,
(G2) $\forall x, y, z\in X$, $(x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z)= y\rightsquigarrow z$.

Proposition 2.2 [29] A pseudo-BCI algebra X is an anti-grouped pseudo-BCI algebra if and only if it satisfies:
$\forall x\in X$, $(x\to 1)\to 1=x$ or $(x\rightsquigarrow 1)\rightsquigarrow 1=x$.

Definition 2.12[29] A filter F of a pseudo-BCI algebra X is called an anti-grouped filter of X if it satisfies
(GF) $\forall x\in X$, $(x\to 1)\to 1\in F$ or $(x\rightsquigarrow 1)\rightsquigarrow 1\in F\Rightarrow x\in F$.

Definition 2.13[29] A filter F of a pseudo-BCI algebra X is called a closed filter of X if it satisfies
(CF) $\forall x\in X$, $x\to 1\in F$.

Definition 2.14[34] A filter F of pseudo-BCI algebra X is said to be regular if it satisfies:

(RF1) $\forall x, y\in X$, $y\in F$ and $x\to y\in F \Rightarrow x\in F$.
(RF2) $\forall x, y\in X$, $y\in F$ and $x\rightsquigarrow y\in F \Rightarrow x\in F$.

Proposition 2.3 [34] Let X be a pseudo-BCI algebra, F a filter of X. Then F is regular if and only if F is anti-grouped and closed.

Definition 2.15[31, 33] A fuzzy set A in pseudo-BCI algebra X is called fuzzy filter of X if it satisfies:

(FF1) $\forall x\in X, \mu_A(x)\leq\mu_A(1)$;

(FF2) $\forall x, y\in X, \min\{\mu_A(x), \mu_A(x\to y)\}\leq\mu_A(y)$;

(FF3) $\forall x, y\in X, \min\{\mu_A(x), \mu_A(x\rightsquigarrow y)\}\leq\mu_A(y)$.

Definition 2.16[31] A fuzzy set A: $X\to[0, 1]$ is called a fuzzy closed filter of pseudo-BCI algebra X if it is a fuzzy filter of X such that:

(FCF) $\mu_A(x\to 1)\geq\mu_A(x), \forall x\in X$.

Definition 2.17[31] A fuzzy set A in pseudo-BCI algebra X is called fuzzy anti-grouped filter of X if it satisfies:

(1) $\forall x\in X, \mu_A(x)\leq\mu_A(1)$;

(2) $\forall x, y, z\in X, \min\{\mu_A(y), \mu_A((x\to y)\to(x\to z))\}\leq\mu_A(z)$;

(3) $\forall x, y, z\in X, \min\{\mu_A(y), \mu_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\leq\mu_A(z)$.

Proposition 2.4[31] Let A be a fuzzy filter of pseudo-BCI algebra X. Then A is a fuzzy anti-grouped filter of X if and only if it satisfies:

$$\forall x\in X, \mu_A(x)\geq\mu_A((x\to 1)\to 1), \mu_A(x)\geq\mu_A((x\rightsquigarrow 1)\rightsquigarrow 1).$$

Definition 2.18[35] A neutrosophic set A in pseudo-BCI algebra X is called a neutrosophic filter in X if it satisfies: $\forall x, y\in X$,

(NSF1) $T_A(x)\leq T_A(1)$, $I_A(x)\leq I_A(1)$ and $F_A(x)\geq F_A(1)$;

(NSF2) $\min\{T_A(x), T_A(x\to y)\}\leq T_A(y)$, $\min\{I_A(x), I_A(x\to y)\}\leq I_A(y)$ and $\max\{F_A(x), F_A(x\to y)\}\geq F_A(y)$;

(NSF3) $\min\{T_A(x), T_A(x\rightsquigarrow y)\}\leq T_A(y)$, $\min\{I_A(x), I_A(x\rightsquigarrow y)\}\leq I_A(y)$ and $\max\{F_A(x), F_A(x\rightsquigarrow y)\}\geq F_A(y)$.

Proposition 2.5[35] Let A be a neutrosophic filter in pseudo-BCI algebra X, then $\forall x, y\in X$,

(NSF4) $x\leq y \Rightarrow T_A(x)\leq T_A(y)$, $I_A(x)\leq I_A(y)$ and $F_A(x)\geq F_A(y)$.

Definition 2.19[35] A neutrosophic set A in pseudo-BCI algebra X is called anti-grouped neutrosophic filter in X if it satisfies: $\forall x, y, z\in X$,

(1) $T_A(x)\leq T_A(1)$, $I_A(x)\leq I_A(1)$ and $F_A(x)\geq F_A(1)$;

(2) $\min\{T_A(y), T_A((x\to y)\to(x\to z))\}\leq T_A(z)$, $\min\{I_A(y), I_A((x\to y)\to(x\to z))\}\leq I_A(z)$ and $\max\{F_A(x), F_A((x\to y)\to(x\to z))\}\geq F_A(z)$;

(3) $\min\{T_A(y), T_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\leq T_A(z)$, $\min\{I_A(y), I_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\leq I_A(z)$ and $\max\{F_A(x), F_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\geq F_A(z)$.

Proposition 2.6[35] Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is a neutrosophic filter in X if and only if A satisfies:

(i) T_A is a fuzzy filter of X;

(ii) I_A is a fuzzy filter of X;

(iii) $1-F_A$ is a fuzzy filter of X, where $(1-F_A)(x) = 1-F_A(x)$, $\forall x\in X$.

Proposition 2.7[35] Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is an anti-grouped neutrosophic filter in X if and only if A satisfies:

(i) T_A is a fuzzy anti-grouped filter of X;

(ii) I_A is a fuzzy anti-grouped filter of X;

(iii) $1-F_A$ is a fuzzy anti-grouped filter of X, where $(1-F_A)(x)=1-F_A(x)$, $\forall x\in X$.

3 Neutrosophic regular filters and neutrosophic closed filters

Definition 3.1 A neutrosophic set A in pseudo-BCI algebra X is called a neutrosophic regular filter in X if it is a neutrosophic filter in X such that: $\forall x, y\in X$,

(NSRF1) $\min\{T_A(y), T_A(x\to y)\}\leq T_A(x)$, $\min\{I_A(y), I_A(x\to y)\}\leq I_A(x)$ and $\max\{F_A(y), F_A(x\to y)\}\geq F_A(x)$;

(NSRF2) $\min\{T_A(y), T_A(x\rightsquigarrow y)\}\leq T_A(x)$, $\min\{I_A(y), I_A(x\rightsquigarrow y)\}\leq I_A(x)$ and $\max\{F_A(y), F_A(x\rightsquigarrow y)\}\geq F_A(x)$.

Definition 3.2 A neutrosophic set A in pseudo-BCI algebra X is called a neutrosophic closed filter in X if it is a neutrosophic filter in X such that: $\forall x\in X$,

(NSCF) $T_A(x\to 1)\geq T_A(x)$, $I_A(x\to 1)\geq I_A(x)$, $F_A(x\to 1)\leq F_A(x)$.

Proposition 3.1 Let A be a neutrosophic regular filter in pseudo-BCI algebra X. Then A is closed.

Proof: Suppose $x\in X$. By Definition 2.9 (2) and Proposition 2.1 (12) we have

$$x\leq(x\to 1)\rightsquigarrow 1=(x\to 1)\to 1.$$

From this and Proposition 2.5 we get

$$T_A(x)\leq T_A((x\to 1)\to 1), I_A(x)\leq I_A((x\to 1)\to 1),$$
$$F_A(x)\geq F_A((x\to 1)\to 1).$$

Moreover, by Definition 2.18 (NSF1) and Definition 3.1 (NSRF1)

$$T_A((x\to 1)\to 1)=\min\{T_A(1), T_A((x\to 1)\to 1)\}\leq T_A(x\to 1),$$
$$I_A((x\to 1)\to 1)=\min\{I_A(1), I_A((x\to 1)\to 1)\}\leq I_A(x\to 1),$$
$$F_A((x\to 1)\to 1)=\max\{F_A(1), F_A((x\to 1)\to 1)\}\geq F_A(x\to 1).$$

Thus,

$$T_A(x)\leq T_A((x\to 1)\to 1)\leq T_A(x\to 1),$$
$$I_A(x)\leq I_A((x\to 1)\to 1)\leq I_A(x\to 1),$$
$$F_A(x)\geq T_A((x\to 1)\to 1)\geq T_A(x\to 1).$$

By Definition 3.2 we know that A is closed.

By Proposition 2.4 and Proposition 2.7 we can get the following proposition.

Proposition 3.2 Let A be a neutrosophic filter of pseudo-BCI algebra X. Then A is an anti-grouped neutrosophic filter of X if and only if it satisfies: $\forall x\in X$,

$$T_A(x)\geq T_A((x\to 1)\to 1), T_A(x)\geq T_A((x\rightsquigarrow 1)\rightsquigarrow 1);$$
$$I_A(x)\geq I_A((x\to 1)\to 1), I_A(x)\geq I_A((x\rightsquigarrow 1)\rightsquigarrow 1);$$
$$F_A(x)\leq F_A((x\to 1)\to 1), F_A(x)\leq F_A((x\rightsquigarrow 1)\rightsquigarrow 1).$$

Proposition 3.3 Let A be a neutrosophic regular filter in pseudo-BCI algebra X. Then A is anti-grouped.

Proof: Suppose $x\in X$. By Definition 2.9 and Proposition 2.1 we have

$$x\to((x\to 1)\to 1)=x\to((x\to 1)\rightsquigarrow 1)=1.$$

From this we get

$$T_A(x\to((x\to 1)\to 1))=T_A(1), I_A(x\to((x\to 1)\to 1))=I_A(1),$$

$$F_A(x\to((x\to1)\to1))=F_A(1).$$

Thus, applying Definition 3.1 (NSRF1) we get

$$T_A(x)\geq\min\{T_A((x\to1)\to1),\ T_A(x\to((x\to1)\to1))\}$$
$$=\min\{T_A((x\to1)\to1),\ T_A(1)\}=T_A((x\to1)\to1),$$
$$I_A(x)\geq\min\{I_A((x\to1)\to1),\ I_A(x\to((x\to1)\to1))\}$$
$$=\min\{I_A((x\to1)\to1),\ I_A(1)\}=I_A((x\to1)\to1),$$
$$F_A(x)\leq\max\{F_A((x\to1)\to1),\ F_A(x\to((x\to1)\to1))\}$$
$$=\max\{F_A((x\to1)\to1),\ F_A(1)\}=F_A((x\to1)\to1).$$

Similarly, we can prove that

$$T_A(x)\geq T_A((x\rightsquigarrow1)\rightsquigarrow1), I_A(x)\geq I_A((x\rightsquigarrow1)\rightsquigarrow1),$$
$$F_A(x)\leq F_A((x\rightsquigarrow1)\rightsquigarrow1).$$

By Proposition 3.2 we know that A is anti-grouped.

Proposition 3.2 Assume that A is both an anti-grouped neutrosophic filter and a neutrosophic closed filter in pseudo-BCI algebra X. Then A satisfies: $\forall x\in X$,

$$T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

Proof: For any $x\in X$, by Definition 3.2 we have

$$T_A(x\to1)\geq T_A(x),\ I_A(x\to1)\geq I_A(x),\ F_A(x\to1)\leq F_A(x).$$

Moreover, $\forall x\in X$, by Definition 2.19 and Definition 3.2,

$$T_A(x)\geq\min\{T_A((x\to1)\to(x\to x)),\ T_A(1)\}$$
$$=\min\{T_A((x\to1)\to1),\ T_A(1)\}$$
$$=T_A((x\to1)\to1)\geq T_A(x\to1),$$
$$I_A(x)\geq\min\{I_A((x\to1)\to(x\to x)),\ I_A(1)\}$$
$$=\min\{I_A((x\to1)\to1),\ I_A(1)\}$$
$$=I_A((x\to1)\to1)\geq I_A(x\to1),$$
$$F_A(x)\leq\max\{F_A((x\to1)\to(x\to x)),\ F_A(1)\}$$
$$=\max\{F_A((x\to1)\to1),\ F_A(1)\}$$
$$=F_A((x\to1)\to1)\leq F_A(x\to1).$$

That is,

$$T_A(x)\geq T_A(x\to1),\ I_A(x)\geq I_A(x\to1),\ F_A(x)\leq F_A(x\to1).$$

Therefore,

$$\forall x\in X,\ T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

Theorem 3.1 Let A be a neutrosophic filter in pseudo-BCI algebra X. Then the following conditions are equivalent:

(i) A is both an anti-grouped neutrosophic filter and a neutrosophic closed filter in X;

(ii) A satisfies: $\forall x\in X$,

$$T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

(iii) A is a neutrosophic regular filter in X.

Proof: (i) $\Rightarrow$ (ii) See Proposition 3.2.

(iii) $\Rightarrow$ (i) See Proposition 3.1 and Proposition 3.3.

(ii) $\Rightarrow$ (iii) Suppose that A satisfies: $\forall x\in X$,

$$T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

For any $x, y\in X$, using Proposition 2.1 (6) we have

$$y\to1\leq(x\to y)\to(x\to1).$$

From this, applying Propostion 2.5,

$$T_A(y\to1)\leq T_A((x\to y)\to(x\to1)),$$
$$I_A(y\to1)\leq I_A((x\to y)\to(x\to1)),$$
$$F_A(y\to1)\geq F_A((x\to y)\to(x\to1)).$$

From these, by Definition 2.18 we get

$$\min\{T_A(y\to1),\ T_A(x\to y)\}$$
$$\leq\min\{T_A((x\to y)\to(x\to1)),\ T_A(x\to y)\}=T_A(x\to1),$$
$$\min\{I_A(y\to1),\ I_A(x\to y)\}$$
$$\leq\min\{I_A((x\to y)\to(x\to1)),\ I_A(x\to y)\}=I_A(x\to1),$$
$$\max\{F_A(y\to1),\ F_A(x\to y)\}$$
$$\geq\max\{F_A((x\to y)\to(x\to1)),\ F_A(x\to y)\}=F_A(x\to1).$$

Moreover, by condition (ii),

$$T_A(y\to1)=T_A(y),\ T_A(x\to1)=T_A(x);$$
$$I_A(y\to1)=I_A(y),\ I_A(x\to1)=I_A(x);$$
$$F_A(y\to1)=F_A(y),\ F_A(x\to1)=F_A(x).$$

Therefore,

$$\min\{T_A(y),\ T_A(x\to y)\}\leq T_A(x),$$
$$\min\{I_A(y),\ I_A(x\to y)\}\leq I_A(x),$$
$$\max\{F_A(y),\ F_A(x\to y)\}\geq F_A(x).$$

Similarly, we can get

$$\min\{T_A(y),\ T_A(x\rightsquigarrow y)\}\leq T_A(x),$$
$$\min\{I_A(y),\ I_A(x\rightsquigarrow y)\}\leq I_A(x),$$
$$\max\{F_A(y),\ F_A(x\rightsquigarrow y)\}\geq F_A(x).$$

By Definition 3.1 we know that A is a neutrosophic regular filter in X.

4 Fuzzy regular filters and neutrosophic filters

Definition 4.1 A fuzzy filter A in pseudo-BCI algebra X is called to be regular if it satisfies:

(FRF1) $\forall x, y\in X$, $\min\{\mu_A(y),\ \mu_A(x\to y)\}\leq\mu_A(x)$;

(FRF2) $\forall x, y\in X$, $\min\{\mu_A(y),\ \mu_A(x\rightsquigarrow y)\}\leq\mu_A(x)$.

Lemma 4.1[9, 33] Let X be a pseudo-BCI algebra. Then a fuzzy set μ: $X\to[0, 1]$ is a fuzzy filter of X if and only if the level set $\mu_t=\{x\in X \mid \mu(x)\geq t\}$ is filter of X for all $t\in Im(\mu)$.

Theorem 4.1 Let X be a pseudo-BCI algebra. Then a fuzzy set μ: $X\to[0, 1]$ is a fuzzy regular filter of X if and only if the level set $\mu_t=\{x\in X \mid \mu(x)\geq t\}$ is regular filter of X for all $t\in Im(\mu)$.

Proof: Assume that μ is fuzzy regular filter of X. By Lemma 4.1, for any $t\in Im(\mu)$, we have

$$\mu_t=\{x\in X \mid \mu(x)\geq t\}\text{ is filter of }X.$$

If $y\in\mu_t$ and $x\to y\in\mu_t$, then

$$\mu(y)\geq t,\ \mu(x\to y)\geq t.$$

From this and Definition 4.1 (FRF1) we get

$$\mu_A(x)\geq\min\{\mu_A(y),\ \mu_A(x\to y)\}\geq t.$$

This means that $x\in\mu_t$. Similarly, we can prove that

$$y\in\mu_t\text{ and }x\rightsquigarrow y\in\mu_t\Rightarrow x\in\mu_t.$$

By Definition 2.14 we know that μ_t is regular filter of X

Conversely, assume that the level set $\mu_t=\{x\in X \mid \mu(x)\geq t\}$ is regular filter of X for all $t\in Im(\mu)$. By Lemma 4.1 we know that μ: $X\to[0, 1]$ is a fuzzy filter of X. Let $x, y\in X$, denote $t_0=\min\{\mu_A(y),\ \mu_A(x\to y)\}$, then $t_0\in Im(\mu)$ and

$$\mu(y)\geq t_0,\ \mu(x\to y)\geq t_0.$$

This means that $y\in\mu_{t_0}$ and $x\to y\in\mu_{t_0}$. Since μ_{t_0} is regular filter of X, by Definition 2.14 we have $x\in\mu_{t_0}$, that is

$$\mu(x) \geq t_0 = \min\{\mu_A(y), \mu_A(x \to y)\}.$$

It follows that Definition 4.1 (FRF1) holds. Similarly, we can prove that $\forall x, y \in X$, $\min\{\mu_A(y), \mu_A(x \rightsquigarrow y)\} \leq \mu_A(x)$. Therefore, μ: $X \to [0, 1]$ is a fuzzy regular filter of X.

Similar to Theorem 4.1 we can get the following proposition (the proofs are omitted).

Proposition 4.1 Let X be a pseudo-BCI algebra. Then a fuzzy set μ: $X \to [0, 1]$ is a fuzzy closed filter of X if and only if the level set $\mu_t = \{ x \in X \mid \mu(x) \geq t\}$ is closed filter of X for all $t \in Im(\mu)$.

By Theorem 6 in [31] we have

Theorem 4.2 Let μ be a fuzzy filter of pseudo-BCI algebra X. Then the following conditions are equivalent:

(i) μ is fuzzy closed anti-grouped filter of X;

(ii) $\forall x \in X$, $\mu_A(x \to 1) = \mu_A(x)$.

(iii) μ is a fuzzy regular filter of X.

Theorem 4.3 Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is a neutrosophic closed filter in X if and only if A satisfies:

(i) T_A is a fuzzy closed filter of X;

(ii) I_A is a fuzzy closed filter of X;

(iii) $1-F_A$ is a fuzzy closed filter of X, where $(1-F_A)(x) = 1-F_A(x)$, $\forall x \in X$.

Proof: Assume that A is a neutrosophic closed filter in X. By Definition 3.2 we have ($\forall x \in X$)

$$T_A(x \to 1) \geq T_A(x),\ I_A(x \to 1) \geq I_A(x),\ F_A(x \to 1) \leq F_A(x).$$

Thus,

$$(1-F_A)(x \to 1) = 1-F_A(x \to 1) \geq 1-F_A(x) = (1-F_A)(x).$$

Therefore, using Definition 2.16, we get that T_A, I_A and $1-F_A$ are fuzzy closed filters of X.

Conversely, assume that T_A, I_A and $1-F_A$ are fuzzy closed filters of X. Then, by Definition 2.16,

$$T_A(x \to 1) \geq T_A(x),\ I_A(x \to 1) \geq I_A(x),$$
$$(1-F_A)(x \to 1) \geq (1-F_A)(x).$$

Thus,

$$F_A(x \to 1) = 1-(1-F_A)(x \to 1) \leq 1-(1-F_A)(x) = F_A(x).$$

Hence, applying Definition 3.2 we get that A is a neutrosophic closed filter A in X.

By Theorem 4.2, Theorem 4.3, Theorem 3.1 and Proposition 2.7 we can get the following results.

Theorem 4.4 Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is a neutrosophic regular filter in X if and only if A satisfies:

(i) T_A is a fuzzy regular filter of X;

(ii) I_A is a fuzzy regular filter of X;

(iii) $1-F_A$ is a fuzzy regular filter of X, where $(1-F_A)(x) = 1-F_A(x)$, $\forall x \in X$.

Theorem 4.5 Let X be a pseudo-BCI algebra, A be a neutrosophic set in X such that $T_A(x) \geq \alpha_0$, $I_A(x) \geq \beta_0$ and $F_A(x) \leq \gamma_0$, $\forall x \in X$, where $\alpha_0 \in Im(T_A)$, $\beta_0 \in Im(I_A)$ and $\gamma_0 \in Im(F_A)$. Then A is a neutrosophic closed filter in X if and only if (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)}$ is closed filter of X for all $\alpha \in Im(T_A)$, $\beta \in Im(I_A)$ and $\gamma \in Im(F_A)$.

Proof: Assume that A is neutrosophic closed filter in X. By Theorem 4.3 and Proposition 4.1, for any $\alpha \in Im(T_A)$, $\beta \in Im(I_A)$ and $\gamma \in Im(F_A)$, we have

$$(T_A)_\alpha = \{x \in X \mid T_A(x) \geq \alpha\},\ (I_A)_\beta = \{x \in X \mid I_A(x) \geq \beta\} \text{ and}$$
$$(1-F_A)_{1-\gamma} = \{x \in X \mid (1-F_A)(x) \geq 1-\gamma\} = \{x \in X \mid F_A(x) \leq \gamma\} \text{ are closed filters of } X.$$

Thus $(T_A)_\alpha \cap (I_A)_\beta \cap (1-F_A)_{1-\gamma}$ is a closed filters of X. Moreover, by Definition 2.8, it is easy to verify that (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)} = (T_A)_\alpha \cap (I_A)_\beta \cap (1-F_A)_{1-\gamma}$. Therefore, $A^{(\alpha, \beta, \gamma)}$ is closed filter of X for all $\alpha \in Im(T_A)$, $\beta \in Im(I_A)$ and $\gamma \in Im(F_A)$.

Conversely, assume that $A^{(\alpha, \beta, \gamma)}$ is closed filter of X for all $\alpha \in Im(T_A)$, $\beta \in Im(I_A)$ and $\gamma \in Im(F_A)$. Since $T_A(x) \geq \alpha_0$, $I_A(x) \geq \beta_0$ and $F_A(x) \leq \gamma_0$, $\forall x \in X$, then

$$(T_A)_\alpha = \{x \in X \mid T_A(x) \geq \alpha\} = (T_A)_\alpha \cap X \cap X$$
$$= (T_A)_\alpha \cap (I_A)_{\beta_0} \cap (1-F_A)_{1-\gamma_0} = A^{(\alpha, \beta_0, \gamma_0)};$$
$$(I_A)_\beta = \{x \in X \mid I_A(x) \geq \beta\} = X \cap (I_A)_\beta \cap X$$
$$= (T_A)_{\alpha_0} \cap (I_A)_\beta \cap (1-F_A)_{1-\gamma_0} = A^{(\alpha_0, \beta, \gamma_0)};$$
$$(1-F_A)_{1-\gamma} = \{x \in X \mid (1-F_A)(x) \geq 1-\gamma\}$$
$$= X \cap X \cap \{x \in X \mid F_A(x) \leq \gamma\}$$
$$= (T_A)_{\alpha_0} \cap (I_A)_{\beta_0} \cap \{x \in X \mid F_A(x) \leq \gamma\} = A^{(\alpha_0, \beta_0, \gamma)}.$$

Thus,

$$(T_A)_\alpha = \{x \in X \mid T_A(x) \geq \alpha\},\ (I_A)_\beta = \{x \in X \mid I_A(x) \geq \beta\} \text{ and}$$
$$(1-F_A)_{1-\gamma} = \{x \in X \mid (1-F_A)(x) \geq 1-\gamma\} = \{x \in X \mid F_A(x) \leq \gamma\} \text{ are closed filters of } X.$$

From this, applying Proposition 4.1, we know that T_A, I_A and $1-F_A$ are fuzzy closed filters of X. By Theorem 4.3 we get that A is neutrosophic closed filter in X.

Similarly, we can get

Lemma 4.2 Let X be a pseudo-BCI algebra, A be a neutrosophic set in X such that $T_A(x) \geq \alpha_0$, $I_A(x) \geq \beta_0$ and $F_A(x) \leq \gamma_0$, $\forall x \in X$, where $\alpha_0 \in Im(T_A)$, $\beta_0 \in Im(I_A)$ and $\gamma_0 \in Im(F_A)$. Then A is a (anti-grouped) neutrosophic filter in X if and only if (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)}$ is (anti-grouped) filter of X for all $\alpha \in Im(T_A)$, $\beta \in Im(I_A)$ and $\gamma \in Im(F_A)$.

Combining Theorem 4.5, Lemma 4.2 and Theorem 3.1 we can get the following theorem.

Theorem 4.6 Let X be a pseudo-BCI algebra, A be a neutrosophic set in X such that $T_A(x) \geq \alpha_0$, $I_A(x) \geq \beta_0$ and $F_A(x) \leq \gamma_0$, $\forall x \in X$, where $\alpha_0 \in Im(T_A)$, $\beta_0 \in Im(I_A)$ and $\gamma_0 \in Im(F_A)$. Then A is a neutrosophic regular filter in X if and only if (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)}$ is regular filter of X for all $\alpha \in Im(T_A)$, $\beta \in Im(I_A)$ and $\gamma \in Im(F_A)$.

Conclusion

The neutrosophic set theory is applied to many scientific fields, and also applied to algebraic structures. This paper applied neutrosophic set theory to pseudo-BCI algebras, and some new notions of neutrosophic regular filter, neutrosophic closed filter and fuzzy regular filter in pseudo-BCI algebras are introduced. In addition to studying the basic properties of these new concepts, this paper also considered the relationships between them, and obtained some necessary and sufficient conditions.

Acknowledgment

This work was supported by National Natural Science Foundation of China (Grant No. 61573240, 61473239).

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Neutrosophic Duplet Semi-Group and Cancellable Neutrosophic Triplet Groups

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Xiaohong Zhang, Florentin Smarandache, Xingliang Liang (2017). Neutrosophic Duplet Semi-Group and Cancellable Neutrosophic Triplet Groups. *Symmetry* 9, 275; DOI: 10.3390/sym9110275

Abstract: The notions of the neutrosophic triplet and neutrosophic duplet were introduced by Florentin Smarandache. From the existing research results, the neutrosophic triplets and neutrosophic duplets are completely different from the classical algebra structures. In this paper, we further study neutrosophic duplet sets, neutrosophic duplet semi-groups, and cancellable neutrosophic triplet groups. First, some new properties of neutrosophic duplet semi-groups are funded, and the following important result is proven: there is no finite neutrosophic duplet semi-group. Second, the new concepts of weak neutrosophic duplet, weak neutrosophic duplet set, and weak neutrosophic duplet semi-group are introduced, some examples are given by using the mathematical software MATLAB (MathWorks, Inc., Natick, MA, USA), and the characterizations of cancellable weak neutrosophic duplet semi-groups are established. Third, the cancellable neutrosophic triplet groups are investigated, and the following important result is proven: the concept of cancellable neutrosophic triplet group and group coincide. Finally, the neutrosophic triplets and weak neutrosophic duplets in BCI-algebras are discussed.

Keywords: neutrosophic duplet; neutrosophic triplet; weak neutrosophic duplet; semi-group; BCI-algebra

1. Introduction

Florentin Smarandache introduced the concept of a neutrosophic set from a philosophical point of view (see [1–3]). The neutrosophic set theory is applied to many scientific fields and also applied to algebraic structures (see [4–10]). Recently, Florentin Smarandache and Mumtaz Ali in [11], for the first time, introduced the notions of a neutrosophic triplet and neutrosophic triplet group. The neutrosophic triplet is agroup of three elements that satisfy certain properties with some binary operation; it is completely different from the classical group in the structural properties. In 2017, Florentin Smarandache wrote the monograph [12] that is present the latest developments in neutrodophic theories, including the neutrosophic triplet, neutrosophic triplet group, neutrosophic duplet, and neutrosophic duplet set.

In this paper, we focus on the neutrosophic duplet, neutrosophic duplet set, and neutrosophic duplet semi-group. We discuss some new properties of the neutrosophic duplet semi-group and investigate the idempotent element in the neutrosophic duplet semi-group. Moreover, we introduce some new concepts to generalize the notion of neutrosophic duplet sets and discuss weak neutrosophic duplets in BCI-algebras (for BCI-algebra and related generalized logical algebra systems, please see [13–26]).

2. Basic Concepts

2.1. Neutrosophic Triplet and Neutrosophic Duplet

Definition 1. *([11,12]) Let N be a set together with a binary operation *. Then, N is called a neutrosophic triplet set if for any a ∈ N, there exist a neutralof "a" called neut(a),different from the classical algebraic unitary element, and an opposite of "a" called anti(a), with neut(a) and anti(a) belonging to N, such that:*

$$a * neut(a) = neut(a) * a = a;$$

$$a * anti(a) = anti(a) * a = neut(a).$$

The elements a, $neut(a)$, and $anti(a)$ are collectively called as a neutrosophic triplet, and we denote it by $(a, neut(a), anti(a))$. By $neut(a)$, we mean neutral of a and, apparently, a is just the first coordinate of a neutrosophic triplet and nota neutrosophic triplet. For the same element "a" in N, there may be more neutrals to it $neut(a)$ and more opposites of it $anti(a)$.

Definition 2. *([11,12]) The element b in (N, *) is the second component, denoted as neut(·), of a neutrosophic triplet, if there exists other elements a and c in N such that a * b = b * a = a and a * c = c * a = b. The formed neutrosophic triplet is (a, b, c).*

Definition 3. *([11,12]) The element c in (N, *) is the third component, denoted as anti(·), of a neutrosophic triplet, if there exists other elements a and b in N such that a * b = b * a = a and a * c = c * a = b. The formed neutrosophic triplet is (a, b, c).*

Definition 4. *([11,12]) Let (N, *) be a neutrosophic triplet set. Then, N is called a neutrosophic triplet group, if the following conditions are satisfied:*

*(1) If (N, *) is well-defined, i.e., for any a, b ∈ N, onehas a * b ∈ N.*
*(2) If (N, *) is associative, i.e., (a * b) * c = a * (b * c) for all a, b, c ∈ N.*

The neutrosophic triplet group, in general, is not a group in the classical algebraic way.

Definition 5. *([11,12]) Let (N, *) be a neutrosophic triplet group. Then, N is called a commutative neutrosophic triplet group if for all a, b ∈ N, we have a * b = b * a.*

Definition 6. *([12]) Let U be a universe of discourse, and a set A ⊆ U, endowed with a well-defined law *.We say that ⟨a, neut(a)⟩, where a, neut(a) ∈ A, is a neutrosophic duplet in A if:*

*(1) neut(a) is different from the unit element of A with respect to the law * (if any);*
*(2) a * neut(a) = neut(a) * a = a;*
*(3) there is no anti(a) ∈ A such that a * anti(a) = anti(a) * a = neut(a).*

Remark 1. *In the above definition, we have A ⊆ U. When A = U, "neutrosophic duplet in A" is simplified as "neutrosophic duplet", without causing confusion.*

Definition 7. *([12]) A neutrosophic duplet set, (D, *), is a set D, endowed with a well-defined binary law *, such that ∀a ∈ D, ∃ a neutrosophic duplet⟨a, neut(a)⟩ such that neut(a) ∈ D. If associative law holds in neutrosophic duplet set (D, *), then call it neutrosophic duplet semi-group.*

Remark 2. *The above definition is different from the original definition of a neutrosophic duplet set in [12]. In fact, the meaning of Theorem IX.2.1 in [12] is not consistent with the original definition of a neutrosophic duplet set. The original definition is modified to ensure that Theorem IX.2.1 in [12] is still correct.*

Remark 3. *In order to include richer structure, the original concept of a neutrosophic triplet is generalized to neutrosophic extended triplet by Florentin Smarandache. For a neutrosophic extended triplet that is a neutrosophic triplet, the neutral of x (called "extended neutral") is allowed to also be equal to the classical* *algebraic unitary element (if any). Therefore, the restriction "different from the classical algebraic unitary element, if any" is released. As a consequence, the "extended opposite" of x is also allowed to be equal to the classical inverse element from a classical group. Thus, a neutrosophic extended triplet is an object of the form (x, neut(x), anti(x)), for $x \in N$, where $neut(x) \in N$ is the extended neutral of x, which can be equal or different from the classical algebraic unitary element, if any, such that: x * neut(x) = neut(x) * x = x, and $anti(x) \in N$ is the extended opposite of x, such that: x * anti(x) = anti(x) * x = neut(x). In this paper, "neutrosophic triplet" means "neutrosophic extended triplet", and "neutrosophic duplet" means "neutrosophic extended duplet".*

2.2. BCI-Algebras

Definition 8. *([15,22]) A BCI-algebra is an algebra (X; $\to$, 1) of type (2,0) in which the following axioms are satisfied:*

(i) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(ii) $x \to x = 1$,
(iii) $1 \to x = x$,
(iv) if $x \to y = y \to x = 1$, then $x = y$.

In any BCI-algebra $(X; \to, 1)$ one can define a relation $\leq$ by putting $x \leq y$ if and only if $x \to y = 1$, then $\leq$ is a partial order on X.

Definition 9. *([16,20]) Let (X; $\to$, 1) be a BCI-algebra. The set $\{x \mid x \leq 1\}$ is called the p-radical (or BCK-part) of X. A BCI-algebra X is called p-semisimple if its p-radical is equal to {1}.*

Definition 10. *([16,20]) A BCI-algebra (X; $\to$, 1) is called associative if*

$$(x \to y) \to z = x \to (y \to z), \forall x,y,z \in X.$$

Proposition 1. *([16]) Let (X; $\to$, 1) be a BCI-algebra. Then the following are equivalent:*

(i) X is associative;
(ii) $x \to 1 = x, \forall x \in X$;
(iii) $x \to y = y \to x, \forall x,y \in X$.

Proposition 2. *([16,24]) Let (X; +, $-$, 1) be anAbel group. Define (X; $\leq$, $\to$, 1), where*

$$x \to y = -x + y,\ x \leq y \text{ if and only if } -x + y = 1, \forall x,y \in X.$$

Then, $(X; \leq, \to, 1)$ is a BCI-algebra.

3. New Properties of Neutrosophic Duplet Semi-Group

For a neutrosophic duplet set $(D, *)$, if $a \in D$, then $neut(a)$ may not be unique. Thus, the symbolic $neut(a)$ sometimes means one and sometimes more than one, which is ambiguous. To this end, this paper introduces the following notations to distinguish:

$neut(a)$: denote any certain one of neutral of a;
$\{neut(a)\}$: denote the set of all neutral of a.

Remark 4. *In order not to cause confusion, we always assume that: for the same a, when multiple neut(a) are present in the same expression, they always are consistent. Of course, if they are neutral of different elements, they refer to different objects (for example, in general, neut(a) is different from neut(b)).*

Proposition 3. *Let (D, *) be a neutrosophic duplet semi-group with respect to * and* $a \in D$*. Then, for any* $x, y \in \{neut(a)\}$, $x * y \in \{neut(a)\}$*. That is,*

$$\{neut(a)\} * \{neut(a)\} \subseteq \{neut(a)\}.$$

Proof. For any $a \in D$, by Definition 7, we have

$$a * neut(a) = a,\ neut(a) * a = a.$$

Assume $x, y \in \{neut(a)\}$, then

$$a * x = x * a = a;\ a * y = y * a = a.$$

From this, using associative law, we can get

$$a * (x * y) = (x * y) * a = a.$$

It follows that $x * y$ is a neutral of a. That is, $x * y \in \{neut(a)\}$. This means that $\{neut(a)\} * \{neut(a)\} \subseteq \{neut(a)\}$. □

Remark 5. *If neut(a) is unique, then*

$$neut(a) * neut(a) = neut(a).$$

But, if $neut(a)$ is not unique, for example, assume $\{neut(a)\} = \{s, t\} \in D$, then $neut(a)$ denote any one of s, t. Thus $neut(a) * neut(a)$represents one of $s * s$, and $t * t$; and $\{neut(a)\} * \{neut(a)\} = \{s * s, s * t, t* s, t * t\}$. Proposition 3 means that $s * s, s * t, t * s, t * t \in \{neut(a)\} = \{s, t\}$, that is,

$$s * s = s, \text{ or } s * s = t;\ s * t = s, \text{ or } s * t = t.$$

$$t * s = s, \text{ or } t * s = t;\ t * t = s, \text{ or } t * t = t.$$

In this case, the equation $neut(a) * neut(a) = neut(a)$ may not hold.

Proposition 4. *Let (D, *) be a neutrosophic duplet semi-group with respect to * and let* $a, b, c \in D$*. Then*

(1) $neut(a) * b = neut(a) * c \Rightarrow a * b = a * c.$
(2) $b * neut(a) = c * neut(a) \Rightarrow b * a = c * a.$

Proof. (1) Assume $neut(a) * b = neut(a) * c$. Then

$$a * (neut(a) * b) = a * (neut(a) * c).$$

By associative law, we have

$$(a * neut(a)) * b = (a * neut(a)) * c.$$

Thus, $a * b = a * c$. That is, (1) holds.
Similarly, we can prove that (2) holds. □

Theorem 1. *Let (D, *) be a commutative neutrosophic duplet semi-group with respect to * and* $a, b \in D$*. Then*

$$neut(a) * neut(b) \in \{neut(a * b)\}.$$

Proof. For any $a, b \in D$, we have

$$a * neut(a) * neut(b) * b = (a * neut(a)) * (neut(b) * b) = a * b.$$

From this and applying the commutativity and associativity of operation * we get

$$(neut(a) * neut(b)) * (a * b) = (a * b) * (neut(a) * neut(b)) = a * b.$$

This means that$neut(a) * neut(b) \in \{neut(a * b)\}$. □

Theorem 2. *Let (D, *) be a neutrosophic duplet set with respect to *. Then there is no idempotent element in D, that is,*

$$\forall a \in D, a * a \neq a.$$

Proof. Assume that there is $a \in D$ such that $a * a = a$. Then $a \in \{neut(a)\}$, and $a \in \{anti(a)\}$, This is a contraction with Definition 6 (3). □

Since the classical algebraic unitary element is idempotent, we have

Corollary 1. *Let (D, *) be a neutrosophic duplet set with respect to *. Then there is no classical unitary element in D, that is, there is no $e \in D$ such that $\forall a \in D, a * e = e * a = a$.*

Theorem 3. *Let (D, *) be a neutrosophic duplet semi-group with respect to *. Then D is infinite. That is, there is no finite neutrosophic duplet semi-group.*

Proof. Assume that D is a finite neutrosophic duplet semi-group with respect to *. Then, for any $a \in D$,

$$a, a * a = a^2, a * a * a = a^3, \ldots, a^n, \ldots \in D.$$

Since D is finite, so there exists natural number m, k such that

$$a^m = a^{m+k}.$$

Case 1: if $k = m$, then $a^m = a^{2m}$, that is, $a^m = a^m * a^m$, a^m is an idempotent element in D, this is a contraction with Theorem 2.

Case 2: if $k > m$, then from $a^m = a^{m+k}$ we can get

$$a^k = a^m * a^{k-m} = a^{m+k} * a^{km} = a^{2k} = a^k * a^k.$$

This means that a^k is an idempotent element in D, this is a contraction with Theorem 2.

Case 3: if $k < m$, then from $a^m = a^{m+k}$ we can get

$$a^m = a^{m+k} = a^m * a^k = a^{m+k} * a^k = a^{m+2k};$$

$$a^m = a^{m+2k} = a^m * a^{2k} = a^{m+k} * a^{2k} = a^{m+3k};$$

$$\ldots \ldots$$

$$a^m = a^{m+mk}.$$

Since m and k are natural numbers, then $mk \geq m$. Therefore, from $a^m = a^{m+mk}$, applying Case 1 or Case 2, we know that there exists an idempotent element in D, this is a contraction with Theorem 2. □

Theorem 4. *Let (D, *) be a neutrosophic duplet semi-group with respect to *and $a \in D$. Then*

$$neut(neut(a)) \in \{neut(a)\}.$$

Proof. For any $a \in D$, by the definition of $neut(\cdot)$, we have

$$neut(a) * neut(neu(a)) = neut(a);$$

$$neut(neut(a)) * neut(a) = neut(a).$$

Then

$$a * (neut(a) * neut(neut(a))) = a * neut(a);$$

$$(neut(neut(a)) * neut(a)) * a = neut(a) * a.$$

wrong, because the asso

$$(b * a) *c = a * c = c, \text{but } b * (a * c) = b * c = b.$$

4. Weak Neutrosophic Duplet Set (and Semi-Group)

From Theorems 3 and 5, we can see that the structure of the neutrosophic duplet semi-group is very scarce. What are the reasons for that? The key reason is that under the original definition of neutrosophic duplet, the idempotent element is not allowed (since it has a corresponding opposite element). In fact, for any idempotent element a, we have $a \in \{neut(a)\}$ and $a \in \{anti(a)\}$, that is, (a, a, a) is a neutrosophic triplet. Therefore, in order for us to study it more widely, we slightly relaxed the condition that allowed such (a, a, a) to exist in a neutrosophic duplet set and introduced a new concept as follows.

Definition 11. *A weak neutrosophic duplet set, (D, *), is a set D, endowed with a well-defined binary law *, such that* $\forall a \in D$, *if a* $\{neut(a)\}$, *then* $\exists a$ *neutrosophic duplet* $\langle a, neut(a)\rangle$ *such that* $neut(a) \in D$. *If the associative law holds in weak neutrosophic duplet set (D, *), then call it a weak neutrosophic duplet semi-group.*

The situation is quite different from that of the neutrosophic duplet semi-group, as there are many finite weak neutrosophic duplet semi-groups. See the following examples.

Example 1. *Let D = {1, 2, 3}. The operation * on D is defined as Table 1. Then, (D, *) is a commutative neutrosophic duplet semi-group.*

Table 1. Weak neutrosophic duplet semi-group (1).

*	1	2	3
1	1	2	3
2	2	2	2
3	3	2	2

In fact, we can verify that (D, *) is a neutrosophic duplet semi-group by MATLAB programming, as shown in Figure 1.

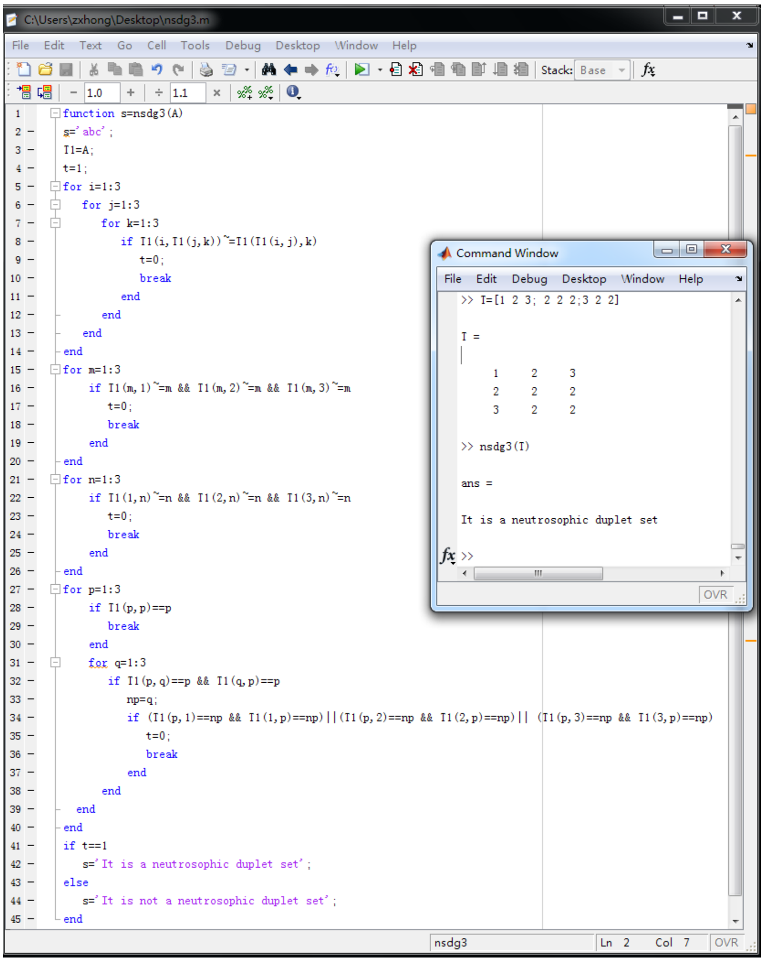

Figure 1. Verity weak neutrosophic duplet semi-group by MATLAB.

Example 2. *Let D = {1, 2, 3}. The operation * on D is defined as Table 2. Then, (D, *) is a non-commutative neutrosophic duplet semi-group.*

Table 2. Weak neutrosophic duplet semi-group (2).

*	1	2	3
1	1	1	1
2	1	2	3
3	3	3	3

In this example, "1", "2", and "3" are idempotent elements in D, and $\{neut(1)\} = \{1, 2\}$,$neut(2) = 2$, $\{neut(3)\} = \{2, 3\}$.

Example 3. *Let D = {1, 2, 3, 4}. The operation * on D is defined as Table 3. Then, (D, *) is a commutative neutrosophic duplet semi-group.*

Table 3. Weak neutrosophic duplet semi-group (3).

*	1	2	3	4
1	3	1	4	4
2	1	2	3	4
3	4	3	4	4
4	4	4	4	4

In this example, "2" and "4" are idempotent elements in D, and $neut(2) = 2$,$\{neut(4)\} = \{1, 2, 3, 4\}$. $neut(1) = 2$, $\{anti(1)\} = \emptyset$; $neut(3) = 2$, $\{anti(3)\} = \emptyset$.

Example 4. *Let D = {1, 2, 3, 4}. The operation * on D is defined as Table 4. Then, (D, *) is a non-commutative neutrosophic duplet semi-group.*

Table 4. Weak neutrosophic duplet semi-group (4).

*	1	2	3	4
1	2	2	3	1
2	2	2	3	2
3	2	2	3	3
4	1	2	3	4

In this example, "2", "3", and "4" are idempotent elements in D, and $neut(1) = 4$,$\{anti(1)\} = \emptyset$.

Now, we explain all of the neutrosophic duplet semi-groups with three elements. In total, we can obtain 50 neutrosophic duplet semi-groups with three elements, some of which may be isomorphic. They are funded by MATLAB programming, as shown in Figure 2.

Definition 12. *A weak neutrosophic duplet semi-group (D, *) is called to be cancellable, if it satisfies*

$$\forall a, b, c \in D, a * b = a * c \Rightarrow b = c;$$

$$\forall a, b, c \in D, b * a = c * a \Rightarrow b = c.$$

The weak neutrosophic duplet semi-groups in Examples 1–4 are not cancellable. We give a cancellable example as follows.

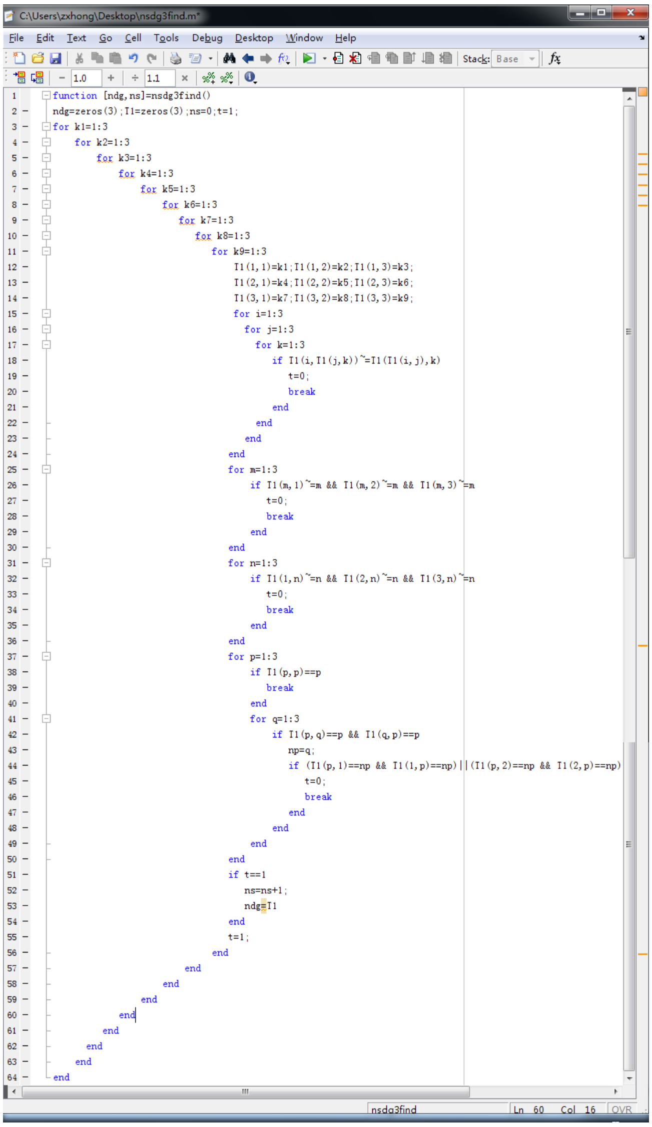

Figure 2. Find weak neutrosophic duplet semi-group by MATLAB

In this example, for any element a in D, and $neut(a) = 0$.

Theorem 6. *Let $(D, *)$ be a cancellable weak neutrosophic duplet semi-group with respect to $*$. Then*

(1) $\forall a \in D$, *$neut(a)$ is unique.*

(2) $\forall a \in D$, $neut(a) * neut(a) = neut(a)$.

(3) $\forall a \in D$, $neut(a) * neut(a) = neut(a * a)$.

(4) $\forall a, b \in D$, $neut(a) = neut(b)$.

Proof. (1) For any $a \in D$, we have

Case 1: if $a \in \{neut(a)\}$, then $a * a = a$. Thus

$$a * a = a = a * neut(a).$$

By Definition 12, we have $a = neut(a)$. This means that $\{neut(a)\} = \{a\}$, that is, $neut(a)$ is unique.
Case 2: if $a \quad \{neut(a)\}$, assume $x, y \in \{neut(a)\}$, then

$$a * x = a = a * y.$$

By Definition 12, we have$x = y$. This means that $|\{neut(a)\}| = 1$, that is, $neut(a)$ is unique.
(2) If $a \in \{neut(a)\}$, then $a * a = a$, by (1) we get $a = neut(a)$, so $neut(a) * neut(a) = neut(a)$.
If $a \quad \{neut(a)\}$, by the same way with Proposition 3, we can prove that

$$\{neut(a)\} * \{neut(a)\} \subseteq \{neut(a)\}.$$

Using (1) we have $neut(a) * neut(a) = neut(a)$.
(3) For any $a \in D$, since (by associative law)

$$(neut(a) * neut(a)) * (a * a) = a * a;$$

$$(a * a) * (neut(a) * neut(a)) = a * a.$$

This means that $neut(a) * neut(a) \in \{neut(a * a)\}$, but by (1) $|\{neut(a)\}| = 1$, thus

$$neut(a) * neut(a) = \mathrm{neut(a * a)}.$$

(4) For any $a, b \in D$, since (by associative law)

$$a * neut(a) * neut(b) *b = a * b.$$

From this, applying Definition 12,

$$neut(a) * neut(b) * b = b.$$

$$neut(a) * neut(b) * b = b = neut(b) * b.$$

Applying Definition 12 again,

$$neut(a) * neut(b) = neut(b).$$

Similarly, we can get

$$neut(a) * neut(b) = neut(a).$$

Hence, $neut(a) = neut(b)$. □

Theorem 7. *Let (D,) be a cancellable weak neutrosophic duplet semi-group with respect to *. If D is a finite set, then D is a single point set, that is,* $|D| = 1$.

Proof. By Theorem 6, we know that $\{neut(a) \mid a \in D\}$ is a single point set. Denote $neut(a) = e$ ($\forall a \in D$).
Assume that D is a finite set, if $|D| \neq 1$, then there exists $x \in D$ such that $x \neq e$. Denote $|D| = n$, $D = \{a_1, a_2, \ldots, a_n\}$. In the table of operation *, consider the line in which the x is located:

$$x * a_1, x * a_2, \ldots, x * a_n$$

Since D is cancellable, then $x * a_1, x * a_2, \ldots, x * a_n$are different from each other. Thus, $\exists a_i$ such that $x * a_i = e$. It follows that $\langle x, neut(x) = e\rangle$ is not a neutrosophic duplet. Applying Definition 11, $x \in \{neut(x)\} = \{e\}$. That is, $x \neq e$. This is a contraction with the hypothesis $x \neq e$. Hence $|D| = 1$.

Applying Theorems 2 and 6, we can get the following theorem. □

Theorem 8. *Let (D, *) be a neutrosophic duplet semi-group with respect to *. Then D is not cancellable. That is, there is no cancellable neutrosophic duplet semi-group.*

5. On Cancellable Neutrosophic Tripet Groups

Definition 13. *A neutrosophic triplet group (D, *) is called to be cancellable, if it satisfies*

$$\forall a, b, c \in D, a * b = a * c \Rightarrow b = c;$$

$$\forall a, b, c \in D, b * a = c * a \Rightarrow b = c.$$

Example 7. *Let D = {1, 2, 3, 4}. The operation * on D is defined as Table 5. Then, (D, *) is a cancellable neutrosophic triplet group.*

Table 5. Cancellable neutrosophic triplet group.

*	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

In this example, $neut(1) = neut(2) = neut(3) = neut(4) = 1$, and $anti(1) = 1$, $anti(2) = 2$, $anti(3) = 3$, $anti(4) = 4$.

Theorem 9. *Let (D, *) be a cancellable neutrosophic triplet group with respect to *. Then*

(1) $\forall a \in D$, *neut(a) is unique.*
(2) $\forall a \in D$, *anti(a) is unique.*
(3) $\forall a, b \in D$, *neut(a) = neut(b).*
(4) *(D, *) is a group, the unit is neut(a),* $\forall a \in D$.

Proof. (1) For any $a \in D$, assume $x, y \in \{neut(a)\}$, then

$$A * x = a = a * y.$$

By Definition 13, we have $x = y$. This means that $|\{neut(a)\}| = 1$, that is, $neut(a)$ is unique.

(2) For any $a \in D$, using (1), $neut(a)$ is unique. Assume $x, y \in \{anti(a)\}$, then

$$a * x = neut(a) = a * y.$$

By Definition 13, we have $x = y$. This means that $|\{anti(a)\}| = 1$, that is, $anti(a)$ is unique.

(3) For any $a, b \in D$, since (by associative law)

$$neut(a) * b = neut(a) * neut(b) * b.$$

From this, applying Definition 13,

$$neut(a) = neut(a) * neut(b).$$

On the other hand, since (by associative law)

$$a * neut(b) = a * (neut(a) * neut(b)).$$

From this, applying Definition 13 again,

$$neut(b) = neut(a) * neut(b).$$

Thus, $neut(a) = neut(b)$.
(4) It follows from (1)~(3). □

Since any group is a cancellable neutrosophic triplet group, by Theorem 9 (3), we have

Theorem 10. *The concepts of neutrosophic triplet group and group coincide.*

The following example shows that there exists a non-cancellable neutrosophic triplet group, in which $(\forall a \in D)$ $neut(a)$ is unique and $anti(a)$ is unique.

Example 8. *Let D = {1, 2, 3, 4}. The operation * on D is defined as Table 6. Then, (D, *) is a non-cancellable neutrosophic triplet group, but (∀a ∈ D) neut(a) is unique and anti(a) is unique.*

Table 6. Non-cancellable neutrosophic triplet group.

*	**1**	**2**	**3**	**4**
1	1	2	3	4
2	1	2	3	4
3	1	2	3	4
4	1	2	3	4

In this example, $neut(1) = anti(1) = 1$, $neut(2) = anti(2) = 2$, $neut(3) = anti(3) = 3$, $neut(4) = anti(4) = 4$.

Definition 14. *A neutrosophic triplet group (D, *) is called to be weak cancellable, if it satisfies*

$$\forall a, b, c \in D, (a * b = a * c \text{ and } b * a = c * a) \Rightarrow b = c.$$

Obviously, acancellable neutrosophic triplet group is weak cancellable, but a weak cancellable neutrosophic triplet group may not be cancellable. In fact, the $(D, *)$ in Example 8 is weak cancellable, but is not cancellable.

Theorem 11. *Let (D, *) be a weak cancellable neutrosophic triplet group with respect to *. Then*

(1) ∀a ∈ D, neut(a) is unique.
(2) ∀a ∈ D, anti(a)is unique.

Proof. (1) For any $a \in D$, assume $x, y \in \{neut(a)\}$, then

$$a * x = a = a * y.$$

$$x * a = a = y * a.$$

By Definition 14, we have $x = y$. This means that $|\{neut(a)\}| = 1$, that is, $neut(a)$ is unique.
(2) For any $a \in D$, using (1), $neut(a)$ is unique. Assume $x, y \in \{anti(a)\}$, then

$$a * x = neut(a) = a * y.$$

$$x * a = neut(a) = y * a.$$

By Definition 14, we have $x = y$. This means that $|\{anti(a)\}| = 1$, that is, $anti(a)$ is unique.

The following example shows that there exists a neutrosophic triplet group in which ($\forall a \in D$) $neut(a)$ is unique and $anti(a)$ is unique, but it is not weak cancellable. □

Example 9. *Let D = {1, 2, 3}. The operation * on D is defined as Table 7. Then, (D, *) is a neutrosophic triplet group, and (∀a ∈ D) neut(a) is unique and anti(a) is unique. However, it is not weak cancellable, since*

$$2 * 1 = 2 * 2, 1 * 2 = 2 * 2, 1 \neq 2.$$

Table 7. Not weak cancellable neutrosophic triplet group.

*	1	2	3
1	1	2	3
2	2	2	3
3	3	3	2

In this example, we have

$$neut(1) = anti(1) = 1, neut(2) = anti(2) = 2, neut(3) = anti(3) = 2.$$

The following example shows that there exists a commutative neutrosophic triplet group which ($\exists a \in D$) $anti(a)$ is not unique.

Example 10. *Consider (Z6, *), where * is classical multiplication. Then, (Z6, *) is a commutative neutrosophic triplet group, the binary operation * is defined in Table 8. For each a ∈ Z6, we have neut(a) in Z6. That is,*

$$neut([0]) = [0], neut([1]) = [1], neut([2]) = [4],$$

$$neut([3]) = [3], neut([4]) = [4], neut([5]) = [1];$$

$$\{anti([0])\} = \{[0], [1], [2], [3], [4], [5]\},$$

$$\{anti([1])\} = \{[1]\},$$

$$\{anti([2])\} = \{[2], [5]\},$$

$$\{anti([3])\} = \{[1], [3], [5]\},$$

$$\{anti([4])\} = \{[1], [4]\},$$

$$\{anti([5])\} = \{[5]\}.$$

Table 8. Cayley table of (Z_6, *).

*	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

6. Neutrosophic Triplets and Weak Neutrosophic Duplets in BCI-Algebras

Now, we discuss BCI-algebra ($X; \rightarrow, 1$).

Theorem 12. *Let ($X; \rightarrow, 1$) be a BCI-algebra. Then*

(1) $\forall x \in X$, *if* $\{neut(x)\} \neq \emptyset$ *and* $y \in \{neut(x)\}$, *then* $x \rightarrow 1 = x$, $y \rightarrow 1 = 1$.

(2) $\forall x \in X$, *if* $\{neut\ (x)\} \neq \varnothing$ *and* $\{anti\ (x)\} \neq \varnothing$, *then* $z \to 1 = x$ *for any* $z \in \{anti\ (x)\}$.

Proof. (1) Assume $y \in \{neut(x)\}$, then

$$X \to y = y \to x = x.$$

Using the properties of BCI-algebras, we have

$$x \to 1 = x \to (y \to y) = y \to (x \to y) = y \to x = x.$$

$$y \to 1 = y \to (x \to x) = x \to (y \to x) = x \to x = 1.$$

(2) Assume $z \in \{anti(x)\}$, then

$$Z \to x = x \to z = neut(x).$$

Using (1) and the properties of BCI-algebras, we have

$$1 = neut(x) \to 1 = (z \to x) \to 1 = (z \to 1) \to (x \to 1) = (z \to 1) \to x.$$

$$1 = neut(x) \to 1 = (x \to z) \to 1 = (x \to 1) \to (z \to 1) = x \to (z \to 1).$$

Hence, $z \to 1 = x$. □

Example 11. *Let* $D = \{a, b, c, 1\}$. *The operation* $\to$ *on* D *is defined as Table 9. Then,* $(D, \to)$ *is a BCI-algebra (it is a dual form of* $I_{4\text{-}2\text{-}2}$ *in [16]), and* $\langle c, 1, c\rangle$ *is a neutrosophic triplet in* $(D, \to)$.

Table 9. Neutrosophic triplet in BCI-algebra.

$\to$	a	b	c	1
a	1	c	c	1
b	c	1	1	c
c	b	a	1	c
1	a	b	c	1

Theorem 13. *Let* $(X; \to, 1)$ *be a BCI-algebra. Then* $(X, \to)$ *is a neutrosophic triplet group if and only if* $(X; \to, 1)$ *is an associative BCI-algebra.*

Proof. Suppose that $(X; \to)$ is a neutrosophic triplet group. Then $\forall x \in X$, $\{neut(x)\} \neq \varnothing$. By Theorem 12, $x \to 1 = x$. Using Proposition 1, $(X; \to, 1)$ is an associative BCI-algebra.

Conversely, suppose that $(X; \to, 1)$ is an associative BCI-algebra. Then $(X; \to, 1)$ is a group. Hence, $(X; \to)$ is a neutrosophic triplet group. □

Example 12. *Let* $D = \{a, b, c, 1\}$. *The operation* $\to$ *on* D *is defined as Table 10. Then,* $(D; \to, 1)$ *is a BCI-algebra (it is a dual form of* $I_{4\text{-}1\text{-}1}$ *in [16]), and* $(D, \to)$ *is a neutrosophic triplet group.*

Table 10. Neutrosophic triplet group and BCI-algebra.

$\to$	a	b	c	1
a	1	c	c	1
b	c	1	1	c
c	b	a	1	c
1	a	b	c	1

Theorem 14. *Let* $(X; \to, 1)$ *be a BCI-algebra. Then* $(X, \to)$ *is not a neutrosophic duplet semi-group.*

7. Conclusions

This paper is focused on the neutrosophic duplet semi-group. We proved some new properties of the neutrosophic duplet semi-group, and proved that there is no finite neutrosophic duplet semi-group. We introduced the new concept of weak neutrosophic duplet semi-groups and gave some examples by MATLAB. Moreover, we investigated cancellable neutrosophic triplet groups and proved that the concept of cancellable neutrosophic triplet group and group coincide. Finally, we discussed neutrosophic triplets and weak neutrosophic duplets in BCI-algebras.

Acknowledgments: This work was supported by National Natural Science Foundation of China (Grant Nos. 61573240, 61473239).

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Further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras

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G. Muhiuddin, Hashem Bordbar, Florentin Smarandache, Young Bae Jun (2018). Further results on (ε, ε,)-neutrosophic subalgebras and ideals in BCK/BCI-algebras. N*eutrosophic Sets and Systems* 20, 36-43.

Abstract: Characterizations of an $(\in, \in)$-neutrosophic ideal are considered. Any ideal in a BCK/BCI-algebra will be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal. The relation between $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic subalgebra in a BCK-algebra is discussed. Conditions for an $(\in, \in)$-neutrosophic subalgebra to be a $(\in, \in)$-neutrosophic ideal are provided. Using a collection of ideals in a BCK/BCI-algebra, an $(\in, \in)$-neutrosophic ideal is established. Equivalence relations on the family of all $(\in, \in)$-neutrosophic ideals are introduced, and related properties are investigated.

Keywords: $(\in, \in)$-neutrosophic subalgebra, $(\in, \in)$-neutrosophic ideal.

1 Introduction

Neutrosophic set (NS) developed by Smarandache [8, 9, 10] introduced neutrosophic set (NS) as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site

http://fs.gallup.unm.edu/neutrosophy.htm.

Jun et al. studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [5] and [7]).

In this paper, we characterize an $(\in, \in)$-neutrosophic ideal in a BCK/BCI-algebra. We show that any ideal in a BCK/BCI-algebra can be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal. We investigate the relation between $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic subalgebra in a BCK-algebra. We provide conditions for an $(\in, \in)$-neutrosophic subalgebra to be a $(\in, \in)$-neutrosophic ideal. Using a collection of ideals in a BCK/BCI-algebra, we establish an $(\in, \in)$-neutrosophic ideal. We discuss equivalence relations on the family of all $(\in, \in)$-neutrosophic ideals, and investigate related properties.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [2] and [3]) and was extensively investigated by several researchers.

By a BCI-*algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-*algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\left(\begin{array}{c} x \leq y \ \Rightarrow\ x * z \leq y * z \\ x \leq y \ \Rightarrow\ z * y \leq z * x \end{array}\right), \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.6}$$

We refer the reader to the books [4, 6] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [9]) is a structure of the form:

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T, A_I, A_F)$ for the neutrosophic set

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$\begin{aligned} T_\in(A_\sim; \alpha) &:= \{x \in X \mid A_T(x) \geq \alpha\}, \\ I_\in(A_\sim; \beta) &:= \{x \in X \mid A_I(x) \geq \beta\}, \\ F_\in(A_\sim; \gamma) &:= \{x \in X \mid A_F(x) \leq \gamma\}. \end{aligned}$$

We say $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are *neutrosophic $\in$-subsets*.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic subalgebra* of X (see [5]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \\ \qquad \Rightarrow\ x * y \in T_\in(A_\sim; \alpha_x \wedge \alpha_y), \\ x \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \\ \qquad \Rightarrow\ x * y \in I_\in(A_\sim; \beta_x \wedge \beta_y), \\ x \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \\ \qquad \Rightarrow\ x * y \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.7}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic ideal* of X (see [7]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x)\ \Rightarrow\ 0 \in T_\in(A_\sim; \alpha_x) \\ x \in I_\in(A_\sim; \beta_x)\ \Rightarrow\ 0 \in I_\in(A_\sim; \beta_x) \\ x \in F_\in(A_\sim; \gamma_x)\ \Rightarrow\ 0 \in F_\in(A_\sim; \gamma_x) \end{array}\right) \tag{2.8}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \\ \qquad \Rightarrow\ x \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \\ \qquad \Rightarrow\ x \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ x * y \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \\ \qquad \Rightarrow\ x \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.9}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

3 $(\in, \in)$-neutrosophic subalgebras and ideals

We first provide characterizations of an $(\in, \in)$-neutrosophic ideal.

Theorem 3.1. *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(\begin{array}{l} A_T(0) \geq A_T(x), \\ A_I(0) \geq A_I(x), \\ A_F(0) \leq A_F(x) \end{array}\right) \tag{3.1}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq A_T(x * y) \wedge A_T(y) \\ A_I(x) \geq A_I(x * y) \wedge A_I(y) \\ A_F(x) \leq A_F(x * y) \vee A_F(y) \end{array}\right) \tag{3.2}$$

Proof. Assume that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Suppose there exist $a, b, c \in X$ be such that $A_T(0) < A_T(a)$, $A_I(0) < A_I(b)$ and $A_F(0) > A_F(c)$. Then $a \in T_\in(A_\sim; A_T(a))$, $b \in I_\in(A_\sim; A_I(b))$ and $c \in F_\in(A_\sim; A_F(c))$. But

$$0 \notin T_\in(A_\sim; A_T(a)) \cap I_\in(A_\sim; A_I(b)) \cap F_\in(A_\sim; A_F(c)).$$

This is a contradiction, and thus $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose that $A_T(x) < A_T(x * y) \wedge A_T(y)$, $A_I(a) < A_I(a * b) \wedge A_I(b)$ and $A_F(c) > A_F(c * d) \vee A_F(d)$ for some $x, y, a, b, c, d \in X$. Taking $\alpha := A_T(x*y) \wedge A_T(y)$, $\beta := A_I(a*b) \wedge A_I(b)$ and $\gamma := A_F(c*d) \vee A_F(d)$ imply that $x*y \in T_\in(A_\sim; \alpha)$, $y \in T_\in(A_\sim; \alpha)$, $a * b \in I_\in(A_\sim; \beta)$, $b \in I_\in(A_\sim; \beta)$, $c * d \in F_\in(A_\sim; \gamma)$ and $d \in F_\in(A_\sim; \gamma)$. But $x \notin T_\in(A_\sim; \alpha)$, $a \notin I_\in(A_\sim; \beta)$ and $c \notin F_\in(A_\sim; \gamma)$. This is impossible, and so (3.2) is valid.

Conversely, suppose $A_\sim = (A_T, A_I, A_F)$ satisfies two conditions (3.1) and (3.2). For any $x, y, z \in X$, let $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ be such that $x \in T_\in(A_\sim; \alpha)$, $y \in I_\in(A_\sim; \beta)$ and

$z \in F_\in(A_\sim; \gamma)$. It follows from (3.1) that $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(y) \geq \beta$ and $A_F(0) \leq A_F(z) \leq \gamma$ and so that $0 \in T_\in(A_\sim; \alpha) \cap I_\in(A_\sim; \beta) \cap F_\in(A_\sim; \gamma)$. Let $a, b, c, d, x, y \in X$ be such that $a * b \in T_\in(A_\sim; \alpha_a)$, $b \in T_\in(A_\sim; \alpha_b)$, $c * d \in I_\in(A_\sim; \beta_c)$, $d \in I_\in(A_\sim; \beta_d)$, $x * y \in F_\in(A_\sim; \gamma_x)$, and $y \in F_\in(A_\sim; \gamma_y)$ for $\alpha_a, \alpha_b, \beta_c, \beta_d \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$. Using (3.2), we have

$$\begin{aligned} A_T(a) &\geq A_T(a * b) \wedge A_T(b) \geq \alpha_a \wedge \alpha_b \\ A_I(c) &\geq A_I(c * d) \wedge A_I(d) \geq \beta_c \wedge \beta_d \\ A_F(x) &\leq A_F(x * y) \vee A_F(y) \leq \gamma_x \vee \gamma_y. \end{aligned}$$

Hence $a \in T_\in(A_\sim; \alpha_a \wedge \alpha_b)$, $c \in I_\in(A_\sim; \beta_c \wedge \beta_d)$ and $x \in F_\in(A_\sim; \gamma_x \vee \gamma_y)$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

Theorem 3.2. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X. Then the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) *The nonempty neutrosophic $\in$-subsets $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X and assume that $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then there exist $x, y, z \in X$ such that $x \in T_\in(A_\sim; \alpha)$, $y \in I_\in(A_\sim; \beta)$ and $z \in F_\in(A_\sim; \gamma)$. It follows from (2.8) that

$$0 \in T_\in(A_\sim; \alpha) \cap I_\in(A_\sim; \beta) \cap F_\in(A_\sim; \gamma).$$

Let $x, y, a, b, u, v \in X$ be such that $x * y \in T_\in(A_\sim; \alpha)$, $y \in T_\in(A_\sim; \alpha)$, $a * b \in I_\in(A_\sim; \beta)$, $b \in I_\in(A_\sim; \beta)$, $u * v \in F_\in(A_\sim; \gamma)$ and $v \in F_\in(A_\sim; \gamma)$. Then

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y) \geq \alpha \wedge \alpha = \alpha \\ A_I(a) &\geq A_I(a * b) \wedge A_I(b) \geq \beta \wedge \beta = \beta \\ A_F(u) &\leq A_F(u * v) \vee A_F(v) \leq \gamma \vee \gamma = \gamma \end{aligned}$$

by (3.2), and so $x \in T_\in(A_\sim; \alpha)$, $a \in I_\in(A_\sim; \beta)$ and $u \in F_\in(A_\sim; \gamma)$. Hence the nonempty neutrosophic $\in$-subsets $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are nonempty and are ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Assume that $A_T(0) < A_T(x)$, $A_I(0) < A_I(y)$ and $A_F(0) > A_F(z)$ for some $x, y, z \in X$. Then $x \in T_\in(A_\sim; A_T(x))$, $y \in I_\in(A_\sim; A_I(y))$ and $z \in F_\in(A_\sim; A_F(z))$, that is, $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are nonempty. But $0 \notin T_\in(A_\sim; A_T(x)) \cap I_\in(A_\sim; A_I(y)) \cap F_\in(A_\sim; A_F(z))$, which is a contradiction since $T_\in(A_\sim; A_T(x))$, $I_\in(A_\sim; A_I(y))$ and $F_\in(A_\sim; A_F(z))$ are ideals of X. Hence $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose that

$$\begin{aligned} A_T(x) &< A_T(x * y) \wedge A_T(y), \\ A_I(a) &< A_I(a * b) \wedge A_I(b), \\ A_F(u) &> A_F(u * v) \vee A_F(v) \end{aligned}$$

for some $x, y, a, b, u, v \in X$. Taking $\alpha := A_T(x * y) \wedge A_T(y)$, $\beta := A_I(a * b) \wedge A_I(b)$ and $\gamma := A_F(u * v) \vee A_F(v)$ imply that $\alpha, \beta \in (0, 1]$, $\gamma \in [0, 1)$, $x * y \in T_\in(A_\sim; \alpha)$, $y \in T_\in(A_\sim; \alpha)$, $a * b \in I_\in(A_\sim; \beta)$, $b \in I_\in(A_\sim; \beta)$, $u * v \in F_\in(A_\sim; \gamma)$ and $v \in F_\in(A_\sim; \gamma)$. But $x \notin T_\in(A_\sim; \alpha)$, $a \notin I_\in(A_\sim; \beta)$ and $u \notin F_\in(A_\sim; \gamma)$. This is a contradiction since $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are ideals of X. Thus

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x * y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x * y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Proposition 3.3. *Every $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the following assertions.*

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \begin{cases} A_T(x) \geq A_T(y) \\ A_I(x) \geq A_I(y) \\ A_F(x) \leq A_F(y) \end{cases} \right), \tag{3.3}$$

$$(\forall x, y, z \in X) \left(x * y \leq z \Rightarrow \begin{cases} A_T(x) \geq A_T(y) \wedge A_T(z) \\ A_I(x) \geq A_I(y) \wedge A_I(z) \\ A_F(x) \leq A_F(y) \vee A_F(z) \end{cases} \right). \tag{3.4}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y), \\ A_I(x) &\geq A_I(x * y) \wedge A_I(y) = A_I(0) \wedge A_I(y) = A_I(y), \\ A_F(x) &\leq A_F(x * y) \vee A_F(y) = A_F(0) \vee A_F(y) = A_F(y) \end{aligned}$$

by Theorem 3.1. Hence (3.3) is valid. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and thus

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y) \\ &\geq (A_T((x * y) * z) \wedge A_T(z)) \wedge A_T(y) \\ &\geq (A_T(0) \wedge A_T(z)) \wedge A_T(y) \\ &\geq A_T(z) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x) &\geq A_I(x * y) \wedge A_I(y) \\ &\geq (A_I((x * y) * z) \wedge A_I(z)) \wedge A_I(y) \\ &\geq (A_I(0) \wedge A_I(z)) \wedge A_I(y) \\ &\geq A_I(z) \wedge A_I(y) \end{aligned}$$

and

$$\begin{aligned}A_F(x) &\leq A_F(x*y) \vee A_F(y) \\ &\leq (A_F((x*y)*z) \vee A_F(z)) \vee A_F(y) \\ &\leq (A_F(0) \vee A_F(z)) \vee A_F(y) \\ &\leq A_F(z) \vee A_F(y)\end{aligned}$$

by Theorem 3.1. □

Theorem 3.4. *Any ideal of a BCK/BCI-algebra X can be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal of X.*

Proof. Let I be an ideal of a BCK/BCI-algebra X and let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X given as follows:

$$A_T : X \to [0,1], \quad x \mapsto \begin{cases} \alpha & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_I : X \to [0,1], \quad x \mapsto \begin{cases} \beta & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_F : X \to [0,1], \quad x \mapsto \begin{cases} \gamma & \text{if } x \in I, \\ 1 & \text{otherwise} \end{cases}$$

where (α, β, γ) is a fixed ordered triple in $(0,1] \times (0,1] \times [0,1)$. Then $T_\in(A_\sim; \alpha) = I$, $I_\in(A_\sim; \beta) = I$ and $F_\in(A_\sim; \gamma) = I$. Obviously, $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Let $x, y \in X$. If $x*y \in I$ and $y \in I$, then $x \in I$. Hence

$$\begin{aligned}A_T(x*y) &= A_T(y) = A_T(x) = \alpha, \\ A_I(x*y) &= A_I(y) = A_I(x) = \beta, \\ A_F(x*y) &= A_F(y) = A_F(x) = \gamma,\end{aligned}$$

and so

$$\begin{aligned}A_T(x) &\geq A_T(x*y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y).\end{aligned}$$

If $x*y \notin I$ and $y \notin I$, then

$$\begin{aligned}A_T(x*y) &= A_T(y) = 0, \\ A_I(x*y) &= A_I(y) = 0, \\ A_F(x*y) &= A_F(y) = 1.\end{aligned}$$

Thus

$$\begin{aligned}A_T(x) &\geq A_T(x*y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y).\end{aligned}$$

If $x*y \in I$ and $y \notin I$, then

$$\begin{aligned}&A_T(x*y) = \alpha \text{ and } A_T(y) = 0, \\ &A_I(x*y) = \beta \text{ and } A_I(y) = 0, \\ &A_F(x*y) = \gamma \text{ and } A_F(y) = 1,\end{aligned}$$

It follows that

$$\begin{aligned}A_T(x) &\geq 0 = A_T(x*y) \wedge A_T(y), \\ A_I(x) &\geq 0 = A_I(x*y) \wedge A_I(y), \\ A_F(x) &\leq 1 = A_F(x*y) \vee A_F(y).\end{aligned}$$

Similarly, if $x*y \notin I$ and $y \in I$, then

$$\begin{aligned}A_T(x) &\geq A_T(x*y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y).\end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. This completes the proof. □

Lemma 3.5 ([5]). *A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x*y) \geq A_T(x) \wedge A_T(y) \\ A_I(x*y) \geq A_I(x) \wedge A_I(y) \\ A_F(x*y) \leq A_F(x) \vee A_F(y) \end{array} \right). \tag{3.5}$$

Theorem 3.6. *In a BCK-algebra, every $(\in, \in)$-neutrosophic ideal is an $(\in, \in)$-neutrosophic subalgebra.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Since $x*y \leq x$ for all $x, y \in X$, it follows from Proposition 3.3 and (3.2) that

$$\begin{aligned}&A_T(x*y) \geq A_T(x) \geq A_T(x*y) \wedge A_T(y) \geq A_T(x) \wedge A_T(y), \\ &A_I(x*y) \geq A_I(x) \geq A_I(x*y) \wedge A_I(y) \geq A_I(x) \wedge A_I(y), \\ &A_F(x*y) \leq A_F(x) \leq A_F(x*y) \vee A_F(y) \leq A_F(x) \vee A_F(y).\end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X by Lemma 3.5. □

The following example shows that the converse of Theorem 3.6 is not true in general.

Example 3.7. Consider a set $X = \{0,1,2,3\}$ with the binary operation $*$ which is given in Table 1.
Then $(X; *, 0)$ is a BCK-algebra (see [6]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2
It is routine to verify that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. We know that $I_\in(A_\sim; \beta)$ is an ideal of X for all $\beta \in (0,1]$. If $\alpha \in (0.3, 0.7]$, then $T_\in(A_\sim; \alpha) = \{0,1,3\}$ is not an ideal of X. Also, if $\gamma \in [0.2, 0.8)$, then $F_\in(A_\sim; \gamma) = \{0,1,3\}$ is not an ideal of X. Therefore $A_\sim = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.2.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 2: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.9	0.2
1	0.7	0.6	0.2
2	0.3	0.6	0.8
3	0.7	0.4	0.2

We give a condition for an $(\in, \in)$-neutrosophic subalgebra to be an $(\in, \in)$-neutrosophic ideal.

Theorem 3.8. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. If $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X that satisfies the condition (3.4), then it is an $(\in, \in)$-neutrosophic ideal of X.*

Proof. Taking $x = y$ in (3.5) and using (III) induce the condition (3.1). Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows from (3.4) that

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x * y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x * y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Theorem 3.9. *Let $\{D_k \mid k \in \Lambda^T \cup \Lambda^I \cup \Lambda^F\}$ be a collection of ideals of a BCK/BCI-algebra X, where Λ^T, Λ^I and Λ^F are nonempty subsets of $[0, 1]$, such that*

$$X = \{D_\alpha \mid \alpha \in \Lambda^T\} \cup \{D_\beta \mid \beta \in \Lambda^I\} \cup \{D_\gamma \mid \gamma \in \Lambda^F\}, \tag{3.6}$$

$$(\forall i, j \in \Lambda^T \cup \Lambda^I \cup \Lambda^F)\,(i > j \Leftrightarrow D_i \subset D_j). \tag{3.7}$$

Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined as follows:

$$\begin{aligned} A_T &: X \to [0,1],\ x \mapsto \bigvee\{\alpha \in \Lambda^T \mid x \in D_\alpha\}, \\ A_I &: X \to [0,1],\ x \mapsto \bigvee\{\beta \in \Lambda^I \mid x \in D_\beta\}, \\ A_F &: X \to [0,1],\ x \mapsto \bigwedge\{\gamma \in \Lambda^F \mid x \in D_\gamma\}. \end{aligned} \tag{3.8}$$

Then $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.

Proof. Let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $T_\in(A_\sim; \alpha) \neq \emptyset$, $I_\in(A_\sim; \beta) \neq \emptyset$ and $F_\in(A_\sim; \gamma) \neq \emptyset$. We consider the following two cases:

$$\alpha = \bigvee\{i \in \Lambda^T \mid i < \alpha\} \text{ and } \alpha \neq \bigvee\{i \in \Lambda^T \mid i < \alpha\}.$$

First case implies that

$$\begin{aligned} x \in T_\in(A_\sim; \alpha) &\Leftrightarrow x \in D_i \text{ for all } i < \alpha \\ &\Leftrightarrow x \in \cap\{D_i \mid i < \alpha\}. \end{aligned} \tag{3.9}$$

Hence $T_\in(A_\sim; \alpha) = \cap\{D_i \mid i < \alpha\}$, which is an ideal of X. For the second case, we claim that $T_\in(A_\sim; \alpha) = \cup\{D_i \mid i \geq \alpha\}$. If $x \in \cup\{D_i \mid i \geq \alpha\}$, then $x \in D_i$ for some $i \geq \alpha$. Thus $A_T(x) \geq i \geq \alpha$, and so $x \in T_\in(A_\sim; \alpha)$. If $x \notin \cup\{D_i \mid i \geq \alpha\}$, then $x \notin D_i$ for all $i \geq \alpha$. Since $\alpha \neq \bigvee\{i \in \Lambda^T \mid i < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Lambda^T = \emptyset$. Hence $x \notin D_i$ for all $i > \alpha - \varepsilon$, which means that if $x \in D_i$ then $i \leq \alpha - \varepsilon$. Thus $A_T(x) \leq \alpha - \varepsilon < \alpha$, and so $x \notin T_\in(A_\sim; \alpha)$. Therefore $T_\in(A_\sim; \alpha) = \cup\{D_i \mid i \geq \alpha\}$ which is an ideal of X since $\{D_k\}$ forms a chain. Similarly, we can verify that $I_\in(A_\sim; \beta)$ is an ideal of X. Finally, we consider the following two cases:

$$\gamma = \bigwedge\{j \in \Lambda^F \mid \gamma < j\} \text{ and } \gamma \neq \bigwedge\{j \in \Lambda^F \mid \gamma < j\}.$$

For the first case, we have

$$\begin{aligned} x \in F_\in(A_\sim; \gamma) &\Leftrightarrow x \in D_j \text{ for all } j > \gamma \\ &\Leftrightarrow x \in \cap\{D_j \mid j > \gamma\}, \end{aligned} \tag{3.10}$$

and thus $F_\in(A_\sim; \gamma) = \cap\{D_j \mid j > \gamma\}$ which is an ideal of X. The second case implies that $F_\in(A_\sim; \gamma) = \cup\{D_j \mid j \leq \gamma\}$. In fact, if $x \in \cup\{D_j \mid j \leq \gamma\}$, then $x \in D_j$ for some $j \leq \gamma$. Thus $A_F(x) \leq j \leq \gamma$, that is, $x \in F_\in(A_\sim; \gamma)$. Hence $\cup\{D_j \mid j \leq \gamma\} \subseteq F_\in(A_\sim; \gamma)$. Now if $x \notin \cup\{D_j \mid j \leq \gamma\}$, then $x \notin D_j$ for all $j \leq \gamma$. Since $\gamma \neq \bigwedge\{j \in \Lambda^F \mid \gamma < j\}$, there exists $\varepsilon > 0$ such that $(\gamma, \gamma+\varepsilon) \cap \Lambda^F$ is empty. Hence $x \notin D_j$ for all $j < \gamma+\varepsilon$, and so if $x \in D_j$, then $j \geq \gamma+\varepsilon$. Thus $A_F(x) \geq \gamma+\varepsilon > \gamma$, and hence $x \notin F_\in(A_\sim; \gamma)$. Thus $F_\in(A_\sim; \gamma) \subseteq \cup\{D_j \mid j \leq \gamma\}$, and therefore $F_\in(A_\sim; \gamma) = \cup\{D_j \mid j \leq \gamma\}$ which is an ideal of X. Consequently, $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.2. □

A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCK/BCI-algebras, then $f(0) = 0$. Given a homomorphism $f : X \to Y$ of BCK/BCI-algebras and a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in Y, we define a neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X, which is called the *induced neutrosophic set*, as follows:

$$\begin{aligned} A^f_T &: X \to [0,1],\ x \mapsto A_T(f(x)), \\ A^f_I &: X \to [0,1],\ x \mapsto A_I(f(x)), \\ A^f_F &: X \to [0,1],\ x \mapsto A_F(f(x)). \end{aligned}$$

Theorem 3.10. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of Y, then the induced neutrosophic set*

$A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X is an $(\in, \in)$-neutrosophic ideal of X.

Proof. For any $x \in X$, we have

$$\begin{aligned}
A^f_T(x) &= A_T(f(x)) \leq A_T(0) = A_T(f(0)) = A^f_T(0),\\
A^f_I(x) &= A_I(f(x)) \leq A_I(0) = A_I(f(0)) = A^f_I(0),\\
A^f_F(x) &= A_F(f(x)) \geq A_F(0) = A_F(f(0)) = A^f_F(0).
\end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned}
&A^f_T(x * y) \wedge A^f_T(y) = A_T(f(x * y)) \wedge A_T(f(y))\\
&= A_T(f(x) * f(y)) \wedge A_T(f(y))\\
&\leq A_T(f(x)) = A^f_T(x),
\end{aligned}$$

$$\begin{aligned}
&A^f_I(x * y) \wedge A^f_I(y) = A_I(f(x * y)) \wedge A_I(f(y))\\
&= A_I(f(x) * f(y)) \wedge A_I(f(y))\\
&\leq A_I(f(x)) = A^f_I(x),
\end{aligned}$$

and

$$\begin{aligned}
&A^f_F(x * y) \vee A^f_F(y) = A_F(f(x * y)) \vee A_F(f(y))\\
&= A_F(f(x) * f(y)) \vee A_F(f(y))\\
&\geq A_F(f(x)) = A^f_F(x).
\end{aligned}$$

Therefore $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Theorem 3.11. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in Y. If the induced neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X is an $(\in, \in)$-neutrosophic ideal of X, then $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of Y.*

Proof. Assume that the induced neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X is an $(\in, \in)$-neutrosophic ideal of X. For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$ since f is onto. Using (3.1), we have

$$\begin{aligned}
A_T(x) &= A_T(f(a)) = A^f_T(a) \leq A^f_T(0) = A_T(f(0)) = A_T(0),\\
A_I(x) &= A_I(f(a)) = A^f_I(a) \leq A^f_I(0) = A_I(f(0)) = A_I(0),\\
A_F(x) &= A_F(f(a)) = A^f_F(a) \geq A^f_F(0) = A_F(f(0)) = A_F(0).
\end{aligned}$$

Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. It follows from (3.2) that

$$\begin{aligned}
A_T(x) &= A_T(f(a)) = A^f_T(a)\\
&\geq A^f_T(a * b) \wedge A^f_T(b)\\
&= A_T(f(a * b)) \wedge A_T(f(b))\\
&= A_T(f(a) * f(b)) \wedge A_T(f(b))\\
&= A_T(x * y) \wedge A_T(y),
\end{aligned}$$

$$\begin{aligned}
A_I(x) &= A_I(f(a)) = A^f_I(a)\\
&\geq A^f_I(a * b) \wedge A^f_I(b)\\
&= A_I(f(a * b)) \wedge A_I(f(b))\\
&= A_I(f(a) * f(b)) \wedge A_I(f(b))\\
&= A_I(x * y) \wedge A_I(y),
\end{aligned}$$

and

$$\begin{aligned}
A_F(x) &= A_F(f(a)) = A^f_F(a)\\
&\leq A^f_F(a * b) \vee A^f_F(b)\\
&= A_F(f(a * b)) \vee A_F(f(b))\\
&= A_F(f(a) * f(b)) \vee A_F(f(b))\\
&= A_F(x * y) \vee A_F(y).
\end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of Y by Theorem 3.1. □

Let $\mathcal{N}_{(\in,\in)}(X)$ be the collection of all $(\in, \in)$-neutrosophic ideals of X and let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Define binary relations $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$ on $\mathcal{N}_{(\in,\in)}(X)$ as follows:

$$\begin{aligned}
A_T \mathcal{R}^\alpha_T B_T &\Leftrightarrow T_\in(A_\sim; \alpha) = T_\in(B_\sim; \alpha)\\
A_I \mathcal{R}^\beta_I B_I &\Leftrightarrow I_\in(A_\sim; \beta) = I_\in(B_\sim; \beta)\\
A_F \mathcal{R}^\gamma_F B_F &\Leftrightarrow F_\in(A_\sim; \gamma) = F_\in(B_\sim; \gamma)
\end{aligned} \tag{3.11}$$

for all $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ in $\mathcal{N}_{(\in,\in)}(X)$.

Clearly $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$ are equivalence relations on $\mathcal{N}_{(\in,\in)}(X)$. For any $A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X)$, let $[A_\sim]_T$ (resp., $[A_\sim]_I$ and $[A_\sim]_F$) denote the equivalence class of $A_\sim = (A_T, A_I, A_F)$ in $\mathcal{N}_{(\in,\in)}(X)$ under $\mathcal{R}^\alpha_T$ (resp., $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$). Denote by $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\alpha_T$, $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\beta_I$ and $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\gamma_F$ the collection of all equivalence classes under $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$, respectively, that is,

$$\begin{aligned}
\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\alpha_T &= \{[A_\sim]_T \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X),\\
\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\beta_I &= \{[A_\sim]_I \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X),\\
\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\gamma_F &= \{[A_\sim]_F \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X).
\end{aligned}$$

Now let $\mathcal{I}(X)$ denote the family of all ideals of X. Define maps f_α, g_β and h_γ from $\mathcal{N}_{(\in,\in)}(X)$ to $\mathcal{I}(X) \cup \{\emptyset\}$ by
$f_\alpha(A_\sim) = T_\in(A_\sim; \alpha)$, $g_\beta(A_\sim) = I_\in(A_\sim; \beta)$ and
$h_\gamma(A_\sim) = F_\in(A_\sim; \gamma)$,
respectively, for all $A_\sim = (A_T, A_I, A_F)$ in $\mathcal{N}_{(\in,\in)}(X)$. Then f_α, g_β and h_γ are clearly well-defined.

Theorem 3.12. *For any $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, the maps f_α, g_β and h_γ are surjective from $\mathcal{N}_{(\in,\in)}(X)$ to $\mathcal{I}(X) \cup \{\emptyset\}$.*

Proof. Let $0_\sim := (0_T, 0_I, 1_F)$ be a neutrosophic set in X where 0_T, 0_I and 1_F are fuzzy sets in X defined by $0_T(x) = 0$, $0_I(x) = 0$ and $1_F(x) = 1$ for all $x \in X$. Obviously, $0_\sim := (0_T, 0_I, 1_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Also, $f_\alpha(0_\sim) = T_\in(0_\sim; \alpha) = \emptyset$, $g_\beta(0_\sim) = I_\in(0_\sim; \beta) = \emptyset$

and $h_\gamma(0_\sim) = F_\in(0_\sim;\gamma) = \emptyset$. For any ideal I of X, let $A_\sim = (A_T, A_I, A_F)$ be the $(\in, \in)$-neutrosophic ideal of X in the proof of Theorem 3.4. Then $f_\alpha(A_\sim) = T_\in(A_\sim;\alpha) = I$, $g_\beta(A_\sim) = I_\in(A_\sim;\beta) = I$ and $h_\gamma(A_\sim) = F_\in(A_\sim;\gamma) = I$. Therefore f_α, g_β and h_γ are surjective. □

Theorem 3.13. *The quotient sets $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_T^\alpha$, $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_I^\beta$ and $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_F^\gamma$ are equivalent to $\mathcal{I}(X)\cup\{\emptyset\}$ for any $\alpha,\beta\in(0,1]$ and $\gamma\in[0,1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X)$. For any $\alpha,\beta\in(0,1]$ and $\gamma\in[0,1)$, define

$$\begin{aligned} &f_\alpha^* : \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_T^\alpha \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_T \mapsto f_\alpha(A_\sim),\\ &g_\beta^* : \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_I^\beta \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_I \mapsto g_\beta(A_\sim),\\ &h_\gamma^* : \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_F^\gamma \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_F \mapsto h_\gamma(A_\sim).\end{aligned}$$

Assume that $f_\alpha(A_\sim) = f_\alpha(B_\sim)$, $g_\beta(A_\sim) = g_\beta(B_\sim)$ and $h_\gamma(A_\sim) = h_\gamma(B_\sim)$ for $B_\sim = (B_T, B_I, B_F) \in \mathcal{N}_{(\in,\in)}(X)$. Then $T_\in(A_\sim;\alpha) = T_\in(B_\sim;\alpha)$, $I_\in(A_\sim;\beta) = I_\in(B_\sim;\beta)$ and $F_\in(A_\sim;\gamma) = F_\in(B_\sim;\gamma)$ which imply that $A_T\mathcal{R}_T^\alpha B_T$, $A_I\mathcal{R}_I^\beta B_I$ and $A_F\mathcal{R}_F^\gamma B_F$. Hence $[A_\sim]_T = [B_\sim]_T$, $[A_\sim]_I = [B_\sim]_I$ and $[A_\sim]_F = [B_\sim]_F$. Therefore f_α^*, g_β^* and h_γ^* are injective. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then $f_\alpha^*([0_\sim]_T) = f_\alpha(0_\sim) = T_\in(0_\sim;\alpha) = \emptyset$, $g_\beta^*([0_\sim]_I) = g_\beta(0_\sim) = I_\in(0_\sim;\beta) = \emptyset$, and $h_\gamma^*([0_\sim]_F) = h_\gamma(0_\sim) = F_\in(0_\sim;\gamma) = \emptyset$. For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then $f_\alpha^*([A_\sim]_T) = f_\alpha(A_\sim) = T_\in(A_\sim;\alpha) = I$, $g_\beta^*([A_\sim]_I) = g_\beta(A_\sim) = I_\in(A_\sim;\beta) = I$, and $h_\gamma^*([A_\sim]_F) = h_\gamma(A_\sim) = F_\in(A_\sim;\gamma) = I$. Hence f_α^*, g_β^* and h_γ^* are surjective, and the proof is over. □

For any $\alpha,\beta\in[0,1]$, we define another relations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ on $\mathcal{N}_{(\in,\in)}(X)$ as follows:

$$\begin{aligned} (A_\sim, B_\sim)\in\mathcal{R}_\alpha \ &\Leftrightarrow\ T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha)\\ &\qquad = T_\in(B_\sim;\alpha)\cap F_\in(B_\sim;\alpha),\\ (A_\sim, B_\sim)\in\mathcal{R}_\beta \ &\Leftrightarrow\ I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta)\\ &\qquad = I_\in(B_\sim;\beta)\cap F_\in(B_\sim;\beta)\end{aligned} \tag{3.12}$$

for all $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ in $\mathcal{N}_{(\in,\in)}(X)$. Then the relations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are also equivalence relations on $\mathcal{N}_{(\in,\in)}(X)$.

Theorem 3.14. *Given $\alpha,\beta\in(0,1)$, we define two maps*

$$\begin{aligned} \varphi_\alpha &: \mathcal{N}_{(\in,\in)}(X)\to\mathcal{I}(X)\cup\{\emptyset\},\\ &\qquad A_\sim \mapsto f_\alpha(A_\sim)\cap h_\alpha(A_\sim),\\ \varphi_\beta &: \mathcal{N}_{(\in,\in)}(X)\to\mathcal{I}(X)\cup\{\emptyset\},\\ &\qquad A_\sim \mapsto g_\beta(A_\sim)\cap h_\beta(A_\sim)\end{aligned} \tag{3.13}$$

for each $A_\sim = (A_T, A_I, A_F)\in\mathcal{N}_{(\in,\in)}(X)$. Then φ_α and φ_β are surjective.

Proof. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then

$$\begin{aligned}\varphi_\alpha(0_\sim) &= f_\alpha(0_\sim)\cap h_\alpha(0_\sim) = T_\in(0_\sim;\alpha)\cap F_\in(0_\sim;\alpha) = \emptyset,\\ \varphi_\beta(0_\sim) &= g_\beta(0_\sim)\cap h_\beta(0_\sim) = I_\in(0_\sim;\beta)\cap F_\in(0_\sim;\beta) = \emptyset.\end{aligned}$$

For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then

$$\begin{aligned}\varphi_\alpha(A_\sim) &= f_\alpha(A_\sim)\cap h_\alpha(A_\sim)\\ &= T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha) = I\end{aligned}$$

and

$$\begin{aligned}\varphi_\beta(A_\sim) &= g_\beta(A_\sim)\cap h_\beta(A_\sim)\\ &= I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta) = I.\end{aligned}$$

Therefore φ_α and φ_β are surjective. □

Theorem 3.15. *For any $\alpha,\beta\in(0,1)$, the quotient sets $\mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha$ and $\mathcal{N}_{(\in,\in)}(X)/\varphi_\beta$ are equivalent to $\mathcal{I}(X)\cup\{\emptyset\}$.*

Proof. Given $\alpha,\beta\in(0,1)$, define two maps φ_α^* and φ_β^* as follows:

$$\begin{aligned}&\varphi_\alpha^* : \mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_{\mathcal{R}_\alpha}\mapsto\varphi_\alpha(A_\sim),\\ &\varphi_\beta^* : \mathcal{N}_{(\in,\in)}(X)/\varphi_\beta \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_{\mathcal{R}_\beta}\mapsto\varphi_\beta(A_\sim).\end{aligned}$$

If $\varphi_\alpha^*([A_\sim]_{\mathcal{R}_\alpha}) = \varphi_\alpha^*([B_\sim]_{\mathcal{R}_\alpha})$ and $\varphi_\beta^*\left([A_\sim]_{\mathcal{R}_\beta}\right) = \varphi_\beta^*\left([B_\sim]_{\mathcal{R}_\beta}\right)$ for all $[A_\sim]_{\mathcal{R}_\alpha}, [B_\sim]_{\mathcal{R}_\alpha}\in\mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha$ and $[A_\sim]_{\mathcal{R}_\beta}, [B_\sim]_{\mathcal{R}_\beta}\in\mathcal{N}_{(\in,\in)}(X)/\varphi_\beta$, then

$$f_\alpha(A_\sim)\cap h_\alpha(A_\sim) = f_\alpha(B_\sim)\cap h_\alpha(B_\sim)$$

and

$$g_\beta(A_\sim)\cap h_\beta(A_\sim) = g_\beta(B_\sim)\cap h_\beta(B_\sim),$$

that is,

$$T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha) = T_\in(B_\sim;\alpha)\cap F_\in(B_\sim;\alpha)$$

and

$$I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta) = I_\in(B_\sim;\beta)\cap F_\in(B_\sim;\beta).$$

Hence $(A_\sim, B_\sim)\in\mathcal{R}_\alpha$ and $(A_\sim, B_\sim)\in\mathcal{R}_\beta$. It follows that $[A_\sim]_{\mathcal{R}_\alpha} = [B_\sim]_{\mathcal{R}_\alpha}$ and $[A_\sim]_{\mathcal{R}_\beta} = [B_\sim]_{\mathcal{R}_\beta}$. Thus φ_α^* and φ_β^* are injective. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then

$$\begin{aligned}\varphi_\alpha^*\left([0_\sim]_{\mathcal{R}_\alpha}\right) &= \varphi_\alpha(0_\sim) = f_\alpha(0_\sim)\cap h_\alpha(0_\sim)\\ &= T_\in(0_\sim;\alpha)\cap F_\in(0_\sim;\alpha) = \emptyset\end{aligned}$$

and

$$\varphi_\beta^*\left([0_\sim]_{\mathcal{R}_\beta}\right) = \varphi_\beta(0_\sim) = g_\beta(0_\sim) \cap h_\beta(0_\sim)$$
$$= I_\in(0_\sim;\beta) \cap F_\in(0_\sim;\beta) = \emptyset.$$

For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then

$$\varphi_\alpha^*\left([A_\sim]_{\mathcal{R}_\alpha}\right) = \varphi_\alpha(A_\sim) = f_\alpha(A_\sim) \cap h_\alpha(A_\sim)$$
$$= T_\in(A_\sim;\alpha) \cap F_\in(A_\sim;\alpha) = I$$

and

$$\varphi_\beta^*\left([A_\sim]_{\mathcal{R}_\beta}\right) = \varphi_\beta(A_\sim) = g_\beta(A_\sim) \cap h_\beta(A_\sim)$$
$$= I_\in(A_\sim;\beta) \cap F_\in(A_\sim;\beta) = I.$$

Therefore φ_α^* and φ_β^* are surjective. This completes the proof. □

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Algebraic Structure of Neutrosophic Duplets in Neutrosophic Rings $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$

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W.B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2018). Algebraic Structure of Neutrosophic Duplets in Neutrosophic Rings <Z U I>, <Q U I> and <R U I>. *Neutrosophic Sets and Systems* 23, 85-95

Abstract: The concept of neutrosophy and indeterminacy I was introduced by Smarandache, to deal with neutralies. Since then the notions of neutrosophic rings, neutrosophic semigroups and other algebraic structures have been developed. Neutrosophic duplets and their properties were introduced by Florentin and other researchers have pursued this study.In this paper authors determine the neutrosophic duplets in neutrosophic rings of characteristic zero. The neutrosophic duplets of $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$; the neutrosophic ring of integers, neutrosophic ring of rationals and neutrosophic ring of reals respectively have been analysed. It is proved the collection of neutrosophic duplets happens to be infinite in number in these neutrosophic rings. Further the collection enjoys a nice algebraic structure like a neutrosophic subring, in case of the duplets collection $\{a-aI|a \in Z\}$ for which $1-I$ acts as the neutral. For the other type of neutrosophic duplet pairs $\{a - aI, 1 - dI\}$ where $a \in R^+$ and $d \in R$, this collection under component wise multiplication forms a neutrosophic semigroup. Several other interesting algebraic properties enjoyed by them are obtained in this paper.

Keywords: Neutrosophic ring; neutrosophic duplet; neutrosophic duplet pairs; neutrosophic semigroup; neutrosophic subring

1 Introduction

The concept of indeterminacy in the real world data was introduced by Florentin Smarandache [1, 2] as Neutrosophy. Existing neutralities and indeterminacies are dealt by the neutrosophic theory and are applied to real world and engineering problems [3, 4, 5]. Neutrosophic algebraic structures were introduced and studied by [6]. Since then several researchers have been pursuing their research in this direction [7, 8, 9, 10, 11, 12]. Neutrosophic rings [9] and other neutrosophic algebraic structures are elaborately studied in [6, 7, 8, 10].

Related theories of neutrosophic triplet, neutrosophic duplet, and duplet set was studied by Smarandache [13]. Neutrosophic duplets and triplets have interested many and they have studied [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Neutrosophic duplet semigroup [18], the neutrosophic triplet group [12], classical group of neutrosophic triplet groups[22] and neutrosophic duplets of $\{Z_{pn}, \times\}$ and $\{Z_{pq}, \times\}$ [23] have been recently studied.

Here we mainly introduce the concept of neutrosophic duplets in case of of neutrosophic rings of characteristic zero and study only the algebraic properties enjoyed by neutrosophic duplets, neutrals and neutrosophic duplet pairs.

In this paper we investigate the neutrosophic duplets of the neutrosophic rings $\langle Z \cup I \rangle, \langle Q \cup I \rangle$ and $\langle R \cup I \rangle$. We prove the duplets for a fixed neutral happens to be an infinite collection and enjoys a nice algebraic structure. In fact the collection of neutrals for fixed duplet happens to be infinite in number and they too enjoy a nice algebraic structure.

This paper is organised into five sections , section one is introductory in nature. Important results in this paper are given in section two of this paper. Neutrosophic duplets of the neutrosophic ring $\langle Z \cup I \rangle$, and its properties are analysed in section three of this paper. In the forth section neutrosophic duplets of the rings $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$; are defined and developed and several theorems are proved. In the final section discussions, conclusions and future research that can be carried out is described.

2 Results

The basic definition of neutrosophic duplet is recalled from [12]. We just give the notations and describe the neutrosophic rings and neutrosophic semigroups [9].

Notation: $\langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}$ is the collection of neutrosophic integers which is a neutrosophic ring of integers. $\langle Q \cup I \rangle = \{a + bI | a, b \in Q, I^2 = I\}$ is the collection of neutrosophic rationals and $\langle R \cup I \rangle = \{a + bI | a, b \in R, I^2 = I\}$ is the collection of neutrosophic reals which are neutrosophic ring of rationals and reals respectively.

Let S be any ring which is commutative and has a unit element 1. Then $\langle S \cup I \rangle = \{a + bI | a, b \in S, I^2 = I, +, \times\}$ be the neutrosophic ring. For more refer [9].

Consider U to be the universe of discourse, and D a set in U, which has a well-defined law $\#$.

Definition 2.1. Consider $\langle a, neut(a) \rangle$, where a, and $neut(a)$ belong to D. It is said to be a neutrosophic duplet if it satisfies the following conditions:

1. $neut(a)$ is not same as the unitary element of D in relation with the law $\#$ (if any);
2. $a\#\ neut(a)$ = neut(a) $\#$ a = a;
3. $anti(a) \notin$ D for which a $\#$ anti(a) = anti(a) $\#$ a = neut(a).

The results proved in this paper are

1. All elements of the form $a - aI$ and $aI - a$ with $1 - I$ as the neutral forms a neutrosophic duplet, $a \in Z^+ \setminus \{0\}$.
2. In fact $B = \{a - aI / a \in Z \setminus \{0\}\} \cup \{0\}$, forms a neutrosophic subring of S.
3. Let $S = \{\langle Q \cup I \rangle, +, \times\}$ be the neutrosophic ring. For every nI with $n \in Q \setminus \{0\}$ we have $a + bI \in \langle Q \cup I \rangle$ with $a + b = 1; a, b \in Q \setminus \{0\}$. such that $\{nI, a + bI\}$ is a neutrosophic duplet.
4. The idempotent $x = 1 - I$ acts as the neutral for infinite collection of elements $a - aI$ where $a \in Q$.

5. For every $a - aI \in S$ where $a \in Q, 1 - dI$ acts as neutrals for $d \in Q$.

6. The ordered pair of neutrosophic duplets $B = \{(nI, m - (m-1)I); n \in R, m \in R \cup \{0\}\}$ forms a neutrosophic semigroup of $S = \langle R \cup I \rangle$ under component wise product.

7. The ordered pair of neutrosophic duplets $D = \{(a - aI, 1 - dI); a \in R^+; d \in R\}$ forms a neutrosophic semigroup under product taken component wise.

3 Neutrosophic duplets of $\langle Z \cup I \rangle$ and its properties

In this section we find the neutrosophic duplets in $\langle Z \cup I \rangle$. Infact we prove there are infinite number of neutrals for any relevant element in $\langle Z \cup I \rangle$. Several interesting results are proved.

First we illustrate some of the neutrosophic duplets in $\langle Z \cup I \rangle$.

Example 3.1. Let S = $\langle Z \cup I \rangle = \{a + bI | a, b \in I, I^2 = I\}$ be the neutrosophic ring. Consider any element $x = 9I \in \langle Z \cup I \rangle$; we see the element $16 - 15I \in \langle Z \cup I \rangle$ is such that $9I \times 16 - 15I = 144I - 135I = 9I = x$. Thus $16 - 15I$ acts as the neutral of $9I$ and $\{9I, 16 - 15I\}$ is a neutrosophic duplet.

Cconsider $15I = y \in \langle Z \cup I \rangle$; $15I \times 16 - 15I = 15I = y$. Thus $\{15I, 16 - 15I\}$ is again a neutrosophic duplet. Let $-9I = s \in \langle Z \cup I \rangle$; $-9I \times 16 - 15I = -144I + 135I = -9I = s$, so $\{-9I, 16 - 15I\}$ is a neutrosophic duplet. Thus $\{\pm 9I, 16 - 15I\}$ happens to be neutrosophic duplets.

Further $nI \in \langle Z \cup I \rangle$ is such that $nI \times 16 - 15I = 16nI - 15nI = nI$. Similarly $-nI \times 16 - 15I = -16nI + 15nI = -nI$. So $\{nI, 16 - 15I\}$ is a neutrosophic duplet for all $n \in Z \setminus \{0\}$. Another natural question which comes to one mind is will $16I - 15$ act as a neutral for nI; $n \in Z \setminus \{0\}$, the answer is yes for $nI \times (16I - 15) = 16nI - 15nI = nI$. Hence the claim.

We call $0I = 0$ as the trivial neutrosophic duplet as $(0, x)$ is a neutrosophic duplet for all $x \in \langle Z \cup I \rangle$.

In view of this example we prove the following theorem.

Theorem 3.2. *Let $S = \langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}$ be a neutrosophic ring. Every $\pm nI \in S; n \in Z \setminus \{0\}$ has infinite number of neutrals of the form*

- $mI - (m-1) = x$
- $m - (m-1)I = y$
- $(m-1) - mI = -x$
- $(m-1)I - mI = -y$

where $m \in Z^+ \setminus \{1, 0\}$.

Proof. Consider $nI \in \langle Z \cup I \rangle$ we see

$$nI \times x = nI[mI - (m-1)] = nnI - nmI + nI = nI.$$

Thus $\{nI, mI - (m-1)\}$ form an infinite collection of neutrosophic duplets for a fixed n and varying $m \in Z^+ \setminus \{0, 1\}$. Proof for other parts (ii), (iii) and (iv) follows by a similar argument. □

Thus in view of the above theorem we can say for any $nI; n \in Z \setminus \{0\}$, n is fixed; we have an infinite collection of neutrals paving way for an infinite collection of neutrosophic duplets contributed by elements $x, y, -x$ and $-y$ given in the theorem. On the other hand for any fixed x or y or $-x$ or $-y$ given in the theorem we have an infinite collection of elements of the form $nI; n \in Z \setminus \{0\}$ such that $\{n, x,$or y or $-x$ or $-y\}$ is a neutrosophic duplet.

Now our problem is to find does these neutrals collection $\{x, y, -x, -y\}$ in theorem satisfy any nice algebraic structure in $\langle Z \cup I\rangle$.

We first illustrate this using some examples before we propose and prove any theorem.

Example 3.3. Let $S = \langle Z \cup I\rangle = \{a + bI | a, b \in Z, I^2 = I\}$ be the ring. $\{S, \times\}$ is a commutative semigroup under product []. Consider the element $x = 5I - 4 \in \langle Z \cup I\rangle$. $5I - 4$ acts as neutral for all elements $nI \in \langle Z \cup I\rangle, n \in Z \setminus \{0\}$. Consider $x \times x = 5I - 4 \times 5I - 4 = 25I - 20I - 20I + 16 = -15I + 16 = x^2$. Now $-15I + 16 \times nI = -15nI + 16nI = nI$. Thus if $\{nI, x\}$ a neutrosophic duplet so is $\{nI, x^2\}$. Consider

$$x^3 = x^2 \times x = (-15I + 16) \times (5I - 4)$$

$$= -75I + 80I + 60I - 64 = 65I - 64 = x^3$$

$$nI \times x^3 = 65nI - 64nI = nI$$

So $\{nI, 65I - 64\} = \{nI, x^3\}$ is a neutrosophic duplet for all $n \in Z \setminus \{0\}$ Consider

$$x^4 = x^3 \times x = 65I - 64 \times 5I - 4$$

$$= 325I - 320I - 260I + 256 = -255I + 256 = x^4$$

Clearly

$$nI \times x^4 = nI \times (-255I + 25) = -255nI + 256nI = nI.$$

So $\{nI, x^4\}$ is a neutrosophic duplet. In fact one can prove for any $nI \in \langle Z \cup I\rangle; n \in Z \setminus \{0\}$ then $x = m - (m-1)I$ is the neutral of nI then $\{nI, x\}, \{nI, x^2\}, \{nI, x^3\}, \ldots, \{nI, x^r\}, \ldots, \{nI, x^t\}; t \in Z^+ \setminus \{0\}$ are all neutrosophic duplets for nI. Thus for any fixed nI there is an infinite collection of neutrals. We see if x is a neutral then the cyclic semigroup generated by x denoted by $\langle x\rangle = \{x, x^2, x^3, \ldots\}$ happens to be a collection of neutrals for $nI \in S$.

Now we proceed onto give examples of other forms of neutrosophic duplets using $\langle Z \cup I\rangle$.

Example 3.4. Let $S = \{\langle Z \cup I\rangle = \{a + bI | a, b \in Z, I^2 = I\}, +, \times\}$ be a neutrosophic ring. We see $x = 1 - I \in S$ such that

$$(1 - I)^2 = 1 - I \times 1 - I = 1 - 2I + I^2 (\because I^2 = I)$$

$$= 1 - I = x.$$

Thus x is an idempotent of S. We see $y = 5 - 5I$ such that

$$y \times x = (5 - 5I) \times (5 - 5I) = 5 - 5I - 5I + 5I = 5 - 5I = y$$

Thus $\{5 - 5I, 1 - I\}$ is a neutrosophic duplets and $1 - I$ is the neutral of $5 - 5I$.

$$y^2 = 5 - 5I \times 5 - 5I = 25 - 25I - 25I + 25I = 25 - 25I$$

We see $\{y^2, 1-I\}$ is again a neutrosophic duplet.

$$y^3 = y \times y^2 = 5 - 5I \times (25 - 25I) = 125 - 125I - 125I + 125I$$

$$= 125 - 125I = y^3$$

Once again $\{y^3, 1-I\}$ is a neutrosophic duplet. In fact we can say for the idempotent $1-I$ the cyclic semigroup $B = \{y, y^2, y^3, \dots\}$ is such that for every element in B, $1-I$ serves as the neutral.

In view of all these we prove the following theorem.

Theorem 3.5. *Let $S = \{\langle Z \cup I \rangle, +, \times\}$ be the neutrosophic ring.*

1. *$1 - I$ is an idempotent of S.*

2. *All elements of the form $a - aI$ and $aI - a$ with $1 - I$ as the neutral forms a neutrosophic duplet, $a \in Z^+ \setminus \{0\}$.*

3. *In fact $B = \{a - aI/a \in Z \setminus \{0\}\} \cup \{0\}$, forms a neutrosophic subring of S.*

Proof. 1. Let $x = 1 - I \in S$ to show x is an idempotent of S, we must show $x \times x = x$. We see $1 - I \times 1 - I = 1 - 2I + I^2$ as $I^2 = I$, we get $1 - I \times 1 - I = 1 - I$; hence the claim.

2. Let $a - aI \in S; a \in Z$. $1 - I$ is the neutral of $a - aI$ as $a - aI \times 1 - I = a - aI - aI + aI = a - aI$. Thus $\{a - aI, 1 - I\}$ is a neutrosophic duplet. On similar lines $aI - a$ will also yield a neutrosophic duplet with $1 - I$. Hence the result (ii).

3. Given $B = \{a - aI | a \in Z\}$. To prove B is a group under +. Let $x = a - aI$ and $y = b - bI \in B$; $x + y = a - aI + b - bI = (a + b) - (a + b)I$ as $a + b \in Z$; $a + b - (a + b)I \in B$. So B is closed under the operation +. When $a = 0$ we get $0 - 0I = \in B$ and $a - aI + 0 = a - aI$. 0 acts as the additive identity of B. For every $a - aI \in B$ we have

$$-(a - aI) = (-a) - (-a)I = -a + aI \in B$$

is such that $a - aI + (-a) + aI = 0$ so every $a - aI$ has an additive inverse. Now we show $\{B, \times\}$ is a semigroup under product $\times$.

$$(a - aI) \times (b - bI) = ab - abI - baI + abI = ab - abI \in B.$$

Thus B is a semigroup under product. Clearly $1 - I \in B$. Now we test the distributive law. let $x = a - aI, y = b - bI$ and $z = c - cI \in B$.

$$(a - aI) \times [b - bI + c - cI] = a - aI \times [(b + c) - (b + c)I$$

$$= a(b + c) - aI(b + c) - (b + c)aI + a(b + c)I = a(b + c) - aI(b + c) \in B$$

Thus $\{B, +, \times\}$ is a neutrosophic subring of S. Finally we prove $\langle Z \cup I \rangle$ has neutrosophic duplets of the form $\{a - aI, 1 + dI\}; d \in Z \setminus \{0\}$.

□

Theorem 3.6. *Let $S = \{\langle Z \cup I\rangle = \{a + bI | a, b \in Z, I^2 = I\}, +, \times\}$ be a neutrosophic ring $a + bI \in S$ contributes to a neutrosophic duplet if and only if $a = -b$.*

Proof. Let $a + bI \in S(a \neq 0, b \neq 0)$ be an element which contributes a neutrosophic duplet with $c + dI \in S$. If $\{a + bI, c + dI\}$ is a neutrosophic duplet then $(a + bI) \times (c + dI) = a + bI$, this implies

$$ac + (bd + ad + bc)I = a + bI.$$

This implies $ac = a$ and $bd + ad + bc = b$. $ac = a$ implies $a(c - 1) = 0$ since $a \neq 0$ we have $c = 1$. Now in $bd + ad + bc = b$ substitute $c = 1$; it becomes $bd + ad + b = b$ which implies $bd + ad = 0$ that is $(b + a)d = 0$; $d \neq 0$ for if $d = 0$ then $c + dI = 1$ acts as a neutral, for all $a + bI \in S$ which is a trivial neutrosophic duplet. Thus $d \neq 0$, which forces $a + b = 0$ or $a = -b$. hence $a + bI = a - aI$. Now we have to find d. We have $(a - aI)(1 + dI) = a - aI + adI - adI = a - aI$.

This is true for any $d \in Z \setminus \{0\}$. Proof of the converse is direct. □

Next we proceed on to study neutrosophic duplets of $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$

4 Neutrosophic Duplets of $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$

In this section we study the neutrosophic duplets of the neutrosophic rings $\langle Q \cup I\rangle = \{a + bI | a, b \in Q, I^2 = I\}$; where Q the field of rationals and $\langle R \cup I\rangle = \{a + bI | a, b, \in R, I^2 = I\}$; where R is the field of reals. We obtain several interesting results in this direction. It is important to note $\langle Z \cup I\rangle \subset \langle Q \cup I\rangle \subset \langle R \cup I\rangle$. Hence all neutrosophic duplets of $\langle Z \cup I\rangle$ will continue to be neutrosophic duplets of $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$. Our analysis pertains to the existence of other neutrosophic duplets as Z is only a ring where as Q and R are fields. We enumerate many interesting properties related to them.

Example 4.1. Let $S = \{\langle Q \cup I\rangle = \{a + bI | a, b \in Q, I^2 = I\}, +, \times\}$ be the neutrosophic ring of rationals. Consider for any $nI \in$S we have the neutral

$$x = \frac{-7I}{9} + \frac{16}{9} \in S,$$

such that

$$nI \times x = nI\left(\frac{-7I}{9} + \frac{16}{9}\right) = nI.$$

Thus for the element nI the neutral is

$$\frac{-7I}{9} + \frac{16}{9} \in S.$$

We make the following observation

$$\frac{-7}{9} + \frac{16}{9} = 1.$$

In fact all elements of the form $a + bI$ in $\langle Q \cup I\rangle$ with $a + b = 1$; $a, b \in Q \setminus \{0\}$ can act as neutrals for nI. Suppose

$$x = \frac{8I}{9} + \frac{1}{9} \in \langle Q \cup I\rangle$$

then for $nI = y$ we see

$$x \times y = nI \times \left(\frac{8I}{9} + \frac{1}{9}\right) = \frac{8In}{9} + \frac{nI}{9} = nI.$$

Take $x = -9I + 10$ we see

$$x \times y = -9I + 0 \times nI = -9In + 10nI = nI$$

and so on.

However we have proved in section 3 of this paper for any $nI \in \langle Z \cup I\rangle$ the collection of all elements $a + bI \in \langle Z \cup I\rangle$ with $a + b = 1; a, b \in Z \setminus \{0\}$ will act as neutrals of nI.

In view of all these we put forth the following theorem.

Theorem 4.2. *Let $S = \{\langle Q \cup I\rangle, +, \times\}$ be the neutrosophic ring. For every nI with $n \in Q \setminus \{0\}$ we have $a + bI \in \langle Q \cup I\rangle$ with $a + b = 1; a, b \in Q \setminus \{0\}$. such that $\{nI, a + bI\}$ is neutrosophic duplet.*

Proof. Given $nI \in \langle Q \cup I\rangle; n \in Q \setminus \{0\}$, we have to show $a + bI$ is a neutral where $a + b = 1, a, b, \in Q \setminus \{0\}$. consider

$$nI \times (a + bI) = anI + bnI = (a + b)nI = nI$$

as $a+b = 1$. Hence for any fixed $nI \in \langle Q \cup I\rangle$ we have an infinite collection of neutrals. Further the number of such neutrosophic duplets are infinite in number for varying n and varying $a, b \in Q \setminus \{0\}$ with $a + b = 1$. Thus the number of neutrosophic duplets in case of neutrosophic ring $\langle Q \cup I\rangle$ contains all the neutrosophic duplets of $\langle Z \cup I\rangle$ and the number of neutrosophic duplets in $\langle Q \cup I\rangle$ is a bigger infinite than that of the neutrosophic duplets in $\langle Z \cup I\rangle$. Further all $a + bI$ where $a, b \in Q \setminus Z$ with $a + b = 1$ happens to contribute to neutrosophic duplets which are not in $\langle Z \cup I\rangle$. □

Now we proceed on to give other types of neutrosopohic duplets in $\langle Q \cup I\rangle$ using $1 - I$ the idempotent which acts as neutral. Consider

$$x = \frac{5}{3} - \frac{5I}{3} \in \langle Q \cup I\rangle$$

let $y = 1 - I$, we find

$$x \times y = \frac{5}{3} - \frac{5I}{3} \times 1 - I = \frac{5}{3} - \frac{5I}{3} - \frac{5I}{3} + \frac{5I}{3} = \frac{5}{3} - \frac{5I}{3} = x.$$

In view of this we propose the following theorem.

Theorem 4.3. *Let $S = \{\langle Q \cup I\rangle = \{a + bI | a, b \in Q, I^2 = I\}, +, \times\}$ be the neutrosophic ring of rationals.*

1. *The idempotent $x = 1 - I$ acts as the neutral for infinite collection of elements $a - aI$ where $a \in Q$.*

2. *For every $a - aI \in S$ where $a \in Q, 1 - dI$ acts as neutrals for $d \in Q$.*

Proof. Consider any $a - aI = x \in \langle Q \cup I\rangle; a \in Q$ we see for $y = 1 - I$ the idempotent in $\langle Q \cup I\rangle$.

$$x \times y = a - aI \times 1 - I = a - aI - aI + aI = a - aI = x.$$

Thus $1-I$ acts as the neutral for $a-aI$; in fact $\{a-aI, 1_I\}$ is a neutrosophic duplet; for all $a \in Q$. Now consider $s = p - pI$ where $p \in Q$ and $r = 1 - dI \in \langle Q \cup I \rangle; d \in Q$.

$$S \times r = p - pI \times 1 - dI = p - pI - pdI + pdI = p - pI = s$$

Thus $\{p - pI, 1 - dI\}$ are neutrosophic duplets for all $p \in Q$ and $d \in Q$. The collection of neutrosophic duplets which are in $\langle Q \cup I \rangle \setminus \{\langle Z \cup I \rangle\}$ is in fact is of infinite cardinality. □

Next we search of other types of neutrosophic duplets in $\{\langle Q \cup I \rangle\}$. Suppose $a + bI \in \langle Q \cup I \rangle$ and let $c + dI$ be the possible neutral for it, we arrive the conditions on a, b, c and d

$$(a + bI) \times (c + dI) = a + bI$$

$$ac + bc + adI + bdI = a + bI$$

$ac = a$ which is possible if and only if $c = 1$. Hence

$$b + ad + bd = b$$

$$ad + bd = 0$$

$$d(a + b) = 0$$

as $d \neq 0$;

$$a = -b.$$

Thus $a + bI = a - aI$ are only possible elements in $\langle Q \cup I \rangle$which can contribute to neutrosophic duplets and the neutrals associated with them is of the form $1 \pm dI$ and $d \in Q \setminus \{0\}$. Thus we can say even in case of R the field of reals and for the associated neutrosophic ring $\langle R \cup I \rangle$. All results are true in case $\langle Q \cup I \rangle$ and $\langle Z \cup I \rangle$; expect $\langle R \cup I \rangle \setminus \langle Q \cup I \rangle$ has infinite duplets and $\langle R \cup I \rangle$ has infinitely many more neutrosophic duplets than $\langle Q \cup I \rangle$.

The following theorem on real neutrsophic rings is both innovative and intersting.

Theorem 4.4. *Let $S = \langle R \cup I \rangle$ be the real neutrosophic ring. The neutrosophic duplets are contributed only by elements of the form nI and $a - aI$ where $n \in R$ and $a \in R^+$ with neutrals $m - (m-1)I$ and $1 - dI; m, d \in R$ respectively.*

Proof. Consider $\{nI, m(m-1)I\}$ the pair

$$nI \times m - (m-1)I = nmI$$

$$-nmI + nI = nI$$

for all $n, m \in R \setminus \{1, 0\}$. Thus $\{nI, m - (m-1)I\}$ is an infinite collection of neutrosophic duplets. We define $(nI, m - (m-1)I)$ as a neutrosophic duplet pair. Consider the pair $\{(a - aI), (1 - dI)\}; a \in R^+, d \in R$. We see

$$a - aI \times 1 - dI = a - aI - daI + adI = a - aI$$

Thus $\{(a - aI), (1 - dI)\}$ forms an infinite collection of neutrosophic duplets. We call $((a - aI), (1 - dI))$ as a neutrosophic duplet pair. Hence the theorem. □

Theorem 4.5. *Let $S = \langle R \cup I \rangle$ be the neutrosophic ring*

1. *The ordered pair of neutrosophic duplets $B = \{(nI, m - (m-1)I); n \in R, m \in R \cup \{0\}\}$ forms a neutrosophic semigroup of $S = \langle R \cup I \rangle$ under component wise product.*
2. *The ordered pair of neutrosophic duplets $D = \{(a - aI, 1 - dI); a \in R^+; d \in R\}$ form a neutrosophic semigroup under product taken component wise.*

Proof. Given $B = \{(nI, m - (m-1)I | n \in R, m \in (R \setminus \{1\})\} \cup (nI, 0) \subseteq (\{\langle R \cup I \rangle\}, \{\langle R \cup I \rangle\})$. To prove B is a neutrosophic semigroup of $(\langle R \cup I \rangle, \langle R \cup I \rangle)$.. For any $x = (nI, (m - (m-1)I)$ and $y = (sI, t - 9t - 1)I) \in B$ we prove $xy = yx \in B$

$$x \times y = xy = (nI, m - (m-1)I \times (sI, t - (t-1)I)$$

$$= (nsI, [m - (m-1)I] \times [t - (t-1)I])$$

$$(nsI, mt - t(m-1)I - m(t-1)I + (m-1)(t-1)I)$$

$$= (nsI, mt - (mt-1)I) \in B$$

It is easily verified $xy = yx$ for all $x, y \in B$. Thus $\{B, \times\}$ is a neutrosophic semigroup of neutrosophic duplet pairs. Consider $x, y \in D$; we show $x \times y \in D$. Let $x = (a - aI, 1 - dI)$ and $y = (b - bI, 1 - cI) \in D$

$$x \times y = (a - aI, 1 - dI) \times (b - bI, 1 - cI)$$

$$= (a - aI \times b - bI, (-aI \times 1 - cI)$$

$$= (ab - abI - abI + abI, 1 - dI - cI + cdI)$$

$$= (ab - abI, 1 - (d + c - cd)I) \in D$$

as $x \times y$ is also in the form of x and y. Hence D the neutrosophic duplet pairs forms a neutrosophic semigroup under component wise product. □

5 Discussions and Conclusions

In this paper the notion of duplets in case neutrosophic rings, $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$, have been introduced and analysed. It is proved that the number of neutrosophic duplets in all these three rings happens to be an infinite collection. We further prove there are infinitely many elements for which $1 - I$ happens to be the neutral. Here we establish the duplet pair $\{a - aI, 1 - dI\}; a \in R^+$ and $d \in R$ happen to be a neutrosophic semigroup under component wise product. The collection $\{a - aI\}$ forms a neutrosophic subring $a \in Z$ or Q or R. For future research we want to analyse whether these neutrosophic rings can have neutrosophic triplets and if that collections enjoy some nice algebraic property. Finally we leave it as an open problem to find some applications of these neutrosophic duplets which form an infinite collection.

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COMMUTATIVE NEUTROSOPHIC TRIPLET GROUP AND NEUTRO-HOMOMORPHISM BASIC THEOREM

Xiaohong Zhang, Florentin Smarandache, Mumtaz Ali, Xingliang Liang

Xiaohong Zhang, Florentin Smarandache, Mumtaz Ali, Xingliang Liang (2018). Commutative Neutrosophic Triplet Group and Neutro-Homomorphism Basic Theorem. *Italian Journal of Pure and Applied Mathematics* 40, 353-375.

Abstract. The neutrosophic triplet is a group of three elements that satisfy certain properties with some binary operations. The neutrosophic triplet group is completely different from the classical group in the structural properties. In this paper, we further study neutrosophic triplet group. First, to avoid confusion, some new symbols are introduced, and several basic properties of neutrosophic triplet group are rigorously proved (because the original proof is flawed), and a result about neutrosophic triplet subgroup is revised. Second, some new properties of commutative neutrosophic triplet group are funded, and a new equivalent relation is established. Third, based on the previous results, the following important proposi-tions are proved: from any commutative neutrosophic triplet group, an Abel group can be constructed; from any commutative neutrosophic triplet group, a BCI-algebra can be constructed.

1. Introduction

From a philosophical point of view, Florentin Smarandache introduced the con-cept of a neutrosophic set (see [12, 13, 14]). The neutrosophic set theory is applied to many scientific fields and also applied to algebraic structures (see [1, 3, 7, 10, 11, 15, 17, 19]). Recently, Florentin Smarandache and Mumtaz Ali in [16], for the first time, introduced the notions of neutrosophic triplet and neu-trosophic triplet group. The neutrosophic triplet is a group of three elements that satisfy certain properties with some binary operation. The neutrosophic triplet group is completely different from the classical group in the structural properties. In 2017, Florentin Smarandache has written the monograph [15] which is present the last developments in neutrodophic theories (including neu-trosophic triplet and neutrosophic triplet group).

In this paper, we further study neutrosophic triplet group. We discuss some new properties of commutative neutrosophic triplet group, and investigate the relationships among commutative neutrosophic triplet group, Abel group (that is, commutative group) and BCI-algebra. Moreover, we establish the quotient structure and neutro-homomorphism basic theorem.

As a guide, it is necessary to give a brief overview of the basic aspects of BCI-algebra and related algebraic systems. In 1966, K. Iseki introduced the concept of BCI-algebra as an algebraic counterpart of the BCI-logic (see [5, 24]). The algebraic structures closely related to BCI algebra are BCK-algebra, BCC-algebra, BZ-algebra, BE-algebra, and so on (see [2, 8, 20, 21, 22, 25]). As a generalization of BCI-algebra, W. A. Dudek and Y. B. Jun [4] introduced the notion of pseudo-BCI algebras. Moreover, pseudo-BCI algebra is also as a generalization of pseudo-BCK algebra (which is close connection with various non-commutative fuzzy logic formal systems, see [18, 22, 23, 24]). Recently, some articles related filter theory of pseudo-BCI algebras are published (see [26, 27, 28, 29]). As non-classical logic algebras, BCI-algebras are closely related to Abel groups (see [9]); similarly, BZ-algebras (pseudo-BCI algebras) are closely related general groups (see [20, 26]), and some results in [9, 20] will be applied in this paper.

2. Some basic concepts

2.1 On neutrosophic triplet group

Definition 2.1 ([16])**.** Let N be a set together with a binary operation $*$. Then, N is called a *neutrosophic triplet set* if for any $a \in N$, there exist a neutral of "a" called $neut(a)$, different from the classical algebraic unitary element, and an opposite of "a" called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$$a * neut(a) = neut(a) * a = a;$$
$$a * anti(a) = anti(a) * a = neut(a).$$

The elements a, $neut(a)$ and $anti(a)$ are collectively called as neutrosophic triplet, and we denote it by $(a, neut(a), anti(a))$. By $neut(a)$, we mean neutral of a and apparently, a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet. For the same element "a" in N, there may be more neutrals to it $neut(a)$ and more opposites of it $anti(a)$.

Definition 2.2 ([16])**.** The element b in $(N, *)$ is the second component, denoted as $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).

Definition 2.3 ([16])**.** The element c in $(N, *)$ is the third component, denoted as $anti(\cdot)$, of a neutrosophic triplet, if there exist other elements a and b in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).

Definition 2.4 ([16])**.** Let $(N, *)$ be a neutrosophic triplet set. Then, N is called a *neutrosophic triplet group*, if the following conditions are satisfied:

(1) If $(N, *)$ is well-defined, i.e. for any $a, b \in N$, one has $a * b \in N$.

(2) If $(N, *)$ is associative, i.e. $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

Definition 2.5 ([16])**.** Let $(N, *)$ be a neutrosophic triplet group. Then, N is called a *commutative neutrosophic triplet group* if for all $a, b \in N$, we have $a * b = b * a$.

Definition 2.6 ([16])**.** Let $(N, *)$ be a neutrosophic triplet group under $*$, and let H be a subset of N. Then, H is called a neutrosophic triplet subgroup of N if H itself is a *neutrosophic triplet group* with respect to $*$.

Remark 2.7. In order to include richer structure, the original concept of neutrosophic triplet is generalized to neutrosophic extended triplet by Florentin Smarandache. A neutrosophic extended triplet is a neutrosophic triplet, defined as above, but where the neutral of x (called "extended neutral") is allowed to also be equal to the classical algebraic unitary element (if any). Therefore, the restriction "different from the classical algebraic unitary element if any" is released. As a consequence, the "extended opposite" of x, is also allowed to be equal to the classical inverse element from a classical group. Thus, a neutrosophic extended triplet is an object of the form $(x, neut(x), anti(x))$, for $x \in N$, where $neut(x) \in N$ is the extended neutral of x, which can be equal or different from the classical algebraic unitary element if any, such that: $x * neut(x) = neut(x) * x = x$, and $anti(x) \in N$ is the extended opposite of x such that: $x * anti(x) = anti(x) * x = neut(x)$. In this paper, "neutrosophic triplet" means that "neutrosophic extended triplet".

2.2 On BCI-algebras

Definition 2.8 ([5, 23]). A BCI-algebra is an algebra $(X; \to, 1)$ of type $(2, 0)$ in which the following axioms are satisfied:

(i) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(ii) $x \to x = 1$,
(iii) $1 \to x = x$,
(iv) if $x \to y = y \to x = 1$, then $x = y$.

In any BCI-algebra $(X; \to, 1)$ one can define a relation $\leq$ by putting $x \leq y$ if and only if $x \to y = 1$, then $\leq$ is a partial order on X.

Definition 2.9 ([9, 26]). Let $(X; \to, 1)$ be a BCI-algebra. The set $\{x|x \leq 1\}$ is called the *p-radical* (or BCK-part) of X. A BCI-algebra X is called *p-semisimple* if its p-radical is equal to $\{1\}$.

Proposition 2.10 ([9]). *Let $(X; \to, 1)$ be a BCI-algebra. Then the following are equivalent:*

(i) X is p-semisimple,
(ii) $x \to 1 = 1 \Rightarrow x = 1$,
(iii) $(x \to 1) \to 1 = x$, $\forall x \in X$,
(iv) $(x \to 1) \to y = (y \to 1) \to x$ for all $x, y \in X$.

Proposition 2.11 ([26]). *Let $(X; \to, 1)$ be a BCI-algebra. Then the following are equivalent:*

(S1) X is p-semisimple,
(S2) $x \to y = 1 \Rightarrow x = y$ for all $x, y \in X$,
(S3) $(x \to y) \to (z \to y) = z \to x$ for all $x, y, z \in X$,
(S4) $(x \to y) \to 1 = y \to x$ for all $x, y \in X$,
(S5) $(x \to y) \to (a \to b) = (x \to a) \to (y \to b)$ for all $x, y, a, b \in X$.

Proposition 2.12 ([9, 26]). *Let $(X; \to, 1)$ be p-semisimple BCI-algebra; define $+$ and $-$ as follows: for all $x, y \in X$,*

$$x + y \overset{def}{=} (x \to 1) \to y, \quad -x \overset{def}{=} x \to 1.$$

Then $(X; +, -, 1)$ is an Abel group.

Proposition 2.13 ([9, 26]). *Let* $(X;+,-,1)$ *be an Abel group. Define* $(X;\leq,\rightarrow,1)$, *where*

$$x\rightarrow y=-x+y,\ x\leq y \textit{ if and only if } -x+y=1,\ \forall x,y\in X.$$

Then, $(X;\leq,\rightarrow,1)$ *is a BCI-algebra.*

3. Some properties of neutrosophic triplet group

As mentioned earlier, for a neutrosophic triplet group $(N,*)$, if $a\in N$, then $neut(a)$ may not be unique, and $anti(a)$ may not be unique. Thus, the symbolic $neut(a)$ sometimes means one and sometimes more than one, which is ambiguous. To this end, this paper introduces the following notations to distinguish:

$neut(a)$: denote any certain one of neutral of a;

$\{neut(a)\}$: denote the set of all neutral of a.

Similarly,

$anti(a)$: denote any certain one of opposite of a;

$\{anti(a)\}$: denote the set of all opposite of a.

Remark 3.1. In order not to cause confusion, we always assume that: (1) for the same a, when multiple $neut(a)$ (or $anti(a)$) are present in the same expression, they are always are consistent. Of course, if they are neutral (or opposite) of different elements, they refer to different objects (for example, in general, $neut(a)$ is different from $neut(b)$). (2) if $neut(a)$ and $anti(a)$ are present in the same expression, then they are match each other.

Proposition 3.2. *Let* $(N,*)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a\in N$. *Then*

$$neut(a)*neut(a)\in\{neut(a)\}.$$

Proof. For any $a\in N$, by Definition 2.1 we have

$$a*neut(a)=a,\ neut(a)*a=a.$$

From this, using associative law, we can get

$$a*(neut(a)*neut(a))=(neut(a)*neut(a))*a=a.$$

By Definition 2.1, it follows that $(neut(a)*neut(a))$ is a neutral of a. That is, $neut(a)*neut(a)\in\{neut(a)\}$. □

Remark 3.3. This proposition is a revised version of Theorem 3.21(1) in [16]. If $neut(a)$ is unique, then they are same. But, if $neut(a)$ is not unique, they are different. For example, assume $\{neut(a)\}=\{s,t\}$, then $neut(a)$ denote any one of s,t. Thus $neut(a)*neut(a)$ represents one of $s*s$, and $t*t$. Moreover, Proposition 3.2 means that $s*s,\ t*t\in\{neut(a)\}=\{s,t\}$, that is,

$$s*s=s,\ \text{or}\ s*s=t;\quad t*t=s,\ \text{or}\ t*t=t.$$

And, in this case, the equation $neut(a)*neut(a)=neut(a)$ means that $s*s=s$, $t*t=t$. So, they are different.

Proposition 3.4. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *If*

$$neut(a) * neut(a) = neut(a).$$

Then

$$neut(a) * anti(a) \in \{anti(a)\};$$
$$anti(a) * neut(a) \in \{anti(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$a * neut(a) = neut(a) * a = a;$$
$$a * anti(a) = anti(a) * a = neut(a).$$

From this, using associative law, we can get

$$a * (neut(a) * anti(a)) = (a * neut(a)) * anti(a) = a * anti(a) = neut(a).$$

And,

$$(neut(a) * anti(a)) * a = neut(a) * (anti(a) * a) = neut(a) * neut(a) = neut(a).$$

By Definition 2.1, it follows that $(neut(a) * anti(a))$ is a opposite of a. That is, $neut(a) * anti(a) \in \{anti(a)\}$. In the same way, we can get $anti(a) * neut(a) \in \{anti(a)\}$. □

Proposition 3.5. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and let* $a, b, c \in N$. *Then*

(1) $a * b = a * c$ *if and only if* $neut(a) * b = neut(a) * c$.

(2) $b * a = c * a$ *if and only if* $b * neut(a) = c * neut(a)$.

Proof. Assume $a * b = a * c$. Then $anti(a) * (a * b) = anti(a) * (a * c)$. By associative law, we have

$$(anti(a) * a) * b = (anti(a) * a) * c.$$

Using Definition 2.1 we get $neut(a) * b = neut(a) * c$.

Conversely, assume $neut(a) * b = neut(a) * c$. Then $a * (neut(a) * b) = a * (neut(a) * c)$. By associative law, we have

$$(a * neut(a)) * b = (a * neut(a)) * c.$$

Using Definition 2.1 we get $a * b = a * c$. That is, (1) holds.

Similarly, we can prove that (2) holds. □

Proposition 3.6. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and let* $a, b, c \in N$.

(1) *If* $anti(a) * b = anti(a) * c$, *then* $neut(a) * b = neut(a) * c$.

(2) *If* $b * anti(a) = c * anti(a)$, *then* $b * neut(a) = c * neut(a)$.

Proof. Assume $anti(a)*b = anti(a)*c$. Then $a*(anti(a)*b) = a*(anti(a)*c)$. By associative law, we have

$$(a * anti(a)) * b = (a * anti(a)) * c.$$

Using Definition 2.1 we get $neut(a) * b = neut(a) * c$. It follows that (1) holds.

Similarly, we can prove that $b * anti(a) = c * anti(a) \Rightarrow b * neut(a) = c * neut(a)$. □

Theorem 3.7. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$neut(neut(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$neut(a) * neut(neut(a)) = neut(a);$$
$$neut(neut(a)) * neut(a) = neut(a).$$

Then

$$a * (neut(a) * neut(neut(a))) = a * neut(a);$$
$$(neut(neut(a)) * neut(a)) * a = neut(a) * a.$$

By associative law and Definition 2.1, we have

$$a * neut(neut(a)) = a;$$
$$neut(neut(a)) * a = a.$$

From this, by Definition 2.1, $neut(neut(a)) \in \{neut(a)\}$. □

Theorem 3.8. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$neut(anti(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$anti(a) * neut(anti(a)) = anti(a);$$
$$neut(anti(a)) * anti(a) = anti(a).$$

Then

$$a * (anti(a) * neut(anti(a))) = a * anti(a);$$
$$(neut(anti(a)) * anti(a)) * a = anti(a) * a.$$

Using associative law and Definition 2.1,

$$neut(a) * neut(anti(a)) = neut(a);$$
$$neut(anti(a)) * neut(a) = neut(a).$$

It follows that $a*neut(anti(a)) = a$, $neut(anti(a))*a = a$. That is, $neut(anti(a)) \in \{neut(a)\}$. □

Theorem 3.9. *Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and $a \in N$. Then*

$$neut(a) * anti(anti(a)) = a.$$

where, $neut(a) \in \{neut(a)\}$, $anti(a) \in \{anti(a)\}$, and $neut(a)$ matches $anti(a)$, that is, $a * anti(a) = anti(a) * a = neut(a)$.

Proof. For any $a \in N$, by Definition 2.1 we have

$$anti(a) * anti(anti(a)) = neut(anti(a)).$$

Then

$$a * (anti(a) * anti(anti(a))) = a * neut(anti(a)).$$
$$(a * anti(a)) * anti(anti(a)) = a * neut(anti(a)).$$
$$neut(a) * anti(anti(a)) = a * neut(anti(a)).$$

On the other hand, by Theorem 3.8, $neut(anti(a)) \in \{neut(a)\}$. By Definition 2.1, it follows that $a*neut(anti(a))=a$. Therefore, $neut(a)*anti(anti(a))=a$. □

Theorem 3.10. *Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and $a \in N$. Then*

$$anti(neut(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$neut(a) * anti(neut(a)) = neut(neut(a));$$
$$anti(neut(a)) * neut(a) = neut(neut(a)).$$

Thus

$$a * (neut(a) * anti(neut(a))) = a * neut(neut(a));$$
$$(anti(neut(a)) * neut(a)) * a = neut(neut(a)) * a.$$

Applying associative law and Definition 2.1,

$$a * anti(neut(a)) = a * neut(neut(a));$$
$$anti(neut(a)) * a = neut(neut(a)) * a.$$

On the other hand, by Theorem 3.7, $neut(neut(a)) \in \{neut(a)\}$. It follows that

$$a * neut(neut(a)) = neut(neut(a)) * a = a.$$

Therefore,

$$a * anti(neut(a))) = anti(neut(a)) * a = a.$$

This means that $anti(neut(a)) \in \{neut(a)\}$. □

Theorem 3.11. *Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and $a, b \in N$. Then*

$$neut(a * a) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$(a * a) * neut(a * a) = a * a.$$

From this and applying the associativity of operation $*$ and Definition 2.1 we get

$$(anti(a) * a) * a * neut(a * a) = (anti(a) * a) * a.$$
$$neut(a) * a * neut(a * a) = neut(a) * a.$$
$$a * neut(a * a) = a.$$

Similarly, we can prove $neut(a * a) * a = a$. This means that $neut(a * a) \in \{neut(a)\}$. $\square$

Now, we note that Proposition 3.18 in [16] is not true.

Example 3.12. Consider $(Z_{10}, \sharp)$, where $\sharp$ is defined as $a \sharp b = 3ab(mod 10)$. Then, $(Z_{10}, \sharp)$ is a neutrosophic triplet group under the binary operation $\sharp$ with Table 1.

Table 1 Cayley table of neutrosophic triplet group $(Z_{10}, \sharp)$

$\sharp$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	3	6	9	2	5	8	1	4	7
2	0	6	2	8	4	0	6	2	8	4
3	0	9	8	7	6	5	4	3	2	1
4	0	2	4	6	8	0	2	4	6	8
5	0	5	0	5	0	5	0	5	0	5
6	0	8	6	4	2	0	8	6	4	2
7	0	1	2	3	4	5	6	7	8	9
8	0	4	8	2	6	0	4	8	2	6
9	0	7	4	1	8	5	2	9	6	3

For each $a \in Z_{10}$, we have $neut(a)$ in Z_{10}. That is,

$$neut(0) = 0, neut(1) = 7, neut(2) = 2, neut(3) = 7, neut(4) = 2,$$
$$neut(5) = 5, neut(6) = 2, neut(7) = 7, neut(8) = 2, neut(9) = 7.$$

Let $H = \{0, 2, 5, 7\}$, then $(H, \sharp)$ is a neutrosophic triplet subgroup of $(Z_{10}, \sharp)$, but

$$anti(5) \in \{1, 3, 5, 7, 9\} \not\subset H,$$
$$anti(0) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \not\subset H.$$

Therefore, Proposition 3.18 in [16] should be revised to the following form.

Proposition 3.13. *Let $(N, *)$ be a neutrosophic triplet group and H be a subset of N. Then H is a neutrosophic triplet subgroup of N if and only if the following conditions hold:*

(1) $a * b \in H$ *for all* $a, b \in H$.

(2) *there exists* $neut(a) \in H$ *for all* $a \in H$.

(3) *there exists* $anti(a) \in H$ *for all* $a \in H$.

4. New properties of commutative neutrosophic triplet group

Theorem 4.1. *Let* $(N,)$ *be a commutative neutrosophic triplet group with respect to* $*$ *and* $a, b \in N$. *Then*

$$\{neut(a)\} * \{neut(b)\} \subseteq \{neut(a * b)\}.$$

Proof. For any $a, b \in N$, by Definition 2.1 and 2.4 we have

$$a * neut(a) * neut(b) * b = (a * neut(a)) * (neut(b) * b) = a * b.$$

From this and applying the commutativity and associativity of operation $*$ we get

$$(neut(a) * neut(b)) * (a * b) = (a * b) * (neut(a) * neut(b)) = a * b.$$

This means that $neut(a)*neut(b) \in \{neut(a*b)\}$, that is, $\{neut(a)\}*\{neut(b)\} \subseteq \{neut(a * b)\}$. □

Proposition 4.2. *Let* $(N, *)$ *be a commutative neutrosophic triplet group with respect to* $*$ *and* $H = \{neut(a) \mid a \in N\}$. *Then* H *is a neutrosophic triplet subgroup of* N *such that* $(\forall a \in N)$ $neut(a) \in H$ *and* $unit(h) \in H$ *for any* $h \in N$.

Proof. For any $h_1, h_2 \in N$, by the definition of H, there exists $a, b \in N$ such that $h_1 = neut(a)$, $h_2 = neut(b)$. Then, by Theorem 4.1 we have

$$h_1 * h_2 = neut(a) * neut(b) \in \{neut(a * b)\} \subseteq H.$$

Moreover, applying Theorem 3.7 and 3.10,

$$neut(h_1) = neut(neut(a)) \in \{neut(a)\} \subseteq H.$$
$$anti(h_1) = anti(neut(a)) \in \{neut(a)\} \subseteq H.$$

Using Proposition 3.13 we know that H is a neutrosophic triplet subgroup of N, and it satisfies

$$(\forall a \in N)\ neut(a) \in H, \text{and } unit(h) \in H \text{ for any } h \in N.$$

□

Theorem 4.3. *Let* $(N, *)$ *be a commutative neutrosophic triplet group with respect to* $*$ *and* $a, b \in N$. *Then*

$$\{anti(a)\} * \{anti(b)\} \subseteq \{anti(a * b)\}.$$

Proof. For any $a, b \in N$, by Definition 2.1 and 2.4 we have

$$a * anti(a) * anti(b) * b = (a * anti(a)) * (anti(b) * b) = neut(a) * neut(b).$$

From this and applying the commutativity and associativity of operation $*$ we get

$$(anti(a) * anti(b))(a * b) = (a * b) * (anti(a) * anti(b)) = neut(a) * neut(b).$$

Applying Theorem 4.1, $neut(a) * neut(b) \in \{neut(a * b)\}$. Hence, by Definition 2.1, $anti(a) * anti(b) \in \{anti(a * b)\}$, that is, $\{anti(a)\} * \{anti(b)\} \subseteq \{anti(a * b)\}$. $\square$

Theorem 4.4. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$. Define binary relation $\approx_{neut}$ on N as following:*

$\forall a, b \in N$, $a \approx_{neut} b$ iff there exists $anti(b) \in \{anti(b)\}$, and $p, q \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in \{neut(q)\}.$$

Then $\approx_{neut}$ is reflexive and symmetric.

Proof. (1) For any $a \in N$, by Proposition 3.2, $neut(a) * neut(a) \in \{neut(a)\}$. Using Definition 2.1 we get

$$a * anti(a) * neut(a) = neut(a) * neut(a) \in \{neut(a)\}.$$

Then, $a \approx_{neut} a$.

(2) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a * anti(b) * neut(p) = neut(q). \tag{C1}$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in neut(p)$, $neut(q) \in \{neut(q)\}$. Using Theorem 3.10, $anti(neut(p)) \in \{neut(p)\}$. So, we denote $anti(neut(p)) = x \in \{neut(p)\}$. Thus,

$$\begin{aligned}
& b * anti(a) * x = b * anti(a) * anti(neut(p)) \\
& = anti(a) * b * anti(neut(p)) && (by\ Definition\ 2.5) \\
& = anti(a) * (neut(b) * anti(anti(b))) * anti(neut(p)) && (by\ Theorem\ 3.9) \\
& = (anti(a) * anti(anti(b)) * anti(neut(p))) * neut(b) && (by\ Definition\ 2.4and\ 2.5) \\
& \in \{anti(a * anti(b) * neut(p))\} * neut(b) && (by\ Theorem\ 4.3) \\
& \subseteq \{anti(neut(q))\} * neut(b) && (by\ the\ above\ result\ (C1)) \\
& \subseteq \{neut(q)\} * neut(b) && (by\ Theorem\ 3.10) \\
& \subseteq \{neut(q * b)\} && (by\ Theorem\ 4.1)
\end{aligned}$$

This means that $b \approx_{neut} a$. $\square$

Definition 4.5. Let $(N, *)$ be a neutrosophic triplet group. Then, N is called a neutrosophic triplet group with condition (AN) if for all $a, b \in N$, we have
(AN) $\{anti(a * b)\} \subseteq \{anti(a)\} * \{anti(b)\}$.

Proposition 4.6. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN) and $a, b \in N$. Then*

$$neut(a * b) \in \{neut(a)\} * \{neut(b)\}.$$

Proof. For any $a, b \in N$, by Definition 4.5, there exists $anti(a) \in \{anti(a)\}$, $anti(b) \in \{anti(b)\}$ such that

$$anti(a * b) = anti(a) * anti(b).$$

Then

$$\begin{aligned} neut(a * b) &= (a * b) * anti(a * b) = (a * b) * (anti(a) * anti(b)) \\ &= (a * anti(a)) * (b * anti(b)) = neut(a) * neut(b). \end{aligned}$$

This means that $neut(a * b) \in \{neut(a)\} * \{neut(b)\}$. □

Lemma 4.7. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN) and $a, b \in N$. If there exists $anti(b) \in \{anti(b)\}$, $p, q \in N$, $neut(p) \in \{neut(p)\}$ and $neut(q) \in \{neut(q)\}$ such that*

$$a * anti(b) * neut(p) = neut(q).$$

Then for any $x \in \{anti(b)\}$, there exists $p_1, q_1 \in N$, $neut(p_1) \in \{neut(p_1)\}$ and $neut(q_1) \in \{neut(q_1)\}$ such that

$$a * x * neut(p_1) = neut(q_1).$$

Proof. For any $x \in \{anti(b)\}$, there exists $y \in \{neut(b)\}$ such that $b * x = x * b = y$. Thus, from $a * anti(b) * neut(p) = neut(q)$ we get

$$\begin{aligned} & a * x * (neut(b) * neut(p)) \\ & = a * x * (anti(b) * b) * neut(p) \\ & = (a * anti(b) * neut(p)) * (x * b) \\ & = neut(q) * y \\ & \in neut(q) * \{neut(b)\} \\ & \subseteq \{neut(q * b)\} \qquad \textit{(by Theorem 4.1)} \end{aligned}$$

Therefore, there exists $p_1, q_1 \in N$, $neut(p_1) \in \{neut(p_1)\}$ and $neut(q_1) \in \{neut(q_1)\}$ such that $a * x * neut(p_1) = neut(q_1)$. □

Theorem 4.8. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as following:*

$\forall a, b \in N$, $a \approx_{neut} b$ iff there exists $anti(b) \in \{anti(b)\}$, $p, q \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in \{neut(q)\}.$$

Then $\approx_{neut}$ is an equivalent relation on N.

Proof. By Theorem 4.4, we only prove that $\approx_{neut}$ is transitive. Assume that $a \approx_{neut} b$ and $b \approx_{neut} c$, then there exists $p, q, r, s \in N$ such that

(C1) $$a * anti(b) * neut(p) = neut(q).$$

(C2) $$b * anti(c) * neut(r) = neut(s).$$

where $anti(b) \in \{anti(b)\}$, $anti(c) \in \{anti(c)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$, $neut(r) \in \{neut(r)\}$, $neut(s) \in \{neut(s)\}$. Using Theorem 3.10 and Theorem 4.1, we have

$$neut(p)*neut(c)*anti(neut(s))\in\{neut(p)\}*\{neut(c)\}*\{neut(s)\}\subseteq\{neut(p*s*c)\}.$$

Denote $y = neut(p) * neut(c) * anti(neut(s)) \in \{neut(p * s * c)\}$, then

$$\begin{aligned}
& a * anti(c) * y = a * anti(c) * neut(p) * neut(c) * anti(neut(s)) \\
& = a * anti(c) * neut(p) * anti(neut(s)) * neut(c) \qquad \text{(by Definition 2.5)} \\
& = a * anti(c) * neut(p) * anti(b * anti(c) * neut(r)) * neut(c) \\
& \qquad \text{(by the above result (C2))} \\
& \in a * anti(c) * neut(p) * \{anti(b) * anti(anti(c)) * anti(neut(r))\} * neut(c) \\
& \qquad \text{(by Definition 4.5)} \\
& \subseteq a * anti(c) * neut(p) * \{anti(b) * c * anti(neut(r))\} \\
& \qquad \text{(by Definition 2.4, 2.5 and Theorem 3.9)} \\
& \subseteq a * neut(p) * \{anti(b) * neut(r) * (anti(c) * c)\} \\
& \qquad \text{(by Theorem 3.10, Definition 2.4 and 2.5)} \\
& = a * neut(p) * \{anti(b) * neut(r) * neut(c)\} \qquad \text{(by Definition 2.1)} \\
& \subseteq \{(a * anti(b) * neut(p)) * neut(r) * neut(c)\} \qquad \text{(by Definition 2.1)} \\
& \subseteq \{neut(q_1)*neut(r)*neut(c)\} \qquad \text{(by the above result (C1) and Lemma 4.7)} \\
& \subseteq \{neut(q_1 * r * c)\} \qquad \text{(by Theorem 4.1)}
\end{aligned}$$

This means that $a \approx_{neut} c$. □

5. Commutative neutrosophic triplet group and Abel group with BCI-algebra

Theorem 5.1. *Let $(N, *)$ be a commutative neutrosophic triplet group condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. Then the following statements are hold:*

(1) $a, b, c \in N$, $a \approx_{neut} b \Rightarrow a * c \approx_{neut} b * c$.

(2) $a \approx_{neut} b \Rightarrow neut(a) \approx_{neut} neut(b)$.

(3) $a \approx_{neut} b \Rightarrow anti(a) \approx_{neut} anti(b)$.

(4) $a, b \in N$, $neut(a) \approx_{neut} neut(b)$.

Proof. (1) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$(C1)$ $$a * anti(b) * neut(p) = neut(q),$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Thus,

$$
\begin{aligned}
& (a*c)*anti(b*c)*neut(p) \\
& \in (a*c)*\{anti(b)\}*\{anti(c)\}*neut(p) \qquad (by\ Definition\ 4.5) \\
& \subseteq \{a*anti(b)*neut(p)\}*\{c*anti(c)\}\ (by\ Definition\ 2.4\ and\ 2.5) \\
& = \{a*anti(b)*neut(p)\}*\{neut(c)\} \qquad (by\ Definition\ 2.1) \\
& \subseteq \{neut(q_1)\}*\{neut(c)\}(by\ the\ above\ result\ (C1)\ and\ Lemma\ 4.7) \\
& \subseteq \{neut(q_1*c)\} \qquad (by\ Theorem\ 4.1)
\end{aligned}
$$

It follows that $a*c \approx_{neut} b*c$.

(2) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a*anti(b)*neut(p) = neut(q).$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Then, applying Theorem 3.8 and Theorem 4.1 we have

$$neut(a)*anti(neut(b))*neut(p) \in \{neut(a)\}*\{neut(b)\}*\{neut(p)\} \subseteq \{neut(a*b*p)\}.$$

This means that $neut(a) \approx_{neut} neut(b)$.

(3) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a*anti(b)*neut(p) = neut(q).$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Using Theorem 3.10,

$$anti(neut(p)) \in \{neut(p)\},\ anti(neut(q)) \in \{neut(q)\}.$$

Applying Theorem 4.3 we have

$$
\begin{aligned}
anti(a)*anti(anti(b))*anti(neut(p)) &\in \{anti(a*anti(b)*neut(p))\} \\
&\subseteq \{anti(neut(q))\} \subseteq \{neut(q)\}.
\end{aligned}
$$

It follows that $anti(a) \approx_{neut} anti(b)$.

(4) $\forall a, b \in N$, since

$$
\begin{aligned}
& neut(a)*anti(neut(b))*neut(a) \\
& \in neut(a)*\{neut(b)\}*neut(a) \qquad (by\ Theorem\ 3.10) \\
& \subseteq \{neut(a*b*a)\} \qquad (by\ Theorem\ 4.1)
\end{aligned}
$$

This means that $neut(a) \approx_{neut} neut(b)$. □

Theorem 5.2. *Let $(N,*)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. Then the quotient $N/\approx_{neut}$ is an Abel group with respect to the following operation:*

$$\forall\ a, b \in N,\ [a]_{neut} \bullet [b]_{neut} = [a*b]_{neut}.$$

where $[a]_{neut}$ is the equivalent class of a, the unit elment of $(N/\approx_{neut}, \bullet)$ is $1_{neut} = [neut(a)]_{neut}$, $\forall a \in N$, $neut(a) \in \{neut(a)\}$.

Proof. By Theorem 5.1 (1) $\sim$ (3) we know that the operation "$\bullet$" is well definition. Obviously, $(N/\approx_{neut}, \bullet)$ is a commutative neutrosophic triplet group.

Moreover, by Theorem 5.1 (4) we get

$$\forall a, b \in N,\ [neut(a)]_{neut} = [neut(b)]_{neut}.$$
$$\forall a, b \in N,\ neut([a]_{neut}) = neut([b]_{neut}).$$

This means that $neut(\cdot)$ is unique. Denote

$$1_{neut} = [neut(a)]_{neut},\ \forall\ a \in N,\ neut(a) \in \{neut(a)\}.$$

Then 1_{neut} is the unit element of $(N/\approx_{neut}, \bullet)$. Moreover, by Theorem 5.1 (3) we get that $anti([a]_{neut})$ is unique, $\forall a \in N$. Therefore, $(N/\approx_{neut}, \bullet)$ is an Abel group. □

Theorem 5.3. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. If define a new operation "$\rightarrow$" on the quotient $N/\approx_{neut}$ as following:*

$$\forall a, b \in N,\ [a]_{neut} \rightarrow [b]_{neut} = [a]_{neut} \bullet anti([b]_{neut}).$$

Then $(N/\approx_{neut}, \rightarrow, 1_{neut})$ is a BCI-algebra, where 1_{neut}=$[neut(a)]_{neut}$, $\forall a \in N$.

Proof. By Theorem 5.2 and Proposition 2.13 we can get the result. □

Example 5.4. Let $N = \{1, 2, 3, 4, 6, 7, 8, 9\}$. The operation $*$ on N is defined as Tables 2. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). For each $a \in N$, we have $neut(a)$ in N. That is,

$$neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7.$$

Moreover, for each $a \in N$, $anti(a)$ in N. That is,

$$anti(1) = 9,\ anti(2) \in \{2, 7\},\ anti(3) = 3,\ anti(4) \in \{1, 6\},$$
$$anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1.$$

It is easy to verify that $N/\approx_{neut}= \{[2]_{neut}, [1]_{neut}, [3]_{neut}, [4]_{neut}\}$ and $(N/\approx_{neut}, \bullet)$ is isomorphism to $(Z_4, +)$, where

$$[2]_{neut} = \{2, 7\},\ [1]_{neut} = \{1, 6\},\ [3]_{neut} = \{3, 8\},\ [4]_{neut} = \{4, 9\}.$$

Table 2 Cayley table of neutrosophic triplet group $(N, *)$

$*$	1	2	3	4	6	7	8	9
1	3	6	9	2	8	1	4	7
2	6	2	8	4	6	2	8	4
3	9	8	7	6	4	3	2	1
4	2	4	6	8	2	4	6	8
6	8	6	4	2	8	6	4	2
7	1	2	3	4	6	7	8	9
8	4	8	2	6	4	8	2	6
9	7	4	1	8	2	9	6	3

Table 3 Cayley table of Abel group $((N/\approx_{neut}, \bullet)$

$\bullet$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$
$[2]_{neut}$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$
$[1]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$	$[2]_{neut}$
$[3]_{neut}$	$[3]_{neut}$	$[4]_{neut}$	$[2]_{neut}$	$[1]_{neut}$
$[4]_{neut}$	$[4]_{neut}$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$

Table 4 Cayley table of Abel group $(Z_4, +)$

+	0	1	3	4
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Example 5.5. Consider $(Z_{10}, \sharp)$, where $\sharp$ is defined as $a \sharp b = 3ab(mod10)$. Then, $(Z_{10}, \sharp)$ is a neutrosophic triplet group with condition (AN), the binary operation $\sharp$ is defined in Table 1. For each $\in Z_{10}$, we have $neut(a)$ in Z_{10}. That is,

$$neut(0) = 0,\ neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(5) = 5,\ neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7.$$

Moreover, for each $a \in Z_{10}$, $anti(a)$ in Z_{10}. That is,

$$anti(0) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},\ anti(1) = 9,\ anti(2) \in \{2, 7\},$$
$$anti(3) = 3,\ anti(4) \in \{1, 6\},\ anti(5) \in \{1, 3, 5, 7, 9\},$$
$$anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1.$$

It is easy to verify that $N/\approx_{neut} = \{1_{neut} = [0]_{neut}\}$ and $(N/\approx_{neut}, \bullet)$ is isomorphism to $\{1\}$,where

$$[0]_{neut} = 1_{neut} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

6. Quotient structure and neutro-homomorphism basic theorem

Definition 6.1 ([16])**.** Let $(N_1, *_1)$ and $(N_2, *_2)$ be two neutrosophic triplet groups. Let $f : N_1 \to N_2$ be a mapping. Then, f is called *neutro-homomorphism* if for all $a, b \in N_1$, we have:

(1) $f(a *_1 b) = f(a) *_2 f(b)$;
(2) $f(neut(a)) = neut(f(a))$;
(3) $f(anti(a)) = anti(f(a))$.

Theorem 6.2. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$, H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then $\approx_H$ is reflexive and symmetric.

Proof. (1) For any $a \in N$, by Proposition 3.2 and the hypothesis ($neut(a) \in H$ for any $a \in N$), we have

$$neut(a) * neut(a) \in \{neut(a)\} \subseteq H.$$

By Definition 2.1 we get

$$a * anti(a) * neut(a) = neut(a) * neut(a) \in H.$$

Then, $a \approx_H a$.

(2) Assume $a \approx_H b$, then there exists $p \in N$ such that

$$(C2) \qquad a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Moreover, by the hypothesis ($anti(a) \in H$ for any $a \in H$), we have

$$(C3) \qquad anti(a * anti(b) * neut(p)) \in H.$$

Using Theorem 3.10, $anti(neut(p)) \in \{neut(p)\}$. So, we denote $anti(neut(p)) = x \in \{neut(p)\}$. Thus,

$b * anti(a) * x$
$= b * anti(a) * anti(neut(p))$
$= anti(a) * b * anti(neut(p))$ (by Definition 2.5)
$= anti(a) * (neut(b) * anti(anti(b))) * anti(neut(p))$ (by Theorem 3.9)
$= (anti(a) * anti(anti(b)) * anti(neut(p))) * neut(b)$ (by Definition 2.4 and 2.5)
$\in \{anti(a * anti(b) * neut(p))\} * neut(b)$ (by Theorem 4.3)
$\subseteq H$ (by (C3), the hypothesis and Proposition 3.13 (1))

This means that $b \approx_H a$. □

Lemma 6.3. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), $a, b \in N$, and H be a neutrosophic triplet subgroup of N such that ($\forall a \in N$) $neut(a) \in H$ and ($\forall a \in H$) $anti(a) \in H$. If there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that*

$$a * anti(b) * neut(p) \in H.$$

Then for any $x \in \{anti(b)\}$, there exists $p_1 \in N$, and $neut(p_1) \in \{neut(p_1)\}$ such that

$$a * x * neut(p_1) \in H.$$

Proof. For any $x \in \{anti(b)\}$, there exists $y \in \{neut(b)\}$ such that $b*x = x*b = y$. Since $(\forall a \in N)\ neut(a) \in H$, then $y \in H$. Thus, from $a*anti(b)*neut(p) \in H$ we get

$$\begin{aligned}
&a * x * (neut(b) * neut(p)) \\
&= a * x * (anti(b) * b) * neut(p) \\
&= (a * anti(b) * neut(p)) * (x * b) \\
&= (a * anti(b * neut(p)) * y \\
&\in H \qquad\qquad (by\ Proposition\ 3.13)
\end{aligned}$$

□

Theorem 6.4. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then $\approx_H$ is an equivalent relation on N.

Proof. By Theorem 6.2, we only prove that $\approx_H$ is transitive. Assume that $a \approx_H b$ and $b \approx_H c$, then there exists $p, r \in N$ and $q, s \in N$ such that

(C3) $$a * anti(b) * neut(p) = q \in H.$$

(C4) $$b * anti(c) * neut(r) = s \in H.$$

where $anti(b) \in \{anti(b)\}$, $anti(c) \in \{anti(c)\}$, $neut(p) \in \{neut(p)\}$, $neut(r) \in \{neut(r)\}$. Using Theorem 4.1 and the hypothesis ($neut(a) \in H$ for any $a \in N$), we have

$$neut(p) * neut(s) * neut(c) \in neut(p * s * c) \subseteq H.$$

Denote $y = neut(p) * neut(s) * neut(c) \in neut(p * s * c)$, then

$a * anti(c) * y$
$= a * anti(c) * neut(p) * neut(s) * neut(c)$
$= a * anti(c) * neut(p) * (s * anti(s)) * neut(c)$ (by Definition 2.1)
$= a * anti(c) * neut(p) * s * anti(b * anti(c) * neut(r)) * neut(c)$
(by the above result (C4))
$\in a*anti(c)*neut(p)*s*\{anti(b)\}*\{anti(anti(c))\}*\{anti(neut(r))\}neut(c)$
(by Definition 4.5)
$= a * anti(c) * neut(p) * s * \{anti(b)\} * c * \{anti(neut(r))\}$ (by Theorem 3.9)
$\subseteq a * anti(c) * neut(p) * s * \{anti(b)\} * c * \{neut(r)\}$ (by Theorem 3.10)
$\subseteq \{a * anti(b) * neut(p)\} * s * (anti(c) * c) * \{neut(r)\}$ (by Definition 2.4 and 2.5)
$\subseteq H * s * neut(c) * \{neut(r)\}$

(by Definition 2.1, the above result (C3) and Lemma 6.3)
$\subseteq H$ (by (C4), the hypothesis and Proposition 3.13 (1))
It follows that $a \approx_H c$. □

Theorem 6.5. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then the following statements are hold:
(1) $a, b, c \in N$, $a \approx_H b \Rightarrow a * c \approx_H b * c$.
(2) $a \approx_H b \Rightarrow neut(a) \approx_H neut(b)$.
(3) $a \approx_H b \Rightarrow anti(a) \approx_H anti(b)$.

Proof. (1) Assume $a \approx_H b$, then there exists $p \in N$ such that

$$(C2) \qquad a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. We have
$(a * c) * anti(b * c) * neut(p)$
$\in (a * c) * \{anti(b)\} * \{anti(c)\} * neut(p)$ (by Definition 4.5)
$\subseteq \{a * anti(b) * neut(p)\} * \{c * anti(c)\}$ (by Definition 2.4 and 2.5)
$= \{a * anti(b) * neut(p)\} * neut(c)$ (by Definition 2.1)
$\in H.$ (by (C2), the hypothesis, Lemma 6.3 and Proposition 3.13 (1))
It follows that $a * c \approx_H b * c$.

(2) Assume $a \approx_H b$, then there exists $p \in N$ such that $a * anti(b) * neut(p) \in H$, where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Applying Theorem 3.8 and Theorem 4.1 we have
$neut(a) * anti(neut(b)) * neut(p) \in neut(a) * \{neut(b)\} * neut(p)$
$\subseteq \{neut(a*b*p)\} \subseteq H.$ (by the hypothesis, $neut(a) \in H$ for any $a \in N$)
It follows that $neut(a) \approx_H neut(b)$.

Assume $a \approx_H b$, then there exists $p \in N$ such that

$$a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Applying the hypothesis ($(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$) and Theorem 3.10,

$$anti(a * anti(b) * neut(p)) \in H.$$
$$anti(neut(p)) \in \{neut(p)\} \subseteq H.$$

Moreover, by Theorem 4.3 we have

$$anti(a) * anti(anti(b)) * anti(neut(p)) \in \{anti(a * anti(b) * neut(p))\} \subseteq H.$$

Hence, $anti(a) \approx_H anti(b)$. □

Theorem 6.6. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as Theorem 6.5. Then the quotient $N/\approx_H$ is a commutative neutrosophic triplet group with respect to the following operation:*

$$\forall a, b \in N,\ [a]_H \bullet [b]_H = [a * b]_H.$$

where $[a]_H$ is the equivalent class of a with respect to $\approx_H$. Moreover, $(N, *)$ is neutron-homomorphism to $(N/\approx_H, \bullet)$ with respect to the following mapping:

$$f:\ N \to N/\approx_H;\ \textit{and}\ \forall a \in N,\ f(a) = [a]_H.$$

Proof. By Theorem 6.5 we know that the operation "$\bullet$" is well definition. Obviously, $(N/\approx_H, \bullet)$ is a commutative neutrosophic triplet group.

By the definitions of operation "$\bullet$" and mapping f we have

$$\forall a, b \in N,\ f(a * b) = [a * b]_H = [a]_H \bullet [b]_H = f(a) \bullet f(b).$$

Moreover, by Theorem 6.5 (2) and (3) we get

$$\forall a \in N, f(neut(a)) = [neut(a)]_H = neut([a]_H) = neut(f(a)).$$
$$\forall a \in N, f(anti(a)) = [anti(a)]_H = anti([a]_H) = anti(f(a)).$$

Therefore, $(N, *)$ is neutron-homomorphism to $(N/\approx_H, \bullet)$ with respect to the mapping f. □

Theorem 6.7. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as Theorem 6.5. If define a new operation "$\to$" on the quotient $N/\approx_H$ as following: $\forall a, b \in N$, $[a]_H \to [b]_H = [a]_H \bullet anti([b]_H)$. Then $(N/\approx_H, \to, 1_H)$ is a BCI-algebra, where $1_H = [neut(a)]_H$, $\forall a \in N$.*

Proof. By Theorem 6.7 and Proposition 2.13 we can get the result. □

Example 6.8. Let $N = \{1, 2, 3, 4, 6, 7, 8, 9\}$. The operation $*$ on N is defined as Tables 2. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). We can get the following equation

$$neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7;$$
$$anti(1) = 9,\ anti(2) \in \{2, 7\},\ anti(3) = 3,\ anti(4) \in \{1, 6\},$$
$$anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1.$$

Denote $H = \{2, 3, 7, 8\}$, it is easy to verify that H is a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Moreover, $N/\approx_H = \{H = [2]_H, [1]_H\}$ and $(N/\approx_H, \bullet)$ is isomorphism to $(Z_2, +)$, where

$$[2]_H = \{2, 3, 7, 8\},\ [1]_H = \{1, 4, 6, 9\}.$$

Table 5 Cayley table of Abel group $(N/\approx_H, \bullet)$

$\bullet$	$[2]_H$	$[1]_H$
$[2]_H$	$[2]_H$	$[1]_H$
$[1]_H$	$[1]_H$	$[2]_H$

Table 6 Cayley table of Abel group $(Z_2, +)$

$+$	0	1
0	0	1
1	1	0

The following example shows that the basic theorem of neutro-homomorphism (Theorem 6.7) is a natural and substantial generalization of the basic theorem of group-homomorphism.

Example 6.9. Let $(N, *)$ be a commutative group. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). Obviously, if H is a subgroup of N, then binary relation $\approx_H$ on N is the relation induced by subgroup H, that is,

$$\forall a, b \in N,\ a \approx_H b \text{ if and only if } a * b^{-1} \in H.$$

Thus, $(N, *)$ is group-homomorphism to $(N/\approx_H, \bullet) = (N/H, \bullet)$.

7. Conclusion

This paper is focus on neutrosophic triplet group. We proved some new properties of (commutative) neutrosophic triplet group, and constructed a new equivalent relation on any commutative neutrosophic triplet group with condition (AN). Based on these results, for the first time, we have described the inner link between commutative neutrosophic triplet group with condition (AN) and Abel group with BCI-algebra. Furthermore, we establish the quotient structure by neutrosophic triplet subgroup, and prove the basic theorem of neutro-homomorphism, which is a natural and substantial generalization of the basic theorem of group-homomorphism. Obviously, these results will play an important role in the further study of neutrosophic triplet group.

Acknowledgment

This work was supported by National Natural Science Foundation of China (Grant No. 61573240).

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Commutative Generalized Neutrosophic Ideals in *BCK*-Algebras

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Rajab Ali Borzooei, Xiaohong Zhang, Florentin Smarandache, Young Bae Jun (2018). Commutative Generalized Neutrosophic Ideals in BCK-Algebras. *Symmetry* 10, 350.
DOI: 10.3390/sym10080350

Abstract: The concept of a commutative generalized neutrosophic ideal in a BCK-algebra is proposed, and related properties are proved. Characterizations of a commutative generalized neutrosophic ideal are considered. Also, some equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras are introduced, and some properties are investigated.

Keywords: (commutative) ideal; generalized neutrosophic set; generalized neutrosophic ideal; commutative generalized neutrosophic ideal

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy set in which the degree of membership is expressed by one function (that is, truth or t). The theory of fuzzy set is applied to many fields, including fuzzy logic algebra systems (such as pseudo-BCI-algebras by Zhang [1]). In 1986, Atanassov introduced the concept of intuitionistic fuzzy set in which there are two functions, membership function (t) and nonmembership function (f). In 1995, Smarandache introduced the new concept of neutrosophic set in which there are three functions, membership function (t), nonmembership function (f) and indeterminacy/neutrality membership function (i), that is, there are three components (t, i, f) = (truth, indeterminacy, falsehood) and they are independent components.

Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [2–10]. Moreover, Zhang et al. studied totally dependent-neutrosophic sets, neutrosophic duplet semi-group and cancellable neutrosophic triplet groups (see [11,12]). Song et al. proposed the notion of generalized neutrosophic set and applied it to BCK/BCI-algebras.

In this paper, we propose the notion of a commutative generalized neutrosophic ideal in a BCK-algebra, and investigate related properties. We consider characterizations of a commutative generalized neutrosophic ideal. Using a collection of commutative ideals in BCK-algebras, we obtain a commutative generalized neutrosophic ideal. We also establish some equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras, and discuss related basic properties of these ideals.

2. Preliminaries

A set X with a constant element 0 and a binary operation $*$ is called a BCI-algebra, if it satisfies $(\forall x, y, z \in X)$:

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $x*y=0, y*x=0 \Rightarrow x=y$.

A BCI-algebra X is called a BCK-algebra, if it satisfies $(\forall x \in X)$:

(V) $0*x=0$,

For any BCK/BCI-algebra X, the following conditions hold $(\forall x,y,z \in X)$:

$$x*0=x, \tag{1}$$
$$x \leq y \Rightarrow x*z \leq y*z, z*y \leq z*x, \tag{2}$$
$$(x*y)*z=(x*z)*y, \tag{3}$$
$$(x*z)*(y*z) \leq x*y \tag{4}$$

where the relation $\leq$ is defined by: $x \leq y \iff x*y=0$. If the following assertion is valid for a BCK-algebra X, $\forall x,y \in X$,

$$x*(x*y)=y*(y*x). \tag{5}$$

then X is called a commutative BCK-algebra.

Assume I is a subset of a BCK/BCI-algebra X. If the following conditions are valid, then we call I is an ideal of X:

$$0 \in I, \tag{6}$$
$$(\forall x \in X)(\forall y \in I)(x*y \in I \Rightarrow x \in I). \tag{7}$$

A subset I of a BCK-algebra X is called a commutative ideal of X if it satisfies (6) and

$$(\forall x,y,z \in X)((x*y)*z \in I, z \in I \Rightarrow x*(y*(y*x)) \in I). \tag{8}$$

Recall that any commutative ideal is an ideal, but the inverse is not true in general (see [7]).

Lemma 1 ([7]). *Let I be an ideal of a BCK-algebra X. Then I is commutative ideal of X if and only if it satisfies the following condition for all x,y in X:*

$$x*y \in I \Rightarrow x*(y*(y*x)) \in I. \tag{9}$$

For further information regarding BCK/BCI-algebras, please see the books [7,13].

Let X be a nonempty set. A fuzzy set in X is a function $\mu : X \to [0,1]$, and the complement of μ, denoted by μ^c, is defined by $\mu^c(x)=1-\mu(x)$, $\forall x \in X$. A fuzzy set μ in a BCK/BCI-algebra X is called a fuzzy ideal of X if

$$(\forall x \in X)(\mu(0) \geq \mu(x)), \tag{10}$$
$$(\forall x,y \in X)(\mu(x) \geq \min\{\mu(x*y),\mu(y))\}. \tag{11}$$

Assume that X is a non-empty set. A neutrosophic set (NS) in X (see [14]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$, $A_I : X \to [0,1]$, and $A_F : X \to [0,1]$. We shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

A generalized neutrosophic set (GNS) in a non-empty set X is a structure of the form (see [15]):

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}$$

where $A_T : X \to [0,1]$, $A_F : X \to [0,1]$, $A_{IT} : X \to [0,1]$, and $A_{IF} : X \to [0,1]$.

We shall use the symbol $A = (A_T, A_{IT}, A_{IF}, A_F)$ for the generalized neutrosophic set

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}.$$

Note that, for every GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X, we have (for all x in X)

$$(\forall x \in X)\, (0 \leq A_T(x) + A_{IT}(x) + A_{IF}(x) + A_F(x) \leq 3)\,.$$

If $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in X, then $\Box A = (A_T, A_{IT}, A_{IT}^c, A_T^c)$ and $\Diamond A = (A_F^c, A_{IF}^c, A_{IF}, A_F)$ are also GNSs in X.

Given a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in a BCK/BCI-algebra X and α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$, we define four sets as follows:

$$\begin{aligned}
&U_A(T, \alpha_T) := \{x \in X \mid A_T(x) \geq \alpha_T\},\\
&U_A(IT, \alpha_{IT}) := \{x \in X \mid A_{IT}(x) \geq \alpha_{IT}\},\\
&L_A(F, \beta_F) := \{x \in X \mid A_F(x) \leq \beta_F\},\\
&L_A(IF, \beta_{IF}) := \{x \in X \mid A_{IF}(x) \leq \beta_{IF}\}.
\end{aligned}$$

A GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in a BCK/BCI-algebra X is called a generalized neutrosophic ideal of X (see [15]) if

$$(\forall x \in X)\begin{pmatrix} A_T(0) \geq A_T(x),\ A_{IT}(0) \geq A_{IT}(x) \\ A_{IF}(0) \leq A_{IF}(x), A_F(0) \leq A_F(x) \end{pmatrix}, \tag{12}$$

$$(\forall x, y \in X)\begin{pmatrix} A_T(x) \geq \min\{A_T(x*y), A_T(y)\} \\ A_{IT}(x) \geq \min\{A_{IT}(x*y), A_{IT}(y)\} \\ A_{IF}(x) \leq \max\{A_{IF}(x*y), A_{IF}(y)\} \\ A_F(x) \leq \max\{A_F(x*y), A_F(y)\} \end{pmatrix}. \tag{13}$$

3. Commutative Generalized Neutrosophic Ideals

Unless specified, X will always represent a BCK-algebra in the following discussion.

Definition 1. *A GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is called a commutative generalized neutrosophic ideal of X if it satisfies the condition* (12) *and*

$$(\forall x, y, z \in X)\begin{pmatrix} A_T(x*(y*(y*x))) \geq \min\{A_T((x*y)*z), A_T(z)\} \\ A_{IT}(x*(y*(y*x))) \geq \min\{A_{IT}((x*y)*z), A_{IT}(z)\} \\ A_{IF}(x*(y*(y*x))) \leq \max\{A_{IF}((x*y)*z), A_{IF}(z)\} \\ A_F(x*(y*(y*x))) \leq \max\{A_F((x*y)*z), A_F(z)\} \end{pmatrix}. \tag{14}$$

Example 1. *Denote $X = \{0, a, b, c\}$. The binary operation $*$ on X is defined in Table 1.*

Table 1. The operation "$*$".

$*$	**0**	***a***	***b***	***c***
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

We can verify that $(X, *, 0)$ is a BCK-algebra (see [7]). Define a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X by Table 2.

Table 2. GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$.

X	$A_T(x)$	$A_{IT}(x)$	$A_{IF}(x)$	$A_F(x)$
0	0.7	0.6	0.1	0.3
a	0.5	0.5	0.2	0.4
b	0.3	0.2	0.4	0.6
c	0.3	0.2	0.4	0.6

Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.

Theorem 1. *Every commutative generalized neutrosophic ideal is a generalized neutrosophic ideal.*

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. $\forall x, z \in X$, we have

$$A_T(x) = A_T(x * (0 * (0 * x))) \geq \min\{A_T((x * 0) * z), A_T(z)\} = \min\{A_T(x * z), A_T(z)\},$$

$$A_{IT}(x) = A_{IT}(x * (0 * (0 * x))) \geq \min\{A_{IT}((x * 0) * z), A_{IT}(z)\} = \min\{A_{IT}(x * z), A_{IT}(z)\},$$

$$A_{IF}(x) = A_{IF}(x * (0 * (0 * x))) \leq \max\{A_{IF}((x * 0) * z), A_{IF}(z)\} = \max\{A_{IF}(x * z), A_{IF}(z)\},$$

and

$$A_F(x) = A_F(x * (0 * (0 * x))) \leq \max\{A_F((x * 0) * z), A_F(z)\} = \max\{A_F(x * z), A_F(z)\}.$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal. □

The following example shows that the inverse of Theorem 1 is not true.

Example 2. *Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is defined in Table 3.*

Table 3. The operation "$*$".

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

We can verify that $(X, *, 0)$ is a BCK-algebra (see [7]). We define a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X by Table 4.

Table 4. GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$.

X	$A_T(x)$	$A_{IT}(x)$	$A_{IF}(x)$	$A_F(x)$
0	0.7	0.6	0.1	0.3
1	0.5	0.4	0.2	0.6
2	0.3	0.5	0.4	0.4
3	0.3	0.4	0.4	0.6
4	0.3	0.4	0.4	0.6

It is routine to verify that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X, but A is not a commutative generalized neutrosophic ideal of X since

$$A_T(2 * (3 * (3 * 2))) = A_T(2) = 0.3 \ngeq \min\{A_T((2 * 3) * 0), A_T(0)\}$$

and/or

$$A_{IF}(2 * (3 * (3 * 2))) = A_{IF}(2) = 0.4 \nleq \max\{A_{IF}((2 * 3) * 0), A_{IF}(0)\}.$$

Theorem 2. *Suppose that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is commutative if and only if it satisfies the following condition.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \leq A_T(x * (y * (y * x))) \\ A_{IT}(x * y) \leq A_{IT}(x * (y * (y * x))) \\ A_{IF}(x * y) \geq A_{IF}(x * (y * (y * x))) \\ A_F(x * y) \geq A_F(x * (y * (y * x))) \end{array} \right). \tag{15}$$

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. Taking $z = 0$ in (14) and using (12) and (1) induces (15).

Conversely, let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a generalized neutrosophic ideal of X satisfying the condition (15). Then

$$A_T(x * (y * (y * x))) \geq A_T(x * y) \geq \min\{A_T((x * y) * z), A_T(z)\},$$

$$A_{IT}(x * (y * (y * x))) \geq A_{IT}(x * y) \geq \min\{A_{IT}((x * y) * z), A_{IT}(z)\},$$

$$A_{IF}(x * (y * (y * x))) \leq A_{IF}(x * y) \leq \max\{A_{IF}((x * y) * z), A_{IF}(z)\}$$

and

$$A_F(x * (y * (y * x))) \leq A_F(x * y) \leq \max\{A_F((x * y) * z), A_F(z)\}$$

for all $x, y, z \in X$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 2 ([15]). *Any generalized neutrosophic ideal $A = (A_T, A_{IT}, A_{IF}, A_F)$ of X satisfies:*

$$(\forall x, y, z \in X) \left(x * y \leq z \Rightarrow \left\{ \begin{array}{l} A_T(x) \geq \min\{A_T(y), A_T(z)\} \\ A_{IT}(x) \geq \min\{A_{IT}(y), A_{IT}(z)\} \\ A_{IF}(x) \leq \max\{A_{IF}(y), A_{IF}(z)\} \\ A_F(x) \leq \max\{A_F(y), A_F(z)\} \end{array} \right. \right). \tag{16}$$

We provide a condition for a generalized neutrosophic ideal to be commutative.

Theorem 3. *For any commutative BCK-algebra, every generalized neutrosophic ideal is commutative.*

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of a commutative BCK-algebra X. Note that

$$\begin{aligned}((x*(y*(y*x)))*((x*y)*z))*z &= ((x*(y*(y*x)))*z)*((x*y)*z)\\ &\leq (x*(y*(y*x)))*(x*y)\\ &= (x*(x*y))*(y*(y*x)) = 0,\end{aligned}$$

thus, $(x*(y*(y*x)))*((x*y)*z) \leq z, \forall x,y,z \in X$. By Lemma 2 we get

$$\begin{aligned}&A_T(x*(y*(y*x))) \geq \min\{A_T((x*y)*z), A_T(z)\},\\ &A_{IT}(x*(y*(y*x))) \geq \min\{A_{IT}((x*y)*z), A_{IT}(z)\},\\ &A_{IF}(x*(y*(y*x))) \leq \max\{A_{IF}((x*y)*z), A_{IF}(z)\},\\ &A_F(x*(y*(y*x))) \leq \max\{A_F((x*y)*z), A_F(z)\}.\end{aligned}$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 3 ([15]). *If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is a generalized neutrosophic ideal of X, then the sets $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$ whenever they are non-empty.*

Theorem 4. *If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is a commutative generalized neutrosophic ideal of X, then the sets $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$ whenever they are non-empty.*

The commutative ideals $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are called *level neutrosophic commutative ideals* of $A = (A_T, A_{IT}, A_{IF}, A_F)$.

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. Thus $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X whenever they are non-empty applying Lemma 3. Suppose that $x,y \in X$ and $x*y \in U_A(T,\alpha_T) \cap U_A(IT,\alpha_{IT})$. Using (15),

$$\begin{aligned}&A_T(x*(y*(y*x))) \geq A_T(x*y) \geq \alpha_T,\\ &A_{IT}(x*(y*(y*x))) \geq A_{IT}(x*y) \geq \alpha_{IT},\end{aligned}$$

and so $x*(y*(y*x)) \in U_A(T,\alpha_T)$ and $x*(y*(y*x)) \in U_A(IT,\alpha_{IT})$. Suppose that $a,b \in X$ and $a*b \in L_A(IF,\beta_{IF}) \cap L_A(F,\beta_F)$. It follows from (15) that $A_{IF}(a*(b*(b*a))) \leq A_{IF}(a*b) \leq \beta_{IF}$ and $A_F(a*(b*(b*a))) \leq A_F(a*b) \leq \beta_F$. Hence $a*(b*(b*a)) \in L_A(IF,\beta_{IF})$ and a*(b*(b*a)) $\in L_A(F,\beta_F)$. Therefore $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X. □

Lemma 4 ([15]). *Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in X and $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X, $\forall\alpha_T$, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X.*

Theorem 5. *Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X such that $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.*

Proof. Let $\alpha_T, \alpha_{IT}, \beta_F, \beta_{IF} \in [0,1]$ be such that the non-empty sets $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X. Then $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X. Hence $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X applying Lemma 4. For any $x,y \in X$, let $A_T(x*y) = \alpha_T$. Then $x*y \in U_A(T,\alpha_T)$, and so $x*(y*(y*x)) \in U_A(T,\alpha_T)$ by (9). Hence $A_T(x*(y*(y*x))) \geq \alpha_T = A_T(x*y)$. Similarly, we can show that

$$(\forall x,y \in X)(A_{IT}(x*(y*(y*x))) \geq A_{IT}(x*y)).$$

For any $x,y,a,b,\in X$, let $A_F(x*y) = \beta_F$ and $A_{IF}(a*b) = \beta_{IF}$. Then $x*y \in L_A(F,\beta_F)$ and $a*b \in L_A(IF,\beta_{IF})$. Using Lemma 1 we have $x*(y*(y*x)) \in L_A(F,\beta_F)$ and $a*(b*(b*a)) \in L_A(IF,\beta_{IF})$. Thus $A_F(x*y) = \beta_F \geq A_F(x*(y*(y*x)))$ and $A_{IF}(a*b) = \beta_{IF} \geq A_{IF}((a*b)*b)$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Theorem 6. *Every commutative generalized neutrosophic ideal can be realized as level neutrosophic commutative ideals of some commutative generalized neutrosophic ideal of X.*

Proof. Given a commutative ideal C of X, define a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ as follows

$$A_T(x) = \begin{cases} \alpha_T & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases} \qquad A_{IT}(x) = \begin{cases} \alpha_{IT} & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{IF}(x) = \begin{cases} \beta_{IF} & \text{if } x \in C, \\ 1 & \text{otherwise,} \end{cases} \qquad A_F(x) = \begin{cases} \beta_F & \text{if } x \in C, \\ 1 & \text{otherwise,} \end{cases}$$

where $\alpha_T, \alpha_{IT} \in (0,1]$ and $\beta_F, \beta_{IF} \in [0,1)$. Let $x,y,z \in X$. If $(x*y)*z \in C$ and $z \in C$, then $x*(y*(y*x)) \in C$. Thus

$$\begin{aligned}
A_T(x*(y*(y*x))) &= \alpha_T = \min\{A_T((x*y)*z), A_T(z)\}, \\
A_{IT}(x*(y*(y*x))) &= \alpha_{IT} = \min\{A_{IT}((x*y)*z), A_{IT}(z)\}, \\
A_{IF}(x*(y*(y*x))) &= \beta_{IF} = \max\{A_{IF}((x*y)*z), A_{IF}(z)\}, \\
A_F(x*(y*(y*x))) &= \beta_F = \max\{A_F((x*y)*z), A_F(z)\}.
\end{aligned}$$

Assume that $(x*y)*z \notin C$ and $z \notin C$. Then $A_T((x*y)*z) = 0$, $A_T(z) = 0$, $A_{IT}((x*y)*z) = 0$, $A_{IT}(z) = 0$, $A_{IF}((x*y)*z) = 1$, $A_{IF}(z) = 1$, and $A_F((x*y)*z) = 1$, $A_F(z) = 1$. It follows that

$$\begin{aligned}
A_T(x*(y*(y*x))) &\geq \min\{A_T((x*y)*z), A_T(z)\}, \\
A_{IT}(x*(y*(y*x))) &\geq \min\{A_{IT}((x*y)*z), A_{IT}(z)\}, \\
A_{IF}(x*(y*(y*x))) &\leq \max\{A_{IF}((x*y)*z), A_{IF}(z)\}, \\
A_F(x*(y*(y*x))) &\leq \max\{A_F((x*y)*z), A_F(z)\}.
\end{aligned}$$

If exactly one of $(x*y)*z$ and z belongs to C, then exactly one of $A_T((x*y)*z)$ and $A_T(z)$ is equal to 0; exactly one of $A_{IT}((x*y)*z)$ and $A_{IT}(z)$ is equal to 0; exactly one of $A_F((x*y)*z)$ and $A_F(z)$ is equal to 1 and exactly one of $A_{IF}((x*y)*z)$ and $A_{IF}(z)$ is equal to 1. Hence

$$\begin{aligned}
A_T(x*(y*(y*x))) &\geq \min\{A_T((x*y)*z), A_T(z)\}, \\
A_{IT}(x*(y*(y*x))) &\geq \min\{A_{IT}((x*y)*z), A_{IT}(z)\}, \\
A_{IF}(x*(y*(y*x))) &\leq \max\{A_{IF}((x*y)*z), A_{IF}(z)\}, \\
A_F(x*(y*(y*x))) &\leq \max\{A_F((x*y)*z), A_F(z)\}.
\end{aligned}$$

It is clear that $A_T(0) \geq A_T(x)$, $A_{IT}(0) \geq A_{IT}(x)$, $A_{IF}(0) \leq A_{IF}(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.

Obviously, $U_A(T, \alpha_T) = C$, $U_A(IT, \alpha_{IT}) = C$, $L_A(F, \beta_F) = C$ and $L_A(IF, \beta_{IF}) = C$. This completes the proof. □

Theorem 7. *Let $\{C_t \mid t \in \Lambda\}$ be a collection of commutative ideals of X such that*

(1) $X = \bigcup_{t\in\Lambda} C_t$,
(2) $(\forall s, t \in \Lambda)\,(s > t \iff C_s \subset C_t)$

where Λ is any index set. Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X given by

$$(\forall x \in X)\begin{pmatrix} A_T(x) = \sup\{t \in \Lambda \mid x \in C_t\} = A_{IT}(x) \\ A_{IF}(x) = \inf\{t \in \Lambda \mid x \in C_t\} = A_F(x) \end{pmatrix}. \tag{17}$$

Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.

Proof. According to Theorem 5, it is sufficient to show that $U(T, t)$, $U(IT, t)$, $L(F, s)$ and $L(IF, s)$ are commutative ideals of X for every $t \in [0, A_T(0) = A_{IT}(0)]$ and $s \in [A_{IF}(0) = A_F(0), 1]$. In order to prove $U(T, t)$ and $U(IT, t)$ are commutative ideals of X, we consider two cases:

(i) $t = \sup\{q \in \Lambda \mid q < t\}$,
(ii) $t \neq \sup\{q \in \Lambda \mid q < t\}$.

For the first case, we have

$$x \in U(T, t) \iff (\forall q < t)(x \in C_q) \iff x \in \bigcap_{q<t} C_q,$$
$$x \in U(IT, t) \iff (\forall q < t)(x \in C_q) \iff x \in \bigcap_{q<t} C_q.$$

Hence $U(T, t) = \bigcap_{q<t} C_q = U(IT, t)$, and so $U(T, t)$ and $U(IT, t)$ are commutative ideals of X. For the second case, we claim that $U(T, t) = \bigcup_{q\geq t} C_q = U(IT, t)$. If $x \in \bigcup_{q\geq t} C_q$, then $x \in C_q$ for some $q \geq t$. It follows that $A_{IT}(x) = A_T(x) \geq q \geq t$ and so that $x \in U(T, t)$ and $x \in U(IT, t)$. This shows that $\bigcup_{q\geq t} C_q \subseteq U(T, t)$ and $\bigcup_{q\geq t} C_q \subseteq U(IT, t)$. Now, suppose $x \notin \bigcup_{q\geq t} C_q$. Then $x \notin C_q$, $\forall q \geq t$. Since $t \neq \sup\{q \in \Lambda \mid q < t\}$, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \Lambda = \emptyset$. Thus $x \notin C_q$, $\forall q > t - \varepsilon$, this means that if $x \in C_q$, then $q \leq t - \varepsilon$. So $A_{IT}(x) = A_T(x) \leq t - \varepsilon < t$, and so $x \notin U(T, t) = U(IT, t)$. Therefore $U(T, t) = U(IT, t) \subseteq \bigcup_{q\geq t} C_q$. Consequently, $U(T, t) = U(IT, t) = \bigcup_{q\geq t} C_q$ which is a commutative ideal of X. Next we show that $L(F, s)$ and $L(IF, s)$ are commutative ideals of X. We consider two cases as follows:

(iii) $s = \inf\{r \in \Lambda \mid s < r\}$,
(iv) $s \neq \inf\{r \in \Lambda \mid s < r\}$.

Case (iii) implies that

$$x \in L(IF, s) \iff (\forall s < r)(x \in C_r) \iff x \in \bigcap_{s<r} C_r,$$
$$x \in U(F, s) \iff (\forall s < r)(x \in C_r) \iff x \in \bigcap_{s<r} C_r.$$

It follows that $L(IF, s) = L(F, s) = \bigcap_{s<r} C_r$, which is a commutative ideal of X. Case (iv) induces $(s, s + \varepsilon) \cap \Lambda = \emptyset$ for some $\varepsilon > 0$. If $x \in \bigcup_{s\geq r} C_r$, then $x \in C_r$ for some $r \leq s$, and so $A_{IF}(x) = A_F(x) \leq r \leq s$, that is, $x \in L(IF, s)$ and $x \in L(F, s)$. Hence $\bigcup_{s\geq r} C_r \subseteq L(IF, s) = L(F, s)$. If $x \notin \bigcup_{s\geq r} C_r$, then $x \notin C_r$

for all $r \leq s$ which implies that $x \notin C_r$ for all $r \leq s+\varepsilon$, that is, if $x \in C_r$ then $r \geq s+\varepsilon$. Hence $A_{IF}(x) = A_F(x) \geq s+\varepsilon > s$, and so $x \notin L(A_{IF}, s) = L(A_F, s)$. Hence $L(A_{IF}, s) = L(A_F, s) = \bigcup_{s \geq r} C_r$ which is a commutative ideal of X. This completes the proof. □

Assume thta $f : X \to Y$ is a homomorphism of BCK/BCI-algebras ([7]). For any GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y, we define a new GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X, which is called the *induced GNS*, by

$$(\forall x \in X) \left(\begin{array}{l} A^f_T(x) = A_T(f(x)),\ A^f_{IT}(x) = A_{IT}(f(x)) \\ A^f_{IF}(x) = A_{IF}(f(x)),\ A^f_F(x) = A_F(f(x)) \end{array} \right). \tag{18}$$

Lemma 5 ([15])**.** *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y is a generalized neutrosophic ideal of Y, then the new GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X.*

Theorem 8. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y is a commutative generalized neutrosophic ideal of Y, then the new GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a commutative generalized neutrosophic ideal of X.*

Proof. Suppose that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y by Theorem 1, and so $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a generalized neutrosophic ideal of Y by Lemma 5. For any $x, y \in X$, we have

$$\begin{aligned} A^f_T(x*(y*(y*x))) &= A_T(f(x*(y*(y*x)))) \\ &= A_T(f(x)*(f(y)*(f(y)*f(x)))) \\ &\geq A_T(f(x)*f(y)) \\ &= A_T(f(x*y)) = A^f_T(x*y), \end{aligned}$$

$$\begin{aligned} A^f_{IT}(x*(y*(y*x))) &= A_{IT}(f(x*(y*(y*x)))) \\ &= A_{IT}(f(x)*(f(y)*(f(y)*f(x)))) \\ &\geq A_{IT}(f(x)*f(y)) \\ &= A_{IT}(f(x*y)) = A^f_{IT}(x*y), \end{aligned}$$

$$\begin{aligned} A^f_{IF}(x*(y*(y*x))) &= A_{IF}(f(x*(y*(y*x)))) \\ &= A_{IF}(f(x)*(f(y)*(f(y)*f(x)))) \\ &\leq A_{IF}(f(x)*f(y)) \\ &= A_{IF}(f(x*y)) = A^f_{IF}(x*y), \end{aligned}$$

and

$$\begin{aligned} A^f_F(x*(y*(y*x))) &= A_F(f(x*(y*(y*x)))) \\ &= A_F(f(x)*(f(y)*(f(y)*f(x)))) \\ &\leq A_F(f(x)*f(y)) \\ &= A_F(f(x*y)) = A^f_F(x*y). \end{aligned}$$

Therefore $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 6 ([15]). *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in Y. If the induced GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X, then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y.*

Theorem 9. *Assume thta $f : X \to Y$ is an onto homomorphism of BCK-algebras and $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in Y. If the induced GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a commutative generalized neutrosophic ideal of X, then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y.*

Proof. Suppose that $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a commutative generalized neutrosophic ideal of X. Then $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a generalized neutrosophic ideal of X, and thus $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y. For any $a, b, c \in Y$, there exist $x, y, z \in X$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$. Thus,

$$\begin{aligned} A_T(a*(b*(b*a))) &= A_T(f(x)*(f(y)*(f(y)*f(x)))) = A_T(f(x*(y*(y*x)))) \\ &= A^f_T(x*(y*(y*x))) \geq A^f_T(x*y) \\ &= A_T(f(x)*f(y)) = A_T(a*b), \end{aligned}$$

$$\begin{aligned} A_{IT}(a*(b*(b*a))) &= A_{IT}(f(x)*(f(y)*(f(y)*f(x)))) = A_{IT}(f(x*(y*(y*x)))) \\ &= A^f_{IT}(x*(y*(y*x))) \geq A^f_{IT}(x*y) \\ &= A_{IT}(f(x)*f(y)) = A_{IT}(a*b), \end{aligned}$$

$$\begin{aligned} A_{IF}(a*(b*(b*a))) &= A_{IF}(f(x)*(f(y)*(f(y)*f(x)))) = A_{IF}(f(x*(y*(y*x)))) \\ &= A^f_{IF}(x*(y*(y*x))) \leq A^f_{IF}(x*y) \\ &= A_{IF}(f(x)*f(y)) = A_{IF}(a*b), \end{aligned}$$

and

$$\begin{aligned} A_F(a*(b*(b*a))) &= A_F(f(x)*(f(y)*(f(y)*f(x)))) = A_F(f(x*(y*(y*x)))) \\ &= A^f_F(x*(y*(y*x))) \leq A^f_F(x*y) \\ &= A_F(f(x)*f(y)) = A_F(a*b). \end{aligned}$$

It follows from Theorem 2 that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y. □

Let $CGNI(X)$ denote the set of all commutative generalized neutrosophic ideals of X and $t \in [0, 1]$. Define binary relations U^t_T, U^t_{IT}, L^t_F and L^t_{IF} on $CGNI(X)$ as follows:

$$\begin{aligned} &(A, B) \in U^t_T \Leftrightarrow U_A(T, t) = U_B(T, t),\ (A, B) \in U^t_{IT} \Leftrightarrow U_A(IT, t) = U_B(IT, t), \\ &(A, B) \in L^t_F \Leftrightarrow L_A(F, t) = L_B(F, t),\ (A, B) \in L^t_{IF} \Leftrightarrow L_A(IF, t) = L_B(IF, t) \end{aligned} \tag{19}$$

for $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_{IF}, B_F)$ in $CGNI(X)$. Then clearly U^t_T, U^t_{IT}, L^t_F and L^t_{IF} are equivalence relations on $CGNI(X)$. For any $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$, let $[A]_{U^t_T}$ (resp., $[A]_{U^t_{IT}}$, $[A]_{L^t_F}$ and $[A]_{L^t_{IF}}$) denote the equivalence class of $A = (A_T, A_{IT}, A_{IF}, A_F)$ modulo U^t_T (resp, U^t_{IT}, L^t_F and L^t_{IF}). Denote by $CGNI(X)/U^t_T$ (resp., $CGNI(X)/U^t_{IT}$, $CGNI(X)/L^t_F$ and $CGNI(X)/L^t_{IF}$) the system of all equivalence classes modulo U^t_T (resp, U^t_{IT}, L^t_F and L^t_{IF}); so

$$CGNI(X)/U^t_T = \{[A]_{U^t_T} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{20}$$

$$CGNI(X)/U^t_{IT} = \{[A]_{U^t_{IT}} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{21}$$

$$CGNI(X)/L^t_F = \{[A]_{L^t_F} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{22}$$

and

$$CGNI(X)/L^t_{IF} = \{[A]_{L^t_{IF}} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{23}$$

respectively. Let $CI(X)$ denote the family of all commutative ideals of X and let $t \in [0,1]$. Define maps

$$f_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto U_A(T,t), \tag{24}$$

$$g_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto U_A(IT,t), \tag{25}$$

$$\alpha_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto L_A(F,t), \tag{26}$$

and

$$\beta_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto L_A(IF,t). \tag{27}$$

Then the definitions of f_t, g_t, α_t and β_t are well.

Theorem 10. *Suppose $t \in (0,1)$, the definitions of f_t, g_t, α_t and β_t are as above. Then the maps f_t, g_t, α_t and β_t are surjective from $CGNI(X)$ to $CI(X) \cup \{\emptyset\}$.*

Proof. Assume $t \in (0,1)$.We know that $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F)$ is in $CGNI(X)$ where $\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}$ and $\mathbf{1}_F$ are constant functions on X defined by $\mathbf{0}_T(x) = 0$, $\mathbf{0}_{IT}(x) = 0$, $\mathbf{1}_{IF}(x) = 1$ and $\mathbf{1}_F(x) = 1$ for all $x \in X$. Obviously $f_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t)$, $g_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t)$, $\alpha_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(F,t)$ and $\beta_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(IF,t)$ are empty. Let $G(\neq \emptyset) \in CGNI(X)$, and consider functions:

$$G_T : X \to [0,1],\ G \mapsto \begin{cases} 1 & \text{if } x \in G\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_{IT} : X \to [0,1],\ G \mapsto \begin{cases} 1 & \text{if } x \in G\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_F : X \to [0,1],\ G \mapsto \begin{cases} 0 & \text{if } x \in G\,, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$G_{IF} : X \to [0,1],\ G \mapsto \begin{cases} 0 & \text{if } x \in G\,, \\ 1 & \text{otherwise.} \end{cases}$$

Then $G_\sim = (G_T, G_{IT}, G_{IF}, G_F)$ is a commutative generalized neutrosophic ideal of X, and $f_t(G_\sim) = U_{G_\sim}(T,t) = G$, $g_t(G_\sim) = U_{G_\sim}(IT,t) = G$, $\alpha_t(G_\sim) = L_{G_\sim}(F,t) = G$ and $\beta_t(G_\sim) = L_{G_\sim}(IF,t) = G$. Therefore f_t, g_t, α_t and β_t are surjective. □

Theorem 11. *The quotient sets*

$$CGNI(X)/U^t_T,\ CGNI(X)/U^t_{IT},\ CGNI(X)/L^t_F \text{ and } CGNI(X)/L^t_{IF}$$

are equipotent to $CI(X) \cup \{\emptyset\}$.

Proof. For $t \in (0,1)$, let f_t^* (resp, g_t^*, α_t^* and β_t^*) be a map from $CGNI(X)/U_T^t$ (resp., $CGNI(X)/U_{IT}^t$, $CGNI(X)/L_F^t$ and $CGNI(X)/L_{IF}^t$) to $CI(X) \cup \{\emptyset\}$ defined by $f_t^*\left([A]_{U_T^t}\right) = f_t(A)$ (resp., $g_t^*\left([A]_{U_{IT}^t}\right) = g_t(A)$, $\alpha_t^*\left([A]_{L_F^t}\right) = \alpha_t(A)$ and $\beta_t^*\left([A]_{L_{IF}^t}\right) = \beta_t(A)$) for all $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$. If $U_A(T,t) = U_B(T,t)$, $U_A(IT,t) = U_B(IT,t)$, $L_A(F,t) = L_B(F,t)$ and $L_A(IF,t) = L_B(IF,t)$ for $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_F, B_{IF})$ in $CGNI(X)$, then $(A,B) \in U_T^t$, $(A,B) \in U_{IT}^t$, $(A,B) \in L_F^t$ and $(A,B) \in L_{IF}^t$. Hence $[A]_{U_T^t} = [B]_{U_T^t}$, $[A]_{U_{IT}^t} = [B]_{U_{IT}^t}$, $[A]_{L_F^t} = [B]_{L_F^t}$ and $[A]_{L_{IF}^t} = [B]_{L_{IF}^t}$. Therefore f_t^* (resp, g_t^*, α_t^* and β_t^*) is injective. Now let $G(\neq \emptyset) \in CGNI(X)$. For $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$, we have

$$f_t^*\left([G_\sim]_{U_T^t}\right) = f_t(G_\sim) = U_{G_\sim}(T,t) = G,$$

$$g_t^*\left([G_\sim]_{U_{IT}^t}\right) = g_t(G_\sim) = U_{G_\sim}(IT,t) = G,$$

$$\alpha_t^*\left([G_\sim]_{L_F^t}\right) = \alpha_t(G_\sim) = L_{G_\sim}(F,t) = G$$

and

$$\beta_t^*\left([G_\sim]_{L_{IF}^t}\right) = \beta_t(G_\sim) = L_{G_\sim}(IF,t) = G.$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$, we have

$$f_t^*\left([\mathbf{0}_\sim]_{U_T^t}\right) = f_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t) = \emptyset,$$

$$g_t^*\left([\mathbf{0}_\sim]_{U_{IT}^t}\right) = g_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t) = \emptyset,$$

$$\alpha_t^*\left([\mathbf{0}_\sim]_{L_F^t}\right) = \alpha_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(F,t) = \emptyset$$

and

$$\beta_t^*\left([\mathbf{0}_\sim]_{L_{IF}^t}\right) = \beta_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(IF,t) = \emptyset.$$

Therefore, f_t^* (resp, g_t^*, α_t^* and β_t^*) is surjective. □

$\forall t \in [0,1]$, define another relations R^t and Q^t on $CGNI(X)$ as follows:

$$(A,B) \in R^t \Leftrightarrow U_A(T,t) \cap L_A(F,t) = U_B(T,t) \cap L_B(F,t)$$

and

$$(A,B) \in Q^t \Leftrightarrow U_A(IT,t) \cap L_A(IF,t) = U_B(IT,t) \cap L_B(IF,t)$$

for any $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_{IF}, B_F)$ in $CGNI(X)$. Then R^t and Q^t are equivalence relations on $CGNI(X)$.

Theorem 12. *Suppose $t \in (0,1)$, consider the following maps*

$$\varphi_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto f_t(A) \cap \alpha_t(A), \tag{28}$$

and

$$\psi_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto g_t(A) \cap \beta_t(A) \tag{29}$$

for each $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$. Then φ_t and ψ_t are surjective.

Proof. Assume $t \in (0,1)$. For $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$,

$$\varphi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap \alpha_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t) \cap L_{\mathbf{0}_\sim}(F,t) = \emptyset$$

and

$$\psi_t(\mathbf{0}_\sim) = g_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t) \cap L_{\mathbf{0}_\sim}(IF,t) = \emptyset.$$

For any $G \in CI(X)$, there exists $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$ such that

$$\varphi_t(G_\sim) = f_t(G_\sim) \cap \alpha_t(G_\sim) = U_{G_\sim}(T,t) \cap L_{G_\sim}(F,t) = G$$

and

$$\psi_t(G_\sim) = g_t(G_\sim) \cap \beta_t(G_\sim) = U_{G_\sim}(IT,t) \cap L_{G_\sim}(IF,t) = G.$$

Therefore φ_t and ψ_t are surjective. □

Theorem 13. *For any $t \in (0,1)$, the quotient sets $CGNI(X)/R^t$ and $CGNI(X)/Q^t$ are equipotent to $CI(X) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$ and define maps

$$\varphi_t^* : CGNI(X)/R^t \to CI(X) \cup \{\emptyset\},\ [A]_{R^t} \mapsto \varphi_t(A)$$

and

$$\psi_t^* : CGNI(X)/Q^t \to CI(X) \cup \{\emptyset\},\ [A]_{Q^t} \mapsto \psi_t(A).$$

If $\varphi_t^*([A]_{R^t}) = \varphi_t^*([B]_{R^t})$ and $\psi_t^*\left([A]_{Q^t}\right) = \psi_t^*\left([B]_{Q^t}\right)$ for all $[A]_{R^t}, [B]_{R^t} \in CGNI(X)/R^t$ and $[A]_{Q^t}, [B]_{Q^t} \in CGNI(X)/Q^t$, then $f_t(A) \cap \alpha_t(A) = f_t(B) \cap \alpha_t(B)$ and $g_t(A) \cap \beta_t(A) = g_t(B) \cap \beta_t(B)$, that is, $U_A(T,t) \cap L_A(F,t) = U_B(T,t) \cap L_B(F,t)$ and $U_A(IT,t) \cap L_A(IF,t) = U_B(IT,t) \cap L_B(IF,t)$. Hence $(A,B) \in R^t$, $(A,B) \in Q^t$. So $[A]_{R^t} = [B]_{R^t}$, $[A]_{Q^t} = [B]_{Q^t}$, which shows that φ_t^* and ψ_t^* are injective. For $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$,

$$\varphi_t^*([\mathbf{0}_\sim]_{R^t}) = \varphi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap \alpha_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(\mathbf{0}_T,t) \cap L_{\mathbf{0}_\sim}(\mathbf{1}_F,t) = \emptyset$$

and

$$\psi_t^*\left([\mathbf{0}_\sim]_{Q^t}\right) = \psi_t(\mathbf{0}_\sim) = g_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(\mathbf{0}_{IT},t) \cap L_{\mathbf{0}_\sim}(\mathbf{1}_{IF},t) = \emptyset.$$

If $G \in CI(X)$, then $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$, and so

$$\varphi_t^*([G_\sim]_{R^t}) = \varphi_t(G_\sim) = f_t(G_\sim) \cap \alpha_t(G_\sim) = U_{G_\sim}(G_T,t) \cap L_{G_\sim}(G_F,t) = G$$

and

$$\psi_t^*\left([G_\sim]_{Q^t}\right) = \psi_t(G_\sim) = g_t(G_\sim) \cap \beta_t(G_\sim) = U_{G_\sim}(G_{IT}, t) \cap L_{G_\sim}(G_{IF}, t) = G.$$

Hence φ_t^* and ψ_t^* are surjective, and the proof is complete. □

4. Conclusions

Based on the theory of generalized neutrosophic sets, we proposed the new concept of commutative generalized neutrosophic ideal in a BCK-algebra, and obtained some characterizations. Moreover, we investigated some homomorphism properties related to commutative generalized neutrosophic ideals.

The research ideas of this paper can be extended to a wide range of logical algebraic systems such as pseudo-BCI algebras (see [1,16]). At the same time, the concept of generalized neutrosophic set involved in this paper can be further studied according to the thought in [11,17], which will be the direction of our next research work.

Funding: This research was funded by the National Natural Science Foundation of China grant number 61573240.

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Neutrosophic Quadruple *BCK*/*BCI*-Algebras

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Young Bae Jun, Seok-Zun Song, Florentin Smarandache, Hashem Bordbar (2018). Neutrosophic Quadruple BCK/BCI-Algebras. *Axioms* 7, 41. DOI: 10.3390/axioms7020041

Abstract: The notion of a neutrosophic quadruple BCK/BCI-number is considered, and a neutrosophic quadruple BCK/BCI-algebra, which consists of neutrosophic quadruple BCK/BCI-numbers, is constructed. Several properties are investigated, and a (positive implicative) ideal in a neutrosophic quadruple BCK-algebra and a closed ideal in a neutrosophic quadruple BCI-algebra are studied. Given subsets A and B of a BCK/BCI-algebra, the set $NQ(A, B)$, which consists of neutrosophic quadruple BCK/BCI-numbers with a condition, is established. Conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra are provided, and conditions for the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra are given. An example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra is provided, and conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra are then discussed.

Keywords: neutrosophic quadruple BCK/BCI-number; neutrosophic quadruple BCK/BCI-algebra; neutrosophic quadruple subalgebra; (positive implicative) neutrosophic quadruple ideal

1. Introduction

The notion of a neutrosophic set was developed by Smarandache [1–3] and is a more general platform that extends the notions of classic sets, (intuitionistic) fuzzy sets, and interval valued (intuitionistic) fuzzy sets. Neutrosophic set theory is applied to a different field (see [4–8]). Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in [9–16]. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [17,18].

In this paper, we will use neutrosophic quadruple numbers based on a set and construct neutrosophic quadruple BCK/BCI-algebras. We investigate several properties and consider ideals and positive implicative ideals in neutrosophic quadruple BCK-algebra, and closed ideals in neutrosophic quadruple BCI-algebra. Given subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, we consider sets $NQ(A, B)$, which consist of neutrosophic quadruple BCK/BCI-numbers with a condition. We provide conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra and for the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. We give an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and we then consider conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by Iséki (see [19,20]).

By a BCI-algebra, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$;
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$;
(III) $(\forall x \in X)\ (x * x = 0)$;
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\ (x * 0 = x) \tag{1}$$
$$(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x) \tag{2}$$
$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y) \tag{3}$$
$$(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y) \tag{4}$$

where $x \leq y$ if and only if $x * y = 0$. Any BCI-algebra X satisfies the following conditions (see [21]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{5}$$
$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)). \tag{6}$$

A BCK-algebra X is said to be *positive implicative* if the following assertion is valid.

$$(\forall x, y, z \in X)\ ((x * z) * (y * z) = (x * y) * z). \tag{7}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0 \in I, \tag{8}$$
$$(\forall x \in X)\ (\forall y \in I)\ (x * y \in I\ \Rightarrow\ x \in I). \tag{9}$$

A subset I of a BCI-algebra X is called a *closed ideal* (see [21]) of X if it is an ideal of X which satisfies

$$(\forall x \in X)(x \in I\ \Rightarrow\ 0 * x \in I). \tag{10}$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* (see [22]) of X if it satisfies (8) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\ y * z \in I\ \Rightarrow\ x * z \in I). \tag{11}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [22]). Note also that a BCK-algebra X is positive implicative if and only if every ideal of X is positive implicative (see [22]).

We refer the reader to the books [21,22] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3. Neutrosophic Quadruple *BCK/BCI*-Algebras

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Definition 1. *Let X be a set. A neutrosophic quadruple X-number is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in X$ and T, I, F have their usual neutrosophic logic meanings.*

The set of all neutrosophic quadruple X-numbers is denoted by $NQ(X)$, that is,

$$NQ(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $NQ(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let X be a BCK/BCI-algebra. We define a binary operation $\odot$ on $NQ(X)$ by

$$(a, xT, yI, zF) \odot (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in NQ(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(X)$ as follows:

$$\begin{aligned} \tilde{x} \ll \tilde{y} &\Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4 \\ \tilde{x} = \tilde{y} &\Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4 \end{aligned}$$

for all $\tilde{x}, \tilde{y} \in NQ(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $NQ(X)$.

Theorem 1. *If X is a BCK/BCI-algebra, then $(NQ(X); \odot, \tilde{0})$ is a BCK/BCI-algebra.*

Proof. Let X be a BCI-algebra. For any $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$, we have

$$\begin{aligned} (\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{z}) &= (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\ &\qquad \odot (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \\ &= ((x_1 * y_1) * (x_1 * z_1), ((x_2 * y_2) * (x_2 * z_2))T, \\ &\qquad ((x_3 * y_3) * (x_3 * z_3))I, ((x_4 * y_4) * (x_4 * z_4))T) \\ &\ll (z_1 * y_1, (z_2 * y_2)T, (z_3 * y_3)I, (z_4 * y_4)F) \\ &= \tilde{z} \odot \tilde{y} \end{aligned}$$

$$\begin{aligned} \tilde{x} \odot (\tilde{x} \odot \tilde{y}) &= (x_1, x_2T, x_3I, x_4F) \odot (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\ &= (x_1 * (x_1 * y_1), (x_2 * (x_2 * y_2))T, (x_3 * (x_3 * y_3))I, (x_4 * (x_4 * y_4))F) \\ &\ll (y_1, y_2T, y_3I, y_4F) \\ &= \tilde{y} \end{aligned}$$

$$\begin{aligned} \tilde{x} \odot \tilde{x} &= (x_1, x_2T, x_3I, x_4F) \odot (x_1, x_2T, x_3I, x_4F) \\ &= (x_1 * x_1, (x_2 * x_2)T, (x_3 * x_3)I, (x_4 * x_4)F) \\ &= (0, 0T, 0I, 0F) = \tilde{0}. \end{aligned}$$

Assume that $\tilde{x} \odot \tilde{y} = \tilde{0}$ and $\tilde{y} \odot \tilde{x} = \tilde{0}$. Then

$$(x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) = (0, 0T, 0I, 0F)$$

and

$$(y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) = (0, 0T, 0I, 0F).$$

It follows that $x_1 * y_1 = 0 = y_1 * x_1$, $x_2 * y_2 = 0 = y_2 * x_2$, $x_3 * y_3 = 0 = y_3 * x_3$ and $x_4 * y_4 = 0 = y_4 * x_4$. Hence, $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, and $x_4 = y_4$, which implies that

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) = (y_1, y_2T, y_3I, y_4F) = \tilde{y}.$$

Therefore, we know that $(NQ(X); \odot, \tilde{0})$ is a BCI-algebra. We call it the *neutrosophic quadruple BCI-algebra*. Moreover, if X is a BCK-algebra, then we have

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) = (0, 0T, 0I, 0F) = \tilde{0}.$$

Hence, $(NQ(X); \odot, \tilde{0})$ is a BCK-algebra. We call it the *neutrosophic quadruple BCK-algebra*. $\square$

Example 1. *If $X = \{0, a\}$, then the neutrosophic quadruple set $NQ(X)$ is given as follows:*

$$NQ(X) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, aF)$, $\tilde{2} = (0, 0T, aI, 0F)$, $\tilde{3} = (0, 0T, aI, aF)$,
$\tilde{4} = (0, aT, 0I, 0F)$, $\tilde{5} = (0, aT, 0I, aF)$, $\tilde{6} = (0, aT, aI, 0F)$, $\tilde{7} = (0, aT, aI, aF)$,
$\tilde{8} = (a, 0T, 0I, 0F)$, $\tilde{9} = (a, 0T, 0I, aF)$, $\tilde{10} = (a, 0T, aI, 0F)$, $\tilde{11} = (a, 0T, aI, aF)$,
$\tilde{12} = (a, aT, 0I, 0F)$, $\tilde{13} = (a, aT, 0I, aF)$, $\tilde{14} = (a, aT, aI, 0F)$, *and* $\tilde{15} = (a, aT, aI, aF)$.

Consider a BCK-algebra $X = \{0, a\}$ with the binary operation $*$, which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	$\boldsymbol{a}$
0	0	0
a	a	0

Then $(NQ(X), \odot, \tilde{0})$ is a BCK-algebra in which the operation $\odot$ is given by Table 2.

Table 2. Cayley table for the binary operation "$\odot$".

$\odot$	$\tilde{0}$	$\tilde{1}$	$\tilde{2}$	$\tilde{3}$	$\tilde{4}$	$\tilde{5}$	$\tilde{6}$	$\tilde{7}$	$\tilde{8}$	$\tilde{9}$	$\tilde{10}$	$\tilde{11}$	$\tilde{12}$	$\tilde{13}$	$\tilde{14}$	$\tilde{15}$
$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{1}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{2}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{3}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{5}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{6}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{7}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{9}$	$\tilde{9}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{10}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$

Table 2. *Cont.*

$\odot$	$\tilde{0}$	$\tilde{1}$	$\tilde{2}$	$\tilde{3}$	$\tilde{4}$	$\tilde{5}$	$\tilde{6}$	$\tilde{7}$	$\tilde{8}$	$\tilde{9}$	$\tilde{10}$	$\tilde{11}$	$\tilde{12}$	$\tilde{13}$	$\tilde{14}$	$\tilde{15}$
$\tilde{11}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{13}$	$\tilde{13}$	$\tilde{12}$	$\tilde{13}$	$\tilde{12}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{14}$	$\tilde{14}$	$\tilde{14}$	$\tilde{12}$	$\tilde{12}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{15}$	$\tilde{15}$	$\tilde{14}$	$\tilde{13}$	$\tilde{12}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$

Theorem 2. *The neutrosophic quadruple set $NQ(X)$ based on a positive implicative BCK-algebra X is a positive implicative BCK-algebra.*

Proof. Let X be a positive implicative BCK-algebra. Then X is a BCK-algebra, so $(NQ(X); \odot, \tilde{0})$ is a BCK-algebra by Theorem 1. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Then

$$(x_i * z_i) * (y_i * z_i) = (x_i * y_i) * z_i$$

for all $i = 1, 2, 3, 4$ since $x_i, y_i, z_i \in X$ and X is a positive implicative BCK-algebra. Hence, $(\tilde{x} \odot \tilde{z}) \odot (\tilde{y} * \tilde{z}) = (\tilde{x} \odot \tilde{y}) \odot \tilde{z}$; therefore, $NQ(X)$ based on a positive implicative BCK-algebra X is a positive implicative BCK-algebra. □

Proposition 1. *The neutrosophic quadruple set $NQ(X)$ based on a positive implicative BCK-algebra X satisfies the following assertions.*

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))\ (\tilde{x} \odot \tilde{y} \ll \tilde{z} \Rightarrow \tilde{x} \odot \tilde{z} \ll \tilde{y} \odot \tilde{z}) \tag{12}$$

$$(\forall \tilde{x}, \tilde{y} \in NQ(X))\ (\tilde{x} \odot \tilde{y} \ll \tilde{y} \Rightarrow \tilde{x} \ll \tilde{y}). \tag{13}$$

Proof. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. If $\tilde{x} \odot \tilde{y} \ll \tilde{z}$, then

$$\tilde{0} = (\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}),$$

so $\tilde{x} \odot \tilde{z} \ll \tilde{y} \odot \tilde{z}$. Assume that $\tilde{x} \odot \tilde{y} \ll \tilde{y}$. Using Equation (12) implies that

$$\tilde{x} \odot \tilde{y} \ll \tilde{y} \odot \tilde{y} = \tilde{0},$$

so $\tilde{x} \odot \tilde{y} = \tilde{0}$, i.e., $\tilde{x} \ll \tilde{y}$. □

Let X be a BCK/BCI-algebra. Given $a, b \in X$ and subsets A and B of X, consider the sets

$$NQ(a, B) := \{(a, aT, yI, zF) \in NQ(X) \mid y, z \in B\}$$

$$NQ(A, b) := \{(a, xT, bI, bF) \in NQ(X) \mid a, x \in A\}$$

$$NQ(A, B) := \{(a, xT, yI, zF) \in NQ(X) \mid a, x \in A; y, z \in B\}$$

$$NQ(A^*, B) := \bigcup_{a \in A} NQ(a, B)$$

$$NQ(A, B^*) := \bigcup_{b \in B} NQ(A, b)$$

and

$$NQ(A \cup B) := NQ(A, 0) \cup NQ(0, B).$$

The set $NQ(A, A)$ is denoted by $NQ(A)$.

Proposition 2. *Let X be a BCK/BCI-algebra. Given $a, b \in X$ and subsets A and B of X, we have*

(1) $NQ(A^*, B)$ *and* $NQ(A, B^*)$ *are subsets of* $NQ(A, B)$.
(1) *If* $0 \in A \cap B$ *then* $NQ(A \cup B)$ *is a subset of* $NQ(A, B)$.

Proof. Straightforward. □

Let X be a BCK/BCI-algebra. Given $a, b \in X$ and subalgebras A and B of X, $NQ(a, B)$ and $NQ(A, b)$ may not be subalgebras of $NQ(X)$ since

$$(a, aT, x_3I, x_4F) \odot (a, aT, u_3I, v_4F) = (0, 0T, (x_3 * u_3)I, (x_4 * v_4)F) \notin NQ(a, B)$$

and

$$(x_1, x_2T, bI, bF) \odot (u_1, u_2T, bI, bF) = (x_1 * u_1, (x_2 * u_2)T, 0I, 0F) \notin NQ(A, b)$$

for $(a,\ aT,\ x_3I,\ x_4F) \in NQ(a, B)$, $(a,\ aT,\ u_3I,\ v_4F) \in NQ(a, B)$, $(x_1,\ x_2T,\ bI,\ bF) \in NQ(A, b)$, and $(u_1, u_2T, bI, bF) \in NQ(A, b)$.

Theorem 3. *If A and B are subalgebras of a BCK/BCI-algebra X, then the set $NQ(A, B)$ is a subalgebra of $NQ(X)$, which is called a neutrosophic quadruple subalgebra.*

Proof. Assume that A and B are subalgebras of a BCK/BCI-algebra X. Let $\tilde{x} = (x_1,\ x_2T,\ x_3I,\ x_4F)$ and $\tilde{y} = (y_1,\ y_2T,\ y_3I,\ y_4F)$ be elements of $NQ(A, B)$. Then $x_1, x_2, y_1, y_2 \in A$ and $x_3, x_4, y_3, y_4 \in B$, which implies that $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $NQ(A, B)$ is a subalgebra of $NQ(X)$. □

Theorem 4. *If A and B are ideals of a BCK/BCI-algebra X, then the set $NQ(A, B)$ is an ideal of $NQ(X)$, which is called a neutrosophic quadruple ideal.*

Proof. Assume that A and B are ideals of a BCK/BCI-algebra X. Obviously, $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1,\ x_2T,\ x_3I,\ x_4F)$ and $\tilde{y} = (y_1,\ y_2T,\ y_3I,\ y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$ and $x_4 * y_4 \in B$. Since $\tilde{y} \in NQ(A, B)$, we have $y_1, y_2 \in A$ and $y_3, y_4 \in B$. Since A and B are ideals of X, it follows that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence, $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$, so $NQ(A, B)$ is an ideal of $NQ(X)$. □

Since every ideal is a subalgebra in a BCK-algebra, we have the following corollary.

Corollary 1. *If A and B are ideals of a BCK-algebra X, then the set $NQ(A, B)$ is a subalgebra of $NQ(X)$.*

The following example shows that Corollary 1 is not true in a BCI-algebra.

Example 2. *Consider a BCI-algebra* $(\mathbb{Z}, -, 0)$. *If we take* $A = \mathbb{N}$ *and* $B = \mathbb{Z}$, *then* $NQ(A, B)$ *is an ideal of* $NQ(\mathbb{Z})$. *However, it is not a subalgebra of* $NQ(\mathbb{Z})$ *since*

$$(2, 3T, -5I, 6F) \odot (3, 5T, 6I, -7F) = (-1, -2T, -11I, 13F) \notin NQ(A, B)$$

for $(2, 3T, -5I, 6F), (3, 5T, 6I, -7F) \in NQ(A, B)$.

Theorem 5. *If A and B are closed ideals of a BCI-algebra X, then the set* $NQ(A, B)$ *is a closed ideal of* $NQ(X)$.

Proof. If A and B are closed ideals of a BCI-algebra X, then the set $NQ(A, B)$ is an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$. Then

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \in NQ(A, B)$$

since $0 * x_1, 0 * x_2 \in A$ and $0 * x_3, 0 * x_4 \in B$. Therefore, $NQ(A, B)$ is a closed ideal of $NQ(X)$. □

Since every closed ideal of a BCI-algebra X is a subalgebra of X, we have the following corollary.

Corollary 2. *If A and B are closed ideals of a BCI-algebra X, then the set* $NQ(A, B)$ *is a subalgebra of* $NQ(X)$.

In the following example, we know that there exist ideals A and B in a BCI-algebra X such that $NQ(A, B)$ is not a closed ideal of $NQ(X)$.

Example 3. *Consider BCI-algebras* $(Y, *, 0)$ *and* $(\mathbb{Z}, -, 0)$. *Then* $X = Y \times \mathbb{Z}$ *is a BCI-algebra (see [21]). Let* $A = Y \times \mathbb{N}$ *and* $B = \{0\} \times \mathbb{N}$. *Then A and B are ideals of X, so* $NQ(A, B)$ *is an ideal of* $NQ(X)$ *by Theorem 4. Let* $((0, 0), (0, 1)T, (0, 2)I, (0, 3)F) \in NQ(A, B)$. *Then*

$$\begin{aligned}
&((0, 0), (0, 0)T, (0, 0)I, (0, 0)F) \odot ((0, 0), (0, 1)T, (0, 2)I, (0, 3)F) \\
&= ((0, 0), (0, -1)T, (0, -2)I, (0, -3)F) \notin NQ(A, B).
\end{aligned}$$

Hence, $NQ(A, B)$ *is not a closed ideal of* $NQ(X)$.

We provide conditions wherethe set $NQ(A, B)$ is a closed ideal of $NQ(X)$.

Theorem 6. *Let A and B be ideals of a BCI-algebra X and let*

$$\Gamma := \{\tilde{a} \in NQ(X) \mid (\forall \tilde{x} \in NQ(X))(\tilde{x} \ll \tilde{a} \ \Rightarrow \ \tilde{x} = \tilde{a})\}.$$

Assume that, if $\Gamma \subseteq NQ(A, B)$, *then* $|\Gamma| < \infty$. *Then* $NQ(A, B)$ *is a closed ideal of* $NQ(X)$.

Proof. If A and B are ideals of X, then $NQ(A, B)$ is an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{a} = (a_1, a_2T, a_3I, a_4F) \in NQ(A, B)$. For any $n \in \mathbb{N}$, denote $n(\tilde{a}) := \tilde{0} \odot (\tilde{0} \odot \tilde{a})^n$. Then $n(\tilde{a}) \in \Gamma$ and

$$\begin{aligned}
n(\tilde{a}) &= (0 * (0 * a_1)^n, (0 * (0 * a_2)^n)T, (0 * (0 * a_3)^n)I, (0 * (0 * a_4)^n)F) \\
&= (0 * (0 * a_1^n), (0 * (0 * a_2^n))T, (0 * (0 * a_3^n))I, (0 * (0 * a_4^n))F) \\
&= \tilde{0} \odot (\tilde{0} \odot \tilde{a}^n).
\end{aligned}$$

Hence,

$$\begin{aligned}
n(\tilde{a}) \odot \tilde{a}^n &= (\tilde{0} \odot (\tilde{0} \odot \tilde{a}^n)) \odot \tilde{a}^n \\
&= (\tilde{0} \odot \tilde{a}^n) \odot (\tilde{0} \odot \tilde{a}^n) \\
&= \tilde{0} \in NQ(A, B),
\end{aligned}$$

so $n(\tilde{a}) \in NQ(A,B)$, since $\tilde{a} \in NQ(A,B)$, and $NQ(A,B)$ is an ideal of $NQ(X)$. Since $|\Gamma| < \infty$, it follows that $k \in \mathbb{N}$ such that $n(\tilde{a}) = (n+k)(\tilde{a})$, that is, $n(\tilde{a}) = n(\tilde{a}) \odot (\tilde{0} \odot \tilde{a})^k$, and thus

$$\begin{aligned} k(\tilde{a}) &= \tilde{0} \odot (\tilde{0} \odot \tilde{a})^k \\ &= (n(\tilde{a}) \odot (\tilde{0} \odot \tilde{a})^k) \odot n(\tilde{a}) \\ &= n(\tilde{a}) \odot n(\tilde{a}) = \tilde{0}, \end{aligned}$$

i.e., $(k-1)(\tilde{a}) \odot (\tilde{0} \odot \tilde{a}) = \tilde{0}$. Since $\tilde{0} \odot \tilde{a} \in \Gamma$, it follows that $\tilde{0} \odot \tilde{a} = (k-1)(\tilde{a}) \in NQ(A,B)$. Therefore, $NQ(A,B)$ is a closed ideal of $NQ(X)$. □

Theorem 7. *Given two elements a and b in a BCI-algebra X, let*

$$A_a := \{x \in X \mid a * x = a\} \text{ and } B_b := \{x \in X \mid b * x = b\}. \tag{14}$$

Then $NQ(A_a, B_b)$ is a closed ideal of $NQ(X)$.

Proof. Since $a * 0 = a$ and $b * 0 = b$, we have $0 \in A_a \cap B_b$. Thus, $\tilde{0} \in NQ(A_a, B_b)$. If $x \in A_a$ and $y \in B_b$, then

$$0 * x = (a * x) * a = a * a = 0 \text{ and } 0 * y = (b * y) * b = b * b = 0. \tag{15}$$

Let $x, y, c, d \in X$ be such that $x, y * x \in A_a$ and $c, d * c \in B_b$. Then

$$(a * y) * a = 0 * y = (0 * y) * 0 = (0 * y) * (0 * x) = 0 * (y * x) = 0$$

and

$$(b * d) * b = 0 * d = (0 * d) * 0 = (0 * d) * (0 * c) = 0 * (d * c) = 0,$$

that is, $a * y \leq a$ and $b * d \leq b$. On the other hand,

$$a = a * (y * x) = (a * x) * (y * x) \leq a * y$$

and

$$b = b * (d * c) = (b * c) * (d * c) \leq b * d.$$

Thus, $a * y = a$ and $b * d = b$, i.e., $y \in A_a$ and $d \in B_b$. Hence, A_a and B_b are ideals of X, and $NQ(A_a, B_b)$ is therefore an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A_a, B_b)$. Then $x_1, x_2 \in A_a$, and $x_3, x_4 \in B_b$. It follows from Equation (15) that $0 * x_1 = 0 \in A_a$, $0 * x_2 = 0 \in A_a$, $0 * x_3 = 0 \in B_b$, and $0 * x_4 = 0 \in B_b$. Hence,

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \in NQ(A_a, B_b).$$

Therefore, $NQ(A_a, B_b)$ is a closed ideal of $NQ(X)$. □

Proposition 3. *Let A and B be ideals of a BCK-algebra X. Then*

$$NQ(A) \cap NQ(B) = \{\tilde{0}\} \;\Leftrightarrow\; (\forall \tilde{x} \in NQ(A))(\forall \tilde{y} \in NQ(B))(\tilde{x} \odot \tilde{y} = \tilde{x}). \tag{16}$$

Proof. Note that $NQ(A)$ and $NQ(B)$ are ideals of $NQ(X)$. Assume that $NQ(A) \cap NQ(B) = \{\tilde{0}\}$. Let

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A) \text{ and } \tilde{y} = (y_1, y_2T, y_3I, y_4F) \in NQ(B).$$

Since $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \ll \tilde{x}$ and $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \ll \tilde{y}$, it follows that $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \in NQ(A) \cap NQ(B) = \{\tilde{0}\}$. Obviously, $(\tilde{x} \odot \tilde{y}) \odot \tilde{x} \in \{\tilde{0}\}$. Hence, $\tilde{x} \odot \tilde{y} = \tilde{x}$.

Conversely, suppose that $\tilde{x} \odot \tilde{y} = \tilde{x}$ for all $\tilde{x} \in NQ(A)$ and $\tilde{y} \in NQ(B)$. If $\tilde{z} \in NQ(A) \cap NQ(B)$, then $\tilde{z} \in NQ(A)$ and $\tilde{z} \in NQ(B)$, which is implied from the hypothesis that $\tilde{z} = \tilde{z} \odot \tilde{z} = \tilde{0}$. Hence $NQ(A) \cap NQ(B) = \{\tilde{0}\}$. □

Theorem 8. *Let A and B be subsets of a BCK-algebra X such that*

$$(\forall a, b \in A \cap B)(K(a, b) \subseteq A \cap B) \tag{17}$$

where $K(a, b) := \{x \in X \mid x * a \leq b\}$. Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.

Proof. If $x \in A \cap B$, then $0 \in K(x, x)$ since $0 * x \leq x$. Hence, $0 \in A \cap B$ by Equation (17), so it is clear that $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Using (II), we have $x_1 \in K(x_1 * y_1, y_1) \subseteq A$, $x_2 \in K(x_2 * y_2, y_2) \subseteq A$, $x_3 \in K(x_3 * y_3, y_3) \subseteq B$, and $x_4 \in K(x_4 * y_4, y_4) \subseteq B$. This implies that $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$. Therefore, $NQ(A, B)$ is an ideal of $NQ(X)$. □

Corollary 3. *Let A and B be subsets of a BCK-algebra X such that*

$$(\forall a, x, y \in X)(x, y \in A \cap B,\ (a * x) * y = 0 \ \Rightarrow\ a \in A \cap B). \tag{18}$$

Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.

Theorem 9. *Let A and B be nonempty subsets of a BCK-algebra X such that*

$$(\forall a, x, y \in X)(x, y \in A \text{ (or } B),\ a * x \leq y \ \Rightarrow\ a \in A \text{ (or } B)). \tag{19}$$

Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.

Proof. Assume that the condition expressed by Equation (19) is valid for nonempty subsets A and B of X. Since $0 * x \leq x$ for any $x \in A$ (or B), we have $0 \in A$ (or B) by Equation (19). Hence, it is clear that $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Note that $x_i * (x_i * y_i) \leq y_i$ for $i = 1, 2, 3, 4$. It follows from Equation (19) that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence,

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B);$$

therefore, $NQ(A, B)$ is an ideal of $NQ(X)$. □

Theorem 10. *If A and B are positive implicative ideals of a BCK-algebra X, then the set $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$, which is called a positive implicative neutrosophic quadruple ideal.*

Proof. Assume that A and B are positive implicative ideals of a BCK-algebra X. Obviously, $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T, \\ ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A, B),\end{aligned}$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A, B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. Since A and B are positive implicative ideals of X, it follows that $x_1 * z_1, x_2 * z_2 \in A$ and $x_3 * z_3, x_4 * z_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A, B),$$

so $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

Theorem 11. *Let A and B be ideals of a BCK-algebra X such that*

$$(\forall x, y, z \in X)((x * y) * z \in A \text{ (or } B) \;\Rightarrow\; (x * z) * (y * z) \in A \text{ (or } B)). \tag{20}$$

Then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.

Proof. Since A and B are ideals of X, it follows from Theorem 4 that $NQ(A, B)$ is an ideal of $NQ(X)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T, \\ ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A, B),\end{aligned}$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A, B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. It follows from Equation (20) that $(x_1 * z_1) * (y_1 * z_1) \in A$, $(x_2 * z_2) * (y_2 * z_2) \in A$, $(x_3 * z_3) * (y_3 * z_3) \in B$, and $(x_4 * z_4) * (y_4 * z_4) \in B$. Since A and B are ideals of X, we get $x_1 * z_1 \in A$, $x_2 * z_2 \in A$, $x_3 * z_3 \in B$, and $x_4 * z_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A, B).$$

Therefore, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

Corollary 4. *Let A and B be ideals of a BCK-algebra X such that*

$$(\forall x, y \in X)((x * y) * y \in A \text{ (or } B) \;\Rightarrow\; x * y \in A \text{ (or } B)). \tag{21}$$

Then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.

Proof. If the condition expressed in Equation (21) is valid, then the condition expressed in Equation (20) is true. Hence, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$ by Theorem 11. □

Theorem 12. *Let A and B be subsets of a BCK-algebra X such that $0 \in A \cap B$ and*

$$((x * y) * y) * z \in A \text{ (or } B\text{)},\ z \in A \text{ (or } B\text{)} \ \Rightarrow\ x * y \in A \text{ (or } B\text{)} \tag{22}$$

for all $x, y, z \in X$. Then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.

Proof. Since $0 \in A \cap B$, it is clear that $\tilde{0} \in NQ(A, B)$. We first show that

$$(\forall x, y \in X)(x * y \in A \text{ (or } B\text{)},\ y \in A \text{ (or } B\text{)} \Rightarrow x \in A \text{ (or } B\text{)}). \tag{23}$$

Let $x, y \in X$ be such that $x * y \in A$ (or B) and $y \in A$ (or B). Then

$$((x * 0) * 0) * y = x * y \in A \text{ (or } B\text{)}$$

by Equation (1), which, based on Equations (1) and (22), implies that $x = x * 0 \in A$ (or B). Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, &((x_2 * y_2) * z_2)T, \\ &((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A, B),\end{aligned}$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A, B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. Note that

$$(((x_i * z_i) * z_i) * (y_i * z_i)) * ((x_i * y_i) * z_i) = 0 \in A \text{ (or } B\text{)}$$

for $i = 1, 2, 3, 4$. Since $(x_i * y_i) * z_i \in A$ for $i = 1, 2$ and $(x_j * y_j) * z_j \in B$ for $j = 3, 4$, it follows from Equation (23) that $((x_i * z_i) * z_i) * (y_i * z_i) \in A$ for $i = 1, 2$, and $((x_j * z_j) * z_j) * (y_j * z_j) \in B$ for $j = 3, 4$. Moreover, since $y_i * z_i \in A$ for $i = 1, 2$, and $y_j * z_j \in B$ for $j = 3, 4$, we have $x_1 * z_1 \in A$, $x_2 * z_2 \in A$, $x_3 * z_3 \in B$, and $x_4 * z_4 \in B$ by Equation (22). Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A, B).$$

Therefore, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

Theorem 13. *Let A and B be subsets of a BCK-algebra X such that $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. Then the set*

$$\Omega_{\tilde{a}} := \{\tilde{x} \in NQ(X) \mid \tilde{x} \odot \tilde{a} \in NQ(A, B)\} \tag{24}$$

is an ideal of $NQ(X)$ for any $\tilde{a} \in NQ(X)$.

Proof. Obviously, $\tilde{0} \in \Omega_{\tilde{a}}$. Let $\tilde{x}, \tilde{y} \in NQ(X)$ be such that $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{a}}$ and $\tilde{y} \in \Omega_{\tilde{a}}$. Then $(\tilde{x} \odot \tilde{y}) \odot \tilde{a} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{a} \in NQ(A, B)$. Since $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$, it follows from Equation (11) that $\tilde{x} \odot \tilde{a} \in NQ(A, B)$ and therefore that $\tilde{x} \in \Omega_{\tilde{a}}$. Hence, $\Omega_{\tilde{a}}$ is an ideal of $NQ(X)$. □

Combining Theorems 12 and 13, we have the following corollary.

Corollary 5. *If A and B are subsets of a BCK-algebra X satisfying $0 \in A \cap B$ and the condition expressed in Equation (22), then the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$.*

Theorem 14. *For any subsets A and B of a BCK-algebra X, if the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$, then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.*

Proof. Since $\tilde{0} \in \Omega_{\tilde{a}}$, we have $\tilde{0} = \tilde{0} \odot \tilde{a} \in NQ(A, B)$. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$ be such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{z}}$ and $\tilde{y} \in \Omega_{\tilde{z}}$. Since $\Omega_{\tilde{z}}$ is an ideal of $NQ(X)$, it follows that $\tilde{x} \in \Omega_{\tilde{z}}$. Hence, $\tilde{x} \odot \tilde{z} \in NQ(A, B)$. Therefore, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

Theorem 15. *For any ideals A and B of a BCK-algebra X and for any $\tilde{a} \in NQ(X)$, if the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$, then $NQ(X)$ is a positive implicative BCK-algebra.*

Proof. Let Ω be any ideal of $NQ(X)$. For any $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$, assume that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in \Omega$ and $\tilde{y} \odot \tilde{z} \in \Omega$. Then $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{z}}$ and $\tilde{y} \in \Omega_{\tilde{z}}$. Since $\Omega_{\tilde{z}}$ is an ideal of $NQ(X)$, it follows that $\tilde{x} \in \Omega_{\tilde{z}}$. Hence, $\tilde{x} \odot \tilde{z} \in \Omega$, which shows that Ω is a positive implicative ideal of $NQ(X)$. Therefore, $NQ(X)$ is a positive implicative BCK-algebra. □

In general, the set $\{\tilde{0}\}$ is an ideal of any neutrosophic quadruple BCK-algebra $NQ(X)$, but it is not a positive implicative ideal of $NQ(X)$ as seen in the following example.

Example 4. *Consider a BCK-algebra $X = \{0, 1, 2\}$ with the binary operation $*$, which is given in Table 3.*

Table 3. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**
0	0	0	0
1	1	0	0
2	2	1	0

Then the neutrosophic quadruple BCK-algebra $NQ(X)$ has 81 elements. If we take $\tilde{a} = (2, 2T, 2I, 2F)$ and $\tilde{b} = (1, 1T, 1I, 1F)$ in $NQ(X)$, then

$$\begin{aligned}(\tilde{a} \odot \tilde{b}) \odot \tilde{b} &= ((2*1)*1, ((2*1)*1)T, ((2*1)*1)I, ((2*1)*1)F) \\ &= (1*1, (1*1)T, (1*1)I, (1*1)F) = (0, 0T, 0I, 0F) = \tilde{0},\end{aligned}$$

and $\tilde{b} \odot \tilde{b} = \tilde{0}$. However,

$$\tilde{a} \odot \tilde{b} = (2*1, (2*1)T, (2*1)I, (2*1)F) = (1, 1T, 1I, 1F) \neq \tilde{0}.$$

Hence, $\{\tilde{0}\}$ is not a positive implicative ideal of $NQ(X)$.

We now provide conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in the neutrosophic quadruple BCK-algebra.

Theorem 16. *Let $NQ(X)$ be a neutrosophic quadruple BCK-algebra. If the set*

$$\Omega(\tilde{a}) := \{\tilde{x} \in NQ(X) \mid \tilde{x} \ll \tilde{a}\} \tag{25}$$

is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$, then $\{\tilde{0}\}$ is a positive implicative ideal of $NQ(X)$.

Proof. We first show that

$$(\forall \tilde{x}, \tilde{y} \in NQ(X))((\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{0} \ \Rightarrow \ \tilde{x} \odot \tilde{y} = \tilde{0}). \tag{26}$$

Assume that $(\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{0}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Then $\tilde{x} \odot \tilde{y} \ll \tilde{y}$, so $\tilde{x} \odot \tilde{y} \in \Omega(\tilde{y})$. Since $\tilde{y} \in \Omega(\tilde{y})$ and $\Omega(\tilde{y})$ is an ideal of $NQ(X)$, we have $\tilde{x} \in \Omega(\tilde{y})$. Thus, $\tilde{x} \ll \tilde{y}$, that is, $\tilde{x} \odot \tilde{y} = \tilde{0}$. Let $\tilde{u} := (\tilde{x} \odot \tilde{y}) \odot \tilde{y}$. Then

$$((\tilde{x} \odot \tilde{u}) \odot \tilde{y}) \odot \tilde{y} = ((\tilde{x} \odot \tilde{y}) \odot \tilde{y}) \odot \tilde{u} = \tilde{0},$$

which implies, based on Equations (3) and (26), that

$$(\tilde{x} \odot \tilde{y}) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{y}) = (\tilde{x} \odot \tilde{y}) \odot \tilde{u} = (\tilde{x} \odot \tilde{u}) \odot \tilde{y} = \tilde{0},$$

that is, $\tilde{x} \odot \tilde{y} \ll (\tilde{x} \odot \tilde{y}) \odot \tilde{y}$. Since $(\tilde{x} \odot \tilde{y}) \odot \tilde{y} \ll \tilde{x} \odot \tilde{y}$, it follows that

$$(\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{x} \odot \tilde{y}. \tag{27}$$

If we put $\tilde{y} = \tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))$ in Equation (27), then

$$\begin{aligned}
\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) &= (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\
&= (\tilde{y} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}) \\
&= ((\tilde{y} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x}) \\
&\ll (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \\
&= ((\tilde{x} \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))))) \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}) \\
&= ((\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot (\tilde{y} \odot \tilde{x})) = \tilde{0},
\end{aligned}$$

so $((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) = \tilde{0}$, that is,

$$((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}) \ll \tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))).$$

Hence,

$$\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) = ((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})). \tag{28}$$

If we use $\tilde{y} \odot \tilde{x}$ instead of $\tilde{x}$ in Equation (28), then

$$\begin{aligned}
\tilde{y} \odot \tilde{x} &= (\tilde{y} \odot \tilde{x}) \odot \tilde{0} \\
&= (\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \\
&= ((\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y})) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\
&= (\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})),
\end{aligned}$$

which, by taking $\tilde{x} = \tilde{y} \odot \tilde{x}$, implies that

$$\begin{aligned}\tilde{y} \odot (\tilde{y} \odot \tilde{x}) &= (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\ &= (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x}).\end{aligned}$$

It follows that

$$\begin{aligned}(\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) &= ((\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\ &\ll (\tilde{x} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\ &= (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}),\end{aligned}$$

so,

$$\begin{aligned}\tilde{y} \odot \tilde{x} &= (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot \tilde{0} \\ &= (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y}) \\ &\ll ((\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y})) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\ &= (\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\ &\ll (\tilde{y} \odot \tilde{x}) \odot \tilde{x}.\end{aligned}$$

Since $(\tilde{y} \odot \tilde{x}) \odot \tilde{x} \ll \tilde{y} \odot \tilde{x}$, it follows that

$$(\tilde{y} \odot \tilde{x}) \odot \tilde{x} = \tilde{y} \odot \tilde{x}. \tag{29}$$

Based on Equation (29), it follows that

$$\begin{aligned}&((\tilde{x} \odot \tilde{z}) * (\tilde{y} \odot \tilde{z})) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &= (((\tilde{x} \odot \tilde{z}) \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &\ll ((\tilde{x} \odot \tilde{z}) \odot \tilde{y}) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &= \tilde{0},\end{aligned}$$

that is, $(\tilde{x} \odot \tilde{z}) * (\tilde{y} \odot \tilde{z}) \ll (\tilde{x} \odot \tilde{y}) \odot \tilde{z}$. Note that

$$\begin{aligned}&((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \odot ((x \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})) \\ &= ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \odot ((x \odot (\tilde{y} \odot \tilde{z})) \odot \tilde{z}) \\ &\ll (\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot (\tilde{y} \odot \tilde{z})) \\ &\ll (\tilde{y} \odot \tilde{z}) \odot \tilde{y} = \tilde{0},\end{aligned}$$

which shows that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \ll (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})$. Hence, $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})$. Therefore, $NQ(X)$ is a positive implicative, so $\{\tilde{0}\}$ is a positive implicative ideal of $NQ(X)$. □

4. Conclusions

We have considered a neutrosophic quadruple BCK/BCI-number on a set and established neutrosophic quadruple BCK/BCI-algebras, which consist of neutrosophic quadruple BCK/BCI-numbers. We have investigated several properties and considered ideal theory in a neutrosophic quadruple BCK-algebra and a closed ideal in a neutrosophic quadruple BCI-algebra. Using subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, we have considered sets $NQ(A, B)$, which consist of neutrosophic quadruple BCK/BCI-numbers with a condition. We have provided conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra, and the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. We have provided an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and we have considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra.

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Interval Neutrosophic Sets with Applications in *BCK*/*BCI*-Algebra

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Young Bae Jun, Seon Jeong Kim, Florentin Smarandache (2018). Interval Neutrosophic Sets with Applications in BCK/BCI-Algebra. *Axioms* 7, 23. DOI: 10.3390/axioms7020023

Abstract: For $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebra in BCK/BCI-algebra is introduced, and their properties and relations are investigated. The notion of interval neutrosophic length of an interval neutrosophic set is also introduced, and related properties are investigated.

Keywords: interval neutrosophic set; interval neutrosophic subalgebra; interval neutrosophic length

1. Introduction

Intuitionistic fuzzy set, which is introduced by Atanassov [1], is a generalization of Zadeh's fuzzy sets [2], and consider both truth-membership and falsity-membership. Since the sum of degree true, indeterminacy and false is one in intuitionistic fuzzy sets, incomplete information is handled in intuitionistic fuzzy sets. On the other hand, neutrosophic sets can handle the indeterminate information and inconsistent information that exist commonly in belief systems in a neutrosophic set since indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent, which is mentioned in [3]. As a formal framework that generalizes the concept of the classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, interval valued intuitionistic fuzzy set and paraconsistent set, etc., the neutrosophic set is developed by Smarandache [4,5], which is applied to various parts, including algebra, topology, control theory, decision-making problems, medicines and in many real-life problems. The concept of interval neutrosophic sets is presented by Wang et al. [6], and it is more precise and more flexible than the single-valued neutrosophic set. The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information, which exists in the real world. BCK-algebra is introduced by Imai and Iséki [7], and it has been applied to several branches of mathematics, such as group theory, functional analysis, probability theory and topology, etc. As a generalization of BCK-algebra, Iséki introduced the notion of BCI-algebra (see [8]).

In this article, we discuss interval neutrosophic sets in BCK/BCI-algebra. We introduce the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebra in BCK/BCI-algebra for $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, and investigate their properties and relations. We also introduce the notion of interval neutrosophic length of an interval neutrosophic set, and investigate related properties.

2. Preliminaries

By a *BCI-algebra*, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x*y)*(x*z))*(z*y) = 0$,
(II) $(x*(x*y))*y = 0$,
(III) $x*x = 0$,
(IV) $x*y = y*x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0*x = 0$ for all $x \in X$, then we say that X is *BCK-algebra*.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y \in S$ for all $x, y \in S$.

The collection of all BCK-algebra and all BCI-algebra are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. In addition, $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [9,10] for further information regarding BCK/BCI-algebra.

By a *fuzzy structure* over a nonempty set X, we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Definition 1 ([11]). *For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a*

- *fuzzy subalgebra of $(X, *, 0)$ with type 1 (briefly, 1-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x*y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{1}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 2 (briefly, 2-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x*y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{2}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 3 (briefly, 3-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x*y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{3}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 4 (briefly, 4-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x*y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{4}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (see [4]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\},$$

where $A_T : X \to [0,1]$ is a truth-membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function.

An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$ (see [3,6]).

3. Interval Neutrosophic Subalgebra

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ and $\mathcal{P}^*([0,1])$ be the family of all subintervals of $[0,1]$ unless otherwise specified.

Definition 2 ([3,6]). *An interval neutrosophic set in a nonempty set X is a structure of the form:*

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\},$$

where

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),$$

which is called interval truth-membership function,

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),$$

which is called interval indeterminacy-membership function, and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),$$

which is called interval falsity-membership function.

For the sake of simplicity, we will use the notation $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ for the interval neutrosophic set

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we consider the following functions:

$$\begin{aligned}
&\mathcal{I}[T]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[T](x)\},\\
&\mathcal{I}[I]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[I](x)\},\\
&\mathcal{I}[F]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[F](x)\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{I}[T]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[T](x)\},\\
&\mathcal{I}[I]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[I](x)\},\\
&\mathcal{I}[F]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[F](x)\}.
\end{aligned}$$

Definition 3. *For any $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is called a $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebra of X if the following assertions are valid.*

(1) *$(X, \mathcal{I}[T]_{\inf})$ is an i-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[T]_{\sup})$ is a j-fuzzy subalgebra of $(X, *, 0)$,*
(2) *$(X, \mathcal{I}[I]_{\inf})$ is a k-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[I]_{\sup})$ is an l-fuzzy subalgebra of $(X, *, 0)$,*
(3) *$(X, \mathcal{I}[F]_{\inf})$ is an m-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[F]_{\sup})$ is an n-fuzzy subalgebra of $(X, *, 0)$.*

Example 1. *Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$, which is given in Table 1 (see [10]).*

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

(1) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ for which $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1])\ \ x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x = 0,\\ (0.3, 0.5] & \text{if } x = 1,\\ [0.2, 0.6) & \text{if } x = 2,\\ [0.1, 0.7] & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.5, 0.8) & \text{if } x = 0, \\ (0.2, 0.7) & \text{if } x = 1, \\ [0.5, 0.6] & \text{if } x = 2, \\ [0.4, 0.8) & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x = 0, \\ (0.2, 0.9) & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ (0.4, 0.7] & \text{if } x = 3. \end{cases}$$

It is routine to verify that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,4), I(1,4), F(1,4))$*-interval neutrosophic subalgebra of* $(X, *, 0)$.

(2) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *be an interval neutrosophic set in* $(X, *, 0)$ *for which* $\mathcal{I}[T], \mathcal{I}[I]$ *and* $\mathcal{I}[F]$ *are given as follows:*

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.1, 0.4) & \text{if } x = 0, \\ (0.3, 0.5) & \text{if } x = 1, \\ [0.2, 0.7] & \text{if } x = 2, \\ [0.4, 0.6) & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} (0.2, 0.5) & \text{if } x = 0, \\ [0.5, 0.8] & \text{if } x = 1, \\ (0.4, 0.5] & \text{if } x = 2, \\ [0.2, 0.6] & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.3, 0.4) & \text{if } x = 0, \\ (0.4, 0.7) & \text{if } x = 1, \\ (0.6, 0.8) & \text{if } x = 2, \\ [0.4, 0.6] & \text{if } x = 3. \end{cases}$$

By routine calculations, we know that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4), I(4,4), F(4,4))$*-interval neutrosophic subalgebra of* $(X, *, 0)$.

Example 2. *Consider a BCI-algebra* $X = \{0, a, b, c\}$ *with the binary operation* $*$, *which is given in Table 2 (see [10]).*

Table 2. Cayley table for the binary operation "$*$".

$*$	**0**	***a***	***b***	***c***
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *be an interval neutrosophic set in* $(X, *, 0)$ *for which* $\mathcal{I}[T], \mathcal{I}[I]$ *and* $\mathcal{I}[F]$ *are given as follows:*

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.3, 0.9) & \text{if } x = 0, \\ (0.7, 0.9) & \text{if } x = a, \\ [0.7, 0.8) & \text{if } x = b, \\ (0.5, 0.8] & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.2, 0.65) & \text{if } x = 0, \\ [0.5, 0.55] & \text{if } x = a, \\ (0.6, 0.65) & \text{if } x = b, \\ [0.5, 0.55) & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} (0.3, 0.6) & \text{if } x = 0, \\ [0.4, 0.6] & \text{if } x = a, \\ (0.4, 0.5] & \text{if } x = b, \\ [0.3, 0.5) & \text{if } x = c. \end{cases}$$

Routine calculations show that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic subalgebra of* $(X, *, 0)$*. However, it is not a* $(T(2,1), I(2,1), F(2,1))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *since*

$$\mathcal{I}[T]_{\inf}(c * a) = \mathcal{I}[T]_{\inf}(b) = 0.7 > 0.5 = \min\{\mathcal{I}[T]_{\inf}(c), \mathcal{I}[T]_{\inf}(a)\}$$

and/or

$$\mathcal{I}[I]_{\inf}(a * c) = \mathcal{I}[I]_{\inf}(b) = 0.6 > 0.5 = \min\{\mathcal{I}[I]_{\inf}(a), \mathcal{I}[I]_{\inf}(c)\}.$$

In addition, it is not a $(T(4,3), I(4,3), F(4,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *since*

$$\mathcal{I}[T]_{\sup}(a * b) = \mathcal{I}[T]_{\sup}(c) = 0.8 < 0.9 = \max\{\mathcal{I}[T]_{\inf}(a), \mathcal{I}[T]_{\inf}(c)\}$$

and/or

$$\mathcal{I}[F]_{\sup}(a * b) = \mathcal{I}[F]_{\sup}(c) = 0.5 < 0.6 = \max\{\mathcal{I}[F]_{\inf}(a), \mathcal{I}[F]_{\inf}(c)\}.$$

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X. We consider the following sets:

$$\begin{aligned} U(\mathcal{I}[T]_{\inf}; \alpha_I) &:= \{x \in X \mid \mathcal{I}[T]_{\inf}(x) \geq \alpha_I\}, \\ L(\mathcal{I}[T]_{\sup}; \alpha_S) &:= \{x \in X \mid \mathcal{I}[T]_{\sup}(x) \leq \alpha_S\}, \\ U(\mathcal{I}[I]_{\inf}; \beta_I) &:= \{x \in X \mid \mathcal{I}[I]_{\inf}(x) \geq \beta_I\}, \\ L(\mathcal{I}[I]_{\sup}; \beta_S) &:= \{x \in X \mid \mathcal{I}[I]_{\sup}(x) \leq \beta_S\}, \end{aligned}$$

and

$$\begin{aligned} U(\mathcal{I}[F]_{\inf}; \gamma_I) &:= \{x \in X \mid \mathcal{I}[F]_{\inf}(x) \geq \gamma_I\}, \\ L(\mathcal{I}[F]_{\sup}; \gamma_S) &:= \{x \in X \mid \mathcal{I}[F]_{\sup}(x) \leq \gamma_S\}, \end{aligned}$$

where α_I, α_S, β_I, β_S, γ_I and γ_S are numbers in $[0,1]$.

Theorem 1. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,4), I(i,4), F(i,4))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{1,3\}$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or subalgebra of* $(X, *, 0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 1-fuzzy subalgebra of X; and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy subalgebra of X. Let $\alpha_I, \alpha_S \in [0,1]$ be such that $U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $L(\mathcal{I}[T]_{\sup}; \alpha_S)$ are nonempty. For any $x, y \in X$, if $x, y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$, then $\mathcal{I}[T]_{\inf}(x) \geq \alpha_I$ and $\mathcal{I}[T]_{\inf}(y) \geq \alpha_I$, and so

$$\mathcal{I}[T]_{\inf}(x * y) \geq \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\} \geq \alpha_I,$$

that is, $x * y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$. If $x, y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$, then $\mathcal{I}[T]_{\sup}(x) \leq \alpha_S$ and $\mathcal{I}[T]_{\sup}(y) \leq \alpha_S$, which imply that

$$\mathcal{I}[T]_{\sup}(x * y) \leq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\} \leq \alpha_S,$$

that is, $x * y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$. Hence, $U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $L(\mathcal{I}[T]_{\sup}; \alpha_S)$ are subalgebra of $(X, *, 0)$ for all α_I, $\alpha_S \in [0, 1]$. Similarly, we can prove that $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all β_I, β_S, γ_I, $\gamma_S \in [0, 1]$. Suppose that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,4), I(3,4), F(3,4))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 3-fuzzy subalgebra of X; and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy subalgebra of X. Let β_I and $\beta_S \in [0, 1]$ be such that $U(\mathcal{I}[I]_{\inf}; \beta_I)$ and $L(\mathcal{I}[I]_{\sup}; \beta_S)$ are nonempty. Let $x, y \in U(\mathcal{I}[I]_{\inf}; \beta_I)$. Then, $\mathcal{I}[I]_{\inf}(x) \geq \beta_I$ and $\mathcal{I}[I]_{\inf}(y) \geq \beta_I$. It follows that

$$\mathcal{I}[I]_{\inf}(x * y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\} \geq \beta_I$$

and so $x * y \in U(\mathcal{I}[I]_{\inf}; \beta_I)$. Thus, $U(\mathcal{I}[I]_{\inf}; \beta_I)$ is a subalgebra of $(X, *, 0)$. If $x, y \in L(\mathcal{I}[I]_{\inf}; \beta_S)$, then $\mathcal{I}[I]_{\inf}(x) \leq \beta_S$ and $\mathcal{I}[I]_{\inf}(y) \leq \beta_S$. Hence,

$$\mathcal{I}[I]_{\inf}(x * y) \leq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\} \leq \beta_S,$$

and so $x * y \in L(\mathcal{I}[I]_{\inf}; \beta_S)$. Thus, $L(\mathcal{I}[I]_{\inf}; \beta_S)$ is a subalgebra of $(X, *, 0)$. Similarly, we can show that $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, γ_I, $\gamma_S \in [0, 1]$. □

Since every 2-fuzzy subalgebra is a 4-fuzzy subalgebra, we have the following corollary.

Corollary 1. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,2), I(i,2), F(i,2))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{1, 3\}$, *then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or subalgebra of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

By a similar way to the proof of Theorem 1, we have the following theorems.

Theorem 2. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,4), I(i,4), F(i,4))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$, *then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or subalgebra of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

Corollary 2. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,2), I(i,2), F(i,2))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$, *then* $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or subalgebra of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

Theorem 3. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(k,1), I(k,1), F(k,1))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $k \in \{1, 3\}$, *then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or subalgebra of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

Corollary 3. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(k,3), I(k,3), F(k,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $k \in \{1, 3\}$, *then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or subalgebra of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

Theorem 4. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(k,1), I(k,1), F(k,1))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $k \in \{2,4\}$, then $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.*

Corollary 4. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(k,3), I(k,3), F(k,3))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $k \in \{2,4\}$, then $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.*

Theorem 5. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are nonempty subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic subalgebra of $(X, *, 0)$.*

Proof. Suppose that $(X, \mathcal{I}[T]_{\inf})$ is not a 1-fuzzy subalgebra of $(X, *, 0)$. Then, there exists $x, y \in X$ such that

$$\mathcal{I}[T]_{\inf}(x * y) < \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\}.$$

If we take $\alpha_I = \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\}$, then $x, y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$, but $x * y \notin U(\mathcal{I}[T]_{\inf}; \alpha_I)$. This is a contradiction, and so $(X, \mathcal{I}[T]_{\inf})$ is a 1-fuzzy subalgebra of $(X, *, 0)$. If $(X, \mathcal{I}[T]_{\sup})$ is not a 4-fuzzy subalgebra of $(X, *, 0)$, then

$$\mathcal{I}[T]_{\sup}(a * b) > \max\{\mathcal{I}[T]_{\sup}(a), \mathcal{I}[T]_{\sup}(b)\}$$

for some $a, b \in X$, and so $a, b \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$ and $a * b \notin L(\mathcal{I}[T]_{\sup}; \alpha_S)$ by taking

$$\alpha_S := \max\{\mathcal{I}[T]_{\sup}(a), \mathcal{I}[T]_{\sup}(b)\}.$$

This is a contradiction, and therefore $(X, \mathcal{I}[T]_{\sup})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. Similarly, we can verify that $(X, \mathcal{I}[I]_{\inf})$ is a 1-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[I]_{\sup})$ is a 4-fuzzy subalgebra of $(X, *, 0)$; and $(X, \mathcal{I}[F]_{\inf})$ is a 1-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[F]_{\sup})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. Consequently, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic subalgebra of $(X, *, 0)$. □

Using the similar method to the proof of Theorem 5, we get the following theorems.

Theorem 6. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ are nonempty subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,1), I(4,1), F(4,1))$-interval neutrosophic subalgebra of $(X, *, 0)$.*

Theorem 7. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are nonempty subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,4), I(4,4), F(4,4))$-interval neutrosophic subalgebra of $(X, *, 0)$.*

Theorem 8. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ are nonempty subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,1), I(1,1), F(1,1))$-interval neutrosophic subalgebra of $(X, *, 0)$.*

4. Interval Neutrosophic Lengths

Definition 4. *Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we define the interval neutrosophic length of $\mathcal{I}$ as an ordered triple $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ where*

$$\mathcal{I}[T]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x),$$
$$\mathcal{I}[I]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x),$$

and

$$\mathcal{I}[F]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x),$$

which are called interval neutrosophic T-length, interval neutrosophic I-length and interval neutrosophic F-length of $\mathcal{I}$, respectively.

Example 3. *Consider the interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, which is given in Example 2. Then, the interval neutrosophic length of $\mathcal{I}$ is given by Table 3.*

Table 3. Interval neutrosophic length of $\mathcal{I}$.

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.6	0.45	0.3
a	0.2	0.05	0.2
b	0.1	0.05	0.1
c	0.3	0.05	0.2

Theorem 9. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,3), I(i,3), F(i,3))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2,4\}$, then $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 3-fuzzy subalgebra of $(X, *, 0)$.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,3), I(2,3), F(2,3))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy subalgebra of X. Thus,

$$\mathcal{I}[T]_{\inf}(x * y) \leq \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},$$
$$\mathcal{I}[I]_{\inf}(x * y) \leq \min\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},$$
$$\mathcal{I}[F]_{\inf}(x * y) \leq \min\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},$$

and

$$\mathcal{I}[T]_{\sup}(x * y) \geq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},$$
$$\mathcal{I}[I]_{\sup}(x * y) \geq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},$$
$$\mathcal{I}[F]_{\sup}(x * y) \geq \max\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},$$

for all $x, y \in X$. It follows that

$$\mathcal{I}[T]_\ell(x * y) = \mathcal{I}[T]_{\sup}(x * y) - \mathcal{I}[T]_{\inf}(x * y) \geq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_\ell(x),$$
$$\mathcal{I}[T]_\ell(x * y) = \mathcal{I}[T]_{\sup}(x * y) - \mathcal{I}[T]_{\inf}(x * y) \geq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_\ell(y),$$
$$\mathcal{I}[I]_\ell(x * y) = \mathcal{I}[I]_{\sup}(x * y) - \mathcal{I}[I]_{\inf}(x * y) \geq \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_\ell(x),$$
$$\mathcal{I}[I]_\ell(x * y) = \mathcal{I}[I]_{\sup}(x * y) - \mathcal{I}[I]_{\inf}(x * y) \geq \mathcal{I}[I]_{\sup}(y) - \mathcal{I}[I]_{\inf}(y) = \mathcal{I}[I]_\ell(y),$$

and

$$\mathcal{I}[F]_\ell(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \geq \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_\ell(x),$$
$$\mathcal{I}[F]_\ell(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \geq \mathcal{I}[F]_{\sup}(y) - \mathcal{I}[F]_{\inf}(y) = \mathcal{I}[F]_\ell(y).$$

Hence,

$$\mathcal{I}[T]_\ell(x*y) \geq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},$$
$$\mathcal{I}[I]_\ell(x*y) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},$$

and

$$\mathcal{I}[F]_\ell(x*y) \geq \max\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\},$$

for all $x, y \in X$. Therefore, $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 3-fuzzy subalgebra of $(X, *, 0)$.

Suppose that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,3), I(4,3), F(4,3))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 4-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy subalgebra of X. Hence,

$$\begin{aligned}
&\mathcal{I}[T]_{\inf}(x*y) \leq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},\\
&\mathcal{I}[I]_{\inf}(x*y) \leq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},\\
&\mathcal{I}[F]_{\inf}(x*y) \leq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},
\end{aligned} \tag{5}$$

and

$$\mathcal{I}[T]_{\sup}(x*y) \geq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},$$
$$\mathcal{I}[I]_{\sup}(x*y) \geq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},$$
$$\mathcal{I}[F]_{\sup}(x*y) \geq \max\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},$$

for all $x, y \in X$. Label (5) implies that

$$\mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(x) \text{ or } \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(y),$$
$$\mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\inf}(x) \text{ or } \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\inf}(y),$$
$$\mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\inf}(x) \text{ or } \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\inf}(y).$$

If $\mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(x)$, then

$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_\ell(x).$$

If $\mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(y)$, then

$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_\ell(y).$$

It follows that $\mathcal{I}[T]_\ell(x*y) \geq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$. Therefore, $(X, \mathcal{I}[T]_\ell)$ is a 3-fuzzy subalgebra of $(X, *, 0)$. Similarly, we can show that $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 3-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 5. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,3), I(i,3), F(i,3))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2,4\}$, then $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 1-fuzzy subalgebra of $(X, *, 0)$.*

Theorem 10. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(3,4), I(3,4), F(3,4))$*-interval neutrosophic subalgebra of* $(X,*,0)$*, then* $(X, \mathcal{I}[T]_\ell)$*,* $(X, \mathcal{I}[I]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are* 4*-fuzzy subalgebra of* $(X,*,0)$*.*

Proof. Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be a $(T(3,4), I(3,4), F(3,4))$-interval neutrosophic subalgebra of $(X,*,0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 3-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy subalgebra of X. Thus,

$$\begin{aligned}
&\mathcal{I}[T]_{\inf}(x*y) \geq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},\\
&\mathcal{I}[I]_{\inf}(x*y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},\\
&\mathcal{I}[F]_{\inf}(x*y) \geq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{I}[T]_{\sup}(x*y) \leq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},\\
&\mathcal{I}[I]_{\sup}(x*y) \leq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},\\
&\mathcal{I}[F]_{\sup}(x*y) \leq \max\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},
\end{aligned} \tag{6}$$

for all $x, y \in X$. It follows from Label (6) that

$$\begin{aligned}
&\mathcal{I}[T]_{\sup}(x*y) \leq \mathcal{I}[T]_{\sup}(x) \text{ or } \mathcal{I}[T]_{\sup}(x*y) \leq \mathcal{I}[T]_{\sup}(y),\\
&\mathcal{I}[I]_{\sup}(x*y) \leq \mathcal{I}[I]_{\sup}(x) \text{ or } \mathcal{I}[I]_{\sup}(x*y) \leq \mathcal{I}[I]_{\sup}(y),\\
&\mathcal{I}[F]_{\sup}(x*y) \leq \mathcal{I}[F]_{\sup}(x) \text{ or } \mathcal{I}[F]_{\sup}(x*y) \leq \mathcal{I}[F]_{\sup}(y).
\end{aligned}$$

Assume that $\mathcal{I}[T]_{\sup}(x*y) \leq \mathcal{I}[T]_{\sup}(x)$. Then,

$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_\ell(x).$$

If $\mathcal{I}[T]_{\sup}(x*y) \leq \mathcal{I}[T]_{\sup}(y)$, then

$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_\ell(y).$$

Hence, $\mathcal{I}[T]_\ell(x*y) \leq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$ for all $x, y \in X$. By a similar way, we can prove that

$$\mathcal{I}[I]_\ell(x*y) \leq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\}$$

and

$$\mathcal{I}[F]_\ell(x*y) \leq \max\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}$$

for all $x, y \in X$. Therefore, $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 4-fuzzy subalgebra of $(X,*,0)$. □

Theorem 11. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(3,2), I(3,2), F(3,2))$*-interval neutrosophic subalgebra of* $(X,*,0)$*, then* $(X, \mathcal{I}[T]_\ell)$*,* $(X, \mathcal{I}[I]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are* 2*-fuzzy subalgebra of* $(X,*,0)$*.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,2), I(3,2), F(3,2))$-interval neutrosophic subalgebra of $(X,*,0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 3-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 2-fuzzy subalgebra of X. Hence,

$$\begin{aligned}
&\mathcal{I}[T]_{\inf}(x*y) \geq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},\\
&\mathcal{I}[I]_{\inf}(x*y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},\\
&\mathcal{I}[F]_{\inf}(x*y) \geq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}[T]_{\sup}(x*y) &\leq \min\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},\\
\mathcal{I}[I]_{\sup}(x*y) &\leq \min\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},\\
\mathcal{I}[F]_{\sup}(x*y) &\leq \min\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},
\end{aligned}$$

for all $x, y \in X$, which imply that

$$\begin{aligned}
\mathcal{I}[T]_{\ell}(x*y) &= \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\ell}(x),\\
\mathcal{I}[T]_{\ell}(x*y) &= \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_{\ell}(y),\\
\mathcal{I}[I]_{\ell}(x*y) &= \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\ell}(x),\\
\mathcal{I}[I]_{\ell}(x*y) &= \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\sup}(y) - \mathcal{I}[I]_{\inf}(y) = \mathcal{I}[I]_{\ell}(y),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}[F]_{\ell}(x*y) &= \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\ell}(x),\\
\mathcal{I}[F]_{\ell}(x*y) &= \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\sup}(y) - \mathcal{I}[F]_{\inf}(y) = \mathcal{I}[F]_{\ell}(y).
\end{aligned}$$

It follows that

$$\mathcal{I}[T]_{\ell}(x*y) \leq \min\{\mathcal{I}[T]_{\ell}(x), \mathcal{I}[T]_{\ell}(y)\},$$

$$\mathcal{I}[I]_{\ell}(x*y) \leq \min\{\mathcal{I}[I]_{\ell}(x), \mathcal{I}[I]_{\ell}(y)\},$$

and

$$\mathcal{I}[F]_{\ell}(x*y) \leq \min\{\mathcal{I}[F]_{\ell}(x), \mathcal{I}[F]_{\ell}(y)\},$$

for all $x, y \in X$. Hence, $(X, \mathcal{I}[T]_{\ell})$, $(X, \mathcal{I}[I]_{\ell})$ and $(X, \mathcal{I}[F]_{\ell})$ are 2-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 6. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(3,2), I(3,2), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$, then $(X, \mathcal{I}[T]_{\ell})$, $(X, \mathcal{I}[I]_{\ell})$ and $(X, \mathcal{I}[F]_{\ell})$ are 4-fuzzy subalgebra of $(X, *, 0)$.*

Theorem 12. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,3), I(3,4), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2, 4\}$, then*

(1) $(X, \mathcal{I}[T]_{\ell})$ *is a 3-fuzzy subalgebra of* $(X, *, 0)$.
(2) $(X, \mathcal{I}[I]_{\ell})$ *is a 4-fuzzy subalgebra of* $(X, *, 0)$.
(3) $(X, \mathcal{I}[F]_{\ell})$ *is a 2-fuzzy subalgebra of* $(X, *, 0)$.

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,3), I(3,4), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$ is a 4-fuzzy subalgebra of X, $(X, \mathcal{I}[T]_{\sup})$ is a 3-fuzzy subalgebra of X, $(X, \mathcal{I}[I]_{\inf})$ is a 3-fuzzy subalgebra of X, $(X, \mathcal{I}[I]_{\sup})$ is a 4-fuzzy subalgebra of X, $(X, \mathcal{I}[F]_{\inf})$ is a 3-fuzzy subalgebra of X, and $(X, \mathcal{I}[F]_{\sup})$ is a 2-fuzzy subalgebra of X. Hence,

$$\mathcal{I}[T]_{\inf}(x*y) \leq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\}, \tag{7}$$

$$\mathcal{I}[T]_{\sup}(x*y) \geq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\}, \tag{8}$$

$$\mathcal{I}[I]_{\inf}(x*y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\}, \tag{9}$$

$$\mathcal{I}[I]_{\sup}(x*y) \leq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\}, \tag{10}$$

$$\mathcal{I}[F]_{\inf}(x*y) \geq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\}, \tag{11}$$

and

$$\mathcal{I}[F]_{\sup}(x * y) \leq \min\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\}, \tag{12}$$

for all $x, y \in X$. Then,

$$\mathcal{I}[T]_{\inf}(x * y) \leq \mathcal{I}[T]_{\inf}(x) \text{ or } \mathcal{I}[T]_{\inf}(x * y) \leq \mathcal{I}[T]_{\inf}(y)$$

by Label (7). It follows from Label (8) that

$$\mathcal{I}[T]_{\ell}(x * y) = \mathcal{I}[T]_{\sup}(x * y) - \mathcal{I}[T]_{\inf}(x * y) \geq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\ell}(x)$$

or

$$\mathcal{I}[T]_{\ell}(x * y) = \mathcal{I}[T]_{\sup}(x * y) - \mathcal{I}[T]_{\inf}(x * y) \geq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_{\ell}(y),$$

and so that $\mathcal{I}[T]_{\ell}(x * y) \geq \max\{\mathcal{I}[T]_{\ell}(x), \mathcal{I}[T]_{\ell}(y)\}$ for all $x, y \in X$. Thus, $(X, \mathcal{I}[T]_{\ell})$ is a 3-fuzzy subalgebra of $(X, *, 0)$. The condition (10) implies that

$$\mathcal{I}[I]_{\sup}(x * y) \leq \mathcal{I}[I]_{\sup}(x) \text{ or } \mathcal{I}[I]_{\sup}(x * y) \leq \mathcal{I}[I]_{\sup}(y). \tag{13}$$

Combining Labels (9) and (13), we have

$$\mathcal{I}[I]_{\ell}(x * y) = \mathcal{I}[I]_{\sup}(x * y) - \mathcal{I}[I]_{\inf}(x * y) \leq \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\ell}(x)$$

or

$$\mathcal{I}[I]_{\ell}(x * y) = \mathcal{I}[I]_{\sup}(x * y) - \mathcal{I}[I]_{\inf}(x * y) \leq \mathcal{I}[I]_{\sup}(y) - \mathcal{I}[I]_{\inf}(y) = \mathcal{I}[I]_{\ell}(y).$$

It follows that $\mathcal{I}[I]_{\ell}(x * y) \leq \max\{\mathcal{I}[I]_{\ell}(x), \mathcal{I}[I]_{\ell}(y)\}$ for all $x, y \in X$. Thus, $(X, \mathcal{I}[I]_{\ell})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. Using Labels (11) and (12), we have

$$\mathcal{I}[F]_{\ell}(x * y) = \mathcal{I}[F]_{\sup}(x * y) - \mathcal{I}[F]_{\inf}(x * y) \leq \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\ell}(x)$$

and

$$\mathcal{I}[F]_{\ell}(x * y) = \mathcal{I}[F]_{\sup}(x * y) - \mathcal{I}[F]_{\inf}(x * y) \leq \mathcal{I}[F]_{\sup}(y) - \mathcal{I}[F]_{\inf}(y) = \mathcal{I}[F]_{\ell}(y),$$

and so $\mathcal{I}[F]_{\ell}(x * y) \leq \min\{\mathcal{I}[F]_{\ell}(x), \mathcal{I}[F]_{\ell}(y)\}$ for all $x, y \in X$. Therefore, $(X, \mathcal{I}[F]_{\ell})$ is a 2-fuzzy subalgebra of $(X, *, 0)$. Similarly, we can prove the desired results for $i = 2$. $\square$

Corollary 7. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,3), I(3,4), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2, 4\}$, then*

(1) *$(X, \mathcal{I}[T]_{\ell})$ is a 1-fuzzy subalgebra of $(X, *, 0)$.*
(2) *$(X, \mathcal{I}[I]_{\ell})$ and $(X, \mathcal{I}[F]_{\ell})$ are 4-fuzzy subalgebra of $(X, *, 0)$.*

By a similar way to the proof of Theorem 12, we have the following theorems.

Theorem 13. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,3), I(3,2), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2, 4\}$, then*

(1) *$(X, \mathcal{I}[T]_{\ell})$ is a 3-fuzzy subalgebra of $(X, *, 0)$.*
(2) *$(X, \mathcal{I}[I]_{\ell})$ and $(X, \mathcal{I}[F]_{\ell})$ are 2-fuzzy subalgebra of $(X, *, 0)$.*

Corollary 8. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,3), I(3,2), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2, 4\}$, then*

(1) *$(X, \mathcal{I}[T]_{\ell})$ is a 1-fuzzy subalgebra of $(X, *, 0)$.*
(2) *$(X, \mathcal{I}[I]_{\ell})$ and $(X, \mathcal{I}[F]_{\ell})$ are 4-fuzzy subalgebra of $(X, *, 0)$.*

Theorem 14. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,2), F(2,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2,4\}$*, then*

(1) $(X, \mathcal{I}[T]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are* 3*-fuzzy subalgebra of* $(X, *, 0)$.
(2) $(X, \mathcal{I}[I]_\ell)$ *is a* 2*-fuzzy subalgebra of* $(X, *, 0)$.

Corollary 9. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,2), F(2,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2,4\}$*, then*

(1) $(X, \mathcal{I}[T]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are* 1*-fuzzy subalgebra of* $(X, *, 0)$.
(2) $(X, \mathcal{I}[I]_\ell)$ *is a* 4*-fuzzy subalgebra of* $(X, *, 0)$.

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Neutrosophic Permeable Values and Energetic Subsets with Applications in *BCK*/*BCI*-Algebras

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Young Bae Jun, Florentin Smarandache, Seok-Zun Song, Hashem Bordbar (2018). Neutrosophic Permeable Values and Energetic Subsets with Applications in BCK/BCI-Algebras. *Mathematics* 6, 74; DOI: 10.3390/math6050074

Abstract: The concept of a $(\in, \in)$-neutrosophic ideal is introduced, and its characterizations are established. The notions of neutrosophic permeable values are introduced, and related properties are investigated. Conditions for the neutrosophic level sets to be energetic, right stable, and right vanished are discussed. Relations between neutrosophic permeable S- and I-values are considered.

Keywords: $(\in, \in)$-neutrosophic subalgebra; $(\in, \in)$-neutrosophic ideal; neutrosophic (anti-)permeable S-value; neutrosophic (anti-)permeable I-value; S-energetic set; I-energetic set

1. Introduction

The notion of neutrosophic set (NS) theory developed by Smarandache (see [1,2]) is a more general platform that extends the concepts of classic and fuzzy sets, intuitionistic fuzzy sets, and interval-valued (intuitionistic) fuzzy sets and that is applied to various parts: pattern recognition, medical diagnosis, decision-making problems, and so on (see [3–6]). Smarandache [2] mentioned that a cloud is a NS because its borders are ambiguous and because each element (water drop) belongs with a neutrosophic probability to the set (e.g., there are types of separated water drops around a compact mass of water drops, such that we do not know how to consider them: in or out of the cloud). Additionally, we are not sure where the cloud ends nor where it begins, and neither whether some elements are or are not in the set. This is why the percentage of indeterminacy is required and the neutrosophic probability (using subsets—not numbers—as components) should be used for better modeling: it is a more organic, smooth, and particularly accurate estimation. Indeterminacy is the zone of ignorance of a proposition's value, between truth and falsehood.

Algebraic structures play an important role in mathematics with wide-ranging applications in several disciplines such as coding theory, information sciences, computer sciences, control engineering, theoretical physics, and so on. NS theory is also applied to several algebraic structures. In particular, Jun et al. applied it to BCK/BCI-algebras (see [7–12]). Jun et al. [8] introduced the notions of energetic subsets, right vanished subsets, right stable subsets, and (anti-)permeable values in BCK/BCI-algebras and investigated relations between these sets.

In this paper, we introduce the notions of neutrosophic permeable S-values, neutrosophic permeable I-values, $(\in, \in)$-neutrosophic ideals, neutrosophic anti-permeable S-values, and neutrosophic anti-permeable I-values, which are motivated by the idea of subalgebras (i.e., S-values) and ideals (i.e., I-values), and investigate their properties. We consider characterizations of $(\in, \in)$-neutrosophic ideals. We discuss conditions for the lower (upper) neutrosophic $\in_\Phi$-subsets to be S- and I-energetic. We provide conditions for a triple (α, β, γ) of numbers to be a neutrosophic (anti-)permeable S- or I-value. We consider conditions for the upper (lower) neutrosophic $\in_\Phi$-subsets to be right stable (right vanished) subsets. We establish relations between neutrosophic (anti-)permeable S- and I-values.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$;
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$;
(III) $(\forall x \in X)\ (x * x = 0)$;
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{1}$$
$$(\forall x, y, z \in X)\,(x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y)\,, \tag{4}$$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following:

$$0 \in I, \tag{5}$$
$$(\forall x, y \in X)\,(x * y \in I,\ y \in I \rightarrow x \in I)\,. \tag{6}$$

We refer the reader to the books [13] and [14] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} = \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} = \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \{1, 2\}\}$ and $\bigwedge\{a_i \mid i \in \{1, 2\}\}$, respectively.

We let X be a nonempty set. A NS in X (see [1]) is a structure of the form

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\},$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we use the symbol $A = (A_T, A_I, A_F)$ for the NS

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

A subset A of a BCK/BCI-algebra X is said to be S-energetic (see [8]) if it satisfies

$$(\forall x, y \in X)\,(x * y \in A \;\Rightarrow\; \{x, y\} \cap A \neq \emptyset)\,. \tag{7}$$

A subset A of a BCK/BCI-algebra X is said to be *I-energetic* (see [8]) if it satisfies

$$(\forall x, y \in X)\,(y \in A \;\Rightarrow\; \{x, y * x\} \cap A \neq \emptyset)\,. \tag{8}$$

A subset A of a BCK/BCI-algebra X is said to be *right vanished* (see [8]) if it satisfies

$$(\forall x, y \in X)\,(x * y \in A \;\Rightarrow\; x \in A)\,. \tag{9}$$

A subset A of a BCK/BCI-algebra X is said to be *right stable* (see [8]) if $A * X := \{a * x \mid a \in A,\ x \in X\} \subseteq A$.

3. Neutrosophic Permeable Values

Given a NS $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$\begin{aligned}
U_T^{\in}(A;\alpha) &= \{x \in X \mid A_T(x) \geq \alpha\},\ U_T^{\in}(A;\alpha)^* = \{x \in X \mid A_T(x) > \alpha\},\\
U_I^{\in}(A;\beta) &= \{x \in X \mid A_I(x) \geq \beta\},\ U_I^{\in}(A;\beta)^* = \{x \in X \mid A_I(x) > \beta\},\\
U_F^{\in}(A;\gamma) &= \{x \in X \mid A_F(x) \leq \gamma\},\ U_F^{\in}(A;\gamma)^* = \{x \in X \mid A_F(x) < \gamma\},\\
L_T^{\in}(A;\alpha) &= \{x \in X \mid A_T(x) \leq \alpha\},\ L_T^{\in}(A;\alpha)^* = \{x \in X \mid A_T(x) < \alpha\},\\
L_I^{\in}(A;\beta) &= \{x \in X \mid A_I(x) \leq \beta\},\ L_I^{\in}(A;\beta)^* = \{x \in X \mid A_I(x) < \beta\},\\
L_F^{\in}(A;\gamma) &= \{x \in X \mid A_F(x) \geq \gamma\},\ L_F^{\in}(A;\gamma)^* = \{x \in X \mid A_F(x) > \gamma\}.
\end{aligned}$$

We say $U_T^{\in}(A;\alpha)$, $U_I^{\in}(A;\beta)$, and $U_F^{\in}(A;\gamma)$ are *upper neutrosophic $\in_\Phi$-subsets* of X, and $L_T^{\in}(A;\alpha)$, $L_I^{\in}(A;\beta)$, and $L_F^{\in}(A;\gamma)$ are *lower neutrosophic $\in_\Phi$-subsets* of X, where $\Phi \in \{T, I, F\}$. We say $U_T^{\in}(A;\alpha)^*$, $U_I^{\in}(A;\beta)^*$, and $U_F^{\in}(A;\gamma)^*$ are *strong upper neutrosophic $\in_\Phi$-subsets* of X, and $L_T^{\in}(A;\alpha)^*$, $L_I^{\in}(A;\beta)^*$, and $L_F^{\in}(A;\gamma)^*$ are *strong lower neutrosophic $\in_\Phi$-subsets* of X, where $\Phi \in \{T, I, F\}$.

Definition 1 ([7]). *A NS $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-neutrosophic subalgebra of X if the following assertions are valid:*

$$\begin{aligned}
x \in U_T^{\in}(A;\alpha_x),\ y \in U_T^{\in}(A;\alpha_y) &\;\Rightarrow\; x * y \in U_T^{\in}(A;\alpha_x \wedge \alpha_y),\\
x \in U_I^{\in}(A;\beta_x),\ y \in U_I^{\in}(A;\beta_y) &\;\Rightarrow\; x * y \in U_I^{\in}(A;\beta_x \wedge \beta_y),\\
x \in U_F^{\in}(A;\gamma_x),\ y \in U_F^{\in}(A;\gamma_y) &\;\Rightarrow\; x * y \in U_F^{\in}(A;\gamma_x \vee \gamma_y),
\end{aligned} \tag{10}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Lemma 1 ([7]). *A NS $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X if and only if $A = (A_T, A_I, A_F)$ satisfies*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x * y) \geq A_T(x) \wedge A_T(y)\\ A_I(x * y) \geq A_I(x) \wedge A_I(y)\\ A_F(x * y) \leq A_F(x) \vee A_F(y) \end{array}\right). \tag{11}$$

Proposition 1. *Every $(\in, \in)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies*

$$(\forall x \in X)\,(A_T(0) \geq A_T(x),\ A_I(0) \geq A_I(x),\ A_F(0) \leq A_F(x))\,. \tag{12}$$

Proof. Straightforward. □

Theorem 1. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then the lower neutrosophic $\in_\Phi$-subsets of X are S-energetic subsets of X, where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, y \in X$ and $\alpha \in (0, 1]$ be such that $x * y \in L_T^{\in}(A; \alpha)$. Then

$$\alpha \geq A_T(x * y) \geq A_T(x) \wedge A_T(y),$$

and thus $A_T(x) \leq \alpha$ or $A_T(y) \leq \alpha$; that is, $x \in L_T^{\in}(A; \alpha)$ or $y \in L_T^{\in}(A; \alpha)$. Thus $\{x, y\} \cap L_T^{\in}(A; \alpha) \neq \emptyset$. Therefore $L_T^{\in}(A; \alpha)$ is an S-energetic subset of X. Similarly, we can verify that $L_I^{\in}(A; \beta)$ is an S-energetic subset of X. We let $x, y \in X$ and $\gamma \in [0, 1)$ be such that $x * y \in L_F^{\in}(A; \gamma)$. Then

$$\gamma \leq A_F(x * y) \leq A_F(x) \vee A_F(y).$$

It follows that $A_F(x) \geq \gamma$ or $A_F(y) \geq \gamma$; that is, $x \in L_F^{\in}(A; \gamma)$ or $y \in L_F^{\in}(A; \gamma)$. Hence $\{x, y\} \cap L_F^{\in}(A; \gamma) \neq \emptyset$, and therefore $L_F^{\in}(A; \gamma)$ is an S-energetic subset of X. □

Corollary 1. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then the strong lower neutrosophic $\in_\Phi$-subsets of X are S-energetic subsets of X, where $\Phi \in \{T, I, F\}$.*

Proof. Straightforward. □

The converse of Theorem 1 is not true, as seen in the following example.

Example 1. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ that is given in Table 1 (see [14]).*

Table 1. Cayley table for the binary operation "$*$".

*	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	1
3	3	2	1	0	2
4	4	1	1	1	0

Let $A = (A_T, A_I, A_F)$ be a NS in X that is given in Table 2.

Table 2. Tabulation representation of $A = (A_T, A_I, A_F)$.

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.8	0.2
1	0.4	0.5	0.7
2	0.4	0.5	0.6
3	0.4	0.5	0.5
4	0.7	0.8	0.2

If $\alpha \in [0.4, 0.6)$, $\beta \in [0.5, 0.8)$, and $\gamma \in (0.2, 0.5]$, then $L_T^{\in}(A; \alpha) = \{1, 2, 3\}$, $L_I^{\in}(A; \beta) = \{1, 2, 3\}$, and $L_F^{\in}(A; \gamma) = \{1, 2, 3\}$ are S-energetic subsets of X. Because

$$A_T(4 * 4) = A_T(0) = 0.6 \ngeq 0.7 = A_T(4) \wedge A_T(4)$$

and/or

$$A_F(3*2) = A_F(1) = 0.7 \nleq 0.6 = A_F(3) \vee A_F(2),$$

it follows from Lemma 1 that $A = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic subalgebra of X.

Definition 2. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. Then (α, β, γ) is called a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$ if the following assertion is valid:*

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in U_T^{\in}(A; \alpha) \;\Rightarrow\; A_T(x) \vee A_T(y) \geq \alpha, \\ x * y \in U_I^{\in}(A; \beta) \;\Rightarrow\; A_I(x) \vee A_I(y) \geq \beta, \\ x * y \in U_F^{\in}(A; \gamma) \;\Rightarrow\; A_F(x) \wedge A_F(y) \leq \gamma \end{array} \right) \tag{13}$$

Example 2. *Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ that is given in Table 3.*

Table 3. Cayley table for the binary operation "$*$".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [14]). Let $A = (A_T, A_I, A_F)$ be a NS in X that is given in Table 4.

Table 4. Tabulation representation of $A = (A_T, A_I, A_F)$.

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.2	0.3	0.7
1	0.6	0.4	0.6
2	0.5	0.3	0.4
3	0.4	0.8	0.5
4	0.7	0.6	0.2

It is routine to verify that $(\alpha, \beta, \gamma) \in (0,2,1] \times (0.3, 1] \times [0, 0.7)$ is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$.

Theorem 2. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ satisfies the following condition:*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x*y) \leq A_T(x) \vee A_T(y) \\ A_I(x*y) \leq A_I(x) \vee A_I(y) \\ A_F(x*y) \geq A_F(x) \wedge A_F(y) \end{array} \right), \tag{14}$$

then (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$.

Proof. Let $x, y \in X$ be such that $x * y \in U_T^{\in}(A; \alpha)$. Then

$$\alpha \leq A_T(x*y) \leq A_T(x) \vee A_T(y).$$

Similarly, if $x * y \in U_I^{\in}(A;\beta)$ for $x, y \in X$, then $A_I(x) \vee A_I(y) \geq \beta$. Now, let $a, b \in X$ be such that $a * b \in U_F^{\in}(A;\gamma)$. Then

$$\gamma \geq A_F(a * b) \geq A_F(a) \wedge A_F(b).$$

Therefore (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$. □

Theorem 3. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ satisfies the following conditions:*

$$(\forall x \in X)\,(A_T(0) \leq A_T(x), A_I(0) \leq A_I(x), A_F(0) \geq A_F(x)) \tag{15}$$

and

$$(\forall x, y \in X)\begin{pmatrix} A_T(x) \leq A_T(x * y) \vee A_T(y) \\ A_I(x) \leq A_I(x * y) \vee A_I(y) \\ A_F(x) \geq A_F(x * y) \wedge A_F(y) \end{pmatrix}, \tag{16}$$

then (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$.

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in U_T^{\in}(A;\alpha)$, $a * b \in U_I^{\in}(A;\beta)$, and $u * v \in U_F^{\in}(A;\gamma)$. Then

$$\begin{aligned}\alpha &\leq A_T(x * y) \leq A_T((x * y) * x) \vee A_T(x) \\ &= A_T((x * x) * y) \vee A_T(x) = A_T(0 * y) \vee A_T(x) \\ &= A_T(0) \vee A_T(x) = A_T(x),\end{aligned}$$

$$\begin{aligned}\beta &\leq A_I(a * b) \leq A_I((a * b) * a) \vee A_I(a) \\ &= A_I((a * a) * b) \vee A_I(a) = A_I(0 * b) \vee A_I(a) \\ &= A_I(0) \vee A_I(a) = A_I(a),\end{aligned}$$

and

$$\begin{aligned}\gamma &\geq A_F(u * v) \geq A_F((u * v) * u) \wedge A_F(u) \\ &= A_F((u * u) * v) \wedge A_F(u) = A_F(0 * v) \wedge A_F(v) \\ &= A_F(0) \wedge A_F(v) = A_F(v)\end{aligned}$$

by Equations (3), (V), (15), and (16). It follows that

$$\begin{aligned}&A_T(x) \vee A_T(y) \geq A_T(x) \geq \alpha, \\ &A_I(a) \vee A_I(b) \geq A_I(a) \geq \beta, \\ &A_F(u) \wedge A_F(v) \leq A_F(u) \leq \gamma.\end{aligned}$$

Therefore (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$. □

Theorem 4. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. If (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$, then upper neutrosophic $\in_\Phi$-subsets of X are S-energetic where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in U_T^{\in}(A;\alpha)$, $a * b \in U_I^{\in}(A;\beta)$, and $u * v \in U_F^{\in}(A;\gamma)$. Using Equation (13), we have $A_T(x) \vee A_T(y) \geq \alpha$, $A_I(a) \vee A_I(b) \geq \beta$, and $A_F(u) \wedge A_F(v) \leq \gamma$. It follows that

$$A_T(x) \geq \alpha \text{ or } A_T(y) \geq \alpha, \text{ that is, } x \in U_T^{\in}(A;\alpha) \text{ or } y \in U_T^{\in}(A;\alpha);$$
$$A_I(a) \geq \beta \text{ or } A_I(b) \geq \beta, \text{ that is, } a \in U_I^{\in}(A;\beta) \text{ or } b \in U_I^{\in}(A;\beta);$$

and

$$A_F(u) \leq \gamma \text{ or } A_F(v) \leq \gamma, \text{ that is, } u \in U_F^{\in}(A;\gamma) \text{ or } v \in U_F^{\in}(A;\gamma).$$

Hence $\{x, y\} \cap U_T^{\in}(A;\alpha) \neq \emptyset$, $\{a, b\} \cap U_I^{\in}(A;\beta) \neq \emptyset$, and $\{u, v\} \cap U_F^{\in}(A;\gamma) \neq \emptyset$. Therefore $U_T^{\in}(A;\alpha)$, $U_I^{\in}(A;\beta)$, and $U_F^{\in}(A;\gamma)$ are S-energetic subsets of X. □

Definition 3. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. Then (α, β, γ) is called a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$ if the following assertion is valid:*

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in L_T^{\in}(A;\alpha) \;\Rightarrow\; A_T(x) \wedge A_T(y) \leq \alpha, \\ x * y \in L_I^{\in}(A;\beta) \;\Rightarrow\; A_I(x) \wedge A_I(y) \leq \beta, \\ x * y \in L_F^{\in}(A;\gamma) \;\Rightarrow\; A_F(x) \vee A_F(y) \geq \gamma \end{array} \right). \tag{17}$$

Example 3. *Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ that is given in Table 5.*

Table 5. Cayley table for the binary operation "$*$".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [14]). Let $A = (A_T, A_I, A_F)$ be a NS in X that is given in Table 6.

Table 6. Tabulation representation of $A = (A_T, A_I, A_F)$.

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.6	0.4
1	0.4	0.5	0.6
2	0.4	0.5	0.6
3	0.5	0.2	0.7
4	0.3	0.3	0.9

It is routine to verify that $(\alpha, \beta, \gamma) \in (0.3, 1] \times (0.2, 1] \times [0, 0.9)$ is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$.

Theorem 5. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X, then (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$.*

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in L_T^{\in}(A;\alpha)$, $a * b \in L_I^{\in}(A;\beta)$, and $u * v \in L_F^{\in}(A;\gamma)$. Using Lemma 1, we have

$$\begin{aligned}
&A_T(x) \wedge A_T(y) \leq A_T(x * y) \leq \alpha,\\
&A_I(a) \wedge A_I(b) \leq A_I(a * b) \leq \beta,\\
&A_F(u) \vee A_F(v) \geq A_F(u * v) \geq \gamma,
\end{aligned}$$

and thus (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$. □

Theorem 6. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. If (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$, then lower neutrosophic $\in_\Phi$-subsets of X are S-energetic where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in L_T^{\in}(A;\alpha)$, $a * b \in L_I^{\in}(A;\beta)$, and $u * v \in L_F^{\in}(A;\gamma)$. Using Equation (17), we have $A_T(x) \wedge A_T(y) \leq \alpha$, $A_I(a) \wedge A_I(b) \leq \beta$, and $A_F(u) \vee A_F(v) \geq \gamma$, which imply that

$$A_T(x) \leq \alpha \text{ or } A_T(y) \leq \alpha, \text{ that is, } x \in L_T^{\in}(A;\alpha) \text{ or } y \in L_T^{\in}(A;\alpha);$$
$$A_I(a) \leq \beta \text{ or } A_I(b) \leq \beta, \text{ that is, } a \in L_I^{\in}(A;\beta) \text{ or } b \in L_I^{\in}(A;\beta);$$

and

$$A_F(u) \geq \gamma \text{ or } A_F(v) \geq \gamma, \text{ that is, } u \in L_F^{\in}(A;\gamma) \text{ or } v \in L_F^{\in}(A;\gamma).$$

Hence $\{x, y\} \cap L_T^{\in}(A;\alpha) \neq \emptyset$, $\{a, b\} \cap L_I^{\in}(A;\beta) \neq \emptyset$, and $\{u, v\} \cap L_F^{\in}(A;\gamma) \neq \emptyset$. Therefore $L_T^{\in}(A;\alpha)$, $L_I^{\in}(A;\beta)$, and $L_F^{\in}(A;\gamma)$ are S-energetic subsets of X. □

Definition 4. *A NS $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$- neutrosophic ideal of X if the following assertions are valid:*

$$(\forall x \in X) \left(\begin{array}{l} x \in U_T^{\in}(A;\alpha) \;\Rightarrow\; 0 \in U_T^{\in}(A;\alpha) \\ x \in U_I^{\in}(A;\beta) \;\Rightarrow\; 0 \in U_I^{\in}(A;\beta) \\ x \in U_F^{\in}(A;\gamma) \;\Rightarrow\; 0 \in U_F^{\in}(A;\gamma) \end{array} \right), \tag{18}$$

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in U_T^{\in}(A;\alpha_x),\, y \in U_T^{\in}(A;\alpha_y) \;\Rightarrow\; x \in U_T^{\in}(A;\alpha_x \wedge \alpha_y) \\ x * y \in U_I^{\in}(A;\beta_x),\, y \in U_I^{\in}(A;\beta_y) \;\Rightarrow\; x \in U_I^{\in}(A;\beta_x \wedge \beta_y) \\ x * y \in U_F^{\in}(A;\gamma_x),\, y \in U_F^{\in}(A;\gamma_y) \;\Rightarrow\; x \in U_F^{\in}(A;\gamma_x \vee \gamma_y) \end{array} \right), \tag{19}$$

for all $\alpha, \beta, \alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma, \gamma_x, \gamma_y \in [0,1)$.

Theorem 7. *A NS $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic ideal of X if and only if $A = (A_T, A_I, A_F)$ satisfies*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(0) \geq A_T(x) \geq A_T(x * y) \wedge A_T(y) \\ A_I(0) \geq A_I(x) \geq A_I(x * y) \wedge A_I(y) \\ A_F(0) \leq A_F(x) \leq A_F(x * y) \vee A_F(y) \end{array} \right). \tag{20}$$

Proof. Assume that Equation (20) is valid, and let $x \in U_T^{\in}(A;\alpha)$, $a \in U_I^{\in}(A;\beta)$, and $u \in U_F^{\in}(A;\gamma)$ for any $x, a, u \in X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$. Hence $0 \in U_T^{\in}(A;\alpha)$, $0 \in U_I^{\in}(A;\beta)$, and $0 \in U_F^{\in}(A;\gamma)$, and thus Equation (18) is valid. Let $x, y, a, b, u, v \in X$ be such that $x * y \in U_T^{\in}(A;\alpha_x)$, $y \in U_T^{\in}(A;\alpha_y)$, $a * b \in U_I^{\in}(A;\beta_a)$, $b \in U_I^{\in}(A;\beta_b)$, $u * v \in U_F^{\in}(A;\gamma_u)$, and $v \in U_F^{\in}(A;\gamma_v)$ for all $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0,1]$

and $\gamma_u, \gamma_v \in [0,1)$. Then $A_T(x*y) \geq \alpha_x$, $A_T(y) \geq \alpha_y$, $A_I(a*b) \geq \beta_a$, $A_I(b) \geq \beta_b$, $A_F(u*v) \leq \gamma_u$, and $A_F(v) \leq \gamma_v$. It follows from Equation (20) that

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y, \\ A_I(a) &\geq A_I(a*b) \wedge A_I(b) \geq \beta_a \wedge \beta_b, \\ A_F(u) &\leq A_F(u*v) \vee A_F(v) \leq \gamma_u \vee \gamma_v. \end{aligned}$$

Hence $x \in U_T^{\in}(A; \alpha_x \wedge \alpha_y)$, $a \in U_I^{\in}(A; \beta_a \wedge \beta_b)$, and $u \in U_F^{\in}(A; \gamma_u \vee \gamma_v)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.

Conversely, let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X. If there exists $x_0 \in X$ such that $A_T(0) < A_T(x_0)$, then $x_0 \in U_T^{\in}(A; \alpha)$ and $0 \notin U_T^{\in}(A; \alpha)$, where $\alpha = A_T(x_0)$. This is a contradiction, and thus $A_T(0) \geq A_T(x)$ for all $x \in X$. Assume that $A_T(x_0) < A_T(x_0 * y_0) \wedge A_T(y_0)$ for some $x_0, y_0 \in X$. Taking $\alpha := A_T(x_0 * y_0) \wedge A_T(y_0)$ implies that $x_0 * y_0 \in U_T^{\in}(A; \alpha)$ and $y_0 \in U_T^{\in}(A; \alpha)$; but $x_0 \notin U_T^{\in}(A; \alpha)$. This is a contradiction, and thus $A_T(x) \geq A_T(x*y) \wedge A_T(y)$ for all $x, y \in X$. Similarly, we can verify that $A_I(0) \geq A_I(x) \geq A_I(x*y) \wedge A_I(y)$ for all $x, y \in X$. Now, suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in U_F^{\in}(A; \gamma)$ and $0 \notin U_F^{\in}(A; \gamma)$ by taking $\gamma = A_F(a)$. This is impossible, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose there exist $a_0, b_0 \in X$ such that $A_F(a_0) > A_F(a_0 * b_0) \vee A_F(b_0)$, and take $\gamma := A_F(a_0 * b_0) \vee A_F(b_0)$. Then $a_0 * b_0 \in U_F^{\in}(A; \gamma)$, $b_0 \in U_F^{\in}(A; \gamma)$, and $a_0 \notin U_F^{\in}(A; \gamma)$, which is a contradiction. Thus $A_F(x) \leq A_F(x*y) \vee A_F(y)$ for all $x, y \in X$. Therefore $A = (A_T, A_I, A_F)$ satisfies Equation (20). □

Lemma 2. *Every $(\in, \in)$-neutrosophic ideal $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; A_T(x) \geq A_T(y),\, A_I(x) \geq A_I(y),\, A_F(x) \leq A_F(y)). \tag{21}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and thus

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y) = A_I(0) \wedge A_I(y) = A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y) = A_F(0) \vee A_F(y) = A_F(y), \end{aligned}$$

by Equation (20). This completes the proof. □

Theorem 8. *A NS $A = (A_T, A_I, A_F)$ in a BCK-algebra X is an $(\in, \in)$-neutrosophic ideal of X if and only if $A = (A_T, A_I, A_F)$ satisfies*

$$(\forall x, y, z \in X)\left(x * y \leq z \;\Rightarrow \begin{cases} A_T(x) \geq A_T(y) \wedge A_T(z) \\ A_I(x) \geq A_I(y) \wedge A_I(z) \\ A_F(x) \leq A_F(y) \vee A_F(z) \end{cases} \right) \tag{22}$$

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X, and let $x, y, z \in X$ be such that $x * y \leq z$. Using Theorem 7 and Lemma 2, we have

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y) \geq A_T(y) \wedge A_T(z), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y) \geq A_I(y) \wedge A_I(z), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y) \leq A_F(y) \vee A_F(z). \end{aligned}$$

Conversely, assume that $A = (A_T, A_I, A_F)$ satisfies Equation (22). Because $0 * x \leq x$ for all $x \in X$, it follows from Equation (22) that

$$\begin{aligned} A_T(0) &\geq A_T(x) \wedge A_T(x) = A_T(x), \\ A_I(0) &\geq A_I(x) \wedge A_I(x) = A_I(x), \\ A_F(0) &\leq A_F(x) \vee A_F(x) = A_F(x), \end{aligned}$$

for all $x \in X$. Because $x * (x * y) \leq y$ for all $x, y \in X$, we have

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x * y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x * y) \vee A_F(y), \end{aligned}$$

for all $x, y \in X$ by Equation (22). It follows from Theorem 7 that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

Theorem 9. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of a BCK/BCI-algebra X, then the lower neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, a, u \in X$, $\alpha, \beta \in (0, 1]$, and $\gamma \in [0, 1)$ be such that $x \in L_T^{\in}(A; \alpha)$, $a \in L_I^{\in}(A; \beta)$, and $u \in L_F^{\in}(A; \gamma)$. Using Theorem 7, we have

$$\begin{aligned} \alpha &\geq A_T(x) \geq A_T(x * y) \wedge A_T(y), \\ \beta &\geq A_I(a) \geq A_I(a * b) \wedge A_I(b), \\ \gamma &\leq A_F(u) \leq A_F(u * v) \vee A_F(v), \end{aligned}$$

for all $y, b, v \in X$. It follows that

$$A_T(x * y) \leq \alpha \text{ or } A_T(y) \leq \alpha, \text{ that is, } x * y \in L_T^{\in}(A; \alpha) \text{ or } y \in L_T^{\in}(A; \alpha);$$

$$A_I(a * b) \leq \beta \text{ or } A_I(b) \leq \beta, \text{ that is, } a * b \in L_T^{\in}(A; \beta) \text{ or } b \in L_T^{\in}(A; \beta);$$

and

$$A_F(u * v) \geq \gamma \text{ or } A_F(v) \geq \gamma, \text{ that is, } u * v \in L_T^{\in}(A; \gamma) \text{ or } v \in L_T^{\in}(A; \gamma).$$

Hence $\{y, x * y\} \cap L_T^{\in}(A; \alpha)$, $\{b, a * b\} \cap L_I^{\in}(A; \beta)$, and $\{v, u * v\} \cap L_F^{\in}(A; \gamma)$ are nonempty, and therefore $L_T^{\in}(A; \alpha)$, $L_I^{\in}(A; \beta)$ and $L_F^{\in}(A; \gamma)$ are I-energetic subsets of X. □

Corollary 2. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of a BCK/BCI-algebra X, then the strong lower neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$.*

Proof. Straightforward. □

Theorem 10. *Let $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0, 1]$. If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of a BCK-algebra X, then*

(1) *the (strong) upper neutrosophic $\in_\Phi$-subsets of X are right stable where $\Phi \in \{T, I, F\}$;*
(2) *the (strong) lower neutrosophic $\in_\Phi$-subsets of X are right vanished where $\Phi \in \{T, I, F\}$.*

Proof. (1) Let $x \in X$, $a \in U_T^{\in}(A; \alpha)$, $b \in U_I^{\in}(A; \beta)$, and $c \in U_F^{\in}(A; \gamma)$. Then $A_T(a) \geq \alpha$, $A_I(b) \geq \beta$, and $A_F(c) \leq \gamma$. Because $a * x \leq a$, $b * x \leq b$, and $c * x \leq c$, it follows from Lemma 2 that $A_T(a * x) \geq A_T(a) \geq \alpha$, $A_I(b * x) \geq A_I(b) \geq \beta$, and $A_F(c * x) \leq A_F(c) \leq \gamma$; that is, $a * x \in U_T^{\in}(A; \alpha)$,

$b * x \in U_I^{\in}(A;\beta)$, and $c * x \in U_F^{\in}(A;\gamma)$. Hence the upper neutrosophic $\in_\Phi$-subsets of X are right stable where $\Phi \in \{T, I, F\}$. Similarly, the strong upper neutrosophic $\in_\Phi$-subsets of X are right stable where $\Phi \in \{T, I, F\}$.

(2) Assume that $x * y \in L_T^{\in}(A;\alpha)$, $a * b \in L_I^{\in}(A;\beta)$, and $c * d \in L_F^{\in}(A;\gamma)$ for any $x, y, a, b, c, d \in X$. Then $A_T(x * y) \leq \alpha$, $A_I(a * b) \leq \beta$, and $A_F(c * d) \geq \gamma$. Because $x * y \leq x$, $a * b \leq a$, and $c * d \leq c$, it follows from Lemma 2 that $\alpha \geq A_T(x * y) \geq A_T(x)$, $\beta \geq A_I(a * b) \geq A_I(a)$, and $\gamma \leq A_F(c * d) \leq A_F(c)$; that is, $x \in L_T^{\in}(A;\alpha)$, $a \in L_I^{\in}(A;\beta)$, and $c \in L_F^{\in}(A;\gamma)$. Therefore the lower neutrosophic $\in_\Phi$-subsets of X are right vanished where $\Phi \in \{T, I, F\}$. In a similar way, we know that the strong lower neutrosophic $\in_\Phi$-subsets of X are right vanished where $\Phi \in \{T, I, F\}$. □

Definition 5. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0, 1]$. Then (α, β, γ) is called a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$ if the following assertion is valid:*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in U_T^{\in}(A;\alpha) \Rightarrow A_T(x * y) \vee A_T(y) \geq \alpha, \\ x \in U_I^{\in}(A;\beta) \Rightarrow A_I(x * y) \vee A_I(y) \geq \beta, \\ x \in U_F^{\in}(A;\gamma) \Rightarrow A_F(x * y) \wedge A_F(y) \leq \gamma \end{array} \right). \tag{23}$$

Example 4. *(1) In Example 2, (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$.*

(2) Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ that is given in Table 7 (see [14]).

Table 7. Cayley table for the binary operation "$*$".

*	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let $A = (A_T, A_I, A_F)$ be a NS in X that is given in Table 8.

Table 8. Tabulation representation of $A = (A_T, A_I, A_F)$.

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.33	0.38	0.77
1	0.44	0.48	0.66
a	0.55	0.68	0.44
b	0.66	0.58	0.44
c	0.66	0.68	0.55

It is routine to check that $(\alpha, \beta, \gamma) \in (0.33, 1] \times (0.38, 1] \times [0, 0.77)$ is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$.

Lemma 3. *If a NS $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition of Equation (14), then*

$$(\forall x \in X)\, (A_T(0) \leq A_T(x),\ A_I(0) \leq A_I(x),\ A_F(0) \geq A_F(x)). \tag{24}$$

Proof. Straightforward. □

Theorem 11. *If a NS $A = (A_T, A_I, A_F)$ in a BCK-algebra X satisfies the condition of Equation (14), then every neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$ is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$.*

Proof. Let (α, β, γ) be a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$. Let $x, y, a, b, u, v \in X$ be such that $x * y \in U_T^{\in}(A; \alpha)$, $a * b \in U_I^{\in}(A; \beta)$, and $u * v \in U_F^{\in}(A; \gamma)$. It follows from Equations (23), (3), (III), and (V) and Lemma 3 that

$$\begin{aligned}\alpha &\le A_T((x*y)*x) \vee A_T(x) = A_T((x*x)*y) \vee A_T(x)\\ &= A_T(0*y) \vee A_T(x) = A_T(0) \vee A_T(x) = A_T(x),\end{aligned}$$

$$\begin{aligned}\beta &\le A_I((a*b)*a) \vee A_I(a) = A_I((a*a)*b) \vee A_I(a)\\ &= A_I(0*b) \vee A_I(a) = A_I(0) \vee A_I(a) = A_I(a),\end{aligned}$$

and

$$\begin{aligned}\gamma &\ge A_F((u*v)*u) \wedge A_F(u) = A_F((u*u)*v) \wedge A_F(u)\\ &= A_F(0*v) \wedge A_F(u) = A_F(0) \wedge A_F(u) = A_F(u).\end{aligned}$$

Hence $A_T(x) \vee A_T(y) \ge A_T(x) \ge \alpha$, $A_I(a) \vee A_I(b) \ge A_I(a) \ge \beta$, and $A_F(u) \wedge A_F(v) \le A_F(u) \le \gamma$. Therefore (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$. $\square$

Given a NS $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, any upper neutrosophic $\in_\Phi$-subsets of X may not be I-energetic where $\Phi \in \{T, I, F\}$, as seen in the following example.

Example 5. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ that is given in Table 9 (see [14]).*

Table 9. Cayley table for the binary operation "$*$".

*	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	1	1	0	0
4	4	2	1	2	0

Let $A = (A_T, A_I, A_F)$ be a NS in X that is given in Table 10.

Table 10. Tabulation representation of $A = (A_T, A_I, A_F)$.

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.75	0.73	0.34
1	0.53	0.45	0.58
2	0.67	0.86	0.34
3	0.53	0.56	0.58
4	0.46	0.56	0.66

Then $U_T^{\in}(A; 0.6) = \{0, 2\}$, $U_I^{\in}(A; 0.7) = \{0, 2\}$, and $U_F^{\in}(A; 0.4) = \{0, 2\}$. Because $2 \in \{0, 2\}$ and $\{1, 2*1\} \cap \{0, 2\} = \emptyset$, we know that $\{0, 2\}$ is not an I-energetic subset of X.

We now provide conditions for the upper neutrosophic $\in_\Phi$-subsets to be I-energetic where $\Phi \in \{T, I, F\}$.

Theorem 12. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0, 1]$. If (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$, then the upper neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, a, u \in X$ and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$ such that $x \in U_T^{\in}(A;\alpha)$, $a \in U_I^{\in}(A;\beta)$, and $u \in U_F^{\in}(A;\gamma)$. Because (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$, it follows from Equation (23) that

$$A_T(x * y) \vee A_T(y) \geq \alpha,\ A_I(a * b) \vee A_I(b) \geq \beta, \text{ and } A_F(u * v) \wedge A_F(v) \leq \gamma$$

for all $y, b, v \in X$. Hence

$$A_T(x * y) \geq \alpha \text{ or } A_T(y) \geq \alpha, \text{ that is, } x * y \in U_T^{\in}(A;\alpha) \text{ or } y \in U_T^{\in}(A;\alpha);$$
$$A_I(a * b) \geq \beta \text{ or } A_I(b) \geq \beta, \text{ that is, } a * b \in U_I^{\in}(A;\beta) \text{ or } b \in U_I^{\in}(A;\beta);$$

and

$$A_F(u * v) \leq \gamma \text{ or } A_F(v) \leq \gamma, \text{ that is, } u * v \in U_F^{\in}(A;\gamma) \text{ or } v \in U_F^{\in}(A;\gamma).$$

Hence $\{y, x * y\} \cap U_T^{\in}(A;\alpha)$, $\{b, a * b\} \cap U_I^{\in}(A;\beta)$, and $\{v, u * v\} \cap U_F^{\in}(A;\gamma)$ are nonempty, and therefore the upper neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$. □

Theorem 13. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ satisfies the following condition:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x) \leq A_T(x * y) \vee A_T(y) \\ A_I(x) \leq A_I(x * y) \vee A_I(y) \\ A_F(x) \geq A_F(x * y) \wedge A_F(y) \end{pmatrix}, \tag{25}$$

then (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$.

Proof. Let $x, a, u \in X$ and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$ such that $x \in U_T^{\in}(A;\alpha)$, $a \in U_I^{\in}(A;\beta)$, and $u \in U_F^{\in}(A;\gamma)$. Using Equation (25), we obtain

$$\alpha \leq A_T(x) \leq A_T(x * y) \vee A_T(y),$$
$$\beta \leq A_I(a) \leq A_I(a * b) \vee A_I(b),$$
$$\gamma \geq A_F(u) \geq A_F(u * v) \wedge A_F(v),$$

for all $y, b, v \in X$. Therefore (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$. □

Combining Theorems 12 and 13, we have the following corollary.

Corollary 3. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ satisfies the condition of Equation (25), then the upper neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$.*

Definition 6. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0,1]$. Then (α, β, γ) is called a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$ if the following assertion is valid:*

$$(\forall x, y \in X) \begin{pmatrix} x \in L_T^{\in}(A;\alpha) \ \Rightarrow \ A_T(x * y) \wedge A_T(y) \leq \alpha, \\ x \in L_I^{\in}(A;\beta) \ \Rightarrow \ A_I(x * y) \wedge A_I(y) \leq \beta, \\ x \in L_F^{\in}(A;\gamma) \ \Rightarrow \ A_F(x * y) \vee A_F(y) \geq \gamma \end{pmatrix}. \tag{26}$$

Theorem 14. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0, 1]$. If $A = (A_T, A_I, A_F)$ satisfies the condition of Equation (19), then (α, β, γ) is a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$.*

Proof. Let $x, a, u \in X$ be such that $x \in L_T^{\in}(A; \alpha)$, $a \in L_I^{\in}(A; \beta)$, and $u \in L_F^{\in}(A; \gamma)$. Then

$$
\begin{aligned}
&A_T(x * y) \wedge A_T(y) \leq A_T(x) \leq \alpha,\\
&A_I(a * b) \wedge A_I(b) \leq A_I(a) \leq \beta,\\
&A_F(u * v) \vee A_F(v) \geq A_F(u) \geq \gamma,
\end{aligned}
$$

for all $y, b, v \in X$ by Equation (20). Hence (α, β, γ) is a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$. □

Theorem 15. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0, 1]$. If (α, β, γ) is a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$, then the lower neutrosophic $\in_\Phi$-subsets of X are I-energetic where $\Phi \in \{T, I, F\}$.*

Proof. Let $x \in L_T^{\in}(A; \alpha)$, $a \in L_I^{\in}(A; \beta)$, and $u \in L_F^{\in}(A; \gamma)$. Then $A_T(x * y) \wedge A_T(y) \leq \alpha$, $A_I(a * b) \wedge A_I(b) \leq \beta$, and $A_F(u * v) \vee A_F(v) \geq \gamma$ for all $y, b, v \in X$ by Equation (26). It follows that

$$A_T(x * y) \leq \alpha \text{ or } A_T(y) \leq \alpha, \text{ that is, } x * y \in L_T^{\in}(A; \alpha) \text{ or } y \in L_T^{\in}(A; \alpha);$$

$$A_I(a * b) \leq \beta \text{ or } A_I(b) \leq \beta, \text{ that is, } a * b \in L_I^{\in}(A; \beta) \text{ or } b \in L_I^{\in}(A; \beta);$$

and

$$A_F(u * v) \geq \gamma \text{ or } A_F(v) \geq \gamma, \text{ that is, } u * v \in L_F^{\in}(A; \gamma) \text{ or } v \in L_F^{\in}(A; \gamma).$$

Hence $\{y, x * y\} \cap L_T^{\in}(A; \alpha)$, $\{b, a * b\} \cap L_I^{\in}(A; \beta)$ and $\{v, u * v\} \cap L_F^{\in}(A; \gamma)$ are nonempty, and therefore the lower neutrosophic $\in_\Phi$-subsets of X are I-energetic where $\Phi \in \{T, I, F\}$. □

Combining Theorems 14 and 15, we obtain the following corollary.

Corollary 4. *Let $A = (A_T, A_I, A_F)$ be a NS in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$, where Λ_T, Λ_I, and Λ_F are subsets of $[0, 1]$. If $A = (A_T, A_I, A_F)$ satisfies the condition of Equation (19), then the lower neutrosophic $\in_\Phi$-subsets of X are I-energetic where $\Phi \in \{T, I, F\}$.*

Theorem 16. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of a BCK-algebra X, then every neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$ is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$.*

Proof. Let (α, β, γ) be a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$. Let $x, y, a, b, u, v \in X$ be such that $x * y \in L_T^{\in}(A; \alpha)$, $a * b \in L_I^{\in}(A; \beta)$, and $u * v \in L_F^{\in}(A; \gamma)$. It follows from Equations (26), (3), (III), and (V) and Proposition 1 that

$$
\begin{aligned}
\alpha &\geq A_T((x * y) * x) \wedge A_T(x) = A_T((x * x) * y) \wedge A_T(x)\\
&= A_T(0 * y) \wedge A_T(x) = A_T(0) \wedge A_T(x) = A_T(x),
\end{aligned}
$$

$$
\begin{aligned}
\beta &\geq A_I((a * b) * a) \wedge A_I(a) = A_I((a * a) * b) \wedge A_I(a)\\
&= A_I(0 * b) \wedge A_I(a) = A_I(0) \wedge A_I(a) = A_I(a),
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma &\leq A_F((u * v) * u) \vee A_F(u) = A_F((u * u) * v) \vee A_F(u)\\
&= A_F(0 * v) \vee A_F(u) = A_F(0) \vee A_F(u) = A_F(u).
\end{aligned}
$$

Hence $A_T(x) \wedge A_T(y) \leq A_T(x) \leq \alpha$, $A_I(a) \wedge A_I(b) \leq A_I(a) \leq \beta$, and $A_F(u) \vee A_F(v) \geq A_F(u) \geq \gamma$. Therefore (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$. $\square$

4. Conclusions

Using the notions of subalgebras and ideals in BCK/BCI-algebras, Jun et al. [8] introduced the notions of energetic subsets, right vanished subsets, right stable subsets, and (anti-)permeable values in BCK/BCI-algebras, as well as investigated relations between these sets. As a more general platform that extends the concepts of classic and fuzzy sets, intuitionistic fuzzy sets, and interval-valued (intuitionistic) fuzzy sets, the notion of NS theory has been developed by Smarandache (see [1,2]) and has been applied to various parts: pattern recognition, medical diagnosis, decision-making problems, and so on (see [3–6]). In this article, we have introduced the notions of neutrosophic permeable S-values, neutrosophic permeable I-values, $(\in, \in)$-neutrosophic ideals, neutrosophic anti-permeable S-values, and neutrosophic anti-permeable I-values, which are motivated by the idea of subalgebras (s-values) and ideals (I-values), and have investigated their properties. We have considered characterizations of $(\in, \in)$-neutrosophic ideals and have discussed conditions for the lower (upper) neutrosophic $\in_\Phi$-subsets to be S- and I-energetic. We have provided conditions for a triple (α, β, γ) of numbers to be a neutrosophic (anti-)permeable S- or I-value, and have considered conditions for the upper (lower) neutrosophic $\in_\Phi$-subsets to be right stable (right vanished) subsets. We have established relations between neutrosophic (anti-)permeable S- and I-values.

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Left (Right)-Quasi Neutrosophic Triplet Loops (Groups) and Generalized BE-Algebras

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Xiaohong Zhang, Xiaoying Wu, Florentin Smarandache, Minghao Hu (2018). Left (Right)-Quasi Neutrosophic Triplet Loops (Groups) and Generalized BE-Algebras. *Symmetry* 10, 241; DOI: 10.3390/sym10070241

Abstract: The new notion of a neutrosophic triplet group (NTG) is proposed by Florentin Smarandache; it is a new algebraic structure different from the classical group. The aim of this paper is to further expand this new concept and to study its application in related logic algebra systems. Some new notions of left (right)-quasi neutrosophic triplet loops and left (right)-quasi neutrosophic triplet groups are introduced, and some properties are presented. As a corollary of these properties, the following important result are proved: for any commutative neutrosophic triplet group, its every element has a unique neutral element. Moreover, some left (right)-quasi neutrosophic triplet structures in BE-algebras and generalized BE-algebras (including CI-algebras and pseudo CI-algebras) are established, and the adjoint semigroups of the BE-algebras and generalized BE-algebras are investigated for the first time.

Keywords: neutrosophic triplet; quasi neutrosophic triplet loop; quasi neutrosophic triplet group; BE-algebra; CI-algebra

1. Introduction

The symmetry exists in the real world, and group theory is a mathematical tool for describing symmetry. At the same time, in order to describe the generalized symmetry, the concept of group is popularized in different ways, for example, the notion of a generalized group is introduced (see [1–4]). Recently, F. Smarandache [5,6] introduced another new algebraic structure, namely: neutrosophic triplet group, which comes from the theory of the neutrosophic set (see [7–11]). As a new extension of the concept of group, the neutrosophic triplet group has attracted the attention of many scholars, and a series of related papers have been published [12–15].

On the other hand, in the last twenty years, the non-classical logics, such as various fuzzy logics, have made great progress. At the same time, the research on non-classical logic algebras that are related to it have also made great achievements [16–26]. As a generalization of BCK-algebra, H.S. Kim and Y.H. Kim [27] introduced the notion of BE-algebra. Since then, some scholars have studied ideals (filters), congruence relations of BE-algebras, and various special BE-algebras have been proposed, these research results are included in the literature [28–31] and monograph [32]. In 2013 and 2016, the new notions of pseudo BE-algebra and commutative pseudo BE-algebra were introduced, and some new properties were obtained [33,34]. Similar to BCI-algebra as a generalization of BCK-algebra, B.L. Meng introduced the concept of CI-algebra, which is as a generalization of BE-algebra, and studied the structures and closed filters of CI-algebras [35–37]. After that, the CI-algebras and their related algebraic structures (such as Q-algebras, pseudo Q-algebras, pseudo CI-algebras, and pseudo BCH-algebras) have been extensively studied [38–46].

This paper will combine the above two directions to study general neutrosophic triplet structures and the relationships between these structures and generalized BE-algebras. On the one hand, we introduce various general neutrosophic triplet structures, such as (*l-l*)-type, (*l-r*)-type, (*r-l*)-type, (*r-r*)-type, (*l-lr*)-type, (*r-lr*)-type, (*lr-l*)-type, and (*lr-r*)-type quasi neutrosophic triplet loops (groups), and investigate their basic properties. Moreover, we get an important corollary, namely: that for any commutative neutrosophic triplet group, its every element has a unique neutral element. On the other hand, we further study the properties of (pseudo) BE-algebras and (pseudo) CI-algebras, and the general neutrosophic triplet structures that are contained in a BE-algebra (CI-algebra) and pseudo BE-algebra (pseudo CI-algebra). Moreover, for the first time, we introduce the concepts of adjoint semigroups of BE-algebras and generalized BE-algebras (including CI-algebras, pseudo BE-algebras, and pseudo CI-algebras) and discuss some interesting topics.

2. Basic Concepts

Definition 1. *([5,6]) Let N be a set together with a binary operation *. Then, N is called a neutrosophic triplet set if, for any a∈N, there exists a neutral of 'a', called neut(a), and an opposite of 'a', called anti(a), with neut(a) and anti(a), belonging to N, such that:*

$$a * neut(a) = neut(a) * a = a;$$

$$a * anti(a) = anti(a) * a = neut(a).$$

It should be noted that *neut*(*a*) and *anti*(*a*) may not be unique here for some $a \in N$. We call (*a*, *neut*(*a*), and *anti*(*a*)) a neutrosophic triplet for the determined *neut*(*a*) and *anti*(*a*).

Remark 1. *In the original definition, the neutral element is different from the unit element in the traditional group theory. The above definition of this paper takes away such restriction, please see the Remark 3 in Ref. [12].*

Definition 2. *([5,6,13]) Let (N, *) be a neutrosophic triplet set.*

*(1) If * is well-defined, that is, for any a, b ∈ N, one has a * b ∈ N. Then, N is called a neutrosophic triplet loop.*
*(2) If N is a neutrosophic triplet loop, and * is associative, that is, (a * b) * c= a * (b * c) for all a, b, c ∈ N. Then, N is called a neutrosophic triplet group.*
*(3) If N is a neutrosophic triplet group, and * is commutative, that is, a * b = b * a for all a, b ∈ N. Then, N is called a commutative neutrosophic triplet group.*

Definition 3. *([27,35,41,42]) A CI-algebra (dual Q-algebra) is an algebra (X; →, 1) of type (2, 0), satisfying the following conditions:*

(i) $x \to x = 1$,
(ii) $1 \to x = x$,
(iii) $x \to (y \to z) = y \to (x \to z)$, *for all* $x, y, z \in X$.

A CI-algebra (X; →, 1)is called a BE-algebra, if it satisfies the following axiom:

(iv) $x \to 1 = 1$, *for all* $x \in X$.

A CI-algebra (X; →, 1)is called a dual BCH-algebra, if it satisfies the following axiom:

(v) $x \to y = y \to x = 1 \Rightarrow x = y$.

A binary relation ≤ on CI-algebra (BE-algebra) *X*, is defined by $x \leq y$ if, and only if, $x \to y = 1$.

Definition 4. *([33,43,45]) An algebra (X; $\to$, $\rightsquigarrow$, 1) of type (2, 2, 0) is called a dual pseudo Q-algebra if, for all x, y, z $\in$ X, it satisfies the following axioms:*

(dpsQ1) $x \to x = x \rightsquigarrow x = 1$,
(dpsQ2) $1 \to x = 1 \rightsquigarrow x = x$,
(dpsQ3) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.

A dual pseudo Q-algebra X is called a pseudo CI-algebra, if it satisfies the following condition:

(psCI) $x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1$.

A pseudo CI-algebra X is called a pseudo BE-algebra, if it satisfies the following condition:

(psBE) $x \to 1 = x \rightsquigarrow 1 = 1$, *for all* $x \in X$.

A pseudo CI-algebra X is called a pseudo BCH-algebra, if it satisfies the following condition:

(psBCH) $x \to y = y \rightsquigarrow x = 1 \Rightarrow x = y$.

In a dual pseudo-Q algebra, one can define the following binary relations:

$$x \leq_{\to} y \Leftrightarrow x \to y = 1.\ x \leq_{\rightsquigarrow} y \Leftrightarrow x \rightsquigarrow y = 1.$$

Obviously, a dual pseudo-Q algebra X is a pseudo CI-algebra if, and only if, $\leq_{\to} = \leq_{\rightsquigarrow}$.

3. Various Quasi Neutrosophic Triplet Loops (Groups)

Definition 5. *Let N be a set together with a binary operation * (that is, (N, *) be a loop) and a $\in$ N.*

*(1) If exist b, c $\in$ N, such that a * b = a and a * c = b, then a is called an NT-element with (r-r)- property;*
*(2) If exist b, c $\in$ N, such that a * b = a and c * a = b, then a is called an NT-element with (r-l)- property;*
*(3) If exist b, c $\in$ N, such that b * a = a and c * a = b, then a is called an NT-element with (l-l)- property;*
*(4) If exist b, c $\in$ N, such that b * a = a and a * c = b, then a is called an NT-element with (l-r)- property;*
*(5) If exist b, c $\in$ N, such that a * b = b * a = a and c * a = b, then a is called an NT-element with (lr-l)-property;*
*(6) If exist b, c $\in$ N, such that a * b = b * a = a and a * c = b, then a is called an NT-element with (lr-r)-property;*
*(7) If exist b, c $\in$ N, such that b * a = a and a * c = c * a = b, then a is called an NT-element with (l-lr)-property;*
*(8) If exist b, c $\in$ N, such that a * b = a and a * c = c * a = b, then a is called an NT-element with (r-lr)-property;*
*(9) If exist b, c $\in$ N, such that a * b = b * a = a and a * c = c * a = b, then a is called an NT-element with (lr-lr)-property.*

It is easy to verify that, (i) if *a* is an NT-element with (*l-lr*)-property, then *a* is an NT-element with (*l-l*)-property and (*l-r*)-property; if *a* is an NT-element with (*lr-l*)-property, then *a* is an NT-element with (*l-l*)-property and (*r-l*)-property; and so on; (ii) a neutrosophic triplet loop (*N*, *) is a neutrosophic triplet group if, and only if, every element in *N* is an NT-element with (*lr-lr*)-property; (iii) if * is commutative, then the above properties coincide. Moreover, the following example shows that (*r-l*)-property and (*r-r*)-property cannot infer to (*r-lr*)-property, and (*r-r*)-property and (*l-lr*)-property cannot infer to (*lr-lr*)-property.

Example 1. *Let N = {a, b, c, d}. The operation * on N is defined as Table 1. Then, (N, *) is a loop, and a is an NT-element with (lr-lr)-property; b is an NT-element with (lr-r)-property; c is an NT-element with (r-l)-property and (r-r)-property, but c is not an NT-element with (r-lr)-property; and d is an NT-element with (r-r)-property and (l-lr)-property, but d is not an NT-element with (lr-lr)-property.*

Table 1. Neutrosophic triplet (NT)-elements in a loop.

*	**a**	**b**	**c**	**d**
a	a	a	a	d
b	c	a	b	c
c	c	b	d	a
d	a	d	b	a

Definition 6. *Let (N, *) be a loop (semi-group). If for every element a in N, a is an NT-element with (r-r)-property, then (N, *) is called (r-r)-quasi neutrosophic triplet loop (group). Similarly, if for every element a in N, a is an NT-element with (r-l)-, (l-l)-, (l-r)-, (lr-l)-, (lr-r)-, (l-lr)-, (r-lr)-property, then (N, *) is called (r-l)-, (l-l)-, (l-r)-, (lr-l)-, (lr-r)-, (l-lr)-, (r-lr)-quasi neutrosophic triplet loop (group), respectively. All of these generalized neutrosophic triplet loops (groups) are collectively known as quasi neutrosophic triplet loops (groups).*

Remark 2. *For quasi neutrosophic triplet loops (groups), we will use the notations like neutrosophic triplet loops (groups), for example, to denote a (r-r)-neutral of 'a' by* $neut_{(r\text{-}r)}(a)$*, denote a (r-r)-opposite of 'a' by* $anti_{(r\text{-}r)}(a)$*, where 'a' is an NT-element with (r-r)-property. If* $neut_{(r\text{-}r)}(a)$ *and* $anti_{(r\text{-}r)}(a)$ *are not unique, then denote the set of all (r-r)-neutral of 'a' by* $\{neut_{(r\text{-}r)}(a)\}$*, denote the set of all (r-r)-opposite of 'a' by* $\{anti_{(r\text{-}r)}(a)\}$*.*

For the loop $(N, *)$ in Example 1, we can verify that $(N, *)$ is a $(r\text{-}r)$-quasi neutrosophic triplet loop, and we have the following:

$$neut_{(r\text{-}r)}(a) = a,\ anti_{(r\text{-}r)}(a) = a;\ neut_{(r\text{-}r)}(b) = c,\ \{anti_{(r\text{-}r)}(b)\} = \{a, d\};$$

$$neut_{(r\text{-}r)}(c) = a,\ anti_{(r\text{-}r)}(c) = d;\ neut_{(r\text{-}r)}(d) = b,\ anti_{(r\text{-}r)}(d) = c.$$

Theorem 1. *If (N, *) is a (l-lr)-quasi neutrosophic triplet group, then (N, *) is a neutrosophic triplet group. Moreover, if (N, *) is a (r-lr)-quasi neutrosophic triplet group, then (N, *) is a neutrosophic triplet group.*

Proof. Suppose that $(N, *)$ is a $(l\text{-}lr)$-quasi neutrosophic triplet group. For any $a \in N$, by Definitions 5 and 6, we have the following:

$$neut_{(l\text{-}lr)}(a) * a = a,\ anti_{(l\text{-}lr)}(a) * a = a * anti_{(l\text{-}lr)}(a) = neut_{(l\text{-}lr)}(a).$$

Here, $neut_{(l\text{-}lr)}(a) \in \{neut_{(l\text{-}lr)}(a)\}$, $anti_{(l\text{-}lr)}(a) \in \{anti_{(l\text{-}lr)}(a)\}$. Applying associative law we get the following:

$$a * neut_{(l\text{-}lr)}(a) = a * (anti_{(l\text{-}lr)}(a) * a) = (a * anti_{(l\text{-}lr)}(a)) * a = neut_{(l\text{-}lr)}(a) * a = a.$$

This means that $neut_{(l\text{-}lr)}(a)$ is a right neutral of 'a'. From the arbitrariness of a, it is known that $(N, *)$ is a neutrosophic triplet group.

Another result can be proved similarly. □

Theorem 2. *Let (N, *) be a (r-lr)-quasi neutrosophic triplet group such that:*

$$(s * p) * a = a * (s * p),\ \forall\, s \in \{neut_{(r\text{-}lr)}(a)\},\ \forall\, p \in \{anti_{(r\text{-}lr)}(a)\}.$$

Then,

(1) *for any* $a \in N$, $s \in \{neut_{(r\text{-}lr)}(a)\} \Rightarrow s * s = s$.
(2) *for any* $a \in N$, $s, t \in \{neut_{(r\text{-}lr)}(a)\} \Rightarrow s * t = t$.
(3) *when * is commutative, for any* $a \in N$, $neut_{(r\text{-}lr)}(a)$ *is unique.*

Proof. (1) Assume $s \in \{neut_{(r\text{-}lr)}(a)\}$, then $a * s = a$, and exist $p \in N$, such that $p * a = a * p = s$. Thus,

$$(s * p) * a = s * (p * a) = s * s,$$

$$a * (s * p) = (a * s) * p = a * p = s.$$

According to the hypothesis, $(s * p) * a = a * (s * p)$, it follows that $s * s = s$.

(2) Assume s, $t \in \{neut_{(r\text{-}lr)}(a)\}$, then $a * s = a$, $a * t = a$, and exist $p, q \in N$, such that $p * a = a * p = s$, $q * a = a * q = t$. Thus,

$$(s * q) * a = s * (q * a) = s * t,$$

$$a * (s * q) = (a * s) * q = a * q = t.$$

According to the hypothesis, $(s * p) * a = a * (s * p)$, it follows that $s * t = t$.

(3) Suppose $a \in N$, s, $t \in \{neut_{(r\text{-}lr)}(a)\}$. Applying Theorem (2) to s and t we have $s * t = t$. Moreover, applying Therorem (2) to t and s we have $t * s = s$. Hence, when * is commutative, $s * t = t * s$. Therefore, $s = t$, that is, $neut_{(r\text{-}lr)}(a)$ is unique. □

Corollary 1. *Let (N, *) be a commutative neutrosophic triplet group. Then neut(a) is unique for any* $a \in N$.

Proof. Since all neutrosophic triplet groups are *(r-lr)*-quasi neutrosophic triplet groups, and * is commutative, then the assumption conditions in Theorem 2 are valid for N, so applying Theorem 2 (3), we get that $neut(a)$ is unique for any $a \in N$. □

The following examples show that the neutral element may be not unique in the neutrosophic triplet loop.

Example 2. *Let N = {1, 2, 3}. Define binary operation * on N as following Table 2. Then, (N, *) is a commutative neutrosophic triplet loop, and {neut(1)} = {1, 2}. Since* $(1 * 3) * 3 \neq 1 * (3 * 3)$*, so (N, *) is not a neutrosophic triplet group.*

Table 2. Commutative neutrosophic triplet loop.

*	1	2	3
1	1	1	2
2	1	2	3
3	2	3	3

Example 3. *Let N = {1, 2, 3, 4}. Define binary operation * on N as following Table 3. Then, (N, *) is a neutrosophic triplet loop, and {neut(4)} = {2, 3}. Since* $(4 * 1) * 1 \neq 4 * (1 * 1)$*, so (N, *) is not a neutrosophic triplet group.*

Table 3. Non-commutative neutrosophic triplet loop.

*	1	2	3	4
1	3	1	1	3
2	4	2	2	4
3	1	3	3	4
4	3	4	4	2

4. Quasi Neutrosophic Triplet Structures in BE-Algebras and CI-Algebras

From the definition of BE-algebra and CI-algebra (see Definition 3), we can see that '1' is a left neutral element of every element, that is, BE-algebras and CI-algebras are directly related to quasi neutrosophic triplet structures. This section will reveal the various internal connections among them.

4.1. BE-Algebras (CI-Algebras) and (l-l)-Quasi Neutrosophic Triplet Loops

Theorem 3. *Let (X; →, 1) be a BE-algebra. Then (X, →) is a (l-l)-quasi neutrosophic triplet loop. And, when* $|X|>1$, *(X, →) is not a (lr-l)-quasi neutrosophic triplet loop with neutral element 1.*

Proof. By Definition 3, for all $x \in X$, $1 \to x = x$ and $x \to x = 1$. According Definition 6, we know that $(X, \to)$ is a (l-l)-quasi neutrosophic triplet loop, such that:

$$1 \in \{neut_{(l\text{-}l)}(x)\},\ x \in \{anti_{(l\text{-}l)}(x)\}, \text{ for any } x \in X.$$

If $|X| > 1$, then exist $x \in X$, such that $x \neq 1$. Using Definition 3 (iv), $x \to 1 = 1 \neq x$, this means that 1 is not a right neutral element of x. Hence, $(X, \to)$ is not a *(lr-l)*-quasi neutrosophic triplet loop with neutral element 1. □

Example 4. *Let X = {a, b, c, 1}. Define binary operation * on N as following Table 4. Then, (X; →, 1) is a BE-algebra, and (X, →) is a (l-l)-quasi neutrosophic triplet loop, such that:*

$$\{neut_{(l\text{-}l)}(a)\} = \{1\}, \{anti_{(l\text{-}l)}(a)\} = \{a, c\}; \{neut_{(l\text{-}l)}(b)\} = \{1\}, \{anti_{(l\text{-}l)}(b)\} = \{b, c\};$$

$$\{neut_{(l\text{-}l)}(c)\} = \{1\}, \{anti_{(l\text{-}l)}(c)\} = \{c\}; \{neut_{(l\text{-}l)}(1)\} = \{1\}, \{anti_{(l\text{-}l)}(1)\} = \{1\}.$$

Table 4. BE-algebra and (l-l)-quasi neutrosophic triplet loop (1).

→	*a*	*b*	*c*	*1*
a	1	b	b	1
b	a	1	a	1
c	1	1	1	1
1	a	b	c	1

Example 5. *Let X = {a, b, c, 1}. Define binary operation * on N as following Table 5. Then, (X; →, 1) is a BE-algebra, and (X, →) is a (l-l)-quasi neutrosophic triplet loop such that:*

$$\{neut_{(l\text{-}l)}(a)\} = \{1\}, \{anti_{(l\text{-}l)}(a)\} = \{a\}; \{neut_{(l\text{-}l)}(b)\} = \{1\}, \{anti_{(l\text{-}l)}(b)\} = \{b\};$$

$$\{neut_{(l\text{-}l)}(c)\} = \{1\}, \{anti_{(l\text{-}l)}(c)\} = \{c\}; \{neut_{(l\text{-}l)}(1)\} = \{1\}, \{anti_{(l\text{-}l)}(1)\} = \{1\}.$$

Table 5. BE-algebra and (l-l)-quasi neutrosophic triplet loop (2).

→	*a*	*b*	*c*	*1*
a	1	b	c	1
b	a	1	c	1
c	a	b	1	1
1	a	b	c	1

Definition 7. *([36]) Let $(X; \to, 1)$ be a CI-algebra and $a \in X$. If for any $x \in X$, $a \to x = 1$ implies $a = x$, then a is called an atom in X. Denote $A(X) = \{a \in X \mid a \text{ is an atom in } X\}$, it is called the singular part of X. A CI-algebra $(X; \to, 1)$ is said to be singular if every element of X is an atom.*

Lemma 1. *([35–37]) If $(X; \to, 1)$ is a CI-algebra, then for all $x, y \in X$:*

(1) $x \to ((x \to y) \to y) = 1$,
(2) $1 \to x = 1$ (or equivalently, $1 \leq x$) implies $x = 1$,
(3) $(x \to y) \to 1 = (x \to 1) \to (y \to 1)$.

Lemma 2. *([36]) Let $(X; \to, 1)$ be a CI-algebra. If $a, b \in X$ are atoms in X, then the following are true:*

(1) $a = (a \to 1) \to 1$,
(2) $(a \to b) \to 1 = b \to a$,
(3) $((a \to b) \to 1) \to 1 = a \to b$,
(4) for any $x \in X$, $(a \to x) \to (b \to x) = b \to a$,
(5) for any $x \in X$, $(a \to x) \to b = (b \to x) \to a$,
(6) for any $x \in X$, $(a \to x) \to (y \to b) = (b \to x) \to (y \to a)$.

Definition 8. *Let $(X; \to, 1)$ be a CI-algebra. If for any $x \in X$, $x \to 1 = x$, then $(X; \to, 1)$ is said to be a strong singular.*

Proposition 1. *If $(X; \to, 1)$ is a strong singular CI-algebra. Then $(X; \to, 1)$ is a singular CI-algebra.*

Proof. For any $x \in X$, assume that $a \to x = 1$, where $a \in X$. By Definition 8, we have $x \to 1 = x$, $a \to 1 = a$. Hence, applying Definition 3,

$$a = a \to 1 = a \to (x \to x) = x \to (a \to x) = x \to 1 = x.$$

By Definition 7, x is an atom. Therefore, $(X; \to, 1)$ is singular CI-algebra. □

Proposition 2. *Let $(X; \to, 1)$ be a CI-algebra. Then $(X; \to, 1)$ is a strong singular CI-algebra if, and only if, $(X; \to, 1)$ is an associative BCI-algebra.*

Proof. Obviously, every associative BCI-algebra is a strong singular CI-algebra (see [36] and Proposition 1 in Ref. [12]).

Assume that $(X; \to, 1)$ is a strong singular CI-algebra.

(1) For any $x, y \in X$, if $x \to y = y \to x = 1$, then, by Definitions 8 and 3, we have the following:

$$x = x \to 1 = x \to (y \to x) = y \to (x \to x) = y \to 1 = y.$$

(2) For any $x, y, z \in X$, by Proposition 1 and Lemma 2 (4), we can get the following:

$$(y \to z) \to ((z \to x) \to (y \to x)) = (y \to z) \to (y \to z) = 1.$$

Combining Proof (1) and (2), we know that $(X; \to, 1)$ is a BCI-algebra. From this, applying Definition 8 and Proposition 1 in Ref. [12], $(X; \to, 1)$ is an associative BCI-algebra. □

Theorem 4. *Let $(X; \to, 1)$ be a CI-algebra. Then, $(X, \to)$ is a (l-l)-quasi neutrosophic triplet loop. Moreover, $(X, \to)$ is a neutrosophic triplet group if, and only if, $(X; \to, 1)$ is a strong singular CI-algebra (associative BCI-algebra).*

Proof. It is similar to the proof of Theorem 3, and we know that $(X, \to)$ is a $(l\text{-}l)$-quasi neutrosophic triplet loop.

If $(X; \to, 1)$ is a strong singular CI-algebra, using Proposition 2, $(X; \to, 1)$ is an associative BCI-algebra. Hence, $\to$ is associative and commutative, it follows that $(X, \to)$ is a neutrosophic triplet group.

Conversely, if $(X, \to)$ is a neutrosophic triplet group, then $\to$ is associative, thus

$$x \to 1 = x \to (x \to x) = (x \to x) \to x = 1 \to x = x.$$

By Definition 8 we know that $(X; \to, 1)$ is a strong singular CI-algebra. □

Example 6. *Let X = {a, b, c, d, e,1}. Define operation → on X, as following Table 6. Then, (X; →, 1) is a CI-algebra, and (X, →) is a (l-l)-quasi neutrosophic triplet loop, such that*

$$\{neut_{(l\text{-}l)}(a)\} = \{1\}, \{anti_{(l\text{-}l)}(a)\} = \{a,b\}; \{neut_{(l\text{-}l)}(b)\} = \{1\}, \{anti_{(l\text{-}l)}(b)\} = \{a,b,c\};$$

$$\{neut_{(l\text{-}l)}(c)\} = \{1\}, \{anti_{(l\text{-}l)}(c)\} = \{c,d,e\}; \{neut_{(l\text{-}l)}(d)\} = \{1\}, \{anti_{(l\text{-}l)}(d)\} = \{d,e\};$$

$$\{neut_{(l\text{-}l)}(e)\}=\{1\}, \{anti_{(l\text{-}l)}(e)\}=\{d,e\}; \{neut_{(l\text{-}l)}(1)\}=\{1\}, \{anti_{(l\text{-}l)}(1)\}=\{1\}.$$

Table 6. CI-algebra and $(l\text{-}l)$-quasi neutrosophic triplet loop.

$\to$	**a**	**b**	**c**	**d**	**e**	**1**
a	1	1	c	c	c	1
b	1	1	c	c	c	1
c	d	1	1	a	b	c
d	c	c	1	1	1	c
e	c	c	1	1	1	c
1	a	b	c	d	e	1

4.2. BE-Algebras (CI-Algebras) and Their Adjoint Semi-Groups

I. Fleischer [16] studied the relationship between BCK-algebras and semigroups, and W. Huang [17] studied the close connection between the BCI-algebras and semigroups. In this section, we have studied the adjoint semigroups of the BE-algebras and CI-algebras, and will give some interesting examples.

For any BE-algebra or CI-algebra $(X; \to, 1)$, and any element a in X, we use p_a to denote the self-map of X defined by the following:

$$p_a\colon X \to X; \mapsto a \to x, \text{ for all } x \in X.$$

Theorem 5. *Let (X; →, 1) be a BE-algebra (or CI-algebra), and M(X) be the set of finite products* $p_a * \ldots * p_b$ *of self-map of X with a, . . . , b ∈X, where * represents the composition operation of mappings. Then, (M(X), *) is a commutative semigroup with identity* p_1.

Proof. Since the composition operation of mappings satisfies the associative law, $(M(X), *)$ is a semigroup. Moreover, since

$$p_1\colon X \to X \mapsto 1 \to x, \text{ for all } x \in X.$$

Applying Definition 3 (ii), we get that $p_1(x)=x$ for any $x \in X$. Hence, $p_1*m= p_1*m=m$ for any $m \in M(X)$.

For any $a, b \in X$, using Definition 3 (iii) we have ($\forall x \in X$) the following:

$$(p_a{*}\, p_b)(x) = p_a(b \to x) = a \to (b \to x) = b \to (a \to x) = p_b(a \to x) = (p_b{*}\, p_a)(x).$$

Therefore, $(M(X), *)$ is a commutative semigroup with identity p_1. □

Now, we call $(M(X), *)$ the adjoint semigroup of X.

Example 7. *Let X = {a, b, c, 1}. Define operation → on X, as following Table 7. Then, (X; →, 1) is a BE-algebra, and*

p_a: $X \to X$; $a \mapsto 1$, $b \mapsto 1$, $c \mapsto 1$, $1 \mapsto 1$. It is abbreviated to $p_a = (1, 1, 1, 1)$.
p_b: $X \to X$; $a \mapsto c$, $b \mapsto 1$, $c \mapsto a$, $1 \mapsto 1$. It is abbreviated to $p_b = (c, 1, a, 1)$.
p_c: $X \to X$; $a \mapsto 1$, $b \mapsto 1$, $c \mapsto 1$, $1 \mapsto 1$. It is abbreviated to $p_c = (1, 1, 1, 1)$.
p_1: $X \to X$; $a \mapsto a$, $b \mapsto b$, $c \mapsto c$, $1 \mapsto 1$. It is abbreviated to $p_1 = (a, b, c, 1)$.

We can verify that $p_a * p_a = p_a$, $p_a * p_b = p_a$, $p_a * p_c = p_a$; $p_b * p_b = (a, 1, c, 1)$, $p_b * p_c = p_c = p_a$; $p_a * (p_b * p_b) = p_a$, $p_b * (p_b * p_b) = p_b$, $p_c * (p_b * p_b) = p_c = p_a$. Denote $p_{bb} = p_b{*}\, p_b = (a, 1, c, 1)$, then $M(X) = \{p_a, p_b, p_{bb}, p_1\}$, and its Cayley table is Table 8. Obviously, $(M(X), *)$ is a commutative neutrosophic triplet group and

$$neut(p_a) = p_a,\ anti(p_a) = p_a;\ neut(p_b) = p_{bb},\ anti(p_b) = p_b;\ neut(p_{bb}) = p_{bb},\ anti(p_{bb}) = p_{bb};\ neut(p_1) = p_1,\ anti(p_1) = p_1.$$

Table 7. BE-algebra.

→	a	b	c	1
a	1	1	1	1
b	c	1	a	1
c	1	1	1	1
1	a	b	c	1

Table 8. Adjoint semigroup of the above BE-algebra.

*	p_a	p_b	p_{bb}	p_1
p_a	p_a	p_a	p_a	p_a
p_b	p_a	p_{bb}	p_b	p_b
p_{bb}	p_a	p_b	p_{bb}	p_{bb}
p_1	p_a	p_b	p_{bb}	p_1

Example 8. *Let X = {a, b, 1}. Define operation → on X, as following Table 9. Then, (X; →, 1) is a CI-algebra, and*

p_a: $X \to X$; $a \mapsto 1$, $b \mapsto a$, $1 \mapsto b$. It is abbreviated to $p_a = (1, a, b)$.
p_b: $X \to X$; $a \mapsto b$, $b \mapsto 1$, $1 \mapsto a$. It is abbreviated to $p_b = (b, 1, a)$.
p_1: $X \to X$; $a \mapsto a$, $b \mapsto b$, $1 \mapsto 1$. It is abbreviated to $p_1 = (a, b, 1)$.

We can verify that $p_a * p_a = p_b$, $p_a * p_b = p_1$; $p_b * p_b = p_a$. Then $M(X) = \{p_a, p_b, p_1\}$ and its Cayley table is Table 10. Obviously, $(M(X), *)$ is a commutative group with identity p_1 and $(p_a)^{-1} = p_b$, $(p_b)^{-1} = p_a$.

Table 9. CI-algebra.

$\rightarrow$	a	b	1
a	1	a	b
b	b	1	a
1	a	b	1

Table 10. Adjoint semigroup of the above CI-algebra.

*	p_a	p_b	p_1
p_a	p_b	p_1	p_a
p_b	p_1	p_a	p_b
p_1	p_a	p_b	p_1

Theorem 6. *Let $(X; \rightarrow, 1)$ be a singular CI-algebra, and $M(X)$ be the adjoint semigroup. Then $(M(X), *)$ is a commutative group with identity p_1, where $M(X) = \{p_a \mid a \in X\}$ and $|M(X)| = |X|$.*

Proof. (1) First, we prove that for any singular CI-algebra, $a \rightarrow (b \rightarrow x) = ((a \rightarrow 1) \rightarrow b) \rightarrow x$, $\forall\, a, b, x \in X$.

In fact, by Definition 7 and Lemma 2, we have the following:

$$\begin{aligned}((a \rightarrow 1) \rightarrow b) \rightarrow x &= ((a \rightarrow 1) \rightarrow b) \rightarrow ((x \rightarrow 1) \rightarrow 1)\\ &= (x \rightarrow 1) \rightarrow (((a \rightarrow 1) \rightarrow b) \rightarrow 1)\\ &= (x \rightarrow 1) \rightarrow (((a \rightarrow 1) \rightarrow 1) \rightarrow (b \rightarrow 1))\\ &= (x \rightarrow 1) \rightarrow (a \rightarrow (b \rightarrow 1))\\ &= a \rightarrow ((x \rightarrow 1) \rightarrow (b \rightarrow 1))\\ &= a \rightarrow (b \rightarrow x).\end{aligned}$$

(2) Second, we prove that for any singular CI-algebra, $a \neq b \Rightarrow p_a \neq p_b$, $\forall\, a, b \in X$.
Assume $p_a = p_b$, $a, b \in X$. Then, for all x in X, $p_a(x) = p_b(x)$. Hence,

$$a \rightarrow b = p_a(b) = p_b(b) = b \rightarrow b = 1.$$

From this, applying Lemma 2 (1) and (6) we get

$$a = (a \rightarrow 1) \rightarrow 1 = (a \rightarrow 1) \rightarrow (a \rightarrow b) = (b \rightarrow 1) \rightarrow (a \rightarrow a) = (b \rightarrow 1) \rightarrow 1 = b.$$

(3) Using Lemma 2 (1), we know that for any $a, b \in X$, there exist $c \in X$, such that $p_a * p_b = p_c$, where $c = (a \rightarrow 1) \rightarrow b$. This means that $M(X) \subseteq \{p_a \mid a \in X\}$. By the definition of $M(X)$, $\{p_a\ a \in X\} \subseteq M(X)$. Hence, $M(X) = \{p_a \mid a \in X\}$.

(4) Using Lemma 2 (2) and (3), we know that $|M(X)| = |X|$. □

5. Quasi Neutrosophic Triplet Structures in Pseudo BE-Algebras and Pseudo CI-Algebras

Like the above Section 4, we can discuss the relationships between pseudo BE-algebras (pseudo CI-algebras) and quasi neutrosophic triplet structures. This section will give some related results and examples, but part of the simple proofs will be omitted.

5.1. Pseudo BE-Algebras (Pseudo CI-Algebras) and (l-l)-Quasi Neutrosophic Triplet Loops

Theorem 7. *Let $(X; \rightarrow, \rightsquigarrow, 1)$ be pseudo BE-algebra. Then $(X, \rightarrow)$ and $(X, \rightsquigarrow)$ are (l-l)-quasi neutrosophic triplet loops. And, when $|X| > 1$, $(X, \rightarrow)$ and $(X, \rightsquigarrow)$ are not (lr-l)-quasi neutrosophic triplet loops with neutral element 1.*

Example 9. *Let $X = \{a, b, c, 1\}$. Define operations $\rightarrow$ and $\rightsquigarrow$ on X as following* Tables 11 and 12. *Then, $(X; \rightarrow, \rightsquigarrow, 1)$ is a pseudo BE-algebra, and $(X, \rightarrow)$ and $(X, \rightsquigarrow)$ are (l-l)-quasi neutrosophic triplet loops.*

Table 11. Pseudo BE-algebra (1).

$\rightarrow$	**a**	**b**	**c**	**1**
a	1	1	b	1
b	a	1	c	1
c	1	1	1	1
1	a	b	c	1

Table 12. Pseudo BE-algebra (2).

$\rightsquigarrow$	**a**	**b**	**c**	**1**
a	1	1	a	1
b	a	1	a	1
c	1	1	1	1
1	a	b	c	1

Definition 9. *([44,46]) Let a be an element of a pseudo CI-algebra $(X; \rightarrow, \rightsquigarrow, 1)$. a is said to be an atom in X if for any $x \in X$, $a \rightarrow x = 1$ implies $a = x$.*

Applying the results in Ref. [44–46] we have the following propositions (the proofs are omitted).

Proposition 3. *If $(X; \rightarrow, \rightsquigarrow, 1)$ is a pseudo CI-algebra, then for all $x, y \in X$*

(1) $x \leq (x \rightarrow y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \rightarrow y$,
(2) $x \leq y \rightarrow z \Leftrightarrow y \leq x \rightsquigarrow z$,
(3) $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightsquigarrow 1)$, $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \rightarrow (y \rightarrow 1)$,
(4) $x \rightarrow 1 = x \rightsquigarrow 1$,
(5) $x \leq y$ *implies* $x \rightarrow 1 = y \rightarrow 1$.

Proposition 4. *Let $(X; \rightarrow, \rightsquigarrow, 1)$ be a pseudo CI-algebra. If $a, b \in X$ are atoms in X, then the following are true:*

(1) $a = (a \rightarrow 1) \rightarrow 1$,
(2) *for any* $x \in X$, $(a \rightarrow x) \rightsquigarrow x = a$, $(a \rightsquigarrow x) \rightarrow x = a$,
(3) *for any* $x \in X$, $(a \rightarrow x) \rightsquigarrow 1 = x \rightarrow a$, $(a \rightsquigarrow x) \rightarrow 1 = x \rightsquigarrow a$,
(4) *for any* $x \in X$, $x \rightarrow a = (a \rightarrow 1) \rightsquigarrow (x \rightarrow 1)$, $x \rightsquigarrow a = (a \rightsquigarrow 1) \rightarrow (x \rightsquigarrow 1)$.

Definition 10. *A pseudo CI-algebra $(X; \rightarrow, \rightsquigarrow, 1)$ is said to be singular if every element of X is an atom. A pseudo CI-algebra $(X; \rightarrow, \rightsquigarrow, 1)$ is said to be strong singular if for any $x \in X$, $x \rightarrow 1 = x = x \rightsquigarrow 1$.*

Proposition 5. *If $(X; \rightarrow, \rightsquigarrow, 1)$ is a strong singular pseudo CI-algebra. Then $(X; \rightarrow, \rightsquigarrow, 1)$ is singular.*

Proof. For any $x \in X$, assume that $a \to x = 1$, where $a \in X$. It follows from Definition 10,

$$x \to 1 = x = x \rightsquigarrow 1,\ a \to 1 = a = a \rightsquigarrow 1.$$

Hence, applying Definition 4 and Proposition 3,

$$a = a \to 1 = a \to (x \rightsquigarrow x) = x \rightsquigarrow (a \to x) = x \rightsquigarrow 1 = x.$$

By Definition 9, x is an atom. Therefore, $(X; \to, \rightsquigarrow, 1)$ is singular pseudo CI-algebra. □

Applying Theorem 3.11 in Ref. [46], we can get the following:

Lemma 3. *Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo CI-algebra. Then the following statements are equivalent*:

(1) $x \to (y \to z) = (x \to y) \to z$, *for all* x, y, z *in* X;
(2) $x \to 1 = x = x \rightsquigarrow 1$, *for every* x *in* X;
(3) $x \to y = x \rightsquigarrow y = y \to x$, *for all* x, y *in* X;
(4) $x \rightsquigarrow (y \rightsquigarrow z) = (x \rightsquigarrow y) \rightsquigarrow z$, *for all* x, y, z *in* X.

Proposition 6. *Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo CI-algebra. Then $(X; \to, \rightsquigarrow, 1)$ is a strong singular pseudo CI-algebra if, and only if, $\to = \rightsquigarrow$ and $(X; \to, 1)$ is an associative BCI-algebra.*

Proof. We know that every associative BCI-algebra is a strong singular pseudo CI-algebra. □

Now, suppose that $(X; \to, 1)$ is a strong singular pseudo CI-algebra. By Definition 10 and Lemma 3 (3), $x \to y = x \rightsquigarrow y$, $\forall x, y \in X$. That is, $\to = \rightsquigarrow$. Hence, $(X; \to, 1)$ is a strong singular CI-algebra. It follows that $(X; \to, 1)$ is an associative BCI-algebra (using Proposition 2).

Theorem 8. *Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo CI-algebra. Then $(X, \to)$ and $(X, \rightsquigarrow)$ are(l-l)-quasi neutrosophic triplet loops. Moreover, $(X, \to)$ and $(X, \rightsquigarrow)$ are neutrosophic triplet groups if, and only if, $(X; \to, \rightsquigarrow, 1)$ is a strong singular pseudo CI-algebra (associative BCI-algebra).*

Proof. Applying Lemma 3, and the proof is omitted. □

5.2. Pseudo BE-Algebras (Pseudo CI-Algebras) and Their Adjoint Semi-Groups

For any pseudo BE-algebra or pseudo CI-algebra $(X; \to, \rightsquigarrow, 1)$ as well as any element a in X, we use $p_a^{\to}$ and $p_a^{\rightsquigarrow}$ to denote the self-map of X, which is defined by the following:

$$p_a^{\to}: X \to X; \mapsto a \to x, \text{ for all } x \in X.$$

$$p_a^{\rightsquigarrow}: X \to X; \mapsto a \rightsquigarrow x, \text{ for all } x \in X.$$

Theorem 9. *Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo BE-algebra (or pseudo CI-algebra), and*

$$M^{\to}(X) = \{\text{finite products } p_a^{\to} * \ldots * p_b^{\to} \text{ of self-map of } X \mid a, \ldots, b \in X\},$$

$$M^{\rightsquigarrow}(X) = \{\text{finite products } p_a^{\rightsquigarrow} * \ldots * p_b^{\rightsquigarrow} \text{ of self-map of } X \mid a, \ldots, b \in X\},$$

$$M(X) = \{\text{finite products } p_a^{\to} \text{ (or } p_a^{\rightsquigarrow}) * \ldots * p_b^{\to} \text{ (or } p_b^{\rightsquigarrow}) \text{ of self-map of } X \mid a, \ldots, b \in X\},$$

where * represents the composition operation of mappings. Then $(M^{\to}(X), *)$, $(M^{\rightsquigarrow}(X), *)$, and $(M(X), *)$ are all semigroups with the identity $p_1 = p_1^{\to} = p_1^{\rightsquigarrow}$.

Proof. It is similar to Theorem 5. □

Now, we call $(M^{\rightarrow}(X), *)$, $(M^{\rightsquigarrow}(X), *)$, *and* $(M(X), *)$ the adjoint semigroups of X.

Example 10. *Let X = {a, b, c, 1}. Define operations* $\rightarrow$ *and* $\rightsquigarrow$ *on X as following* Tables 13 and 14. *Then,* $(X; \rightarrow, \rightsquigarrow, 1)$ *is a pseudo BE-algebra, and*

$$p_a{}^{\rightarrow} = (1, b, b, 1), p_b{}^{\rightarrow} = (a, 1, c, 1), p_c{}^{\rightarrow} = (1, 1, 1, 1), p_1{}^{\rightarrow} = (a, b, c, 1).$$

We can verify the following:

$$p_a{}^{\rightarrow} * p_a{}^{\rightarrow} = p_a{}^{\rightarrow}, p_a{}^{\rightarrow} * p_b{}^{\rightarrow} = (1, 1, b, 1), p_a{}^{\rightarrow} * p_c{}^{\rightarrow} = p_c{}^{\rightarrow}, p_a{}^{\rightarrow} * p_1{}^{\rightarrow} = p_a{}^{\rightarrow};$$

$$p_b{}^{\rightarrow} * p_a{}^{\rightarrow} = p_c{}^{\rightarrow}, p_b{}^{\rightarrow} * p_b{}^{\rightarrow} = p_b{}^{\rightarrow}, p_b{}^{\rightarrow} * p_c{}^{\rightarrow} = p_c{}^{\rightarrow}, p_b{}^{\rightarrow} * p_1{}^{\rightarrow} = p_b{}^{\rightarrow};$$

$$p_c{}^{\rightarrow} * p_a{}^{\rightarrow} = p_c{}^{\rightarrow}, p_c{}^{\rightarrow} * p_b{}^{\rightarrow} = p_c{}^{\rightarrow}, p_c{}^{\rightarrow} * p_c{}^{\rightarrow} = p_c{}^{\rightarrow}, p_c{}^{\rightarrow} * p_1{}^{\rightarrow} = p_c{}^{\rightarrow};$$

$$p_1{}^{\rightarrow} * p_a{}^{\rightarrow} = p_a{}^{\rightarrow}, p_1{}^{\rightarrow} * p_b{}^{\rightarrow} = p_b{}^{\rightarrow}, p_1{}^{\rightarrow} * p_c{}^{\rightarrow} = p_c{}^{\rightarrow}, p_1{}^{\rightarrow} * p_1{}^{\rightarrow} = p_1{}^{\rightarrow}.$$

Denote $p_{ab}{}^{\rightarrow} = p_a{}^{\rightarrow} * p_b{}^{\rightarrow} = (1, 1, b, 1)$, then $p_{ab}{}^{\rightarrow} * p_a{}^{\rightarrow} = p_c{}^{\rightarrow}$, $p_{ab}{}^{\rightarrow} * p_b{}^{\rightarrow} = p_{ab}{}^{\rightarrow}$, $p_{ab}{}^{\rightarrow} * p_{ab}{}^{\rightarrow} = p^{\rightarrow}$, $p_{ab}{}^{\rightarrow} * p_c{}^{\rightarrow} = p_c{}^{\rightarrow}$. Hence, $M^{\rightarrow}(X) = \{p_a{}^{\rightarrow}, p_b{}^{\rightarrow}, p_{ab}{}^{\rightarrow}, p_c{}^{\rightarrow}, p_1{}^{\rightarrow}\}$ and its Cayley table is Table 15. Obviously, $(M^{\rightarrow}(X), *)$ is a non-commutative semigroup, but it is not a neutrosophic triplet group.

Table 13. Pseudo BE-algebra and adjoint semigroups (1).

$\rightarrow$	a	b	c	1
a	1	b	b	1
b	a	1	c	1
c	1	1	1	1
1	a	b	c	1

Table 14. Pseudo BE-algebra and adjoint semigroups (2).

$\rightsquigarrow$	a	b	c	1
a	1	b	c	1
b	a	1	a	1
c	1	1	1	1
1	a	b	c	1

Table 15. Pseudo BE-algebra and adjoint semigroups (3).

*	$p_a{}^{\rightarrow}$	$p_b{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_1{}^{\rightarrow}$
$p_a{}^{\rightarrow}$	$p_a{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_a{}^{\rightarrow}$
$p_b{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_b{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_b{}^{\rightarrow}$
$p_{ab}{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$
$p_c{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_c{}^{\rightarrow}$
$p_1{}^{\rightarrow}$	$p_a{}^{\rightarrow}$	$p_b{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_c{}^{\rightarrow}$	$p_1{}^{\rightarrow}$

Similarly, we can verify that

$$p_a{}^{\rightsquigarrow} = (1, b, c, 1), p_b{}^{\rightsquigarrow} = (a, 1, a, 1), p_c{}^{\rightsquigarrow} = (1, 1, 1, 1), p_1{}^{\rightsquigarrow} = (a, b, c, 1).$$

$$p_a{}^{\rightsquigarrow} * p_a{}^{\rightsquigarrow} = p_a{}^{\rightsquigarrow}, p_a{}^{\rightsquigarrow} * p_b{}^{\rightsquigarrow} = p_a{}^{\rightsquigarrow} * p_c{}^{\rightsquigarrow} = (1, 1, 1, 1), p_a{}^{\rightsquigarrow} * p_1{}^{\rightsquigarrow} = p_a{}^{\rightsquigarrow};$$

$$p_b{}^{\rightsquigarrow} * p_a{}^{\rightsquigarrow} = (1, 1, a, 1), p_b{}^{\rightsquigarrow} * p_b{}^{\rightsquigarrow} = p_b{}^{\rightsquigarrow}, p_b{}^{\rightsquigarrow} * p_c{}^{\rightsquigarrow} = p_c{}^{\rightsquigarrow}, p_b{}^{\rightsquigarrow} * p_1{}^{\rightsquigarrow} = p_b{}^{\rightsquigarrow};$$

$$p_c^{\rightsquigarrow} * p_a^{\rightsquigarrow} = p_c^{\rightsquigarrow}, p_c^{\rightsquigarrow} * p_b^{\rightsquigarrow} = p_c^{\rightsquigarrow}, p_c^{\rightsquigarrow} * p_c^{\rightsquigarrow} = p_c^{\rightsquigarrow}, p_c^{\rightsquigarrow} * p_1^{\rightsquigarrow} = p_c^{\rightsquigarrow}.$$

Denote $p_{ba}^{\rightsquigarrow} = p_b^{\rightsquigarrow} * p_a^{\rightsquigarrow} = (1, 1, a, 1)$, then $p_{ba}^{\rightsquigarrow} * p_a^{\rightsquigarrow} = p_{ba}^{\rightsquigarrow}$, $p_a^{\rightsquigarrow} * p_{ba}^{\rightsquigarrow} = p_c^{\rightsquigarrow}$; $p_{ba}^{\rightsquigarrow} * p_b^{\rightsquigarrow} = p_c^{\rightsquigarrow}$, $p_b^{\rightsquigarrow} * p_{ba}^{\rightsquigarrow} = p_{ba}^{\rightsquigarrow}$; $p_{ba}^{\rightsquigarrow} * p_{ba}^{\rightsquigarrow} = p_c^{\rightsquigarrow}$; $p_{ba}^{\rightsquigarrow} * p_c^{\rightsquigarrow} = p_c^{\rightsquigarrow}$, $p_c^{\rightsquigarrow} * p_{ba}^{\rightsquigarrow} = p_c^{\rightsquigarrow}$. Hence, $M^{\rightsquigarrow}(X) = \{p_a^{\rightsquigarrow}, p_b^{\rightsquigarrow}, p_{ba}^{\rightsquigarrow}, p_c^{\rightsquigarrow}, p_1^{\rightsquigarrow}\}$ and its Cayley table is Table 16. Obviously, $(M^{\rightsquigarrow}(X), *)$ is a non-commutative semigroup, but it is not a neutrosophic triplet group.

Table 16. Pseudo BE-algebra and adjoint semigroups (4).

$*$	$p_a^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_1^{\rightsquigarrow}$
$p_a^{\rightsquigarrow}$	$p_a^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_a^{\rightsquigarrow}$
$p_b^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$
$p_{ba}^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$
$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$
$p_1^{\rightsquigarrow}$	$p_a^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightsquigarrow}$	$p_1^{\rightsquigarrow}$

Now, we consider $M(X)$. Since

$$p_c^{\rightarrow} = (1, 1, 1, 1) = p_c^{\rightsquigarrow}, p_1^{\rightarrow} = (a, b, c, 1) = p_1^{\rightsquigarrow};$$

$$p_a^{\rightarrow} * p_a^{\rightsquigarrow} = p_a^{\rightarrow}, p_a^{\rightsquigarrow} * p_a^{\rightarrow} = p_a^{\rightarrow};$$

$$p_a^{\rightarrow} * p_b^{\rightsquigarrow} = (1, 1, 1, 1) = p_c^{\rightarrow}, p_b^{\rightsquigarrow} * p_a^{\rightarrow} = (1, 1, 1, 1) = p_c^{\rightarrow};$$

$$p_a^{\rightsquigarrow} * p_b^{\rightarrow} = p_b^{\rightarrow} * p_a^{\rightsquigarrow} = (1, 1, c, 1);$$

$$p_a^{\rightsquigarrow} * p_{ab}^{\rightarrow} = p_{ab}^{\rightarrow}, p_{ab}^{\rightarrow} * p_a^{\rightsquigarrow} = p_{ab}^{\rightarrow}; p_b^{\rightarrow} * p_b^{\rightsquigarrow} = p_b^{\rightsquigarrow}, p_b^{\rightsquigarrow} * p_b^{\rightarrow} = p_b^{\rightsquigarrow};$$

$$p_{ab}^{\rightarrow} * p_b^{\rightsquigarrow} = (1, 1, 1, 1) = p_c^{\rightarrow}, p_b^{\rightsquigarrow} * p_{ab}^{\rightarrow} = (1, 1, 1, 1) = p_c^{\rightarrow};$$

$$p_a^{\rightarrow} * p_{ba}^{\rightsquigarrow} = (1, 1, 1, 1) = p_c^{\rightarrow}, p_{ba}^{\rightsquigarrow} * p_a^{\rightarrow} = (1, 1, 1, 1) = p_c^{\rightarrow};$$

$$p_b^{\rightarrow} * p_{ba}^{\rightsquigarrow} = p_{ba}^{\rightsquigarrow}, p_{ba}^{\rightsquigarrow} * p_b^{\rightarrow} = p_{ba}^{\rightsquigarrow};$$

$$p_{ab}^{\rightarrow} * p_{ba}^{\rightsquigarrow} = (1, 1, 1, 1) = p_c^{\rightarrow}, p_{ba}^{\rightsquigarrow} * p_{ab}^{\rightarrow} = (1, 1, 1, 1) = p_c^{\rightarrow}.$$

Denote $p = (1, 1, c, 1)$, then $M(X) = \{p_a^{\rightarrow}, p_a^{\rightsquigarrow}, p_b^{\rightarrow}, p_b^{\rightsquigarrow}, p_{ab}^{\rightarrow}, p_{ba}^{\rightsquigarrow}, p, p_c^{\rightarrow}, p_1^{\rightarrow}\}$, and Table 17 is its Cayley table (it is a non-commutative semigroup, but it is not a neutrosophic triplet group).

Table 17. Pseudo BE-algebra and adjoint semigroups (5).

$*$	$p_a^{\rightarrow}$	$p_a^{\rightsquigarrow}$	$p_b^{\rightarrow}$	$p_b^{\rightsquigarrow}$	$p_{ab}^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	p	$p_c^{\rightarrow}$	$p_1^{\rightarrow}$
$p_a^{\rightarrow}$	$p_a^{\rightarrow}$	$p_a^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_c^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_c^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_c^{\rightarrow}$	$p_a^{\rightarrow}$
$p_a^{\rightsquigarrow}$	$p_a^{\rightarrow}$	$p_a^{\rightsquigarrow}$	p	$p_c^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	p	$p_c^{\rightarrow}$	$p_a^{\rightsquigarrow}$
$p_b^{\rightarrow}$	$p_c^{\rightarrow}$	p	$p_b^{\rightarrow}$	$p_b^{\rightsquigarrow}$	$p_c^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	p	$p_c^{\rightarrow}$	$p_b^{\rightarrow}$
$p_b^{\rightsquigarrow}$	$p_c^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_c^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightarrow}$	$p_b^{\rightsquigarrow}$
$p_{ab}^{\rightarrow}$	$p_c^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_c^{\rightarrow}$	$p_{ab}^{\rightarrow}$
$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	$p_c^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$
p	$p_c^{\rightarrow}$	p	p	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	p	$p_c^{\rightarrow}$	p
$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$	$p_c^{\rightarrow}$
$p_1^{\rightarrow}$	$p_a^{\rightarrow}$	$p_a^{\rightsquigarrow}$	$p_b^{\rightarrow}$	$p_b^{\rightsquigarrow}$	$p_{ab}^{\rightarrow}$	$p_{ba}^{\rightsquigarrow}$	p	$p_c^{\rightarrow}$	$p_1^{\rightarrow}$

The following example shows that the adjoint semigroups of a pseudo BE-algebra may be a commutative neutrosophic triplet group.

Example 11. *Let X= {a, b, c, d, 1}. Define operations* $\rightarrow$ *and* $\rightsquigarrow$ *on X as* Tables 18 and 19. *Then, (X;* $\rightarrow$, $\rightsquigarrow$, *1) is a pseudo BE-algebra, as well as the following:*

$$p_a{}^{\rightarrow} = (1, c, c, 1, 1), p_b{}^{\rightarrow} = (d, 1, 1, d, 1), p_c{}^{\rightarrow} = (d, 1, 1, d, 1), p_d{}^{\rightarrow} = (1, c, c, 1, 1), p_1{}^{\rightarrow} = (a, b, c, d, 1).$$

We can verify the following:

$$p_a{}^{\rightarrow} * p_a{}^{\rightarrow} = p_a{}^{\rightarrow}, p_a{}^{\rightarrow} * p_b{}^{\rightarrow} = p_a{}^{\rightarrow} * p_c{}^{\rightarrow} = (1, 1, 1, 1, 1), p_a{}^{\rightarrow} * p_d{}^{\rightarrow} = p_a{}^{\rightarrow}, p_a{}^{\rightarrow} * p_1{}^{\rightarrow} = p_a{}^{\rightarrow};$$

$$p_b{}^{\rightarrow} * p_a{}^{\rightarrow} = (1, 1, 1, 1, 1), p_b{}^{\rightarrow} * p_b{}^{\rightarrow} = p_b{}^{\rightarrow} * p_c{}^{\rightarrow} = p_b{}^{\rightarrow}, p_b{}^{\rightarrow} * p_d{}^{\rightarrow} = (1, 1, 1, 1, 1), p_b{}^{\rightarrow} * p_1{}^{\rightarrow} = p_b{}^{\rightarrow};$$

$$p_c{}^{\rightarrow} * p_a{}^{\rightarrow} = (1, 1, 1, 1, 1), p_c{}^{\rightarrow} * p_b{}^{\rightarrow} = p_c{}^{\rightarrow} * p_c{}^{\rightarrow} = p_c{}^{\rightarrow}, p_c{}^{\rightarrow} * p_d{}^{\rightarrow} = (1, 1, 1, 1, 1), p_c{}^{\rightarrow} * p_1{}^{\rightarrow} = p_b{}^{\rightarrow};$$

$$p_d{}^{\rightarrow} * p_a{}^{\rightarrow} = p_d{}^{\rightarrow}, p_d{}^{\rightarrow} * p_b{}^{\rightarrow} = p_d{}^{\rightarrow} * p_c{}^{\rightarrow} = (1, 1, 1, 1, 1), p_d{}^{\rightarrow} * p_d{}^{\rightarrow} = p_d{}^{\rightarrow}, p_d{}^{\rightarrow} * p_1{}^{\rightarrow} = p_d{}^{\rightarrow}.$$

Denote $p_{ab}{}^{\rightarrow} = p_a{}^{\rightarrow} * p_b{}^{\rightarrow} = (1, 1, 1, 1, 1)$, then $p_{ab}{}^{\rightarrow} * p_a{}^{\rightarrow} = p_{ab}{}^{\rightarrow} * p_b{}^{\rightarrow} = p_{ab}{}^{\rightarrow} * p_c{}^{\rightarrow} = p_{ab}{}^{\rightarrow} * p_d{}^{\rightarrow} = p_{ab}{}^{\rightarrow} * p_{ab}{}^{\rightarrow} = p_{ab}{}^{\rightarrow} * p_1{}^{\rightarrow} = p_{ab}{}^{\rightarrow}$. Hence, $M^{\rightarrow}(X) = \{p_a{}^{\rightarrow}, p_b{}^{\rightarrow}, p_{ab}{}^{\rightarrow}, p_1{}^{\rightarrow}\}$ and its Cayley table is Table 20. Obviously, $(M^{\rightarrow}(X), *)$ is a commutative neutrosophic triplet group.

Table 18. Pseudo BE-algebra and commutative neutrosophic triplet groups (1).

$\rightarrow$	**a**	**b**	**c**	**d**	**1**
a	1	c	c	1	1
b	d	1	1	d	1
c	d	1	1	d	1
d	1	c	c	1	1
1	a	b	c	d	1

Table 19. Pseudo BE-algebra and commutative neutrosophic triplet groups (2).

$\rightsquigarrow$	**a**	**b**	**c**	**d**	**1**
a	1	b	c	1	1
b	d	1	1	d	1
c	d	1	1	d	1
d	1	b	c	1	1
1	a	b	c	d	1

Table 20. Pseudo BE-algebra and commutative neutrosophic triplet groups (3).

*	$p_a{}^{\rightarrow}$	$p_b{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_1{}^{\rightarrow}$
$p_a{}^{\rightarrow}$	$p_a{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_a{}^{\rightarrow}$
$p_b{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_b{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_b{}^{\rightarrow}$
$p_{ab}{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$
$p_1{}^{\rightarrow}$	$p_a{}^{\rightarrow}$	$p_b{}^{\rightarrow}$	$p_{ab}{}^{\rightarrow}$	$p_1{}^{\rightarrow}$

Similarly, we can verify the following:

$$p_a{}^{\rightsquigarrow} = (1, b, c, 1, 1), p_b{}^{\rightsquigarrow} = (d, 1, 1, d, 1), p_c{}^{\rightsquigarrow} = (d, 1, 1, d, 1), p_d{}^{\rightsquigarrow} = (1, b, c, 1, 1), p_1{}^{\rightsquigarrow} = (a, b, c, d, 1).$$

$$p_a{}^{\rightsquigarrow} * p_a{}^{\rightsquigarrow} = p_a{}^{\rightsquigarrow}, p_a{}^{\rightsquigarrow} * p_b{}^{\rightsquigarrow} = p_a{}^{\rightsquigarrow} * p_c{}^{\rightsquigarrow} = (1, 1, 1, 1, 1), p_a{}^{\rightsquigarrow} * p_d{}^{\rightsquigarrow} = p_a{}^{\rightsquigarrow};$$

$$p_b{}^{\rightsquigarrow} * p_a{}^{\rightsquigarrow} = (1, 1, 1, 1, 1), p_b{}^{\rightsquigarrow} * p_b{}^{\rightsquigarrow} = p_b{}^{\rightsquigarrow} * p_c{}^{\rightsquigarrow} = p_b{}^{\rightsquigarrow}, p_b{}^{\rightsquigarrow} * p_d{}^{\rightsquigarrow} = (1, 1, 1, 1, 1).$$

Denote $p_{ab}{}^{\rightsquigarrow} = p_a{}^{\rightsquigarrow} * p_b{}^{\rightsquigarrow} = (1, 1, 1, 1, 1)$, then $M^{\rightsquigarrow}(X) = \{p_a{}^{\rightsquigarrow}, p_b{}^{\rightsquigarrow}, p_{ab}{}^{\rightsquigarrow}, p_1{}^{\rightsquigarrow}\}$ and its Cayley table is Table 21. Obviously, $(M^{\rightsquigarrow}(X), *)$ is a commutative neutrosophic triplet group.

Table 21. Pseudo BE-algebra and commutative neutrosophic triplet groups (4).

*	$p_a^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_1^{\rightsquigarrow}$
$p_a^{\rightsquigarrow}$	$p_a^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_a^{\rightsquigarrow}$
$p_b^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$
$p_{ab}^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$
$p_1^{\rightsquigarrow}$	$p_a^{\rightsquigarrow}$	$p_b^{\rightsquigarrow}$	$p_{ab}^{\rightsquigarrow}$	$p_1^{\rightsquigarrow}$

Now, we consider $M(X)$. Since the following:

$$p_b^{\rightarrow} = p_c^{\rightarrow} = (d, 1, 1, d, 1) = p_b^{\rightsquigarrow} = p_c^{\rightsquigarrow}, p_a^{\rightarrow} = p_d^{\rightarrow} = (1, c, c, 1, 1), p_a^{\rightsquigarrow} = p_d^{\rightsquigarrow} = (1, b, c, 1, 1);$$

$$p_a^{\rightarrow} * p_a^{\rightsquigarrow} = p_a^{\rightarrow}, p_a^{\rightsquigarrow} * p_a^{\rightarrow} = p_a^{\rightarrow}; p_a^{\rightarrow} * p_b^{\rightsquigarrow} = (1, 1, 1, 1, 1) = p_{ab}^{\rightarrow} = p_{ab}^{\rightsquigarrow}, p_b^{\rightsquigarrow} * p_a^{\rightarrow} = (1, 1, 1, 1, 1).$$

Hence, $M(X) = \{p_a^{\rightarrow}, p_a^{\rightsquigarrow}, p_b^{\rightarrow}, p_{ab}^{\rightarrow}, p_1^{\rightarrow}\}$, and Table 22 is its Cayley table (it is a commutative neutrosophic triplet group).

Table 22. Pseudo BE-algebra and commutative neutrosophic triplet groups (5).

*	$p_a^{\rightarrow}$	$p_a^{\rightsquigarrow}$	$p_b^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_1^{\rightarrow}$
$p_a^{\rightarrow}$	$p_a^{\rightarrow}$	$p_a^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_a^{\rightarrow}$
$p_a^{\rightsquigarrow}$	$p_a^{\rightarrow}$	$p_a^{\rightsquigarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_a^{\rightsquigarrow}$
$p_b^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_b^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_b^{\rightarrow}$
$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_{ab}^{\rightarrow}$
$p_1^{\rightarrow}$	$p_a^{\rightarrow}$	$p_a^{\rightsquigarrow}$	$p_b^{\rightarrow}$	$p_{ab}^{\rightarrow}$	$p_1^{\rightarrow}$

Remark 3. *Through the discussions of Examples 10 and 11 above, we get the following important revelations: (1) ($M^{\rightarrow}(X)$, *), ($M^{\rightsquigarrow}(X)$, *), and ($M(X)$, *) are usually three different semi-groups; (2) ($M^{\rightarrow}(X)$, *) and ($M^{\rightsquigarrow}(X)$, *) are all sub-semi-groups of ($M(X)$, *), which can also be proved from their definitions; (3) ($M^{\rightarrow}(X)$, *), ($M^{\rightsquigarrow}(X)$, *), and ($M(X)$, *) may be neutrosophic triplet groups. Under what circumstances they will become neutrosophic triplet groups, will be examined in the next study.*

6. Conclusions

In this paper, the concepts of neutrosophic triplet loops (groups) are further generalized, and some new concepts of generalized neutrosophic triplet structures are proposed, including (*l*-*l*)-type, (*l*-*r*)-type, (*r*-*l*)-type, (*r*-*r*)-type, (*l*-*lr*)-type, (*r*-*lr*)-type, (*lr*-*l*)-type, and (*lr*-*r*)-type quasi neutrosophic triplet loops (groups), and their basic properties are discussed. In particular, as a corollary of these new properties, an important result is proved. For any commutative neutrosophic triplet group, its every element has only one neutral element. At the same time, the BE-algebras and its various extensions (including CI-algebras, pseudo BE-algebras, and pseudo CI-algebras) have been studied, and some related generalized neutrosophic triplet structures that are contained in these algebras are presented. Moreover, the concept of adjoint semigroups of (generalized) BE-algebras are proposed for the first time, abundant examples are given, and some new results are obtained.

Acknowledgments: This work was supported by the National Natural Science Foundation of China (Grant Nos. 61573240, 61473239).

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Certain Notions of Neutrosophic Topological *K*-Algebras

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Muhammad Akram, Hina Gulzar, Florentin Smarandache, Said Broumi (2018). Certain Notions of Neutrosophic Topological K-Algebras. *Mathematics* 6, 234; DOI: 10.3390/math6110

Abstract: The concept of neutrosophic set from philosophical point of view was first considered by Smarandache. A single-valued neutrosophic set is a subclass of the neutrosophic set from a scientific and engineering point of view and an extension of intuitionistic fuzzy sets. In this research article, we apply the notion of single-valued neutrosophic sets to K-algebras. We introduce the notion of single-valued neutrosophic topological K-algebras and investigate some of their properties. Further, we study certain properties, including C_5-connected, super connected, compact and Hausdorff, of single-valued neutrosophic topological K-algebras. We also investigate the image and pre-image of single-valued neutrosophic topological K-algebras under homomorphism.

Keywords: K-algebras; single-valued neutrosophic sets; homomorphism; compactness; C_5-connectedness

1. Introduction

A new kind of logical algebra, known as K-algebra, was introduced by Dar and Akram in [1]. A K-algebra is built on a group G by adjoining the induced binary operation on G. The group G is particularly of the type in which each non-identity element is not of order 2. This algebraic structure is, in general, non-commutative and non-associative with right identity element [1–3]. Akram et al. [4] introduced fuzzy K-algebras. They then developed fuzzy K-algebras with other researchers worldwide. The concepts and results of K-algebras have been broadened to the fuzzy setting frames by applying Zadeh's fuzzy set theory and its generalizations, namely, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, bipolar fuzzy sets and vague sets [5]. In handling information regarding various aspects of uncertainty, non-classical logic is considered to be a more powerful tool than the classical logic. It has become a strong mathematical tool in computer science, medical, engineering, information technology, etc. In 1998, Smarandache [6] introduced neutrosophic set as a generalization of intuitionistic fuzzy set [7]. A neutrosophic set is identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) functions. To apply neutrosophic set in real-life problems more conveniently, Smarandache [6] and Wang et al. [8] defined single-valued neutrosophic sets which takes the value from the subset of $[0, 1]$. Thus, a single-valued neutrosophic set is an instance of neutrosophic set.

Algebraic structures have a vital place with vast applications in various areas of life. Algebraic structures provide a mathematical modeling of related study. Neutrosophic set theory has also been applied to many algebraic structures. Agboola and Davazz introduced the concept of neutrosophic BCI/BCK-algebras and discuss elementary properties in [9]. Jun et al. introduced the notion of (ϕ, ψ) neutrosophic subalgebra of a BCK/BCI-algebra [10]. Jun et al. [11] defined interval neutrosophic sets on BCK/BCI-algebra [11]. Jun et al. [12] proposed neutrosophic positive implicative N-ideals and study their extension property [12] Several set theories and their topological structures have been introduced by many researchers to deal with uncertainties. Chang [13] was the first to introduce the notion of fuzzy topology. Later, Lowan [14], Pu and Liu [15], and Chattopadhyay and Samanta [16] introduced other concepts related to fuzzy topology. Coker [17] introduced the notion of intuitionistic fuzzy topology as a generalization of fuzzy topology. Salama and Alblowi [18] defined the topological structure of neutrosophic set theory. Akram and Dar [19] introduced the concept of fuzzy topological K-algebras. They extended their work on intuitionistic fuzzy topological K-algebras [20]. In this paper, we introduce the notion of single-valued neutrosophic topological K-algebras and investigate some of their properties. Further, we study certain properties, including C_5-connected, super connected, compact and Hausdorff, of single-valued neutrosophic topological K-algebras. We also investigate the image and pre-image of single-valued neutrosophic topological K-algebras under homomorphism.

2. Preliminaries

The notion of K-algebra was introduced by Dar and Akram in [1].

Definition 1. *[1] Let $(G, \cdot, e)$ be a group in which each non-identity element is not of order 2. A K-algebra is a structure $\mathcal{K} = (G, \cdot, \odot, e)$ over a particular group G, where $\odot$ is an induced binary operation $\odot : G \times G \to G$ is defined by $\odot(s, t) = s \odot t = s.t^{-1}$, and satisfy the following conditions:*

(i) $(s \odot t) \odot (s \odot u) = (s \odot ((e \odot u) \odot (e \odot t))) \odot s$;
(ii) $s \odot (s \odot t) = (s \odot (e \odot t) \odot s$;
(iii) $s \odot s = e$;
(iv) $s \odot e = s$; *and*
(v) $e \odot s = s^{-1}$

for all $s, t, u \in G$. The homomorphism between two K-algebras $\mathcal{K}_1$ and $\mathcal{K}_2$ is a mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ such that, for all $u, v \in \mathcal{K}_1$, $f(u \odot v) = f(u) \odot f(v)$.

In [6], Smarandache initiated the idea of neutrosophic set theory which is a generalization of intuitionistic fuzzy set theory. Later, Smarandache and Wang et al. introduced a single-valued neutrosophic set (SNS) as an instance of neutrosophic set in [8].

Definition 2. *[8] Let Z be a space of points with a general element $s \in Z$. A SNS $\mathcal{A}$ in Z is equipped with three membership functions: truth membership function ($\mathcal{T}_\mathcal{A}$), indeterminacy membership function ($\mathcal{I}_\mathcal{A}$) and falsity membership function($\mathcal{F}_\mathcal{A}$), where $\forall\ s \in Z$, $\mathcal{T}_\mathcal{A}(s)$, $\mathcal{I}_\mathcal{A}(s)$, $\mathcal{F}_\mathcal{A}(s) \in [0, 1]$. There is no restriction on the sum of these three components. Therefore, $0 \leq \mathcal{T}_\mathcal{A}(s) + \mathcal{I}_\mathcal{A}(s) + \mathcal{F}_\mathcal{A}(s) \leq 3$.*

Definition 3. *[8] A single-valued neutrosophic empty set ($\emptyset_{SN}$) and single-valued neutrosophic whole set (1_{SN}) on Z is defined as:*

- $\emptyset_{SN}(u) = \{u \in Z : (u, 0, 0, 1)\}$.
- $1_{SN}(u) = \{u \in Z : (u, 1, 1, 0)\}$.

Definition 4. *[8] If f is a mapping from a set Z_1 into a set Z_2, then the following statements hold:*

(i) Let $\mathcal{A}$ be a SNS in Z_1 and $\mathcal{B}$ be a SNS in Z_2, then the pre-image of $\mathcal{B}$ is a SNS in Z_1, denoted by $f^{-1}(\mathcal{B})$, defined as:
$f^{-1}(\mathcal{B}) = \{z_1 \in Z_1 : f^{-1}(\mathcal{T}_\mathcal{B})(z_1) = \mathcal{T}_\mathcal{B}(f(z_1)), f^{-1}(\mathcal{I}_\mathcal{B})(z_1) = \mathcal{I}_\mathcal{B}(f(z_1)), f^{-1}(\mathcal{F}_\mathcal{B})(z_1) = \mathcal{F}_\mathcal{B}(f(z_1))\}$.

(ii) *Let* $\mathcal{A} = \{z_1 \in Z_1 : \mathcal{T}_{\mathcal{A}}(z_1), \mathcal{I}_{\mathcal{A}}(z_1), \mathcal{F}_{\mathcal{A}}(z_1)\}$ *be a SNS in* Z_1 *and* $\mathcal{B} = \{z_2 \in Z_2 : \mathcal{T}_{\mathcal{B}}(z_2), \mathcal{I}_{\mathcal{B}}(z_2), \mathcal{F}_{\mathcal{B}}(z_2)\}$ *be a SNS in* Z_2*. Under the mapping* f*, the image of* $\mathcal{A}$ *is a SNS in* Z_2*, denoted by* $f(\mathcal{A})$*, defined as:* $f(\mathcal{A}) = \{z_2 \in Z_2 : f_{\sup}(\mathcal{T}_{\mathcal{A}})(z_2), f_{\sup}(\mathcal{I}_{\mathcal{A}})(z_2), f_{\inf}(\mathcal{F}_{\mathcal{A}})(z_2)\}$*, where for all* $z_2 \in Z_2$.

$$f_{\sup}(\mathcal{T}_{\mathcal{A}})(z_2) = \begin{cases} \sup_{z_1 \in f^{-1}(z_2)} \mathcal{T}_{\mathcal{A}}(Z_1), & \text{if } f^{-1}_{(z_2)} \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{\sup}(\mathcal{I}_{\mathcal{A}})(z_2) = \begin{cases} \sup_{z_1 \in f^{-1}(z_2)} \mathcal{I}_{\mathcal{A}}(Z_1), & \text{if } f^{-1}_{(z_2)} \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{\inf}(\mathcal{F}_{\mathcal{A}})(z_2) = \begin{cases} \inf_{z_1 \in f^{-1}(z_2)} \mathcal{F}_{\mathcal{A}}(Z_1), & \text{if } f^{-1}_{(z_2)} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

We formulate the following proposition.

Proposition 1. *Let* $f : Z_1 \to Z_2$ *and* $\mathcal{A}$, $(\mathcal{A}_j, j \in J)$ *be a SNS in* Z_1 *and* $\mathcal{B}$ *be a SNS in* Z_2*. Then,* f *possesses the following properties:*

(i) *If* f *is onto , then* $f(1_{SN}) = 1_{SN}$.
(ii) $f(\emptyset_{SN}) = \emptyset_{SN}$.
(iii) $f^{-1}(1_{SN}) = 1_{SN}$.
(iv) $f^{-1}(\emptyset_{SN}) = \emptyset_{SN}$.
(v) *If* f *is onto, then* $f(f^{-1}(\mathcal{B}) = \mathcal{B}$.
(vi) $f^{-1}(\bigcup_{i=1}^{n} \mathcal{A}_i) = \bigcup_{i=1}^{n} f^{-1}(\mathcal{A}_i)$.

3. Neutrosophic Topological K-algebras

Definition 5. *Let* Z *be a nonempty set. A collection* χ *of single-valued neutrosophic sets (SNSs) in* Z *is called a single-valued neutrosophic topology (SNT) on* Z *if the following conditions hold:*

(a) $\emptyset_{SN}, 1_{SN} \in \chi$
(b) *If* $\mathcal{A}, \mathcal{B} \in \chi$*, then* $\mathcal{A} \bigcap \mathcal{B} \in \chi$
(c) *If* $\mathcal{A}_i \in \chi$*,* $\forall i \in I$*, then* $\bigcup_{i \in I} \mathcal{A}_i \in \chi$

The pair (Z, χ) *is called a single-valued neutrosophic topological space (SNTS). Each member of* χ *is said to be* χ*-open or single-valued neutrosophic open set (SNOS) and compliment of each open single-valued neutrosophic set is a single-valued neutrosophic closed set (SNCS). A discrete topology is a topology which contains all single-valued neutrosophic subsets of* Z *and indiscrete if its elements are only* $\emptyset_{SN}$*,* 1_{SN}.

Definition 6. *Let* $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ *be a single-valued neutrosophic set in* $\mathcal{K}$*. Then,* $\mathcal{A}$ *is called a single-valued neutrosophic K-subalgebra of* $\mathcal{K}$ *if following conditions hold for* $\mathcal{A}$*:*

(i) $\mathcal{T}_{\mathcal{A}}(e) \geq \mathcal{T}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(e) \geq \mathcal{I}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(e) \leq \mathcal{F}_{\mathcal{A}}(s)$.
(ii) $\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}_{\mathcal{A}}(s), \mathcal{T}_{\mathcal{A}}(t)\}$,
$\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(t)\}$,
$\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(t)\} \; \forall \, s, t \in \mathcal{K}$.

Example 1. *Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Caley's table for $\odot$ is given as:*

$\odot$	e	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
e	e	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x
x	x	e	x^8	x^7	x^6	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^8	x^7	x^6	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^8	x^7	x^6	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^8	x^7	x^6	x^5
x^5	x^5	x^4	x^3	x^2	x	e	x^8	x^7	x^6
x^6	x^6	x^5	x^4	x^3	x^2	x	e	x^8	x^7
x^7	x^7	x^6	x^5	x^4	x^3	x^2	x	e	x^8
x^8	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	e

If we define a single-valued neutrosophic set $\mathcal{A}, \mathcal{B}$ in $\mathcal{K}$ such that:

$$\mathcal{A} = \{(e, 0.4, 0.5, 0.8), (s, 0.3, 0.4, 0.7)\},$$
$$\mathcal{B} = \{(e, 0.3, 0.4, 0.8), (s, 0.2, 0.3, 0.6)\}$$

$\forall\, s \neq e \in G$.

According to Definition 5, the family $\{\emptyset_{SN}, 1_{SN}, \mathcal{A}, \mathcal{B}\}$ of SNSs of K-algebra is a SNT on $\mathcal{K}$. We define a SNS $\mathcal{A} = \{\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}}\}$ in $\mathcal{K}$ such that $\mathcal{T}_{\mathcal{A}}(e) = 0.7, \mathcal{I}_{\mathcal{A}}(e) = 0.5, \mathcal{F}_{\mathcal{A}}(e) = 0.2$, $\mathcal{T}_{\mathcal{A}}(s) = 0.2, \mathcal{I}_{\mathcal{A}}(s) = 0.4, \mathcal{F}_{\mathcal{A}}(s) = 0.6$. Clearly, $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is a SN K-subalgebra of $\mathcal{K}$.

Definition 7. *Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra and let $\chi_{\mathcal{K}}$ be a topology on $\mathcal{K}$. Let $\mathcal{A}$ be a SNS in $\mathcal{K}$ and let $\chi_{\mathcal{K}}$ be a topology on $\mathcal{K}$. Then, an induced single-valued neutrosophic topology on $\mathcal{A}$ is a collection or family of single-valued neutrosophic subsets of $\mathcal{A}$ which are the intersection with $\mathcal{A}$ and single-valued neutrosophic open sets in $\mathcal{K}$ defined as $\chi_{\mathcal{A}} = \{\mathcal{A} \cap F : F \in \chi_{\mathcal{K}}\}$. Then, $\chi_{\mathcal{A}}$ is called single-valued neutrosophic induced topology on $\mathcal{A}$ or relative topology and the pair $(\mathcal{A}, \chi_{\mathcal{A}})$ is called an induced topological space or single-valued neutrosophic subspace of $(\mathcal{K}, \chi_{\mathcal{K}})$.*

Definition 8. *Let $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$ be two SNTSs and let $f : (\mathcal{K}_1, \chi_1) \to (\mathcal{K}_2, \chi_2)$. Then, f is called single-valued neutrosophic continuous if following conditions hold:*

(i) For each SNS $\mathcal{A} \in \chi_2$, $f^{-1}(\mathcal{A}) \in \chi_1$.
(ii) For each SN K-subalgebra $\mathcal{A} \in \chi_2$, $f^{-1}(\mathcal{A})$ is a SN K-subalgebra $\in \chi_1$.

Definition 9. *Let $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$ be two SNTSs and let $(\mathcal{A}, \chi_{\mathcal{A}})$ and $(\mathcal{B}, \chi_{\mathcal{B}})$ be two single-valued neutrosophic subspaces over $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$. Let f be a mapping from $(\mathcal{K}_1, \chi_1)$ into $(\mathcal{K}_2, \chi_2)$, then f is a mapping from $(\mathcal{A}, \chi_{\mathcal{A}})$ to $(\mathcal{B}, \chi_{\mathcal{B}})$ if $f(\mathcal{A}) \subset \mathcal{B}$.*

Definition 10. *Let f be a mapping from $(\mathcal{A}, \chi_{\mathcal{A}})$ to $(\mathcal{B}, \chi_{\mathcal{B}})$. Then, f is relatively single-valued neutrosophic continuous if for every SNOS $Y_{\mathcal{B}}$ in $\chi_{\mathcal{B}}$, $f^{-1}(Y_{\mathcal{B}}) \cap \mathcal{A} \in \chi_{\mathcal{A}}$.*

Definition 11. *Let f be a mapping from $(\mathcal{A}, \chi_{\mathcal{A}})$ to $(\mathcal{B}, \chi_{\mathcal{B}})$. Then, f is relatively single-valued neutrosophic open if for every SNOS $X_{\mathcal{A}}$ in $\chi_{\mathcal{A}}$, the image $f(X_{\mathcal{A}}) \in \chi_{\mathcal{B}}$.*

Proposition 2. *Let $(\mathcal{A}, \chi_{\mathcal{A}})$ and $(\mathcal{B}, \chi_{\mathcal{B}})$ be single-valued neutrosophic subspaces of $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are K-algebras. If f is a single-valued neutrosophic continuous function from $\mathcal{K}_1$ to $\mathcal{K}_2$ and $f(\mathcal{A}) \subset \mathcal{B}$. Then, f is relatively single-valued neutrosophic continuous function from $\mathcal{A}$ into $\mathcal{B}$.*

Definition 12. *Let $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$ be two SNTSs. A mapping $f : (\mathcal{K}_1, \chi_1) \to (\mathcal{K}_2, \chi_2)$ is called a single-valued neutrosophic homomorphism if following conditions hold:*

(i) f *is a one-one and onto function.*
(ii) f *is a single-valued neutrosophic continuous function from* $\mathcal{K}_1$ *to* $\mathcal{K}_2$.
(iii) f^{-1} *is a single-valued neutrosophic continuous function from* $\mathcal{K}_2$ *to* $\mathcal{K}_1$.

Theorem 1. *Let* $(\mathcal{K}_1, \chi_1)$ *be a SNTS and* $(\mathcal{K}_2, \chi_2)$ *be an indiscrete SNTS on K-algebras* $\mathcal{K}_1$ *and* $\mathcal{K}_2$*, respectively. Then, each function* f *defined as* $f : (\mathcal{K}_1, \chi_1) \to (\mathcal{K}_2, \chi_2)$ *is a single-valued neutrosophic continuous function from* $\mathcal{K}_1$ *to* $\mathcal{K}_2$*. If* $(\mathcal{K}_1, \chi_1)$ *and* $(\mathcal{K}_2, \chi_2)$ *be two discrete SNTSs* $\mathcal{K}_1$ *and* $\mathcal{K}_2$*, respectively, then each homomorphism* $f : (\mathcal{K}_1, \chi_1) \to (\mathcal{K}_2, \chi_2)$ *is a single values neutrosophic continuous function from* $\mathcal{K}_1$ *to* $\mathcal{K}_2$.

Proof. Let f be a mapping defined as $f : \mathcal{K}_1 \to \mathcal{K}_2$. Let χ_1 be SNT on $\mathcal{K}_1$ and χ_2 be SNT on $\mathcal{K}_2$, where $\chi_2 = \{\emptyset_{SN}, 1_{SN}\}$. We show that $f^{-1}(\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_1$, i.e., for each $\mathcal{A} \in \chi_2$, $f^{-1}(\mathcal{A}) \in \chi_1$. Since $\chi_2 = \{\emptyset_{SN}, 1_{SN}\}$, then for any $u \in \chi_1$, consider $\emptyset_{SN} \in \chi_2$ such that $f^{-1}(\emptyset_{SN})(u) = \emptyset_{SN}(f(u)) = \emptyset_{SN}(u)$.

Therefore, $(f^{-1}(\emptyset_{SN})) = \emptyset_{SN} \in \chi_1$. Likewise, $(f^{-1}(1_{SN})) = 1_{SN} \in \chi_1$. Hence, f is a SN continuous function from $\mathcal{K}_1$ to $\mathcal{K}_2$.

Now, for the second part of the theorem, where both χ_1 and χ_2 are SNTSs on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and $f : (\mathcal{K}_1, \chi_1) \to (\mathcal{K}_2, \chi_2)$ is a homomorphism. Therefore, for all $\mathcal{A} \in \chi_2$ and $f^{-1}\mathcal{A} \in \chi_1$, where f is not a usual inverse homomorphism. To prove that $f^{-1}(\mathcal{A})$ is a single-valued neutrosophic K-subalgebra in of $\mathcal{K}_1$. Let for $u, v \in \mathcal{K}_1$,

$$\begin{aligned}
f^{-1}(\mathcal{T}_\mathcal{A})(u \odot v) &= \mathcal{T}_\mathcal{A}(f(u \odot v)) \\
&= \mathcal{T}_\mathcal{A}(f(u) \odot f(v)) \\
&\geq \min\{\mathcal{T}_\mathcal{A}(f(u)) \odot \mathcal{T}_(f(v))\} \\
&= \min\{f^{-1}(\mathcal{T}_\mathcal{A})(u), f^{-1}(\mathcal{T}_\mathcal{A})(v)\}, \\
f^{-1}(\mathcal{I}_\mathcal{A})(u \odot v) &= \mathcal{I}_\mathcal{A}(f(u \odot v)) \\
&= \mathcal{I}_\mathcal{A}(f(u) \odot f(v)) \\
&\geq \min\{\mathcal{I}_\mathcal{A}(f(u)) \odot \mathcal{I}_(f(v))\} \\
&= \min\{f^{-1}(\mathcal{I}_\mathcal{A})(u), f^{-1}(\mathcal{I}_\mathcal{A})(v)\}, \\
f^{-1}(\mathcal{F}_\mathcal{A})(u \odot v) &= \mathcal{F}_\mathcal{A}(f(u \odot v)) \\
&= \mathcal{F}_\mathcal{A}(f(u) \odot f(v)) \\
&\leq \max\{\mathcal{F}_\mathcal{A}(f(u)) \odot \mathcal{F}_(f(v))\} \\
&= \max\{f^{-1}(\mathcal{F}_\mathcal{A})(u), f^{-1}(\mathcal{F}_\mathcal{A})(v)\}.
\end{aligned}$$

Hence, f is a single-valued neutrosophic continuous function from $\mathcal{K}_1$ to $\mathcal{K}_2$. □

Proposition 3. *Let* χ_1 *and* χ_2 *be two SNTSs on* $\mathcal{K}$*. Then, each homomorphism* $f : (\mathcal{K}, \chi_1) \to (\mathcal{K}, \chi_2)$ *is a single-valued neutrosophic continuous function.*

Proof. Let $(\mathcal{K}, \chi_1)$ and $(\mathcal{K}, \chi_2)$ be two SNTSs, where $\mathcal{K}$ is a K-algebra. To prove the above result, it is enough to show that result is false for a particular topology. Let $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A},)$ and $\mathcal{B} = (\mathcal{T}_\mathcal{B}, \mathcal{I}_\mathcal{B}, \mathcal{F}_\mathcal{B})$ be two SNSs in $\mathcal{K}$. Take $\chi_1 = \{\emptyset_{SN}, 1_{SN}, \mathcal{A}\}$ and $\chi_2 = \{\emptyset_{SN}, 1_{SN}, \mathcal{B}\}$. If $f : (\mathcal{K}, \chi_1) \to (\mathcal{K}, \chi_2)$, defined by $f(u) = e \odot u$, for all $u \in \mathcal{K}$, then f is a homomorphism. Now, for $u \in \mathcal{A}$, $v \in \chi_2$, $(f^{-1}(\mathcal{B}))(u) = \mathcal{B}(f(u)) = \mathcal{B}(e \odot u) = \mathcal{B}(u)$,
$\forall\, u \in \mathcal{K}$, i.e., $f^{-1}(\mathcal{B}) = \mathcal{B}$. Therefore, $(f^{-1}(\mathcal{B})) \notin \chi_1$. Hence, f is not a single-valued neutrosophic continuous mapping. □

Definition 13. *Let* $\mathcal{K} = (G, \cdot, \odot, e)$ *be a K-algebra and* χ *be a SNT on* $\mathcal{K}$*. Let* $\mathcal{A}$ *be a single-valued neutrosophic K-algebra (K-subalgebra) of* $\mathcal{K}$ *and* $\chi_\mathcal{A}$ *be a SNT on* $\mathcal{A}$*. Then,* $\mathcal{A}$ *is said to be a single-valued neutrosophic topological K-algebra (K-subalgebra) on* $\mathcal{K}$ *if the self mapping* $\rho_a : (\mathcal{A}, \chi_\mathcal{A}) \to (\mathcal{A}, \chi_\mathcal{A})$ *defined as* $\rho_a(u) = u \odot a$, $\forall a \in \mathcal{K}$*, is a relatively single-valued neutrosophic continuous mapping.*

Theorem 2. *Let χ_1 and χ_2 be two SNTSs on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a homomorphism such that $f^{-1}(\chi_2) = \chi_1$. If $\mathcal{A} = \{\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A}\}$ is a single-valued neutrosophic topological K-algebra of $\mathcal{K}_2$, then $f^{-1}(\mathcal{A})$ is a single-valued neutrosophic topological K-algebra of $\mathcal{K}_1$.*

Proof. Let $\mathcal{A} = \{\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A}\}$ be a single-valued neutrosophic topological K-algebra of $\mathcal{K}_2$. To prove that $f^{-1}(\mathcal{A})$ be a single-valued neutrosophic topological K-algebra of $\mathcal{K}_1$. Let for any $u, v \in \mathcal{K}_1$,

$$\begin{aligned}
\mathcal{T}_{f^{-1}(\mathcal{A})}(u \odot v) &= \mathcal{T}_\mathcal{A}(f(u \odot v)) \\
&\geq \min\{\mathcal{T}_\mathcal{A}(f(u)), \mathcal{T}_\mathcal{A}(f(v))\} \\
&= \min\{\mathcal{T}_{f^{-1}(\mathcal{A})}(u), \mathcal{T}_{f^{-1}(\mathcal{A})}(v)\}, \\
\mathcal{I}_{f^{-1}(\mathcal{A})}(u \odot v) &= \mathcal{I}_\mathcal{A}(f(u \odot v)) \\
&\geq \min\{\mathcal{I}_\mathcal{A}(f(u)), \mathcal{I}_\mathcal{A}(f(v))\} \\
&= \min\{\mathcal{I}_{f^{-1}(\mathcal{A})}(u), \mathcal{I}_{f^{-1}(\mathcal{A})}(v)\}, \\
\mathcal{F}_{f^{-1}(\mathcal{A})}(u \odot v) &= \mathcal{F}_\mathcal{A}(f(u \odot v)) \\
&\leq \max\{\mathcal{F}_\mathcal{A}(f(u)), \mathcal{F}_\mathcal{A}(f(v))\} \\
&= \max\{\mathcal{F}_{f^{-1}(\mathcal{A})}(u), \mathcal{F}_{f^{-1}(\mathcal{A})}(v)\}.
\end{aligned}$$

Hence, $f^{-1}(\mathcal{A})$ is a single-valued neutrosophic K-algebra of $\mathcal{K}_1$.

Now, we prove that $f^{-1}(\mathcal{A})$ is single-valued neutrosophic topological K-algebra of $\mathcal{K}_1$. Since f is a single-valued neutrosophic continuous function, then by proposition 3.1, f is also a relatively single-valued neutrosophic continuous function which maps $(f^{-1}(\mathcal{A}), \chi_{f^{-1}(\mathcal{A})})$ to $(\mathcal{A}, \chi_\mathcal{A})$.

Let $a \in \mathcal{K}_1$ and Y be a SNS in $\chi_\mathcal{A}$, and let X be a SNS in $\chi_{f^{-1}(\mathcal{A})}$ such that

$$f^{-1}(Y) = X. \tag{1}$$

We are to prove that $\rho_a : (f^{-1}(\mathcal{A}), \chi_{f^{-1}(\mathcal{A})}) \to (f^{-1}(\mathcal{A}), \chi_{f^{-1}(\mathcal{A})})$ is relatively single-valued neutrosophic continuous mapping, then for any $a \in \mathcal{K}_1$, we have

$$\begin{aligned}
\mathcal{T}_{\rho_a^{-1}(X)}(u) &= \mathcal{T}_{(X)}(\rho_a(u)) = \mathcal{T}_{(X)}(u \odot a) \\
&= \mathcal{T}_{f^{-1}(Y)}(u \odot a) = \mathcal{T}_{(Y)}(f(u \odot a)) \\
&= \mathcal{T}_{(Y)}(f(u) \odot f(a)) = \mathcal{T}_{(Y)}(\rho_{f(a)}(f(u))) \\
&= \mathcal{T}_{\rho^{-1}f(a)Y}(f(u)) = \mathcal{T}_{f^{-1}}(\rho_{f(a)}^{-1}(Y)(u)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{\rho_a^{-1}(X)}(u) &= \mathcal{I}_{(X)}(\rho_a(u)) = \mathcal{I}_{(X)}(u \odot a) \\
&= \mathcal{I}_{f^{-1}(Y)}(u \odot a) = \mathcal{I}_{(Y)}(f(u \odot a)) \\
&= \mathcal{I}_{(Y)}(f(u) \odot f(a)) = \mathcal{I}_{(Y)}(\rho_{f(a)}(f(u))) \\
&= \mathcal{I}_{\rho^{-1}f(a)Y}(f(u)) = \mathcal{I}_{f^{-1}}(\rho_{f(a)}^{-1}(Y)(u)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{\rho_a^{-1}(X)}(u) &= \mathcal{F}_{(X)}(\rho_a(u)) = \mathcal{F}_{(X)}(u \odot a) \\
&= \mathcal{F}_{f^{-1}(Y)}(u \odot a) = \mathcal{F}_{(Y)}(f(u \odot a)) \\
&= \mathcal{F}_{(Y)}(f(u) \odot f(a)) = \mathcal{F}_{(Y)}(\rho_{f(a)}(f(u))) \\
&= \mathcal{F}_{\rho^{-1}f(a)Y}(f(u)) = \mathcal{F}_{f^{-1}}(\rho_{f(a)}^{-1}(Y)(u)).
\end{aligned}$$

It concludes that $\rho_a^{-1}(X) = f^{-1}(\rho_{f(a)}^{-1}(Y))$. Thus, $\rho_a^{-1}(X) \cap f^{-1}(\mathcal{A}) = f^{-1}(\rho_{f(a)}^{-1}(Y)) \cap f^{-1}(\mathcal{A})$ is a SNS in $f^{-1}(\mathcal{A})$ and a SNS in $\chi_{f^{-1}(\mathcal{A})}$. Hence, $f^{-1}(\mathcal{A})$ and a single-valued neutrosophic topological K-algebra of $\mathcal{K}$. Hence, the proof. □

Theorem 3. *Let $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$ be two SNTSs on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and let f be a bijective homomorphism of $\mathcal{K}_1$ into $\mathcal{K}_2$ such that $f(\chi_1) = \chi_2$. If $\mathcal{A}$ is a single-valued neutrosophic topological K-algebra of $\mathcal{K}_1$, then $f(\mathcal{A})$ is a single-valued neutrosophic topological K-algebra of $\mathcal{K}_2$.*

Proof. Suppose that $\mathcal{A} = \{\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A}\}$ is a SN topological K-algebra of $\mathcal{K}_1$. To prove that $f(\mathcal{A})$ is a single-valued neutrosophic topological K-algebra of $\mathcal{K}_2$, let, for $u, v \in \mathcal{K}_2$,

$$f(\mathcal{A}) = (f_{\sup}(\mathcal{T}_{\mathcal{A}})(v), f_{\sup}(\mathcal{I}_{\mathcal{A}})(v), f_{\inf}(\mathcal{F}_{\mathcal{A}})(v)).$$

Let $a_o \in f^{-1}(u), b_o \in f^{-1}(v)$ such that

$$\begin{array}{l}
\sup_{x\in f^{-1}(u)} \mathcal{T}_{\mathcal{A}}(x) = \mathcal{T}_{\mathcal{A}}(a_o), \sup_{x\in f^{-1}(v)} \mathcal{T}_{\mathcal{A}}(x) = \mathcal{T}_{\mathcal{A}}(b_o), \\
\sup_{x\in f^{-1}(u)} \mathcal{I}_{\mathcal{A}}(x) = \mathcal{I}_{\mathcal{A}}(a_o), \sup_{x\in f^{-1}(v)} \mathcal{I}_{\mathcal{A}}(x) = \mathcal{I}_{\mathcal{A}}(b_o), \\
\inf_{x\in f^{-1}(u)} \mathcal{F}_{\mathcal{A}}(x) = \mathcal{F}_{\mathcal{A}}(a_o), \inf_{x\in f^{-1}(v)} \mathcal{F}_{\mathcal{A}}(x) = \mathcal{F}_{\mathcal{A}}(b_o).
\end{array}$$

Now,

$$\begin{aligned}
\mathcal{T}_{f(\mathcal{A})}(u \odot v) &= \sup_{x\in f^{-1}(u\odot v)} \mathcal{T}_{\mathcal{A}}(x) \\
&\geq \mathcal{T}_{\mathcal{A}}(a_o, b_o) \\
&\geq \min\{\mathcal{T}_{\mathcal{A}}(a_o), \mathcal{T}_{\mathcal{A}}(b_o)\} \\
&= \min\{\sup_{x\in f^{-1}(u)} \mathcal{T}_{\mathcal{A}}(x), \sup_{x\in f^{-1}(v)} \mathcal{T}_{\mathcal{A}}(x)\} \\
&= \min\{\mathcal{T}_{f(\mathcal{A})}(u), \mathcal{T}_{f(\mathcal{A})}(v)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{f(\mathcal{A})}(u \odot v) &= \sup_{x\in f^{-1}(u\odot v)} \mathcal{I}_{\mathcal{A}}(x) \\
&\geq \mathcal{I}_{\mathcal{A}}(a_o, b_o) \\
&\geq \min\{\mathcal{I}_{\mathcal{A}}(a_o), \mathcal{I}_{\mathcal{A}}(b_o)\} \\
&= \min\{\sup_{x\in f^{-1}(u)} \mathcal{I}_{\mathcal{A}}(x), \sup_{x\in f^{-1}(v)} \mathcal{I}_{\mathcal{A}}(x)\} \\
&= \min\{\mathcal{I}_{f(\mathcal{A})}(u), \mathcal{I}_{f(\mathcal{A})}(v)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{f(\mathcal{A})}(u \odot v) &= \inf_{x\in f^{-1}(u\odot v)} \mathcal{F}_{\mathcal{A}}(x) \\
&\leq \mathcal{F}_{\mathcal{A}}(a_o, b_o) \\
&\leq \max\{\mathcal{F}_{\mathcal{A}}(a_o), \mathcal{F}_{\mathcal{A}}(b_o)\} \\
&= \max\{\inf_{x\in f^{-1}(u)} \mathcal{F}_{\mathcal{A}}(x), \inf_{x\in f^{-1}(v)} \mathcal{F}_{\mathcal{A}}(x)\} \\
&= \max\{\mathcal{F}_{f(\mathcal{A})}(u), \mathcal{F}_{f(\mathcal{A})}(v)\}.
\end{aligned}$$

Hence, $f(\mathcal{A})$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}_2$. Now, we prove that the self mapping $\rho_b : (f(\mathcal{A}), \chi_f(\mathcal{A})) \to (f(\mathcal{A}), \chi_f(\mathcal{A}))$, defined by $\rho_b(v) = v \odot b$, for all $b \in \mathcal{K}_2$, is a relatively single-valued neutrosophic continuous mapping. Let $Y_{\mathcal{A}}$ be a SNS in $\chi_{\mathcal{A}}$, there exists a SNS "Y" in χ_1 such that $Y_{\mathcal{A}} = Y \cap \mathcal{A}$. We show that for a SNS in $\chi_{f(\mathcal{A})}$,

$$\rho^{-1}{}_b(Y_{f(\mathcal{A})}) \cap f(\mathcal{A}) \in \chi_{f(\mathcal{A})}$$

Since f is an injective mapping, then $f(Y_{\mathcal{A}}) = f(Y \cap \mathcal{A}) = f(Y) \cap f(\mathcal{A})$ is a SNS in $\chi_{f(\mathcal{A})}$ which shows that f is relatively single-valued neutrosophic open. In addition, f is surjective, then for all $b \in \mathcal{K}_2, a = f(b)$, where $a \in \mathcal{K}_1$.

Now,

$$\begin{aligned}
\mathcal{T}_{f^{-1}(\rho^{-1}{}_b(Y_{f(\mathcal{A})}))}(u) &= \mathcal{T}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(\mathcal{A})}))}(u) \\
&= \mathcal{T}_{\rho^{-1}{}_{f(a)}(Y_{f(\mathcal{A})})}(f(u)) \\
&= \mathcal{T}_{(Y_{f(\mathcal{A})})}(\rho_{f(a)}(f(u))) \\
&= \mathcal{T}_{(Y_{f(\mathcal{A})})}(f(u) \odot f(a)) \\
&= \mathcal{T}_{f^{-1}(Y_{f(\mathcal{A})})}(u \odot a) \\
&= \mathcal{T}_{f^{-1}(Y_{f(\mathcal{A})})}(\rho_a(u)) \\
&= \mathcal{T}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(\mathcal{A})}))(u),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{f^{-1}(\rho^{-1}{}_b(Y_{f(\mathcal{A})}))}(u) &= \mathcal{I}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(\mathcal{A})}))}(u) \\
&= \mathcal{I}_{\rho^{-1}{}_{f(a)}(Y_{f(\mathcal{A})})}(f(u)) \\
&= \mathcal{I}_{(Y_{f(\mathcal{A})})}(\rho_{f(a)}(f(u))) \\
&= \mathcal{I}_{(Y_{f(\mathcal{A})})}(f(u) \odot f(a)) \\
&= \mathcal{I}_{f^{-1}(Y_{f(\mathcal{A})})}(u \odot a) \\
&= \mathcal{I}_{f^{-1}(Y_{f(\mathcal{A})})}(\rho_a(u)) \\
&= \mathcal{I}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(\mathcal{A})}))(u),
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{f^{-1}(\rho^{-1}{}_b(Y_{f(\mathcal{A})}))}(u) &= \mathcal{F}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(\mathcal{A})}))}(u) \\
&= \mathcal{F}_{\rho^{-1}{}_{f(a)}(Y_{f(\mathcal{A})})}(f(u)) \\
&= \mathcal{F}_{(Y_{f(\mathcal{A})})}(\rho_{f(a)}(f(u))) \\
&= \mathcal{F}_{(Y_{f(\mathcal{A})})}(f(u) \odot f(a)) \\
&= \mathcal{F}_{f^{-1}(Y_{f(\mathcal{A})})}(u \odot a) \\
&= \mathcal{F}_{f^{-1}(Y_{f(\mathcal{A})})}(\rho_a(u)) \\
&= \mathcal{F}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(\mathcal{A})}))(u).
\end{aligned}$$

This implies that $f^{-1}(\rho^{-1}_{(b)}((Y_{f(\mathcal{A})}))) = \rho^{-1}_{(a)}(f^{-1}(Y_{(\mathcal{A})}))$. Since $\rho_a : (\mathcal{A}, \chi_{\mathcal{A}}) \to (\mathcal{A}, \chi_{\mathcal{A}})$ is relatively single-valued neutrosophic continuous mapping and f is relatively single-valued neutrosophic continues mapping from $(\mathcal{A}, \chi_{\mathcal{A}})$ into $(f(\mathcal{A}), \chi_{f(\mathcal{A})})$, $f^{-1}(\rho^{-1}_{(b)}((Y_{f(\mathcal{A})}))) \cap \mathcal{A} = \rho^{-1}_{(a)}(f^{-1}(Y_{(\mathcal{A})})) \cap \mathcal{A}$ is a SNS in $\chi_{\mathcal{A}}$. Hence, $f(f^{-1}(\rho_{(b)}((Y_{f(\mathcal{A})}))) \cap \mathcal{A}) = \rho^{-1}_{(b)}(Y_{f(\mathcal{A})}) \cap f(\mathcal{A})$ is a SNS in $\chi_{\mathcal{A}}$, which completes the proof. □

Example 2. *Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Caley's table for $\odot$ is given in Example 1. We define a SNS as:*

$$\mathcal{A} = \{(e, 0.4, 0.5, 0.8), (s, 0.3, 0.4, 0.6)\},$$
$$\mathcal{B} = \{(e, 0.3, 0.4, 0.8), (s, 0.2, 0.3, 0.6)\},$$

for all $s \neq e \in G$, where $\mathcal{A}, \mathcal{B} \in [0, 1]$. The collection $\chi_{\mathcal{K}} = \{\emptyset_{SN}, 1_{SN}, \mathcal{A}, \mathcal{B}\}$ of SNSs of $\mathcal{K}$ is a SNT on $\mathcal{K}$ and $(\mathcal{K}, \chi_{\mathcal{K}})$ is a SNTS. Let $\mathcal{C}$ be a SNS in $\mathcal{K}$, defined as:

$$\mathcal{C} = \{(e, 0.7, 0.5, 0.2), (s, 0.5, 0.4, 0.6)\}, \forall s \neq e \in G.$$

Clearly, $\mathcal{C}$ is a single-valued neutrosophic K-subalgebra of $\mathcal{K}$. By direct calculations relative topology $\chi_{\mathcal{C}}$ is obtained as $\chi_{\mathcal{C}} = \{\emptyset_{\mathcal{A}}, 1_{\mathcal{A}}, \mathcal{A}\}$. Then, the pair $(\mathcal{C}, \chi_{\mathcal{C}})$ is a single-valued neutrosophic subspace of $(\mathcal{K}, \chi_{\mathcal{K}})$. We show that $\mathcal{C}$ is a single-valued neutrosophic topological K-subalgebra of $\mathcal{K}$, i.e., the self mapping $\rho_a : (\mathcal{C}, \chi_{\mathcal{C}}) \to (\mathcal{C}, \chi_{\mathcal{C}})$ defined by $\rho_a(u) = u \odot a, \forall a \in \mathcal{K}$ is relatively single-valued neutrosophic continuous mapping, i.e., for a SNOS $\mathcal{A}$ in $(\mathcal{C}, \chi_{\mathcal{C}})$, $\rho_a^{-1}(\mathcal{A}) \cap \mathcal{C} \in \chi_{\mathcal{C}}$. Since ρ_a is homomorphism, then $\rho_a^{-1}(\mathcal{A}) \cap \mathcal{C} = \mathcal{A} \in \chi_{\mathcal{C}}$. Therefore, $\rho_a : (\mathcal{C}, \chi_{\mathcal{C}}) \to (\mathcal{C}, \chi_{\mathcal{C}})$ is relatively single-valued neutrosophic continuous mapping. Hence, $\mathcal{C}$ is a single-valued neutrosophic topological K-algebra of $\mathcal{K}$.

Example 3. *Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Caley's table for $\odot$ is given in Example 3.1. We define a SNS as:*

$$\begin{aligned}
\mathcal{A} &= \{(e, 0.4, 0.5, 0.8), (s, 0.3, 0.4, 0.6)\},\\
\mathcal{B} &= \{(e, 0.3, 0.4, 0.8), (s, 0.2, 0.3, 0.6)\},\\
\mathcal{D} &= \{(e, 0.2, 0.1, 0.3), (s, 0.1, 0.1, 0.5)\},
\end{aligned}$$

for all $s \neq e \in G$, where $\mathcal{A}, \mathcal{B} \in [0,1]$. The collection $\chi_1 = \{\emptyset_{SN}, 1_{SN}, \mathcal{D}\}$ and $\chi_2 = \{\emptyset_{SN}, 1_{SN}, \mathcal{A}, \mathcal{B}\}$ of SNSs of $\mathcal{K}$ are SNTs on $\mathcal{K}$ and $(\mathcal{K}, \chi_1)$, $(\mathcal{K}, \chi_2)$ be two SNTSs. Let $\mathcal{C}$ be a SNS in $(\mathcal{K}, \chi_2)$, defined as:

$$\mathcal{C} = \{(e, 0.7, 0.5, 0.2), (s, 0.5, 0.4, 0.6)\}, \forall s \neq e \in G.$$

Now, Let $f : (\mathcal{K}, \chi_1) \to (\mathcal{K}, \chi_2)$ be a homomorphism such that $f^{-1}(\chi_2) = \chi_1$ (we have not consider $\mathcal{K}$ to be distinct), then, by Proposition 3, f is a single-valued neutrosophic continuous function and f is also relatively single-valued neutrosophic continues mapping from $(\mathcal{K}, \chi_1)$ into $(\mathcal{K}, \chi_2)$. Since $\mathcal{C}$ is a SNS in $(\mathcal{K}, \chi_2)$ and with relative topology $\chi_{\mathcal{C}} = \{\emptyset_{\mathcal{A}}, 1_{\mathcal{A}}, \mathcal{A}\}$ is also a single-valued neutrosophic topological K-algebra of $(\mathcal{K}, \chi_2)$. We prove that $f^{-1}(\mathcal{C})$ is a single-valued neutrosophic topological K-algebra in $(\mathcal{K}, \chi_1)$. Since f is a continuous function, then, by Definition 8, $f^{-1}(\mathcal{C})$ is a single-valued neutrosophic K-subalgebra in $(\mathcal{K}, \chi_1)$. To prove that $f^{-1}(c)$ is a single-valued neutrosophic topological K-algebra, then for $b \in \mathcal{K}_1$ take

$$\rho_b : (f^{-1}(\mathcal{C}), \chi_{f^{-1}(\mathcal{C})}) \to (f^{-1}(\mathcal{C}), \chi_{f^{-1}(\mathcal{C})}),$$

for $\mathcal{A} \in \chi_{f^{-1}(C)}, \rho_b^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{C}) \in \chi_{f^{-1}(C)}$ which shows that $f^{-1}(\mathcal{C})$ is a single-valued neutrosophic topological K-algebra in $(\mathcal{K}, \chi_1)$. Similarly, we can show that $f(\mathcal{C})$ is a a single-valued neutrosophic topological K-algebra in $(\mathcal{K}, \chi_2)$ by considering a bijective homomorphism.

Definition 14. *Let χ be a SNT on $\mathcal{K}$ and $(\mathcal{K}, \chi)$ be a SNTS. Then, $(\mathcal{K}, \chi)$ is called single-valued neutrosophic C_5-disconnected topological space if there exist a SNOS and SNCS $\mathcal{H}$ such that $\mathcal{H} = (\mathcal{T}_{\mathcal{H}}, \mathcal{I}_{\mathcal{H}}, \mathcal{F}_{\mathcal{H}},) \neq 1_{SN}$ and $\mathcal{H} = (\mathcal{T}_{\mathcal{H}}, \mathcal{I}_{\mathcal{H}}, \mathcal{F}_{\mathcal{H}},) \neq \emptyset_{SN}$, otherwise $(\mathcal{K}, \chi)$ is called single-valued neutrosophic C_5-connected.*

Example 4. *Every indiscrete SNT space on $\mathcal{K}$ is C_5-connected.*

Proposition 4. *Let $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$ be two SNTSs and $f : (\mathcal{K}_1, \chi_1) \to (\mathcal{K}_2, \chi_2)$ be a surjective single-valued neutrosophic continuous mapping. If $(\mathcal{K}_1, \chi_1)$ is a single-valued neutrosophic C_5-connected space, then $(\mathcal{K}_2, \chi_2)$ is also a single-valued neutrosophic C_5-connected space.*

Proof. Suppose on contrary that $(\mathcal{K}_2, \chi_2)$ is a single-valued neutrosophic C_5-disconnected space. Then, by Definition 14, there exist both SNOS and SNCS $\mathcal{H}$ be such that $\mathcal{H} \neq 1_{SN}$ and $\mathcal{H} \neq \emptyset_{SN}$. Since f is a single-valued neutrosophic continuous and onto function, so $f^{-1}(\mathcal{H}) = 1_{SN}$ or $f^{-1}(\mathcal{H}) = \emptyset_{SN}$, where $f^{-1}(\mathcal{H})$ is both SNOS and SNCS. Therefore,

$$\mathcal{H} = f(f^{-1}(\mathcal{H})) = f(1_{SN}) = 1_{SN} \tag{2}$$

and

$$\mathcal{H} = f(f^{-1}(\mathcal{H})) = f(\emptyset_{SN}) = \emptyset_{SN}, \tag{3}$$

a contradiction. Hence, $(\mathcal{K}_2, \chi_2)$ is a single-valued neutrosophic C_5-connected space. □

Corollary 1. *Let χ be a SNT on $\mathcal{K}$. Then, $(\mathcal{K}, \chi)$ is called a single-valued neutrosophic C_5-connected space if and only if there does not exist a single-valued neutrosophic continuous map $f : (\mathcal{K}, \chi) \to (\mathcal{F}_T, \chi_T)$ such that $f \neq 1_{SN}$ and $f \neq \emptyset_{SN}$*

Definition 15. *Let $\mathcal{A} = \{\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A}\}$ be a SNS in $\mathcal{K}$. Let χ be a SNT on $\mathcal{K}$. The interior and closure of $\mathcal{A}$ in $\mathcal{K}$ is defined as:*

$\mathcal{A}^{Int}$: The union of SNOSs which contained in $\mathcal{A}$.
$\mathcal{A}^{Clo}$: The intersection of SNCSs for which $\mathcal{A}$ is a subset of these SNCSs.

Remark 1. *Being union of SNOS $\mathcal{A}^{Int}$ is a SNO and $\mathcal{A}^{Clo}$ being intersection of SNCS is SNC.*

Theorem 4. *Let $\mathcal{A}$ be a SNS in a SNTS $(\mathcal{K}, \chi)$. Then, $\mathcal{A}^{Int}$ is such an open set which is the largest open set of $\mathcal{K}$ contained in $\mathcal{A}$.*

Corollary 2. *$\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a SNOS in $\mathcal{K}$ if and only if $\mathcal{A}^{Int} = \mathcal{A}$ and $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ is a SNCS in $\mathcal{K}$ if and only if $\mathcal{A}^{Clo} = \mathcal{A}$.*

Proposition 5. *Let $\mathcal{A}$ be a SNS in $\mathcal{K}$. Then, following results hold for $\mathcal{A}$:*

(i) $(1_{SN})^{Int} = 1_{SN}$.
(ii) $(\emptyset_{SN})^{Clo} = \emptyset_{SN}$.
(iii) $\overline{(\mathcal{A})}^{Int} = \overline{(\mathcal{A})^{Clo}}$.
(iv) $\overline{(\mathcal{A})}^{Clo} = \overline{(\mathcal{A})^{Int}}$.

Definition 16. *Let $\mathcal{K}$ be a K-algebra and χ be a SNT on $\mathcal{K}$. A SNOS $\mathcal{A}$ in $\mathcal{K}$ is said to be single-valued neutrosophic regular open if*

$$\mathcal{A} = (\mathcal{A}^{Clo})^{Int}. \tag{4}$$

Remark 2. *Every SNOS which is regular is single-valued neutrosophic open and every single-valued neutrosophic closed and open set is a single-valued neutrosophic regular open.*

Definition 17. *A single-valued neutrosophic super connected K-algebra is such a K-algebra in which there does not exist a single-valued neutrosophic regular open set $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ such that $\mathcal{A} \neq \emptyset_{SN}$ and $\mathcal{A} \neq 1_{SN}$. If there exists such a single-valued neutrosophic regular open set $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ such that $\mathcal{A} \neq \emptyset_{SN}$ and $\mathcal{A} \neq 1_{SN}$, then K-algebra is said to be a single-valued neutrosophic super disconnected.*

Example 5. *Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Caley's table for $\odot$ is given in Example 1 We define a SNS as:*

$$\mathcal{A} = \{(e, 0.2, 0.3, 0.8), (s, 0.1, 0.2, 0.6)\}.$$

Let $\chi_\mathcal{K} = \{\emptyset_{SN}, 1_{SN}, \mathcal{A}\}$ be a SNT on $\mathcal{K}$ and let $\mathcal{B} = \{(e, 0.3, 0.3, 0.8), (s, 0.2, 0.2, 0.6)\}$ be a SNS in $\mathcal{K}$. here

SNOSs : $\emptyset_{SN} = \{0, 0, 1\}, 1_{SN} = \{1, 1, 0\}, \mathcal{A} = \{(e, 0.2, 0.3, 0.8), (s, 0.1, 0.2, 0.6)\}$.
SNCSs : $(\emptyset_{SN})^c = (\{0, 0, 1\})^c = (\{1, 1, 0\}) = 1_{SN}, (1_{SN})^c = (\{1, 1, 0\})^c = (\{0, 0, 1\}) = \emptyset_{SN}$,
$(\mathcal{A})^c = (\{(e, 0.2, 0.3, 0.8), (s, 0.1, 0.2, 0.6)\})^c = (\{(e, 0.8, 0.3, 0.2), (s, 0.6, 0.2, 0.1)\}) = \mathcal{A}'$ *(say)*.

Then, closure of $\mathcal{B}$ is the intersection of closed sets which contain $\mathcal{B}$. Therefore,

$$\mathcal{A}^{'} = \mathcal{B}^{Clo}. \tag{5}$$

Now, interior of $\mathcal{B}$ is the union of open sets which contain in $\mathcal{B}$. Therefore,

$$\emptyset_{SN} \bigcup \mathcal{A} = \mathcal{A}$$

$$\mathcal{A} = \mathcal{B}^{Int}. \tag{6}$$

Note that $(\mathcal{B}^{Clo})^{Clo} = \mathcal{B}^{Clo}$. Now, if we consider a SNS $\mathcal{A} = \{(e, 0.2, 0.3, 0.8), (s, 0.1, 0.2, 0.6)\}$ in a K-algebra $\mathcal{K}$ and if $\chi_{\mathcal{K}} = \{\emptyset_{SN}, 1_{SN}, \mathcal{A}\}$ is a SNT on $\mathcal{K}$. Then, $(\mathcal{A})^{Clo} = \mathcal{A}$ and $(\mathcal{A})^{Int} = \mathcal{A}$. Consequently,

$$\mathcal{A} = (\mathcal{A}^{Clo})^{Int}, \tag{7}$$

which shows that $\mathcal{A}$ is a SN regular open set in K-algebra $\mathcal{K}$. Since $\mathcal{A}$ is a SN regular open set in $\mathcal{K}$ and $\mathcal{A} \neq \emptyset_{SN}, \mathcal{A} \neq 1_{SN}$, then, by Definition 17, K-algebra $\mathcal{K}$ is a single-valued neutrosophic supper disconnected K-algebra.

Proposition 6. *Let $\mathcal{K}$ be a K-algebra and let $\mathcal{A}$ be a SNOS. Then, the following statements are equivalent:*

(i) *A K-algebra is single-valued neutrosophic super connected.*
(ii) $(\mathcal{A})^{Clo} = 1_{SN}$*, for each SNOS* $\mathcal{A} \neq \emptyset_{SN}$.
(iii) $(\mathcal{A})^{Int} = \emptyset_{SN}$*, for each SNCS* $\mathcal{A} \neq 1_{SN}$.
(iv) *There do not exist SNOSs $\mathcal{A}, \mathcal{F}$ such that $\mathcal{A} \subseteq \overline{\mathcal{F}}$ and $\mathcal{A} \neq \emptyset_{SN} \neq \mathcal{F}$ in K-algebra $\mathcal{K}$.*

Definition 18. *Let $(\mathcal{K}, \chi)$ be a SNTS, where $\mathcal{K}$ is a K-algebra. Let S be a collection of SNOSs in $\mathcal{K}$ denoted by $S = \{(\mathcal{T}_{\mathcal{A}_j}, \mathcal{I}_{\mathcal{A}_j}, \mathcal{F}_{\mathcal{A}_j}) : j \in J\}$. Let $\mathcal{A}$ be a SNOS in $\mathcal{K}$. Then, S is called a single-valued neutrosophic open covering of $\mathcal{A}$ if $\mathcal{A} \subseteq \bigcup S$.*

Definition 19. *Let $\mathcal{K}$ be a K-algebra and $(\mathcal{K}, \chi)$ be a SNTS. Let L be a finite sub-collection of S. If L is also a single-valued neutrosophic open covering of $\mathcal{A}$, then it is called a finite sub-covering of S and $\mathcal{A}$ is called single-valued neutrosophic compact if each single-valued neutrosophic open covering S of $\mathcal{A}$ has a finite sub-cover. Then, $(\mathcal{K}, \chi)$ is called compact K-algebra.*

Remark 3. *If either $\mathcal{K}$ is a finite K-algebra or χ is a finite topology on $\mathcal{K}$, i.e., consists of finite number of single-valued neutrosophic subsets of $\mathcal{K}$, then the SNT $(\mathcal{K}, \chi)$ is a single-valued neutrosophic compact topological space.*

Proposition 7. *Let $(\mathcal{K}_1, \chi_1)$ and $(\mathcal{K}_2, \chi_2)$ be two SNTSs and f be a single-valued neutrosophic continuous mapping from $\mathcal{K}_1$ into $\mathcal{K}_2$. Let $\mathcal{A}$ be a SNS in $(\mathcal{K}_1, \chi_1)$. If $\mathcal{A}$ is single-valued neutrosophic compact in $(\mathcal{K}_1, \chi_1)$, then $f(\mathcal{A})$ is single-valued neutrosophic compact in $(\mathcal{K}_2, \chi_2)$.*

Proof. Let $f : (\mathcal{K}_1, \chi_1) \to (\mathcal{K}_2, \chi_2)$ be a single-valued neutrosophic continuous function. Let $\acute{S} = (f^{-1}(\mathcal{A}_j : j \in J))$ be a single-valued neutrosophic open covering of $\mathcal{A}$ since $\mathcal{A}$ be a SNS in $(\mathcal{K}_1, \chi_1)$. Let $\acute{L} = (\mathcal{A}_j : j \in J)$ be a single-valued neutrosophic open covering of $f(\mathcal{A})$. Since $\mathcal{A}$ is compact, then there exists a single-valued neutrosophic finite sub-cover $\bigcup\limits_{j=1}^{n} f^{-1}(\mathcal{A}_j)$ such that

$$\mathcal{A} \subseteq \bigcup_{j=1}^{n} f^{-1}(\mathcal{A}_j)$$

We have to prove that there also exists a finite sub-cover of $\acute{L}$ for $f(\mathcal{A})$ such that

$$f(\mathcal{A}) \subseteq \bigcup_{j=1}^{n} (\mathcal{A}_j)$$

Now,

$$\mathcal{A} \subseteq \bigcup_{j=1}^{n} f^{-1}(\mathcal{A}_j)$$

$$f(\mathcal{A}) \subseteq f(\bigcup_{j=1}^{n} f^{-1}(\mathcal{A}_j))$$

$$f(\mathcal{A}) \subseteq \bigcup_{j=1}^{n} (f(f^{-1}(\mathcal{A}_j)))$$

$$f(\mathcal{A}) \subseteq \bigcup_{j=1}^{n} (\mathcal{A}_j).$$

Hence, $f(\mathcal{A})$ is single-valued neutrosophic compact in $(\mathcal{K}_2, \chi_2)$. □

Definition 20. *A single-valued neutrosophic set $\mathcal{A}$ in a K-algebra $\mathcal{K}$ is called a single-valued neutrosophic point if*

$$\mathcal{T}_{\mathcal{A}}(v)= \begin{cases} \alpha \in (0,1], & \text{if } v{=}u \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_{\mathcal{A}}(v)= \begin{cases} \beta \in (0,1], & \text{if } v{=}u \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{F}_{\mathcal{A}}(v)= \begin{cases} \gamma \in [0,1), & \text{if } v{=}u \\ 0, & \text{otherwise,} \end{cases}$$

with support u and value (α, β, γ), denoted by $u(\alpha, \beta, \gamma)$. This single-valued neutrosophic point is said to "belong to" a SNS $\mathcal{A}$, written as $u(\alpha, \beta, \gamma) \in \mathcal{A}$ if $\mathcal{T}_{\mathcal{A}}(u) \geq \alpha, \mathcal{I}_{\mathcal{A}}(u) \geq \beta, \mathcal{F}_{\mathcal{A}}(u) \leq \gamma$ and said to be "quasi-coincident with" a SNS $\mathcal{A}$, written as $u(\alpha, \beta, \gamma)\ q\ \mathcal{A}$ if $\mathcal{T}_{\mathcal{A}}(u) + \alpha > 1, \mathcal{I}_{\mathcal{A}}(u) + \beta > 1, \mathcal{F}_{\mathcal{A}}(u) + \gamma < 1$.

Definition 21. *Let $\mathcal{K}$ be a K-algebra and let $(\mathcal{K}, \chi)$ be a SNTS. Then, $(\mathcal{K}, \chi)$ is called a single-valued neutrosophic Hausdorff space if and only if, for any two distinct single-valued neutrosophic points $u_1, u_2 \in \mathcal{K}$, there exist SNOSs $\mathcal{B}_1 = (\mathcal{T}_{\mathcal{B}_1}, \mathcal{I}_{\mathcal{B}_1}, \mathcal{F}_{\mathcal{B}_1}), \mathcal{B}_2 = (\mathcal{T}_{\mathcal{B}_2}, \mathcal{I}_{\mathcal{B}_2}, \mathcal{F}_{\mathcal{B}_2})$ such that $u_1 \in \mathcal{B}_1, u_2 \in \mathcal{B}_2$, i.e.,*

$$\mathcal{T}_{\mathcal{B}_1}(u_1) = 1, \mathcal{I}_{\mathcal{B}_1}(u_1) = 1, \mathcal{F}_{\mathcal{B}_1}(u_1) = 0,$$
$$\mathcal{T}_{\mathcal{B}_2}(u_2) = 1, \mathcal{I}_{\mathcal{B}_2}(u_2) = 1, \mathcal{F}_{\mathcal{B}_2}(u_2) = 0$$

and satisfy the condition that $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset_{SN}$. Then, $(\mathcal{K}, \chi)$ is called single-valued neutrosophic Hausdorff space and K-algebra is said to ba a Hausdorff K-algebra. In fact, $(\mathcal{K}, \chi)$ is a Hausdorff K-algebra.

Example 6. *Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra and let $(\mathcal{K}, \chi_{\mathcal{K}})$ be a SNTS on $\mathcal{K}$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Caley's table for $\odot$ is given in Example 1. We define two SNSs as $\mathcal{A} = \{(e, 1, 1, 0), (s, 0, 0, 1)\}$. $\mathcal{B} = \{(e, 0, 0, 1), (s, 1, 1, 0)\}$. Consider a single-valued neutrosophic point for $e \in \mathcal{K}$ such that*

$$\mathcal{T}_{\mathcal{A}}(e)= \begin{cases} 0.3, & \text{if } e{=}u \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_{\mathcal{A}}(e)= \begin{cases} 0.2, & \text{if } e{=}u \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{F}_{\mathcal{A}}(e)=\begin{cases} 0.4, & \text{if } e{=}u \\ 0, & \text{otherwise.} \end{cases}$$

Then, $e(0.3, 0.2, 0.4)$ is a single-valued neutrosophic point with support e and value $(0.3, 0.2, 0.4)$. This single-valued neutrosophic point belongs to SNS "A" but not SNS "B".

Now, for all $s \neq e \in \mathcal{K}$

$$\mathcal{T}_{\mathcal{B}}(s)=\begin{cases} 0.5, & \text{if } s{=}u \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_{\mathcal{B}}(s)=\begin{cases} 0.4, & \text{if } s{=}u \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{F}_{\mathcal{B}}(s)=\begin{cases} 0.3, & \text{if } s{=}u \\ 0, & \text{otherwise.} \end{cases}$$

Then, $s(0.5, 0.4, 0.3)$ is a single-valued neutrosophic point with support s and value $(0.5, 0.4, 0.3)$. This single-valued neutrosophic point belongs to SNS "B" but not SNS "A". Thus, $e(0.3, 0.2, 0.4) \in \mathcal{A}$ and $e(0.3, 0.2, 0.4) \notin \mathcal{B}$, $s(0.5, 0.4, 0.3) \in \mathcal{B}$ and $s(0.5, 0.4, 0.3) \notin \mathcal{A}$ and $\mathcal{A} \bigcap \mathcal{B} = \emptyset_{SN}$. Thus, K-algebra is a Hausdorff K-algebra and $(\mathcal{K}, \chi_{\mathcal{K}})$ is a Hausdorff topological space.

Theorem 5. *Let $(\mathcal{K}_1, \chi_1)$, $(\mathcal{K}_2, \chi_2)$ be two SNTSs. Let f be a single-valued neutrosophic homomorphism from $(\mathcal{K}_1, \chi_1)$ into $(\mathcal{K}_2, \chi_2)$. Then, $(\mathcal{K}_1, \chi_1)$ is a single-valued neutrosophic Hausdorff space if and only if $(\mathcal{K}_2, \chi_2)$ is a single-valued neutrosophic Hausdorff K-algebra.*

Proof. Let $(\mathcal{K}_1, \chi_1)$, $(\mathcal{K}_2, \chi_2)$ be two SNTSs. Let $\mathcal{K}_1$ be a single-valued neutrosophic Hausdorff space, then, according to the Definition 21, there exist two SNOSs X and Y for two distinct single-valued neutrosophic points $u_1, u_2 \in \chi_2$ also $a, b \in \mathcal{K}_1 (a \neq b)$ such that $X \bigcap Y = \emptyset_{SN}$.
Now, for $w \in \mathcal{K}_1$, consider $(f^{-1}(u_1))(w) = u_1(f^{-1}(w))$, where $u_1(f^{-1}(w)) = s \in (0, 1]$ if $w = f^{-1}(a)$, otherwise 0. That is, $(f^{-1}(u_1))(w) = ((f^{-1}(u))_1(w))$. Therefore, we have $f^{-1}(u_1) = (f^{-1}(u))_1$. Similarly, $f^{-1}(u_2) = (f^{-1}(u))_2$. Now, since f^{-1} is a single-valued neutrosophic continuous mapping from $\mathcal{K}_2$ into $\mathcal{K}_1$, there exist two SNOSs $f(X)$ and $f(Y)$ of u_1 and u_2, respectively, such that $f(X) \bigcap f(Y) = f(\emptyset_{SN}) = \emptyset_{SN}$. This implies that $\mathcal{K}_2$ is a single-valued neutrosophic Hausdorff K-algebra. The converse part can be proved similarly. □

Theorem 6. *Let f be a single-valued neutrosophic continuous function which is both one-one and onto, where f is a mapping from a single-valued neutrosophic compact K-algebra $\mathcal{K}_1$ into a single-valued neutrosophic Hausdorff K-algebra $\mathcal{K}_2$. Then, f is a homomorphism.*

Proof. Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a single-valued neutrosophic continuous bijective function from single-valued neutrosophic compact K-algebra $\mathcal{K}_1$ into a single-valued neutrosophic Hausdorff K-algebra $\mathcal{K}_2$. Since f is a single-valued neutrosophic continuous mapping from $\mathcal{K}_1$ into $\mathcal{K}_2$, f is a homomorphism. Since f is bijective, we only prove that f is single-valued neutrosophic closed. Let $\mathcal{D} = (\mathcal{T}_{\mathcal{D}}, \mathcal{I}_{\mathcal{D}}, \mathcal{F}_{\mathcal{D}})$ be a single-valued neutrosophic closed in $\mathcal{K}_1$. If $\mathcal{D} = \emptyset_{SN}$ is single-valued neutrosophic closed in $\mathcal{K}_1$, then $f(\mathcal{D}) = \emptyset_{SN}$ is single-valued neutrosophic closed in $\mathcal{K}_2$. However, if $\mathcal{D} \neq \emptyset_{SN}$, then $\mathcal{D}$ will be a single-valued neutrosophic compact, being subset of a single-valued neutrosophic compact K-algebra. Then, $f(\mathcal{D})$, being single-valued neutrosophic continuous image of a single-valued neutrosophic compact K-algebra, is also single-valued neutrosophic compact. Therefore, $\mathcal{K}_2$ is closed, which implies that mapping f is closed. Thus, f is a homomorphism. □

4. Conclusions

Non-classical logic is considered as a powerful tool for inspecting uncertainty and indeterminacy found in real world problems. Being a great extension of classical logic, neutrosophic set theory is considered as a useful mathematical tool to cope up with uncertainties in science, technology, and computer science. We have used this mathematical model with a topological structure to investigate the uncertainty in K-algebras. We have introduced the notion of single-valued neutrosophic topological K-algebras and presented certain concepts, including continuous function between two topological on K-algebras, relatively continuous function and homomorphism. We have investigated the image and pre-image of single-valued neutrosophic topological K-algebras under this homomorphism. We have proposed some conclusive concepts, including single-valued neutrosophic compact K-algebras and single-valued neutrosophic Hausdorff K-algebras. We plan to extend our study to: (i) single-valued neutrosophic soft topological K-algebras; and (ii) bipolar neutrosophic soft topological K-algebras.

For other notations and terminologies, readers are referred to [21–26].

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Study on the Development of Neutrosophic Triplet Ring and Neutrosophic Triplet Field

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Mumtaz Ali, Florentin Smarandache, Mohsin Khan (2018). Study on the Development of Neutrosophic Triplet Ring and Neutrosophic Triplet Field. *Mathematics* 6, 46; DOI: 10.3390/math6040046

Abstract: Rings and fields are significant algebraic structures in algebra and both of them are based on the group structure. In this paper, we attempt to extend the notion of a neutrosophic triplet group to a neutrosophic triplet ring and a neutrosophic triplet field. We introduce a neutrosophic triplet ring and study some of its basic properties. Further, we define the zero divisor, neutrosophic triplet subring, neutrosophic triplet ideal, nilpotent integral neutrosophic triplet domain, and neutrosophic triplet ring homomorphism. Finally, we introduce a neutrosophic triplet field.

Keywords: ring; field; neutrosophic triplet; neutrosophic triplet group; neutrosophic triplet ring; neutrosophic triplet field

1. Introduction

The concept of a ring first arose from attempts to prove Fermat's last theorem [1], starting with Richard Dedekind in the 1880s. After contributions from other fields, mainly number theory, the notion of a ring was generalized and firmly established during the 1920s by Emmy Noether and Wolfgang Krull [2] Modern ring theory, a very active mathematical discipline, studies rings in their own right. To explore rings, mathematicians have devised various notions to break rings into smaller, more understandable pieces, such as ideals, quotient rings, and simple rings. In addition to these abstract properties, ring theorists also make various distinctions between the theories of commutative rings and noncommutative rings, the former belonging to algebraic number theory and algebraic geometry. A particularly rich theory has been developed for a certain special class of commutative rings, known as fields, which lies within the realm of field theory. Likewise, the corresponding theory for noncommutative rings, that of noncommutative division rings, constitutes an active research interest for noncommutative ring theorists. Since the discovery of a mysterious connection between noncommutative ring theory and geometry during the 1980s by Alain Connes [3–5], noncommutative geometry has become a particularly active discipline in ring theory.

The foundation of the subject (i.e., the mapping from subfields to subgroups and vice versa) is set up in the context of an absolutely general pair of fields. In addition to the clarification that normally accompanies such a generalization, there are useful applications to infinite algebraic extensions and to the Galois Theory of differential equations [6]. There is also a logical simplicity to the procedure: everything hinges on a pair of estimates of field degrees and subgroup indices. One might describe it as a further step in the Dedekind–Artin linearization [7].

An early contributor to the theory of noncommutative rings was the Scottish mathematician Wedderburn who, in 1905, proved "Wedderburn's Theorem", namely that every finite division ring is commutative and so is a field [8]. It was only around the 1930s that the theories of commutative and noncommutative rings came together and that their ideas began to influence each other.

Neutrosophy is a new branch of philosophy which studies the nature, origin, and scope of neutralities as well as their interaction with ideational spectra. The concept of neutrosophic logic and a neutrosophic set was first introduced by Florentin Smarandache [9] in 1995, where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F such that this neutrosophic logic is called an extension of fuzzy logic, especially to intuitionistic fuzzy logic [10]. The generalization of classical sets [9], fuzzy sets [11], and intuitionistic fuzzy sets [10], etc., is in fact the neutrosophic set. This mathematical tool is used to handle problems consisting of uncertainty, imprecision, indeterminacy, inconsistency, incompleteness, and falsity. By utilizing the idea of neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache studied neutrosophic algebraic structures [12–14] by inserting a literal indeterminate element "I", where $I^2 = I$, in the algebraic structure and then combining "I" with each element of the structure with respect to the corresponding binary operation, denoted *. They call it the neutrosophic element, and the generated algebraic structure is then termed as a neutrosophic algebraic structure. Some other neutrosophic algebraic structures can be seen as neutrosophic fields [15], neutrosophic vector spaces [16], neutrosophic groups [17], neutrosophic bigroups [17], neutrosophic N-groups [15], neutrosophic semigroups [12], neutrosophic bisemigroups [12], neutrosophic N-semigroups [12], neutrosophic loops [12], neutrosophic biloops [12], neutrosophic N-loop [12], neutrosophic groupoids [12] and neutrosophic bigroupoids [12] and so on.

In this paper, we introduce the neutrosophic triplet ring. Further, we define the neutrosophic triplet zero divisor, neutrosophic triplet subring, neutrosophic triplet ideal, nilpotent neutrosophic triplet, integral neutrosophic triplet domain, and neutrosophic triplet ring homomorphism. Finally, we introduce a neutrosophic triplet field. The rest of the paper is organized as follows. After the literature review in Section 1 and basic concepts in Section 2, we introduce the neutrosophic triplet ring in Section 3. Section 4 is about the introduction of the integral neutrosophic triplet domain with some of its interesting properties, and is also where we develop the neutrosophic triplet ring homomorphism. In Section 5, we study neutrosophic triplet fields. Conclusions are given in Section 6.

2. Basic Concepts

In this section, all definitions and examples have been taken from [18] to provide some basic concepts about neutrosophic triplets and neutrosophic triplet groups.

Definition 1. *Let N be a set together with a binary operation $*$. Then N is called a neutrosophic triplet set if for any $a \in N$, there exists a neutral of "a" called $neut(a)$, different from the classical algebraic unitary element, and an opposite of "a" called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that*

$$a * neut(a) = neut(a) * a = a$$

and

$$a * anti(a) = anti(a) * a = neut(a).$$

The element a, $neut(a)$, and *$anti(a)$ are collectively called a neutrosophic triplet and we denote it by $(a, neut(a), anti(a))$. By $neut(a)$, we mean the neutral of a, and a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet [18].*

For the same element "a" in N, there may be more than one neutral $neut(a)$ and more than one opposite $anti(a)$.

Definition 2. *The element b in $(N, *)$ is the second component, denoted by $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c)* [12].

Definition 3. *The element c in* $(N,*)$ *is the third component, denoted by* $anti(\cdot)$ *of a neutrosophic triplet, if there exist other elements a and b in N such that* $a * b = b * a = a$ *and* $a * c = c * a = b$. *The formed neutrosophic triplet is* (a, b, c) [12].

Example 1. *Consider* Z_6 *under multiplication modulo 6, where*

$$Z_6 = \{0, 1, 2, 3, 4, 5\}.$$

Then the element 2 gives rise to a neutrosophic triplet because $neut(2) = 4 \neq 1$, *as* $2 \times 4 = 4 \times 2 = 8 \equiv 2(\text{mod}6)$. *Also,* $anti(2) = 2$ *because* $2 \times 2 = 4$. *Thus* $(2, 4, 2)$ *is a neutrosophic triplet. Similarly 4 gives rise to a neutrosophic triplet because* $neut(4) = anti(4) = 4$ *So* $(4, 4, 4)$ *is a neutrosophic triplet. However, 3 does not give rise to a neutrosophic triplet as* $neut(3) = 5$ *but* $anti(3)$ *does not exist in* Z_6, *and lastly, 0 gives rise to a trivial neutrosophic triplet as* $neut(0) = anti(0) = 0$. *The trivial neutrosophic triplet is denoted by* $(0, 0, 0)$ [12].

Definition 4. *Let* $(N,*)$ *be a neutrosophic triplet set. Then N is called a neutrosophic triplet group if the following conditions are satisfied* [12].

1. *If* $(N,*)$ *is well defined, i.e., for any* $a, b \in N$, *one has* $a * b \in N$.
2. *If* $(N,*)$ *is associative, i.e.,* $(a * b) * c = a * (b * c)$ *for all* $a, b, c \in N$.

The neutrosophic triplet group, in general, is not a group in the classical algebraic sense. We consider the neutrosophic neutrals as replacing the classical unitary element, and the neutrosophic opposites as replacing the classical inverse elements.

Example 2. *Consider* $(Z_{10}, \#)$, *where # is defined as* $a\#b = 3ab(\text{mod}10)$. *Then* $(Z_{10}, \#)$ *is a neutrosophic triplet group under the binary operation #, as shown in* Table 1 [18].

Table 1. Cayley table of neutrosophic triplet group $(Z_{10}, \#)$.

#	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	0	0	0	0	0	0	0	0
1	0	3	6	9	2	5	8	1	4	7
2	0	6	2	8	4	0	6	2	8	4
3	0	9	8	7	6	5	4	3	2	1
4	0	2	4	6	8	0	2	4	6	8
5	0	5	0	5	0	5	0	5	0	5
6	0	8	6	4	2	0	8	6	4	2
7	0	1	2	3	4	5	6	7	8	9
8	0	4	8	2	6	0	4	8	2	6
9	0	7	4	1	8	5	2	9	6	3

It is also associative, i.e.,

$$(a\#b)\#c = a\#(b\#c).$$

Now we take the LHS to prove the RHS.

$$(a\#b)\#c = (3ab)\#c$$

$$= 3(3ab)c = 9abc$$

$$= 3a(3bc) = 3a(b\#c)$$

$$= a\#(b\#c)$$

For each $a \in Z_{10}$*, we have* $neut(a)$ *in* Z_{10}.

That is, $neut(0) = 0$*,* $neut(1) = 7$*,* $neut(2) = 2$*,* $neut(3) = 7$*,* $neut(4) = 2$*, and so on.*

Similarly, for each $a \in Z_{10}$*, we have* $anti(a)$ *in* Z_{10}.

That is, $anti(0) = 0$*,* $anti(1) = 9$*,* $anti(2) = 2$*,* $anti(3) = 3$*,* $anti(4) = 1$*, and so on. Thus* $(Z_{10}, \#)$ *is a neutrosophic triplet group with respect to* # [12].

3. Neutrosophic Triplet Rings

In this section, we introduce neutrosophic triplet rings and study some of their basic properties and notions.

Notations 1. *Since the neutrosophic triplet ring and the neutrosophic triplet field are algebraic structures endowed with two internal laws * and #, in order to avoid any confusion, we use the following notation:* $neut*(x)$ and $anti*(x)$ *for the neutrals and anti's, respectively, of the element* x *with respect to the law * and* $neu\#(x)$ *and* $ant\#(x)$ *for the neutrals and anti's, respectively, of the element* x *with respect to the law #.*

Definition 5. *Let* $(NTR, *, \#)$ *be a set together with two binary operations* $*$ *and #. Then NTR is called a neutrosophic triplet ring if the following conditions hold:*

1. $(NTR, *)$ *is a commutative neutrosophic triplet group with respect to* $*$;
2. $(NTR, \#)$ *is well defined and associative;*
3. $a\#(b * c) = (a\#b) * (a\#c)$ *and* $(b * c)\#a = (b\#a) * (c\#a)$ *for all* $a, b, c \in NTR$.

Remark 1. *An NTR in general is not a classical ring.*

Definition 6. *Let* $(NTR, *, \#)$ *be a neutrosophic triplet ring and let* $a \in NTR$*. We call the structure a unitary neutrosophic triplet ring (UNTR) if each element a has a* $neut^{\#}(a)$.

Definition 7. *Let* $(NTR, *, \#)$ *be a neutrosophic triplet ring. We call the structure a commutative unitary neutrosophic triplet ring if it is a UNTR and # is commutative.*

Definition 8. *Let* $(NTR, *, \#)$ *be a neutrosophic triplet ring and let* $0 \neq a \in NTR$*. If there exists a nonzero element* $b \in NTR$ *such that* $b\#a = 0$*, then b is called a left zero divisor of a. Similarly, an element* $b \in NTR$ *is called a right zero divisor of a if* $a\#b = 0$.

A zero divisor of an element is one which is both a left zero divisor and a right zero divisor of that element.

Theorem 1. *Let NTR be a commutative neutrosophic triplet ring and* $a, b \in NTR$ *such that a, b,* $neut^{\#}(a)$*,* $neut^{\#}(b)$*,* $neut(a\#b)$*, and* $anti^{\#}(a\#b)$ *are cancellable and that* $neut^{\#}(a)$*,* $neut^{\#}(b)$ *and* $anti^{\#}(a)$*,* $anti^{\#}(b)$ *do exist in NTR. Then*

1. $neut^{\#}(a)\#neut^{\#}(b) = neut^{\#}(a\#b)$*; and*
2. $anti^{\#}(a)\#anti^{\#}(b) = anti^{\#}(a\#b)$.

Proof.

(1) Consider the left-hand side, with $neut^{\#}(a)\#neut^{\#}(b)$. Multiply by a to the left and by b to the right; then we have

$$a\#neut^{\#}(a)\#neut^{\#}(b)\#b = (a\#neut^{\#}(a))\#(neut^{\#}(b)\#b) = a\#b,$$

since # is associativeNow we consider the right-hand side; we have $neut^{\#}(a\#b)$. Multiplying by a to the left and by b to the right, we have

$$a\#neut^{\#}(a\#b)\#b = (a\#b)\#neut^{\#}(a\#b) = a\#b,$$

since # is associative and commutative,

Thus, LHS = $a\#b = a\#b$ = RHS.

(2) Considering the left-hand side, we have $anti^{\#}(a)\#anti^{\#}(b)$.

Multiplying by a to the left and by b to the right, we have

$$a\#anti^{\#}(a(\#anti^{\#}(b)\#b = (a\#anti^{\#}(a))\#(anti^{\#}(b)\#b) = a\#b.$$

Now consider the right-hand side, where we have $anti^{\#}(a\#b)$.

Multiplying by a to the left and by b to the right, we have $a\#anti^{\#}(a\#b)\#b = (a\#b)\#anti^{\#}(a\#b) = a\#b$, since # is associative and commutative, □

Definition 9. *Let $(NTR, *, \#)$ be a neutrosophic triplet ring and let S be a subset of NTR. Then S is called a neutrosophic triplet subring of NTR if $(S, *, \#)$ is a neutrosphic triplet ring.*

Definition 10. *Let $(NTR, *, \#)$ be a neutrosophic triplet ring and I be a subset of NTR. Then I is called a neutrosophic triplet ideal of NTR if the following conditions are satisfied.*

1. $(I, *)$ is a neutrosophic triplet subgroup of $(NTR, *)$; and
2. For all $x \in I$ and $r \in NTR$, $x\#r \in I$ and $r\#x \in I$.

Theorem 2. *Every neutrosophic triplet ideal is trivially a neutrosophic triplet subring, but the converse is not true in general.*

Remark 2. *Let $(NTR, *, \#)$ be a neutrosophic triplet ring and let $a \in NTR$. Then the following are true.*

1. *neut**(a) and *anti**(a) in general are not unique in NTR.
2. *neut*#(a) and *anti*#(a) (if they exist for some element a) in general are not unique in NTR.

Definition 11. *Let NTR be a neutrosophic triplet ring and let $a \in NTR$. Then a is called a nilpotent element if $a^n = 0$, for some positive integer $n > 1$.*

Theorem 3. *Let NTR be a commutative neutrosophic triplet ring and let $a \in NTR$. If a is a nilpotent, the following are true.*

1. $(neut * (a))^n = neut * (0)$*; and*
2. $(anti * (a))^n = anti * (0)$.

Proof.

(1) Suppose that a is a nilpotent in a neutrosophic triplet ring NTR. Then, by definition, $a^n = 0$ for some positive integer $n > 1$.

We prove by mathematical induction.

We can show that $neut * (a) * neut * (a) = neut * (a * b)$ and $anti * (a) * anti * (a) = anti * (a * b)$ in the same way as we did in Theorem 1 above by just replacing the law * by #.

Now we make $a = b$, so we get $neut * (a)^2 = neut * (a) * neut * (a) = neut(a^2)$.

We assume, by mathematical induction, that our equality is true for any positive integer up to $n-1$, and we need to prove it for n.

Now we consider left-hand side of 1:

$$(neut * (a))^n = (neut * (a)) * (neut * (a))^{n-1} = neut * (a * a^{n-1}) = neut * (a^n) = neut * (0).$$

This completes the proof.

The proof of (2) is similar to that of (1) □

4. Integral Neutrosophic Triplet Domain and Neutrosophic Triplet Ring Homomorphism

Section 4 is about the introduction of the integral neutrosophic triplet domain and some of its interesting properties. Moreover, in this section, we develop a neutrosophic triplet ring homomorphism.

Definition 12. *Let $(NTR, *, \#)$ be a neutrosophic triplet ring. Then NTR is called a commutative neutrosophic triplet ring if $a\#b = b\#a$ for all $a, b \in NTR$.*

Definition 13. *A commutative neutrosophic triplet ring NTR is called an integral neutrosophic triplet domain if for all $a, b \in NTR$, $a\#b = 0$ implies $a = 0$ or $b = 0$.*

Theorem 4. *Let NTR be an integral neutrosophic triplet domain. Then the following are true for all a, b ∈ NTR.*

1. *If $neut^{\#}(a)$ and $neut^{\#}(b)$ do exist, then $neut^{\#}(a)\#neut^{\#}(b) = 0$ implies $neut^{\#}(a) = 0$ or $neut^{\#}(b) = 0$;*
2. *If $anti^{\#}(a)$ and $anti^{\#}(b)$ do exist, then $anti^{\#}(a)\#anti^{\#}(b) = 0$ implies $anti^{\#}(a) = 0$ or $anti^{\#}(b) = 0$.*

Proof.

(1) Obvious, since NTR is an integral neutrosophic triplet domain, and $neut^{\#}(a)$ and $neut^{\#}(b)$ belong to NTR.

(2) Obvious, since NTR is an integral neutrosophic triplet domain, and $anti^{\#}(a)$ and $anti^{\#}(b)$ belong to NTR. □

Proposition 1. *A commutative neutrosophic triplet ring NTR is an integral neutrosophic triplet domain if, and only if, whenever $a, b, c \in NTR$ such that $a\#b = a\#c$ and $a \neq 0$, then $b = c$.*

Proof. Suppose that NTR is an integral neutrosophic triplet domain and let $a, b, c \in NTR$. Since $a \neq 0$ and $a \in NTR$, a is not a zero divisor, so a is cancellable, i.e.,

$$\text{a\#b} = a\#\text{c} \Rightarrow b = c.$$

Reciprocally, let $a \in NTR$, such that $a \neq 0$; then, by hypothesis, a is cancellable, so a is not a zero divisor. NTR is an integral neutrosophic triplet domain. □

Definition 14. *Let $(NTR_1, *, \#)$ and $(NTR_2, \oplus, \otimes)$ be two neutrosophic triplet rings. Let $f : NTR_1 \to NTR_2$ be a mapping. Then f is called a neutrosophic triplet ring homomorphism if the following conditions are true.*

1. *$f(a * b) = f(a) \oplus f(b)$, for all $a, b \in NTR_1$.*
2. *$f(a\#b) = f(a) \otimes f(b)$, for all $a, b \in NTR_1$.*
3. *$f(neut * (a)) = neut^{\oplus}(f(a)), foralla \in NTR_1$.*
4. *$f(anti * (a)) = anti^{\oplus}(f(a)), foralla \in NTR_1$.*

5. Neutrosophic Triplet Fields

In this section, we study neutrosophic triplet fields and some of their interesting properties.

Definition 15. *Let $(NTR, *, \#)$ be a neutrosophic triplet set together with two binary operations $*$ and $\#$. Then $(NTR, *, \#)$ is called a neutrosophic triplet field if the following conditions hold.*

1. *$(NTR, *)$ is a commutative neutrosophic triplet group with respect to *.*
2. *$(NTR, \#)$ is a neutrosophic triplet group with respect to #.*
3. *$a\#(b * c) = (a\#b) * (a\#c)$ and $(b * c)\#a = (b\#a) * (c\#a)$ for all $a, b, c \in NTF$.*

Example 3. *Let X be a set and let $P(X)$ be the power set of X. Then $(P(X), \cup, \cap)$ is a neutrosophic triplet field since $neut(A) = A$ and $anti(A) = A$ for all $A \in P(X)$ with respect to both $\cup$ and $\cap$.*

Proposition 2. *A neutrosophic triplet field NTF always has an $anti(a)$ for every $a \in NTF$ with respect to both laws * and #.*

Proof. The proof is straightforward. □

Theorem 5. *A neutrosophic triplet ring is not in general a neutrosophic triplet field.*

Counterexample:

$$NTR = (\{1, 2\}, *, \#)$$

*	1	2
1	2	1
2	1	2

Neutrosophic triplets: $(1, 2, 1), (2, 2, 2), (\{1, 2\}, *)$ is a commutative NTG.

#	1	2
1	1	1
2	1	1

$(\{1, 2\}, \#)$ is well defined, associative, and commutative.

For the element 2 there is no $neut^{\#}(2)$ and, consequently, no $anti^{\#}(2)$.

Therefore, *NTR* = ({1,2},#) is a neutrosophic triplet commutative semigroup, but not a neutrosophic triplet group.

In conclusion, *NTR* = ([1], *, #) is a neutrosophic triplet commutative ring, but it is not a neutrosophic triplet field.

Theorem 6. *A neutrosophic triplet field NTF is not in general an integral neutrosophic triplet domain NTD.*

Proof. Consider the NTF $N = (\{0, 5\}, *, \#)$, where $0 * 0 = 0, 0 * 5 = 5 * 0 = 5, 5 * 5 = 5$. The neutrosophic triplets with respect to * are $(0, 0, 0)$ and $(5, 0, 5)$. Hence, we get $5 * 5 = 0$.

Also $0\#0 = 0\#5 = 5\#0 = 5$ and $5\#5 = 0$. The neutrosophic triplets with respect to # are $(0, 5, 0)$ and $(5, 0, 5)$.

As we can see, $5\#5 = 0$.

Therefore, this is a NTF which is not an integral neutrosophic triplet domain. □

Theorem 7. *Assume that $f : NTR_1 \to NTR_2$ is a neutrosophic triplet ring homomorphism. The following then hold.*

1. *If S is a neutrosophic triplet subring $NTR_1(*,\#)$, then $f(S)$ is a neutrosophic triplet subring of $NTR_2(\oplus,\otimes)$.*
2. *If U is a neutrosophic triplet subring of NTR_2, then $f^{-1}(U)$ is a neutrosophic triplet subring of NTR_1.*
3. *If I is a neutrosophic triplet ideal of NTR_2, then $f^{-1}(I)$ is a neutrosophic triplet ideal of NTR_1.*
4. *If f is onto, and J is an ideal of NTR_1, then $f(j)$ is an ideal of NTR_2.*

Proof.

(1) If S is a neutrosophic triplet subring $NTR_1(*,\#)$, then $f(S)$ is a neutrosophic triplet subring of $NTR_2(\oplus,\otimes)$.

Let $a,b \in S$, then $a * b \in S, neut*(a) \in S, anti*(a) \in S$.

Then $f(a), f(b) \in f(S)$ and $f(a*b) \in f(S)$, but $f(a*b) = f(a) \oplus f(b)$, since f is a homomorphism. Thus, we have proved that if $f(a),\ f(b) \in f(S)$, then $f(a) \oplus f(b) \in f(S)$.

Since *neut**(a) and *anti**(a) $\in S$, $f(neut(a))$ and $f(anti(a)) \in f(S)$ since f is a homomorphism.

But $f(neut{*}(a)) = neut^{\oplus} f(a)$, and $f(anti{*}(a)) = anti^{\oplus} f(a)$.

Therefore, if $f(a) \in f(S)$, then $neut^{\oplus} f(a) = f(neut*(a)) \in f(S)$ and, similarly,

$$anti^{\oplus} f(a) = f(anti*(a)) \in f(S).$$

Now, if $a,b \in S$, then $a\#b \in S$. Since $a\#b \in S$, $f(a\#b) \in f(S)$.

But $f(a\#b) = f(a) \otimes f(b)$.

Therefore, if $f(a),\ f(b) \in S$, then $f(a) \otimes f(b) = f(a\#b) = f(S)$.

(2) Let $c,d \in U$. Then $f^{-1}(c), f^{-1}(d) \in f^{-1}(U)$. Also $c \oplus d \in U$, hence

$$f^{-1}(c \oplus d) \in f^{-1}(U),$$

$$f^{-1}(c) * f^{-1}(d) \in f^{-1}(U).$$

But

$$f^{-1}(c) * f^{-1}(d) = f^{-1}(c \oplus d),$$

because if we apply f on both sides we get

$$f\left(f^{-1}(c) * f^{-1}(d)\right) = f\left(f^{-1}(c \oplus d)\right),$$

or

$$f\left(f^{-1}(c)\right) \oplus f\left(f^{-1}(d)\right) = c \oplus d$$

or

$$c \oplus d = c \oplus d.$$

Similarly,

$$f^{-1}(c)\#f^{-1}(d) \in f^{-1}(U).$$

But

$$f^{-1}(c)\#f^{-1}(d) = f^{-1}(c \otimes d),$$

because if we apply f on both sides, we get

$$f\left(f^{-1}(c)\#f^{-1}(d)\right) = f(f^{-1}(c \otimes d)),$$

$$\text{or } f\left(f^{-1}(c)\right) \otimes f\left(f^{-1}(d)\right) = c \otimes d,$$

$$c \otimes d = c \otimes d.$$

Since $c \in U$, we have $neut^{\oplus}(c)$ and $anti^{\oplus}(c) \in U, f^{-1}(neut^{\oplus}(c)) = neut^{*}\left(f^{-1}(c)\right)$ and $f^{-1}(anti^{\oplus}(c)) = anti^{*}\left(f^{-1}(c)\right)$.

We prove them by applying f on both sides for each equality.

$$f(f^{-1}(neut^{\oplus}(c))) = f(neut^{*}(f^{-1}((c))),$$

$$\text{or } neut^{\oplus}(c) = neut^{\oplus}\left(f\left(f^{-1}(c)\right)\right),$$

$$\text{or } neut^{\oplus}(c) = neut^{\oplus}(c).$$

Similarly,

$$f(f^{-1}(anti^{\oplus}(c))) = f(anti^{*}(f^{-1}((c))),$$

or

$$anti^{\oplus}(c) = anti^{\oplus}\left(f\left(f^{-1}(c)\right)\right)$$

or

$$anti^{\oplus}(c) = anti^{\oplus}(c)$$

(3) Let $i \in I$ and $r \in NTR_2$. Then, $i \oplus r \in I$, and therefore, $f^{-1}(i \oplus r) \in f^{-1}(I)$.

$$f^{-1}(i) \in f^{-1}(I) \text{ and } f^{-1}(r) \in NTR_1.$$

We prove that

$$f^{-1}(i) * f^{-1}(r) = f^{-1}(i \oplus r).$$

Applying f to both sides, we get

$$f(f^{-1}(i) * f^{-1}(r) = f\left(f^{-1}(i \oplus r)\right);$$

$$f(f^{-1}(i)) \oplus f\left(f^{-1}(r)\right) = i \oplus r;$$

$$i \oplus r = i + r.$$

Therefore, if $i \in I,\ r \in NTR_2$, then $i \oplus r \in f^{-1}(I)$.

(4) Let $j \in f(J)$ and $r \in NTR_2$. Since f is onto, then $\exists h \in J \subset NTR_1$ such that $f(h) = j$ and $\exists\, s \in NTR_1$ such that $f(s) = r$. We need to prove that $j \oplus r \in f(J)$.

Applying f^{-1} to both sides, we get

$$f^{-1}\ (j \oplus r) \in f^{-1}(f(J)),$$

or

$$f^{-1}(j) * f^{-1}(r) \in J$$

or

$$h * s \in J$$

which is true, since $h \in J$, which is an ideal in NTR_1, while $s \in NTR_1$. $\square$

6. Conclusions

In this paper, we presented the neutrosophic triplet ring. Further, we presented the zero divisor, neutrosophic triplet subring, neutrosophic triplet ideal, nilpotent, integral neutrosophic triplet domain, and neutrosophic triplet ring homomorphism. Finally, we presented the neutrosophic triplet field. In the future, we can develop neutrosophic triplet vector spaces, neutrosophic modules, and neutrosophic triplet near rings, and so on.

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Positive implicative BMBJ-neutrosophic ideals in BCK-algebras

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Rajab Ali Borzooei, M. Mohseni Takallo, Florentin Smarandache, Young Bae Jun (2018). Positive implicative BMBJ-neutrosophic ideals in BCK-algebras. *Neutrosophic Sets and Systems* 23, 126-141

Abstract: The concepts of a positive implicative BMBJ-neutrosophic ideal is introduced, and several properties are investigated. Conditions for an MBJ-neutrosophic set to be a (positive implicative) BMBJ-neutrosophic ideal are provided. Relations between BMBJ-neutrosophic ideal and positive implicative BMBJ-neutrosophic ideal are discussed. Characterizations of positive implicative BMBJ-neutrosophic ideal are displayed.

Keywords: MBJ-neutrosophic set; BMBJ-neutrosophic ideal; positive implicative BMBJ-neutrosophic ideal.

1 Introduction

In 1965, L.A. Zadeh [18] introduced the fuzzy set in order to handle uncertainties in many real applications. In 1983, K. Atanassov introdued the notion of intuitionistic fuzzy set as a generalization of fuzzy set. As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is initiated by Smarandache ([13], [14] and [15]). Neutrosophic set is applied to many branchs of sciences. In the aspect of algebraic structures, neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [1], [3], [4], [5], [6], [11], [12], [16] and [17]. In [9], the notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, and it is applied to BCK/BCI-algebras. Mohseni et al. [9] introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigated related properties. Jun and Roh [7] applied the notion of MBJ-neutrosophic sets to ideals of BCK/BI-algebras, and introduced the concept of MBJ-neutrosophic ideals in BCK/BCI-algebras.

In this article, we introduce the concepts of a positive implicative BMBJ-neutrosophic ideal, and investigate several properties. We provide conditions for an MBJ-neutrosophic set to be a (positive implicative) BMBJ-neutrosophic ideal, and discussed relations between BMBJ-neutrosophic ideal and positive implicative BMBJ-neutrosophic ideal. We consider characterizations of positive implicative BMBJ-neutrosophic ideal.

2 Preliminaries

By a BCI-*algebra*, we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0,\ y * x = 0\ \Rightarrow\ x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-*algebra.*

Every BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$
$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I\ \Rightarrow\ x \in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* of X (see [8]) if it satisfies (2.5) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\ y * z \in I\ \Rightarrow\ x * z \in I)\,. \tag{2.7}$$

Note from [8] that a subset I of a BCK-algebra X is a positive implicative ideal of X if and only if it is an ideal of X which satisfies the condition

$$(\forall x, y \in X)((x * y) * y \in I\ \Rightarrow\ x * y \in I). \tag{2.8}$$

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}
&\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],\\
&\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right],\\
&\tilde{a}_1 \succeq \tilde{a}_2\ \Leftrightarrow\ a_1^- \geq a_2^-,\ a_1^+ \geq a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [14]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function.

We refer the reader to the books [2, 8] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

Let X be a non-empty set. By an ***MBJ-neutrosophic set*** in X (see [9]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a ***BMBJ-neutrosophic ideal*** of X (see [10]) if it satisfies

$$(\forall x \in X)(M_A(x) + B_A^-(x) \leq 1,\, B_A^+(x) + J_A(x) \leq 1), \tag{2.9}$$

$$(\forall x \in X)\begin{pmatrix} M_A(0) \geq M_A(x) \\ B_A^-(0) \leq B_A^-(x) \\ B_A^+(0) \geq B_A^+(x) \\ J_A(0) \leq J_A(x) \end{pmatrix}, \tag{2.10}$$

and

$$(\forall x, y \in X) \left(\begin{array}{l} M_A(x) \geq \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} \\ B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} \\ J_A(x) \leq \max\{J_A(x * y), J_A(y)\} \end{array} \right). \tag{2.11}$$

3 Positive implicative BMBJ-neutrosophic ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a *positive implicative BMBJ-neutrosophic ideal* of X if it satisfies (2.9), (2.10) and

$$(\forall x, y, z \in X) \left(\begin{array}{l} M_A(x * z) \geq \min\{M_A((x * y) * z), M_A(y * z)\} \\ B_A^-(x * z) \leq \max\{B_A^-((x * y) * z), B_A^-(y * z)\} \\ B_A^+(x * z) \geq \min\{B_A^+((x * y) * z), B_A^+(y * z)\} \\ J_A(x * z) \leq \max\{J_A((x * y) * z), J_A(y * z)\} \end{array} \right). \tag{3.1}$$

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.71	$[0.04, 0.09]$	0.22
1	0.61	$[0.03, 0.08]$	0.55
2	0.51	$[0.02, 0.06]$	0.55
3	0.41	$[0.01, 0.03]$	0.77
4	0.31	$[0.02, 0.05]$	0.99

$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X.

Theorem 3.3. *Every positive implicative BMBJ-neutrosophic ideal is a BMBJ-neutrosophic ideal.*

Proof. The condition (2.11) is induced by taking $z = 0$ in (3.1) and using (2.1). Hence every positive implica-tive BMBJ-neutrosophic ideal is a BMBJ-neutrosophic ideal. □

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.4. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 4.

Table 4: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.6	$[0.04, 0.09]$	0.3
1	0.5	$[0.03, 0.08]$	0.7
2	0.5	$[0.03, 0.08]$	0.7
3	0.3	$[0.01, 0.03]$	0.5

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. But it is not a positive implicative MBJ-neutrosophic ideal of X since

$$M_A(2 * 1) = 0.5 < 0.6 = \min\{M_A((2 * 1) * 1), M_A(1 * 1)\},$$

Lemma 3.5. *Every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.*

$$(\forall x, y \in X)\left(x \leq y \ \Rightarrow \begin{cases} M_A(x) \geq M_A(y), B_A^-(x) \leq B_A^-(y), \\ B_A^+(x) \geq B_A^+(y), J_A(x) \leq J_A(y) \end{cases}\right). \tag{3.2}$$

Proof. Assume that $x \leq y$ for all $x, y \in X$. Then $x * y = 0$, and so

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = \min\{M_A(0), M_A(y)\} = M_A(y),$$

$$B_A^-(x) \leq \max\{B_A^-(x*y), B_A^-(y)\} = \max\{B_A^-(0), B_A^-(y)\} = B_A^-(y),$$

$$B_A^+(x) \geq \min\{B_A^+(x*y), B_A^+(y)\} = \min\{B_A^+(0), B_A^+(y)\} = B_A^+(y),$$

and

$$J_A(x) \leq \max\{J_A(x*y), J_A(y)\} = \max\{J_A(0), J_A(y)\} = J_A(y).$$

This completes the proof. □

We provide conditions for a BMBJ-neutrosophic ideal to be a positive implicative BMBJ-neutrosophic ideal.

Theorem 3.6. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative BMBJ-neutrosophic ideal of X if and only if it is a BMBJ-neutrosophic ideal of X and satisfies the following condition.*

$$(\forall x, y \in X) \left(\begin{array}{l} M_A(x*y) \geq M_A((x*y)*y) \\ B_A^-(x*y) \leq B_A^-((x*y)*y) \\ B_A^+(x*y) \geq B_A^+((x*y)*y) \\ J_A(x*y) \leq J_A((x*y)*y) \end{array} \right). \tag{3.3}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. If z is replaced by y in (3.1), then

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A((x*y)*y), M_A(y*y)\} \\ &= \min\{M_A((x*y)*y), M_A(0)\} = M_A((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^-(x*y) &\leq \max\{B_A^-((x*y)*y), B_A^-(y*y)\} \\ &= \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+((x*y)*y), B_A^+(y*y)\} \\ &= \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A((x*y)*y), J_A(y*y)\} \\ &= \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y) \end{aligned}$$

for all $x, y \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of X satisfying the condition (3.3). Since

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$\begin{aligned} M_A((x*y)*z) \leq M_A(((x*z)*z)*(y*z)), \\ B_A^-((x*y)*z) \geq B_A^-(((x*z)*z)*(y*z)), \\ B_A^+((x*y)*z) \leq B_A^+(((x*z)*z)*(y*z)), \\ J_A((x*y)*z) \geq J_A(((x*z)*z)*(y*z)) \end{aligned} \tag{3.4}$$

for all $x, y, z \in X$. Using (3.3), (2.11) and (3.4), we have

$$\begin{aligned} M_A(x*z) &\geq M_A((x*z)*z) \geq \min\{M_A(((x*z)*z)*(y*z)), M_A(y*z)\} \\ &\geq \min\{M_A((x*y)*z), M_A(y*z)\}, \end{aligned}$$

$$\begin{aligned} B_A^-(x*z) &\leq B_A^-((x*z)*z) \leq \max\{B_A^-(((x*z)*z)*(y*z)), B_A^-(y*z)\} \\ &\leq \max\{B_A^-((x*y)*z), B_A^-(y*z)\}, \end{aligned}$$

$$\begin{aligned} B_A^+(x*z) &\geq B_A^+((x*z)*z) \geq \min\{B_A^+(((x*z)*z)*(y*z)), B_A^+(y*z)\} \\ &\geq \min\{B_A^+((x*y)*z), B_A^+(y*z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x*z) &\leq J_A((x*z)*z) \leq \max\{J_A(((x*z)*z)*(y*z)), J_A(y*z)\} \\ &\leq \max\{J_A((x*y)*z), J_A(y*z)\} \end{aligned}$$

for all $x, y, z \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned} U(M_A; t) &:= \{x \in X \mid M_A(x) \geq t\}, \\ L(B_A^-; \alpha^-) &:= \{x \in X \mid B_A^-(x) \leq \alpha^-\}, \\ U(B_A^+; \alpha^+) &:= \{x \in X \mid B_A^+(x) \geq \alpha^+\}, \\ L(J_A; s) &:= \{x \in X \mid J_A(x) \leq s\} \end{aligned}$$

where $t, s, \alpha^-, \alpha^+ \in [0, 1]$.

Lemma 3.7 ([10]). *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a BMBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.*

Theorem 3.8. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative BMBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.*

Proof. Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, B\tilde{\ }_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.3. It follows from Lemma 3.7 that the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. Let

$x, y, a, b, c, d, u, v \in X$ be such that $(x * y) * y \in U(M_A; t)$, $(a * b) * b \in L(B_A^-; \alpha^-)$, $(c * d) * d \in U(B_A^+; \alpha^+)$ and $(u * v) * v \in L(J_A; s)$. Using Theorem 3.6, we have

$$\begin{aligned}
&M_A(x * y) \geq M_A((x * y) * y) \geq t, \text{ that is, } x * y \in U(M_A; t),\\
&B_A^-(a * b) \leq B_A^-((a * b) * b) \leq \alpha^-, \text{ that is, } a * b \in L(B_A^-; \alpha^-),\\
&B_A^+(c * d) \geq B_A^+((c * d) * d) \geq \alpha^+, \text{ that is, } c * d \in U(B_A^+; \alpha^+),\\
&J_A(u * v) \leq J_A((u * v) * v) \leq s, \text{ that is, } u * v \in L(J_A; s).
\end{aligned}$$

Therefore $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.

Conversely, suppose that the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. Then $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. It follows from Lemma 3.7 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. Assume that $M_A(x_0 * y_0) < M_A((x_0 * y_0) * y_0) = t_0$ for some $x_0, y_0 \in X$. Then $(x_0 * y_0) * y_0 \in U(M_A; t_0)$ and $x_0 * y_0 \notin U(M_A; t_0)$, which is a contradiction. Thus $M_A(x * y) \geq M_A((x * y) * y)$ for all $x, y \in X$. Similarly, we have $B_A^+(x * y) \geq B_A^+((x * y) * y)$ for all $x, y \in X$. If there exist $a_0, b_0 \in X$ such that $J_A(a_0 * b_0) > J_A((a_0 * b_0) * b_0) = s_0$, then $(a_0 * b_0) * b_0 \in L(J_A; s_0)$ and $a_0 * b_0 \notin L(J_A; s_0)$. This is impossible, and thus $J_A(a * b) \leq J_A((a * b) * b)$ for all $a, b \in X$. By the similar way, we know that $B_A^-(a * b) \leq B_A^-((a * b) * b)$ for all $a, b \in X$. It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Theorem 3.9. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is positive implicative if and only if it satisfies the following condition.*

$$(\forall x, y, z \in X) \left(\begin{array}{l} M_A((x * z) * (y * z)) \geq M_A((x * y) * z),\\ B_A^-((x * z) * (y * z)) \leq B_A^-((x * y) * z),\\ B_A^+((x * z) * (y * z)) \geq B_A^+((x * y) * z),\\ J_A((x * z) * (y * z)) \leq J_A((x * y) * z). \end{array} \right) \tag{3.5}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.3, and satisfies the condition (3.3) by Theorem 3.6. Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$\begin{aligned}
&M_A((x * y) * z) \leq M_A(((x * (y * z)) * z) * z),\\
&B_A^-((x * y) * z) \geq B_A^-(((x * (y * z)) * z) * z),\\
&B_A^+((x * y) * z) \leq B_A^+(((x * (y * z)) * z) * z),\\
&J_A((x * y) * z) \geq J_A(((x * (y * z)) * z) * z)
\end{aligned} \tag{3.6}$$

for all $x, y, z \in X$. Using (2.3), (3.3) and (3.6), we have

$$\begin{aligned} M_A((x*z)*(y*z)) &= M_A((x*(y*z))*z) \\ &\geq M_A(((x*(y*z))*z)*z) \\ &\geq M_A((x*y)*z), \end{aligned}$$

$$\begin{aligned} B_A^-((x*z)*(y*z)) &= B_A^-((x*(y*z))*z) \\ &\leq B_A^-(((x*(y*z))*z)*z) \\ &\leq B_A^-((x*y)*z), \end{aligned}$$

$$\begin{aligned} B_A^+((x*z)*(y*z)) &= B_A^+((x*(y*z))*z) \\ &\geq B_A^+(((x*(y*z))*z)*z) \\ &\geq B_A^+((x*y)*z), \end{aligned}$$

and

$$\begin{aligned} J_A((x*z)*(y*z)) &= J_A((x*(y*z))*z) \\ &\leq J_A(((x*(y*z))*z)*z) \\ &\leq J_A((x*y)*z). \end{aligned}$$

Hence (3.5) is valid.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of X which satisfies the condition (3.5). If we put $z = y$ in (3.5) and use (III) and (2.1), then we obtain the condition (3.3). Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X by Theorem 3.6. □

Theorem 3.10. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X if and only if it satisfies the condition* (2.9), (2.10) *and*

$$(\forall x, y, z \in X) \left(\begin{array}{l} M_A(x*y) \geq \min\{M_A(((x*y)*y)*z), M_A(z)\}, \\ B_A^-(x*y) \leq \max\{B_A^-(((x*y)*y)*z), B_A^-(z)\}, \\ B_A^+(x*y) \geq \min\{B_A^+(((x*y)*y)*z), B_A^+(z)\}, \\ J_A(x*y) \leq \max\{J_A(((x*y)*y)*z), J_A(z)\}. \end{array} \right) \tag{3.7}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X (see Theorem 3.3), and so the conditions (2.9) and (2.10) are valid. Using (2.11), (III), (2.1), (2.3) and (3.5), we have

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A((x*y)*z), M_A(z)\} \\ &= \min\{M_A(((x*z)*y)*(y*y)), M_A(z)\} \\ &\geq \min\{M_A(((x*z)*y)*y), M_A(z)\} \\ &= \min\{M_A(((x*y)*y)*z), M_A(z)\}, \end{aligned}$$

$$\begin{aligned} B_A^-(x*y) &\leq \max\{B_A^-((x*y)*z), B_A^-(z)\} \\ &= \max\{B_A^-(((x*z)*y)*(y*y)), B_A^-(z)\} \\ &\leq \max\{B_A^-(((x*z)*y)*y), B_A^-(z)\} \\ &= \max\{B_A^-(((x*y)*y)*z), B_A^-(z)\}, \end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+((x*y)*z), B_A^+(z)\} \\ &= \min\{B_A^+(((x*z)*y)*(y*y)), B_A^+(z)\} \\ &\geq \min\{B_A^+(((x*z)*y)*y), B_A^+(z)\} \\ &= \min\{B_A^+(((x*y)*y)*z), B_A^+(z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A((x*y)*z), J_A(z)\} \\ &= \max\{J_A(((x*z)*y)*(y*y)), J_A(z)\} \\ &\leq \max\{J_A(((x*z)*y)*y), J_A(z)\} \\ &= \max\{J_A(((x*y)*y)*z), J_A(z)\} \end{aligned}$$

for all $x, y, z \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which satisfies conditions (2.9), (2.10) and (3.7). Then

$$M_A(x) = M_A(x*0) \geq \min\{M_A(((x*0)*0)*z), M_A(z)\} = \min\{M_A(x*z), M_A(z)\},$$

$$B_A^-(x) = B_A^-(x*0) \leq \max\{B_A^-(((x*0)*0)*z), B_A^-(z)\} = \max\{B_A^-(x*z), B_A^-(z)\},$$

$$B_A^+(x) = B_A^+(x*0) \geq \min\{B_A^+(((x*0)*0)*z), B_A^+(z)\} = \min\{B_A^+(x*z), B_A^+(z)\},$$

and

$$J_A(x) = J_A(x*0) \leq \max\{J_A(((x*0)*0)*z), J_A(z)\} = \max\{J_A(x*z), J_A(z)\}$$

for all $x, z \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. Taking $z = 0$ in (3.7) and using (2.1) and (2.10) imply that

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A(((x*y)*y)*0), M_A(0)\} \\ &= \min\{M_A((x*y)*y), M_A(0)\} = M_A((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^-(x*y) &\leq \max\{B_A^-(((x*y)*y)*0), B_A^-(0)\} \\ &= \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+(((x*y)*y)*0), B_A^+(0)\} \\ &= \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A(((x*y)*y)*0), J_A(0)\} \\ &= \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y) \end{aligned}$$

for all $x, y \in X$. It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Proposition 3.11. *Every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.*

$$x*y \leq z \ \Rightarrow \begin{cases} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ B_A^-(x) \leq \max\{B_A^-(y), B_A^-(z)\}, \\ B_A^+(x) \geq \min\{B_A^+(y), B_A^+(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\} \end{cases} \tag{3.8}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $x*y \leq z$. Then

$$M_A(x*y) \geq \min\{M_A((x*y)*z), M_A(z)\} = \min\{M_A(0), M_A(z)\} = M_A(z),$$

$$B_A^-(x*y) \leq \max\{B_A^-((x*y)*z), B_A^-(z)\} = \max\{B_A^-(0), B_A^-(z)\} = B_A^-(z),$$

$$B_A^+(x*y) \geq \min\{B_A^+((x*y)*z), B_A^+(z)\} = \min\{B_A^+(0), B_A^+(z)\} = B_A^+(z),$$

and

$$J_A(x*y) \leq \max\{J_A((x*y)*z), J_A(z)\} = \max\{J_A(0), J_A(z)\} = J_A(z).$$

It follows that

$$M_A(x) \geq \min\{M_A(x*y), M_A(y)\} \geq \min\{M_A(y), M_A(z)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x*y), B_A^-(y)\} \leq \max\{B_A^-(y), B_A^-(z)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x*y), B_A^+(y)\} \geq \min\{B_A^+(y), B_A^+(z)\},$$

and

$$J_A(x) \leq \max\{J_A(x*y), J_A(y)\} \leq \max\{J_A(y), J_A(z)\}.$$

This completes the proof. □

We provide conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in BCK/BCI-algebras.

Theorem 3.12. *Every MBJ-neutrosophic set in X satisfying* (2.9), (2.10) *and* (3.8) *is a BMBJ-neutrosophic ideal of X.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X satisfying (2.9), (2.10) and (3.8). Note that $x * (x * y) \leq y$ for all $x, y \in X$. It follows from (3.8) that

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\},$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. □

Theorem 3.13. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a BMBJ-neutrosophic ideal of X if and only if (M_A, B_A^-) and (B_A^+, J_A) are intuitionistic fuzzy ideals of X.*

Proof. Straightforward. □

Theorem 3.14. *Given an ideal I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by*

$$M_A(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad B_A^-(x) = \begin{cases} \alpha^- & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases}$$
$$B_A^+(x) = \begin{cases} \alpha^+ & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases}$$

where $t, \alpha^+ \in (0, 1]$ and $s, \alpha^- \in [0, 1)$ with $t + \alpha^- \leq 1$ and $s + \alpha^+ \leq 1$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X such that $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$.

Proof. It is clear that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ satisfies the condition (2.9) and $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$. Let $x, y \in X$. If $x * y \in I$ and $y \in I$, then $x \in I$ and so

$$\begin{aligned}
&M_A(x) = t = \min\{M_A(x * y), M_A(y)\} \\
&B_A^-(x) = \alpha^- = \max\{B_A^-(x * y), B_A^-(y)\}, \\
&B_A^+(x) = \alpha^+ = \min\{B_A^+(x * y), B_A^+(y)\}, \\
&J_A(x) = s = \max\{J_A(x * y), J_A(y)\}.
\end{aligned}$$

If any one of $x * y$ and y is contained in I, say $x * y \in I$, then $M_A(x * y) = t$, $B_A^-(x * y) = \alpha^-$, $J_A(x * y) = s$, $M_A(y) = 0$, $B_A^-(y) = 1$, $B_A^+(y) = 0$ and $J_A(y) = 1$. Hence

$$\begin{aligned} &M_A(x) \geq 0 = \min\{t, 0\} = \min\{M_A(x * y), M_A(y)\} \\ &B_A^-(x) \leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\ &B_A^+(x) \geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\ &J_A(x) \leq 1 = \max\{s, 1\} = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If $x * y \notin I$ and $y \notin I$, then $M_A(x * y) = 0 = M_A(y)$, $B_A^-(x * y) = 1 = B_A^-(y)$, $B_A^+(x * y) = 0 = B_A^+(y)$ and $J_A(x * y) = 1 = J_A(y)$. It follows that

$$\begin{aligned} &M_A(x) \geq 0 = \min\{M_A(x * y), M_A(y)\} \\ &B_A^-(x) \leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\ &B_A^+(x) \geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\ &J_A(x) \leq 1 = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

It is obvious that $M_A(0) \geq M_A(x)$, $B_A^-(0) \leq B_A^-(x)$, $B_A^+(0) \geq B_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. □

Lemma 3.15. *For any non-empty subset I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.14. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X, then I is an ideal of X.*

Proof. Obviously, $0 \in I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then $M_A(x * y) = t = M_A(y)$, $B_A^-(x * y) = \alpha^- = B_A^-(y)$, $B_A^+(x * y) = \alpha^+ = B_A^+(y)$ and $J_A(x * y) = s = J_A(y)$. Thus

$$\begin{aligned} &M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = t, \\ &B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} = \alpha^-, \\ &B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} = \alpha^+, \\ &J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = s, \end{aligned}$$

and hence $x \in I$. Therefore I is an ideal of X. □

Theorem 3.16. *For any non-empty subset I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.14. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X, then I is a positive implicative ideal of X.*

Proof. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X and satisfies (3.3) by Theorem 3.6. It follows from Lemma 3.15 that I is an ideal of X. Let $x, y \in X$ be such that $(x * y) * y \in I$. Then

$$\begin{aligned} &M_A(x * y) \geq M_A((x * y) * y) = t, B_A^-(x * y) \leq B_A^-((x * y) * y) = \alpha^-, \\ &B_A^+(x * y) \geq B_A^+((x * y) * y) = \alpha^+, J_A(x * y) \leq J_A((x * y) * y) = s, \end{aligned}$$

and so $x * y \in I$. Therefore I is a positive implicative ideal of X. □

Proposition 3.17. *Every positive implicative BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following condition.*

$$(((x * y) * y) * a) * b = 0 \;\Rightarrow \begin{cases} M_A(x * y) \geq \min\{M_A(a), M_A(b)\}, \\ B_A^-(x * y) \leq \max\{B_A^-(a), B_A^-(b)\}, \\ B_A^+(x * y) \geq \min\{B_A^+(a), B_A^+(b)\}, \\ J_A(x * y) \leq \max\{J_A(a), J_A(b)\} \end{cases} \tag{3.9}$$

for all $x, y, a, b \in X$.

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X (see Theorem 3.3). Let $a, b, x, y \in X$ be such that $(((x * y) * y) * a) * b = 0$. Then

$$M_A(x * y) \geq M_A((x * y) * y) \geq \min\{M_A(a), M_A(b)\},$$

$$B_A^-(x * y) \leq \tilde{B}_A((x * y) * y) \leq \max\{B_A^-(a), B_A^-(b)\},$$

$$B_A^+(x * y) \geq B_A^+((x * y) * y) \geq \min\{B_A^+(a), B_A^+(b)\},$$

and $J_A(x * y) \leq J_A((x * y) * y) \leq \max\{J_A(a), J_A(b)\}$ by Theorem 3.6 and Proposition 3.11. Hence (3.9) is valid. □

Theorem 3.18. *If an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X satisfies the conditions (2.9) and (3.9), then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which satisfies the conditions (2.9) and (3.9). It is clear that the condition (2.10) is induced by the condition (3.9). Let $x, a, b \in X$ be such that $x * a \leq b$. Then $(((x * 0) * 0) * a) * b = 0$, and so

$$M_A(x) = M_A(x * 0) \geq \min\{M_A(a), M_A(b)\},$$

$$B_A^-(x) = B_A^-(x * 0) \leq \max\{B_A^-(a), B_A^-(b)\},$$

$$B_A^+(x) = B_A^+(x * 0) \geq \min\{B_A^+(a), B_A^+(b)\},$$

and

$$J_A(x) = J_A(x * 0) \leq \max\{J_A(a), J_A(b)\}$$

by (2.1) and (3.9). Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.12. Since $(((x * y) * y) * ((x * y) * y)) * 0 = 0$ for all $x, y \in X$, we have

$$M_A(x * y) \geq \min\{M_A((x * y) * y), M_A(0)\} = M_A((x * y) * y),$$

$$B_A^-(x*y) \leq \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y),$$

$$B_A^+(x*y) \geq \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y),$$

and

$$J_A(x*y) \leq \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y)$$

by (3.9). It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

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Neutrosophic Hesitant Fuzzy Subalgebras and Filters in Pseudo-BCI Algebras

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Songtao Shao, Xiaohong Zhang, Chunxin Bo, Florentin Smarandache (2018). Neutrosophic Hesitant Fuzzy Subalgebras and Filters in Pseudo-BCI Algebras. *Symmetry* 10, 174; DOI: 10.3390/sym10050174

Abstract: The notions of the neutrosophic hesitant fuzzy subalgebra and neutrosophic hesitant fuzzy filter in pseudo-BCI algebras are introduced, and some properties and equivalent conditions are investigated. The relationships between neutrosophic hesitant fuzzy subalgebras (filters) and hesitant fuzzy subalgebras (filters) is discussed. Five kinds of special sets are constructed by a neutrosophic hesitant fuzzy set, and the conditions for the two kinds of sets to be filters are given. Moreover, the conditions for two kinds of special neutrosophic hesitant fuzzy sets to be neutrosophic hesitant fuzzy filters are proved.

Keywords: pseudo-BCI algebra; hesitant fuzzy set; neutrosophic set; filter

1. Introduction

G. Georgescu and A. Iogulescu presented pseudo-BCKalgebras, which was an extension of the famous BCK algebra theory. In [1], the notion of the pseudo-BCI algebra was introduced by W.A. Dudek and Y.B. Jun. They investigated some properties of pseudo-BCI algebras. In [2], Y.B. Jun et al. presented the concept of the pseudo-BCI ideal in pseudo-BCI algebras and researched its characterizations. Then, some classes of pseudo-BCI algebras and pseudo-ideals (filters) were studied; see [3–14].

In 1965, Zadeh introduced fuzzy set theory [15]. In the study of modern fuzzy logic theory, algebraic systems played an important role, such as [16–22]. In 2010, Torra introduced hesitant fuzzy set theory [23]. The hesitant fuzzy set was a useful tool to express peoples' hesitancy in real life, and uncertainty problems were resolved. Furthermore, hesitant fuzzy sets have been applied to decision making and algebraic systems [24–31]. As a generalization of fuzzy set theory, Smarandache introduced neutrosophic set theory [32]; the neutrosophic set theory is a useful tool to deal with indeterminate and inconsistent decision information [33,34]. The neutrosophic set includes the truth membership, indeterminacy membership and falsity membership. Then, Wang et al. [35,36] introduced the interval neutrosophic set and single-valued neutrosophic set. Ye [37] introduced the single-valued neutrosophic hesitant fuzzy set as an extension of the single-valued neutrosophic set and hesitant fuzzy set. Recently, the neutrosophic triplet structures were introduced and researched [38–40].

In this paper, some preliminary concepts in pseudo-BCI algebras, hesitant fuzzy set theory and neutrosophic set theory are briefly reviewed in Section 2. In Section 3, the notion of neutrosophic hesitant fuzzy subalgebras in pseudo-BCI algebras is introduced. The relationships between neutrosophic hesitant fuzzy subalgebras and hesitant fuzzy subalgebras are investigated. Five kinds of special sets are constructed. Some properties are studied. Third, the two kinds of sets to be filters are given. In Section 4, the concept of neutrosophic hesitant fuzzy filters in pseudo-BCI algebras is proposed. The equivalent conditions of the neutrosophic hesitant fuzzy filters in the construction of hesitant fuzzy filters are given. The conditions for two kinds of special neutrosophic hesitant fuzzy sets to be neutrosophic hesitant fuzzy filters are given.

2. Preliminaries

Let us review some fundamental notions of pseudo-BCI algebra and interval-valued hesitant fuzzy filter in this section.

Definition 1. *([13]) A pseudo-BCI algebra is a structure $(X; \rightarrow, \hookrightarrow, 1)$, where "$\rightarrow$" and "$\hookrightarrow$" are binary operations on X and "1" is an element of X, verifying the axioms: $\forall x, y, z \in X$,*

(1) $(y \rightarrow z) \rightarrow ((z \rightarrow x) \hookrightarrow (y \rightarrow x)) = 1$, $(y \hookrightarrow z) \hookrightarrow ((z \hookrightarrow x) \rightarrow (y \hookrightarrow x)) = 1$;
(2) $x \rightarrow ((x \rightarrow y) \hookrightarrow y) = 1$, $x \hookrightarrow ((x \hookrightarrow y) \rightarrow y) = 1$;
(3) $x \rightarrow x = 1$;
(4) $x \rightarrow y = y \rightarrow x = 1 \Longrightarrow x = y$;
(5) $x \rightarrow y = 1 \Longleftrightarrow x \hookrightarrow y = 1$.

If $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCI algebra satisfying $\forall x, y \in X$, $x \rightarrow y = x \hookrightarrow y$, then $(X; \rightarrow, 1)$ is a BCI algebra. If $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCI algebra satisfying $\forall x \in X$, $x \rightarrow 1 = 1$, then $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCK algebra.

Remark 1. *([1]) In any pseudo-BCI algebra $(X; \rightarrow, \hookrightarrow)$, we can define a binary relation '$\leq$' by putting:*

$$x \leq y \text{ if and only if } x \rightarrow y \text{ (or } x \hookrightarrow y).$$

Proposition 1. *([13]) Let $(X; \rightarrow, \hookrightarrow)$ be a pseudo-BCI algebra, then X satisfies the following properties, $\forall x, y, z \in X$,*

(1) $1 \leq x \Rightarrow x = 1$;
(2) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z, y \hookrightarrow z \leq x \hookrightarrow z$;
(3) $x \leq y, y \leq z \Rightarrow x \leq z$;
(4) $x \hookrightarrow (y \rightarrow z) = y \rightarrow (x \hookrightarrow z)$;
(5) $x \leq y \rightarrow z \Rightarrow y \leq x \hookrightarrow z$;
(6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), x \hookrightarrow y \leq (z \hookrightarrow x) \hookrightarrow (z \hookrightarrow y)$;
(7) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y, z \hookrightarrow x \leq z \hookrightarrow y$;
(8) $1 \rightarrow x = x, 1 \hookrightarrow x = x$;
(9) $((y \rightarrow x) \hookrightarrow x) \rightarrow x = y \rightarrow x, ((y \hookrightarrow x) \rightarrow x) \hookrightarrow x = y \hookrightarrow x$;
(10) $x \rightarrow y \leq (y \rightarrow x) \hookrightarrow 1, x \hookrightarrow y \leq (y \hookrightarrow x) \rightarrow 1$;
(11) $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \hookrightarrow (y \hookrightarrow 1), (x \hookrightarrow y) \hookrightarrow 1 = (x \hookrightarrow 1) \rightarrow (y \rightarrow 1)$;
(12) $x \rightarrow 1 = x \hookrightarrow 1$.

Definition 2. *([13]) A subset F of a pseudo-BCI algebra X is called a filter of X if it satisfies:*
(F1) $1 \in F$;
(F2) $x \in F, x \rightarrow y \in F \Rightarrow y \in F$;
(F3) $x \in F, x \hookrightarrow y \in F \Rightarrow y \in F$.

Definition 3. *([1]) By a pseudo-BCI subalgebra of a pseudo-BCI algebra X, we mean a subset S of X that satisfies $\forall x, y \in S$, $x \rightarrow y \in S, x \hookrightarrow y \in S$.*

Definition 4. *([12]) A pseudo-BCK algebra is called a type-2 positive implicative if it satisfies:*

$$x \rightarrow (y \hookrightarrow z) = (x \rightarrow y) \hookrightarrow (x \rightarrow z),$$
$$x \hookrightarrow (y \rightarrow z) = (x \hookrightarrow y) \rightarrow (x \hookrightarrow z).$$

If X is a type-2 positive implicative pseudo-BCK algebra, then $x \to y = x \hookrightarrow y$ for all $x \in X$.

Definition 5. *([23]) Let X be a reference set. A hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that returns a subset of $[0,1]$ when it is applied to X, i.e.,*

$$A = \{(x, h_A(x)) | x \in X\}.$$

where $h_A(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$. $h_A(x)$ is called a hesitant fuzzy element, a basis unit of the hesitant fuzzy set.

Example 1. *Let $X = \{a, b, c\}$ be a reference set, $h_A(a) = [0.1, 0.2]$, $h_A(b) = [0.3, 0.6]$, $h_A(c) = [0.7, 0.8]$. Then, A is considered as a hesitant fuzzy set,*

$$A = \{(a, [0.1, 0.2]), (b, [0.3, 0.6]), (c, [0.7, 0.8])\}.$$

Definition 6. *([13]) A fuzzy set $\mu : X \to [0,1]$ is called a fuzzy pseudo-filter (fuzzy filter) of a pseudo-BCI algebra X if it satisfies:*
(FF1) $\mu(1) \geq \mu(x)$, $\forall x \in X$;
(FF2) $\mu(y) \geq \mu(x \to y) \wedge \mu(x)$, $\forall x, y \in X$;
(FF3) $\mu(y) \geq \mu(x \hookrightarrow y) \wedge \mu(x)$, $\forall x, y \in X$.

Definition 7. *([32]) Let X be a non-empty fixed set, a neutrosophic set A on X is defined as:*

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0,1]$, denoting the truth, indeterminacy and falsity membership degree of the element $x \in X$, respecting, and satisfying the limit: $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 8. *([34]) Let X be a fixed set; a neutrosophic hesitant fuzzy set N on X is defined as*

$$N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\},$$

in which $\tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x) \in P([0,1])$, denoting the possible truth membership hesitant degrees, indeterminacy membership hesitant degrees and falsity membership hesitant degrees of $x \in X$ to the set N, respectively, with the conditions $0 \leq \delta, \gamma, \eta \leq 1$ and $0 \leq \delta^+ + \gamma^+ + \eta^+ \leq 3$, where $\gamma \in \tilde{t}_N(x)$, $\delta \in \tilde{i}_N(x)$, $\eta \in \tilde{f}_N(x)$, $\gamma^+ \in \bigcup_{\gamma \in \tilde{t}_N(x)} max\{\gamma\}$, $\delta^+ \in \bigcup_{\delta \in \tilde{i}_N(x)} max\{\delta\}$, $\eta^+ \in \bigcup_{\eta \in \tilde{f}_N(x)} max\{\eta\}$ for $x \in X$.

Example 2. *Let $X = \{a, b, c\}$ be a reference set, $h_A(a) = ([0.4, 0.5], [0.1, 0.2], [0.2, 0.4])$, $h_A(b) = ([0.5, 0.6], \{0.2, 0.3\}, [0.3, 0.4])$, $h_A(c) = ([0.5, 0.8], [0.2, 0.4], \{0.3, 0.5\})$. Then, A is considered as a neutrosophic hesitant fuzzy set,*

$$A = \{(a, [0.4, 0.5], [0.1, 0.2], [0.2, 0.4]), (b, [0.5, 0.6], \{0.2, 0.3\}, [0.3, 0.4]), (c, [0.5, 0.8], [0.2, 0.4], \{0.3, 0.5\})\}.$$

Conveniently, $N(x) = \{\tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)\}$ is called a neutrosophic hesitant fuzzy element, which is denoted by the simplified symbol $N(x) = \{\tilde{t}_N, \tilde{i}_N, \tilde{f}_N\}$.

Definition 9. *([34]) Let $N_1 = \{\tilde{t}_{N_1}, \tilde{i}_{N_1}, \tilde{f}_{N_1}\}$ and $N_2 = \{\tilde{t}_{N_2}, \tilde{i}_{N_2}, \tilde{f}_{N_2}\}$ be two neutrosophic hesitant fuzzy sets, then:*

$$N_1 \cup N_2 = \{\tilde{t}_{N_1} \cup \tilde{t}_{N_2}, \tilde{i}_{N_1} \cap \tilde{i}_{N_2}, \tilde{f}_{N_1} \cap f_{N_2}\};$$
$$N_1 \cap N_2 = \{\tilde{t}_{N_1} \cap \tilde{t}_{N_2}, \tilde{i}_{N_1} \cup \tilde{i}_{N_2}, \tilde{f}_{N_1} \cup f_{N_2}\}.$$

3. Neutrosophic Hesitant Fuzzy Subalgebras of Pseudo-BCI Algebras

In the following, let X be a pseudo-BCI algebra, unless otherwise specified.

Definition 10. *A hesitant fuzzy set $A = \{(x, h_A(x)) | x \in X\}$ is called a hesitant fuzzy pseudo-subalgebra (hesitant fuzzy subalgebra) of X if it satisfies:*
(HFS2) $h_A(x) \cap h_A(y) \subseteq h_A(x \to y)$, $\forall x, y \in X$;
(HFS3) $h_A(x) \cap h_A(y) \subseteq h_A(x \hookrightarrow y)$, $\forall x, y \in X$.

Definition 11. *A neutrosophic hesitant fuzzy set $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is called a neutrosophic hesitant fuzzy pseudo-subalgebra (neutrosophic hesitant fuzzy subalgebra) of X if it satisfies:*
(1) $\tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \to y)$, $\tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \hookrightarrow y)$, $\forall x, y \in X$;
(2) $\tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \to y)$, $\tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \hookrightarrow y)$, $\forall x, y \in X$;
(3) $\tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \to y)$, $\tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \hookrightarrow y)$, $\forall x, y \in X$.

Example 3. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 1 and 2.*

Table 1. $\to$.

$\to$	a	b	c	d	1
a	1	c	1	1	1
b	d	1	1	1	1
c	d	c	1	1	1
d	c	c	c	1	1
1	a	b	c	d	1

Table 2. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	d	1	1	1
b	d	1	1	1	1
c	d	d	1	1	1
d	c	b	c	1	1
1	a	b	c	d	1

Then, $(X; \to, \hookrightarrow, 1)$ is a pseudo-BCI algebra. Let:

$$N = \{(1, [0,1], \{0, \tfrac{1}{16}\}, [0, \tfrac{1}{6}]), (a, [\tfrac{1}{3}, \tfrac{1}{4}], [0, \tfrac{1}{2}], [0, \tfrac{5}{6}]), (b, [0, \tfrac{1}{2}], [0, \tfrac{2}{3}], [0, \tfrac{2}{3}]), \\ (c, [\tfrac{1}{3}, \tfrac{2}{3}], [0, \tfrac{1}{6}], [0, \tfrac{1}{5}]), (d, [\tfrac{1}{3}, 1], [0, \tfrac{1}{3}], [0, \tfrac{1}{5}])\}.$$

then, N is a neutrosophic hesitant fuzzy subalgebra of X.

Considering three hesitant fuzzy sets $H_{\tilde{t}_N}$, $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ by:

$$H_{\tilde{t}_N} = \{(x, \tilde{t}_N(x)) | x \in X\}, H_{\tilde{i}_N} = \{(x, 1 - \tilde{i}_N(x)) | x \in X\}, H_{\tilde{f}_N} = \{(x, 1 - \tilde{f}_N(x)) | x \in X\}.$$

Therefore, $H_{\tilde{t}_N}$ is called a generated hesitant fuzzy set by function $\tilde{t}_N(x)$; $H_{\tilde{i}_N}$ is called a generated hesitant fuzzy set by function $\tilde{i}_N(x)$; $H_{\tilde{f}_N}$ is called a generated hesitant fuzzy set by function $\tilde{f}_N(x)$.

Theorem 1. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(y), \tilde{f}_N(x)) | x \in X\}$ be a neutrosophic hesitant fuzzy set on X. Then, N is a neutrosophic hesitant fuzzy subalgebra of X if and only if it satisfies the conditions: $\forall x \in X$, $H_{\tilde{t}_N}$ and $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ are hesitant fuzzy subalgebras of X.*

Proof. Necessity: (i) By Definition 10 and Definition 11, we can obtain that $H_{\tilde{t}_N}$ is a hesitant fuzzy subalgebra of X.

(ii) $\forall x,y \in X$, $(1-\tilde{i}_N(x)) \cap (1-\tilde{i}_N(y)) = 1-(\tilde{i}_N(x)\cup\tilde{i}_N(y)) \subseteq 1-\tilde{i}_N(x\to y), (1-\tilde{i}_N(x)) \cap (1-\tilde{i}_N(y)) = 1-(\tilde{i}_N(x)\cup\tilde{i}_N(y)) \subseteq 1-\tilde{i}_N(x\hookrightarrow y)$.

Similarly, $(1-\tilde{f}_N(x))\cap(1-\tilde{f}_N(y)) \subseteq 1-\tilde{f}_N(x\to y), (1-\tilde{f}_N(x))\cap(1-\tilde{f}_N(y)) \subseteq 1-\tilde{f}_N(x\to y)$. Therefore, $\forall x \in X$, $H_{\tilde{i}_N} = \{(x, 1-\tilde{i}(x)) | x \in X\}$ and $H_{\tilde{f}_N} = \{(x, 1-\tilde{f}_N(x)) | x \in X\}$ are hesitant fuzzy subalgebras of X.

Sufficiency: (i) Let $x,y \in H_{\tilde{t}_N}$. Obviously, $\tilde{t}_N(x)\cap\tilde{t}_N(y) \subseteq \tilde{t}_N(x\to y), \tilde{t}_N(x)\cap\tilde{t}_N(y) \subseteq \tilde{t}_N(x\hookrightarrow y)$.

(ii) Let $x,y \in H_{\tilde{i}_N}$. By Definition 10, we have $(1-\tilde{i}_N(x))\cap(1-\tilde{i}_N(y)) \subseteq 1-\tilde{i}_N(x\to y), (1-\tilde{i}_N(x))\cap(1-\tilde{i}_N(y)) \subseteq 1-\tilde{i}_N(x\to y)$, thus $\tilde{i}_N(x)\cup\tilde{i}_N(y) \supseteq \tilde{i}_N(x\to y), \tilde{i}_N(x)\cup\tilde{i}_N(y) \supseteq \tilde{i}_N(x\hookrightarrow y)$.

Similarly, Let $x,y \in H_{\tilde{f}_N}$; we have $\tilde{f}_N(x)\cup\tilde{f}_N(y) \supseteq \tilde{f}_N(x\to y), \tilde{f}_N(x)\cup\tilde{f}_N(y) \supseteq \tilde{f}(x\hookrightarrow y)$.

That is, N is a neutrosophic hesitant fuzzy subalgebra of X. □

Theorem 2. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ be a neutrosophic hesitant fuzzy set on X. Then, the following conditions are equivalent:*

(1) $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy subalgebra of X;

(2) $\forall \lambda_1, \lambda_2, \lambda_3 \in P([0,1])$, the nonempty hesitant fuzzy level sets $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are subalgebras of X, where $P([0,1])$ is the power set of $[0,1]$,

$$\begin{aligned}
H_{\tilde{t}_N}(\lambda_1) &= \{x \in X | \lambda_1 \subseteq \tilde{t}_N(x)\},\\
H_{\tilde{i}_N}(\lambda_2) &= \{x \in X | \lambda_2 \subseteq 1-\tilde{i}_N(x)\},\\
H_{\tilde{f}_N}(\lambda_3) &= \{x \in X | \lambda_3 \subseteq 1-\tilde{f}_N(x)\}.
\end{aligned}$$

Proof. (1)⇒(2) Suppose $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are nonempty sets. If $x,y \in H_{\tilde{t}_N}(\lambda_1)$, then $\lambda_1 \subseteq \tilde{t}_N(x), \lambda_1 \subseteq \tilde{t}_N(y)$. Since N is a neutrosophic hesitant fuzzy subalgebra of X, by Definition 11, we can obtain:

$$\lambda_1 \subseteq \tilde{t}_N(x)\cap\tilde{t}_N(y) \subseteq \tilde{t}_N(x\to y), \lambda_1 \subseteq \tilde{t}_N(x)\cap\tilde{t}_N(y) \subseteq \tilde{t}_N(x\hookrightarrow y);$$

then $x\to y, x\hookrightarrow y \in H_{\tilde{t}_N}(\lambda_1)$, $H_{\tilde{t}_N}(\lambda_1)$ is a subalgebra of X.

If $x,y \in H_{\tilde{i}_N}(\lambda_2)$, then $\lambda_2 \subseteq 1-\tilde{i}_N(x), \lambda_2 \subseteq 1-\tilde{i}_N(y)$. Since N is a neutrosophic hesitant fuzzy subalgebra of X, by Definition 11, we can obtain:

$$\begin{aligned}
\lambda_2 &\subseteq (1-\tilde{i}_N(x))\cap(1-\tilde{i}_N(y)) = 1-(\tilde{i}_N(x)\cup\tilde{i}_N(y)) \subseteq 1-\tilde{i}_N(x\to y),\\
\lambda_2 &\subseteq (1-\tilde{i}_N(x))\cap(1-\tilde{i}_N(y)) = 1-(\tilde{i}_N(x)\cup\tilde{i}_N(y)) \subseteq 1-\tilde{i}_N(x\hookrightarrow y);
\end{aligned}$$

Thus, $x\to y, x\hookrightarrow y \in H_{\tilde{i}_N}(\lambda_2)$, $H_{\tilde{i}_N}(\lambda_2)$ is a subalgebra of X.

Similarly, we can obtain then that $H_{\tilde{f}_N}(\lambda_3)$ is a subalgebra of X.

(2)⇒(1) Suppose that $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are nonempty subalgebras of X, $\forall \lambda_1, \lambda_2, \lambda_3 \in P([0,1])$. Let $x,y \in X$ with $\tilde{t}_N(x) = \mu_1, \tilde{t}_N(y) = \mu_2$. Let $\mu_1 \cap \mu_2 = \lambda_1$. Therefore, we have $x,y \in H_X^{(1)}(\lambda_1)$. Since $H_X^{(1)}(\lambda_1)$ is a subalgebra, we can obtain $x\to y, x\hookrightarrow y \in H_{\tilde{t}_N}(\lambda_1)$. Hence, we can obtain:

$$\tilde{t}_N(x)\cap\tilde{t}_N(y) \subseteq \tilde{t}_N(x\to y), \tilde{t}_N(x)\cap\tilde{t}_N(y) \subseteq \tilde{t}_N(x\hookrightarrow y);$$

Let $x,y \in X$ with $\tilde{i}(x) = \mu_3, \tilde{i}(y) = \mu_4$. Let $(1-\mu_3)\cap(1-\mu_4) = \lambda_2$. Then, we have $x,y \in H_{\tilde{i}_N}(\lambda_2)$. Since $H_{\tilde{i}_N}(\lambda_2)$ is a subalgebra, we can obtain $x\to y, x\hookrightarrow y \in H_{\tilde{f}_N}(\lambda_2)$. Hence, we can obtain $(1-\tilde{i}_N(x))\cap(1-\tilde{i}_N(y)) = 1-(\tilde{i}_N(x)\cup\tilde{i}_N(y)) = \lambda_2 \subseteq 1-\tilde{i}_N(x\to y)$, $(1-\tilde{i}_N(x))\cap(1-\tilde{i}_N(y)) = 1-(\tilde{i}_N(x)\cup\tilde{i}_N(y)) = \lambda_2 \subseteq 1-\tilde{i}_N(x\hookrightarrow y)$. Then, we have $\tilde{i}_N(x)\cup\tilde{i}_N(y) \supseteq \tilde{i}_N(x\to y)$, $\tilde{i}_N(x)\cup\tilde{i}_N(y) \supseteq \tilde{i}_N(x\hookrightarrow y)$.

Similarly, let $x,y \in X$ with $\tilde{f}_N(x) = \mu_5, \tilde{f}_N(y) = \mu_6$; we can obtain $\tilde{f}_N(x)\cup\tilde{f}_N(y) \supseteq \tilde{f}_N(x\to y)$, $\tilde{f}_N(x)\cup\tilde{f}_N(y) \supseteq \tilde{f}_N(x\hookrightarrow y)$.

Thus, N is a neutrosophic hesitant fuzzy subalgebra of X. □

Definition 12. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ be a neutrosophic hesitant fuzzy set on X. $X_N^{(1)}(a^k, b)$, $X_N^{(2)}(a^k, b)$, $X_N^{(3)}(a^k, b)$, $X_N^{(4)}(a^k, b)$, $X_N^{(5)}(a)$ are called generated subsets by N: $\forall a, b \in X$, $k \in \mathbb{N}$,*

$$\begin{aligned}
X_N^{(1)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k * (b * x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k * (b * x)) = \tilde{i}_N(1), \tilde{f}_N(a^k * (b * x)) = \tilde{f}_N(1)\}; \\
X_N^{(2)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k \to (b \hookrightarrow x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k \to (b \hookrightarrow x)) = \tilde{t}_N(1), \tilde{f}_N(a^k \to (b \hookrightarrow x)) = \tilde{f}_N(1)\}; \\
X_N^{(3)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k \hookrightarrow (b \to x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k \hookrightarrow (b \to x)) = \tilde{t}_N(1), \tilde{f}_N(a^k \hookrightarrow (b \to x)) = \tilde{f}_N(1)\}; \\
X_N^{(4)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k \to (b \to x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k \to (b \to x)) = \tilde{i}_N(1), \tilde{f}_N(a^k \to (b \to x)) = \tilde{f}_N(1), \\
&\tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{t}_N(1), \tilde{i}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{i}_N(1), \tilde{f}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{f}_N(1)\}; \\
X_N^{(5)}(a) = \{x \in X | &\tilde{t}_N(a) \subseteq \tilde{t}_N(x), \\
&\tilde{i}_N(a) \supseteq \tilde{i}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x)\}.
\end{aligned}$$

where "a" appears "k" times, "$*$" represents any binary operation "$\to$" or "$\hookrightarrow$" on X,

$$\begin{aligned}
a^k * (b * x) &= a * (a * (\cdots (a * (b * x)) \cdots)); \\
a^k \to (b \hookrightarrow x)) &= a \to (a \to (\cdots (a \to (b \hookrightarrow x)) \cdots)); \\
a^k \hookrightarrow (b \to x)) &= a \hookrightarrow (a \hookrightarrow (\cdots (a \hookrightarrow (b \to x)) \cdots)); \\
a^k \to (b \to x)) &= a \to (a \to (\cdots (a \to (b \to x)) \cdots)); \\
a^k \hookrightarrow (b \hookrightarrow x) &= a \hookrightarrow (a \hookrightarrow (\cdots (a \hookrightarrow (b \hookrightarrow x)) \cdots)).
\end{aligned}$$

Theorem 3. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ be a neutrosophic hesitant fuzzy set on X. If N satisfies the following conditions:*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{t}_N(x \hookrightarrow y) = \tilde{t}_N(x) \cup \tilde{t}_N(y), \forall x, y \in X$;
(2) $\tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{i}_N(x \hookrightarrow y) = \tilde{i}_N(x) \cap \tilde{i}_N(y), \forall x, y \in X$;
(3) $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x \hookrightarrow y) = \tilde{f}_N(x) \cap \tilde{f}_N(y), \forall x, y \in X$;

then $X_N^{(1)}(a^k, b) = X$, $k \in \mathbb{N}$.

Proof. By Proposition 1, we can obtain $\forall x \in X$,

$$\begin{aligned}
&\tilde{t}_N(a^k * (b * x) = \tilde{t}_N(1 \hookrightarrow (a^k * (b * x))) \\
=&\tilde{t}_N(1) \cup \tilde{t}_N(a^k * (b * x))) = \tilde{t}_N(1). \\
&\tilde{i}_N(a^k * (b * x)) = \tilde{i}_N(1 \hookrightarrow (a^k * (b * x))) \\
=&\tilde{i}_N(1) \cap \tilde{t}_N(a^k * (b * x))) = \tilde{i}_N(1). \\
&\tilde{f}_N(a^k * (b * x)) = \tilde{f}_N(1 \hookrightarrow (a^k * (b * x))) \\
=&\tilde{f}_N(1) \cap \tilde{t}_N(a^k * (b * x))) = \tilde{f}_N(1).
\end{aligned}$$

Thus, $x \in X_N^{(1)}(a^k, b)$, $X \subseteq X_N^{(1)}(a^k, b)$.

Conversely, it is easy to check that $X_N^{(1)}(a^k, b) \subseteq X$.

Finally, we can obtain $X = X_N^{(1)}(a^k, b)$. □

Corollary 1. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ be a neutrosophic hesitant fuzzy set on X. If N satisfies the following conditions:*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{t}_N(x \to y) = \tilde{t}_N(x) \cup \tilde{t}_N(y), \forall x, y \in X$;

(2) $\tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{i}_N(x \to y) = \tilde{i}_N(x) \cap \tilde{i}_N(y), \forall x, y \in X$*;*
(3) $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x \to y) = \tilde{f}_N(x) \cap \tilde{f}_N(y), \forall x, y \in X$*;*
then $X_N^{(1)}(a^k, b) = X, k \in \mathbb{N}$.

Theorem 4. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X. N *satisfies the following conditions:*
(1) $\tilde{t}_N(1) \supseteq \tilde{t}_N(x), \tilde{i}_N(1) \subseteq \tilde{i}_N(x), \tilde{f}_N(1) \subseteq \tilde{f}_N(x), \forall x \in X$*;*
(2) $x \hookrightarrow y = 1 \Rightarrow \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y), \forall x, y \in X$.
If $\forall a, b, c \in X, k \in \mathbb{N}, b \leq c$*, then* $X_N^{(2)}(a^k, c) \subseteq X_N^{(2)}(a^k, b)$.

Proof: Let $x \in X_N^{(2)}(a^k, c)$. If $b \leq c$, by Proposition 1, we can obtain:

$$\begin{aligned}\tilde{t}_N(1) =& \tilde{t}_N(a^k \to (c \hookrightarrow x)) \\ =& \tilde{t}_N(c \hookrightarrow (a^k \to x)) \\ \subseteq& \tilde{t}_N(b \hookrightarrow (a^k \to x)) \\ =& \tilde{t}_N(a^k \to (b \hookrightarrow x)).\end{aligned}$$

Similarly, we can obtain:

$$\tilde{i}_N(a^k \to (b \hookrightarrow x)) \subseteq \tilde{i}_N(a^k \to (c \hookrightarrow x)) \subseteq \tilde{i}_N(1);$$
$$\tilde{f}_N(a^k \to (b \hookrightarrow x)) \subseteq \tilde{f}_N(a^k \to (c \hookrightarrow x)) \subseteq \tilde{f}_N(1).$$

That is, $x \in X_N^{(2)}(a^k, b)$, $X_N^{(2)}(a^k, c) \subseteq X_N^{(2)}(a^k, b)$.

Corollary 2. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X. N *satisfies the following conditions:*
(1) $\tilde{t}_N(1) \supseteq \tilde{t}_N(x), \tilde{i}_N(1) \subseteq \tilde{i}_N(x), \tilde{f}_N(1) \subseteq \tilde{f}_N(x), \forall x \in X$*;*
(2) $x \to y = 1 \Rightarrow \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y), \forall x, y \in X$.
If $\forall a, b, c \in X, k \in \mathbb{N}, b \leq c$*, then* $X_N^{(3)}(a^k, c) \subseteq X_N^{(3)}(a^k, b)$.

The following example shows that $X_N^{(4)}(a^k, b)$ may not be a filter of X.

Example 4. *Let* $X = \{a, b, c, d, 1\}$ *with two binary operations in Tables 3 and 4.*

Table 3. $\to$.

$\to$	a	b	c	d	1
a	1	1	1	1	1
b	d	1	1	1	1
c	d	c	1	1	1
d	c	c	c	1	1
1	a	b	c	d	1

Table 4. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	d	1	1	1
b	d	1	1	1	1
c	d	d	1	1	1
d	c	b	c	1	1
1	a	b	c	d	1

Then, $(X; \to, \hookrightarrow, 1)$ *is a pseudo-BCI algebra. Let:*

$$N = \{(1, [0,1], [\tfrac{1}{6}, \tfrac{1}{5}], [0, \tfrac{1}{5}]), (a, [\tfrac{1}{3}, \tfrac{1}{4}], [0, \tfrac{5}{6}], [0, \tfrac{3}{4}]), (b, [0, \tfrac{1}{2}], [\tfrac{1}{6}, \tfrac{3}{4}], [0, \tfrac{1}{3}]),$$
$$(c, [\tfrac{1}{3}, \tfrac{2}{3}], [0, \tfrac{3}{5}], [0, \tfrac{1}{4}]), (d, [\tfrac{1}{3}, 1], [\tfrac{1}{6}, \tfrac{1}{3}], [0, \tfrac{5}{6}])\}.$$

then $X_N^{(4)}(c,d) = \{a, c, d, 1\}$ *is not a filter of* X*. Since* $c \to b = c \in X_N^{(4)}(c,d)$*, but* $b \notin X_N^{(4)}(c,d)$*.*

Theorem 5. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X*. Let* X *be a type-2 positive implicative pseudo-BCK algebra. If functions* $\tilde{t}_N(x), \tilde{i}_N(x)$ *and* $\tilde{f}_N(x)$ *are injective, then* $X_N^{(4)}(a^k, b)$ *is a filter of* X *for all* $a, b \in X, k \in \mathbb{N}$*.*

Proof. (1) If X is a pseudo-BCK algebra, then by Definition 1 and Proposition 1, we can obtain $1 \in X_N^{(4)}(a^k, b)$.

(2) Let $x, y \in X$ with $x, x \to y \in X_N^{(4)}(a^k, b)$. Thus, $a^k \hookrightarrow (b \hookrightarrow x) = 1, a^k \hookrightarrow (b \hookrightarrow (x \to y)) = 1$. Since functions $\tilde{t}_N, \tilde{i}_N$ and $\tilde{f}_N$ are injective, by Definition 5, we have:

$$\begin{aligned}
\tilde{t}_N(1) &= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow (x \to y))) \\
&= \tilde{t}_N(a^k \hookrightarrow ((b \hookrightarrow x) \to (b \hookrightarrow y))) \\
&= \tilde{t}_N((a^k \hookrightarrow (b \hookrightarrow x)) \to (a^k \hookrightarrow (b \hookrightarrow y))) \\
&= \tilde{t}_N(1 \to (a^k \hookrightarrow (b \hookrightarrow y))) \\
&= \tilde{t}_N(a^k \hookrightarrow ((b \hookrightarrow y)).
\end{aligned}$$

Similarly, we can obtain $\tilde{i}_N(a^k \hookrightarrow ((b \hookrightarrow y)) = \tilde{i}_N(1), \tilde{f}_N(a^k \hookrightarrow ((b \hookrightarrow y)) = \tilde{f}_N(1)$. Thus, we have $y \in X_N^{(4)}(a^k, b)$.

(3) Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in X_N^{(4)}(a^k, b)$; we have $y \in X_N^{(4)}(a^k, b)$. □

This means that $X_N^{(4)}(a^k, b)$ is a filter of X for all $a, b \in X, k \in \mathbb{N}$.

Theorem 6. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X)\}$ *be a neutrosophic hesitant fuzzy set on* X*. Let* X *be a type-2 positive implicative pseudo-BCK algebra. If functions* $\tilde{t}_N(x), \tilde{i}_N(x)$ *and* $\tilde{f}_N(x)$ *satisfy the following identifies:* $\forall x, y \in X$*,*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{i}_N(x) \supseteq i_N(1), \tilde{f}_N(x) \supseteq f_N(1)$*;*

(2) $\tilde{t}_N(x \to y) = \tilde{t}_N(x) \cap \tilde{t}_N(y), \tilde{i}_N(x \to y) = \tilde{i}_N(x) \cup \tilde{i}_N(y), \tilde{f}_N(x \to y) = \tilde{f}_N(x) \cup \tilde{f}_N(y)$*;*

(3) $\tilde{t}_N(x \hookrightarrow y) = \tilde{t}_N(x) \cap \tilde{t}_N(y), \tilde{i}_N(x \hookrightarrow y) = \tilde{i}_N(x) \cup \tilde{i}_N(y), \tilde{f}_N(x \hookrightarrow y) = \tilde{f}_N(x) \cup \tilde{f}_N(y)$*;*

then $X_N^{(4)}(a^k, b)$ *is a filter of* X *for all* $a, b \in X, k \in \mathbb{N}$*.*

Proof. (1) If X is a pseudo-BCK algebra, by Definition 1 and Proposition 1, $1 \in X_N^{(4)}(a^k, b)$.

(2) Let $x, y \in X$ with $x, x \to y \in X_N^{(4)}(a^k, b)$. We have $\tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{t}_N(1), \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow (x \to y))) = \tilde{t}_N(1)$. By Definition 5, we have:

$$\begin{aligned}
\tilde{t}_N(1) &= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow (x \to y))) \\
&= \tilde{t}_N(a^k \hookrightarrow ((b \hookrightarrow x) \to (b \hookrightarrow y))) \\
&= \tilde{t}_N((a^k \hookrightarrow (b \hookrightarrow x)) \to (a^k \hookrightarrow (b \hookrightarrow y))) \\
&= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow x)) \cap \tilde{t}(a^k \hookrightarrow (b \hookrightarrow y)) \\
&= \tilde{t}_N(1) \cap \tilde{t}(a^k \hookrightarrow (b \hookrightarrow y)) \\
&= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow y)).
\end{aligned}$$

Similarly, we can obtain $\tilde{i}_N(a^k \hookrightarrow (b \hookrightarrow y)) = \tilde{i}_N(1), \tilde{f}_N(a^k \hookrightarrow (b \hookrightarrow y)) = \tilde{f}_N(1)$. Thus, we have $y \in X_N^{(4)}(a^k, b)$.

(3) Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in X_N^{(4)}(a^k, b)$; we have $y \in X_N^{(4)}(a^k, b)$.

This means that $X_N^{(4)}(a^k, b)$ is a filter of X for all $a, b \in X, k \in \mathbb{N}$. $\square$

Theorem 7. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ be a neutrosophic hesitant fuzzy set on X and F be a filter of X. If functions $\tilde{t}_N(x), \tilde{i}_N(x)$ and $\tilde{f}_N(x)$ are injective, then $\bigcup X_N^{(4)}(a^k, b) = F$ for all $a, b \in F, k \in \mathbb{N}$.*

Proof. (1) Let $x \in \bigcup X_N^{(4)}(a^k, b)$. By Definition 12, we have $\tilde{t}_N(a \to (a^{k-1} \to (b \to x))) = \tilde{t}_N(1), \tilde{i}_N(a \to (a^{k-1} \to (b \to x))) = \tilde{i}_N(1), \tilde{f}_N(a \to (a^{k-1} \to (b \to x))) = \tilde{f}_N(1)$. Since F is a filter of X and $\tilde{t}_N, \tilde{i}_N, \tilde{f}_N$ are injective, thus we can obtain $a \to (a^{k-1} \to (b \to x)) = 1$ and $a^{k-1} \to (b \to x) \in F$. Continuing, we can obtain $b \to x \in F$. Since $b \in F$, thus $x \in F, \bigcup X_N^{(4)}(a^k, b) \subseteq F$.

(2) Let $x \in F$. When $a = 1, b = x$, we can obtain $\tilde{t}_N(1^k \to (x \to x)) = \tilde{t}_N(1^k \hookrightarrow (x \hookrightarrow x)) = \tilde{t}_N(1)$. Similarly, we have $\tilde{i}_N(1^k \to (x \to x)) = \tilde{i}_N(1^k \hookrightarrow (x \hookrightarrow x)) = \tilde{i}_N(1), \tilde{f}_N(1^k \to (x \to x)) = \tilde{f}_N(1^k \hookrightarrow (x \hookrightarrow x)) = \tilde{f}_N(1)$. Thus, we have $F \subseteq \bigcup X_N^{(4)}(a^k, b)$.

This means that $\bigcup X_N^{(4)}(a^k, b) = F$ for all $a, b \in F, k \in \mathbb{N}$. $\square$

Theorem 8. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ be a neutrosophic hesitant fuzzy set on X.*

(1) If $X_N^{(5)}(a)$ is a filter of X, then N satisfies: $\forall x, y \in X$,

(i) $\tilde{t}_N(a) \subseteq \tilde{t}_N(x \to y) \cap \tilde{t}_N(x), \tilde{i}_N(a) \supseteq \tilde{i}_N(x \to y) \cup \tilde{i}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x \to y) \cup \tilde{f}_N(x) \Rightarrow \tilde{t}_N(a) \subseteq \tilde{t}_N(y), \tilde{i}_N(a) \supseteq \tilde{i}_N(y), \tilde{f}_N(a) \supseteq \tilde{f}_N(y)$;

(ii) $\tilde{t}_N(a) \subseteq \tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x), \tilde{i}_N(a) \supseteq \tilde{i}_N(x \hookrightarrow y) \cup \tilde{i}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x \hookrightarrow y) \cup \tilde{f}_N(x) \Rightarrow \tilde{t}_N(a) \subseteq \tilde{t}_N(y), \tilde{i}_N(a) \supseteq \tilde{i}_N(y), \tilde{f}_N(a) \supseteq \tilde{f}_N(y)$.

(2) If N satisfies Conditions (i), (ii) and $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{f}_N(x) \supseteq \tilde{f}_N(1)$ for all $x, y \in X$, then $X_N^{(5)}(a)$ is a filter of X.

Proof. (1) (i) Let $x, y \in X$ with $\tilde{t}_N(a) \subseteq \tilde{t}_N(x \to y) \cap \tilde{t}_N(x), \tilde{i}_N(a) \supseteq \tilde{i}_N(x \to y) \cup \tilde{i}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x \to y) \cup \tilde{f}_N(x)$; we have $x \in X_N^{(5)}(a), x \to y \in X_N^{(5)}(a)$. Since $X_N^{(5)}(a)$ is a filter, thus we can have $y \in X_N^{(5)}(a), \tilde{t}_N(a) \subseteq \tilde{t}_N(y), \tilde{i}_N(a) \supseteq \tilde{i}_N(y), \tilde{f}_N(a) \supseteq \tilde{f}_N(y)$.

(ii) Similarly, we know that (ii) is correct.

(2) Since $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{f}_N(x) \supseteq \tilde{f}_N(1)$ for all $x \in X$, thus $1 \in X_N^{(5)}(a)$. Let $x, y \in X$ with $x, x \to y \in X_N^{(5)}(a)$; we can obtain $\tilde{t}_N(a) \subseteq \tilde{t}_N(x), \tilde{t}_N(a) \subseteq \tilde{t}_N(x \to y), \tilde{i}_N(a) \supseteq \tilde{i}_N(x), \tilde{i}_N(a) \supseteq \tilde{i}_N(x \to y), \tilde{f}_N(a) \supseteq \tilde{f}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x \to y)$. By Condition (i), we have $\tilde{t}_N(a) \subseteq \tilde{t}_N(y), \tilde{i}_N(a) \supseteq \tilde{i}_N(y), \tilde{f}_N(a) \supseteq \tilde{f}_N(y)$. Thus, we can obtain $y \in X_N^{(5)}(a)$. Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in X_N^{(5)}(a)$, by Condition (1)(ii); we can obtain $y \in X_N^{(5)}(a)$.

This means that $X_N^{(5)}(a)$ is a filter of X. $\square$

4. Neutrosophic Hesitant Fuzzy Filters of Pseudo-BCI Algebras

In the following, let X be a pseudo-BCI algebra, unless otherwise specified.

Definition 13. *([22]) A hesitant fuzzy set $A = \{(x, h_A(x))|x \in X\}$ is called a hesitant fuzzy pseudo-filter (briefly, hesitant fuzzy filter) of X if it satisfies:*

(HFF1) $h_A(x) \subseteq h_A(1), \forall x \in X$;

(HFF2) $h_A(x) \cap h_A(x \to y) \subseteq h_A(y), \forall x, y \in X$;

(HFF3) $h_A(x) \cap h_A(x \hookrightarrow y) \subseteq h_A(y), \forall x, y \in X$.

Definition 14. *A neutrosophic hesitant fuzzy set $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X\}$ is called a neutrosophic hesitant fuzzy pseudo-filter (neutrosophic hesitant fuzzy filter) of X if it satisfies:*

(NHFF1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{f}_N(x) \supseteq \tilde{f}_N(1), \forall x \in X$;

(NHFF2) $\tilde{t}_N(x \to y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x \to y) \cup \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x \to y) \cup \tilde{f}_N(x) \supseteq \tilde{f}_N(y), \forall x, y \in X$;

(NHFF3) $\tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x \hookrightarrow y) \cup \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x \hookrightarrow y) \cup \tilde{f}_N(x) \supseteq \tilde{f}_N(y)$, $\forall x, y \in X$.

A neutrosophic hesitant fuzzy set $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ is called a neutrosophic hesitant fuzzy closed filter of X if it is a neutrosophic hesitant fuzzy filter such that:

$$\tilde{t}_N(x \to 1) \supseteq \tilde{t}_N(x), \tilde{i}_N(x \to 1) \subseteq \tilde{i}_N(x), \tilde{f}_N(x \to 1) \subseteq \tilde{f}_N(x).$$

Example 5. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 5 and 6. Then, $(X; \to, \hookrightarrow, 1)$ is a pseudo-BCI algebra. Let:*

$$N = \{(1, [0,1], [0, \tfrac{3}{7}], [0, \tfrac{1}{10}]), (a, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]), (b, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]), (c, [0, \tfrac{1}{3}], [0, \tfrac{3}{5}], [0, \tfrac{1}{4}]), (d, [0, \tfrac{3}{4}]), [0, \tfrac{3}{6}], [0, \tfrac{1}{5}])\}.$$

Then, N is a neutrosophic hesitant fuzzy filter of X.

Table 5. $\to$.

$\to$	a	b	c	d	1
a	1	1	1	1	1
b	c	1	1	1	1
c	a	b	1	d	1
d	b	b	c	1	1
1	a	b	c	d	1

Table 6. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	1	1	1	1
b	d	1	1	1	1
c	b	b	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Theorem 9. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(y), \tilde{f}_N(x))|x \in X\}$ be a neutrosophic hesitant fuzzy set on X. Then, N is a neutrosophic hesitant fuzzy filter of X if and only if it satisfies the following conditions: $\forall x \in X$, $H_{\tilde{t}_N}$, $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ are hesitant fuzzy filters of X.*

Proof. Necessity: If N is a neutrosophic hesitant fuzzy filter:

(1) Obviously, $H_{\tilde{t}_N}$ is a hesitant fuzzy filter of X.

(2) By Definition 14, we have $(1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{i}_N(1)), 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(x \to y)) = (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(x \to y)) \subseteq (1 - \tilde{i}_N(y))$; similarly, by Definition 14, we have $(1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(x \hookrightarrow y)) \subseteq (1 - \tilde{i}_N(y))$. Thus, $H_{\tilde{i}_N}$ is hesitant fuzzy filter of X.

(3) Similarly, we have that $H_{\tilde{f}_N}$ is a hesitant fuzzy filter of X.

Sufficiency: If $H_{\tilde{t}_N}$, $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ are hesitant fuzzy filters of X. It is easy to prove that $\tilde{t}_N(x)$, $\tilde{i}_N(x)$, $\tilde{f}_N(x)$ satisfies Definition 14. Therefore, $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X\}$ is a neutrosophic hesitant fuzzy filter of X. □

Theorem 10. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X\}$ be a neutrosophic hesitant fuzzy set on X. Then, the following are equivalent:*

(1) $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X\}$ is a neutrosophic hesitant fuzzy filter of X;

(2) $\forall \lambda_1, \lambda_2, \lambda_3 \in P([0,1])$, the nonempty hesitant fuzzy level sets $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are filters of X, where $P([0,1])$ is the power set of $[0,1]$,

$$\begin{aligned} H_{\tilde{t}_N}(\lambda_1) &= \{x \in X | \lambda_1 \subseteq \tilde{t}_N(x)\}; \\ H_{\tilde{i}_N}(\lambda_2) &= \{x \in X | \lambda_2 \subseteq 1 - \tilde{i}_N(x)\}; \\ H_{\tilde{f}_N}(\lambda_3) &= \{x \in X | \lambda_3 \subseteq 1 - \tilde{f}_N(x)\}. \end{aligned}$$

Proof. (1)⇒(2) (i) Suppose $H_{\tilde{t}_N}(\lambda_1) \neq \emptyset$. Let $x \in H_{\tilde{t}_N}(\lambda_1)$, then $\lambda_1 \subseteq \tilde{t}_N(x)$. Since N is a neutrosophic hesitant fuzzy filter of X, by Definition 14, we have $\lambda_1 \subseteq \tilde{t}_N(x) \subseteq \tilde{t}_N(1)$. Thus, $1 \in H_{\tilde{t}_N}(\lambda_1)$.

Let $x, y \in X$ with $x, x \to y \in H_{\tilde{t}_N}(\lambda_1)$, then $\lambda_1 \subseteq \tilde{t}_N(x), \lambda_1 \subseteq \tilde{t}_N(x \to y)$. Since N is a neutrosophic hesitant fuzzy filter of X, by Definition 14, we have $\lambda_1 \subseteq \tilde{t}_N(x \to y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$. Thus $y \in H_{\tilde{t}_N}(\lambda_1)$. Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in H_{\tilde{t}_N}(\lambda_1)$. We have $y \in H_{\tilde{t}_N}(\lambda_1)$.

Thus, we can obtain that $H_{\tilde{t}_N}(\lambda_1)$ is a filter of X.

(ii) Suppose $H_{\tilde{i}_N}(\lambda_2) \neq \emptyset$. Let $x \in H_{\tilde{i}_N}(\lambda_2)$, then $\lambda_2 \subseteq 1 - \tilde{i}_N(x)$. Since N is a neutrosophic hesitant fuzzy filter of X, we have $\tilde{i}_N(1) \subseteq \tilde{i}_N(x)$. Thus, $\lambda_2 \subseteq 1 - \tilde{i}_N(x) \subseteq 1 - \tilde{i}_N(1), 1 \in H_{\tilde{i}_N}(\lambda_2)$.

Let $x, y \in X$ with $x, x \to y \in H_{\tilde{i}_N}(\lambda_2)$, then $\lambda_2 \subseteq 1 - \tilde{i}_N(x), \lambda_2 \subseteq 1 - \tilde{i}_N(x \to y)$. Since N is a neutrosophic hesitant fuzzy filter of X, we have $\tilde{i}_N(x \to y) \cup \tilde{i}_N(x) \supseteq \tilde{i}_N(y)$. Thus, $1 - (\tilde{i}_N(x \to y) \cup \tilde{i}_N(x)) = (1 - \tilde{i}_N(x \to y)) \cap (1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{i}_N(y)), \lambda_2 \subseteq (1 - \tilde{i}_N(y)), y \in H_{\tilde{i}_N}(\lambda_2)$. Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in H_{\tilde{i}_N}(\lambda_2)$. We have $y \in H_{\tilde{i}_N}(\lambda_2)$.

Thus, we can obtain that $H_{\tilde{i}_N}(\lambda_2)$ is a filter of X.

(iii) We have that $H_{\tilde{f}_N}(\lambda_3)$ is a filter of X. The progress of proof is similar to (ii).

(2)⇒(1) Suppose $H_{\tilde{t}_N}(\lambda_1) \neq \emptyset, H_{\tilde{i}_N}(\lambda_2) \neq \emptyset, H_{\tilde{f}_N}(\lambda_3) \neq \emptyset$ for all $\lambda_1, \lambda_2, \lambda_3 \in P([0,1])$.

(i′) Let $x \in X$ with $\tilde{t}_N(x) = \mu_1$. Let $\lambda_1 = \mu_1$. Since $H_{\tilde{t}_N}(\lambda_1)$ is a filter of X, we have $1 \in H_{\tilde{t}_N}(\lambda_1)$. Thus, $\lambda_1 = \mu_1 = \tilde{t}_N(x) \subseteq \tilde{t}_N(1)$.

Let $x, y \in X$ with $\tilde{t}_N(x) = \mu_1, \tilde{t}_N(x \to y) = \mu_4$. Let $\mu_1 \cap \mu_4 = \lambda_1$. Since $H_{\tilde{t}_N}(\lambda_1)$ is a filter of X for all $\lambda_1 \in P([0,1])$, we have $y \in H_{\tilde{t}_N}(\lambda_1)$. Thus, $\lambda_1 = \tilde{t}_N(x) \cap \tilde{t}_N(x \to y) \subseteq \tilde{t}_N(y)$.

Similarly, let $x, y \in X$ with $\tilde{t}_N(x) = \mu_1, \tilde{t}_N(x \hookrightarrow y) = \mu_4'$. We can obtain $\tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$.

(ii′) Let $x \in X$ with $\tilde{i}_N(x) = \mu_2$. Let $\lambda_2 = 1 - \mu_2$. Since $H_{\tilde{i}_N}(\lambda_2)$ is a filter of X for all $\lambda_2 \in P([0,1])$, we have $1 \in H_{\tilde{i}_N}(\lambda_2), \lambda_2 \subseteq 1 - \tilde{i}_N(1)$. Thus, $1 - \lambda_2 = \mu_2 = \tilde{i}_N(x) \supseteq \tilde{i}_N(1)$.

Let $x, y \in X$ with $\tilde{i}_N(x) = \mu_2, \tilde{i}_N(x \to y) = \mu_5$. Let $(1 - \mu_2) \cap (1 - \mu_5) = \lambda_2$. Since $H_{\tilde{i}_N}(\lambda_2)$ is a filter of X for all $\lambda_2 \in P([0,1])$, we have $y \in H_{\tilde{i}_N}(\lambda_2), \lambda_2 \subseteq 1 - \tilde{i}_N(y)$. Thus, $\lambda_2 = (1 - \mu_2) \cap (1 - \mu_5) = (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(x \to y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(x \to y)) \subseteq (1 - \tilde{i}_N(y)), \tilde{i}_N(x) \cup \tilde{i}_N(x \to y) \supseteq \tilde{i}_N(y)$.

Similarly, let $x, y \in X$ with $\tilde{i}_N(x) = \mu_2, \tilde{i}_N(x \hookrightarrow y) = \mu_5'$; we have $\tilde{i}_N(x) \cup \tilde{i}_N(x \hookrightarrow y) \supseteq \tilde{i}_N(y)$.

(iii′) Similarly, we can obtain $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x) \cup \tilde{f}_N(x \to y) \supseteq \tilde{f}_N(y), \tilde{f}_N(x) \cup \tilde{f}_N(x \hookrightarrow y) \supseteq \tilde{f}_N(y)$.

Therefore, $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy filter of X. □

Definition 15. *$N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy set on X. Define a neutrosophic hesitant fuzzy set $N^* = \{(x, \tilde{t}^*_N(x), \tilde{i}^*_N(x), \tilde{f}^*_N(x)) | x \in X\}$ by:*

$$\tilde{t}^*_N : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \tilde{t}_N(x), & x \in H_{\tilde{t}_N}(\lambda_1) \\ \varphi_1, & x \notin H_{\tilde{t}_N}(\lambda_1) \end{cases}$$

$$\tilde{i}^*_N : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \tilde{i}_N(x), & x \in H_{\tilde{i}_N}(\lambda_2) \\ 1 - \varphi_2, & x \notin H_{\tilde{i}_N}(\lambda_2) \end{cases}$$

$$\tilde{f}^*_N : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \tilde{f}_N(x), & x \in H_{\tilde{f}_N}(\lambda_3) \\ 1 - \varphi_3, & x \notin H_{\tilde{f}_N}(\lambda_3) \end{cases}$$

where $\lambda_1, \lambda_2, \lambda_3, \varphi_1, \varphi_2, \varphi_3 \in P([0,1])$, $\varphi_1 \subseteq \lambda_1, \varphi_2 \subseteq \lambda_2, \varphi_3 \subseteq \lambda_3$. *Then,* N^* *is called a generated neutrosophic hesitant fuzzy set by hesitant fuzzy level sets* $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2)$ *and* $H_{\tilde{f}_N}(\lambda_3)$.

Theorem 11. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy filter of* X. *Then,* N^* *is a neutrosophic hesitant fuzzy filter of* X.

Proof. (1) If N is a neutrosophic hesitant fuzzy filter of X, by Theorem 10, we know that $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are filters of X. Thus, $1 \in H_{\tilde{t}_N}(\lambda_1), 1 \in H_{\tilde{i}_N}(\lambda_2), 1 \in H_{\tilde{f}_N}(\lambda_3)$, $\tilde{t}^*_N(1) = \tilde{t}_N(1) \supseteq \tilde{t}^*_N(x)$, $\tilde{i}^*_N(1) = \tilde{i}_N(1) \subseteq \tilde{i}^*_N(x)$, $\tilde{f}^*_N(1) = \tilde{f}_N(1) \subseteq \tilde{f}^*_N(x)$, $\forall x \in X$

(2) (i) Let $x, y \in X$ with $x, x \to y \in H_{\tilde{t}_N}(\lambda_1)$. By Theorem 9, Theorem 10 and Definition 15, we know $\lambda_1 \subseteq \tilde{t}^*_N(x \to y) \cap \tilde{t}^*_N(x) = \tilde{t}_N(x \to y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y) = \tilde{t}^*_N(y)$.

Let $x, y \in X$ with $x, x \to y \in H_{\tilde{i}_N}(\lambda_2)$. By Theorem 9, Theorem 10 and Definition 15, we know $\lambda_2 \subseteq (1 - \tilde{i}^*_N(x \to y)) \cap (1 - \tilde{t}^*_N(x)) = (1 - \tilde{i}_N(x \to y)) \cap (1 - \tilde{t}_N(x)) = 1 - (\tilde{i}_N(x \to y) \cup \tilde{i}_N(x)) \subseteq 1 - \tilde{i}_N(y) = 1 - \tilde{t}^*_N(y)$. Thus, we have $1 - \lambda_2 \supseteq \tilde{i}^*_N(x \to y) \cup \tilde{i}^*_N(x) = \tilde{i}_N(x \to y) \cup \tilde{i}_N(x) \supseteq i_N(y) = \tilde{i}^*_N(y)$.

Similarly, let $x, y \in X$ with $x, x \to y \in H_{\tilde{f}_N}(\lambda_3)$; we have $1 - \lambda_3 \supseteq \tilde{f}^*_N(x \to y) \cup \tilde{f}^*_N(x) = \tilde{f}_N(x \to y) \cup \tilde{f}_N(x) \supseteq f_N(y) = \tilde{f}^*_N(y)$.

(ii) Let $x, y \in X$ with $x \notin H_{\tilde{t}_N}(\lambda_1)$ or $x \to y \notin H_{\tilde{t}_N}(\lambda_1)$. By Definition 15, we have $\tilde{t}^*_N(x) = \varphi_1$ or $\tilde{t}^*_N(x \to y) = \varphi_1$. Thus, we can obtain $\tilde{t}^*_N(x) \cap \tilde{t}^*_N(x \to y) = \varphi_1 \subseteq \tilde{t}^*_N(y)$.

Let $x, y \in X$ with $x \notin H_{\tilde{i}_N}(\lambda_2)$ or $x \to y \notin H_{\tilde{i}_N}(\lambda_2)$. By Definition 15, we have $\tilde{i}^*_N(x) = 1 - \varphi_2$ or $\tilde{i}^*_N(x \to y) = 1 - \varphi_2$. Since $1 - \lambda_2 \subseteq 1 - \varphi_2$; thus, we can obtain $\tilde{i}^*_N(x) \cup \tilde{i}^*_N(x \to y) = 1 - \varphi_2 \supseteq \tilde{t}^*_N(y)$.

Similarly, let $x, y \in X$ with $x \notin H_{\tilde{f}_N}(\lambda_3)$ or $x \to y \notin H_{\tilde{f}_N}(\lambda_3)$; we have $\tilde{f}^*(x) \cup \tilde{f}^*(x \to y) = 1 - \varphi_3 \supseteq \tilde{f}^*(y)$.

(3) We can obtain $\tilde{t}^*(x) \cap \tilde{t}^*(x \hookrightarrow y) \subseteq \tilde{t}^*(y)$, $\tilde{i}^*(x) \cup \tilde{i}^*(x \hookrightarrow y) \supseteq \tilde{i}^*(y)$, $\tilde{f}^*(x) \cup \tilde{f}^*(x \hookrightarrow y) \supseteq \tilde{f}^*(y)$. The process of proof is similar to (2).

Thus N^* is a neutrosophic hesitant fuzzy filter of X. □

Theorem 12. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy filter of* X. *Then,* N *satisfies the following properties,* $\forall x, y, z \in X$,

(1) $x \leq y \Rightarrow \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y)$;

(2) $\tilde{t}_N(x \to z) \supseteq \tilde{t}_N(x \to (y \hookrightarrow z)) \cap \tilde{t}_N(y), \tilde{t}_N(x \hookrightarrow z) \supseteq \tilde{t}_N(x \hookrightarrow (y \to z)) \cap \tilde{t}_N(y)$;
$\tilde{i}_N(x \to z) \subseteq \tilde{i}_N(x \to (y \hookrightarrow z)) \cup \tilde{i}_N(y), \tilde{i}_N(x \hookrightarrow z) \subseteq \tilde{i}_N(x \hookrightarrow (y \to z)) \cup \tilde{i}_N(y)$;
$\tilde{f}_N(x \to z) \subseteq \tilde{f}_N(x \to (y \hookrightarrow z)) \cup \tilde{f}_N(y), \tilde{f}_N(x \hookrightarrow z) \subseteq \tilde{f}_N(x \hookrightarrow (y \to z)) \cup \tilde{f}_N(y)$;

(3) $\tilde{t}_N((x \to y) \hookrightarrow y) \supseteq \tilde{t}_N(x), \tilde{t}_N((x \hookrightarrow y) \to y) \supseteq \tilde{t}_N(x)$;
$\tilde{i}_N((x \to y) \hookrightarrow y) \subseteq \tilde{i}_N(x), \tilde{i}_N((x \hookrightarrow y) \to y) \subseteq \tilde{i}_N(x)$;
$\tilde{f}_N((x \to y) \hookrightarrow y) \subseteq \tilde{f}_N(x), \tilde{f}_N((x \hookrightarrow y) \to y) \subseteq \tilde{f}_N(x)$;

(4) $z \leq x \to y \Rightarrow \tilde{t}_N(x) \cap \tilde{t}_N(z) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \cup \tilde{i}_N(z) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$;
$z \leq x \hookrightarrow y \Rightarrow \tilde{t}_N(x) \cap \tilde{t}_N(z) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \cup \tilde{i}_N(z) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$.

Proof. (1) Let $x, y \in X$ with $x \leq y$. By Proposition 1, we know $x \to y = 1$ (or $x \hookrightarrow y = 1$). If N is a neutrosophic hesitant fuzzy filter of X, by Definition 14, we have $\tilde{t}_N(x) = \tilde{t}_N(1) \cap \tilde{t}_N(x) = \tilde{t}_N(x \to y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$ ($\tilde{t}_N(x) = \tilde{t}_N(1) \cap \tilde{t}_N(x) = \tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$). Thus, $\tilde{t}_N(x) \subseteq \tilde{t}_N(y)$.

Similarly, we have $\tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y)$.

(2) By Proposition 1, Definition 14, we know, $\forall x, y, z \in X$,

$$\tilde{t}_N(x \to z) \supseteq \tilde{t}_N(y \hookrightarrow (x \to z)) \cap \tilde{t}_N(y) = \tilde{t}_N(x \to (y \hookrightarrow z)) \cap \tilde{t}_N(y),$$
$$\tilde{t}_N(x \hookrightarrow z) \supseteq \tilde{t}_N(y \to (x \hookrightarrow z)) \cap \tilde{t}_N(y) = \tilde{t}_N(x \hookrightarrow (y \to z)) \cap \tilde{t}_N(y).$$

Similarly, we have, $\forall x, y, z \in X$:

$$\tilde{i}_N(x \to z) \subseteq \tilde{i}_N(x \to (y \hookrightarrow z)) \cup \tilde{i}_N(y), \tilde{i}_N(x \hookrightarrow z) \subseteq \tilde{i}_N(x \hookrightarrow (y \to z)) \cup \tilde{i}_N(y);$$
$$\tilde{f}_N(x \to z) \subseteq \tilde{f}_N(x \to (y \hookrightarrow z)) \cup \tilde{f}_N(y), \tilde{f}_N(x \hookrightarrow y) \subseteq \tilde{f}_N(x \hookrightarrow (y \to z)) \cup \tilde{f}_N(y).$$

(3) By Definition 1 and Definition 14, with regard to the function $\tilde{t}_N(x)$, we can obtain, $\forall x, y \in X$,

$$\begin{aligned}\tilde{t}_N((x \to y) \hookrightarrow y) &\supseteq \tilde{t}_N(x \to ((x \to y) \hookrightarrow y)) \cap \tilde{t}_N(x) \\ &= \tilde{t}_N((x \to y) \hookrightarrow (x \to y)) \cap \tilde{t}_N(x) \\ &= \tilde{t}_N(1) \cap \tilde{t}_N(x) \\ &= \tilde{t}_N(x).\end{aligned}$$

Similarly, we have $\tilde{t}_N((x \hookrightarrow y) \to y) \supseteq \tilde{t}_N(x)$.

With regard to the function $\tilde{i}_N(x)$, we can obtain, $\forall x, y \in X$,

$$\begin{aligned}\tilde{i}_N((x \to y) \hookrightarrow y) &\subseteq \tilde{i}_N(x \to ((x \to y) \hookrightarrow y)) \cup \tilde{i}_N(x) \\ &= \tilde{i}_N((x \to y) \hookrightarrow (x \to y)) \cup \tilde{i}_N(x) \\ &= \tilde{i}_N(1) \cup \tilde{i}_N(x) \\ &= \tilde{i}_N(x).\end{aligned}$$

Similarly, we have $\tilde{i}_N((x \hookrightarrow y) \to y) \subseteq \tilde{i}_N(x)$.

Similarly, with regard to the function $\tilde{f}_N(x)$, we can obtain $\tilde{f}_N((x \to y) \hookrightarrow y) \subseteq \tilde{f}_N(x)$, $\tilde{f}_N((x \hookrightarrow y) \to y) \subseteq \tilde{f}_N(x)$.

(4) Let $x, y, z \in X$ with $z \leq x \to y$. By Remark 1 and Definition 14, we can obtain:

$$\begin{aligned}\tilde{t}_N(x) \cap \tilde{t}_N(z) &= \tilde{t}_N(x) \cap (\tilde{t}_N(1) \cap \tilde{t}_N(z)) \\ &= \tilde{t}_N(x) \cap (\tilde{t}_N(z \hookrightarrow (x \to y)) \cap \tilde{t}_N(z)) \\ &\subseteq \tilde{t}_N(x) \cap \tilde{t}_N(x \to y), \\ &\subseteq \tilde{t}_N(y). \\ \tilde{i}_N(x) \cup \tilde{i}_N(z) &= \tilde{i}_N(x) \cup (\tilde{i}_N(1) \cup \tilde{i}_N(z)) \\ &= \tilde{i}_N(x) \cup (\tilde{i}_N(z \to (x \to y)) \cup \tilde{i}_N(z)) \\ &\supseteq \tilde{i}_N(x) \cup \tilde{i}_N(x \to y), \\ &\supseteq \tilde{i}_N(y).\end{aligned}$$

Similarly, we can obtain $\tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$.

Let $x, y, z \in X$ with $z \leq x \hookrightarrow y$. We can obtain $\tilde{t}_N(x) \cap \tilde{t}_N(z) \subseteq \tilde{t}_N(y)$, $\tilde{i}_N(x) \cup \tilde{i}_N(z) \supseteq \tilde{i}_N(y)$, $\tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$. The process of the proof is similar to the above. □

Theorem 13. *A neutrosophic hesitant fuzzy set $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X)\}$ is a neutrosophic hesitant fuzzy filter of X if and only if hesitant fuzzy sets $H_{\tilde{t}_N}, H_{\tilde{i}_N}, H_{\tilde{f}_N}$ satisfy the following conditions, respectively.*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{t}_N(x \to (y \hookrightarrow z)) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \to z), \tilde{t}_N(x \hookrightarrow (y \to z)) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \hookrightarrow z), \forall x, y, z \in X$;

(2) $\tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{i}_N(x \to (y \hookrightarrow z)) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \to z), \tilde{i}_N(x \hookrightarrow (y \to z)) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \hookrightarrow z), \forall x, y, z \in X$;

(3) $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x \to (y \hookrightarrow z)) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \to z), \tilde{f}_N(x \hookrightarrow (y \to z)) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \hookrightarrow z), \forall x, y, z \in X$.

Proof. **Necessity:** By Theorem 9, Theorem 12 and Definition 14, (1)$\sim$(3) holds.

Sufficiency: (1) $\forall x, y, z \in X$, by Proposition 1, we can obtain $\tilde{t}_N(y) = \tilde{t}_N(1 \to y) \supseteq \tilde{t}_N(1 \to (x \hookrightarrow y)) \cap \tilde{t}_N(x) = \tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x)$ and $\tilde{t}_N(y) = \tilde{t}_N(1 \hookrightarrow y) \supseteq \tilde{t}_N(1 \hookrightarrow (x \to y)) \cap \tilde{t}_N(x) = \tilde{t}_N(x \to y) \cap \tilde{t}_N(x)$. We have $\tilde{i}_N(x) \supseteq \tilde{i}_N(1)$ for all $x \in X$. Thus, $H_{\tilde{t}_N}$ is a hesitant fuzzy filter of X.

(2) $\forall x, y, z \in X$, by Proposition 1, we can obtain $\tilde{i}_N(y) = \tilde{i}_N(1 \to y) \subseteq \tilde{i}_N(1 \to (x \hookrightarrow y)) \cup \tilde{i}_N(x) = \tilde{i}_N(x \hookrightarrow y) \cup \tilde{i}_N(x)$; thus, we have $(1 - \tilde{i}_N(x \hookrightarrow y)) \cap (1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{i}_N(y))$.

Similarly, we can have $(1 - \tilde{i}_N(x \to y)) \cap (1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{i}_N(y))$.

It is easy to obtain $(1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{t}_N(1))$ for all $x \in X$. Thus, $H_{\tilde{i}_N}$ is a hesitant fuzzy filter of X.

(3) We have that $H_{\tilde{f}_N}$ is a hesitant fuzzy filter of X. The process of the proof is similar (2).

Therefore, $H_{\tilde{t}_N}, H_{\tilde{i}_N}, H_{\tilde{f}_N}$ are hesitant fuzzy filters of X. By Theorem 9, we know that N is a neutrosophic hesitant fuzzy filter of X. □

Theorem 14. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ be a neutrosophic hesitant fuzzy filter of X. Then:*

$$\prod_{k=1}^{n} x_k \to y = 1 \Rightarrow \tilde{t}_N(y) \supseteq \bigcap_{k=1}^{n} \tilde{t}_N(x_k), \tilde{i}_N(y) \subseteq \bigcup_{i=k}^{n} \tilde{i}_N(x_k), \tilde{f}_N(y) \subseteq \bigcup_{k=1}^{n} \tilde{f}_N(x_k).$$

where $n \in \mathbb{N}$,

$$\prod_{k=1}^{n} x_k \to y = x_n \to (x_{n-1} \to (\cdots (x_1 \to y) \cdots)).$$

Proof. If N is a neutrosophic hesitant fuzzy filter of X:

(i) By Theorem 12, we know that $\tilde{t}_N(x_1) \subseteq \tilde{t}_N(y), \tilde{i}_N(x_1) \supseteq \tilde{i}_N(y), \tilde{f}_N(x_1) \supseteq \tilde{f}_N(y)$ for $n = 1$.

(ii) By Theorem 12, we know that $\tilde{t}_N(x_2) \subseteq \tilde{t}_N(x_1 \to y), \tilde{i}_N(x_2) \supseteq \tilde{i}_N(x_1 \to y), \tilde{f}_N(x_2) \supseteq \tilde{f}_N(x_1 \to y)$ for $n = 2$. By Definition 14, we have $\tilde{t}_N(x_1) \cap \tilde{t}_N(x_1 \to y) \subseteq \tilde{t}_N(y), \tilde{i}_N(x_1) \cup \tilde{i}_N(x_1 \to y) \supseteq \tilde{i}_N(y), \tilde{f}_N(x_1) \cup \tilde{f}_N(x_1 \to y) \supseteq \tilde{f}_N(y)$. Thus, $\tilde{t}_N(x_1) \cap \tilde{t}_N(x_2) \subseteq \tilde{t}_N(y), \tilde{i}_N(x_1) \cup \tilde{i}_N(x_2) \supseteq \tilde{i}_N(y), \tilde{f}_N(x_1) \cup \tilde{f}_N(x_2) \supseteq \tilde{f}_N(y)$.

(iii) Suppose that the above formula is true for $n = j$; thus, $\prod_{k=1}^{j} x_k \to y = 1, \forall x_j, \cdots, x_1, y \in X$, and we can obtain $\bigcap_{k=1}^{j} \tilde{t}_N(x_k) \subseteq \tilde{t}_N(y), \bigcup_{k=1}^{j} \tilde{i}_N(x_k) \supseteq \tilde{i}_N(y), \bigcup_{k=1}^{j} \tilde{f}_N(x_k) \supseteq \tilde{f}_N(y)$. Therefore, suppose that $\prod_{k=1}^{j+1} x_k \to y = 1, \forall x_{j+1}, \cdots, x_1, y \in X$, then we have $\bigcap_{k=2}^{j+1} \tilde{t}_N(x_k) \subseteq \tilde{t}_N(x_1 \to y), \bigcup_{k=2}^{j+1} \tilde{i}_N(x_k) \supseteq \tilde{i}_N(x_1 \to y), \bigcup_{k=2}^{j+1} \tilde{f}_N(x_k) \supseteq \tilde{f}_N(x_1 \to y)$. By Definition 14, we can obtain:

$$\begin{aligned}
\tilde{t}_N(y) &\supseteq \tilde{t}_N(x_1) \cap \tilde{t}_N(x_1 \to y) \supseteq \tilde{t}_N(x_1) \cap (\bigcap_{k=2}^{j+1} \tilde{t}_N(x_k)) = \bigcap_{k=1}^{j+1} \tilde{t}_N(x_k), \\
\tilde{i}_N(y) &\subseteq \tilde{i}_N(x_1) \cup \tilde{i}_N(x_1 \to y) \subseteq \tilde{i}_N(x_1) \cup (\bigcup_{k=2}^{j+1} \tilde{i}_N(x_k)) = \bigcup_{k=1}^{j+1} \tilde{i}_N(x_k), \\
\tilde{f}_N(y) &\subseteq \tilde{f}_N(x_1) \cup \tilde{f}_N(x_1 \to y) \subseteq \tilde{f}_N(x_1) \cup (\bigcup_{k=2}^{j+1} \tilde{f}_N(x_k)) = \bigcup_{k=1}^{j+1} \tilde{f}_N(x_k),
\end{aligned}$$

which complete the proof. □

Corollary 3. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)))|x \in X)\}$ be a neutrosophic hesitant fuzzy filter of X. Then:*

$$\prod_{k=1}^{n} x_k * y = 1 \Rightarrow \tilde{t}_N(y) \supseteq \bigcap_{k=1}^{n} \tilde{t}_N(x_k), \tilde{i}_N(y) \subseteq \bigcup_{k=1}^{n} \tilde{i}_N(x_k), \tilde{f}_N(y) \subseteq \bigcup_{k=1}^{n} \tilde{f}_N(x_k).$$

where "$*$" represents any binary operation "$\to$" or "$\hookrightarrow$" on X, $n \in \mathbb{N}$,

$$\prod_{k=1}^{n} x_k * y = x_n * (x_{n-1} * (\cdots (x_1 * y) \cdots)).$$

Theorem 15. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ be a neutrosophic hesitant fuzzy filter of X and X be a pseudo-BCK algebra, then N is a neutrosophic hesitant fuzzy subalgebra of X.*

Proof. If $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ is a neutrosophic hesitant fuzzy filter of X, then we can obtain $\forall x, y \in X$,

$$\begin{aligned}
\tilde{t}_N(x \to y) &\supseteq \tilde{t}_N(y \hookrightarrow (x \to y)) \cap \tilde{t}_N(y) \\
&= \tilde{t}_N(x \to (y \hookrightarrow y)) \cap \tilde{t}_N(y) \\
&= \tilde{t}_N(x \to 1) \cap \tilde{t}_N(y) \\
&\supseteq \tilde{t}_N(x) \cap \tilde{t}_N(y). \\
\tilde{i}_N(x \to y) &\subseteq \tilde{i}_N(y \hookrightarrow (x \to y)) \cup \tilde{i}_N(y) \\
&= \tilde{i}_N(x \to (y \hookrightarrow y)) \cup \tilde{i}_N(y) \\
&= \tilde{i}_N(x \to 1) \cup \tilde{i}_N(y) \\
&\subseteq \tilde{i}_N(x) \cup \tilde{i}_N(y).
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_N(x \to y) &\subseteq \tilde{f}_N(y \hookrightarrow (x \to y)) \cup \tilde{f}(y) \\
&= \tilde{f}_N(x \to (y \hookrightarrow y)) \cup \tilde{f}_N(y) \\
&= \tilde{f}_N(x \to 1) \cup \tilde{f}_N(y) \\
&\subseteq \tilde{f}_N(x) \cup \tilde{f}_N(y).
\end{aligned}$$

Similarly, we can obtain $\tilde{t}_N(x \hookrightarrow y) \supseteq \tilde{t}_N(x) \cap \tilde{t}_N(y), \tilde{i}_N(x \hookrightarrow y) \subseteq \tilde{i}_N(x) \cup \tilde{i}_N(y), \tilde{f}_N(x \hookrightarrow y) \subseteq \tilde{f}_N(x) \cup \tilde{f}_N(y)$. Thus, N is a neutrosophic hesitant fuzzy subalgebra of X. □

Theorem 16. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ be a neutrosophic hesitant fuzzy closed filter of X. Then, N is a neutrosophic hesitant fuzzy subalgebra of X.*

Proof. The process of proof is similar to Theorem 15. □

If $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ is a neutrosophic hesitant fuzzy subalgebra of X, then N may not be a neutrosophic hesitant fuzzy filter of X.

Example 6. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 1 and 2. Then, $(X; \to, \hookrightarrow, 1)$ is a pseudo-BCI algebra. N is a neutrosophic hesitant fuzzy subalgebra of X. However, N is not a neutrosophic hesitant fuzzy filter of X. Since $\tilde{t}(b \to a) \cap \tilde{t}(b) = [\frac{1}{3}, \frac{1}{2}]$, $\tilde{t}(a) = [\frac{1}{3}, \frac{1}{4}]$, we cannot obtain $\tilde{t}(b \to a) \cap \tilde{t}(b) \subseteq \tilde{t}(a)$.*

Definition 16. *$N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ is a neutrosophic hesitant fuzzy set on X. Define a neutrosophic hesitant fuzzy set $N^{(a,b)} = \{(x, \tilde{t}_N^{(a,b)}(x), \tilde{i}_N^{(a,b)}(x), \tilde{f}_N^{(a,b)}(x))|x \in X\}$ by $\forall a, b \in X$,*

$$\tilde{t}_N^{(a,b)} : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \psi_1, & a \to (b \to x) = 1, a \hookrightarrow (b \hookrightarrow x) = 1; \\ \psi_2, & otherwise: \end{cases}$$

$$\tilde{i}_N^{(a,b)} : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \psi_3, & a \to (b \to x) = 1, a \hookrightarrow (b \hookrightarrow x) = 1; \\ \psi_4, & otherwise: \end{cases}$$

$$\tilde{f}_N^{(a,b)} : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \psi_5, & a \to (b \to x) = 1, a \hookrightarrow (b \hookrightarrow x) = 1; \\ \psi_6, & otherwise: \end{cases}$$

where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6 \in P([0,1])$, $\psi_1 \supseteq \psi_2, \psi_3 \subseteq \psi_4, \psi_5 \subseteq \psi_6$. Then, $N^{(a,b)}$ is called a generated neutrosophic hesitant fuzzy set.

A generated neutrosophic hesitant fuzzy set $N^{(a,b)}$ may not be a neutrosophic hesitant fuzzy filter of X.

Example 7. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 1 and 2. Then, $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCI algebra. N is a neutrosophic hesitant fuzzy set of X. However, $N^{(a,b)}$ is not a neutrosophic hesitant fuzzy filter of X. Since $\tilde{t}^{(1,a)}(a \rightarrow b) \cap \tilde{t}^{(1,a)}(a) = [0,1]$, $\tilde{t}^{(1,a)}(b) = [\frac{1}{3}, \frac{2}{3}]$, we cannot obtain $\tilde{t}^{(1,a)}(a \rightarrow b) \cap \tilde{t}^{(1,a)}(a) \subseteq \tilde{t}^{(1,a)}(b)$.*

Theorem 17. *Let X be a pseudo-BCK algebra. If X is a type-2 positive implicative pseudo-BCK algebra, then $N^{(a,b)}$ is a neutrosophic hesitant fuzzy filter of X for all $a, b \in X$.*

Proof. If X is a pseudo-BCK algebra, (1) by Definition 1 and Proposition 1, we can obtain $a \rightarrow (b \rightarrow 1) = 1$ $(a \hookrightarrow (b \hookrightarrow 1) = 1)$. $\tilde{t}_N^{(a,b)}(1) = \psi_1 \supseteq \tilde{t}_N^{(a,b)}(x), \tilde{i}_N^{(a,b)}(1) = \psi_3 \subseteq \tilde{i}_N^{(a,b)}(x), \tilde{f}_N^{(a,b)}(1) = \psi_5 \subseteq \tilde{f}_N^{(a,b)}(x)$ for all $x \in X$.

(2) (i) Let $x, y \in X$ with $a \rightarrow (b \rightarrow x) \neq 1$ or $a \hookrightarrow (b \hookrightarrow x) \neq 1$ or $a \rightarrow (b \rightarrow (x \rightarrow y)) \neq 1$ or $a \hookrightarrow (b \hookrightarrow (x \rightarrow y)) \neq 1$. Thus, we can obtain:

$$\tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \rightarrow y) = \psi_2 \subseteq \tilde{t}_N^{(a,b)}(y), \tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \hookrightarrow y) = \psi_2 \subseteq \tilde{t}_N^{(a,b)}(y);$$
$$\tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \rightarrow y) = \psi_4 \supseteq \tilde{i}_N^{(a,b)}(y), \tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \hookrightarrow y) = \psi_4 \supseteq \tilde{i}_N^{(a,b)}(y);$$
$$\tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \rightarrow y) = \psi_6 \supseteq \tilde{f}_N^{(a,b)}(y), \tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \hookrightarrow y) = \psi_6 \supseteq \tilde{f}_N^{(a,b)}(y).$$

(ii) Let $x, y \in X$ with $a \rightarrow (b \rightarrow x) = 1$, $a \hookrightarrow (b \hookrightarrow x) = 1$ and $a \rightarrow (b \rightarrow (x \rightarrow y)) = 1$, $a \hookrightarrow (b \hookrightarrow (x \hookrightarrow y)) = 1$. Then, by Proposition 1 and Definition 4, we can obtain:

$$\begin{aligned}
&\tilde{t}_N^{(a,b)}(a \hookrightarrow (b \hookrightarrow y)) \\
=&\tilde{t}_N^{(a,b)}(1 \rightarrow (a \hookrightarrow (b \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}((a \hookrightarrow (b \hookrightarrow x)) \rightarrow (a \hookrightarrow (b \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \hookrightarrow ((b \hookrightarrow x) \rightarrow (b \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \hookrightarrow (b \hookrightarrow (x \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(1).
\end{aligned}$$

$$\begin{aligned}
&\tilde{t}_N^{(a,b)}(a \rightarrow (b \rightarrow y)) \\
=&\tilde{t}_N^{(a,b)}(1 \hookrightarrow (a \rightarrow (b \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(((a \rightarrow (b \rightarrow x)) \hookrightarrow (a \rightarrow (b \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \rightarrow ((b \rightarrow x) \hookrightarrow (b \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \rightarrow (b \rightarrow (x \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(1).
\end{aligned}$$

Therefore, we can obtain,

$$\tilde{t}_N^{(a,b)}(y) = \psi_1 = \tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \rightarrow y), \tilde{t}_N^{(a,b)}(y) = \psi_1 = \tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \hookrightarrow y).$$

Similarly, we can obtain,

$$\tilde{i}_N^{(a,b)}(y) = \psi_3 = \tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \rightarrow y), \tilde{i}_N^{(a,b)}(y) = \psi_3 = \tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \hookrightarrow y);$$
$$\tilde{f}_N^{(a,b)}(y) = \psi_5 = \tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \rightarrow y), \tilde{f}_N^{(a,b)}(y) = \psi_5 = \tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \hookrightarrow y).$$

This means that $N^{(a,b)}$ is a neutrosophic hesitant fuzzy filter of X. □

Example 8. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 7 and 8. Then, $(X; \rightarrow, \hookrightarrow, 1)$ is a type-2 positive implicative pseudo-BCI algebra. Let N be a neutrosophic hesitant fuzzy set. We take b, c as* *an example; thus, we have* $\{b, c, d, 1\}$ *satisfy* $d \to (c \to x) = 1, d \hookrightarrow (c \hookrightarrow x) = 1$. *Let* $\psi_1 = [0.1, 0.4]$, $\psi_2 = [0.2, 0.3]$, $\psi_3 = [0.4, 0.5]$, $\psi_4 = [0.3, 0.6]$, $\psi_5 = [0.2, 0.8]$, $\psi_6 = [0.1, 0.9]$,

$$N^{(d,c)} = \{(1, \psi_1, \psi_3, \psi_5), (a, \psi_2, \psi_4, \psi_6), (b, \psi_1, \psi_3, \psi_5), (c, \psi_1, \psi_3, \psi_5), (e, \psi_1, \psi_3, \psi_5)\} = \{(1, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8]), (a, [0.2, 0.3], [0.3, 0.6], [0.1, 0.9]), (b, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8]), (c, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8]), (d, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8])\}.$$

Then, we can obtain that $N^{(d,c)}$ *is a neutrosophic hesitant fuzzy filter of X.*

Table 7. $\to$.

$\to$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	1	1	1
c	a	d	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Table 8. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	1	1	1
c	a	d	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Theorem 18. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}(x), \tilde{f}(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy filter of X. Then,* $X_N^{(5)}(a) = \{x | \tilde{t}_N(a) \subseteq \tilde{t}_N(x), \tilde{i}_N(a) \supseteq \tilde{i}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x)\}$ *is a filter of X for all* $a \in X$.

Proof. (1) Let $x, y \in X$ with $x, x \to y \in X_N^5(a)$. Then, we have $\tilde{t}_N(a) \subseteq \tilde{t}_N(x), \tilde{t}_N(a) \subseteq \tilde{t}_N(x \to y)$. Since $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy filter, thus we have $\tilde{t}_N(a) \subseteq \tilde{t}_N(x) \cap \tilde{t}_N(x \to y) \subseteq \tilde{t}_N(y) \subseteq \tilde{t}_N(1)$. Similarly, we can get $\tilde{i}_N(a) \supseteq \tilde{i}_N(x) \cup \tilde{i}(x \to y) \supseteq \tilde{i}_N(y) \supseteq \tilde{i}_N(1), \tilde{f}_N(a) \supseteq \tilde{f}_N(x) \cup \tilde{f}_N(x \to y) \supseteq \tilde{f}_N(y) \supseteq \tilde{f}_N(1)$.

(2) Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in X_N^{(5)}(a)$; we have $\tilde{t}_N(a) \subseteq \tilde{t}_N(x) \cap \tilde{t}_N(x \hookrightarrow y) \subseteq \tilde{t}_N(y) \subseteq \tilde{t}_N(1), \tilde{i}_N(a) \supseteq \tilde{i}_N(x) \cup \tilde{i}_N(x \hookrightarrow y) \supseteq \tilde{i}_N(y) \supseteq \tilde{i}_N(1), \tilde{f}_N(a) \supseteq \tilde{f}_N(x) \cup \tilde{f}_N(x \hookrightarrow y) \supseteq \tilde{f}_N(y) \supseteq \tilde{f}_N(1)$.

This means that $X_N^{(5)}(a)$ satisfies the conditions of Definition 2 (F1), (F2) and (F3); $X_N^{(5)}(a)$ is a filter of X. $\square$

Example 9. *Let* $X = \{a, b, c, d, 1\}$ *with two binary operations in Tables 5 and 6. Then,* $(X; \to, \hookrightarrow, 1)$ *is a pseudo-BCI algebra. Let:*

$$N = \{(1, [0, 1], [0, \tfrac{3}{7}], [0, \tfrac{1}{10}]), (a, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]), (b, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]), (c, [0, \tfrac{1}{3}], [0, \tfrac{3}{5}], [0, \tfrac{1}{4}]), (d, [0, \tfrac{3}{4}]), [0, \tfrac{3}{6}], [0, \tfrac{1}{5}])\}.$$

Then, N is a neutrosophic hesitant fuzzy filter of X. Let $X_N^{(5)}(c) = \{c, d, 1\}$. *It is easy to get that* $X_N^{(5)}(a)$ *is a filter.*

5. Conclusions

In this paper, the neutrosophic hesitant fuzzy set theory was applied to pseudo-BCI algebra, and the neutrosophic hesitant fuzzy subalgebras (filters) in pseudo-BCI algebras were developed. The relationships between neutrosophic hesitant fuzzy subalgebras (filters) and hesitant fuzzy subalgebras (filters) was discussed, and some properties were demonstrated. In future work, different types of neutrosophic hesitant fuzzy filters will be defined and discussed.

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A Classical Group of Neutrosophic Triplet Groups Using $\{Z_{2p}, \times\}$

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W.B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2018). A Classical Group of Neutrosophic Triplet Groups Using (Z2p, X). *Symmetry* 10, 194; DOI: 10.3390/sym10060194

Abstract: In this paper we study the neutrosophic triplet groups for $a \in Z_{2p}$ and prove this collection of triplets $(a, neut(a), anti(a))$ if trivial forms a semigroup under product, and semi-neutrosophic triplets are included in that collection. Otherwise, they form a group under product, and it is of order $(p-1)$, with $(p+1, p+1, p+1)$ as the multiplicative identity. The new notion of pseudo primitive element is introduced in Z_{2p} analogous to primitive elements in Z_p, where p is a prime. Open problems based on the pseudo primitive elements are proposed. Here, we restrict our study to Z_{2p} and take only the usual product modulo $2p$.

Keywords: neutrosophic triplet groups; semigroup; semi-neutrosophic triplets; classical group of neutrosophic triplets; S-semigroup of neutrosophic triplets; pseudo primitive elements

1. Introduction

Fuzzy set theory was introduced by Zadeh in [1] and was generalized to the Intuitionistic Fuzzy Set (IFS) by Atanassov [2]. Real-world, uncertain, incomplete, indeterminate, and inconsistent data were presented philosophically as a neutrosophic set by Smarandache [3], who also studied the notion of neutralities that exist in all problems. Many [4–7] have studied neutralities in neutrosophic algebraic structures. For more about this literature and its development, refer to [3–10].

It has not been feasible to relate this neutrosophic set to real-world problems and the engineering discipline. To implement such a set, Wang et al. [11] introduced a Single-Valued Neutrosophic Set (SVNS), which was further developed into a Double Valued Neutrosophic Set (DVNS) [12] and a Triple Refined Indeterminate Neutrosophic Set (TRINS) [13]. These sets are capable of dealing with the real world's indeterminate data, and fuzzy sets and IFSs are not.

Smarandache [14] presents recent developments in neutrosophic theories, including the neutrosophic triplet, the related triplet group, the neutrosophic duplet, and the duplet set. The new, innovative, and interesting notion of the neutrosophic triplet group, which is a group of three elements, was introduced by Florentin Smarandache and Ali [10]. Since then, neutrosophic triplets have been a field of interest that many researchers have worked on [15–22]. In [21], cancellable neutrosophic triplet groups were introduced, and it was proved that it coincides with the group. The paper also discusses weak neutrosophic duplets in BCI algebras. Notions such as the neutrosophic triplet coset and its connection with the classical coset, neutrosophic triplet quotient groups, and neutrosophic triplet normal subgroups were defined and studied by [20].

Using the notion of neutrosophic triplet groups introduced in [10], which is different from classical groups, several interesting structural properties are developed and defined in this paper. Here, we study the neutrosophic triplet groups using only $\{Z_{2p}, \times\}$, p is a prime and the operation $\times$ is product modulo $2p$. The properties as a neutrosophic triplet group under the inherited operation $\times$

is studied. This leads to the definition of a semi-neutrosophic triplet. However, it has been proved that semi-neutrosophic triplets form a semigroup under $\times$, but the neutrosophic triplet groups, which are nontrivial and are not semi-neutrosophic triplets, form a classical group of neutrosophic triplets under $\times$.

This paper is organized into five sections. Section 2 provides basic concepts. In Section 3, we study neutrosophic triplets in the case of Z_{2p}, where p is an odd prime. Section 4 defines the semi-neutrosophic triplet and shows several interesting properties associated with the classical group of neutrosophic triplets. The final section provides the conclusions and probable applications.

2. Basic Concepts

We recall here basic definitions from [10].

Definition 1. *Consider $(S, \times)$ to be a nonempty set with a closed binary operation. S is called a neutrosophic triplet set if for any $x \in S$ there will exist a neutral of x called $neut(x)$, which is different from the algebraic unitary element (classical), and an opposite of x called $anti(x)$, with both $neut(x)$ and $anti(x)$ belonging to S such that*

$$x * neut(x) = neut(x) * x = x$$

and

$$x * anti(x) = anti(x) * x = neut(x).$$

The elements x, $neut(x)$, and $anti(x)$ are together called a neutrosophic triplet group, denoted by $(x, neut(x), anti(x))$.

$neut(x)$ denotes the neutral of x. x is the first coordinate of a neutrosophic triplet group and not a neutrosophic triplet. y is the second component, denoted by $neut(x)$, of a neutrosophic triplet if there are elements x and $z \in S$ such that $x * y = y * x = x$ and $x * z = z * x = y$. Thus, (x, y, z) is the neutrosophic triplet.

We know that $(neut(x), neut(x), neut(x))$ is a neutrosophic triplet group. Let $\{S, *\}$ be the neutrosophic triplet set. If $(S, *)$ is well defined and for all $x, y \in S$, $x * y \in S$, and $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$, then $\{S, *\}$ is defined as the neutrosophic triplet group. Clearly, $\{S, *\}$ is not a group in the classical sense.

In the following section, we define the notion of a semi-neutrosophic triplet, which is different from neutrosophic duplets and the classical group of neutrosophic triplets of $\{Z_{2p}, \times\}$, and derive some of its interesting properties.

3. The Classical Group of Neutrosophic Triplet Groups of $\{Z_{2p}, \times\}$ and Its Properties

Here we define the classical group of neutrosophic triplets using $\{Z_{2p}, \times\}$, where p is an odd prime. The collection of all nontrivial neutrosophic triplet groups forms a classical group under the usual product modulo $2p$, and the order of that group is $p - 1$. We also derive interesting properties of such groups.

We will first illustrate this situation with some examples.

Example 1. *Let $S = \{Z_{22}, \times\}$ be the semigroup under $\times$ modulo 22. Clearly, 11 and 12 are the only idempotents or neutral elements of Z_{22}. The idempotent $11 \in Z_{22}$ yields only a trivial neutrosophic triplet $(11, 11, 11)$ for $11 \times 21 = 11$, where 21 is a unit in Z_{22}. The other nontrivial neutrosophic triplets associated with the neutral element 12 are $H = \{(2, 12, 6), (6, 12, 2), (4, 12, 14), (14, 12, 4), (16, 12, 20), (20, 12, 16), (12, 12, 12), (10, 12, 10), (8, 12, 18), (18, 12, 8)\}$. It is easily verified that $\{H, \times\}$ is a classical group of order 10 under component-wise multiplication modulo 22, with $(12, 12, 12)$ as the identity element. $(12, 12, 12) \times (12, 12, 12) = (12, 12, 12)$ product modulo 22. Likewise,*

$$(2, 12, 6) \times (2, 12, 6) = (4, 12, 14),$$

$$
\begin{gathered}
\textit{and } (2,12,6)\times(4,12,14)=(8,12,18)\,; \\
(2,12,6)\times(8,12,18)=(16,12,20)\,, \\
\textit{and } (2,12,6)\times(16,12,20)=(10,12,10)\,; \\
(10,12,10)\times(2,12,6)=(20,12,16)\,, \\
\textit{and } (2,12,6)\times(20,12,16)=(18,12,8)\,; \\
(2,12,6)\times(18,12,8)=(14,12,4)\,, \\
\textit{and } (2,12,6)\times(14,12,4)=(6,12,2)\,; \\
(6,12,2)\times(2,12,6)=(12,12,12)\,, \\
\textit{and } (2,12,6)^{10}=(12,12,12)\,.
\end{gathered}
$$

Thus, H is a cyclic group of order 10.

Example 2. *Let $S=\{Z_{14},\times\}$ be the semigroup under product modulo* 14. *The neutral elements or idempotents of Z_{14} are* 7 *and* 8. *The neutrosophic triplets are*

$$H=\{(2,8,4)\,,(4,8,2)\,,(6,8,6)\,,(10,8,12)\,,(12,8,10)\,,(8,8,8)\},$$

associated with the neutral element 8. *H is a classical group of order* 6. *Clearly,*

$$
\begin{gathered}
(10,8,12)\times(10,8,12)=(2,8,4), \\
(10,8,12)\times(2,8,4)=(6,8,6), \\
(10,8,12)\times(6,8,6)=(4,8,2), \\
(10,8,12)\times(4,8,2)=(12,8,10), \textit{ and} \\
(10,8,12)\times(12,8,10)=(8,8,8).
\end{gathered}
$$

Thus, H is generated by $(10,8,12)$ *as* $(10,8,12)^6=(8,8,8)$, *and* $(8,8,8)$ *is the multiplicative identity of the classical group of neutrosophic triplets.*

Example 3. *Let $S=\{Z_{38},\times\}$ be the semigroup under product modulo* 38. $19,20\in Z_{38}$ *are the idempotents of Z_{38}.*

$$
\begin{gathered}
H=\{(2,20,10)\,,(10,20,2)\,,(4,20,24)\,,(24,20,4)\,,(20,20,20)\,,(8,20,12)\,, \\
(12,20,8)\,,(16,20,6)\,,(6,20,16)\,,(32,20,22)\,,(22,20,32)\,,(18,20,18)\,, \\
(34,20,14)\,,(14,20,34)\,,(26,20,28)\,,(28,20,26)\,,(30,2036)\,,(36,20,30)\}
\end{gathered}
$$

is the classical group of neutrosophic triplets with $(20,20,20)$ *as the identity element of H.*

In view of all these example, we have the following results.

Theorem 1. *Every semigroup $\{Z_{2p},\times\}$, where p is an odd prime, has only two idempotents: p and $p+1$.*

Proof. Clearly, p is a prime of the form $2n+1$ in Z_{2p}.

$$
\begin{aligned}
p^2=(2n+1)^2 &= 4n^2+4n+1 \\
&= 4n^2+2n+2n+1 \\
&= 4n^2+2n+p \\
&= 2n\,(2n+1)+p \\
&= 2np+p \\
&= p.
\end{aligned}
$$

Thus, p is an idempotent in Z_{2p}. Consider $p+1 \in Z_{2p}$:

$$\begin{aligned}(p+1)^2 &= p^2+2p+1\\ &= p^2+1\\ &= p+1 \quad \text{as} \quad p^2=p.\end{aligned}$$

Thus, p and $p+1$ are the only idempotents of Z_{2p}. In fact, Z_{2p} has no other nontrivial idempotent.

Let $x \in Z_{2p}$ be an idempotent. This implies that x must be even as all odd elements other than p are units.

Let $x = 2n$ (where n is an integer), and $2 < n < p-1$ such that $x^2 = 4n^2 = x = 2n$, which implies that $2n\,(2n-1) = 0$.

This is zero only if $2n - 1 = p$ as $2n - 1$ is odd. Otherwise, $2n = 0$, which is not possible, as n is even and n is not equal to 0, $x \neq 0$, so $2n - 1 = p$. That is, $x = 2n = p + 1$ is the only possibility. Otherwise, $x = 0$, which is a contradiction.

Thus, Z_{2p} has only two idempotents, p and $p+1$. □

Theorem 2. *Let G = $\{Z_{2p}, \times\}$, where p is an odd prime, be the semigroup under $\times$, product modulo $2p$.*

1. *If $a \in Z_{2p}$ has neut (a) and anti (a), then a is even.*
2. *The only nontrivial neutral element is $p+1$ for all a, which contributes to neutrosophic triplet groups in G.*

Proof. Let a in G be such that $a \times neut\,(a) = a$ if a is odd and $a \neq p$. Then a^{-1} exists in Z_{2p} and we have $neut\,(a) = 1$, but $neut\,(a) \neq 1$ by definition. Hence the result is true.

Further, we know $neut\,(a) \times neut\,(a) = neut\,(a)$, that is $neut\,(a)$ is an idempotent. This is possible if and only if $a = p+1$ or p.

Clearly, $a = p$ is ruled out because $ap = 0$ for all even a in Z_{2p}, hence the claim.

Thus, $neut\,(a) = p+1$ is the only neutral element for all relevant a in Z_{2p}. □

Definition 2. *Let $\{Z_{2p}, \times\}$ be the semigroup under multiplication modulo $2p$, where p is an odd prime. $H = \{(a, neut\,(a), anti\,(a))\,|a \in 2Z_{2p} \setminus \{0\}\}$. $\{H, \times\}$ is the collection of all neutrosophic triplet groups. H has the multiplicative identity $(p+1, p+1, p+1)$ under the component-wise product modulo $2p$. H is defined as the classical group of neutrosophic triplets.*

We have already given examples of them. It is important to mention this definition is valid only for Z_{2p} under the product modulo $2p$ where p is an odd prime.

Example 4. *Let $S = \{Z_{46}, \times\}$ be the semigroup under product modulo 46. Let*

$$\begin{gathered}H = \{(24,24,24), (2,24,12), (12,24,2), (4,24,6), (6,24,4), (8,24,26),\\ (26,24,8), (16,24,36), (36,24,16), (32,24,18), (18,24,32), (22,24,22),\\ (10,24,30), (14,24,28), (28,24,14), (30,24,10), (20,24,38), (38,24,20),\\ (34,24,44), (44,24,34), (40,24,42), (42,24,40)\}\end{gathered}$$

be the classical group of neutrosophic triplets, with $(24,24,24)$ as the identity under $\times$. $o\,(H) = 22$.

In view of all of this, we have to define the following for Z_{2p}.

Definition 3. *Let $\{Z_{2p}, \times\}$ be the semigroup under product modulo $2p$, where p is an odd prime. Let $K = \{2, 4, \ldots, 2p-2\}$ be the set of all even elements of Z_{2p}. For $p+1 \in K$, $x \times p+1 = x, \forall\, x \in K$. There also exists a $y \in K$ such that $y^{p-1} = p+1$. We define this y as the pseudo primitive element of $K \subseteq Z_{2p}$.*

Note: We can define pseudo primitive elements only for Z_{2p} where p is an odd prime and not for any Z_n, where n is an even integer that is analogous to primitive elements in Z_p, where p is a prime.

We will illustrate this situation with some examples.

Example 5. *Let $\{Z_6, \times\}$ be the modulo semigroup. For $K = \{2,4\}$, 2 is the pseudo primitive element of $K \subseteq Z_6$.*

Example 6. *Let $\{Z_{14}, \times\}$ be the modulo semigroup under product $\times$, modulo 14. Consider $K = \{2,4,6,8,10,12\} \subseteq Z_{14}$. Then 10 is the pseudo primitive element of $K \subseteq Z_{14}$.*

Example 7. *Let $\{Z_{34}, \times\}$ be the semigroup under product modulo integer 34. 10 is the pseudo primitive element of $K = \{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32\} \subseteq Z_{34}$.*

Similarly, for $\{Z_{38}, \times\}$, 10 is the pseudo primitive element of $K = 2Z_{38} \setminus \{0\} \subseteq Z_{38}$.

However, in the case of Z_{22}, Z_{58}, and Z_{26}, 2 is the pseudo primitive element for these semigroups.

We leave it as an open problem to find the number of such pseudo primitive elements of $K = \{2,4,6,\ldots,2(p-1)\}$ of Z_{2p}.

We have the following theorem.

Theorem 3. *Let $S = \{Z_{2p}, \times\}$ be the semigroup under product modulo $2p$, where p is an odd prime.*

1. *$K = \{2,4,\ldots,2p-2\} \subseteq Z_{2p}$ has a pseudo primitive element $x \in K$ with $x^{p-1} = p+1$, where $p+1$ is the multiplicative identity of K.*
2. *K is a cyclic group under $\times$ of order $p-1$ generated by that x, and $p+1$ is the identity element of K.*
3. *S is a Smarandache semigroup.*

Proof. Consider Z_{2p}, where p is an odd prime. Let $K = \{2,4,6,\ldots,2p-2\} \subseteq Z_{2p}$. For any $x \in K$, $(p+1)x = px + x = x$ is$px = 0(mod\ 2p)$, where x is even. Thus, $p+1$ is the identity element of Z_{2p}. There is a $x \in K$ such that $x^{p-1} = p+1$ using the principle of $2p \equiv 0$, where x is even. This x is the pseudo primitive element of K.

This $x \in K$ proves part (2) of the claim.

Since K is a group under $\times$ and $K \subseteq \{Z_{2p}, \times\}$, by the definition of Smarandache semigroup [4], S is an S-semigroup, so (3) is true. □

Next, we prove that the following theorem for our research pertains to the classical group of neutrosophic triplets and their structure.

Theorem 4. *Let $S = \{Z_{2p}, \times\}$ be the semigroup. Then*

$$H = \{(a, neut(a), anti(a))\,|a \in 2Z_{2p} \setminus \{0\}\},$$

is the classical group of neutrosophic triplets, which is cyclic and of the order $p-1$.

Proof. Clearly, from the earlier theorem, $K = 2Z_{2p} \setminus \{0\}$ is a cyclic group of the order $p-1$, and $p+1$ acts as the identity element of K.

$H = \{(a, neut(a), anti(a))\,|a \in K\}$ is a neutrosophic triplet groups collection and $neut(a) = p+1$ acts as the identity and is the unique element (neutral element) for all $a \in K$.

$(neut(a), neut(a), neut(a)) = (p+1, p+1, p+1)$ acts as the unique identity element of every neutrosophic triplet group h in H.

Since $K \subseteq Z_{2p} \setminus \{0\}$ is a cyclic group of order $p-1$ with $p+1$ as the identity element of K, we have $H = \{(a, neut\,(a)\,, anti\,(a))\,|a \in K\}$, to be cyclic. If $x \in K$ is such that $x^{p-1} = p+1$, then that neutrosophic triplet group element $(x, p+1, anti(x))$ in H will generate H as a cyclic group of order $p-1$ as $a \times anti(a) = neut(a)$.

Hence, H is a cyclic group of order $p-1$. □

Next, we proceed to describe the semi-neutrosophic triplets in the following section.

4. Semi-Neutrosophic Triplets and Their Properties

In this section, we define the notion of semi-neutrosophic triplet groups and trivial neutrosophic triplet groups and show some interesting results.

Example 8. *Let $\{Z_{26}, \times\} = S$ be the semigroup under product modulo 26.*

We see that $13 \in Z_{26}$ is an idempotent, but $13 \times 25 = 13$, where 25 is a unit of Z_{26}. Therefore, for this 25, we cannot find anti(13), but $13 \times 13 = 13$ is an idempotent, and $(13, 13, 13)$ is a neutrosophic triplet group. We do not accept it as a neutrosophic triplet, as it cannot yield any other nontrivial triplet other than $(13, 13, 13)$.

Further, the authors of [10] defined $(0, 0, 0)$ as a trivial neutrosophic triplet group.

Definition 4. *Let $S = \{Z_{2p}, \times\}$ be the semigroup under product modulo $2p$. $p \in Z_{2p}$ is an idempotent of Z_{2p}. However, p is not a neutrosophic triplet group as $p \times (2p-1) = 2p - p = p$. Hence, $(p, neut(p), anti(p)) = (p, p, p)$ is defined as a semi-neutrosophic triplet group.*

Proposition 1. *Let $S = \{Z_{2p}, \times\}$ be the semigroup under product modulo $2p$. (p, p, p) is the semi-neutrosophic triplet group of Z_{2p}.*

Proof. This is obvious from the definition and the fact $p^2 = p$ in Z_{2p} under product modulo $2p$. □

Example 9. *Let $S = \{Z_{46}, \times\}$ be the semigroup under product modulo 46. $T = \{(23, 23, 23), (0, 0, 0)\}$ is the semi-neutrosophic triplet group and the zero neutrosophic triplet group. Clearly, T is a semigroup under $\times$, and T is defined as the semigroup of semi-neutrosophic triplet groups of order two as $(23, 23, 23) \times (23, 23, 23) = (23, 23, 23)$. $K = \{(a, neut(a), anti(a)) \,|\, a \in 2Z_{46} \setminus \{0\} = \{2, 4, 6, 8, 10, 12, 14, 16, \ldots, 42, 44\}\}$ is a classical group of neutrosophic triplets.*

Let $P = \langle K \cup T \rangle = K \cup T$. For every $x \in K$ and for every $y \in T$, $x \times y = y \times x = (0, 0, 0)$.

Thus, P is a semigroup under product, and P is defined as the semigroup of neutrosophic triplets.

Further, we define T as the annihilating neutrosophic triplet semigroup of the classical group of neutrosophic triplets.

Definition 5. *Let $S = \{Z_{2p}, \times\}$, where p is an odd prime, be the semigroup under product modulo $2p$. Let $K = \{(a, neut(a), anti(a)) \,|\, a \in 2Z_{2p} \setminus \{0\}, \times\}$ be the classical group of neutrosophic triplets. Let $T = \{(p, p, p), (0, 0, 0)\}$ be the semigroup of semi-neutrosophic triplets (as a minomer, we call the trivial neutrosophic triplet $(0, 0, 0)$ as a semi-neutrosophic triplet). Clearly, $\langle T \cup K \rangle = T \cup K = P$ is defined as the semigroup of neutrosophic triplets with $o(P) = o(T) + o(K) = p - 1 + 2 = p + 1$.*

Further, T is defined as the annihilating semigroup of the classical group of neutrosophic triplets K.

We have seen examples of classical group of neutrosophic triplets, and we have defined and studied this only for Z_{2p} under the product modulo $2p$ for every odd prime p.

In the following section, we identify open problems and probable applications of these concepts.

5. Discussions and Conclusions

This paper studies the neutrosophic triplet groups introduced by [10] only in the case of $\{Z_{2p}, \times\}$, where p is an odd prime, under product modulo $2p$. We have proved the triplets of Z_{2p} are contributed only by elements in $2Z_{2p} \setminus \{0\} = \{2, 4, \ldots, 2p-2\}$, and these triplets under product form a group of order $p-1$, defined as the classical group of neutrosophic triplets.

Further, the notion of pseudo primitive element is defined for elements $K_1 = 2Z_{2p} \setminus \{0\} = \{2, 4, 6, \ldots, 2p-2\} \subseteq Z_{2p}$. This K_1 is a cyclic group of order $p-1$ with $p+1$ as its multiplicative identity. Based on this,

$$K = \{(a, neut(a), anti(a)) \,|a \in K_1, \times\}$$

is proved to be a cyclic group of order $p-1$.

We suggest the following problems:

1. How many pseudo primitive elements are there in $\{Z_{2p}, \times\}$, where p is an odd prime?
2. Can $\{Z_n, \times\}$, where n is any composite number different from $2p$, have pseudo primitive elements? If so, which idempotent serves as the identity?

For future research, one can apply the proposed neutrosophic triplet group to SVNS and develop it for the case of DVNS or TRINS. These neutrosophic triplet groups can be applied to problems where $neut(a)$ and $anti(a)$ are fixed once a is chosen, and vice v ersa. It can be realized as a special case of Single Valued Neutrosophic Sets (SVNSs) where neutral is always fixed. For every a in K_1, the other factor $anti(a)$ is automatically fixed, thereby eliminating the arbitrariness in determining $anti(a)$; however, there is only one case in which $a = anti(a)$. The set $2Z_{2p} \setminus \{0\}$ can be used to model this sort of problem and thereby reduce the arbitrariness in determining $anti(a)$, which is an object of future study.

Abbreviations

The following abbreviations are used in this manuscript:

SVNS	Single Valued Neutrosophic Set
DVNS	Double Valued Neutrosophic Set
TRINS	Triple Refined Indeterminate Neutrosophic Set
IFS	Intuitionistic Fuzzy Set

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Commutative falling neutrosophic ideals in BCK-algebras

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Young Bae Jun, Florentin Smarandache, Mehmat Ali Ozturk (2018). Commutative falling neutrosophic ideals in BCK-algebras. *Neutrosophic Sets and Systems* 20, 44-53

Abstract: The notions of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of a commutative $(\in, \in)$-neutrosophic ideal are obtained. Relations between commutative $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic ideal are discussed. Conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal are established. Relations between commutative $(\in, \in)$-neutrosophic ideal, falling neutrosophic ideal and commutative falling neutrosophic ideal are considered. Conditions for a falling neutrosophic ideal to be commutative are provided.

Keywords: (commutative) $(\in, \in)$-neutrosophic ideal; neutrosophic random set; neutrosophic falling shadow; (commutative) falling neutrosophic ideal.

1 Introduction

Neutrosophic set (NS) developed by Smarandache [11, 12, 13] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun, Borumand Saeid and Öztürk studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [6] and [10]). Goodman [2] pointed out the equivalence of a fuzzy set and a class of random sets in the study of a unified treatment of uncertainty modeled by means of combining probability and fuzzy set theory. Wang and Sanchez [16] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. The mathematical structure of the theory of falling shadows is formulated in [17]. Tan et al. [14, 15] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [7] considered a fuzzy subalgebra and a fuzzy ideal as the falling shadow of the cloud of the subalgebra and ideal. Jun et al. [8] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal.

In this paper, we introduce the concepts of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of a commutative $(\in, \in)$-neutrosophic ideal, and discuss relations between a commutative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal, and consider relations between a commutative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a commutative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be commutative.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [3] and [4]) and was extensively investigated by several researchers.

By a BCI-*algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-*algebra*. Any BCK/BCI-algebra X

satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} x \le y \;\Rightarrow\; x * z \le y * z \\ x \le y \;\Rightarrow\; z * y \le z * x \end{array}\right), \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X if it satisfies (2.5) and

$$(x * y) * z \in I,\; z \in I \;\Rightarrow\; x * (y * (y * x)) \in I \tag{2.7}$$

for all $x, y, z \in X$.

Observe that every commutative ideal is an ideal, but the converse is not true (see [9]).

We refer the reader to the books [5, 9] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [12]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$T_\in(A; \alpha) := \{x \in X \mid A_T(x) \ge \alpha\},$$
$$I_\in(A; \beta) := \{x \in X \mid A_I(x) \ge \beta\},$$
$$F_\in(A; \gamma) := \{x \in X \mid A_F(x) \le \gamma\}.$$

We say $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are *neutrosophic* $\in$-*subsets*.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic subalgebra* of X (see [6]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_\in(A; \alpha_x),\; y \in T_\in(A; \alpha_y) \\ \qquad \Rightarrow\; x * y \in T_\in(A; \alpha_x \wedge \alpha_y), \\ x \in I_\in(A; \beta_x),\; y \in I_\in(A; \beta_y) \\ \qquad \Rightarrow\; x * y \in I_\in(A; \beta_x \wedge \beta_y), \\ x \in F_\in(A; \gamma_x),\; y \in F_\in(A; \gamma_y) \\ \qquad \Rightarrow\; x * y \in F_\in(A; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.8}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic ideal* of X (see [10]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_\in(A; \alpha_x) \;\Rightarrow\; 0 \in T_\in(A; \alpha_x) \\ x \in I_\in(A; \beta_x) \;\Rightarrow\; 0 \in I_\in(A; \beta_x) \\ x \in F_\in(A; \gamma_x) \;\Rightarrow\; 0 \in F_\in(A; \gamma_x) \end{array}\right) \tag{2.9}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A; \alpha_x),\; y \in T_\in(A; \alpha_y) \\ \qquad \Rightarrow\; x \in T_\in(A; \alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A; \beta_x),\; y \in I_\in(A; \beta_y) \\ \qquad \Rightarrow\; x \in I_\in(A; \beta_x \wedge \beta_y) \\ x * y \in F_\in(A; \gamma_x),\; y \in F_\in(A; \gamma_y) \\ \qquad \Rightarrow\; x \in F_\in(A; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.10}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

In what follows, let X and $\mathcal{P}(X)$ denote a BCK/BCI-algebra and the power set of X, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\}, \tag{2.11}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.12}$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a *hyper-measurable structure* on X if $\mathcal{B}$ is a σ-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on X, a *neutrosophic random set* on X (see [8]) is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which ξ_T, ξ_I and ξ_F are mappings from Ω to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \left(\begin{array}{l} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{array} \right). \tag{2.13}$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, consider functions:

$$\begin{aligned} &\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)). \end{aligned}$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on X, and we call it a *neutrosophic falling shadow* (see [8]) of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *neutrosophic cloud* (see [8]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in X. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned} &\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha), \\ &\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta), \\ &\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma) \end{aligned}$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the *neutrosophic cut-cloud* (see [8]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *falling neutrosophic subalgebra* (resp., *falling neutrosophic ideal*) of X (see [8]).

3 Commutative $(\in, \in)$-neutrosophic ideals

Definition 3.1. A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK-algebra X is called a *commutative* $(\in, \in)$*-neutrosophic ideal* of X if it satisfies the condition (2.9) and

$$\begin{aligned} &(x*y)*z \in T_\in(A; \alpha_x),\ z \in T_\in(A; \alpha_y) \\ &\qquad \Rightarrow\ x*(y*(y*x)) \in T_\in(A; \alpha_x \wedge \alpha_y) \\ &(x*y)*z \in I_\in(A; \beta_x),\ z \in I_\in(A; \beta_y) \\ &\qquad \Rightarrow\ x*(y*(y*x)) \in I_\in(A; \beta_x \wedge \beta_y) \\ &(x*y)*z \in F_\in(A; \gamma_x),\ z \in F_\in(A; \gamma_y) \\ &\qquad \Rightarrow\ x*(y*(y*x)) \in F_\in(A; \gamma_x \vee \gamma_y) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Example 3.2. Consider a set $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 1.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2

Table 2: Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.9	0.2
1	0.3	0.6	0.8
2	0.3	0.6	0.8
3	0.5	0.4	0.7

It is routine to verify that $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X.

Theorem 3.3. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK-algebra X, the following are equivalent.*

(1) *The non-empty $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X) \left(\begin{array}{l} A_T(0) \geq A_T(x) \\ A_I(0) \geq A_I(x) \\ A_F(0) \leq A_F(x) \end{array} \right) \tag{3.2}$$

and for all $x, y, z \in X$,

$$\begin{aligned} &A_T(x*(y*(y*x))) \\ &\qquad \geq A_T((x*y)*z) \wedge A_T(z) \\ &A_I(x*(y*(y*x))) \\ &\qquad \geq A_I((x*y)*z) \wedge A_I(z) \\ &A_F(x*(y*(y*x))) \\ &\qquad \leq A_F((x*y)*z) \vee A_F(z) \end{aligned} \tag{3.3}$$

Proof. Assume that the non-empty $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. If $A_T(0) < A_T(a)$ for some $a \in X$, then $a \in T_\in(A; A_T(a))$ and $0 \notin T_\in(A; A_T(a))$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Similarly,

$A_I(0) \geq A_I(x)$ for all $x \in X$. Suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in F_\in(A; A_F(a))$ and $0 \notin F_\in(A; A_F(a))$. This is a contradiction, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore (3.2) is valid. Assume that there exist $a, b, c \in X$ such that

$$A_T(a * (b * (b * a))) < A_T((a * b) * c) \wedge A_T(c).$$

Taking $\alpha := A_T((a * b) * c) \wedge A_T(c)$ implies that $(a * b) * c \in T_\in(A; \alpha)$ and $c \in T_\in(A; \alpha)$ but $a * (b * (b * a)) \notin T_\in(A; \alpha)$, which is a contradiction. Hence

$$A_T(x * (y * (y * x))) \geq A_T((x * y) * z) \wedge A_T(z)$$

for all $x, y, z \in X$. By the similar way, we can verify that

$$A_I(x * (y * (y * x))) \geq A_I((x * y) * z) \wedge A_I(z)$$

for all $x, y, z \in X$. Now suppose there are $x, y, z \in X$ such that

$$A_F(x * (y * (y * x))) > A_F((x * y) * z) \vee A_F(z) := \gamma.$$

Then $(x*y)*z \in F_\in(A; \gamma)$ and $z \in F_\in(A; \gamma)$ but $x*(y*(y*x)) \notin F_\in(A; \gamma)$, a contradiction. Thus

$$A_F(x * (y * (y * x))) \leq A_F((x * y) * z) \vee A_F(z)$$

for all $x, y, z \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying two conditions (3.2) and (3.3). Assume that $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Let $x \in T_\in(A; \alpha)$, $a \in I_\in(A; \beta)$ and $u \in F_\in(A; \gamma)$ for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$ by (3.2). It follows that $0 \in T_\in(A; \alpha)$, $0 \in I_\in(A; \beta)$ and $0 \in F_\in(A; \gamma)$. Let $a, b, c \in X$ be such that $(a * b) * c \in T_\in(A; \alpha)$ and $c \in T_\in(A; \alpha)$ for $\alpha \in (0, 1]$. Then

$$A_T(a * (b * (b * a))) \geq A_T((a * b) * c) \wedge A_T(c) \geq \alpha$$

by (3.3), and so $a * (b * (b * a)) \in T_\in(A; \alpha)$. If $(x * y) * z \in I_\in(A; \beta)$ and $z \in I_\in(A; \beta)$ for all $x, y, z \in X$ and $\beta \in (0, 1]$, then $A_I((x * y) * z) \geq \beta$ and $A_I(z) \geq \beta$. Hence the condition (3.3) implies that

$$A_I(x * (y * (y * x))) \geq A_I((x * y) * z) \wedge A_I(z) \geq \beta,$$

that is, $x * (y * (y * x)) \in I_\in(A; \beta)$. Finally, suppose that

$$(x * y) * z \in F_\in(A; \gamma) \text{ and } z \in F_\in(A; \gamma)$$

for all $x, y, z \in X$ and $\gamma \in (0, 1]$. Then $A_F((x * y) * z) \leq \gamma$ and $A_F(z) \leq \gamma$, which imply from the condition (3.3) that

$$A_F(x * (y * (y * x))) \leq A_F((x * y) * z) \vee A_F(z) \leq \gamma.$$

Hence $x * (y * (y * x)) \in F_\in(A; \gamma)$. Therefore the non-empty $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. □

Theorem 3.4. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X if and only if the non-empty neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

Proof. Let $A = (A_T, A_I, A_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of X and assume that $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then there exist $x, y, z \in X$ such that $x \in T_\in(A; \alpha)$, $y \in I_\in(A; \beta)$ and $z \in F_\in(A; \gamma)$. It follows from (2.9) that $0 \in T_\in(A; \alpha)$, $0 \in I_\in(A; \beta)$ and $0 \in F_\in(A; \gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that

$$\begin{aligned} &(x * y) * z \in T_\in(A; \alpha),\ z \in T_\in(A; \alpha), \\ &(a * b) * c \in I_\in(A; \beta),\ c \in I_\in(A; \beta), \\ &(u * v) * w \in F_\in(A; \gamma),\ w \in F_\in(A; \gamma). \end{aligned}$$

Then

$$\begin{aligned} &x * (y * (y * x)) \in T_\in(A; \alpha \wedge \alpha) = T_\in(A; \alpha), \\ &a * (b * (b * a)) \in I_\in(A; \beta \wedge \beta) = I_\in(A; \beta), \\ &u * (v * (v * u)) \in F_\in(A; \gamma \vee \gamma) = F_\in(A; \gamma) \end{aligned}$$

by (2.10). Hence the non-empty neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are nonempty and are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Obviously, (2.9) is valid. Let $x, y, z \in X$ and $\alpha_x, \alpha_y \in (0, 1]$ be such that $(x * y) * z \in T_\in(A; \alpha_x)$ and $z \in T_\in(A; \alpha_y)$. Then $(x * y) * z \in T_\in(A; \alpha)$ and $z \in T_\in(A; \alpha)$ where $\alpha = \alpha_x \wedge \alpha_y$. Since $T_\in(A; \alpha)$ is a commutative ideal of X, it follows that

$$x * (y * (y * x)) \in T_\in(A; \alpha) = T_\in(A; \alpha_x \wedge \alpha_y).$$

Similarly, if $(x * y) * z \in I_\in(A; \beta_x)$ and $z \in I_\in(A; \beta_y)$ for all $x, y, z \in X$ and $\beta_x, \beta_y \in (0, 1]$, then

$$x * (y * (y * x)) \in I_\in(A; \beta_x \wedge \beta_y).$$

Now, suppose that $(x*y)*z \in F_\in(A; \gamma_x)$ and $z \in F_\in(A; \gamma_y)$ for all $x, y, z \in X$ and $\gamma_x, \gamma_y \in [0, 1)$. Then $(x * y) * z \in F_\in(A; \gamma)$ and $z \in F_\in(A; \gamma)$ where $\gamma = \gamma_x \vee \gamma_y$. Hence

$$x * (y * (y * x)) \in F_\in(A; \gamma) = F_\in(A; \gamma_x \vee \gamma_y)$$

since $F_\in(A; \gamma)$ is a commutative ideal of X. Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

Corollary 3.5. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A = (A_T, A_I, A_F)$ is a commuta-*

tive $(\in, \in)$*-neutrosophic ideal of* X *if and only if it satisfies two conditions* (3.2) *and* (3.3).

Proposition 3.6. *Every commutative* $(\in, \in)$*-neutrosophic ideal* $A = (A_T, A_I, A_F)$ *of a* BCK*-algebra* X *satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in T_\in(A; \alpha) \\ \quad \Rightarrow x * (y * (y * x)) \in T_\in(A; \alpha) \\ x * y \in I_\in(A; \beta) \\ \quad \Rightarrow x * (y * (y * x)) \in I_\in(A; \beta) \\ x * y \in F_\in(A; \gamma) \\ \quad \Rightarrow x * (y * (y * x)) \in F_\in(A; \gamma) \end{array} \right) \tag{3.4}$$

for all $\alpha, \beta \in (0, 1]$ *and* $\gamma \in [0, 1)$.

Proof. It is induced by taking $z = 0$ in (3.1). □

Theorem 3.7. *Every commutative* $(\in, \in)$*-neutrosophic ideal of a* BCK*-algebra* X *is an* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. Let $A = (A_T, A_I, A_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Assume that

$$\begin{array}{l} x * y \in T_\in(A; \alpha_x),\, y \in T_\in(A; \alpha_y), \\ a * b \in I_\in(A; \beta_a),\, b \in I_\in(A; \beta_b), \\ c * d \in F_\in(A; \gamma_c),\, d \in F_\in(A; \gamma_d) \end{array}$$

for all $x, y, a, b, c, d \in X$. Using (2.1), we have

$$\begin{array}{l} (x * 0) * y = x * y \in T_\in(A; \alpha_x), \\ (a * 0) * b = a * b \in I_\in(A; \beta_a), \\ (c * 0) * d = c * d \in F_\in(A; \gamma_c). \end{array}$$

It follows from (3.1), (2.1) and (V) that

$$\begin{array}{l} x = x * 0 = x * (0 * (0 * x)) \in T_\in(A; \alpha_x \wedge \alpha_y), \\ a = a * 0 = a * (0 * (0 * a)) \in I_\in(A; \beta_a \wedge \beta_b), \\ c = c * 0 = c * (0 * (0 * c)) \in F_\in(A; \gamma_c \vee \gamma_d). \end{array}$$

Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 4

Table 4: Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.66	0.77	0.27
1	0.55	0.45	0.37
2	0.33	0.66	0.47
3	0.33	0.45	0.67
4	0.33	0.45	0.67

Routine calculations show that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. But it is not a commutative $(\in, \in)$-neutrosophic ideal of X since $(2 * 3) * 0 \in T_\in(A; 0.6)$ and $0 \in T_\in(A; 0.5)$ but $2 * (3 * (3 * 2)) \notin T_\in(A; 0.5 \wedge 0.6)$, $(1 * 3) * 2 \in I_\in(A; 0.55)$ and $2 \in I_\in(A; 0.63)$ but $1 * (3 * (3 * 1)) \notin I_\in(A; 0.55 \wedge 0.63)$, and/or $(2 * 3) * 0 \in F_\in(A; 0.43)$ and $0 \in F_\in(A; 0.39)$ but $2 * (3 * (3 * 2)) \notin F_\in(A; 0.43 \vee 0.39)$.

We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal.

Theorem 3.9. *Let* $A = (A_T, A_I, A_F)$ *be an* $(\in, \in)$*-neutrosophic ideal of a* BCK*-algebra* X *in which the condition* (3.4) *is valid. Then* $A = (A_T, A_I, A_F)$ *is a commutative* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X and $x, y, z \in X$ be such that $(x * y) * z \in T_\in(A; \alpha_x)$ and $z \in T_\in(A; \alpha_y)$ for $\alpha_x, \alpha_y \in (0, 1]$. Then $x * y \in T_\in(A; \alpha_x \wedge \alpha_y)$ since $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. It follows from (3.4) that $x * (y * (y * x)) \in T_\in(A; \alpha_x \wedge \alpha_y)$. Similarly, if $(x * y) * z \in I_\in(A; \beta_x)$ and $z \in I_\in(A; \beta_y)$, then $x * (y * (y * x)) \in I_\in(A; \beta_x \wedge \beta_y)$. Let $a, b, c \in X$ and $\gamma_a, \gamma_b \in [0, 1)$ be such that $(a * b) * c \in F_\in(A; \gamma_a)$ and $c \in F_\in(A; \gamma_a)$. Then $a * b \in F_\in(A; \gamma_a \vee \gamma_b)$, which implies from (3.4) that $a * (b * (b * a)) \in F_\in(A; \gamma_a \vee \gamma_b)$. Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

Lemma 3.10. *Every* $(\in, \in)$*-neutrosophic ideal* $A = (A_T, A_I, A_F)$ *of a* BCK*-algebra* X *satisfies:*

$$\begin{array}{l} y, z \in T_\in(A; \alpha) \;\Rightarrow\; x \in T_\in(A; \alpha) \\ y, z \in I_\in(A; \beta) \;\Rightarrow\; x \in I_\in(A; \beta) \\ y, z \in F_\in(A; \gamma) \;\Rightarrow\; x \in F_\in(A; \gamma) \end{array} \tag{3.5}$$

for all $\alpha, \beta \in [0, 1)$, $\gamma \in (0, 1]$ *and* $x, y, z \in X$ *with* $x * y \leq z$.

Proof. For any $\alpha, \beta \in [0, 1)$, $\gamma \in (0, 1]$ and $x, y, z \in X$ with $x * y \leq z$, let $y, z \in T_\in(A; \alpha)$, $y, z \in I_\in(A; \beta)$ and $y, z \in F_\in(A; \gamma)$. Then

$$(x * y) * z = 0 \in T_\in(A; \alpha) \cap I_\in(A; \beta) \cap F_\in(A; \gamma)$$

by (2.9). It follows from (2.10) that

$$x * y \in T_\in(A;\alpha) \cap I_\in(A;\beta) \cap F_\in(A;\gamma)$$

and so that

$$x \in T_\in(A;\alpha) \cap I_\in(A;\beta) \cap F_\in(A;\gamma).$$

Thus (3.5) is valid. □

Theorem 3.11. *In a commutative BCK-algebra, every $(\in, \in)$-neutrosophic ideal is a commutative $(\in, \in)$-neutrosophic ideal.*

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a commutative BCK-algebra X. Let $x, y, z \in X$ be such that

$$(x * y) * z \in T_\in(A;\alpha_x) \cap I_\in(A;\beta_x) \cap F_\in(A;\gamma_x)$$

and

$$z \in T_\in(A;\alpha_y) \cap I_\in(A;\beta_y) \cap F_\in(A;\gamma_y)$$

for $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$. Note that

$$\begin{aligned}
&((x * (y * (y * x))) * ((x * y) * z)) * z \\
&= ((x * (y * (y * x))) * z) * ((x * y) * z) \\
&\leq (x * (y * (y * x))) * (x * y) \\
&= (x * (x * y)) * (y * (y * x)) \\
&= 0
\end{aligned}$$

by (2.3), (2.4) and (III), which implies that

$$(x * (y * (y * x))) * ((x * y) * z) \leq z.$$

It follows from Lemma 3.10 that

$$x * (y * (y * x)) \in T_\in(A;\alpha_x) \cap I_\in(A;\beta_x) \cap F_\in(A;\gamma_x).$$

Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

4 Commutative falling neutrosophic ideals

Definition 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *commutative falling neutrosophic ideal* of X if $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$.

Example 4.2. Consider a set $X = \{0,1,2,3,4\}$ with the binary operation $*$ which is given in Table 5

Table 5: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.25), \\ \{0,4\} & \text{if } t \in [0.25, 0.55), \\ \{0,1,2\} & \text{if } t \in [0.55, 0.85), \\ \{0,3,4\} & \text{if } t \in [0.85, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,3\} & \text{if } t \in [0.45, 0.75), \\ \{0,1,2,4\} & \text{if } t \in [0.75, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,4\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } t \in (0.3, 0.5], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are commutative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.3 & \text{if } x \in \{1,2\}, \\ 0.4 & \text{if } x = 3, \\ 0.45 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x \in \{0,1,2\}, \\ 0.3 & \text{if } x = 3, \\ 0.25 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5 & \text{if } x \in \{1,2,4\}, \\ 0.3 & \text{if } x = 3. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set

$\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\Omega(x;\xi_T) := \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\},$$
$$\Omega(x;\xi_I) := \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\},$$
$$\Omega(x;\xi_F) := \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}.$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$ (see [8]).

Proposition 4.3. *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X, then*

$$\begin{aligned}
&\Omega((x*y)*z;\xi_T) \cap \Omega(z;\xi_T) \\
&\qquad \subseteq \Omega(x*(y*(y*x));\xi_T) \\
&\Omega((x*y)*z;\xi_I) \cap \Omega(z;\xi_I) \\
&\qquad \subseteq \Omega(x*(y*(y*x));\xi_I) \\
&\Omega((x*y)*z;\xi_F) \cap \Omega(z;\xi_F) \\
&\qquad \subseteq \Omega(x*(y*(y*x));\xi_F)
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\Omega(x*(y*(y*x));\xi_T) \subseteq \Omega((x*y)*z;\xi_T) \\
\Omega(x*(y*(y*x));\xi_I) \subseteq \Omega((x*y)*z;\xi_I) \\
\Omega(x*(y*(y*x));\xi_F) \subseteq \Omega((x*y)*z;\xi_F)
\end{aligned} \tag{4.2}$$

for all $x, y, z \in X$.

Proof. Let

$$\omega_T \in \Omega((x*y)*z;\xi_T) \cap \Omega(z;\xi_T),$$
$$\omega_I \in \Omega((x*y)*z;\xi_I) \cap \Omega(z;\xi_I),$$
$$\omega_F \in \Omega((x*y)*z;\xi_F) \cap \Omega(z;\xi_F)$$

for all $x, y, z \in X$. Then

$$(x*y)*z \in \xi_T(\omega_T) \text{ and } z \in \xi_T(\omega_T),$$
$$(x*y)*z \in \xi_I(\omega_I) \text{ and } z \in \xi_I(\omega_I),$$
$$(x*y)*z \in \xi_F(\omega_F) \text{ and } z \in \xi_F(\omega_F).$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X, it follows from (2.7) that

$$x*(y*(y*x)) \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$$

and so that

$$\omega_T \in \Omega(x*(y*(y*x));\xi_T),$$
$$\omega_I \in \Omega(x*(y*(y*x));\xi_I),$$
$$\omega_F \in \Omega(x*(y*(y*x));\xi_F).$$

Hence (4.1) is valid. Now let

$$\omega_T \in \Omega(x*(y*(y*x));\xi_T),$$
$$\omega_I \in \Omega(x*(y*(y*x));\xi_I),$$
$$\omega_F \in \Omega(x*(y*(y*x));\xi_F)$$

for all $x, y, z \in X$. Then

$$x*(y*(y*x)) \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Note that

$$\begin{aligned}
&((x*y)*z)*(x*(y*(y*x))) \\
&= ((x*y)*(x*(y*(y*x))))*z \\
&\leq ((y*(y*x))*y)*z = ((y*y)*(y*x))*z \\
&= (0*(y*x))*z = 0*z = 0,
\end{aligned}$$

which yields

$$\begin{aligned}
&((x*y)*z)*(x*(y*(y*x))) \\
&= 0 \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).
\end{aligned}$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals and hence ideals of X, it follows that

$$(x*y)*z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Hence

$$\omega_T \in \Omega((x*y)*z;\xi_T),$$
$$\omega_I \in \Omega((x*y)*z;\xi_I),$$
$$\omega_F \in \Omega((x*y)*z;\xi_F).$$

Therefore (4.2) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.3}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.4}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [7]) where θ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

For any subset A of X and $g_T, g_I, g_F \in \mathcal{F}(X)$, consider the followings:

$$A_T^g := \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\},$$
$$A_I^g := \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\},$$
$$A_F^g := \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\}$$

and

$$\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\},$$
$$\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\},$$
$$\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}.$$

Then $A_T^g, A_I^g, A_F^g \in \mathcal{A}$ (see [8]).

Theorem 4.4. *If K is a commutative ideal of a BCK-algebra X, then*

$$\xi_T(\omega_T) = \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in K\},$$
$$\xi_I(\omega_I) = \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in K\},$$
$$\xi_F(\omega_F) = \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in K\}$$

are commutative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is a commutative ideal of a BCK-algebra X. Since $\theta(\omega_T) = 0 \in K$, $\theta(\omega_I) = 0 \in K$ and $\theta(\omega_F) = 0 \in K$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, we have $\theta \in \xi_T(\omega_T)$, $\theta \in \xi_I(\omega_I)$ and $\theta \in \xi_F(\omega_F)$. Let $f_T, g_T, h_T \in \mathcal{F}(X)$ be such that

$$(f_T \circledast g_T) \circledast h_T \in \xi_T(\omega_T) \text{ and } h_T \in \xi_T(\omega_T).$$

Then

$$(f_T(\omega_T) * g_T(\omega_T)) * h_T(\omega_T) = ((f_T \circledast g_T) \circledast h_T)(\omega_T) \in K$$

and $h_T(\omega_T) \in K$. Since K is a commutative ideal of X, it follows from (2.7) that

$$(f_T \circledast (g_T \circledast (g_T \circledast f_T)))(\omega_T)$$
$$= f_T(\omega_T) * (g_T(\omega_T) * (g_T(\omega_T) * f_T(\omega_T))) \in K,$$

that is, $f_T \circledast (g_T \circledast (g_T \circledast f_T)) \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is a commutative ideal of $\mathcal{F}(X)$. Similarly, we can verify that $\xi_I(\omega_I)$ is a commutative ideal of $\mathcal{F}(X)$. Now, let $f_F, g_F, h_F \in \mathcal{F}(X)$ be such that $(f_F \circledast g_F) \circledast h_F \in \xi_F(\omega_F)$ and $h_F \in \xi_F(\omega_F)$. Then

$$(f_F(\omega_F) * g_F(\omega_F)) * h_F(\omega_F)$$
$$= ((f_F \circledast g_F) \circledast h_F)(\omega_F) \in K$$

and $h_F(\omega_F) \in K$. Then

$$(f_F \circledast (g_F \circledast (g_F \circledast f_F)))(\omega_F)$$
$$= f_F(\omega_F) * (g_F(\omega_F) * (g_F(\omega_F) * f_F(\omega_F))) \in K,$$

and so $f_F \circledast (g_F \circledast (g_F \circledast f_F)) \in \xi_F(\omega_F)$. Hence $\xi_F(\omega_F)$ is a commutative ideal of $\mathcal{F}(X)$. This completes the proof. □

Theorem 4.5. *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every commutative $(\in, \in)$-neutrosophic ideal of a BCK-algebra is a commutative falling neutrosophic ideal.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of X. Then $T_\in(\tilde{H};\alpha)$, $I_\in(\tilde{H};\beta)$ and $F_\in(\tilde{H};\gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_\in(\tilde{H};\alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_\in(\tilde{H};\beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_\in(\tilde{H};\gamma)$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X. □

The converse of Theorem 4.5 is not true as seen in the following example.

Example 4.6. Consider a set $X = \{0,1,2,3,4\}$ with the binary operation $*$ which is given in Table 6

Table 6: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $(X;*,0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0,0.2), \\ \{0,2\} & \text{if } t \in [0.2,0.55), \\ \{0,2,4\} & \text{if } t \in [0.55,0.75), \\ \{0,1,2,3\} & \text{if } t \in [0.75,1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0,0.34), \\ \{0,4\} & \text{if } t \in [0.34,0.66), \\ \{0,1,4\} & \text{if } t \in [0.66,0.78), \\ X & \text{if } t \in [0.78,1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.87,1], \\ \{0,2\} & \text{if } t \in (0.76,0.87], \\ \{0,4\} & \text{if } t \in (0.58,0.76], \\ \{0,2,4\} & \text{if } t \in (0.33,0.58], \\ X & \text{if } t \in [0,0.33]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are commutative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.8 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.68 & \text{if } x = 1, \\ 0.22 & \text{if } x \in \{2,3\}, \\ 0.66 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.67 & \text{if } x \in \{1, 3\}, \\ 0.31 & \text{if } x = 2, \\ 0.24 & \text{if } x = 4. \end{cases}$$

But $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not a commutative $(\in, \in)$-neutrosophic ideal of X since

$$(3 * 4) * 2 \in T_\in(\tilde{H}; 0.4) \text{ and } 2 \in T_\in(\tilde{H}; 0.6),$$

but $3 * (4 * (4 * 3)) = 3 \notin T_\in(\tilde{H}; 0.4)$.

We provide relations between a falling neutrosophic ideal and a commutative falling neutrosophic ideal .

Theorem 4.7. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X, then it is a falling neutrosophic ideal of X.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a commutative falling neutrosophic ideal of a BCK-algebra X. Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$. Thus $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X. □

The following example shows that the converse of Theorem 4.7 is not true in general.

Example 4.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 7

Table 7: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 3\} & \text{if } t \in [0, 0.27), \\ \{0, 1, 2, 3\} & \text{if } t \in [0.27, 0.66), \\ \{0, 1, 2, 4\} & \text{if } t \in [0.67, 1], \end{cases}$$

$$\xi_I : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 3\} & \text{if } t \in [0, 0.35), \\ \{0, 1, 2, 4\} & \text{if } t \in [0.35, 1], \end{cases}$$

and

$$\xi_F : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.84, 1], \\ \{0, 3\} & \text{if } t \in (0.76, 0.84], \\ \{0, 1, 2, 4\} & \text{if } t \in (0.58, 0.76], \\ X & \text{if } t \in [0, 0.58]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are ideals of X for all $t \in [0, 1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic ideal of X. But it is not a commutative falling neutrosophic ideal of X because if $\alpha \in [0, 0.27)$, $\beta \in [0, 0.35)$ and $\gamma \in (0.76, 0.84]$, then $\xi_T(\alpha) = \{0, 3\}$, $\xi_I(\beta) = \{0, 3\}$ and $\xi_F(\gamma) = \{0, 3\}$ are not commutative ideals of X respectively.

Since every ideal is commutative in a commutative BCK-algebra, we have the following theorem.

Theorem 4.9. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a commutative BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a commutative falling neutrosophic ideal of X.*

Corollary 4.10. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. For any BCK-algebra X which satisfies one of the following assertions*

$$(\forall x, y \in X)(x \le y \;\Rightarrow\; x \le y * (y * x)), \tag{4.5}$$

$$(\forall x, y \in X)(x \le y \;\Rightarrow\; x = y * (y * x)), \tag{4.6}$$

$$(\forall x, y \in X)(x * (x * y) = y * (y * (x * (x * y)))), \tag{4.7}$$

$$(\forall x, y, z \in X)(x, y \le z, z * y \le z * x \;\Rightarrow\; x \le y), \tag{4.8}$$

$$(\forall x, y, z \in X)(x \le z, z * y \le z * x \;\Rightarrow\; x \le y), \tag{4.9}$$

let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a commutative falling neutrosophic ideal of X.

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On Neutrosophic Crisp Topology via *N*-Topology

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Riad K. Al-Hamido, T. Gharibah, S. Jafari, Florentin Smarandache (2018). On Neutrosophic Crisp Topology via N-Topology. *Neutrosophic Sets and Systems* 23, 96-109

Abstract. In this paper, we extend the neutrosophic crisp topological spaces into N–neutrosophic crisp topological spaces (N_{nc}-topological space). Moreover, we introduced new types of open and closed sets in N–neutrosophic crisp topological spaces. We also present N_{nc}-semi (open) closed sets, N_{nc}-preopen (closed) sets and N_{nc}-α-open (closed) sets and investigate their basic properties.

Keywords: N_{nc}-topology, N–neutrosophic crisp topological spaces, N_{nc}-semi (open) closed sets, N_{nc}-preopen (closed) sets, N_{nc}-α-open (closed) sets, $N_{nc}int(A)$, $N_{nc}cl(A)$.

Introduction

The concept of non-rigid (fuzzy) sets introduced in 1965 by L. A. Zadeh [11] which revolutionized the field of logic and set theory. Since the need for supplementing the classical two-valued logic with respect to notions with rigid extension engendered the concept of fuzzy set. Soon after its advent, this notion has been utilized in different fields of research such as, decision-making problems, modelling of mental processes, that is, establishing a theory of fuzzy algorithms, control theory, fuzzy graphs, fuzzy automatic machine etc., and in general topology. Three years after the presence of the concept of fuzzy set, Chang [3] introduced and developed the theory of fuzzy topological spaces. Many researchers focused on this theory and they developed it further in different directions. Then another new notion called intuitionistic fuzzy set was established by Atanassov [2] in 1983. Coker [4] introduced the notion of intuitionistic fuzzy topological space. F. Smarandache introduced the concepts of neutrosophy and neutrosophic set ([7], [8]). A. A. Salama and S. A. Alblowi [5] introduced the notions of neutrosophic crisp set and neutrosophic crisp topological space. In 2014, A.A. Salama, F. Smarandache and V. Kroumov [6] presented the concept of neutrosophic crisp topological space (NCTS). W. Al-Omeri [1] also investigated neutrosophic crisp sets in the context of neutrosophic crisp topological Spaces. The geometric existence of N -topology was given by M. Lellis Thivagar et al. [10], which is a nonempty set equipped with N-arbitrary topologies. The notion of N_n-open (closed) sets and N-neutrosophic topological spaces are introduced by M. Lellis Thivagar, S. Jafari,V. Antonysamy and V. Sutha Devi. [9]

In this paper, we explore the possibility of expanding the concept of neutrosophic crisp topological spaces into N-neutrosophic crisp topological spaces (N_{nc}-topological space). Further, we develop the concept of open (closed) sets, semiopen (semiclosed) sets, preopen (preclosed) sets and α-open (α-closed) sets in the context of N-neutrosophic crisp topological spaces and investigate some of their basic properties.

1.Preliminaries

In this section, we discuss some basic definitions and properties of N -topological spaces and neutrosophic crisp topological spaces which are useful in sequel.

Definition 1.1. [6] Let X be a non-empty fixed set. A neutrosophic crisp set (NCS) A is an object having the form $A = \{A_1, A_2, A_3\}$, where A_1, A_2 and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$.

Definition 1.2. [6] Types of *NCSs* ϕ_N and X_N in X are as follows:

1. ϕ_N may be defined in many ways as an NCS as follows:

   1. $\phi_N = (\phi, \phi, X)$ or
   2. $\phi_N = (\phi, X, X)$ or
   3. $\phi_N = (\phi, X, \phi)$ or
   4. $\phi_N = (\phi, \phi, \phi)$.

2. X_N may be defined in many ways as an NCS, as follows:

   1. $X_N = (X, \phi, \phi)$ or
   2. $X_N = (X, X, \phi)$ or
   3. $X_N = (X, X, X)$.

Definition 1.3. [6] Let X be a nonempty set, and the NCSs A and B be in the form $A = \{A_1, A_2, A_3\}, B = \{B_1, B_2, B_3\}$. Then we may consider two possible definitions for subset $A \subseteq B$ which may be defined in two ways:

1. $A \subseteq B \Leftrightarrow A_1 \subseteq B_1,\ A_2 \subseteq B_2\ and\ B_3 \subseteq A_3$.
2. $A \subseteq B \Leftrightarrow A_1 \subseteq B_1,\ B_2 \subseteq A_2\ and\ B_3 \subseteq A_3$.

Definition 1.4. [6] Let X be a non-empty set and the NCSs A and B in the form $A = \{A_1, A_2, A_3\},\ B = \{B_1, B_2, B_3\}$. Then:

1. $A \cap B$ may be defined in two ways as an NCS as follows:

$i)\quad A \cap B = (A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3)$
$ii)\quad A \cap B = (A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3)$.

2. $A \cup B$ may be defined in two ways as an NCS, as follows:

$i)\quad A \cup B = (A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3)$
$ii)\quad A \cup B = (A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3)$.

Definition 1.5. [6] A neutrosophic crisp topology (NCT) on a non-empty set X is a family Γ of neutrosophic crisp subsets in X satisfying the following axioms:

1. $\phi_N, X_N \in \Gamma$.
2. $A_1 \bigcap A_2 \in \Gamma$, for any A_1 and $A_2 \in \Gamma$.
3. $\bigcup A_j \in \Gamma, \forall \{A_j : j \in J\} \subseteq \Gamma$.

The pair (X, Γ) is said to be a neutrosophic crisp topological space (NCTS) in X. Moreover, the elements in Γ are said to be neutrosophic crisp open sets (NCOS). A neutrosophic crisp set F is closed (NCCS) if and only if its complement F^c is an open neutrosophic crisp set.

Definition 1.6. [6] Let X be a non-empty set, and the NCSs A be in the form

$A = \{A_1, A_2, A_3\}$. *Then* A^c may be defined in three ways as an *NCS*:

$i)\ A^c = < A_1^c, A_2^c, A_3^c >$ or
$ii)\ A^c = < A_3, A_2, A_1 >$ or
$iii)\ A^c = < A_3, A_2^c, A_1 >$.

2.*N*

2.N_{nc}-Topological Spaces

In this section, we introduce N–neutrosophic crisp topological spaces (N_{nc}-topological space) and discuss their basic properties. Moreover, we introduced new types of open and closed sets in the context of N_{nc}-topological spaces.

Definition 2.1: Let X be a non-empty set. Then ${}_{nc}\tau_1$, ${}_{nc}\tau_2$, ..., ${}_{nc}\tau_N$ are N-arbitrary crisp topologies defined on X and the collection

$$N_{nc}\tau = \{G \subseteq X : G = (\bigcup_{i=1}^{N} A_i) \cup (\bigcap_{i=1}^{N} B_i) \in N_{nc}\tau, \quad A_i, B_i \in_{nc} \tau_i\}$$

is called N_{nc}-topology on X if the following axioms are satisfied:

1. $\phi_N, X_N \in N_{nc}\tau$.

2. $\bigcup_{i=1}^{\infty} G_i \in N_{nc}\tau \quad for\ all\ \{G_i\}_{i=1}^{\infty} \in N_{nc}\tau$.

3. $\bigcap_{i=1}^{n} G_i \in N_{nc}\tau \quad for\ all\ \{G_i\}_{i=1}^{n} \in N_{nc}\tau.$

Then (X, $N_{nc}\tau$) is called N_{nc}-topological space on X. The elements of $N_{nc}\tau$ are known as

N_{nc}-open (N_{nc}- OS) sets on X and its complement is called N_{nc}-closed (N_{nc}- CS) sets on X. The elements of X are known as N_{nc}-sets (N_{nc}- S) on X.

Remark 2.2: Considering N = 2 in Definition 2.1, we get the required definition of bi-neutrosophic crisp topology on X. The pair (X, $2_{nc}\tau$) is called a bi-neutrosophic crisp topological space on X.

Remark 2.3: Considering N = 3 in Definition 2.1, we get the required definition of tri-neutrosophic crisp topology on X. The pair (X, $3_{nc}\tau$) is called a tri-neutrosophic crisp topological space on X.

Example 2.4:

$X = \{1,2,3,4\},\ {}_{nc}\tau_1 = \{\phi_N, X_N, \mathrm{A}\},\ {}_{nc}\tau_2 = \{\phi_N, X_N, \mathrm{B}\}, {}_{nc}\tau_3 = \{\phi_N, X_N\}$
$\mathrm{A} =< \{3\},\{2,4\},\{1\} >, B =< \{1\},\{2\},\{2,3\} >,$
$\mathrm{A} \cup \mathrm{B} =< \{1,3\},\{2,4\},\varnothing >, A \cap B =< \varnothing,\{2\},\{1,2,3\} >,$ Then we get

$3_{nc}\tau = \{\varnothing_N, X_N, A, B, A \cup B, A \cap B\}$

which is a tri-neutrosophic crisp topology on *X*. The pair (*X*, $3_{nc}\tau$) is called a tri-neutrosophic crisp topological space on *X*.

Example 2.5:

$X = \{1,2,3,4\},\ {}_{nc}\tau_1 = \{\phi_N, X_N, \mathrm{A}\},\ {}_{nc}\tau_2 = \{\phi_N, X_N, \mathrm{B}\}$
$\mathrm{A} =< \{3\},\{2,4\},\{1\} >, B =< \{1\},\{2\},\{2,3\} >,$
$\mathrm{A} \cup \mathrm{B} =< \{1,3\},\{2,4\},\varnothing >, A \cap B =< \varnothing,\{2\},\{1,2,3\} >,$ Then

$2_{nc}\tau = \{\varnothing_N, X_N, A, B, A \cup B, A \cap B\}$

which is a bi-neutrosophic crisp topology on X. The pair $(X, 2_{nc}\tau)$ is called a bi-neutrosophic crisp topological space on X.

Definition 2.6: Let $(X, N_{nc}\tau)$ be a N_{nc}-topological space on X and A be an N_{nc}-set on X then the $N_{nc}int(A)$ and $N_{nc}cl(A)$ are respectively defined as

(i) $N_{nc}int(A) = \cup \{G : G \subseteq A$ and G is a N_{nc}-open set in $X\}$.

(ii) $N_{nc}cl(A) = \cap \{F: A \subseteq F$ and F is a N_{nc}-closed set in $X\}$.

Proposition 2.7: Let $(X, N_{nc}\tau)$ be any N_{nc}-topological space. If A and B are any two N_{nc}-sets in $(X, N_{nc}\tau)$, so the N_{nc}-closure operator satisfies the following properties:

(i) $A \subseteq N_{nc}cl(A)$.

(ii) $A \subseteq B \Rightarrow N_{nc}cl(A) \subseteq N_{nc}cl(B)$.

(iii) $N_{nc}cl(A \cup B) = N_{nc}cl(A) \cup N_{nc}cl(B)$.

Proof

(i) $N_{nc}cl(A) = \cap \{G : G$ is a N_{nc}-closed set in X and $A \subseteq G\}$. Thus, $A \subseteq N_{nc}cl(A)$.

(ii) $N_{nc}cl(B) = \cap \{G : G$ is a N_{nc}-closed set in X and $B \subseteq G\} \supseteq \cap\{G :$ G is a N_{nc}-closed set in X and $A \subseteq G\} \supseteq N_{nc}cl(A)$. Thus, $N_{nc}cl(A) \subseteq N_{nc}cl(B)$.

(iii) $N_{nc}cl(A \cup B) = \cap\{G : G$ is a N_k-closed set in X and $A \cup B \subseteq G\} =$ $(\cap\{G : G$ is a N_{nc}-closed set in X and $A \subseteq G\}) \cup (\cap\{G : G$ is a N_{nc}-closed set in X and $B \subseteq G\}) = N_{nc}cl(A) \cup N_{nc}cl(B)$. Thus, $N_{nc}cl(A \cup B) = N_{nc}cl(A) \cup N_{nc}cl(B)$.

Proposition 2.8: Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A and B are any two N_{nc}-sets in (X, $N_{nc}\tau$), then the $N_{nc}int(A)$ operator satisfies the following properties:

(i) $N_{nc}int(A) \subseteq A$.

(ii) $A \subseteq B \Rightarrow N_{nc}int(A) \subseteq N_{nc}int(B)$.

(iii) $N_{nc}int(A \cap B) = N_{nc}int(A) \cap N_{nc}int(B)$.

(iv) $(N_{nc}cl(A))^c = N_{nc}int(A)^c$.

(v) $(N_{nc}int(A))^c = N_{nc}cl(A)^c$.

Proof

(i) $N_{nc}int(A) = \cup$ {G: G is an N_{nc}-open set in X and $G \subseteq A$ }. Thus, $N_{nc}int(A) \subseteq A$.

(ii) $N_{nc}int(B) = \cup$ {G: G is a N_{nc}-open set in X and $G \subseteq B$ }$\supseteq \cup${ G : G is an N_{nc}-open set in X and $G \subseteq A$ } $\supseteq N_{nc}int(A)$. Thus, $N_{nc}int(A) \subseteq N_{nc}int(B)$.

(iii) $N_{nc}int(A \cap B) = \cup\{G : G$ is an N_{nc}-open set in X and $A \cap B \supseteq G\}$ $= (\cup\{G : G$ is a N_{nc}-open set in X and $A \supseteq G\}) \cap (\cup\{G : G$ is an N_{nc}-open set in X and $B \supseteq G\}) = N_{nc}int(A) \cap N_{nc}int(B)$. Thus, $N_{nc}int(A \cap B) = N_{nc}int(A) \cap N_{nc}int(B)$.

(iv) $N_{nc}cl(A) = \cap$ $\{G$: G is an N_{nc}-closed set in X and $A \subseteq G\}$, $(N_{nc}cl(A))^c = \cup\{G^c : G^c$ is an N_{nc}-open set in X and $A^c \supseteq G^c\} = N_{nc}int(A)^c$. Thus, $(N_{nc}cl(A))^c = N_{nc}int(A)^c$.

(v) $N_{nc}int(A) = \cup$ $\{G$: G is an N_{nc}-open set in X and $A \supseteq G\}$, $(N_{nc}int(A))^c = \cap\{G^c : G^c$ is a N_{nc}-closed set in X and $A^c \supseteq G^c\} = N_{nc}cl(A)^c$. Thus, $(N_{nc}int(A))^c = N_{nc}cl(A)^c$.

Proposition 2.9:

Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A is a N_{nc}-sets in (X, $N_{nc}\tau$), the following properties are true:

(i) N_{nc}cl(A) = A iff A is a N_{nc}-closed set.

(ii) N_{nc}int(A) = A iff A is a N_{nc}-open set.

(iii) N_{nc}cl(A) is the smallest N_{nc}-closed set containing A.

(iv) N_{nc}int(A) is the largest N_{nc}-open set contained in A.

Proof: (i), (ii), (iii) and (iv) are obvious.

3.New open setes *in N_{nc}*-Topological Spaces

Definition 3.1: Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. Let A be an N_{nc}-set in (X, $N_{nc}\tau$). Then A is said to be:

(i) A N_{nc}-preopen set (N_{nc}-P-OS) if A⊆ $N_{nc}int(N_{nc}cl(A))$. The complement of an N_{nc}-preopen set is called an N_{nc}-preopen set in X. The family of all N_{nc}-P-OS (resp. N_{nc}-P-CS) of X is denoted by (N_{nc}POS(X)) (resp. N_{nc}PCS).

(ii) An N_{nc}-semiopen set (N_{nc}-S-OS) if A⊆ $N_{nc}cl(N_{nc}int(A))$. The complement of a N_{nc}-semiopen set is called a N_{nc}-semiopen set in X. The family of all N_{nc}-S-OS (resp. N_{nc}-S-CS) of X is denoted by (N_{nc}POS(X)) (resp. N_{nc}PCS).

(iii) A N_{nc}-α-open set (N_{nc}-α-OS) if A⊆ $N_{nc}int\ (N_{nc}cl(N_{nc}int(A)))$. The complement of a N_{nc}-α-open set is called a N_{nc}-α-open set in X. The family of all N_{nc}-α-OS (resp. N_{nc}-α-CS) of X is denoted by (N_{nc}αOS(X)) (resp. N_{nc}αCS).

Example 3.2:

$X=\{a,b,c,d\}$, ${}_{nc}\tau_1=\{\phi_N, X_N, \mathrm{A}\}$, ${}_{nc}\tau_2=\{\phi_N, X_N, \mathrm{B}\}$

$A =< \{a\},\{b\},\{c\} >, B =< \{a\},\{b,d\},\{c\} >$, then we have $2_{nc}\tau = \{\varnothing_N, X_N, A, B\}$

which is a bi-neutrosophic crisp topology on *X*. Then the pair (*X*, $2_{nc}\tau$) is a bi-neutrosophic crisp topological space on *X*. If $H =< \{a,b\},\{c\},\{d\} >$, then H is a N_{nc}-P-OS but not N_{nc}-α-OS. It is clear that H^c is a N_{nc}-P-CS. A is a N_{nc}-S-OS. It is clear that A^c is a N_{nc}-S-CS. A is a N_{nc}-α-OS. It is clear that A^c is a N_{nc}-α-CS.

Definition 3.3: Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X and A be a N_{nc}-set on X then

(i) N_{nc}-P-int(A) = ∪ {G: G ⊆ A and G is a N_{nc}-P-OS in X}.

(ii) N_{nc}-P-cl(A) = ∩ {F: A ⊆ F and F is a N_{nc}-P-CS in X}.

(iii) N_{nc}-S-int(A) = ∪ {G: G ⊆ A and G is a N_{nc}-S-OS in X}.

(iv) N_{nc}-S-cl(A) = ∩ {F: A ⊆ F and F is a N_{nc}-S-CS in X}.

(v) N_{nc}-α-int(A) = ∪ {G: G ⊆ A and G is a N_{nc}-α-OS in X}.

(vi) N_{nc}-α-cl(A) = ∩ {F: A ⊆ F and F is a N_{nc}-α-CS in X}.

In Proposition 3.4 and Proposition 3.5, by the notion N_{nc}-k-cl(A)(N_{nc}-k-int(A)), we mean N_{nc}-P-cl(A)(N_{nc}-P-int(A)) (if k = p), N_{nc}-S-cl(A)(N_{nc}-S-int(A)) (if k = S) and N_{nc}-α-cl(A)(N_{nc}-α-int(A)) (if k = α).

Proposition 3.4: Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A and B are any two N_{nc}-sets in (X, $N_{nc}\tau$), then the N_{nc}-S-closure operator satisfies the following properties:

(i) A ⊆ N_{nc}-k-cl(A).

(ii) N_{nc}-k-int(A) ⊆ A.

(iii) A ⊆ B ⇒ N_{nc}-k-cl(A)⊆ N_{nc}-k-cl(B).

(iv) $A \subseteq B \Rightarrow N_{nc}$-k-int(A) $\subseteq N_{nc}$-k-int(B).

(v) N_{nc}-k-cl $(A \cup B) = N_{nc}$-k-cl(A)$\cup N_{nc}$-k-cl(B).

(vi) N_{nc}-k-int $(A \cap B) = N_{nc}$-k-int(A) $\cap$ N_{nc}-k-int(B).

(vii) $(N_{nc}\text{-k-cl}(A))^c = N_{nc}\text{-k-cl}(A)^c$.

(viii) $(N_{nc}\text{-k-int}(A))^c = N_{nc}\text{-k-int}(A)^c$.

Proposition 3.5:

Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A is an N_{nc}-sets in (X, $N_{nc}\tau$). Then the following properties are true:

(i) N_{nc}-k-cl(A)= A iff A is a N_{nc}-k-closed set.

(ii) N_{nc}-k-int(A)= A iff A is a N_{nc}-k-open set.

(iii) N_{nc}-k-cl(A) is the smallest N_{nc}-k-closed set containing A.

(iv) N_{nc}-k-int(A) is the largest N_{nc}-k-open set contained in A.

Proof: (i), (ii), (iii) and (iv) are obvious.

Proposition 3.6:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X. Then the following statements hold in whcih the equality of each statement are not true:

(i) Every N_{nc}- OS (resp. N_{nc}- CS) is a N_{nc}-α-OS (resp. N_{nc}-α-CS).

(ii) Every N_{nc}-α-OS (resp. N_{nc}-α-CS) is a N_{nc}-S-OS (resp. N_{nc}-S-CS).

(iii) Every N_{nc}-α-OS (resp. N_{nc}-α-CS) is a N_{nc}-P-OS (resp. N_{nc}-P-CS).

Proposition 3.7:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X, then the following statements hold, and the equality of each statement are not true:

(i) Every N_{nc}-OS (resp. N_{nc}-CS) is a N_{nc}-S-OS (resp. N_{nc}-S-CS).

(ii) Every N_{nc}-OS (resp. N_{nc}-CS) is a N_{nc}-P-OS (resp. N_{nc}-P-CS).

Proof.

(i) Suppose that A is a N_{nc}-OS. Then A=$N_{nc}int(A)$, and so $A\subseteq N_{nc}cl$(A)= $N_{nc}cl(N_{nc}int(A))$. so that A is a N_{nc}-S-OS.

(ii) Suppose that A is a N_{nc}-OS. Then A=$N_{nc}int(A)$, and since $A\subseteq N_{nc}cl$(A) so $A=N_{nc}int(A) \subseteq N_{nc}int(N_{nc}cl(A))$. so that A is a N_{nc}-P-OS.

Proposition 3.8:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X and A a N_{nc}-set on X. Then A is an N_{nc}-α-OS (resp. N_{nc}-α-CS) iff A is a N_{nc}-S-OS (resp. N_{nc}-S-CS) and N_{nc}-P-OS (resp. N_{nc}-P-CS).

Proof. The necessity condition follows from the Definition 3.1. Suppose that A is both a N_{nc}-S-OS and a N_{nc}-P-OS. Then A$\subseteq$ $N_{nc}cl(N_{nc}int(A))$, and hence $N_{nc}cl$(A)$\subseteq$ $N_{nc}cl(N_{nc}cl(N_{nc}int(A)))$= $N_{nc}cl(N_{nc}int(A))$.

It follows that A$\subseteq$ $N_{nc}int(N_{nc}cl(A))\subseteq$ $N_{nc}int(N_{nc}cl(N_{nc}int(A)))$, so that A is a N_{nc}-α-OS.

Proposition 3.9:

Let (X, $N_{nc}\tau$) be an N_{nc}-topological space on X and A an N_{nc}-set on X. Then A is an N_{nc}-α-CS iff A is an N_{nc}-S-CS and N_{nc}-P-CS.

Proof. The proof is straightforward.

Theorem 3.10:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X and A a N_{nc}-set on X. If B is a N_{nc}-S-OS such that B $\subseteq$A$\subseteq$ N_{nc}int(N_{nc}cl (A)), then A is a N_{nc}-α-OS .

Proof. Since B is a N_{nc}-S-OS, we have B $\subseteq$ N_{nc}int(N_{nc}cl (A)) . Thus, A $\subseteq$ N_{nc}int(N_{nc}cl (B))$\subseteq$ N_{nc}int(N_{nc}cl (N_{nc}cl (N_{nc}int((B))))$\subseteq$ N_{nc}int(N_{nc}cl (N_{nc}int((B))))

$\subseteq N_{nc}int(N_{nc}cl\ (N_{nc}int(\ (A))))$ and therefore *A is a* N_{nc}-α-OS.

Theorem 3.11:

Let $(X, N_{nc}\tau)$ be an N_{nc}-topological space on X and A be an N_{nc}-set on X. Then $A \in N_{nc}\alpha OS(X))$ iff there exists an N_{nc}- OS H such that $H \subseteq A \subseteq N_{nc}int\ (N_{nc}cl\ (A))$.

Proposition 3.12:

The union of any family of $N_{nc}\alpha OS(X)$ is a $N_{nc}\alpha OS(X)$.

Proof. The proof is straightforward.

Remark 3.13:

The following diagram shows the relations among the different types of weakly neutrosophic crisp open sets that were studied in this paper:

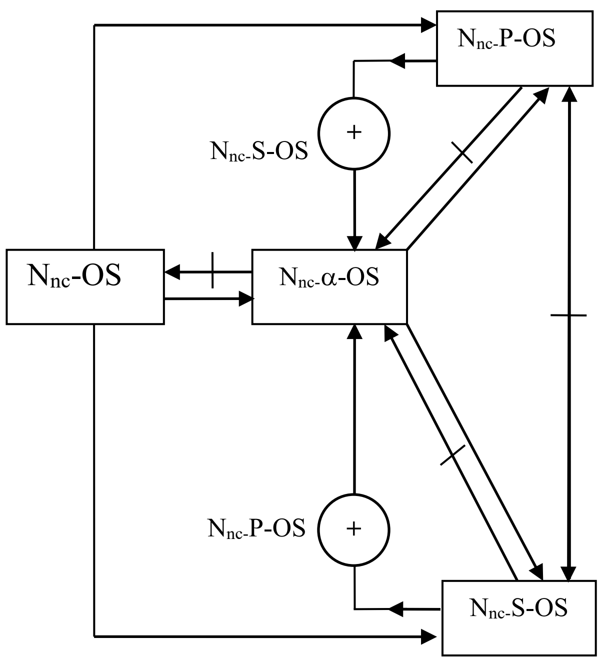

Diagram (3.1)

Conclusion

In this work, we have introduced some new notions of N-neutrosophic crisp open (closed) sets called N_{nc}-semi (open) closed sets, N_{nc}-preopen (closed) sets, and N_{nc}-α-open (closed) sets and studied some of their basic properties in the context of neutrosophic crisp topological spaces. The neutrosophic crisp semi-α-closed sets can be used to derive a new decomposition of neutrosophic crisp continuity.

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Neutrosophic Rare α-Continuity

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R. Dhavaseelan, S. Jafari, R. M. Latif, Florentin Smarandache (2018). Neutrosophic Rare α-Continuity. *New Trends in Neutrosophic Theory and Applications* II, 336-344

ABSTRACT

In this paper, we introduce the concepts of neutrosophic rare α-continuous, neutrosophic rarely continuous, neutrosophic rarely pre-continuous, neutrosophic rarely semi-continuous are introduced and studied in light of the concept of rare set in neutrosophic setting.

KEYWORDS: Neutrosophic rare set; neutrosophic rarely α-continuous; neutrosophic rarely pre-continuous; neutrosophic almost α-continuous; neutrosophic weekly α-continuous; neutrosophic rarely semi-continuous.

1 INTRODUCTION AND PRELIMINARIES

The study of fuzzy sets was initiated by Zadeh (1965). Thereafter the paper of Chang (1968) paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Currently Fuzzy Topology has been observed to be very beneficial in fixing many realistic problems. Several mathematicians have tried almost all the pivotal concepts of General Topology for extension to the fuzzy settings. In 1981, Azad gave fuzzy version of the concepts given by Levine 1961; 1963 and thus initiated the study of weak forms of several notions in fuzzy topological spaces. Popa (1979) introduced the notion of rare continuity as a generalization of weak continuity (Levine, 1961) which has been further investigated by Long and Herrington (1982) and Jafari (1995; 1997). Noiri (1987) introduced and investigated weakly α-continuity as a generalization of weak continuity. He also introduced and investigated almost α-continuity (Noiri, 1988). The concepts of Rarely α-continuity was introduced by Jafari (2005). The concepts of fuzzy rare α-continuity and intuitionistic fuzzy rare α-continuity were introduced by Dhavaseelan and Jafari (n.d.-b, n.d.-c). After the advent of the concepts of neutrosophy and neutrosophic set introduced by Smarandachethe (1999; 2002), the concepts of neutrosophic crisp set and neutrosophic crisp topological spaces were introduced by Salama and Alblowi (2012).

The purpose of the present paper is to introduce and study the concepts of neutrosophic rare α-continuous functions, neutrosophic rarely continuous functions, neutrosophic rarely pre-continuous functions and neutrosophic rarely semi-continuous functions in light of the concept of rare set in a neutrosophic setting.

Definition 1.1. Let X be a nonempty fixed set. A neutrosophic set [briefly NS] A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function ($\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership ($\gamma_A(x)$), respectively, of each element $x \in X$ to the set A.

Remark 1.1. (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.2. Let X be a nonempty set and the neutrosophic sets A and B in the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$; [complement of A]

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$;

(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$;

(f) $[\,]A = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle : x \in X\}$;

(g) $\langle\rangle A = \{\langle x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.3. Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$;

(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we must introduce the neutrosophic sets 0_N and 1_N in X as follows:

Definition 1.4. $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.5. (Dhavaseelan & Jafari, n.d.-a) A neutrosophic topology (briefly NT) on a nonempty set X is a family T of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X, T) or simply X is called a neutrosophic topological space (briefly NTS) and each neutrosophic set in T is called a neutrosophic open set (briefly NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (briefly NCS) in X.

Definition 1.6. (Dhavaseelan & Jafari, n.d.-a) Let A be a neutrosophic set in a neutrosophic topological space X. Then

$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and $G \subseteq A\}$ is called the neutrosophic interior of A;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$ is called the neutrosophic closure of A.

Definition 1.7. (Dhavaseelan & Jafari, n.d.-a) Let X be a nonempty set. If r, t, s be real standard or non standard subsets of $]0^-, 1^+[$, then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point(briefly NP)in X given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,t,s}$, where r denotes the degree of membership value , t the degree of indeterminacy and s the degree of non-membership value of $x_{r,t,s}$.

Definition 1.8. (Dhavaseelan & Jafari, n.d.-b) An intuitionistic fuzzy set R is called intuitionistic fuzzy rare set if $IFint(R) = 0_\sim$.

Definition 1.9. (Dhavaseelan & Jafari, n.d.-b) An intuitionistic fuzzy set R is called intuitionistic fuzzy nowhere dense set if $IFint(IFcl(R)) = 0_\sim$.

2 MAIN RESULTS

Definition 2.1. A neutrosophic set A in a neutrosophic topological space (X, T) is called

1) a neutrosophic semiopen set (briefly NSOS) if $A \subseteq Ncl(Nint(A))$.

2) a neutrosophic α open set (briefly $N\alpha OS$) if $A \subseteq Nint(Ncl(Nint(A)))$.

3) a neutrosophic preopen set (briefly NPOS) if $A \subseteq Nint(Ncl(A))$.

4) a neutrosophic regular open set (briefly NROS) if $A = Nint(Ncl(A))$.

5) a neutrosophic semipreopen or β open set (briefly $N\beta OS$) if $A \subseteq Ncl(Nint(Ncl(A)))$.

A neutrosophic set A is called a neutrosophic semiclosed set, neutrosophic α-closed set, neutrosophic preclosed set, neutrosophic regular closed set and neutrosophic β-closed set (briefly NSCS, NαCS, NPCS, NRCS and NβCS, resp.), if the complement of A is a neutrosophic semiopen set, neutrosophic α-open set, neutrosophic preopen set, neutrosophic regular open set, and neutrosophic β-open set, respectively.

Definition 2.2. Let a neutrosophic set A of a neutrosophic topological space (X, T). Then neutrosophic α-closure of A (briefly $Ncl_\alpha(A)$) is defined as $Ncl_\alpha(A) = \bigcap\{K|$ K is a neutrosophic α closed set in X and $A \subseteq K\}$.

Definition 2.3. (Jun & Song, 2005) Let a neutrosophic set A of a neutrosophic topological space (X, T). Then neutrosophic α interior of A (briefly $Nint_\alpha(A)$) is defined as $Nint_\alpha(A) = \bigcup\{K|\ K$ is a neutrosophic α open set in X and $K \subseteq A\}$.

Definition 2.4. A neutrosophic set R is called neutrosophic rare set if $Nint(R) = 0_N$.

Definition 2.5. A neutrosophic set R is called neutrosophic nowhere dense set if $Nint(Ncl(R)) = 0_N$.

Definition 2.6. Let (X, T) and (Y, S) be two neutrosophic topological spaces. A function $f : (X, T) \to (Y, S)$ is called

(i) neutrosophic α-continuous if for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic open set G in Y containing $f(x_{r,t,s})$, there exists a neutrosophic α open set U in X such that $f(U) \leq G$.

(ii) neutrosophic almost α-continuous if for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic open set G containing $f(x_{r,t,s})$, there exists a neutrosophic α open set U such that $f(U) \leq Nint(Ncl(G))$.

(iii) neutrosophic weakly α-continuous if for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic open set G containing $f(x_{r,t,s})$, there exists a neutrosophic α open set U such that $f(U) \leq Ncl(G)$.

Definition 2.7. Let (X, T) and (Y, S) be two neutrosophic topological spaces. A function $f : (X, T) \to (Y, S)$ is called

(i) neutrosophic rarely α-continuous if for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic open set G in (Y,S) containing $f(x_{r,t,s})$, there exist a neutrosophic rare set R with $G \cap Ncl(R) = 0_{_N}$ and neutrosophic α open set U in (X,T) such that $f(U) \leq G \cup R$.

(ii) neutrosophic rarely continuous if for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic open set G in (Y,S) containing $f(x_{r,t,s})$, there exist a neutrosophic rare set R with $G \cap Ncl(R) = 0_{_N}$ and neutrosophic open set U in (X,T) such that $f(U) \leq G \cup R$.

(iii) neutrosophic rarely precontinuous if for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic open set G in (Y,S) containing $f(x_{r,t,s})$, there exist a neutrosophic rare set R with $G \cap Ncl(R) = 0_{_N}$ and neutrosophic preopen set U in (X,T) such that $f(U) \leq G \cup R$.

(iv) neutrosophic rarely semi-continuous if for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic open set G in (Y,S) containing $f(x_{r,t,s})$, there exist a neutrosophic rare set R with $G \cap Ncl(R) = 0_{_N}$ and neutrosophic semiopen set U in (X,T) such that $f(U) \leq G \cup R$.

Example 2.1. Let $X = \{a,b,c\}$. Define the neutrosophic sets A, B and C as follows:
$A = \langle x, (\frac{a}{0}, \frac{b}{0}, \frac{c}{1}), (\frac{a}{0}, \frac{b}{0}, \frac{c}{1}), (\frac{a}{1}, \frac{b}{1}, \frac{c}{0})\rangle$, $B = \langle x, (\frac{a}{1}, \frac{b}{0}, \frac{c}{0}), (\frac{a}{1}, \frac{b}{0}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{1}, \frac{c}{1})\rangle$ and
$C = \langle x, (\frac{a}{0}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{1}, \frac{b}{0}, \frac{c}{1})\rangle$. Then $T = \{0_{_N}, 1_{_N}, C\}$ and $S = \{0_{_N}, 1_{_N}, A, B, A \cup B\}$ are neutrosophic topologies on X. Let (X,T) and (X,S) be neutrosophic topological spaces. Define $f : (X,T) \to (X,S)$ as a identity function. Clearly f is neutrosophic rarely α-continuous.

Proposition 2.1. Let (X,T) and (Y,S) be any two neutrosophic topological spaces. For a function $f : (X,T) \to (Y,S)$ the following statements are equivalents:

(i) The function f is neutrosophic rarely α-continuous at $x_{r,t,s}$ in (X,T).

(ii) For each neutrosophic open set G containing $f(x_{r,t,s})$, there exists a neutrosophic α open set U in (X,T) such that $Nint(f(U) \cap \overline{G}) = 0_{_N}$.

(iii) For each neutrosophic open set G containing $f(x_{r,t,s})$, there exists a neutrosophic α open set U in (X,T) such that $Nint(f(U)) \leq Ncl(G)$.

(iv) For each neutrosophic open set G in (Y,S) containing $f(x_{r,t,s})$, there exists a neutrosophic rare set R with $G \cap Ncl(R) = 0_{_N}$ such that $x_{r,t,s} \in Nint_{\alpha}(f^{-1}(G \cup R))$.

(v) For each neutrosophic open set G in (Y,S) containing $f(x_{r,t,s})$, there exists a neutrosophic rare set R with $Ncl(G) \cap R = 0_{_N}$ such that $x_{r,t,s} \in Nint_{\alpha}(f^{-1}(Ncl(G) \cup R))$

(vi) For each neutrosophic regular open set G in (Y,S) containing $f(x_{r,t,s})$, there exists a neutrosophic rare set R with $Ncl(G) \cap R = 0_{_N}$ such that $x_{r,t,s} \in Nint_{\alpha}(f^{-1}(G \cup R))$

Proof. $(i) \Rightarrow (ii)$ Let G be a neutrosophic open set in (Y, S) containing $f(x_{r,t,s})$. By $f(x_{r,t,s}) \in G \leq Nint(Ncl(G))$ and $Nint(Ncl(G))$ containing $f(x_{r,t,s})$, there exists a neutrosophic rare set R with $Nint(Ncl(G)) \cap Ncl(R) = 0_{_N}$ and a neutrosophic α open set U in (X, T) containing $x_{r,t,s}$ such that $f(U) \leq Nint(Ncl(G)) \cup R$. We have $Nint(f(U) \cap \overline{G})Nint(\overline{G}) \leq Nint(Ncl(G) \cup R) \cap \overline{(Ncl(G))} \leq Ncl(G) \cup Nint(R) \cap \overline{(Ncl(G))} = 0_{_N}$.

$(ii) \Rightarrow (iii)$ Obvious.

$(iii) \Rightarrow (i)$ Let G be a neutrosophic open set in (Y, S) containing $f(x_{r,t,s})$. Then by (iii), there exists a neutrosophic α-open set U containing $x_{r,t,s}$ such that $Nint(f(U) \leq Ncl(G)$. We have $f(U) = (f(U) \cap \overline{(Nint(f(U)))}) \cup Nint(f(U)) < (f(U) \cap \overline{(Nint(f(U)))}) \cup Ncl(G) = (f(U) \cap \overline{(Nint(f(U)))}) \cup G \cup (Ncl(G) \cap \overline{G}) = (f(U) \cap \overline{(Nint(f(U)))} \cap \overline{G}) \cup G \cup (Ncl(G) \cap \overline{G})$. Set $R_1 = f(U) \cap \overline{(Nint(f(U)))} \cap \overline{G}$ and $R_2 = Ncl(G) \cap \overline{G}$. Then R_1 and R_2 are neutrosophic rare sets. More $R = R_1 \cup R_2$ is a neutrosophic set such that $Ncl(R) \cap G = 0_{_N}$ and $f(U) \leq G \cup R$. This show that f is neutrosophic rarely α-continuous.

$(i) \Rightarrow (iv)$ Suppose that G be a neutrosophic open set in (Y, S) containing $f(x_{r,t,s})$. Then there exists a neutrosophic rare set R with $G \cap Ncl(R) = 0_{_N}$ and U be a neutrosophic α-open set in (X, T) containing $x_{r,t,s}$ such that $f(U) \leq G \cup R$. It follows that $x_{r,t,s} \in U \leq f^{-1}(G \cup R)$. This implies that $x_{r,t,s} \in Nint_\alpha(f^{-1}(G \cup R))$.

$(iv) \Rightarrow (v)$ Suppose that G be a neutrosophic open set in (Y, S) containing $f(x_{r,t,s})$. Then there exists a neutrosophic rare set R with $G \cap Ncl(R) = 0_{_N}$ such that $x_{r,t,s} \in Nint_\alpha(f^{-1}(G \cup R))$. Since $G \cap Ncl(R) = 0_{_N}, R \leq \overline{G}$, where $\overline{G} = \overline{(Ncl(G))} \cup (Ncl(G) \cap \overline{G})$. Now, we have $R \leq R \cup \overline{(Ncl(G))} \cup (Ncl(G) \cap \overline{G})$. Now, $R_1 = R \cap \overline{(Ncl(G))}$. It follows that R_1 is a neutrosophic rare set with $Ncl(G) \cap R_1 = 0_{_N}$. Therefore $x_{r,t,s} \in Nint_\alpha(f^{-1}(G \cup R)) \leq Nint_\alpha(f^{-1}(G \cup R_1))$.

$(v) \Rightarrow (vi)$ Assume that G be a neutrosophic regular open set in (Y, S) containing $f(x_{r,t,s})$. Then there exists a neutrosophic rare set R with $Ncl(G) \cap R = 0_{_N}$ such that $x_{r,t,s} \in Nint_\alpha(f^{-1}(Ncl(G) \cup R))$. Now $R_1 = R \cup (Ncl(G) \cup \overline{G})$. It follows that R_1 is a neutrosophic rare set and $(G \cap Ncl(R_1)) = 0_{_N}$. Hence $x_{r,t,s} \in Nint_\alpha(f^{-1}(Ncl(G) \cup R)) = Nint_\alpha(f^{-1}(G \cup (Ncl(G) \cap \overline{G})) \cup R) = Nint_\alpha(f^{-1}(G \cup R_1))$. Therefore $x_{r,t,s} \in Nint_\alpha(f^{-1}(G \cup R_1))$.

$(vi) \Rightarrow (ii)$ Let G be a neutrosophic open set in (Y, S) containing $f(x_{r,t,s})$. By $f(x_{r,t,s}) \in G \leq Nint(Ncl(G))$ and the fact that $Nint(Ncl(G))$ is a neutrosophic regular open in (Y, S), there exists a neutrosophic rare set R and $Nint(Ncl(G)) \cap Ncl(R) = 0_{_N}$, such that $x_{r,t,s} \in Nint_\alpha(f^{-1}(Nint(Ncl(G)) \cup R)$. Let $U = Nint_\alpha(f^{-1}(Nint(Ncl(G)) \cup R)$. Hence U is a neutrosophic α-open set in (X, T) containing $x_{r,t,s}$ and therefore $f(U) \leq Nint(Ncl(G)) \cup R$. Hence, we have $Nint(f(U) \cap \overline{G}) = 0_{_N}$. □

Proposition 2.2. Let (X, T) and (Y, S) be any two neutrosophic topological space. Then a function $f : (X, T) \to (Y, S)$ is a neutrosophic rarely α-continuous if and only if $f^{-1}(G) \leq Nint_\alpha(f^{-1}(G \cup R))$ for every neutrosophic open set G in (Y, S), where R is a neutrosophic rare set with $Ncl(R) \cap G = 0_{_N}$.

Proof. Suppose that G be a neutrosophic rarely α-open set in (Y, S) containing $f(x_{r,t,s})$. Then $G \cap Ncl(R) = 0_N$ and U be a neutrosophic α-open set in (X, T) containing $x_{r,t,s}$, such that $f(U) \leq G \cup R$. It follows that $x_{r,t,s} \in U \leq f^{-1}(G \cup R)$. This implies that $f^{-1}(G) \leq Nint_\alpha(f^{-1}(G \cup R))$. □

Definition 2.8. A function $f : (X, T) \to (Y, S)$ is neutrosophic $I\alpha$-continuous at $x_{r,t,s}$ in (X, T) if for each neutrosophic open set G in (Y, S) containing $f(x_{r,t,s})$, there exists a neutrosophic α-open set U containing $x_{r,t,s}$, such that $Nint(f(U)) \leq G$.

If f has this property at each neutrosophic point $x_{r,t,s}$ in (X, T), then we say that f is neutrosophic $I\alpha$-continuous on (X, T).

Example 2.2. Let $X = \{a, b, c\}$. Define the neutrosophic sets A and B as follows:
$A = \langle x, (\frac{a}{0}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{1}, \frac{b}{0}, \frac{c}{1})\rangle$ and $B = \langle x, (\frac{a}{1}, \frac{b}{0}, \frac{c}{0}), (\frac{a}{1}, \frac{b}{0}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{1}, \frac{c}{1})\rangle$. Then $T = \{0_N, 1_N, A\}$ and $S = \{0_N, 1_N, B\}$ are neutrosophic topologies on X. Let (X, T) and (X, S) be neutrosophic topological spaces. Let $f : (X, T) \to (X, S)$ as defined by $f(a) = f(b) = b$ and $f(c) = c$ is neutrosophic $I\alpha$-continuous.

Proposition 2.3. Let (Y, S) be a neutrosophic regular space. Then the function $f : (X, T) \to (Y, S)$ is neutrosophic $I\alpha$ continuous on X if and only if f is neutrosophic rarely α-continuous on X.

Proof. $\Rightarrow$ It is obvious.
$\Leftarrow$ Let f be neutrosophic rarely α-continuous on (X, T). Suppose that $f(x_{r,t,s}) \in G$, where G is a neutrosophic open set in (Y, S) and a neutrosophic point $x_{r,t,s}$ in X. By the neutrosophic regularity of (Y, S), there exists a neutrosophic open set G_1 in (Y, S) such that G_1 containing $f(x_{r,t,s})$ and $Ncl(G_1) \leq G$. Since f is neutrosophic rarely α-continuous, then there exists a neutrosophic α open set U, such that $Nint(f(U)) \leq Ncl(G_1)$. This implies that $Nint(f(U)) \leq G$ which means that f is neutrosophic $I\alpha$-continuous on X. □

Definition 2.9. A function $f : (X, T) \to (Y, S)$ is called neutrosophic pre-α-open if for every neutrosophic α-open set U in X such that $f(U)$ is a neutrosophic α-open in Y.

Proposition 2.4. If a function $f : (X, T) \to (Y, S)$ is a neutrosophic pre-α-open and neutrosophic rarely α-continuous then f is neutrosophic almost α-continuous.

Proof. suppose that a neutrosophic point $x_{r,t,s}$ in X and a neutrosophic open set G in Y, containing $f(x_{r,t,s})$. Since f is neutrosophic rarely α-continuous at $x_{r,t,s}$, then there exists a neutrosophic α-open set U in X such that $Nint(f(U)) \subset Ncl(G)$. Since f is neutrosophic pre-α-open, we have $f(U)$ in Y. This implies that $f(U) \subset Nint(Ncl(Nint(f(U)))) \subset Nint(Ncl(G))$. Hence f is neutrosophic almost α-continuous. □

For a function $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$.

Proposition 2.5. Let $f : (X, T) \to (Y, S)$ be any function. If the $g : X \to X \times Y$ of f is neutrosophic rarely α-continuous then f is also neutrosophic rarely α-continuous.

Proof. Suppose that a neutrosophic point $x_{r,t,s}$ in X and a neutrosophic open set W in Y, containing $g(x_{r,t,s})$. It follows that there exists neutrosophic open sets 1_X and V in X and Y respectively, such that $(x_{r,t,s}, f(x_{r,t,s})) \in 1_X \times V \subset W$. Since f is neutrosophic rarely α-continuous, there exists a neutrosophic α-open set G such that $Nint(f(G)) \subset Ncl(V)$. Let $E = 1_X \cap G$. It follows that E be a neutrosophic α-open set in X and we have $Nint(g(E)) \subset Nint(1_X \times f(G)) \subset 1_X \times Ncl(V) \subset Ncl(W)$. Therefore g is neutrosophic rarely α-continuous. □

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Neutrosophic Semi-Continuous Multifunctions

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R. Dhavaseelan, S. Jafari, N. Rajesh, Florentin Smarandache (2018). Neutrosophic Semi-Continuous Multifunctions. *New Trends in Neutrosophic Theory and Applications* II, 345-354

ABSTRACT

In this paper we introduce the concepts of neutrosophic upper and neutrosophic lower semi-continuous multifunctions and study some of their basic properties.

KEYWORDS: Neutrosophic topological space, semi-continuous multifunctions.

1 INTRODUCTION

There is no doubt that the theory of multifunctions plays an important role in functional analysis and fixed point theory. It also has a wide range of applications in economic theory, decision theory, non-cooperative games, artificial intelligence, medicine and information sci-ences. Inspired by the research works of Smarandache (1999; 2001; 2007), we introduce and study the notions of neutrosophic upper and neutrosophic lower semi-continuous mul-tifunctions in this paper. Further, we present some characterizations and properties of such notions.

2 PRELIMINARIES

Throughout this paper, by (X, τ) or simply by X we will mean a topological space in the classical sense, and (Y, τ_1) or simply Y will stand for a neutrosophic topological space as defined by Salama and Alblowi (2012).

Definition 1. *Smarandache (1999, 2001, 2007) Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $A =< x, \mu_A(x), \sigma_A(x), \gamma_A(x) >$, where $\mu_A(x)$, $\sigma_A(x)$ and $\gamma_A(x)$ are represent the degree of member ship function, the degree of indeterminacy, and the degree of non-membership, respectively of each element $x \in X$ to the set A.*

Definition 2. *(Salama & Alblowi, 2012) A neutrosophic topology on a nonempty set X is a family τ of neutrosophic subsets of X which satisfies the following three conditions:*

1. $0, 1 \in \tau$,

2. *If* $g, h \in \tau$, *their* $g \wedge h \in \tau$,

3. *If* $f_i \in \tau$ *for each* $i \in I$, *then* $\vee_{i \in I} f_i \in \tau$.

The pair (X, τ) is called a neutrosophic topological space.

Definition 3. *Members of* τ *are called neutrosophic open sets, denoted by* $NO(X)$*, and complement of neutrosophic open sets are called neutrosophic closed sets, where the complement of a neutrosophic set* A*, denoted by* A^c*, is* $1 - A$*.*

Neutrosophic sets in Y will be denoted by $\lambda, \gamma, \delta, \rho$, etc., and although subsets of X will be denoted by A, B, U, V, etc. A neutrosophic point in Y with support $y \in Y$ and value $\alpha (0 < \alpha \leq 1)$ is denoted by y_α. A neutrosophic set λ in Y is said to be quasi-coincident (q-coincident) with a neutrosophic set μ, denoted by $\lambda q \mu$, if and only if there exists $y \in Y$ such that $\lambda(y) + \mu(y) > 1$. A neutrosophic set λ of Y is called a neutrosophic neighbourhood of a fuzzy point y_α in Y if there exists a neutrosophic open set μ in Y such that $y_\alpha \in \mu \leq \lambda$. The intersection of all neutrosophic closed sets of Y containing λ is called the neutrosophic closure of λ and is denoted by $\mathrm{Cl}(\lambda)$. The union of all neutrosophic open sets contained in λ is called the neutrosophic interior of λ and is denoted by $\mathrm{Int}(\lambda)$. The family of all open sets of a topological space X is denoted by $O(X)$ and $O(X, x)$ denoted the family $\{A \in O(X) | x \in A\}$, where x is a point of X.

Definition 4. *Let* (X, τ) *be a topological space in the classical sense and* (Y, τ_1) *be an neutrosophic topological space.* $F : (X, \tau) \to (Y, \tau_1)$ *is called a neutrosophic multifunction if and only if for each* $x \in X, F(x)$ *is a neutrosophic set in* Y*.*

Definition 5. *For a neutrosophic multifunction* $F : (X, \tau) \to (Y, \tau_1)$*, the upper inverse* $F^+(\lambda)$ *and lower inverse* $F^-(\lambda)$ *of a neutrosophic set* λ *in* Y *are defined as follows:*
$F^+(\lambda) = \{x \in X | F(x) \leq \lambda\}$ *and* $F^-(\lambda) = \{x \in X | F(x) q \lambda\}$.

Lemma 1. *For a neutrosophic multifunction* $F : (X, \tau) \to (Y, \tau_1)$*, we have* $F^-(1 - \lambda) = X - F^+(\lambda)$*, for any neutrosophic set* λ *in* Y*.*

Definition 5. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$, the upper inverse $F^+(\lambda)$ and lower inverse $F^-(\lambda)$ of a neutrosophic set λ in Y are defined as follows:*
$F^+(\lambda) = \{x \in X | F(x) \leq \lambda\}$ *and* $F^-(\lambda) = \{x \in X | F(x) q \lambda\}$.

Lemma 1. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$, we have $F^-(1 - \lambda) = X - F^+(\lambda)$, for any neutrosophic set λ in Y.*

3 NEUTROSOPHIC SEMICONTINUOUS MULTI–FUNCTIONS

Definition 6. *A neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ is said to be*

1. *neutrosophic upper semicontinuous at a point $x \in X$ if for each $\lambda \in NO(Y)$ containing $F(x)$ (therefore, $F(x) \leq \lambda$), there exists $U \in O(X, x)$ such that $F(U) \leq \lambda$ (therefore $U \subset F^+(\lambda)$).*

2. *neutrosophic lower semicontinuous at a point $x \in X$ if for each $\lambda \in NO(Y)$ with $F(x) q \lambda$, there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$.*

3. *neutrosophic upper semicontinuous (neutrosophic lower semicontinuous) if it is neutrosophic upper semicontinuous (neutrosophic lower semicontinuous) at each point $x \in X$.*

Theorem 1. *The following assertions are equivalent for a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$:*

1. *F is neutrosophic upper semicontinuous;*

2. *For each point x of X and each neutrosophic neighbourhood λ of $F(x)$, $F^+(\lambda)$ is a neighbourhood of x;*

3. *For each point x of X and each neutrosophic neighbourhood λ of $F(x)$, there exists a neighbourhood U of x such that $F(U) \leq \lambda$;*

4. *$F^+(\lambda) \in O(X)$ for oeach $\lambda \in NO(Y)$;*

5. *$F^-(\delta)$ is a closed set in X for each neutrosophic closed set δ of Y;*

6. *$\mathrm{Cl}(F^-(\mu)) \subseteq F^-(\mathrm{Cl}(\mu))$ for each neutrosophic set μ of Y.*

Proof. (1)⇒(2) Let $x \in X$ and μ be a neutrosophic neighbourhood of $F(x)$. Then there exists $\lambda \in NO(Y)$ such that $F(x) \leq \lambda \leq \mu$, By (1), there exists $U \in O(X, x)$ such that $F(U) \leq \lambda$. Therefore $x \in U \subseteq F^+(\mu)$ and hence $F^+(\mu)$ is a neighbourhood of x.
(2)⇒(3) Let $x \in X$ and λ be a neutrosophic neighbourhood of $F(x)$. Put $U = F^+(\lambda)$. Then

by (2), U is neighbourhood of x and $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$.
(3)⇒(4) Let $\lambda \in NO(Y)$, we want to show that $F^+(\lambda) \in O(X)$. So let $x \in F^+(\lambda)$. Then there exists a neighbourhood G of x such that $F(G) \leq \lambda$. Therefore for some $U \in O(X,x), U \subseteq G$ and $F(U) \leq \lambda$. Therefore we get $x \in U \subseteq F^+(\lambda)$ and hence $F^+(\lambda) \in O(X)$.
(4)⇒(5) Let δ be a neutrosophic closed set in Y. So, we have $X \backslash F^-(\delta) = F^+(1-\delta) \in O(X)$ and hence $F^-(\delta)$ is closed set in X.
(5)⇒(6) Let μ be any neutrosophic set in Y. Since $\mathrm{Cl}(\mu)$ is neutrosophic closed set in Y, $F^-(\mathrm{Cl}(\mu))$ is closed set in X and $F^-(\mu) \subseteq F^-(\mathrm{Cl}(\mu))$. Therefore, we obtain $\mathrm{Cl}(F^-(\mu)) \subseteq F^-(\mathrm{Cl}(\mu))$.
(6)⇒(1) Let $x \in X$ and $\lambda \in NO(Y)$ with $F(x) \leq \lambda$. Now $F^-(1-\lambda) = \{x \in X | F(x) q (1-\lambda)\}$. So, for x not belongs to $F^-(1-\lambda)$. Then, we must have $F(x) \hbar (1-\lambda)$ and this implies $F(x) \leq 1-(1-\lambda) = \lambda$ which is true. Therefore $x \notin F^-(1-\lambda)$ by (6), $x \notin \mathrm{Cl}(F^-(1-\lambda))$ and there exists $U \in O(X,x)$ such that $U \cap F^-(1-\lambda) = \emptyset$. Therefore, we obtain $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$. This proves F is neutrosophic upper semicontinuous. □

Theorem 2. *The following statements are equivalent for a neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$:*

1. *F is neutrosophic lower semicontinuous;*
2. *For each $\lambda \in NO(Y)$ and each $x \in F^-(\lambda)$, there exists $U \in O(X,x)$ such that $U \subseteq F^-(\lambda)$;*
3. *$F^-(\lambda) \in O(X)$ for every $\lambda \in NO(Y)$.*
4. *$F^+(\delta)$ is a closed set in X for every neutrosophic closed set δ of Y;*
5. *$\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(\mu))$ for every neutrosophic set μ of Y;*
6. *$F(\mathrm{Cl}(A)) \leq \mathrm{Cl}(F(A))$ for every subset A of X;*

Proof. (1)⇒(2) Let $\lambda \in NO(Y)$ and $x \in F^-(\lambda)$ with $F(x) q \lambda$. Then by properties–1, there exists $U \in O(X,x)$ such that $U \subseteq F^-(\lambda)$.
(2)⇒(3) Let $\lambda \in NO(Y)$ adn $x \in F^-(\lambda)$. Then by (2), there exists $U \in O(X,x)$ such that $U \subseteq F^-(\lambda)$. Therefore, we have $x \in U \subseteq \mathrm{Cl}\,\mathrm{Int}(U) \subseteq \mathrm{Cl}\,\mathrm{Int}(F^-(\lambda))$ and hence $F^-(\lambda) \in O(X)$.
(3)⇒(4) Let δ be a neutrosophic closed in Y. So we have $X \backslash F^+(\delta) = F^-(1-\delta) \in O(X)$ and hence $F^+(\delta)$ is closed set in X.
(4)⇒(5) Let μ be any neutrosophic set in Y. Since $\mathrm{Cl}(\mu)$ is neutrosophic closed set in Y, then by (4), we have $F^+(\mathrm{Cl}(\mu))$ is closed set in X and $F^+(\mu) \subseteq F^+(\mathrm{Cl}(\mu))$. Therefore, we obtain $\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(\mu))$.
(5)⇒(6) Let A be any subset of X. By (5), $\mathrm{Cl}(A) \subseteq \mathrm{Cl}\, F^+(F(A)) \subseteq F^+(\mathrm{Cl}(F(A)))$.

Therefore we obtain $\mathrm{Cl}(A) \subseteq F^+(\mathrm{Cl}\, F(A))$. This implies that $F(\mathrm{Cl}(A)) \leq \mathrm{Cl}\, F(A)$.
(6)$\Rightarrow$(5) Let μ be any neutrosophic set in Y. By (6), $F(\mathrm{Cl}\, F^+(\mu)) \leq \mathrm{Cl}(F(F^+(\mu)))$ and hence $\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(F(F^+(\mu)))) \subseteq F^+(\mathrm{Cl}(\mu))$. Therefore $\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(\mu))$.
(5)$\Rightarrow$(1) Let $x \in X$ and $\lambda \in NO(Y)$ with $F(x)q\lambda$. Now, $F^+(1-\lambda) = \{x \in X | F(x) \leq 1-\lambda\}$. So, for x not belongs to $F^+(1-\lambda)$, then we have $F(x) \nleq 1-\lambda$ and this implies that $F(x)q\lambda$. Therefore, $x \notin F^+(1-\lambda)$. Since $1-\lambda$ is neutrosophic closed set in Y, by (5), $x \notin \mathrm{Cl}(F^+(1-\lambda))$ and there exists $U \in O(X,x)$ such that $\emptyset = U \cap F^+(1-\lambda) = U \cap (X \backslash F^-(\lambda))$. Therefore, we obtain $U \subseteq F^-(\lambda)$. This proves F is neutrosophic lower semicontinuous. □

Definition 7. *For a given neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$, a neutrosophic multifunction $\mathrm{Cl}(F) : (X,\tau) \to (Y,\tau_1)$ is defined as $(\mathrm{Cl}\, F)(x) = \mathrm{Cl}\, F(x)$ for each $x \in X$.*

We use $\mathrm{Cl}\, F$ and the following Lemma to obtain a characterization of lower neutrosophic semicontinuous multifunction.

Lemma 2. *If $F : (X,\tau) \to (Y,\tau_1)$ is a neutrosophic multifunction, then $(\mathrm{Cl}\, F)^-(\lambda) = F^-(\lambda)$ for each $\lambda \in NO(Y)$.*

Proof. Let $\lambda \in NO(Y)$ and $x \in (\mathrm{Cl}\, F)^-(\lambda)$. This means that $(\mathrm{Cl}\, F)(x)q\lambda$. Since $\lambda \in NO(Y)$, we have $F(x)q\lambda$ and hence $x \in F^-(\lambda)$. Therefore $(\mathrm{Cl}\, F)^-(\lambda) \subseteq F^-(\lambda) - - - (*)$.

Conversely, let $x \in F^-(\lambda)$ since $\lambda \in NO(Y)$ then $F(x)q\lambda \subseteq (\mathrm{Cl}\, F)(x)q\lambda$ and hence $x \in (\mathrm{Cl}\, F)^-(\lambda)$. Therefore $F^-(\lambda) \subseteq (\mathrm{Cl}\, F)^-(\lambda) - - - -(**)$.
From $(*)$ and $(**)$, we get $(\mathrm{Cl}\, F)^-(\lambda) = F^-(\lambda)$. □

Theorem 3. *A neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$ is neutrosophic lower semicontinuous if and only if $\mathrm{Cl}\, F : (X,\tau) \to (Y,\tau_1)$ is neutrosophic lower semicontinuous.*

Proof. Suppose F is neutrosophic lower semicontinuous. Let $\lambda \in NO(Y)$ and $F(x)q\lambda$. This means that $x \in F^-(\lambda)$. Then there exists $U \in O(X,x)$ such that $U \subseteq F^-(\lambda)$. Therefore, we have $x \in U \subseteq \mathrm{Int}(U) \subseteq \mathrm{Int}\, F^-(\lambda)$ and hence $F^-(\lambda) \in O(X)$. Then by Lemma 2, we have $U \subseteq F^-(\lambda) = (\mathrm{Cl}\, F)^-(\lambda)$ and $(\mathrm{Cl}\, F)^-(\lambda) \in O(X)$, and hence $(\mathrm{Cl}\, F)(x)q\lambda$. Therefore $\mathrm{Cl}\, F$ is fuzy lower semicontinuous. Conversely, suppose $\mathrm{Cl}\, F$ is neutrosophic lower semicontinuous. If for each $\lambda \in NO(Y)$ with $(\mathrm{Cl}\, F)(x)q\lambda$ and $x \in (\mathrm{Cl}\, F)^-(\lambda)$ then there exists $U \in O(X,x)$ such that $U \subseteq (\mathrm{Cl}\, F)^-(\lambda)$. By Lemma 2 and Theorem 2, we have $U \subseteq (\mathrm{Cl}\, F^-(\lambda)) = F^-(\lambda)$ and $F^-(\lambda) \in O(X)$. Therefore F is neutrosophic lower semicontinuous. □

Definition 8. *Given a family $\{F_i : (X,\tau) \to (Y,\sigma) : i \in I\}$ of neutrosophic multifunctions, we define the union $\underset{i\in I}{\vee} F_i$ and the intersection $\underset{i\in I}{\wedge} F_i$ as follows: $\underset{i\in I}{\vee} F_i : (X,\tau) \to (Y,\sigma)$, $(\underset{i\in I}{\vee} F_i)(x) = \underset{i\in I}{\vee} F_i(x)$ and $\underset{i\in I}{\wedge} F_i : (X,\tau) \to (Y,\sigma)$, $(\underset{i\in I}{\wedge} F_i)(x) = \underset{i\in I}{\wedge} F_i(x)$.*

Theorem 4. *If $F_i : X \to Y$ are neutrosophic upper semi-continuous multifunctions for $i = 1,2,...,n$, then $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic upper semi-continuous multifunction.*

Proof. Let A be a neutrosophic open set of Y. We will show that $(\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A) = \{x \in X : \overset{n}{\underset{i\in I}{\vee}} F_i(x) \subset A\}$ is open in X. Let $x \in (\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$. Then $F_i(x) \subset A$ for $i = 1, 2, ..., n$. Since $F_i : X \to Y$ is neutrosophic upper semi-continuous multifunction for $i = 1, 2, ..., n$, then there exists an open set U_x containing x such that for all $z \in U_x$, $F_i(z) \subset A$. Let $U = \overset{n}{\underset{i\in I}{\cup}} U_x$. Then $U \subset (\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$. Thus, $(\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$ is open and hence $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic upper semi-continuous multifunction. □

Lemma 3. *Let $\{A_i\}_{i\in I}$ be a family of neutrosophic sets in a neutrosophic topological space X. Then a neutrosophic point x is quasi-coincident with $\vee A_i$ if and only if there exists an $i_0 \in I$ such that xqA_{i_0}.*

Theorem 5. *If $F_i : X \to Y$ are neutrosophic lower semi-continuous multifunctions for $i = 1, 2, ..., n$, then $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic lower semi-continuous multifunction.*

Proof. Let A be a neutrosophic open set of Y. We will show that $(\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A) = \{x \in X : (\overset{n}{\underset{i\in I}{\vee}} F_i)(x)qA\}$ is open in X. Let $x \in (\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$. Then $(\overset{n}{\underset{i\in I}{\vee}} F_i)(x)qA$ and hence $F_{i0}(x)qA$ for an i_0. Since $F_i : X \to Y$ is neutrosophic lower semi-continuous multifunction, there exists an open set U_x containing x such that for all $z \in U$, $F_{i0}(z)qA$. Then $(\overset{n}{\underset{i\in I}{\vee}} F_i)(z)qA$ and hence $U \subset (\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$. Thus, $(\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$ is open and hence $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic lower semi-continuous multifunction. □

Theorem 6. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

1. *$F_i = F_{|U_i}$ is a neutrosophic lower semi-continuous multifunction for all $i \in I$,*
2. *F is neutrosophic lower semi-continuous.*

Proof. (1) ⇒ (2): Let $x \in X$ and A be a neutrosophic open set in Y with $x \in F^-(A)$. Since $\{U_i : i \in I\}$ is an open cover for X, then $x \in U_{i0}$ for an $i_0 \in I$. We have $F(x) = F_{i0}(x)$ and hence $x \in F^-_{i0}(A)$. Since $F_{|U_i0}$ is neutrosophic lower semi-continuous, there exists an open set $B = G \cap U_{i0}$ in U_{i0} such that $x \in B$ and $F^-(A) \cap U_{i_0} = F_{|U_i}(A) \supset B = G \cap U_{i0}$, where G is open in X. We have $x \in B = G \cap U_{i0} \subset F^-_{|U_i0}(A) = F^-(A) \cap U_{i0} \subset F^-(A)$. Hence, F is neutrosophic lower semi-continuous.
(2) ⇒ (1): Let $x \in X$ and $x \in U_i$. Let A be a neutrosophic open set in Y with $F_i(x)qA$. Since F is lower semi-continuous and $F(x) = F_i(x)$, there exists an open set U containing x such that $U \subset F^-(A)$. Take $B = U_i \cap U$. Then B is open in U_i containing x. We have $B \subset F^-i(A)$. Thus F_i is a neutrosophic lower semi-continuous. □

Theorem 7. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

1. $F_i = F_{|U_i}$ *is a neutrosophic upper semi-continuous multifunction for all* $i \in I$,

2. F *is neutrosophic upper semi-continuous.*

Proof. It is similar to that of Theorem 6. □

Remark 8. *A subset* A *of a topological space* (X, τ) *can be considered as a neutrosophic set with characteristic function defined by*

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let (Y, σ) *be a neutrosophic topological space. The neutrosophic sets of the form* $A \times B$ *with* $A \in \tau$ *and* $B \in \sigma$ *form a basis for the product neutrosophic topology* $\tau \times \sigma$ *on* $X \times Y$, *where for any* $(x, y) \in X \times Y$, $(A \times B)(x, y) = min\{A(x), B(y)\}$.

Definition 9. *For a neutrosophic multifunction* $F : (X, \tau) \to (Y, \sigma)$, *the neutrosophic graph multifunction* $G_F : X \to X \times Y$ *of* F *is defined by* $G_F(x) = x_1 \times F(x)$ *for every* $x \in X$.

Theorem 9. *If the neutrosophic graph multifunction* G_F *of a neutrosophic multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is neutrosophic lower semi-continuous, then* F *is neutrosophic lower semi-continuous.*

Proof. Suppose that G_F is neutrosophic lower semi-continuous and $x \in X$. Let A be a neutrosophic open set in Y such that $F(x)qA$. Then there exists $y \in Y$ such that $(F(x))(y) + A(y) > 1$. Then $(G_F(x))(x, y) + (X \times A)(x, y) = (F(x))(y) + A(y) > 1$. Hence, $G_F(x)q(X \times A)$. Since G_F is neutrosophic lower semi-continuous, there exists an open set B in X such that $x \in B$ and $G_F(b)q(X \times A)$ for all $b \in B$. Let there exists $b_0 \in B$ such that $F(b_0)qA$. Then for all $y \in Y$, $(F(b_0))(y) + A(y) < 1$. For any $(a, c) \in X \times Y$, we have $(G_F(b_0))(a, c) \subset (F(b_0))(c)$ and $(X \times A)(a, c) \subset A(c)$. Since for all $y \in Y$, $(F(b_0))(y) + A(y) < 1$, $(G_F(b_0))(a, c) + (X \times A)(a, c) < 1$. Thus, $G_F(b_0)q(X \times A)$, where $b_0 \in B$. This is a contradiction since $G_F(b)q(X \times A)$ for all $b \in B$. Hence, F is neutrosophic lower semi-continuous. □

Theorem 10. *If the neutrosophic graph multifunction* G_F *of a neutrosophic multifunction* $F : X \to Y$ *is neutrosophic upper semi-continuous, then* F *is neutrosophic upper semi-continuous.*

Proof. Suppose that G_F is neutrosophic upper semi-continuous and let $x \in X$. Let A be neutrosophic open in Y with $F(x) \subset A$. Then $G_F(x) \subset X \times A$. Since G_F is neutrosophic upper semi-continuous, there exists an open set B containing x such that $G_F(B) \subset X \times A$. For any $b \in B$ and $y \in Y$, we have $(F(b))(y) = (G_F(b))(b, y) \subset (X \times A)(b, y) = A(y)$. Then $(F(b))(y) \subset A(y)$ for all $y \in Y$. Thus, $F(b) \subset A$ for any $b \in B$. Hence, F is neutrosophic upper semi-continuous. □

Theorem 11. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction. Then the following are equivalent:*

1. *F is neutrosophic lower semi-continuous,*
2. *For any $x \in X$ and any net $(x_i)_{i\in I}$ converging to x in X and each neutrosophic open set B in Y with $x \in F^-(B)$, the net $(x_i)_{i\in I}$ is eventually in $F^-(B)$.*

Proof. (1) $\Rightarrow$ (2): Let (x_i) be a net converging to x in X and B be any neutrosophic open set in Y with $x \in F^-(B)$. Since F is neutrosophic lower semi-continuous, there exists an open set $A \subset X$ containing x such that $A \subset F^-(B)$. Since $x_i \to x$, there exists an index $i_0 \in I$ such that $x_i \in A$ for every $i \geq i_0$. We have $x_i \in A \subset F^-(B)$ for all $i \geq i_0$. Hence, $(x_i)_{i\in I}$ is eventually in $F^-(B)$.
(2) $\Rightarrow$ (1): Suppose that F is not neutrosophic lower semi-continuous. There exists a point x and a neutrosophic open set A with $x \in F^-(A)$ such that $B \not\subseteq F^-(A)$ for any open set $B \subset X$ containing x. Let $x_i \in B$ and $x_i \notin F^-(A)$ for each open set $B \subset X$ containing x. Then the neighborhood net (x_i) converges to x but $(x_i)_{i\in I}$ is not eventually in $F^-(A)$. This is a contradiction. □

Theorem 12. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction. Then the following are equivalent:*

1. *F is neutrosophic upper semi-continuous,*
2. *For any $x \in X$ and any net (x_i) converging to x in X and any neutrosophic open set B in Y with $x \in F^+(B)$, the net (x_i) is eventually in $F^+(B)$.*

Proof. The proof is similar to that of Theorem 11. □

Theorem 13. *The set of all points of X at which a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is not neutrosophic upper semi-continuous is identical with the union of the frontier of the upper inverse image of neutrosophic open sets containing $F(x)$.*

Proof. Suppose F is not neutrosophic upper semi-continuous at $x \in X$. Then there exists a neutrosophic open set A in Y containing $F(x)$ such that $A \cap (X \backslash F^+(B)) \neq \emptyset$ for every open set A containing x. We have $x \in \mathrm{Cl}(X \backslash F^+(B)) = X \backslash \mathrm{Int}(F^+(B))$ and $x \in F^+(B)$. Thus, $x \in Fr(F^+(B))$. Conversely, let B be a neutrosophic open set in Y containing $F(x)$ with $x \in Fr(F^+(B))$. Suppose that F is neutrosophic upper semi-continuous at x. There exists an open set A containing x such that $A \subset F^+(B)$. We have $x \in \mathrm{Int}(F^+(B))$. This is a contradiction. Thus, F is not neutrosophic upper semi-continuous at x. □

Theorem 14. *The set of all points of X at which a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is not neutrosophic lower semi-continuous is identical with the union of the frontier of the lower inverse image of neutrosophic closed sets which are quasi-coincident with $F(x)$.*

Proof. It is similar to that of Theorem 13. □

Definition 10. *A neutrosophic set λ of a neutrosophic topological space Y is said to be neutrosophic compact relative to Y if every cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of λ by neutrosophic open sets of Y has a finite subcover $\{\lambda_i\}_{i=1}^n$ of λ.*

Definition 11. *A neutrosophic set λ of a neutrosophic topological space Y is said to be neutrosophic Lindelof relative to Y if every cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of λ by neutrosophic open sets of Y has a countable subcover $\{\lambda_n\}_{n\in N}$ of λ.*

Definition 12. *A neutrosophic topological space Y is said to be neutrosophic compact if χ_Y (characteristic function of Y) is neutrosophic compact relative to Y.*

Definition 13. *A neutrosophic topological space Y is said to be neutrosophic Lindelof if χ_Y (characteristic function of Y) is neutrosophic Lindelof relative to Y.*

Definition 14. *A neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$ is said to be punctually neutrosophic compact (resp. punctually neutrosophic Lindelof) if for each $x \in X, F(x)$ is neutrosophic compact (resp. neutrosophic Lindelof).*

Theorem 15. *Let the neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$ be a neutrosophic upper semicontinuous and F is punctually neutrosophic compact. If A is compact relative to X, then $F(A)$ is neutrosophic compact relative to Y.*

Proof. Let $\{\lambda_\alpha|\alpha\in\Delta\}$ be any cover of $F(Z)$ by neutrosophic copen sets of Y. We claim that $F(A)$ is neutrosophic compact relative to Y. For each $x\in A$, there exists a finite subset $\Delta(x)$ of Δ such that $F(x)\leq \cup\{\lambda_\alpha|\alpha\in\Delta(x)\}$. Put $\lambda(x)=\cup\{\lambda_\alpha|\alpha\in\Delta(x)\}$. Then $F(x)\leq \lambda(x)\in NO(Y)$ and there exists $U(x)\in O(X,x)$ such that $F(U(x))\leq\lambda(x)$. Since $\{U(x)|x\in A\}$ is an open cover of A there exists a finite number of A, say, $x_1,x_2,..,x_n$ such that $A\subseteq\cup\{U(x_i)|i=1,2,..,n\}$. Therefore we obtain $F(A)\leq F(\overset{n}{\underset{i=1}{\cup}}U(x_i))\leq \overset{n}{\underset{i=1}{\cup}}F(U(x_i))\leq \overset{n}{\underset{i=1}{\cup}}\lambda(x_i)\leq \overset{n}{\underset{i=1}{\cup}}(\underset{\alpha\in\Delta(x_i)}{\cup}\lambda_\alpha)$. This shows that $F(A)$ is neutrosophic compact relative to Y. □

Theorem 16. *Let the neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$ be a neutrosophic upper semicontinuous and F is punctually neutrosophic Lindelof. If A is Lindelof relative to X, then $F(A)$ is neutrosophic Lindelof relative to Y.*

Proof. The proof is similar to that of Theorem 15 □

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On the Classification of Bol-Moufang Type of Some Varieties of Quasi Neutrosophic Triplet Loop (Fenyves BCI-Algebras)

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Abstract: In this paper, Bol-Moufang types of a particular quasi neutrosophic triplet loop (BCI-algebra), chritened Fenyves BCI-algebras are introduced and studied. 60 Fenyves BCI-algebras are introduced and classified. Amongst these 60 classes of algebras, 46 are found to be associative and 14 are found to be non-associative. The 46 associative algebras are shown to be Boolean groups. Moreover, necessary and sufficient conditions for 13 non-associative algebras to be associative are also obtained: p-semisimplicity is found to be necessary and sufficient for a F_3, F_5, F_{42} and F_{55} algebras to be associative while quasi-associativity is found to be necessary and sufficient for F_{19}, F_{52}, F_{56} and F_{59} algebras to be associative. Two pairs of the 14 non-associative algebras are found to be equivalent to associativity (F_{52} and F_{55}, and F_{55} and F_{59}). Every BCI-algebra is naturally an F_{54} BCI-algebra. The work is concluded with recommendations based on comparison between the behaviour of identities of Bol-Moufang (Fenyves' identities) in quasigroups and loops and their behaviour in BCI-algebra. It is concluded that results of this work are an initiation into the study of the classification of finite Fenyves' quasi neutrosophic triplet loops (FQNTLs) just like various types of finite loops have been classified. This research work has opened a new area of research finding in BCI-algebras, vis-a-vis the emergence of 540 varieties of Bol-Moufang type quasi neutrosophic triplet loops. A 'Cycle of Algebraic Structures' which portrays this fact is provided.

Keywords: quasigroup; loop; BCI-algebra; Bol-Moufang; quasi neutrosophic loops; Fenyves identities

1. Introduction

BCK-algebras and BCI-algebras are abbreviated as two B-algebras. The former was raised in 1966 by Imai and Iseki [1], Japanese mathematicians, and the latter was put forward in the same year by Iseki [2]. The two algebras originated from two different sources: set theory and propositional calculi.

There are some systems which contain the only implicational functor among logical functors, such as the system of weak positive implicational calculus, BCK-system and BCI-system. Undoubtedly, there are common properties among those systems. We know that there are close relationships between the notions of the set difference in set theory and the implication functor in logical systems. For example, we have the following simple inclusion relations in set theory:

$$(A - B) - (A - C) \subseteq C - B, \qquad A - (A - B) \subseteq B.$$

These are similar to the propositional formulas in propositional calculi:

$$(p \to q) \to ((q \to r) \to (p \to r)), \qquad p \to ((p \to q) \to q),$$

which raise the following questions: What are the most essential and fundamental properties of these relationships? Can we formulate a general algebra from the above consideration? How will we find an axiom system to establish a good theory of general algebras? Answering these questions, K.Iseki formulated the notions of two B-algebras in which BCI-algebras are a wider class than BCK-algebras. Their names are taken from BCK and BCI systems in combinatory logic.

BCI-Algebras are very interesting algebraic structures that have generated wide interest among pure mathematicians.

1.1. BCI-algebra, Quasigroups, Loops and the Fenyves Identities

We start with some definitions and examples of some varieties of quasi neutrosophic triplet loop.

Definition 1. *A triple $(X, *, 0)$ is called a BCI-algebra if the following conditions are satisfied for any $x, y, z \in X$:*

1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $x * 0 = x$;
3. $x * y = 0$ *and* $y * x = 0 \Longrightarrow x = y$.

We call the binary operation $*$ on X the multiplication on X, and the constant 0 in X the zero element of X. We often write X instead of $(X, *, 0)$ for a BCI-algebra in brevity. Juxtaposition xy will at times be used for $x * y$ and will have preference over $*$ i.e., $xy * z = (x * y) * z$.

Example 1. *Let S be a set. Let 2^S be the power set of S, $-$ the set difference and $\emptyset$ the empty set. Then $(2^S, -, \emptyset)$ is a BCI-algebra.*

Example 2. *Suppose $(G, \cdot, e)$ is an abelian group with e as the identity element. Define a binary operation $*$ on G by putting $x * y = xy^{-1}$. Then $(G, *, e)$ is a BCI-algebra.*

Example 3. $(\mathbb{Z}, -, 0)$ *and* $(\mathbb{R} - \{0\}, \div, 1)$ *are BCI-algebras.*

Example 4. *Let S be a set. Let 2^S be the power set of S, $\triangle$ the symmetric difference and $\emptyset$ the empty set. Then $(2^S, \triangle, \emptyset)$ is a BCI-algebra.*

The following theorems give necessary and sufficient conditions for the existence of a BCI-algebra.

Theorem 1. *(Yisheng [3])*

Let X be a non-empty set, $*$ a binary operation on X and 0 a constant element of X. Then $(X, *, 0)$ is a BCI-algebra if and only if the following conditions hold:

1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $(x * (x * y)) * y = 0$;
3. $x * x = 0$;
4. $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 2. *A BCI-algebra $(X, *, 0)$ is called a BCK-algebra if $0 * x = 0$ for all $x \in X$.*

Definition 3. *A BCI-algebra $(X, *, 0)$ is called a Fenyves BCI-algebra if it satisfies any of the identities of Bol-Moufang type.*

The identities of Bol-Moufang type are given below:

F_1: $xy * zx = (xy * z)x$
F_2: $xy * zx = (x * yz)x$ (Moufang identity)
F_3: $xy * zx = x(y * zx)$
F_4: $xy * zx = x(yz * x)$ (Moufang identity)
F_5: $(xy * z)x = (x * yz)x$
F_6: $(xy * z)x = x(y * zx)$ (extra identity)
F_7: $(xy * z)x = x(yz * x)$
F_8: $(x * yz)x = x(y * zx)$
F_9: $(x * yz)x = x(yz * x)$
F_{10}: $x(y * zx) = x(yz * x)$
F_{11}: $xy \cdot xz = (xy * x)z$
F_{12}: $xy * xz = (x * yx)z$
F_{13}: $xy * xz = x(yx * z)$ (extra identity)
F_{14}: $xy * xz = x(y * xz)$
F_{15}: $(xy * x)z = (x * yx)z$
F_{16}: $(xy * x)z = x(yx * z)$
F_{17}: $(xy * x)z = x(y * xz)$ (Moufang identity)
F_{18}: $(x * yx)z = x(yx * z)$
F_{19}: $(x * yx)z = x(y * xz)$ (left Bol identity)
F_{20}: $x(yx * z) = x(y * xz)$
F_{21}: $yx * zx = (yx * z)x$
F_{22}: $yx * zx = (y * xz)x$ (extra identity)
F_{23}: $yx * zx = y(xz * x)$
F_{24}: $yx * zx = y(x * zx)$
F_{25}: $(yx * z)x = (y * xz)x$
F_{26}: $(yx * z)x = y(xz * x)$ (right Bol identity)
F_{27}: $(yx * z)x = y(x * zx)$ (Moufang identity)
F_{28}: $(y * xz)x = y(xz * x)$
F_{29}: $(y * xz)x = y(x * zx)$
F_{30}: $y(xz * x) = y(x * zx)$
F_{31}: $yx * xz = (yx * x)z$
F_{32}: $yx * xz = (y * xx)z$
F_{33}: $yx * xz = y(xx * z)$
F_{34}: $yx * xz = y(x * xz)$
F_{35}: $(yx * x)z = (y * xx)z$
F_{36}: $(yx * x)z = y(xx * z)$ (RC identity)
F_{37}: $(yx * x)z = y(x * xz)$ (C identity)
F_{38}: $(y * xx)z = y(xx * z)$
F_{39}: $(y * xx)z = y(x * xz)$ (LC identity)
F_{40}: $y(xx * z) = y(x * xz)$
F_{41}: $xx * yz = (x * xy)z$ (LC identity)
F_{42}: $xx * yz = (xx * y)z$
F_{43}: $xx * yz = x(x * yz)$
F_{44}: $xx * yz = x(xy * z)$
F_{45}: $(x * xy)z = (xx * y)z$
F_{46}: $(x * xy)z = x(x * yz)$ (LC identity)
F_{47}: $(x * xy)z = x(xy * z)$
F_{48}: $(xx * y)z = x(x * yz)$ (LC identity)
F_{49}: $(xx * y)z = x(xy * z)$
F_{50}: $x(x * yz) = x(xy * z)$
F_{51}: $yz * xx = (yz * x)x$
F_{52}: $yz * xx = (y * zx)x$
F_{53}: $yz * xx = y(zx * x)$ (RC identity)
F_{54}: $yz * xx = y(z * xx)$
F_{55}: $(yz * x)x = (y * zx)x$
F_{56}: $(yz * x)x = y(zx * x)$ (RC identity)
F_{57}: $(yz * x)x = y(z * xx)$ (RC identity)
F_{58}: $(y * zx)x = y(zx * x)$
F_{59}: $(y * zx)x = y(z * xx)$
F_{60}: $y(zx * x) = y(z * xx)$

Consequent upon this definition, there are 60 varieties of Fenyves BCI-algebras. Here are some examples of Fenyves' BCI-algebras:

Example 5. *Let us assume the BCI-algebra $(G, *, e)$ in Example 2. Then $(G, *, e)$ is an F_8-algebra, F_{19}-algebra, F_{29}-algebra, F_{39}-algebra, F_{46}-algebra, F_{52}-algebra, F_{54}-algebra, F_{59}-algebra.*

Example 6. *Let us assume the BCI-algebra $(2^S, -, \emptyset)$ in Example 1. Then $(2^S, -, \emptyset)$ is an F_3-algebra, F_5-algebra, F_{21}-algebra, F_{29}-algebra, F_{42}-algebra, F_{46}-algebra, F_{54}-algebra and F_{55}-algebra.*

Example 7. *The BCI-algebra $(2^S, \triangle, \emptyset)$ in Example 4 is associative.*

Example 8. *By considering the direct product of the BCI-algebras $(G, *, e)$ and $(2^S, -, \emptyset)$ of Example 2 and Example 1 respectively, we have a BCI-algebra $\left(G \times 2^S, (*, -), (e, \emptyset)\right)$ which is a F_{29}-algebra and a F_{46}-algebra.*

Remark 1. *Direct products of sets of BCI-algebras will result in BCI-algebras which are F_i-algebra for distinct i's.*

Definition 4. *A BCI-algebra $(X, *, 0)$ is called associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.*

Definition 5. *A BCI-algebra $(X, *, 0)$ is called p-semisimple if $0 * (0 * x) = x$ for all $x \in X$.*

Theorem 2. *(Yisheng [3]) Suppose that $(X, *, 0)$ is a BCI-algebra. Define a binary relation $\leqslant$ on X by which $x \leqslant y$ if and only if $x * y = 0$ for any $x, y \in X$. Then $(X, \leqslant)$ is a partially ordered set with 0 as a minimal element (meaning that $x \leqslant 0$ implies $x = 0$ for any $x \in X$).*

Definition 6. *A BCI-algebra $(X, *, 0)$ is called quasi-associative if $(x * y) * z \leq x * (y * z)$ for all $x, y, z \in X$.*

The following theorems give equivalent conditions for associativity, quasi-associativity and p-semisimplicity in a BCI-algebra:

Theorem 3. *(Yisheng [3])*
Given a BCI-algebra X, the following are equivalent $x, y, z \in X$:

1. *X is associative.*
2. $0 * x = x$.
3. $x * y = y * x \; \forall \; x, y \in X$.

Theorem 4. *(Yisheng [3])*
Let X be a BCI-algebra. Then the following conditions are equivalent for any $x, y, z, u \in X$:

1. *X is p-semisimple*
2. $(x * y) * (z * u) = (x * z) * (y * u)$.
3. $0 * (y * x) = x * y$.
4. $(x * y) * (x * z) = z * y$.
5. $z * x = z * y$ *implies* $x = y$. *(the left cancellation law i.e., LCL)*
6. $x * y = 0$ *implies* $x = y$.

Theorem 5. *(Yisheng [3])*
Given a BCI-algebra X, the following are equivalent for all $x, y \in X$:

1. *X is quasi-associative.*
2. $x * (0 * y) = 0$ *implies* $x * y = 0$.
3. $0 * x = 0 * (0 * x)$.
4. $(0 * x) * x = 0$.

Theorem 6. *(Yisheng [3])*
A triple $(X, *, 0)$ is a BCI-algebra if and only if there is a partial ordering $\leqslant$ on X such that the following conditions hold for any $x, y, z \in X$:

1. $(x * y) * (x * z) \leqslant z * y$;
2. $x * (x * y) \leqslant y$;
3. $x * y = 0$ *if and only if* $x \leqslant y$.

Theorem 7. *(Yisheng [3])*
Let X be a BCI-algebra. X is p-semisimple if and only if one of the following conditions holds for any $x, y, z \in X$:

1. $x * z = y * z$ *implies* $x = y$. *(the right cancellation law i.e., RCL)*
2. $(y * x) * (z * x) = y * z$.
3. $(x * y) * (x * z) = 0 * (y * z)$.

Theorem 8. *(Yisheng [3])*
Let X be a BCI-algebra. X is p-semisimple if and only if one of the following conditions holds for any $x, y \in X$:

1. $x * (0 * y) = y$.
2. $0 * x = 0 \implies x = 0$.

Theorem 9. *(Yisheng [3]) Suppose that $(X, *, 0)$ is a BCI-algebra. X is associative if and only if X is p-semisimple and X is quasi-associative.*

Theorem 10. *(Yisheng [3]) Suppose that $(X, *, 0)$ is a BCI-algebra. Then $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.*

Remark 2. *In Theorem 9, quasi-associativity in BCI-algebra plays a similar role to that which weak associativity (i.e., the F_i identities) plays in quasigroup and loop theory.*

We now move on to quasigroups and loops.

Definition 7. *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L. If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If in a groupoid $(L, \cdot)$, the equations:*

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. If in a quasigroup $(L, \cdot)$, there exists a unique element e called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

Definition 8. *Let $(L, \cdot)$ be a groupoid.*
The left nucleus of L is the set $N_\lambda(L, \cdot) = N_\lambda(L) = \{a \in L : ax \cdot y = a \cdot xy \ \forall\ x, y \in L\}$.
The right nucleus of L is the set $N_\rho(L, \cdot) = N_\rho(L) = \{a \in L : y \cdot xa = yx \cdot a \ \forall\ x, y \in L\}$.
The middle nucleus of L is the set $N_\mu(L, \cdot) = N_\mu(L) = \{a \in L : ya \cdot x = y \cdot ax \ \forall\ x, y \in L\}$.
The nucleus of L is the set $N(L, \cdot) = N(L) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot)$.
The centrum of L is the set $C(L, \cdot) = C(L) = \{a \in L : ax = xa \ \forall\ x \in L\}$.
The center of L is the set $Z(L, \cdot) = Z(L) = N(L, \cdot) \cap C(L, \cdot)$.

In the recent past, and up to now, identities of Bol-Moufang type have been studied on the platform of quasigroups and loops by Fenyves [4], Phillips and Vojtechovsky [5], Jaiyeola [6–8], Robinson [9], Burn [10–12], Kinyon and Kunen [13] as well as several other authors.

Since the late 1970s, BCI and BCK algebras have been given a lot of attention. In particular, the participation in the research of polish mathematicians Tadeusz Traczyk and Andrzej Wronski as well as Australian mathematician William H. Cornish, in addition to others, is causing this branch of algebra to develop rapidly. Many interesting and important results are constantly discovered. Now, the theory of BCI-algebras has been widely spread to areas such as general theory which include congruences, quotient algebras, BCI-Homomorphisms, direct sums and direct products, commutative BCK-algebras, positive implicative and implicative BCK-algebras, derivations of BCI-algebras, and ideal theory of BCI-algebras ([1,14–17]).

1.2. BCI-Algebras as a Quasi Neutrosophic Triplet Loop

Consider the following definition.

Definition 9. *(Quasi Neutrosophic Triplet Loops (QNTL), Zhang et al. [18])*
Let $(X, *)$ be a groupoid.

1. *If there exist $b, c \in X$ such that $a * b = a$ and $a * c = b$, then a is called an NT-element with (r-r)-property. If every $a \in X$ is an NT-element with (r-r)-property, then, $(X, *)$ is called a (r-r)-quasi NTL.*
2. *If there exist $b, c \in X$ such that $a * b = a$ and $c * a = b$, then a is called an NT-element with (r-l)-property. If every $a \in X$ is an NT-element with (r-l)-property, then, $(X, *)$ is called a (r-l)-quasi NTL.*
3. *If there exist $b, c \in X$ such that $b * a = a$ and $c * a = b$, then a is called an NT-element with (l-l)-property. If every $a \in X$ is an NT-element with (l-l)-property, then, $(X, *)$ is called a (l-l)-quasi NTL.*

4. *If there exist $b, c \in X$ such that $b * a = a$ and $a * c = b$, then a is called an NT-element with (l-r)-property. If every $a \in X$ is an NT-element with (l-r)-property, then, $(X, *)$ is called a (l-r)-quasi NTL.*
5. *If there exist $b, c \in X$ such that $a * b = b * a = a$ and $a * c = b$, then a is called an NT-element with (lr-r)-property. If every $a \in X$ is an NT-element with (lr-r)-property, then, $(X, *)$ is called a (lr-r)-quasi NTL.*
6. *If there exist $b, c \in X$ such that $a * b = b * a = a$ and $c * a = b$, then a is called an NT-element with (lr-l)-property. If every $a \in X$ is an NT-element with (lr-l)-property, then, $(X, *)$ is called a (lr-l)-quasi NTL.*
7. *If there exist $b, c \in X$ such that $a * b = a$ and $a * c = c * a = b$, then a is called an NT-element with (r-lr)-property. If every $a \in X$ is an NT-element with (r-lr)- property, then, $(X, *)$ is called a (r-lr)-quasi NTL.*
8. *If there exist $b, c \in X$ such that $b * a = a$ and $a * c = c * a = b$, then a is called an NT-element with (l-lr)-property. If every $a \in X$ is an NT-element with (l-lr)-property, then, $(X, *)$ is called a (l-lr)-quasi NTL.*
9. *If there exist $b, c \in X$ such that $a * b = b * a = a$ and $a * c = c * a = b$, then a is called an NT-element with (lr-lr)-property. If every $a \in X$ is an NT-element with (lr-lr)-property, then, $(X, *)$ is called a (lr-lr)-quasi NTL.*

Consequent upon Definition 9 and the 60 Fenyves identities F_i, $1 \leq i \leq 60$, there are 60 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) for each of the nine varieties of QNTLs in Definition 9. Thereby making it 540 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) in all. A BCI-algebra is a (r-r)-QNT, (r-l)-QNTL and (r-lr)-QNTL. Thus, any F_i BCI-algebra, $1 \leq i \leq 60$ belongs to at least one of the following varieties of Fenyves quasi neutrosophic triplet loops: (r-r)-QNTL, (r-l)-QNTL and (r-lr)-QNTL which we refer to as (r-r)-FQNTL, (r-l)-FQNTL and (r-lr)-FQNTL respectively. Any associative QNTL will be called quasi neutrosophic triplet group (QNTG).

The variety of quasi neutrosophic triplet loop is a generalization of neutrosophic triplet group (NTG) which was originally introduced by Smarandache and Ali [19]. Neutrosophic triplet set (NTS) is the foundation of neutrosophic triplet group. New results and developments on neutrosophic triplet groups and neutrosophic triplet loop have been reported by Zhang et al. [18,20,21], and Smarandache and Jaiyéọlá [22,23].

It must be noted that triplets are not connected at all with intuitionistic fuzzy set. Neutrosophic set [24] is a generalization of intuitionistic fuzzy set (a generalization of fuzzy set). In Intuitionistic fuzzy set, an element has a degree of membership and a degree of non-membership, and the deduction of the sum of these two from 1 is considered the hesitant degree of the element. These intuitionistic fuzzy set components are dependent (viz. [25–28]). In the neutrosophic set, an element has three independent degrees: membership (truth-t), indeterminacy (i), and non-membership (falsity-f), and their sum is up to 3. However, the current paper utilizes the neutrosophic triplets, which are not defined in intuitionistic fuzzy set, since there is no neutral element in intuitionistic fuzzy sets. In a neutrosophic triplet set $(X, *)$, for each element $x \in X$ there exists a neutral element denoted $neut(x) \in X$ such that $x * neut(x) = neut(x) * x = x$, and an opposite of x denoted $anti(x) \in X$ such that $anti(x) * x = x * anti(x) = neut(x)$. Thus, the triple $(x, neut(x), anti(x))$ is called a neutrosophic triplet which in the philosophy of 'neutrosophy', can be algebraically harmonized with (t, i, f) in neutrosophic set and then extended for neutrosophic hesitant fuzzy [29] set as proposed for (t, i, f)-neutrosophic structures [30]. Unfortunately, such harmonization is not readily defined in intuitionistic fuzzy sets.

Theorem 11. *(Zhang et al. [18]) A (r-lr)-QNTG or (l-lr)-QNTG is a NTG.*

This present study looks at Fenyves identities on the platform of BCI-algebras. The main objective of this study is to classify the Fenyves BCI-algebras into associative and non-associative types. It will also be shown that some Fenyves identities play the roles of quasi-associativity and p-semisimplicity , vis-a-vis Theorem 9 in BCI-algebras.

2. Main Results

We shall first clarify the relationship between a BCI-algebra, a quasigroup and a loop.

Theorem 12.

1. *A BCI algebra X is a quasigroup if and only if it is p-semisimple.*
2. *A BCI algebra X is a loop if and only if it is associative.*
3. *An associative BCI algebra X is a Boolean group.*

Proof. We use Theorem 3, Theorem 7 and Theorem 4.

1. From Theorems 7 and 4, p-semisimplicity is equivalent to the left and right cancellation laws, which consequently implies that X is a quasigroup if and only if it is p-semisimple.
2. One of the axioms that a BCI-algebra satisfies is $x * 0 = x$ for all $x \in X$. So, 0 is already the right identity element. Now, from Theorem 3, associativity is equivalent to $0 * x = x$ for all $x \in X$. So, 0 is also the left identity element of X. The conclusion follows.
3. In a BCI-algebra, $x * x = 0$ for all $x \in X$. And 0 is the identity element of X. Hence, every element is the inverse of itself.

□

Lemma 1. *Let $(X, *, 0)$ be a BCI-algebra.*

1. $0 \in N_\rho(X)$.
2. $0 \in N_\lambda(X), N_\mu(X)$ *implies X is quasi-associative.*
3. *If $0 \in N_\lambda(X)$, then the following are equivalent:*
   (a) *X is p-semisimple.*
   (b) $xy = 0y \cdot x$ *for all* $x, y \in L$.
   (c) $xy = 0x \cdot y$ *for all* $x, y \in L$.
4. *If $0 \in N_\lambda(X)$ or $0 \in N_\mu(X)$, then X is p-semisimple if and only if X is associative.*
5. *If $0 \in N(X)$, then X is p-semisimple if and only if X is associative.*
6. *If $(X, *, 0)$ is a BCK-algebra, then*
   (a) $0 \in N_\lambda(X)$.
   (b) $0 \in N_\mu(X)$ *implies X is a trivial BCK-algebra.*
7. *The following are equivalent:*
   (a) *X is associative.*
   (b) $x \in N_\lambda(X)$ *for all* $x \in X$.
   (c) $x \in N_\rho(X)$ *for all* $x \in X$.
   (d) $x \in N_\mu(X)$ *for all* $x \in X$.
   (e) $0 \in C(X)$.
   (f) $x \in C(X)$ *for all* $x \in X$.
   (g) $x \in Z(X)$ *for all* $x \in X$.
   (h) $0 \in Z(X)$.
   (i) *X is a (lr-r)-QNTL.*
   (j) *X is a (lr-l)-QNTL.*
   (k) *X is a (lr-lr)-QNTL*
8. *If $(X, *, 0)$ is a BCK-algebra and $0 \in C(X)$, then X is a trivial BCK-algebra.*

Proof. This is routine by simply using the definitions of nuclei, centrum, center of a BCI-algebra and QNTL alongside Theorems 3–10 appropriately. □

Remark 3. *Based on Theorem 11, since an associative BCI-algebra is a (r-lr)-QNTG, then, an associative BCI-algebra is a NTG. This corroborates the importance of the study of non-associative BCI-algebra i.e., weak associative laws (F_i-identities) in BCI-algebra, as mentioned earlier in the objective of this work.*

Theorem 13. *Let $(X, *, 0)$ be a BCI-algebra. If X is any of the following Fenyves BCI-algebras, then X is associative.*

1. *F_1-algebra*
2. *F_2-algebra*
3. *F_4-algebra*
4. *F_6-algebra*
5. *F_7-algebra*
6. *F_9-algebra*
7. *F_{10}-algebra*
8. *F_{11}-algebra*
9. *F_{12}-algebra*
10. *F_{13}-algebra*
11. *F_{14}-algebra*
12. *F_{15}-algebra*
13. *F_{16}-algebra*
14. *F_{17}-algebra*
15. *F_{18}-algebra*
16. *F_{20}-algebra*
17. *F_{22}-algebra*
18. *F_{23}-algebra*
19. *F_{24}-algebra*
20. *F_{25}-algebra*
21. *F_{26}-algebra*
22. *F_{27}-algebra*
23. *F_{28}-algebra*
24. *F_{30}-algebra*
25. *F_{31}-algebra*
26. *F_{32}-algebra*
27. *F_{33}-algebra*
28. *F_{34}-algebra*
29. *F_{35}-algebra*
30. *F_{36}-algebra*
31. *F_{37}-algebra*
32. *F_{38}-algebra*
33. *F_{40}-algebra*
34. *F_{41}-algebra*
35. *F_{43}-algebra*
36. *F_{44}-algebra*
37. *F_{45}-algebra*
38. *F_{47}-algebra*
39. *F_{48}-algebra*
40. *F_{49}-algebra*
41. *F_{50}-algebra*
42. *F_{51}-algebra*
43. *F_{53}-algebra*
44. *F_{57}-algebra*
45. *F_{58}-algebra*
46. *F_{60}-algebra*

Proof.

1. Let X be an F_1-algebra. Then $xy * zx = (xy * z)x$. With $z = y$, we have $xy * yx = (xy * y)x$ which implies $xy * yx = (xy * x)y = (xx * y)y = (0 * y)y = 0 * (y * y)$ (since $0 \in N_\lambda(X)$; this is achieved by putting $y = x$ in the F_1 identity) $= 0 * 0 = 0$. This implies $xy * yx = 0$. Now replacing x with y, and y with x in the last equation gives $yx * xy = 0$ implying that $x * y = y * x$ as required.
2. Let X be an F_2-algebra. Then $xy * zx = (x * yz)x$. With $y = z$, we have $xz * zx = (x * zz)x = (x * 0) * x = x * x = 0$ implying that $xz * zx = 0$. Now replacing x with z, and z with x in the last equation gives $zx * xz = 0$ implying that $x * z = z * x$ as required.
3. Let X be a F_4-algebra. Then, $xy * zx = x(yz * x)$. Put $y = x$ and $z = 0$, then you get $0 * 0x = x$ which means X is p-semisimple. Put $x = 0$ and $y = 0$ to get $0z = 0 * 0z$ which implies that X is quasi-associative (Theorem 5). Thus, by Theorem 9, X is associative.
4. Let X be an F_6-algebra. Then, $(xy * z)x = x(y * zx)$. Put $x = y = 0$ to get $0z = 0 * 0z$ which implies that X is quasi-associative (Theorem 5). Put $y = 0$ and $z = x$, then we have $0 * x = x$. Thus, X is associative.
5. Let X be an F_7-algebra. Then $(xy * z)x = x(yz * x)$. With $z = 0$, we have $xy * x = x(y * x)$. Put $y = x$ in the last equation to get $xx * x = (x * xx)$ implying $0 * x = x$.
6. Let X be an F_9-algebra. Then $(x * yz)x = x(yz * x)$. With $z = 0$, we have $(x * y) * x = x(y * x)$. Put $y = x$ in the last equation to get $(x * x)x = x(x * x)$ implying $0 * x = x$.
7. Let X be an F_{10}-algebra. Then, $x(y * zx) = x(yz * x)$. Put $y = x = z$, then we have $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$. which means that X is p-semisimple (Theorem 8(2)). Hence, X has the LCL by Theorem 4. Thence, the F_{10} identity $x(y * zx) = x(yz * x) \Rightarrow y * zx = yz * x$ which means that X is associative.
8. Let X be an F_{11}-algebra. Then $xy * xz = (xy * x)z$. With $y = 0$, we have $x * xz = xx * z$. Put $z = x$ in the last equation to get $x = 0 * x$ as required.
9. Let X be an F_{12}-algebra. Then $xy * xz = (x * yx)z$. With $z = 0$, we have $xy * x = x * yx$. Put $y = x$ in the last equation to get $xx * x = x * xx$ implying $0 * x = x$ as required.
10. Let X be an F_{13}-algebra. Then $xy * xz = x(yx * z)$. With $z = 0$, we have $(x * y)x = x * yx$ which implies $(x * x)y = x * yx$ which implies $0 * y = x * yx$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
11. Let X be an F_{14}-algebra. Then $xy * xz = x(y * xz)$. With $z = 0$, we have $xy * x = x * yx$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
12. Let X be an F_{15}-algebra. Then $(xy * x)z = (x * yx)z$. With $z = 0$, we have $(xy * x) = (x * yx)$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
13. Let X be an F_{16}-algebra. Then $(xy * x)z = x(yx * z)$. With $z = 0$, we have $(xy * x) = (x * yx)$. Put $y = x$ in the last equation to get $0 * x = x$ as required.

14. Let X be an F_{17}-algebra. Then $(xy*x)z = x(y*xz)$. With $z=0$, we have $(xy*x) = x(y*x)$. Put $y=x$ in the last equation to get $0*x = x$ as required.
15. Let X be an F_{18}-algebra. Then $(x*yx)z = x(yx*z)$. With $y=0$, we have $(x*0x)z = x(0x*z)$. Since $0 \in N_\lambda(X)$ and $0 \in N_\mu(X)$, (these are obtained by putting $x=0$ and $x=y$ respectively in the F_{18}-identity), the last equation becomes $(x0*x)z = x(0*xz) = x0*xz = x*xz$ which implies $0*z = x*xz$. Put $x=z$ in the last equation to get $0*z = z$ as required.
16. This is similar to the proof for F_{10}-algebra.
17. Let X be an F_{22}-algebra. Then $yx*zx = (y*xz)x$. Put $y=x, z=0$, then $0x = 0*0x$ which implies that X is quasi-associative. By Theorem 10, the F_{22} identity implies that $yx*zx = yx*xz$. Substitute $x=0$ to get $yz = y*0z$. Now, put $y=z$ in this to get $z*0z = 0$. So, $0z = 0 \Rightarrow z = 0$. Hence, X is p-semisimple (Theorem 8(2)). Thus, by Theorem 9, X is associative.
18. Let X be an F_{23}-algebra. Then $yx*zx = y(xz*x)$. With $z=0$, we have $yx*0x = y(x*x)$ which implies $yx*0x = y$. Since $0 \in N_\mu(X)$, (this is obtained by putting $z=x$ in the F_{23}-identity), the last equation becomes $(yx*0)*x = y$ which implies $(yx*x) = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
19. Let X be an F_{24}-algebra. Then $yx*zx = y(x*zx)$. With $z=0$, we have $yx*0x = y(x*0x)$. Since $0 \in N_\mu(X)$,(this is obtained by putting $x=0$ in the F_{24}-identity), the last equation becomes $((yx)0*x) = y(x0*x)$ which implies $yx*x = y$. Put $y=x$ in the last equation to get $0*y = y$ as required.
20. Let X be an F_{25}-algebra. Then $(yx*z)x = (y*xz)x$. Put $x=0$, then $yz = y*0z$. Substitute $z=y$, then $y*0y = 0$. So, $0y = 0 \Rightarrow y = 0$. Hence, X is p-semisimple (Theorem 8(2)). Hence, X has the RCL by Theorem 7. Thence, the F_{25} identity $(yx*z)x = (y*xz)x$ implies $yx*z = y*xz$. Thus, X is associative.
21. Let X be an F_{26}-algebra. Then $(yx*z)x = y(xz*x)$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
22. Let X be an F_{27}-algebra. Then $(yx*z)x = y(x*zx)$. Put $z=x=y$, then $0x*x = 0$ which implies X is quasi-associative. Put $x=0$ and $y=z$ to get $z*0z = 0$. So, $0z = 0 \Rightarrow z = 0$. Hence, X is p-semisimple (Theorem 8(2)). Thus, by Theorem 9, X is associative.
23. Let X be an F_{28}-algebra. Then $(y*xz)x = y(xz*x)$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
24. The proof of this is similar to the proof for F_{10}-algebra.
25. Let X be an F_{31}-algebra. Then $yx*xz = (yx*x)z$. By Theorem 10, the F_{31} identity becomes F_{25} identity which implies that X is associative.
26. Let X be an F_{32}-algebra. Then $yx*xz = (y*xx)z$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
27. Let X be an F_{33}-algebra. Then $yx*xz = y(xx*z)$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
28. Let X be an F_{34}-algebra. Then $yx*xz = y(x*xz)$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
29. Let X be an F_{35}-algebra. Then $(yx*x)z = (y*xx)z$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
30. Let X be an F_{36}-algebra. Then $(yx*x)z = y(xx*z)$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
31. Let X be an F_{37}-algebra. Then $(yx*x)z = y(x*xz)$. With $z=0$, we have $yx*x = y$. Put $x=y$ in the last equation to get $0*y = y$ as required.
32. Let X be an F_{38}-algebra. Then, $yz = y*0z$. Put $z=y$, then $y*0y = 0$. So, $0y = 0 \Rightarrow y = 0$. Hence, X is p-semisimple (Theorem 8(2)). Now, put $y=x$, then $xz = x*0z$. Now, substitute $x=0$ to get $0z = 0*0z$ which means that X is quasi-associative. Thus, by Theorem 9, X is associative.
33. Let X be an F_{40}-algebra. By the F_{40} identity, $y*0z = y(x*xz)$. Put $z=x=y$ to get $0*0x = 0$. So, $0x = 0 \Rightarrow x = 0$. Hence, X is p-semisimple (Theorem 8(2)). Thus, X has the LCL by Theorem 4. Thence, the F_{40} identity $y(xx*z) = y(x*xz)$ becomes $0*z = x*xz$. Substituting $z=x$, we get $0x = x$ which means that X is associative.

34. Let X be an F_{41}-algebra. Then $xx * yz = (x * xy)z$. With $z = 0$, we have $0 * y = x * xy$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
35. Let X be an F_{43}-algebra. Then $xx * yz = x(x * yz)$. With $z = 0$, we have $0 * y = x(x * y)$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
36. Let X be an F_{44}-algebra. Then $xx * yz = x(xy * z)$. With $z = 0$, we have $0 * y = x(x * y)$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
37. Let X be an F_{45}-algebra. Then $(x * xy)z = (xx * y)z$. With $z = 0$, we have $x * xy = 0 * y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
38. Let X be an F_{47}-algebra. Then $(x * xy)z = x(xy * z)$. With $y = 0$, we have $0 * z = x(x * z)$. Put $x = z$ in the last equation to get $0 * z = z$ as required.
39. Let X be an F_{48}-algebra. Then $(xx * y)z = x(x * yz)$. With $z = 0$, we have $0 * y = x * xy$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
40. Let X be an F_{49}-algebra. Then $(xx * y)z = x(xy * z)$. With $y = 0$, we have $0 * z = x * xz$. Put $x = z$ in the last equation to get $0 * z = z$ as required.
41. This is similar to the proof for F_{10}-algebra.
42. Let X be an F_{51}-algebra. Then $yz * xx = (yz * x)x$. With $z = 0$, we have $y = (y * x)x$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
43. Let X be an F_{53}-algebra. Then $yz * xx = y(zx * x)$ which becomes $yz = y(zx * x)$. Put $z = x$ to get $yx = y * 0x$. Substituting $y = x$, we get $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$, which means that X is p-semisimple (Theorem 8(2)). Now, put $y = 0$ in $yx = y * 0x$ to get $0x = 0 * 0x$. Hence, X is quasi-associative. Thus, X is associative.
44. Let X be an F_{57}-algebra. Then $(yz * x)x = y(z * xx)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
45. Let X be an F_{58}-algebra. Then $(y * zx)x = y(zx * x)$. Put $y = x = z$ to get $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$, which means that X is p-semisimple (Theorem 8(2)). Now, put $z = x, y = 0$ to get $0x = 0 * 0x$. Hence, X is quasi-associative. Thus, X is associative.
46. Let X be an F_{60}-algebra. Then $y(zx * x) = y(z * xx)$. Put $y = x = z$ to get $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$, which means that X is p-semisimple (Theorem 8(2)). Hence, X has the LCL by Theorem 4. Thence, the F_{10} identity becomes $zx * x = z * xx$. Now, substitute $z = x$ to get $0x = x$. Thus, X is associative.

□

Corollary 1. *Let $(X, *, 0)$ be a BCI-algebra. If X is any of the following Fenyves' BCI-algebras, then $(X, *)$ is a Boolean group.*

1. *F_1-algebra*
2. *F_2-algebra*
3. *F_4-algebra*
4. *F_6-algebra*
5. *F_7-algebra*
6. *F_9-algebra*
7. *F_{10}-algebra*
8. *F_{11}-algebra*
9. *F_{12}-algebra*
10. *F_{13}-algebra*
11. *F_{14}-algebra*
12. *F_{15}-algebra*
13. *F_{16}-algebra*
14. *F_{17}-algebra*
15. *F_{18}-algebra*
16. *F_{20}-algebra*
17. *F_{22}-algebra*
18. *F_{23}-algebra*
19. *F_{24}-algebra*
20. *F_{25}-algebra*
21. *F_{26}-algebra*
22. *F_{27}-algebra*
23. *F_{28}-algebra*
24. *F_{30}-algebra*
25. *F_{31}-algebra*
26. *F_{32}-algebra*
27. *F_{33}-algebra*
28. *F_{34}-algebra*
29. *F_{35}-algebra*
30. *F_{36}-algebra*
31. *F_{37}-algebra*
32. *F_{38}-algebra*
33. *F_{40}-algebra*
34. *F_{41}-algebra*
35. *F_{43}-algebra*
36. *F_{44}-algebra*
37. *F_{45}-algebra*
38. *F_{47}-algebra*
39. *F_{48}-algebra*
40. *F_{49}-algebra*
41. *F_{50}-algebra*
42. *F_{51}-algebra*
43. *F_{53}-algebra*
44. *F_{57}-algebra*
45. *F_{58}-algebra*
46. *F_{60}-algebra*

Proof. This follows from Theorems 12 and 13. □

Theorem 14. *Let $(X, *, 0)$ be a BCI-algebra.*

1. *Let X be an F_3-algebra. X is associative if and only if $x(x * zx) = xz$ if and only if X is p-semisimple.*
2. *Let X be an F_5-algebra. X is associative if and only if $(xy * x)x = yx$.*
3. *Let X be an F_{21}-algebra. X is associative if and only if $(yx * x)x = x * y$.*
4. *Let X be an F_{42}-algebra. X is associative if and only if X is p-semisimple.*

5. *Let X be an F_{55}-algebra. X is associative if and only if* $[(y * x) * x] * x = x * y$.
6. (a) *X is an F_5-algebra and p-semisimple if and only if X is associative.*
   (b) *Let X be an F_8-algebra. X is associative if and only if* $x(y * zx) = yz$.
7. *Let X be an F_{19}-algebra. X is associative if and only if quasi-associative.*
8. *X is an F_{39}-algebra and obeys* $y(x * xz) = zy$ *if and only if X is associative.*
9. *Let X be a F_{46}-algebra. X is associative if and only if* $0(0 * 0x) = x$.
10. (a) *X is an F_{52}-algebra and F_{55}-algebra if and only if X is associative.*
    (b) *X is an F_{52}-algebra and obeys* $(y * zx)x = zy$ *if and only if X is associative.*
    (c) *X is an F_{55}-algebra and p-semisimple if and only if X is associative.*
    (d) *Let X be an F_{52}-algebra. X is associative if and only if X is quasi-associative.*
11. (a) *X is an F_{59}-algebra and F_{55}-algebra if and only if X is associative.*
    (b) *X is an F_{52}-algebra and obeys* $(y * zx)x = zy$ *if and only if X is associative.*
    (c) *Let X be a F_{56}-algebra. X is associative if and only if X is quasi-associative.*
    (d) *Let X be an F_{59}-algebra. X is associative if and only if X is quasi-associative.*

Proof.

1. Suppose X is a F_3-algebra. Then, $xy * zx = x(y * zx)$. Put $y = x$ to get $0 * zx = x(x * zx)$. Substituting $x = 0$, we have $0z = 0 * 0z$ which means X is quasi-associative. Going by Theorem 9, X is associative if and only if X is p-semisimple. Furthermore, by Theorem 4(3) and $0 * zx = x(x * zx)$, an F_3-algebra X is associative if and only if $xy = x(x * zx)$.
2. Suppose X is associative. Then $0 * x = x$. X is F_5 implies $(xy * z)x = (x * yz)x$. With $z = x$, we have $(xy * x)x = (x * yx)x \Rightarrow (xy * x)x = (x * x)yx \Rightarrow (xy * x)x = 0 * yx \Rightarrow (xy * x)x = yx$ as required. Conversely, suppose $(xy * x)x = yx$. Put $z = x$ in $(xy * z)x = (x * yz)x$ to get $(xy * x)x = (x * yx)x \Rightarrow (xy * x)x = (x * x)yx \Rightarrow (xy * x)x = 0 * yx \Rightarrow yx = 0 * yx$ (since $(xy * x)x = yx$). So, X is associative.
3. Suppose X is associative. Then $x * y = y * x$. X is F_{21} implies $yx * zx = (yx * z)x$. With $z = x$, we have $(yx * x)x = y * x = x * y$ as required. Conversely, suppose $(yx * x)x = x * y$. Put $z = x$ in F_{21} to get $(yx * x)x = y * x$. So, $x * y = y * x$ as required.
4. Suppose X is associative. Then $0 * z = z$. X is F_{42} implies $xx * yz = (xx * y)z$. With $y = 0$, we have $0 * 0z = 0 * z = z$ as required. Conversely, suppose $0 * 0z = z$. Put $y = 0$ in F_{42} to get $0 * 0z = 0 * z$. So, $0 * z = z$ as required.
5. Suppose X is associative. Then $x * y = y * x$. X is F_{55} implies $[(y * z) * x] * x = [y * (z * x)] * x$. With $z = x$, we have $[(y * x) * x] * x = y * x = x * y$ as required. Conversely, suppose $[(y * x) * x] * x = x * y$. Put $z = x$ in F_{55} to get $y * x = [(y * x) * x] * x = x * y$. So, $y * x = x * y$ as required.

The proofs of 6 to 11 follow by using the concerned F_i and F_j identities (plus p-simplicity by Theorem 12 in some cases) to get an F_k which is equivalent to associativity by Theorem 13 or which is not equivalent to associativity by 1 to 5 of Theorem 14. □

3. Summary, Conclusions and Recommendations

In this work, we have been able to construct examples of Fenyves' BCI-algebras. We have also obtained the basic algebraic properties of Fenyves' BCI-algebras. Furthermore, we have categorized the Fenyves' BCI-algebras into a 46 member associative class (as captured in Theorem 13). Members of this class include F_1, F_2, F_4, F_6, F_7, F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{20}, F_{22}, F_{23}, F_{24}, F_{25}, F_{26}, F_{27}, F_{28}, F_{30}, F_{31}, F_{32}, F_{33}, F_{34}, F_{35}, F_{36}, F_{37}, F_{38}, F_{40}, F_{41}, F_{43}, F_{44}, F_{45}, F_{47}, F_{48}, F_{49}, F_{50}, F_{51}, F_{53}, F_{57}, F_{58}, F_{60}-algebras; and a 14 member non-associative class. Those Fenyves identities that are equivalent to associativity in BCI-algebras are denoted by ✓ in the fifth column of Table 1. For those that belong to the non-associative class, we have been able to obtain conditions under which they would be associative (as reflected in Theorem 14). This class includes F_3, F_5, F_8, F_{19}, F_{21},F_{29}, F_{39} , F_{42}, F_{46}, F_{52}, F_{54}, F_{55}, F_{56}, F_{59}-algebras. In Table 1 which summarizes the results, members of this class are identified by the symbol '‡'.

Other researchers who have studied Fenyves' identities on the platform of loops, namely Phillips and Vojtechovsky [5], Jaiyeola [6], Kinyon and Kunen (2004) found Moufang (F_2, F_4, F_{17}, F_{27}), extra

(F_6, F_{13}, F_{22}), F_9, F_{15}, left Bol (F_{19}), right Bol (F_{26}), Moufang (F_4, F_{27}), F_{30}, F_{35}, F_{36}, C (F_{37}), F_{38}, F_{39}, F_{40}, LC(F_{39}, F_{41}, F_{46}, F_{48}), F_{42}, F_{43}, F_{45}, F_{51}, RC(F_{36}, F_{53}, F_{56}, F_{57}), F_{54}, and F_{60} Fenyves' identities not to be equivalent to associativity in loops. Interestingly, in our study, some of these identities, particularly the extra identity (F_6, F_{13}, F_{22}), F_7, F_9, F_{15}, F_{17}, right Bol (F_{26}), Moufang (F_4, F_{27}), F_{30}, F_{35}, F_{38}, F_{40}, RC (F_{36}, F_{53}, F_{57}), C (F_{37}), LC (F_{41}, F_{48}), F_{43}, F_{45}, F_{51} and F_{60} have been found to be equivalent to associativity in BCI-algebras. In addition, the aforementioned researchers found F_1, F_3, F_5, F_7, F_8, F_{10}, F_{11}, F_{12}, F_{14}, F_{16}, F_{18}, F_{20}, F_{21}, F_{23}, F_{24}, F_{25}, F_{28}, F_{29}, F_{31}, F_{32}, F_{33}, F_{34}, F_{44}, F_{47}, F_{49}, F_{50}, F_{52}, F_{55}, F_{58} and F_{59} identities to be equivalent to associativity in loops. We have also found some (F_7, F_{10}, F_{11}, F_{12}, F_{14}, F_{16}, F_{18},F_{20}, F_{23}, F_{24}, F_{25}, F_{28}, F_{31}, F_{32}, F_{33}, F_{44}, F_{47}, F_{49}, F_{50}, F_{58}) of these identities to be equivalent to associativity in BCI-algebras while some others (F_3, F_5, F_8, F_{20}, F_{21}, F_{29},F_{55}, F_{59}) were not equivalent to associativity in BCI-algebras.

In loop theory, it is well known that:

- A loop is an extra loop if and only if the loop is both a Moufang loop and a C-loop.
- A loop is a Moufang loop if and only if the loop is both a right Bol loop and a left Bol-loop.
- A loop is a C-loop if and only if the loop is both a RC-loop and a LC-loop.

In this work, we have been able to establish (as stated below) somewhat similar results for a few of the Fenyves' identities in a BCI-algebra X:

- X is an F_i-algebra and F_j-algebra if and only if X is associative, for the pairs: $i = 52$, $j = 55$, $i = 59$, $j = 55$.

Fenyves [31], and Phillips and Vojtěchovský [32,33] found some of the 60 F_i identities to be equivalent to associativity in quasigroups and loops (i.e., groups), and others to describe weak associative laws such as extra, Bol, Moufang, central, flexible laws in quasigroups and loops. Their results are summarised in the second, third and fourth columns of Table 1 with the use of ✓. In this paper, we went further to establish that 46 Fenyves' identities are equivalent to associativity in BCI-algebras while 14 Fenyves' identities are not equivalent to associativity in BCI-algebras. These two categories are denoted by ✓ and ‡ in the fifth column of Table 1.

After the works of [31–33], the authors in [34–38] did an extension by investigating and classifying various generalized forms of the identities of Bol-Moufang types in quasigroups and one sided/two sided loops into associative and non-associative categories. This answered a question originally posed in [39] and also led to the study of one of the newly discovered generalized Bol-Moufang types of loop in Jaiyéọlá et al. [40]. While all the earlier mentioned research works on Bol-Moufang type identities focused on quasigroups and loop, this paper focused on the study of Bol-Moufang type identities (Fenyves' identities) in special types of groupoids (BCI-algebra and quasi neutrosophic triplet loops) which are not necessarily quasigroups or loops (as proved in Theorem 12). Examples of such well known varieties of groupoids were constructed by Ilojide et al. [41], e.g., Abel-Grassmann's groupoid.

The results of this work are an initiation into the study of the classification of finite Fenyves' quasi neutrosophic triplet loops (FQNTLs) just like various types of finite loops have been classified (e.g., Bol loops, Moufang loops and FRUTE loops). In fact, a library of finite Moufang loops of small order is available in the GAPS-LOOPS package [42]. It will be intriguing to have such a library of FQNTLs.

Overall, this research work (especially for the non-associative F_i's) has opened a new area of research findings in BCI-algebras and Bol-Moufang type quasi neutrosophic triplet loops as shown in Figure 1.

Table 1. Characterization of Fenyves Identities in Quasigroups, Loops and BCI-Algebras by Associativity.

Fenyves Identity	$F_i \equiv ASS$ Inaloop	$F_i \not\equiv ASS$ Inaloop	Quassigroup $\Rightarrow$ Loop	$F_i + BCI \Rightarrow ASS$
F_1	✓		✓	✓
F_2		✓	✓	✓
F_3	✓		✓	‡
F_4		✓		✓
F_5	✓			‡
F_6		✓	✓	✓
F_7	✓			✓
F_8	✓			‡
F_9		✓		✓
F_{10}	✓			✓
F_{11}	✓		✓	✓
F_{12}	✓		✓	✓
F_{13}		✓	✓	✓
F_{14}	✓			✓
F_{15}		✓		✓
F_{16}	✓			✓
F_{17}		✓	✓	✓
F_{18}	✓		✓	✓
F_{19}		✓		‡
F_{20}	✓			✓
F_{21}	✓		✓	‡
F_{22}		✓	✓	✓
F_{23}	✓			✓
F_{24}	✓			✓
F_{25}	✓			✓
F_{26}		✓		✓
F_{27}		✓	✓	✓
F_{28}	✓		✓	✓
F_{29}	✓			‡
F_{30}		✓		✓
F_{31}	✓		✓	✓
F_{32}	✓		✓	✓
F_{33}	✓			✓
F_{34}	✓			✓
F_{35}		✓		✓
F_{36}		✓		✓
F_{37}		✓		✓
F_{38}		✓	✓	✓

Table 1. *Cont.*

Fenyves Identity	$F_i \equiv ASS$ Inaloop	$F_i \not\equiv ASS$ Inaloop	Quassigroup $\Rightarrow$ Loop	$F_i + BCI \Rightarrow ASS$
F_{39}		✓		‡
F_{40}		✓		✓
F_{41}		✓	✓	✓
F_{42}		✓		‡
F_{43}		✓		✓
F_{44}	✓			✓
F_{45}		✓		✓
F_{46}		✓		‡
F_{47}	✓		✓	✓
F_{48}		✓		✓
F_{49}	✓			✓
F_{50}	✓			✓
F_{51}		✓		✓
F_{52}	✓			‡
F_{53}		✓	✓	✓
F_{54}		✓		‡
F_{55}	✓			‡
F_{56}		✓		‡
F_{57}		✓		✓
F_{58}	✓		✓	✓
F_{59}	✓			‡
F_{60}		✓		✓

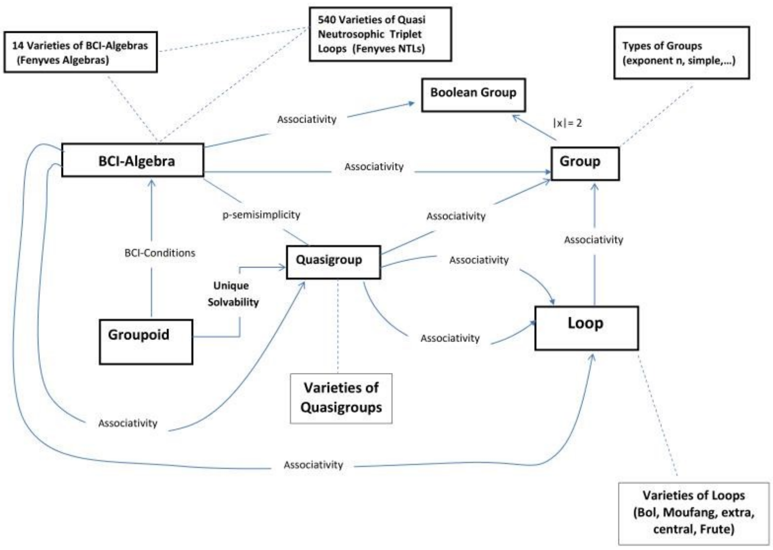

Figure 1. New Cycle of Algebraic Structures.

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New Soft Set Based Class of Linear Algebraic Codes

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Mumtaz Ali, Huma Khan, Le Hoang Son, Florentin Smarandache, W. B. Vasantha Kandasamy (2018). New Soft Set Based Class of Linear Algebraic Codes. *Symmetry* 10, 510; DOI: 10.3390/sym10100510

Abstract: In this paper, we design and develop a new class of linear algebraic codes defined as soft linear algebraic codes using soft sets. The advantage of using these codes is that they have the ability to transmit m-distinct messages to m-set of receivers simultaneously. The methods of generating and decoding these new classes of soft linear algebraic codes have been developed. The notion of soft canonical generator matrix, soft canonical parity check matrix, and soft syndrome are defined to aid in construction and decoding of these codes. Error detection and correction of these codes are developed and illustrated by an example.

Keywords: linear algebraic code; soft set theory; soft linear algebraic code; soft communication; soft syndrome; soft codewords; soft generator matrix

1. Introduction

Shannon [1,2] published an historic paper that marked the beginning of both error correcting codes and information theory. Since then, several researchers have developed and designed codes like BCH codes [3,4], self-dual codes [5], maximum distance codes [6], Hamming distance of linear codes [7], and codes over Z_m [8,9]. However fuzzy codes and distance properties was developed by [10]. For literature used in this paper on coding theory, see Reference [11].

In this paper, we define soft linear codes using soft sets. Soft sets [12] are generalization of fuzzy sets introduced in [13]. Fuzzy sets work on membership degree whose range varies from Reference [0, 1] and soft sets deal with uncertainty in a parametric way. Thus, a soft set is a parameterized family of sets and the boundary of the set depends on the parameters. Since then, soft sets [14] have been developed to neutrosophic soft sets [15], soft neutrosophic groups [16], soft neutrosophic algebraic structures, and their generalization [17–20]. Relationship among soft sets and fuzzy sets was studied in Reference [20,21]. Here, for the first time, soft set theory has been used in the construction of algebraic codes, which we choose to call as soft algebraic linear codes.

This paper is organized into six sections. Section 1 is introductory in nature. All basic concepts to make this paper a self-contained one are given in Section 2. Section 3 introduces the new notion of algebraic soft codes and defines and describes some related properties of them. Soft parity check matrix and soft generator matrix are introduced in Section 3. Section 4 describes decoding, error detection and error correction of the soft linear algebraic codes. Section 5 gives the soft communication model and brings out the difference between the linear algebraic codes and soft linear algebraic codes. Section 6 gives the conclusions based on our study and probable future research for any researcher.

2. Fundamental Notions

In this section the basic concepts needed to make this paper a self-contained one is given. This section is divided into two subsections. Section 2.1 describes the basic concepts about the linear algebraic codes and their related properties and Section 2.2 gives the definition and a few properties of soft sets.

2.1. Algebraic Linear Codes and Their Properties

All the basic concepts, definition and properties of algebraic linear codes are taken from Reference [11]. The fundamental algebraic structure used in the definition of linear algebraic codes are vector spaces and vector subspaces defined over a finite field F. Throughout this paper, we only consider the finite field $Z_2 = \{0, 1\}$, the finite field of characteristic two. We use F to denote Z_2.

Definition 1. *Let V be a set of elements on which a binary operation called addition, '+' is defined. Let F be a field. An operation product or multiplication, denoted by '.', between the elements in F and the elements in V is defined. The set V is called a vector space over the field F if it satisfies the following conditions:*

1. *V is a commutative group under addition.*
2. *For any element a in F and any element v in V, a.v = v.a is in V.*
3. *Distributive law: For any u and v in V and for any a,b ∈ F*

$$a.(u + v) = a.u + a.v;\ (a + b).v = a.v + b.v.$$

4. *Associative law: For any v in V and any a and b in F; (a.b).v = a.(b.v).*
5. *Let 1 be the unit element of F. Then for any v in V, 1.v = v and 0.v = 0 for 0 ∈ F and '$\bar{0}$' is the zero vector of V. We call a proper subset U of V (U ⊂ V) to be a vector subspace of V over F if U itself is a vector space over F.*

Definition 2. *A block code of length n with 2^k codewords is called a linear code, denoted by C (n, k), if and only if its 2^k codewords form a k-dimensional subspace of the vector space V^n of all the n tuples over the field GF(2).*

The method for generating these C(n, k) codes using the generator matrix G is as follows. G is given in the following:

$$G = \begin{bmatrix} g_{00} & g_{01} & g_{02} & \cdots & g_{0,n-1} \\ g_{10} & g_{11} & g_{12} & \cdots & g_{1,n-1} \\ \vdots & \vdots & & & \\ g_{k-1,0} & g_{k-1,1} & g_{k-1,2} & & g_{k-1,n-1} \end{bmatrix}$$

$g_{i,j} \in Z_2 = F$; for $0 \le i \le k-1$ and $0 \le j \le n-1$. Consider $u = (u_0\ u_1\ \ldots\ u_{k-1})$, the message to be encoded, the corresponding codeword v is given by v = u.G. Every codeword v in C (n, k) is a linear combination of k codewords.

The error detection and error correction of these codes is given in Reference [11]. If the generator matrix G in the standard form is $G = (A; I_{k \times k})$, then parity check matrix H can be got in the standard form as $H = (I_{n-k \times n-k}; A^T)$. The generator matrix can be in any other form, and then the parity check matrix can be found out by the usual methods given in Reference [11].

The syndrome of the received codeword y, denoted by $s(y) = yH^T$ is obtained from the parity check matrix H. Thus, the parity check matrix H of a code helps to detect the error from the received word. The error correcting capacity of a code depends on the metric that is used over the code. The most basic metric, namely the Hamming metric of the code is defined as follows:

Definition 3. *For any two vectors $x = (x_1 \ldots x_n)$ and $y = (y_1 \ldots y_n)$ in V^n, the n dimensional vector space over the field $F = Z_2$, the Hamming distance $d(x, y)$ and the Hamming weight $w(x)$ are defined as follows:*

$$d(x, y) = |\{x_i{:}x_i \neq y_i;\ x_i \in x;\ y_i \in y\}|$$

$$w(x) = |\{x_i{:}x_i \neq 0;\ x_i \in x\}|.$$

Definition 4. *The minimum distance d_{min} of a code $C(n, k)$ is defined as*

$$d_{min} = \min_{\substack{x, y \in C \\ x \neq y}} d(x, y).$$

The coset leader method used for error correction, makes use of the standard array for syndrome decoding as described in Reference [11].

2.2. Soft Set Theory

The soft set theory which is a generalization of fuzzy set theory was proposed by Reference [12].
While this part X concerns to an inceptive domain, $P(X)$ is the power set of X, V is called a set of parameters, or $D \subset V$. The soft set theory defined by Reference [12] is given below.

Definition 5. *The set (f, D) is said to be a soft set of X where a mapping of f is given by $f{:}D \rightarrow P(X)$.*

In other words, a soft set over X is a parameterized family of subsets of the universe X. For $d \in D$, $f(d)$ can be considered as the set of d-elements of the soft set (f, D), or as the set of d-approximate elements of the soft set.

Let (f, D) and (g, E) be two soft sets over X, (f, D) is called a soft subset of (g, E) if $D \subseteq E$ and $f(s) \subseteq g(s)$, for all $s \in D$. This relationship is denoted by $(f, D) \subset (g, E)$. Similarly, (f, D) is called a soft superset of (g, E) if (g, E) is a soft subset of (f, D) which is denoted by $(f, D) \supset \bigcap (g, E)$. If $(f, D) \subseteq (g, E)$ and $(g, E) \subseteq (f, D)$, the two soft sets are said to be equal.

3. Algebraic Soft Linear Codes and Their Properties

In this section the concept of soft linear code and algebraic soft linear code of type 1 are proposed and notion of soft generator matrix and soft parity check matrix are introduced.

Definition 6. *Let $F = Z_2$; be the field of characteristic two. Let $W = F \times \ldots \times F = F^m$, be a vector space over the field F of dimension m. $P(W)$ be the power set of W. (f, D) is said to be a soft algebraic linear code over F if and only if $f(d)$ is a linear algebraic code of W for all $d \in D$; $D \subset V$, where V is the set of parameters.*

It is to be noted that not all vector subspaces of W, forms a linear algebraic code. Further, the soft algebraic linear code does not in general include all linear algebraic codes of W.

Example 1. *Let $W = F^3$ be a vector space over the field F. (f, D) is a soft linear code over W where $f(D) = \{f(d_1), f(d_2)\}$ with*

$$f(d_1) = \{000, 111\} \text{ and } f(d_2) = \{000, 110, 101, 011\}.$$

Clearly {000, 111} and {000, 110, 101, 011} are linear algebraic codes. {{000, 000}, {000, 110}, {000, 101}, {000, 011}, {111, 000}, {111, 110}, {111, 101}, and {111, 011}} is the set of soft codewords of (f, D). There are 8 soft codewords for the soft code (f, D).

In view of this example we define soft codeword as follows:

Definition 7. *Let W = F × . . . × F = F^m, be a vector space over the field F of dimension m. P(W) be the power set of W. (f, D) be a soft algebraic linear code over F. Let f(D) = {f(d_1), . . . , f(d_t)} where each f(d_i); $1 \leq i \leq t$ is a linear algebraic code of W. Each t-tuple {x_1, x_2, . . . , x_t}; $x_i \in$ f(d_i); $1 \leq i \leq t$ is defined as the soft codeword of the soft algebraic code (f, D). We have |f(d_1)| × |f(d_2)| × . . . × |f(d_t)| number of soft codewords for this (f, D).*

In the above example, the soft dimension (*f*, *D*) = {1, 2}, that is the number of linearly independent codewords of the linear algebraic code associated with $f(d_1)$ and $f(d_2)$, respectively.

We have the following definition in view of this.

Definition 8. *Let W = F^m, be a vector space over the field F of dimension m. (f, D) be a soft algebraic linear code over F. Let f(D) = {f(d_1), . . . , f(d_t)} where each f(d_i); $1 \leq i \leq t$ is a linear algebraic code of W. Here each f(d_i) $\in$ f(D) is a linear algebraic code and dimension of f(d_i) is n_i where n_i is the number of linear independent elements of f(d_i). The soft dimension of (f, D) = {n_1, . . . , n_t} and the number of soft codewords of (f, D) is |f(d_1)| × |f(d_2)| × . . . × |f(d_t)| where $1 \leq i \leq t$.*

Definition 9. *Let (f, D) be the same as in above Definition 8. (f, D) is called soft code of type 1, if the dimension of (f, D) = {n_1, n_2, . . . , n_t} is such that n_1 = n_2 = . . . = n_t.*

In the following we give an example of soft code of type 1.

Example 2. *Let (f, D) be a soft code in W = F × . . . × F = F^5 over the field F. Consider*

f(d_1) = {00000, 11111, 10110, 01001},
f(d_2) = {00000, 11111, 11001, 00110},
f(d_3) = {00000, 11111, 00111, 11000}, and
f(d_4) = {00000, 11111, 11100, 00011}.

The soft dimension of (f, D) is {2, 2, 2, 2}. Hence (f, D) is a soft code of type 1.

Theorem 1. *Every soft algebraic linear code of type 1 is trivially a soft algebraic linear code but the converse is not true.*

Proof. The result follows from the definition of soft code of type 1. For the converse, result follows from Example 1, where the dimensions of $f(d_1)$ and $f(d_2)$ are different. □

Now we proceed on to define the soft generator matrix for soft linear algebraic code (*f*, *D*).

Definition 10. *Let (f, D) be a soft linear algebraic code as in Definition 8, where f(D) = {f(d_1), . . . , f(d_t)}. We know that associated with each f(d_i) we have an algebraic code of dimension n_i. Let G_i; $1 \leq i \leq t$ be the generator matrix associated with this algebraic code associated with f(d_i). Then we define the soft generator matrix G_s as the t-matrix given by G_s = [G_1|G_2| . . . |G_t]. If the each generator matrix G_i of the soft generator matrix G_s is represented in the standard form then the soft generator matrix G_s is known as soft canonical generator matrix and is denoted by G_s*.*

Example 3. *The soft generator matrix of the soft linear code of type 1 given in Example 2 is as follows:*

$$G_s = [G_1|\,G_2|G_3|G_4] = \left[\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}\middle|\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}\middle|\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}\middle|\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}\right]$$

where G_i is the generator matrix of the algebraic code associated with f(d_i); i = 1, 2, 3, 4; clearly this G_S is not the soft canonical generator matrix.

The following example gives the soft canonical generator matrix for the soft linear code.

Example 4. *Suppose (f, D) be a soft code over* $W = F^5$, *where*

$$f(d_1) = \{00000, 10010, 01001, 00110, 11011, 10100, 01111, 11101\};$$
$$f(d_2) = \{00000, 11111, 10110, 01001\}$$

are algebraic linear codes with standard generator matrix G_1 *and* G_2 *where*

$$G_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, G_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The soft canonical generator bi-matrix of (f, D) is:

$$G_s^* = \left[\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \right].$$

Now we proceed onto to define soft parity check matrix and soft canonical parity check matrix for a soft linear algebraic code.

Definition 11. *Consider (f, D) as in Definition 8. Let* $f(D) = \{f(d_1), \ldots, f(d_t)\}$ *where each* $f(d_i)$ *is linear algebraic code, let* H_i $(1 \leq i \leq t)$ *be the parity check matrix associated with each linear algebraic code. Then* $H_S = \{H_1 \mid H_2 \mid \ldots \mid H_t\}$ *is the soft parity check matrix associated with the soft linear algebraic code.*

If each H_i *is taken in the standard form then the corresponding soft parity check matrix* H_s^* *is defined as the soft canonical parity check matrix of the soft algebraic linear code.*

Now, in the following section, we give a method to determine soft errors in received codewords and how the soft error corrections are carried out.

4. Soft Linear Algebraic Decoding Algorithms

During transmission over any medium, the transmitted codeword can get corrupted with errors. The process of identifying these errors from the received codeword is known as error detection and the process of correcting the errors and obtaining the correct codeword is known as error correction. In this section, we introduce the notion the soft decoding algorithm, error detection, and error correction for soft linear algebraic codes. The method of soft syndrome decoding is proposed.

First, we proceed on to define the notion of coset and soft coset leader. The definition of coset and coset leader for any linear algebraic code can be had from Reference [11].

We now define the coset leaders as elements in each of the cosets with the least weight. For any code, $C = C(n, k)$ is as follows as the algebraic code is a subspace of W so is a subgroup of W.

Coset Leaders	**Codewords as cosets of C**	**Syndromes**
$e_0 = 0$	$x_1, \ldots, x_m$	$s = 0$
e_1	$e_1 + x_1, \ldots, e_1 + x_m$	e_1H^T
e_2	$e_2 + x_1, \ldots, e_2 + x_m$	e_2H^T
$\vdots$	$\vdots$	$\vdots$
e_p	$e_p + x_1, \ldots, e_p + x_m$	e_pH^T

where e_i's are coset leaders. Syndrome of e_i, $s(e_i) = e_iH^T$; $0 \leq i \leq t$.

H is the parity check matrix of the linear algebraic code C. The coset leader method is used for error correction by making use of the standard array for syndrome decoding [11].

Definition 12. *Let (f, D) be the soft linear code as given in Definition 8. Let $H_s = (H_1 \mid H_2 \mid \ldots \mid H_t)$ be the soft parity check matrix of (f, D). Suppose y is the received soft message, the soft syndrome of y is defined as s(y) = y H_s^T; if $s(y) \neq (o)$ then we say the soft codeword has soft error.*

Now, we proceed on to analogously describe the syndrome decoding method for soft linear algebraic codes.

Let $W = F^m$ be a vector space of dimension n over $F = Z_2$. Let (f, D) be a soft algebraic code with $f(D) = (f(d_1), \ldots, f(d_t))$ where each $f(d_i)$; $1 \leq i \leq t$; is a linear algebraic code over W. Any soft codeword in (f, D) will be of the form $x = (x_1, x_2, \ldots, x_t)$ where $x_i \in f(d_i)$ and $x_i = \left(y_1^i, \ldots, y_m^i\right)$ a m-tuple for which it will have k_i message symbols; $1 \leq i \leq t$.

If z is a received message we have to first find out if z has any error and if z has error we have to correct it. Now to check for error we find the soft syndrome $s(z) = z\mathrm{H}^{\mathrm{T}} = z\left(H_1^T \middle| H_2^T \middle| \ldots \middle| H_t^T\right)$ where each H_i is the parity check matrix of the linear algebraic code associated with $f(d_i)$; $1 \leq i \leq t$.

If $s(z) \neq (0)$ we have an error. This error is defined as the soft error and $s(z)$ is defined as soft syndrome of the soft codeword z received. This procedure of finding out whether the received soft codeword is correct or not; it is termed as soft error detection.

Now, we proceed on to correct the soft error as $s(z) \neq (0)$; some soft error has occurred during transmission. We can build an analogous table for error correction or standard array for soft syndrome decoding. Soft coset leaders in the case of soft codes will be carried out in an analogous way, which will be described by an example.

Example 5. *Let (f, D) be a soft code defined in Example 1. The soft parity check matrix of (f, D) be*

$$H = [H_1 | \, H_2] = \left[\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \middle| \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right].$$

The transpose of H is as follows,

$$H^T = \left[\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right].$$

And the soft coset leaders of (f, D) are

$$e_0 = \{000, 000\}, e_1 = \{100, 100\}, e_2 = \{010, 010\} \text{ and } e_3 = \{001, 001\}$$

The Table 1 of soft syndrome decoding is as follows:

Table 1. Soft Syndrome Coding.

Soft Coset Leaders	Soft Codewords as Cosets of *(f, D)*	Soft Syndromes
e_0 = {000, 000}	{000, 000}, (000, 110}, {000, 101}, {000, 011}, {111, 000}, {111, 110}, {111, 101}, {111, 011}	e_0H^T = {00, 0}
e_1 = {100, 100}	{100, 100}, (100, 010}, {100, 001}, {100, 111}, {011, 100}, {111, 010}, {011, 001}, {011, 111}	e_1H^T = {10, 1}
e_2 = {010, 010}	{010, 010}, (010, 100}, {010, 111}, {010, 001}, {101, 010}, {101, 100}, {101, 111}, {101, 001}	e_2H^T = {11, 1}
e_3 = {001, 001}	{001, 001}, (001, 111}, {001, 100}, {001, 010}, {110, 001}, {110, 111}, {110, 100}, {110, 010}	e_3H^T = {01, 1}

Theorem 2. *Suppose (f, D) be a soft linear algebraic code over a field F, given in Definition 8, any element received codeword, which has some error* $y = (y_1, \ldots, y_t)$; $y_i \in W = F^m$; $1 \leq i \leq t$; *then there is a soft codeword nearest to y given by* $x = y +$ *soft coset leader* e_i *of the soft code (f, D).*

Proof. Let (f, D) be a soft linear algebraic code over a field F with H as the soft parity check matrix. Let $y = (y_1, \ldots, y_t)$ be the received codeword, we find the soft syndrome;

$$s(y) = yH^T = (y_1 \ldots y_t)\left[H_1^T \middle| H_2^T \middle| \ldots \middle| H_t^T \right]$$

where $s(y) = (0)$ implies that there is no error, so y is the correct codeword. If $s(y) \neq (0)$, then we work as follows: First, we find all the soft linear algebraic coset of the soft linear algebraic code (f, D) for soft set-based syndrome decoding, and then find the appropriate soft linear algebraic coset leaders e_i from the collection of coset leaders using the one analogues Table 1. Then, for all soft coset leaders we calculate the soft set-based syndrome and make a table of soft linear algebraic coset leaders with their soft set-based syndromes. For decoding a soft linear algebraic codeword y, we can merely find the soft set-based syndrome of the soft linear algebraic codeword and then compare soft coset leader syndrome with their soft set-based syndrome. After the comparison, we add the soft decoded word to the soft linear algebraic coset leader. Thus, y is soft decoded as $x = y + e_i$; e_i is the soft coset leader and x is the corrected word. □

5. Soft Set-Based Communication Transmission and Comparison of Soft Linear Algebraic Codes and Linear Algebraic Codes

In this section, we propose a soft set-based communication transmission. The following proposed model comprises of a soft linear algebraic encoder that is an approximated collection of encoders. Hence, if (f, D) is a soft code; in D corresponding to each parameter d, in the soft encoder we have an encoder. Moreover, we have a soft linear algebraic decoder that is the collection of decoders; hence, to each parameter in D, we have a decoder in the soft linear algebraic decoder. In parameter set $A = (a_1, \ldots, a_m)$, there are m parameters. A soft set-based communication transmission reduces to classical communication transmission if we have $m = 1$. The model of soft set-based communication transmission is given in the following Figure 1.

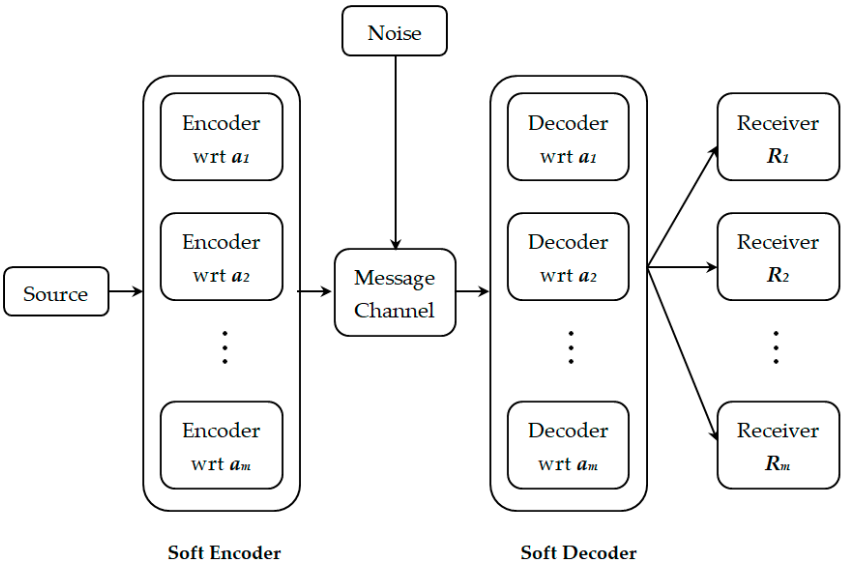

Figure 1. Soft Communication transmission model.

The major difference among linear code and soft linear code is that for the soft linear code every soft code word has some attributes or concept, i.e., each soft code word is distinguished by some attributes, but the linear algebraic codes do not enjoy this property. Thus, one can work on the attributes of soft code words, for example an attribute "d_i" can have some attribute that can trick the hackers. Therefore, the soft linear codes can be more secure as compared to the classical linear codes due to the parameterization. The soft linear codes have a different distinct structure. Soft linear code is a collection of subspaces, whereas a linear code is only one subspace. Each subspace relies on the set of parameters that are used. Hence, soft linear codes are more generalized in comparison to the linear codes.

Linear codes can transfer only one message to a receiver whereas soft linear codes can simultaneously transmit m-well defined messages to m-set of receivers. The time taken for transmitting m-messages to m-receivers will take at least m unit of time in case of linear algebraic codes, whereas in case of soft algebraic codes the time taken will be only the time taken to transmit a single message, since the m-messages are transmitted simultaneously. The latest methodology makes use of bi-matrices and is more generalized uses with the perception of m-matrices. Clearly, this concept of soft algebraic code saves time. In soft decoding procedure, one can decode a set of code words (soft code word) at a time while it is not feasible in case of linear algebraic codewords decoding procedure.

6. Conclusions

There is an important role of algebraic codes in the minimization of data delinquency, which is generated by deficiencies, i.e., inference, noise channel, and crosstalk. In this paper, we have proposed the latest notions of soft linear algebraic codes for the first time by using the soft set. This latest class of codes can remit simultaneously m-messages to the m-people. Therefore, these new codes can save both time and economy. Soft parity check matrix (parity check m-matrix) and soft generator matrix (generator m-matrix) were defined. Decoding of soft linear codes was done using soft syndrome decoding techniques. The channel transmission is also illustrated. Finally, the major difference and comparison of soft linear codes with classical linear codes are presented.

Even though the proposed code has some advantages over the classical ones, it still has limitations in dealing with the multichannel coding problem, rank metrics, etc. Therefore, for future study, we wish to implement neutrosophic soft sets in algebraic linear codes. Further introduction of soft code with rank metric [22] and construction of T-direct soft codes [23] may be helpful to tackle the multichannel coding problem, which is left for researchers in coding theory. The general case based on N-soft sets and others [24–36] will be developed as well.

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Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set

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Florentin Smarandache (2018). Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set. *Neutrosophic Sets and Systems*, 22, 168-170

Abstract. In this paper, we generalize the soft set to the hypersoft set by transforming the function F into a multi-attribute function. Then we introduce the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set.

Keywords: Plithogeny; Plithogenic Set; Soft Set; Hypersoft Set; Plithogenic Hypersoft Set; Multi-argument Function.

1 Introduction

We generalize the soft set to the **hypersoft set** by transforming the function F into a **multi-argument function**.

Then we make the distinction between the types of Universes of Discourse: **crisp**, **fuzzy**, **intuitionistic fuzzy**, **neutrosophic**, and respectively **plithogenic**.

Similarly, we show that a hypersoft set can be **crisp**, **fuzzy**, **intuitionistic fuzzy**, **neutrosophic**, or **plithogenic**.

A detailed numerical example is presented for all types.

2 Definition of Soft Set [1]

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$, and A a set of attributes. Then, the pair $(F, \mathcal{U})$, where

$$F: A \longrightarrow \mathcal{P}(\mathcal{U}) \tag{1}$$

is called a **Soft Set** over $\mathcal{U}$.

3 Definition of Hypersoft Set

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$.

Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$.

Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where:

$$F: A_1 \times A_2 \times \dots \times A_n \longrightarrow \mathcal{P}(\mathcal{U}) \tag{2}$$

is called a **Hypersoft Set** over $\mathcal{U}$.

4 Particular case

For $n = 2$, we obtain the Γ–Soft Set [2].

5 Types of Universes of Discourses

5.1. A **Universe of Discourse** $\mathcal{U}_C$ is called **Crisp** if $\forall x \in \mathcal{U}_C$, x belongs 100% to $\mathcal{U}_C$, or x's membership (T_x) with respect to $\mathcal{U}_C$ is 1. Let's denote it $x(1)$.

5.2. A **Universe of Discourse** $\mathcal{U}_F$ is called **Fuzzy** if $\forall x \in \mathcal{U}_c$, x partially belongs to $\mathcal{U}_F$, or $T_x \subseteq [0, 1]$, where T_x may be a subset, an interval, a hesitant set, a single-value, etc. Let's denote it by $x(T_x)$.

5.3. A **Universe of Discourse** $\mathcal{U}_{IF}$ is called **Intuitionistic Fuzzy** if $\forall x \in \mathcal{U}_{IF}$, x partially belongs (T_x) and partially doesn't belong (F_x) to $\mathcal{U}_{IF}$, or $T_x, F_x \subseteq [0, 1]$, where T_x and F_x may be subsets, intervals, hesitant sets, single-values, etc. Let's denote it by $x(T_x, F_x)$.

5.4. A **Universe of Discourse** $\mathcal{U}_N$ is called **Neutrosophic** if $\forall x \in \mathcal{U}_N$, x partially belongs (T_x), partially its membership is indeterminate (I_x), and partially it doesn't belong (F_x) to $\mathcal{U}_N$, where $T_x, I_x, F_x \subseteq [0, 1]$, may be subsets, intervals, hesitant sets, single-values, etc. Let's denote it by $x(T_x, I_x, F_x)$.

5.5. A **Universe of Discourse** $\mathcal{U}_P$ **over a set** V **of attributes' values**, where $V = \{v_1, v_2, \dots, v_n\}, n \geq 1$, is called Plithogenic, if $\forall x \in \mathcal{U}_P$, x belongs to $\mathcal{U}_P$ in the degree $d_x^0(v_i)$ with respect to the attribute value v_i, for all

$i \in \{1, 2, \dots, n\}$. Since the degree of membership $d_x^0(v_i)$ may be crisp, fuzzy, intuitionistic fuzzy, or neutrosophic, the Plithogenic Universe of Discourse can be Crisp, Fuzzy, Intuitionistic Fuzzy, or respectively Neutrosophic.

Consequently, a Hypersoft Set over a Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic / or Plithogenic Universe of Discourse is respectively called **Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic /** or **Plithogenic Hypersoft Set**.

6 Numerical Example

Let $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$ and a set $\mathcal{M} = \{x_1, x_3\} \subset \mathcal{U}$.

Let the attributes be: a_1 = size, a_2 = color, a_3 = gender, a_4 = nationality, and their attributes' values respectively:

Size = A_1 ={small, medium, tall},
Color = A_2 ={white, yellow, red, black},
Gender = A_3 ={male, female},
Nationality = A_4 ={American, French, Spanish, Italian, Chinese}.

Let the function be:

$$F: A_1 \times A_2 \times A_3 \times A_4 \longrightarrow \mathcal{P}(\mathcal{U}). \tag{3}$$

Let's assume:

$F(\{\text{tall, white, female, Italian}\}) = \{x_1, x_3\}$.

With respect to the set $\mathcal{M}$, one has:

6.1 Crisp Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(1), x_3(1)\}, \tag{4}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ *all together*, x_1 belongs 100% to the set $\mathcal{M}$; similarly x_3.

6.2 Fuzzy Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(0.6), x_3(0.7)\}, \tag{5}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ *all together*, x_1 belongs 60% to the set $\mathcal{M}$; similarly, x_3 belongs 70% to the set $\mathcal{M}$.

6.3 Intuitionistic Fuzzy Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(0.6, 0.1), x_3(0.7, 0.2)\}, \tag{6}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ *all together*, x_1 belongs 60% and 10% it does not belong to the set $\mathcal{M}$; similarly, x_3 belongs 70% and 20% it does not belong to the set $\mathcal{M}$.

6.4 Neutrosophic Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(0.6, 0.2, 0.1), x_3(0.7, 0.3, 0.2)\}, \tag{7}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ all together, x_1 belongs 60% and its indeterminate-belongness is 20% and it doesn't belong 10% to the set $\mathcal{M}$; similarly, x_3 belongs 70% and its indeterminate-belongness is 30% and it doesn't belong 20%.

6.5 Plithogenic Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \begin{Bmatrix} x_1\left(d_{x_1}^0(\text{tall}), d_{x_1}^0(\text{white}), d_{x_1}^0(\text{female}), d_{x_1}^0(\text{Italian})\right), \\ x_2\left(d_{x_2}^0(\text{tall}), d_{x_2}^0(\text{white}), d_{x_2}^0(\text{female}), d_{x_2}^0(\text{Italian})\right) \end{Bmatrix}, \tag{8}$$

where $d_{x_1}^0(\alpha)$ means the degree of appurtenance of element x_1 to the set $\mathcal{M}$ with respect to the attribute value α; and similarly $d_{x_2}^0(\alpha)$ means the degree of appurtenance of element x_2 to the set $\mathcal{M}$ with respect to the attribute value α; where $\alpha \in \{\text{tall, white, female, Italian}\}$.

Unlike the Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic Hypersoft Sets [where the degree of appurtenance of an element x to the set $\mathcal{M}$ is *with respect to all attribute values* ***tall, white, female, Italian*** *together* (as a whole), therefore a degree of appurtenance *with respect to a set of attribute values*], the Plithogenic Hypersoft Set is a **refinement** of Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic Hypersoft Sets [since the degree of appurtenance of an element x to the set $\mathcal{M}$ is with respect to each single attribute value].

But the Plithogenic Hypersoft St is also **combined** with each of the above, since the degree of degree of appurtenance of an element x to the set $\mathcal{M}$ with respect to each single attribute value may be: crisp, fuzzy, intuitionistic fuzzy, or neutrosophic.

7 Classification of Plithogenic Hypersoft Sets

7.1 Plithogenic Crisp Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **crisp**:

$d_x^0(\alpha) = 0$ (nonappurtenance), or 1 (appurtenance).

In our example:

$$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) = \{x_1(1,1,1,1), x_3(1,1,1,1)\}. \quad (9)$$

7.2 Plithogenic Fuzzy Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **fuzzy**:

$d_x^0(\alpha) \in \mathcal{P}([0,1])$, power set of $[0,1]$,

where $d_x^0(\cdot)$ may be a subset, an interval, a hesitant set, a single-valued number, etc.

In our example, for a single-valued number:

$$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) = \{x_1(0.4, 0.7, 0.6, 0.5), x_3(0.8, 0.2, 0.7, 0.7)\}. \quad (10)$$

7.3 Plithogenic Intuitionistic Fuzzy Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **intuitionistic fuzzy**:

$d_x^0(\alpha) \in \mathcal{P}([0,1]^2)$, power set of $[0,1]^2$,

where similarly $d_x^0(\alpha)$ may be: a Cartesian product of subsets, of intervals, of hesitant sets, of single-valued numbers, etc.

In our example, for single-valued numbers:

$$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) = \left\{ \begin{matrix} x_1(0.4,0.3)(0.7,0.2)(0.6,0.0)(0.5,0.1) \\ x_3(0.8,0.1)(0.2,0.5)(0.7,0.0)(0.7,0.4) \end{matrix} \right\}. \quad (11)$$

7.4 Plithogenic Neutrosophic Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **neutrosophic**:

$d_x^0(\alpha) \in \mathcal{P}([0,1]^3)$, power set of $[0,1]^3$,

where $d_x^0(\alpha)$ may be: a triple Cartesian product of subsets, of intervals, of hesitant sets, of single-valued numbers, etc.

In our example, for single-valued numbers:

$$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) = \left\{ \begin{matrix} x\_1\ [(0.4,0.1,0.3)(0.7,0.0,0.2)(0.6,0.3,0.0)(0.5,0.2,0.1)] \\ x\_3\ [(0.8,0.1,0.1)(0.2,0.4,0.5)(0.7,0.1,0.0)(0.7,0.5,0.4)] \end{matrix} \right\}. \quad (12)$$

Conclusion & Future Research

For all types of plithogenic hypersoft sets, the aggregation operators (union, intersection, complement, inclusion, equality) have to be defined and their properties found.

Applications in various engineering, technical, medical, social science, administrative, decision making and so on, fields of knowledge of these types of plithogenic hypersoft sets should be investigated.

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Algebraic Structures of Neutrosophic Triplets, Neutrosophic Duplets, or Neutrosophic Multisets

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Florentin Smarandache, Xiaohong Zhang, Mumtaz Ali (2019). Algebraic Structures of Neutrosophic Triplets, Neutrosophic Duplets, or Neutrosophic Multisets. *Symmetry* 11, 171. DOI: 10.3390/sym11020171

Neutrosophy (1995) is a new branch of philosophy that studies triads of the form (<A>, <neutA>, <antiA>), where <A> is an entity (i.e., element, concept, idea, theory, logical proposition, etc.), <antiA> is the opposite of <A>, while <neutA> is the neutral (or indeterminate) between them, i.e., neither <A> nor <antiA> [1].

Based on neutrosophy, the neutrosophic triplets were founded; they have a similar form: (x, neut(x), anti(x), that satisfy some axioms, for each element x in a given set [2–4].

The book *Algebraic Structures of Neutrosophic Triplets, Neutrosophic Duplets, or Neutrosophic Multisets* contains the successful invited submissions [5–56] to a special issue of *Symmetry*, reporting on state-of-the-art and recent advancements of neutrosophic triplets, neutrosophic duplets, neutrosophic multisets, and their algebraic structures—that have been defined recently in 2016, but have gained interest from world researchers, and several papers have been published in first rank international journals.

The topics approached in the 52 papers included in this book are: neutrosophic sets; neutrosophic logic; generalized neutrosophic set; neutrosophic rough set; multigranulation neutrosophic rough set (MNRS); neutrosophic cubic sets; triangular fuzzy neutrosophic sets (TFNSs); probabilistic single-valued (interval) neutrosophic hesitant fuzzy set; neutro-homomorphism; neutrosophic computation; quantum computation; neutrosophic association rule; data mining; big data; oracle Turing machines; recursive enumerability; oracle computation; interval number; dependent degree; possibility degree; power aggregation operators; multi-criteria group decision-making (MCGDM); expert set; soft sets; LA-semihypergroups; single valued trapezoidal neutrosophic number; inclusion relation; Q-linguistic neutrosophic variable set; vector similarity measure; cosine measure; Dice measure; Jaccard measure; VIKOR model; potential evaluation; emerging technology commercialization; 2-tuple linguistic neutrosophic sets (2TLNSs); TODIM model; Bonferroni mean; aggregation operator; NC power dual MM (NCPDMM) operator; fault diagnosis; defuzzification; simplified neutrosophic weighted averaging operator; linear and non-linear neutrosophic number; de-neutrosophication methods; neutro-monomorphism; neutro-epimorphism; neutro-automorphism; fundamental neutro-homomorphism theorem; neutro-isomorphism theorem; quasi neutrosophic triplet loop; quasi neutrosophic triplet group; BE-algebra; cloud model; Maclaurin symmetric mean; pseudo-BCI algebra; hesitant fuzzy set; photovoltaic plan; decision-making trial and evaluation laboratory (DEMATEL); Choquet integral; fuzzy measure; clustering algorithm; and many more.

In the opening paper [5] of this book, the authors introduce refined concepts for neutrosophic quantum computing such as neutrosophic quantum states and transformation gates, neutrosophic Hadamard matrix, coherent and decoherent superposition states, entanglement and measurement notions based on neutrosophic quantum states. They also give some observations using these principles, and present a number of quantum computational matrix transformations based on neutrosophic logic, clarifying quantum mechanical notions relying on neutrosophic states. The paper is intended to extend the work of Smarandache [57–59] by introducing a mathematical framework for neutrosophic quantum computing and presenting some results.

The second paper [6] introduces oracle Turing machines with neutrosophic values allowed in the oracle information and then give some results when one is permitted to use neutrosophic sets and logic in relative computation. The authors also introduce a method to enumerate the elements of a neutrosophic subset of natural numbers.

In the third paper [7], a new approach and framework based on the interval dependent degree for MCGDM problems with SNSs is proposed. Firstly, the simplified dependent function and distribution function are defined. Then, they are integrated into the interval dependent function which contains interval computing and distribution information of the intervals. Subsequently, the interval transformation operator is defined to convert SNNs into intervals, and then the interval dependent function for SNNs is deduced. Finally, an example is provided to verify the feasibility and effectiveness of the proposed method, together with its comparative analysis. In addition, uncertainty analysis, which can reflect the dynamic change of the final result caused by changes in the decision makers' preferences, is performed in different distribution function situations. That increases the reliability and accuracy of the result.

Neutrosophic triplet structure yields a symmetric property of truth membership on the left, indeterminacy membership in the center and false membership on the right, as do points of object, center and image of reflection. As an extension of a neutrosophic set, the Q-neutrosophic set is introduced in the subsequent paper [8] to handle two-dimensional uncertain and inconsistent situations. The authors extend the soft expert set to the generalized Q-neutrosophic soft expert set by incorporating the idea of a soft expert set to the concept of a Q-neutrosophic set and attaching the parameter of fuzzy set while defining a Q-neutrosophic soft expert set. This pattern carries the benefits of Q-neutrosophic sets and soft sets, enabling decision makers to recognize the views of specialists with no requirement for extra lumbering tasks, thus making it exceedingly reasonable for use in decision-making issues that include imprecise, indeterminate and inconsistent two-dimensional data. Some essential operations, namely subset, equal, complement, union, intersection, AND and OR operations, and additionally several properties relating to the notion of a generalized Q-neutrosophic soft expert set are characterized. Finally, an algorithm on a generalized Q-neutrosophic soft expert set is proposed and applied to a real-life example to show the efficiency of this notion in handling such problems.

In the following paper [9], the authors extend the idea of a neutrosophic triplet set to non-associative semihypergroups and define neutrosophic triplet LA-semihypergroup. They discuss some basic results and properties, and provide an application of the proposed structure in football.

Single valued trapezoidal neutrosophic numbers (SVTNNs) are very useful tools for describing complex information, because of their advantage in describing the information completely, accurately and comprehensively for decision-making problems [60]. In the next paper [10], a method based on SVTNNs is proposed for dealing with MCGDM problems. Firstly, the new operation SVTNNs are developed for avoiding evaluation information aggregation loss and distortion. Then the possibility degrees and comparison of SVTNNs are proposed from the probability viewpoint for ranking and comparing the single valued trapezoidal neutrosophic information reasonably and accurately. Based on the new operations and possibility degrees of SVTNNs, the single valued trapezoidal neutrosophic power average (SVTNPA) and single valued trapezoidal neutrosophic power geometric (SVTNPG) operators are proposed to aggregate the single valued trapezoidal neutrosophic information. Furthermore, based on the developed aggregation operators, a single valued trapezoidal neutrosophic MCGDM method is developed. Finally, the proposed method is applied to solve the practical problem of the most appropriate green supplier selection and the rank results compared with the previous approach demonstrate the proposed method's effectiveness.

After the neutrosophic set (NS) was proposed [58], NS was used in many uncertainty problems. The single-valued neutrosophic set (SVNS) is a special case of NS that can be used to solve real-word problems. The next paper [11] mainly studies multigranulation neutrosophic rough sets (MNRSs) and their applications in multi-attribute group decision-making. Firstly, the existing definition of neutrosophic rough set (the authors call it type-I neutrosophic rough set (NRSI) in this paper) is analyzed, and then the definition of type-II neutrosophic rough set (NRSII), which is similar to NRSI, is given and its properties are studied. Secondly, a type-III neutrosophic rough set (NRSIII) is proposed and its differences from NRSI and NRSII are provided. Thirdly, single granulation NRSs are extended to multigranulation NRSs, and the type-I multigranulation neutrosophic rough set (MNRSI) is studied. The type-II multigranulation neutrosophic rough set (MNRSII) and type-III multigranulation neutrosophic rough set (MNRSIII) are proposed and their different properties are outlined. Finally, MNRSIII in two universes is proposed and an algorithm for decision-making based on MNRSIII is provided. A car ranking example is studied to explain the application of the proposed model.

Since language is used for thinking and expressing habits of humans in real life, the linguistic evaluation for an objective thing is expressed easily in linguistic terms/values. However, existing linguistic concepts cannot describe linguistic arguments regarding an evaluated object in two-dimensional universal sets (TDUSs). To describe linguistic neutrosophic arguments in decision making problems regarding TDUSs, the next article [12] proposes a Q-linguistic neutrosophic variable set (Q-LNVS) for the first time, which depicts its truth, indeterminacy, and falsity linguistic values independently corresponding to TDUSs, and vector similarity measures of Q-LNVSs. Thereafter, a linguistic neutrosophic MADM approach by using the presented similarity measures, including the cosine, Dice, and Jaccard measures, is developed under Q-linguistic neutrosophic setting. Lastly, the applicability and effectiveness of the presented MADM approach is presented by an illustrative example under Q-linguistic neutrosophic setting.

In the following article [13], the authors combine the original VIKOR model with a triangular fuzzy neutrosophic set [61] to propose the triangular fuzzy neutrosophic VIKOR method. In the extended method, they use the triangular fuzzy neutrosophic numbers (TFNNs) to present the criteria values in MCGDM problems. Firstly, they summarily introduce the fundamental concepts, operation formulas and distance calculating method of TFNNs. Then they review some aggregation operators of TFNNs. Thereafter, they extend the original VIKOR model to the triangular fuzzy neutrosophic environment and introduce the calculating steps of the TFNNs VIKOR method, the proposed method which is more reasonable and scientific for considering the conflicting criteria. Furthermore, a numerical example for potential evaluation of emerging technology commercialization is presented to illustrate the new method, and some comparisons are also conducted to further illustrate advantages of the new method.

Another paper [14] in this book aims to extend the original TODIM (Portuguese acronym for interactive multi-criteria decision making) method to the 2-tuple linguistic neutrosophic fuzzy environment [62] to propose the 2TLNNs TODIM method. In the extended method, the authors use 2-tuple linguistic neutrosophic numbers (2TLNNs) to present the criteria values in multiple attribute group decision making (MAGDM) problems. Firstly, they briefly introduce the definition, operational laws, some aggregation operators, and the distance calculating method of 2TLNNs. Then, the calculation steps of the original TODIM model are presented in simplified form. Thereafter, they extend the original TODIM model to the 2TLNNs environment to build the 2TLNNs TODIM model, the proposed method, which is more reasonable and scientific in considering the subjectivity of the decision makers' (DMs') behaviors and the dominance of each alternative over others. Finally, a numerical example for the safety assessment of a construction project is proposed to illustrate the new method, and some comparisons are also conducted to further illustrate the advantages of the new method.

The power Bonferroni mean (PBM) operator is a hybrid structure and can take the advantage of a power average (PA) operator, which can reduce the impact of inappropriate data given by the prejudiced decision makers (DMs) and Bonferroni mean (BM) operator, which can take into account the correlation between two attributes. In recent years, many researchers have extended the PBM operator to handle fuzzy information. The Dombi operations of T-conorm (TCN) and T-norm (TN), proposed by Dombi, have the supremacy of outstanding flexibility with general parameters. However, in the existing literature, PBM and the Dombi operations have not been combined for the above advantages for interval-neutrosophic sets (INSs) [63]. In the following paper [15], the authors define some operational laws for interval neutrosophic numbers (INNs) based on Dombi TN and TCN and discuss several desirable properties of these operational rules. Secondly, they extend the PBM operator based on Dombi operations to develop an interval-neutrosophic Dombi PBM (INDPBM) operator, an interval-neutrosophic weighted Dombi PBM (INWDPBM) operator, an interval-neutrosophic Dombi power geometric Bonferroni mean (INDPGBM) operator and an interval-neutrosophic weighted Dombi power geometric Bonferroni mean (INWDPGBM) operator, and discuss several properties of these aggregation operators. Then they develop a MADM method, based on these proposed aggregation operators, to deal with interval neutrosophic (IN) information. An illustrative example is provided to show the usefulness and realism of the proposed MADM method.

The neutrosophic cubic set (NCS) is a hybrid structure [64], which consists of INS [63] (associated with the undetermined part of information associated with entropy) and SVNS [60] (associated with the determined part of information). NCS is a better tool to handle complex DM problems with INS and SVNS. The main purpose of the next article [16] is to develop some new aggregation operators for cubic neutrosophic numbers (NCNs), which is a basic member of NCS. Taking the advantages of Muirhead mean (MM) operator and PA operator, the power Muirhead mean (PMM) operator is developed and is scrutinized under NC information. To manage the problems upstretched, some new NC aggregation operators, such as the NC power Muirhead mean (NCPMM) operator, weighted NC power Muirhead mean (WNCPMM) operator, NC power dual Muirhead mean (NCPMM) operator and weighted NC power dual Muirhead mean (WNCPDMM) operator are proposed and related properties of these proposed aggregation operators are conferred. The important advantage of the developed aggregation operator is that it can remove the effect of awkward data and it considers the interrelationship among aggregated values at the same time. Finally, a numerical example is given to show the effectiveness of the developed approach.

Smarandache defined a neutrosophic set [57] to handle problems involving incompleteness, indeterminacy, and awareness of inconsistency knowledge, and have further developed neutrosophic soft expert sets. In the next paper [17] of this book, this concept is further expanded to generalized neutrosophic soft expert set (GNSES). The authors then define its basic operations of complement, union, intersection, AND, OR, and study some related properties, with supporting proofs. Subsequently, they define a GNSES-aggregation operator to construct an algorithm for a GNSES decision-making method, which allows for a more efficient decision process. Finally, they apply the algorithm to a decision-making problem, to illustrate the effectiveness and practicality of the proposed concept. A comparative analysis with existing methods is done and the result affirms the flexibility and precision of the proposed method.

In the next paper [18], the authors define the neutrosophic valued (and generalized or G) metric spaces for the first time. Besides, they determine a mathematical model for clustering the neutrosophic big data sets using G-metric. Furthermore, relative weighted neutrosophic-valued distance and weighted cohesion measure are defined for neutrosophic big data set [65]. A very practical method for data analysis of neutrosophic big data is offered, although neutrosophic data type (neutrosophic big data) are in massive and detailed form when compared with other data types.

Bol-Moufang types of a particular quasi neutrosophic triplet loop (BCI-algebra), christened Fenyves BCI-algebras, are introduced and studied in another paper [19] of this book. 60 Fenyves BCI-algebras are introduced and classified. Amongst these 60 classes of algebras, 46 are found to be associative and 14 are found to be non-associative. The 46 associative algebras are shown to be Boolean groups. Moreover, necessary and sufficient conditions for 13 non-associative algebras to be associative are also obtained: p-semisimplicity is found to be necessary and sufficient for a F3, F5, F42, and F55 algebras to be associative while quasi-associativity is found to be necessary and sufficient for F19, F52, F56, and F59 algebras to be associative. Two pairs of the 14 non-associative algebras are found to be equivalent to associativity (F52 and F55, and F55 and F59). Every BCI-algebra is naturally a F54 BCI-algebra. The work is concluded with recommendations based on comparison between the behavior of identities of Bol-Moufang (Fenyves' identities) in quasigroups and loops and their behavior in BCI-algebra. It is concluded that results of this work are an initiation into the study of the classification of finite Fenyves' quasi neutrosophic triplet loops (FQNTLs) just like various types of finite loops have been classified. This research work has opened a new area of research finding in BCI-algebras, vis-a-vis the emergence of 540 varieties of Bol-Moufang type quasi neutrosophic triplet loops. A 'cycle of algebraic structures' which portrays this fact is provided.

The uncertainty and concurrence of randomness are considered when many practical problems are dealt with. To describe the aleatory uncertainty and imprecision in a neutrosophic environment and prevent the obliteration of more data, the concept of the probabilistic single-valued (interval) neutrosophic hesitant fuzzy set is introduced in the next paper [20]. By definition, the probabilistic single-valued neutrosophic hesitant fuzzy set (PSVNHFS) is a special case of the probabilistic interval neutrosophic hesitant fuzzy set (PINHFS). PSVNHFSs can satisfy all the properties of PINHFSs. An example is given to illustrate that PINHFS compared to PSVNHFS is more general. Then, PINHFS is the main research object. The basic operational relations of PINHFS are studied, and the comparison method of probabilistic interval neutrosophic hesitant fuzzy numbers (PINHFNs) is proposed. Then, the probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) and the probability interval neutrosophic hesitant fuzzy weighted geometric (PINHFWG) operators are presented. Some basic properties are investigated. Next, based on the PINHFWA and PINHFWG operators, a decision-making method under a probabilistic interval neutrosophic hesitant fuzzy circumstance is established. Finally, the authors apply this method to the issue of investment options. The validity and application of the new approach is demonstrated.

Competition among different universities depends largely on the competition for talent. Talent evaluation and selection is one of the main activities in human resource management (HRM) which is critical for university development [21]. Firstly, linguistic neutrosophic sets (LNSs) are introduced to better express multiple uncertain information during the evaluation procedure. The authors further merge the power averaging operator with LNSs for information aggregation and propose a LN-power weighted averaging (LNPWA) operator and a LN-power weighted geometric (LNPWG) operator. Then, an extended technique for order preference by similarity to ideal solution (TOPSIS) method is developed to solve a case of university HRM evaluation problem. The main contribution and novelty of the proposed method rely on that it allows the information provided by different DMs to support and reinforce each other which is more consistent with the actual situation of university HRM evaluation. In addition, its effectiveness and advantages over existing methods are verified through sensitivity and comparative analysis. The results show that the proposal is capable in the domain of university HRM evaluation and may contribute to the talent introduction in universities.

The concept of a commutative generalized neutrosophic ideal in a BCK-algebra is proposed, and related properties are proved in another paper [22] of this book. Characterizations of a commutative generalized neutrosophic ideal are considered. Also, some equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras are introduced, and some properties are investigated.

Fault diagnosis is an important issue in various fields and aims to detect and identify the faults of systems, products, and processes. The cause of a fault is complicated due to the uncertainty of the actual environment. Nevertheless, it is difficult to consider uncertain factors adequately with many traditional methods. In addition, the same fault may show multiple features and the same feature might be caused by different faults. In the next paper [23], a neutrosophic set based fault diagnosis method based on multi-stage fault template data is proposed to solve this problem. For an unknown fault sample whose fault type is unknown and needs to be diagnosed, the neutrosophic set based on multi-stage fault template data is generated, and then the generated neutrosophic set is fused via the simplified neutrosophic weighted averaging (SNWA) operator. Afterwards, the fault diagnosis results can be determined by the application of defuzzification method for a defuzzying neutrosophic set. Most kinds of uncertain problems in the process of fault diagnosis, including uncertain information and inconsistent information, could be handled well with the integration of multi-stage fault template data and the neutrosophic set. Finally, the practicality and effectiveness of the proposed method are demonstrated via an illustrative example.

The notions of neutrosophy, neutrosophic algebraic structures, neutrosophic duplet and neutrosophic triplet were introduced by Florentin Smarandache [57]. In another paper [24] of this book, some neutrosophic duplets are studied. A particular case is considered, and the complete characterization of neutrosophic duplets are given. Some open problems related to neutrosophic duplets are proposed.

In the next paper [25], the authors provide an application of neutrosophic bipolar fuzzy sets applied to daily life's problem related with the HOPE foundation, which is planning to build a children's hospital. They develop the theory of neutrosophic bipolar fuzzy sets, which is a generalization of bipolar fuzzy sets. After giving the definition they introduce some basic operation of neutrosophic bipolar fuzzy sets and focus on weighted aggregation operators in terms of neutrosophic bipolar fuzzy sets. They define neutrosophic bipolar fuzzy weighted averaging (NBFWA) and neutrosophic bipolar fuzzy ordered weighted averaging (NBFOWA) operators. Next they introduce different kinds of similarity measures of neutrosophic bipolar fuzzy sets. Finally, as an application, the authors give an algorithm for the multiple attribute decision making problems under the neutrosophic bipolar fuzzy environment by using the different kinds of neutrosophic bipolar fuzzy weighted/fuzzy ordered weighted aggregation operators with a numerical example related with HOPE foundation.

In the following paper [26], the authors introduce the concept of neutrosophic numbers from different viewpoints [57–65]. They define different types of linear and non-linear generalized triangular neutrosophic numbers which are very important for uncertainty theory. They introduce the de-neutrosophication concept for neutrosophic number for triangular neutrosophic numbers. This concept helps to convert a neutrosophic number into a crisp number. The concepts are followed by two applications, namely in an imprecise project evaluation review technique and a route selection problem.

In classical group theory, homomorphism and isomorphism are significant to study the relation between two algebraic systems. Through the next article [27], the authors propose neutro-homomorphism and neutro-isomorphism for the neutrosophic extended triplet group (NETG) which plays a significant role in the theory of neutrosophic triplet algebraic structures. Then, they define neutro-monomorphism, neutro-epimorphism, and neutro-automorphism. They give and prove some theorems related to these structures. Furthermore, the Fundamental homomorphism theorem for the NETG is given and some special cases are discussed. First and second neutro-isomorphism theorems are stated. Finally, by applying homomorphism theorems to neutrosophic extended triplet algebraic structures, the authors have examined how closely different systems are related.

It is an interesting direction to study rough sets from a multi-granularity perspective. In rough set theory, the multi-particle structure was represented by a binary relation. The next paper [28] considers a new neutrosophic rough set model, multi-granulation neutrosophic rough set (MGNRS). First, the concept of MGNRS on a single domain and dual domains was proposed. Then, their properties and operators were considered. The authors obtained that MGNRS on dual domains will degenerate into MGNRS on a single domain when the two domains are the same. Finally, a kind of special multi-criteria group decision making (MCGDM) problem was solved based on MGNRS on dual domains, and an example was given to show its feasibility.

As a new generalization of the notion of the standard group, the notion of the NTG is derived from the basic idea of the neutrosophic set and can be regarded as a mathematical structure describing generalized symmetry. In the next paper [29], the properties and structural features of NTG are studied in depth by using theoretical analysis and software calculations (in fact, some important examples in the paper are calculated and verified by mathematics software, but the related programs are omitted). The main results are obtained as follows: (1) by constructing counterexamples, some mistakes in the some literatures are pointed out; (2) some new properties of NTGs are obtained, and it is proved that every element has a unique neutral element in any neutrosophic triplet group; (3) the notions of NT-subgroups, strong NT-subgroups, and weak commutative neutrosophic triplet groups (WCNTGs) are introduced, the quotient structures are constructed by strong NT-subgroups, and a homomorphism theorem is proved in weak commutative neutrosophic triplet groups.

The aim of the following paper [30] is to introduce some new operators for aggregating single-valued neutrosophic (SVN) information and to apply them to solve the multi-criteria decision-making (MCDM) problems. The single-valued neutrosophic set, as an extension and generalization of an intuitionistic fuzzy set, is a powerful tool to describe the fuzziness and uncertainty [60], and MM is a well-known aggregation operator which can consider interrelationships among any number of arguments assigned by a variable vector. In order to make full use of the advantages of both, the authors introduce two new prioritized MM aggregation operators, such as the SVN prioritized MM (SVNPMM) and SVN prioritized dual MM (SVNPDMM) under an SVN set environment. In addition, some properties of these new aggregation operators are investigated and some special cases are discussed. Furthermore, the authors propose a new method based on these operators for solving the MCDM problems. Finally, an illustrative example is presented to testify the efficiency and superiority of the proposed method by comparing it with the existing method.

Making predictions according to historical values has long been regarded as common practice by many researchers. However, forecasting solely based on historical values could lead to inevitable over-complexity and uncertainty due to the uncertainties inside, and the random influence outside, of the data. Consequently, finding the inherent rules and patterns of a time series by eliminating disturbances without losing important details has long been a research hotspot. In the following paper [31], the authors propose a novel forecasting model based on multi-valued neutrosophic sets to find fluctuation rules and patterns of a time series. The contributions of the proposed model are: (1) using a multi-valued neutrosophic set (MVNS) to describe the fluctuation patterns of a time series, the model could represent the fluctuation trend of up, equal, and down with degrees of truth, indeterminacy, and falsity which significantly preserve details of the historical values; (2) measuring the similarities of different fluctuation patterns by the Hamming distance could avoid the confusion caused by incomplete information from limited samples; and (3) introducing another related time series as a secondary factor to avoid warp and deviation in inferring inherent rules of historical values, which could lead to more comprehensive rules for further forecasting. To evaluate the performance of the model, the authors explore the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) as the major factor, and the Dow Jones Index as the secondary factor to facilitate the predicting of the TAIEX. To show the universality of the model, they apply the proposed model to forecast the Shanghai Stock Exchange Composite Index (SHSECI) as well.

The new notion of a neutrosophic triplet group (NTG) proposed by Smarandache is a new algebraic structure different from the classical group. The aim of the next paper [32] is to further expand this new concept and to study its application in related logic algebra systems. Some new notions of left (right)-quasi neutrosophic triplet loops and left (right)-quasi neutrosophic triplet groups are introduced, and some properties are presented. As a corollary of these properties, the following important result are proved: for any commutative neutrosophic triplet group, its every element has a unique neutral element. Moreover, some left (right)-quasi neutrosophic triplet structures in BE-algebras and generalized BE-algebras (including CI-algebras and pseudo CI-algebras) are established, and the adjoint semigroups of the BE-algebras and generalized BE-algebras are investigated for the first time.

In a neutrosophic triplet set, there is a neutral element and antielement for each element. In the following study [33], the concept of neutrosophic triplet partial metric space (NTPMS) is given and the properties of NTPMS are studied. The authors show that both classical metric and neutrosophic triplet metric (NTM) are different from NTPM. Also, they show that NTPMS can be defined with each NTMS. Furthermore, the authors define a contraction for NTPMS and give a fixed point theory (FPT) for NTPMS. The FPT has been revealed as a very powerful tool in the study of nonlinear phenomena.

Another paper [34] of this book presents a modified Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) with maximizing deviation method based on the SVNS model [60]. A SVNS is a special case of a neutrosophic set which is characterized by a truth, indeterminacy, and falsity membership function, each of which lies in the standard interval of [0,1]. An integrated weight measure approach that takes into consideration both the objective and subjective weights of the attributes is used. The maximizing deviation method is used to compute the objective weight of the attributes, and the non-linear weighted comprehensive method is used to determine the combined weights for each attributes. The use of the maximizing deviation method allows our proposed method to handle situations in which information pertaining to the weight coefficients of the attributes are completely unknown or only partially known. The proposed method is then applied to a multi-attribute decision-making (MADM) problem. Lastly, a comprehensive comparative studies is presented, in which the performance of our proposed algorithm is compared and contrasted with other recent approaches involving SVNSs in literature.

One of the most significant competitive strategies for organizations is sustainable supply chain management (SSCM). The vital part in the administration of a sustainable supply chain is the sustainable supplier selection, which is a multi-criteria decision-making issue, including many conflicting criteria. The valuation and selection of sustainable suppliers are difficult problems due to vague, inconsistent, and imprecise knowledge of decision makers. In the literature on supply chain management for measuring green performance, the requirement for methodological analysis of how sustainable variables affect each other, and how to consider vague, imprecise and inconsistent knowledge, is still unresolved. The next research [35] provides an incorporated multi-criteria decision-making procedure for sustainable supplier selection problems (SSSPs). An integrated framework is presented via interval-valued neutrosophic sets to deal with vague, imprecise and inconsistent information that exists usually in real world. The analytic network process (ANP) is employed to calculate weights of selected criteria by considering their interdependencies. For ranking alternatives and avoiding additional comparisons of analytic network processes, the TOPSIS is used. The proposed framework is turned to account for analyzing and selecting the optimal supplier. An actual case study of a dairy company in Egypt is examined within the proposed framework. Comparison with other existing methods is implemented to confirm the effectiveness and efficiency of the proposed approach.

The concept of interval neutrosophic sets has been studied [63] and the introduction of a new kind of set in topological spaces called the interval valued neutrosophic support soft set is suggested in the next paper [36]. The authors also study some of its basic properties. The main purpose of the paper is to give the optimum solution to decision-making in real life problems the using interval valued neutrosophic support soft set.

In inconsistent and indeterminate settings, as a usual tool, the NCS containing single-valued neutrosophic numbers [60] and interval neutrosophic numbers [64] can be applied in decision-making to present its partial indeterminate and partial determinate information. However, a few researchers have studied neutrosophic cubic decision-making problems, where the similarity measure of NCSs is one of the useful measure methods. For the following work [37] in this book, the authors propose the Dice, cotangent, and Jaccard measures between NCSs, and indicate their properties. Then, under an NCS environment, the similarity measures-based decision-making method of multiple attributes is developed. In the decision-making process, all the alternatives are ranked by the similarity measure of each alternative and the ideal solution to obtain the best one. Finally, two practical examples are applied to indicate the feasibility and effectiveness of the developed method.

In real-world diagnostic procedures, due to the limitation of human cognitive competence, a medical expert may not conveniently use some crisp numbers to express the diagnostic information, and plenty of research has indicated that generalized fuzzy numbers play a significant role in describing complex diagnostic information. To deal with medical diagnosis problems based on generalized fuzzy sets (FSs), the notion of single-valued neutrosophic multisets (SVNMs) [60] is firstly used to express the diagnostic information [38]. Then the model of probabilistic rough sets (PRSs) over two universes is applied to analyze SVNMs, and the concepts of single-valued neutrosophic rough multisets (SVNRMs) over two universes and probabilistic rough single-valued neutrosophic multisets (PRSVNMs) over two universes are introduced. Based on SVNRMs over two universes and PRSVNMs over two universes, single-valued neutrosophic probabilistic rough multisets (SVNPRMs) over two universes are further established. Next, a three-way decision model by virtue of SVNPRMs over two universes in the context of medical diagnosis is constructed. Finally, a practical case study along with a comparative study are carried out to reveal the accuracy and reliability of the constructed three-way decisions model.

The next article [39] is based on new developments on a NTG and applications earlier introduced in 2016 by Smarandache and Ali. NTG sprang up from neutrosophic triplet set X: a collection of triplets (b,neut(b),anti(b)) for an b∈X that obeys certain axioms (existence of neutral(s) and opposite(s)). Some results that are true in classical groups are investigated in NTG and shown to be either universally true in NTG or true in some peculiar types of NTG. Distinguishing features between an NTG and some other algebraic structures such as: generalized group (GG), quasigroup, loop, and group are investigated. Some neutrosophic triplet subgroups (NTSGs) of a neutrosophic triplet group are studied. Applications of the neutrosophic triplet set, and our results on NTG in relation to management and sports, are highlighted and discussed.

Neutrosophic cubic sets [64] are the more generalized tool by which one can handle imprecise information in a more effective way as compared to fuzzy sets and all other versions of fuzzy sets. Neutrosophic cubic sets have the more flexibility, precision and compatibility to the system as compared to previous existing fuzzy models. On the other hand, the graphs represent a problem physically in the form of diagrams and matrices, etc., which is very easy to understand and handle. Therefore, the authors of the subsequent paper [40] apply the neutrosophic cubic sets to graph theory in order to develop a more general approach where they can model imprecise information through graphs. One of very important futures of two neutrosophic cubic sets is the R-union that R-union of two neutrosophic cubic sets is again a neutrosophic cubic set. Since the purpose of this new model is to capture the uncertainty, the authors provide applications in industries to test the applicability of the defined model based on present time and future prediction which is the main advantage of neutrosophic cubic sets.

Thereafter, another paper [41] presents a deciding technique for robotic dexterous hand configurations. This algorithm can be used to decide on how to configure a robotic hand so it can grasp objects in different scenarios. Receiving as input from several sensor signals that provide information on the object's shape, the DSmT decision-making algorithm passes the information through several steps before deciding what hand configuration should be used for a certain object and task. The proposed decision-making method for real time control will decrease the feedback time between the command and grasped object, and can be successfully applied on robot dexterous hands. For this, the authors have used the Dezert–Smarandache theory which can provide information even on contradictory or uncertain systems.

The study [42] that follows introduces simplified neutrosophic linguistic numbers (SNLNs) to describe online consumer reviews in an appropriate manner. Considering the defects of studies on SNLNs in handling linguistic information, the cloud model is used to convert linguistic terms in SNLNs to three numerical characteristics. Then, a novel simplified neutrosophic cloud (SNC) concept is presented, and its operations and distance are defined. Next, a series of simplified neutrosophic cloud aggregation operators are investigated, including the simplified neutrosophic clouds Maclaurin symmetric mean (SNCMSM) operator, weighted SNCMSM operator, and generalized weighted SNCMSM operator. Subsequently, a MCDM model is constructed based on the proposed aggregation operators. Finally, a hotel selection problem is presented to verify the effectiveness and validity of our developed approach.

In recent years, typhoon disasters have occurred frequently and the economic losses caused by them have received increasing attention. The next study [43] focuses on the evaluation of typhoon disasters based on the interval neutrosophic set theory. An interval neutrosophic set (INS) [63] is a subclass of a NS [57]. However, the existing exponential operations and their aggregation methods are primarily for the intuitionistic fuzzy set. So, this paper mainly focus on the research of the exponential operational laws of INNs in which the bases are positive real numbers and the exponents are interval neutrosophic numbers. Several properties based on the exponential operational law are discussed in detail. Then, the interval neutrosophic weighted exponential aggregation (INWEA) operator is used to aggregate assessment information to obtain the comprehensive risk assessment. Finally, a multiple attribute decision making (MADM) approach based on the INWEA operator is introduced and applied to the evaluation of typhoon disasters in Fujian Province, China. Results show that the proposed new approach is feasible and effective in practical applications.

In the coming paper [44] of this book, the authors study the neutrosophic triplet groups for a∈Z2p and prove this collection of triplets (a,neut(a),anti(a)) if trivial forms a semigroup under product, and semi-neutrosophic triplets are included in that collection. Otherwise, they form a group under product, and it is of order (p−1), with (p+1,p+1,p+1) as the multiplicative identity. The new notion of pseudo primitive element is introduced in Z2p analogous to primitive elements in Zp, where p is a prime. Open problems based on the pseudo primitive elements are proposed. The study is restricted to Z2p and take only the usual product modulo 2p.

Fuzzy graph theory plays an important role in the study of the symmetry and asymmetry properties of fuzzy graphs. With this in mind, in the next paper [45], the authors introduce new neutrosophic graphs called complex neutrosophic graphs of type 1 (abbr. CNG1). They then present a matrix representation for it and study some properties of this new concept. The concept of CNG1 is an extension of the generalized fuzzy graphs of type 1 (GFG1) and generalized single-valued neutrosophic graphs of type 1 (GSVNG1). The utility of the CNG1 introduced here is applied to a multi-attribute decision making problem related to Internet server selection.

The purpose of the subsequent paper [46] is to study new algebraic operations and fundamental properties of totally dependent-neutrosophic sets and totally dependent-neutrosophic soft sets. Firstly, the in-coordination relationships among the original inclusion relations of totally dependent-neutrosophic sets (called type-1 and typ-2 inclusion relations in this paper) and union (intersection) operations are analyzed, and then type-3 inclusion relation of totally dependent-neutrosophic sets and corresponding type-3 union, type-3 intersection, and complement operations are introduced. Secondly, the following theorem is proved: all totally dependent-neutrosophic sets (based on a certain universe) determined a generalized De Morgan algebra with respect to type-3 union, type-3 intersection, and complement operations. Thirdly, the relationships among the type-3 order relation, score function, and accuracy function of totally dependent-neutrosophic sets are discussed. Finally, some new operations and properties of totally dependent-neutrosophic soft sets are investigated, and another generalized De Morgan algebra induced by totally dependent-neutrosophic soft sets is obtained.

In the recent years, school administrators often come across various problems while teaching, counseling, and promoting and providing other services which engender disagreements and interpersonal conflicts between students, the administrative staff, and others. Action learning is an effective way to train school administrators in order to improve their conflict-handling styles. In the next paper [47], a novel approach is used to determine the effectiveness of training in school administrators who attended an action learning course based on their conflict-handling styles. To this end, a Rahim Organization Conflict Inventory II (ROCI-II) instrument is used that consists of both the demographic information and the conflict-handling styles of the school administrators. The proposed method uses the neutrosophic set (NS) and support vector machines (SVMs) to construct an efficient classification scheme neutrosophic support vector machine (NS-SVM). The neutrosophic c-means (NCM) clustering algorithm is used to determine the neutrosophic memberships and then a weighting parameter is calculated from the neutrosophic memberships. The calculated weight value is then used in SVM as handled in the fuzzy SVM (FSVM) approach. Various experimental works are carried in a computer environment out to validate the proposed idea. All experimental works are simulated in a MATLAB environment with a five-fold cross-validation technique. The classification performance is measured by accuracy criteria. The prediction experiments are conducted based on two scenarios. In the first one, all statements are used to predict if a school administrator is trained or not after attending an action learning program. In the second scenario, five independent dimensions are used individually to predict if a school administrator is trained or not after attending an action learning program. According to the obtained results, the proposed NS-SVM outperforms for all experimental works.

The notions of the neutrosophic hesitant fuzzy subalgebra and neutrosophic hesitant fuzzy filter in pseudo-BCI algebras are introduced, and some properties and equivalent conditions are investigated in the next paper [48]. The relationships between neutrosophic hesitant fuzzy subalgebras (filters) and hesitant fuzzy subalgebras (filters) are discussed. Five kinds of special sets are constructed by a neutrosophic hesitant fuzzy set, and the conditions for the two kinds of sets to be filters are given. Moreover, the conditions for two kinds of special neutrosophic hesitant fuzzy sets to be neutrosophic hesitant fuzzy filters are proved.

To solve the problems related to inhomogeneous connections among the attributes, the authors of the following paper [49] introduce a novel multiple attribute group decision-making (MAGDM) method based on the introduced linguistic neutrosophic generalized weighted partitioned Bonferroni mean operator (LNGWPBM) for linguistic neutrosophic numbers (LNNs). First of all, inspired by the merits of the generalized partitioned Bonferroni mean (GPBM) operator and LNNs, they combine the GPBM operator and LNNs to propose the linguistic neutrosophic GPBM (LNGPBM) operator, which supposes that the relationships are heterogeneous among the attributes in MAGDM. In addition, aimed at the different importance of each attribute, the weighted form of the LNGPBM operator is investigated. Then, the authors discuss some of its desirable properties and special examples accordingly. Finally, they propose a novel MAGDM method on the basis of the introduced LNGWPBM operator, and illustrate its validity and merit by comparing it with the existing methods.

Based on the multiplicity evaluation in some real situations, the next paper [50] firstly introduces a single-valued neutrosophic multiset (SVNM) as a subclass of neutrosophic multiset (NM) to express the multiplicity information and the operational relations of SVNMs. Then, a cosine measure between SVNMs and weighted cosine measure between SVNMs are presented to measure the cosine degree between SVNMs, and their properties are investigated. Based on the weighted cosine measure of SVNMs, a multiple attribute decision-making method under a SVNM environment is proposed, in which the evaluated values of alternatives are taken in the form of SVNMs. The ranking order of all alternatives and the best one can be determined by the weighted cosine measure between every alternative and the ideal alternative. Finally, an actual application on the selecting problem illustrates the effectiveness and application of the proposed method.

Rooftop distributed photovoltaic projects have been quickly proposed in China because of policy promotion. Before, the rooftops of the shopping mall had not been occupied, and it was urged to have a decision-making framework to select suitable shopping mall photovoltaic plans. However, a traditional MCDM method failed to solve this issue at the same time, due to the following three defects: the interactions problems between the criteria, the loss of evaluation information in the conversion process, and the compensation problems between diverse criteria. In the subsequent paper [51], an integrated MCDM framework is proposed to address these problems. First of all, the compositive evaluation index is constructed, and the application of DEMATEL method helped analyze the internal influence and connection behind each criterion. Then, the interval-valued neutrosophic set is utilized to express the imperfect knowledge of experts group and avoid the information loss. Next, an extended elimination et choice translation reality (ELECTRE) III method is applied, and it succeed in avoiding the compensation problem and obtaining the scientific result. The integrated method used maintained symmetry in the solar photovoltaic (PV) investment. Last but not least, a comparative analysis using Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) method and VIKOR method is carried out, and alternative plan X1 ranks first at the same. The outcome certified the correctness and rationality of the results obtained in this study.

In the next paper [52], by utilizing the concept of a neutrosophic extended triplet (NET), the authors define the neutrosophic image, neutrosophic inverse-image, neutrosophic kernel, and the NET subgroup. The notion of the neutrosophic triplet coset and its relation with the classical coset are defined and the properties of the neutrosophic triplet cosets are given. Furthermore, the neutrosophic triplet normal subgroups, and neutrosophic triplet quotient groups are studied.

The following paper [53] in the book proposes novel skin lesion detection based on neutrosophic clustering and adaptive region growing algorithms applied to dermoscopic images, called NCARG. First, the dermoscopic images are mapped into a neutrosophic set domain using the shearlet transform results for the images. The images are described via three memberships: true, indeterminate, and false memberships. An indeterminate filter is then defined in the neutrosophic set for reducing the indeterminacy of the images. A neutrosophic c-means clustering algorithm is applied to segment the dermoscopic images. With the clustering results, skin lesions are identified precisely using an adaptive region growing method. To evaluate the performance of this algorithm, a public data set (ISIC 2017) is employed to train and test the proposed method. Fifty images are randomly selected for training and 500 images for testing. Several metrics are measured for quantitatively evaluating the performance of NCARG. The results establish that the proposed approach has the ability to detect a lesion with high accuracy, 95.3% average value, compared to the obtained average accuracy, 80.6%, found when employing the neutrosophic similarity score and level set (NSSLS) segmentation approach.

Every organization seeks to set strategies for its development and growth and to do this, it must take into account the factors that affect its success or failure. The most widely used technique in strategic planning is SWOT analysis. SWOT examines strengths (S), weaknesses (W), opportunities (O), and threats (T), to select and implement the best strategy to achieve organizational goals. The chosen strategy should harness the advantages of strengths and opportunities, handle weaknesses, and avoid or mitigate threats. SWOT analysis does not quantify factors (i.e., strengths, weaknesses, opportunities, and threats) and it fails to rank available alternatives. To overcome this drawback, the authors of the next paper [54] integrate it with the analytic hierarchy process (AHP). The AHP is able to determine both quantitative and the qualitative elements by weighting and ranking them via comparison matrices. Due to the vague and inconsistent information that exists in the real world, they apply the proposed model in a neutrosophic environment. A real case study of Starbucks Company is presented to validate the model.

Big Data is a large-sized and complex dataset, which cannot be managed using traditional data processing tools. The mining process of big data is the ability to extract valuable information from these large datasets. Association rule mining is a type of data mining process, which is intended to determine interesting associations between items and to establish a set of association rules whose support is greater than a specific threshold. The classical association rules can only be extracted from binary data where an item exists in a transaction, but it fails to deal effectively with quantitative attributes, through decreasing the quality of generated association rules due to sharp boundary problems. In order to overcome the drawbacks of classical association rule mining, the authors of the following research [55] propose a new neutrosophic association rule algorithm. The algorithm uses a new approach for generating association rules by dealing with membership, indeterminacy, and non-membership functions of items, conducting to an efficient decision-making system by considering all vague association rules. To prove the validity of the method, they compare the fuzzy mining and the neutrosophic mining [65]. The results show that the proposed approach increases the number of generated association rules.

The INS is a subclass of the NS and a generalization of the interval-valued intuitionistic fuzzy set (IVIFS), which can be used in real engineering and scientific applications. The last paper [56] in the book aims at developing new generalized Choquet aggregation operators for INSs, including the generalized interval neutrosophic Choquet ordered averaging (G-INCOA) operator and generalized interval neutrosophic Choquet ordered geometric (G-INCOG) operator. The main advantages of the proposed operators can be described as follows: (i) during decision-making or analyzing process, the positive interaction, negative interaction or non-interaction among attributes can be considered by the G-INCOA and G-INCOG operators; (ii) each generalized Choquet aggregation operator presents a unique comprehensive framework for INSs, which comprises a bunch of existing interval neutrosophic aggregation operators; (iii) new multi-attribute decision making (MADM) approaches for INSs are established based on these operators, and decision makers may determine the value of λ by different MADM problems or their preferences, which makes the decision-making process more flexible; (iv) a new clustering algorithm for INSs are introduced based on the G-INCOA and G-INCOG operators, which proves that they have the potential to be applied to many new fields in the future.

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Neutrosophic Hedge Algebras

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Florentin Smarandache (2019). Neutrosophic Hedge Algebras. *Broad Research in Artificial Intelligence and Neuroscience* 10(3), 117-123

Abstract

We introduce now for the first time the neutrosophic hedge algebras as an extension of classical hedge algebras, together with an application of neutrosophic hedge algebras.

1. Introduction

The classical hedge algebras deal with linguistic variables. In neutrosophic environment we have introduced the neutrosophic linguistic variables. We have defined neutrosophic partial relationships between single-valued neutrosophic numbers. Neutrosophic operations are used in order to aggregate the neutrosophic linguistic values.

2. Materials and Methods

We introduce now, for the first time, the Neutrosophic Hedge Algebras, as extension of classical Hedge Algebras.

Let's consider a Linguistic Variable:

with $Dom(x)$ as the word domain of x, whose each element is a word (label), or string of words.

Let $\mathcal{A}$ be an attribute that describes the value of each element $x \in Dom(x)$, as follows:

$\mathcal{A}: Dom(x) \to [0,1]^3$. (1)

$\mathcal{A}(x)$ is the neutrosophic value of x with respect to this attribute:

$A(x) = \langle t_x, i_x, f_x \rangle$, (2)

where $t_x, i_x, f_x \in [0,1]$, such that

- t_x means the degree of value of x;
- i_x means the indeterminate degree of value of x;
- f_x means the degree of non-value of x.

We may also use the notation: $x\langle t_x, i_x, f_x \rangle$.

A neutrosophic partial relationship $\leq_N$ on $Dom(x)$, defined as follows:

$x\langle t_x, i_x, f_x \rangle \leq_N y\langle t_y, i_y, f_y \rangle$, (3)

if and only if $t_x \leq t_y$, and $i_x \geq i_y$, $f_x \geq f_y$.

Therefore, $(Dom(x), \leq_N)$ becomes a neutros-ophic partial order set (or neutrosophic poset), and $\leq_N$ is called a neutrosophic inequality.

Let $C = \{0, w, 1\}$ be a set of constants, $C \subset Dom(x)$, where:

- 0 = the least element, or $0_{\langle 0,1,1 \rangle}$;
- w = the neutral (middle) element, or $w_{\langle 0.5,0.5,0.5 \rangle}$;
- and 1 = the greatest element, or $1_{\langle 1,0,0 \rangle}$.

Let G be a word-set of two *neutrosophic generators*, $G \subset Dom(x)$, qualitatively a negative primary neutrosophic term (denoted g^-), and the other one that is qualitatively a positive primary neutrosophic term (denoted g^+), such that:

BRAIN – Broad Research in Artificial Intelligence and Neuroscience
Volume 10, Issue 3 (September, 2019), ISSN 2067-3957

$0 \leq_N g^- \leq_N w \leq_N g^+ \leq_N 1,$ (4)

or transcribed using the neutrosophic com-ponents:

$0_{\langle 0,1,1\rangle} \leq_N g^-_{\langle t_{g^-}, i_{g^-}, f_{g^-}\rangle} \leq_N w_{\langle 0.5,0.5,0.5\rangle}$
$\leq_N g^+_{\langle t_{g^+}, i_{g^+}, f_{g^+}\rangle} \leq_N 1_{\langle 1,0,0\rangle},$

where

- $0 \leq t_{g^-} \leq 0.5 \leq t_{g^+} \leq 1$ (here there are classical inequalities)
- $1 \geq i_{g^-} \geq 0.5 \geq i_{g^+} \geq 0$, and
- $1 \geq f_{g^-} \geq 0.5 \geq f_{g^+} \geq 0$.

Let $H \subset Dom(x)$ be the set of *neutrosophic hedges*, regarded as unary operations. Each hedge h∈H is a functor, or comparative particle for adjectives and adverbs as in the natural language (English).

h:Dom(x)→Dom(x)
x→h(x). (5)

Instead of h(x) one easily writes hx to be closer to the natural language.

By associating the neutrosophic components, one has:

h_⟨t_h,i_h,f_h ⟩ x_⟨t_x,i_x,f_x ⟩ .

A hedge applied to x may increase, decrease, or approximate the neutrosophic value of the element x.

There also exists a neutrosophic identity I∈Dom(x), denoted I_⟨0,0,0⟩ that does not hange on the elements:

I_⟨0,0,0⟩ x_⟨t_x,i_x,f_x ⟩ .

In most cases, if a hedge increases / decreases the neutrosophic value of an element x situated above the neutral element w, the same hedge does the opposite, decreases / increases the neutrosophic value of an element y situated below the neutral element w.

And reciprocally.

If a hedge approximates the neutrosophic value, by diminishing it, of an element x situated above the neutral element w, then it approximates the neutrosophic value, by enlarging it, of an element y situated below the neutral element w.

Let's refer the hedges with respect to the upper part (⊔), above the neutral element, since for the lower part (L) it will automatically be the opposite effect.

We split de set of hedges into three disjoint subsets:

H_⊔^+ = the hedges that increase the neutrosophic value of the upper elements;
H_⊔^- = the hedges that decrease the neutrosophic value of the upper elements;
H_⊔^∼ = the hedges that approximate the neutrosophic value of the upper elements.

Notations: Let $x = x_{\sqcup} \cup w \cup x_L$, where $x_{\sqcup}$ cons-titutes the upper element set, while x_L the lower element subset, w the neutral element. $x_{\sqcup}$ and x_L are disjoint two by two.

3. Operations on Neutrosophic Components

Let $\langle t_1, i_1, f_1\rangle$, $\langle t_2, i_2, f_2\rangle$ neutrosophic numbers.

Then:

$$t_1 + t_2 = \begin{cases} t_1 + t_2, \text{if } t_1 + t_2 \leq 1; \\ 1, \text{if } t_1 + t_2 > 1; \end{cases} \quad (6)$$

and

$$t_1 - t_2 = \begin{cases} 0, \text{if } t_1 - t_2 < 0; \\ t_1 - t_2, \text{if } t_1 - t_2 \geq 0. \end{cases} \quad (7)$$

Similarly for i_1 and f_1:

$$i_1 + i_2 = \begin{cases} i_1 + i_2, \text{if } i_1 + i_2 \leq 1; \\ 1, \text{if } i_1 + i_2 > 1; \end{cases} \quad (8)$$

$$i_1 - i_2 = \begin{cases} 0, \text{if } i_1 - i_2 < 0; \\ i_1 - i_2, \text{if } i_1 - i_2 \geq 0. \end{cases} \quad (9)$$

and

$$f_1 + f_2 = \begin{cases} f_1 + f_2, \text{if } f_1 + f_2 \leq 1; \\ 1, \text{if } f_1 + f_2 > 1; \end{cases} \quad (10)$$

$$f_1 - f_2 = \begin{cases} 0, \text{if } f_1 - f_2 < 0; \\ f_1 - f_2, \text{if } f_1 - f_2 \geq 0. \end{cases} \quad (11)$$

4. Neutrosophic Hedge-Element Operators

We define the following operators:

4.1. Neutrosophic Increment

Hedge $\boxed{\uparrow}$ Element = $\langle t_1, i_1, f_1 \rangle \boxed{\uparrow} \langle t_2, i_2, f_2 \rangle = \langle t_2 + t_1, i_2 - i_1, f_2 - f_1 \rangle$, (12)

meaning that the first triplet increases the second.

4.2. Neutrosophic Decrement

Hedge $\boxed{\downarrow}$ Element = $\langle t_1, i_1, f_1 \rangle \boxed{\downarrow} \langle t_2, i_2, f_2 \rangle = \langle t_2 - t_1, i_2 + i_1, f_2 + f_1 \rangle$, (13)

meaning that the first triplet decreases the second.

4.3. Theorem 1

The neutrosophic increment and decrement operators are non-commutattive.

5. Neutrosophic Hedge-Hedge Operators

Hedge $\boxed{\uparrow}$ Hedge = $\langle t_1, i_1, f_1 \rangle \boxed{\uparrow} \langle t_2, i_2, f_2 \rangle = \langle t_1 + t_2, i_1 + i_2, f_1 + f_2 \rangle$ (14)

Hedge $\boxed{\downarrow}$ Hedge = $\langle t_1, i_1, f_1 \rangle \boxed{\downarrow} \langle t_2, i_2, f_2 \rangle = \langle t_1 - t_2, i_1 - i_2, f_1 - f_2 \rangle$ (15)

6. Neutrosophic Hedge Operators

Let $x_\sqcup \langle t_{x_\sqcup}, i_{x_\sqcup}, f_{x_\sqcup} \rangle \in Dom(x)$ i.e. $x_\sqcup$ is an upper element of $Dom(x)$, and

- $h_\sqcup^+ \langle t_{h_\sqcup^+}, i_{h_\sqcup^+}, f_{h_\sqcup^+} \rangle \in H_\sqcup^+$,
- $h_\sqcup^- \langle t_{h_\sqcup^-}, i_{h_\sqcup^-}, f_{h_\sqcup^-} \rangle \in H_\sqcup^-$,
- $h_\sqcup^\sim \langle t_{h_\sqcup^\sim}, i_{h_\sqcup^\sim}, f_{h_\sqcup^\sim} \rangle \in H_\sqcup^\sim$,

then $h_\sqcup^+$ applied to $x_\sqcup$ gives

$(h_\sqcup^+ x_\sqcup) \langle t_{x_\sqcup}, i_{x_\sqcup}, f_{x_\sqcup} \rangle \boxed{\uparrow} \langle t_{h_\sqcup^+}, i_{h_\sqcup^+}, f_{h_\sqcup^+} \rangle$,

and $h_\sqcup^-$ applied to $x_\sqcup$ gives

$(h_\sqcup^- x_\sqcup) \langle t_{x_\sqcup}, i_{x_\sqcup}, f_{x_\sqcup} \rangle \boxed{\downarrow} \langle t_{h_\sqcup^-}, i_{h_\sqcup^-}, f_{h_\sqcup^-} \rangle$,

and $h_\sqcup^\sim$ applied to $x_\sqcup$ gives

$(h_\sqcup^\sim x_\sqcup) \langle t_{x_\sqcup}, i_{x_\sqcup}, f_{x_\sqcup} \rangle \boxed{\downarrow} \langle t_{h_\sqcup^\sim}, i_{h_\sqcup^\sim}, f_{h_\sqcup^\sim} \rangle$.

Now, let $x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \in Dom(x_L)$, i.e. x_L is a lower element of $Dom(x)$. Then, $h_{\sqcup}^{+}$ applied to x_L gives:

$h_{\sqcup}^{+}x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \boxed{\downarrow} \langle t_{h_{\sqcup}^{+}}, i_{h_{\sqcup}^{+}}, f_{h_{\sqcup}^{+}}\rangle$,

and $h_{\sqcup}^{-}$ applied to x_L gives:

$h_{\sqcup}^{-}x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \boxed{\uparrow} \langle t_{h_{\sqcup}^{-}}, i_{h_{\sqcup}^{-}}, f_{h_{\sqcup}^{-}}\rangle$,

and $h_{\sqcup}^{\sim}$ applied to x_L gives:

$h_{\sqcup}^{\sim}x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \boxed{\uparrow} \langle t_{h_{\sqcup}^{\sim}}, i_{h_{\sqcup}^{\sim}}, f_{h_{\sqcup}^{\sim}}\rangle$.

In the same way, we may apply many increasing, decreasing, approximate or other type of hedges to the same upper or lower element

$h_{\sqcup_n}^{+}h_{\sqcup_{n-1}}^{-}h_{\sqcup}^{v} \dots h_{\sqcup_1}^{+}x$,

generating new elements in $Dom(x)$.

The hedges may be applied to the constants as well.

6.1. Theorem 2

A hedge applied to another hedge wekeans or stengthens or approximates it.

6.2. Theorem 3

If $h_{\sqcup}^{+} \in H_{\sqcup}^{+}$ and $x_{\sqcup} \in Dom(x_{\sqcup})$, then $h_{\sqcup}^{+}x_{\sqcup} \geq x_{\sqcup}$.
If $h_{\sqcup}^{-} \in H_{\sqcup}^{-}$ and $x_{\sqcup} \in Dom(x_{\sqcup})$, then $h_{\sqcup}^{-}x_{\sqcup} \geq x_{\sqcup}$.
If $h_{\sqcup}^{+} \in H_{\sqcup}^{+}$ and $x_L \in Dom(x_L)$, then $h_{\sqcup}^{+}x_L \leq_N x_L$.
If $h_{\sqcup}^{-} \in H_{\sqcup}^{-}$ and $x_L \in Dom(x_L)$, then $h_{\sqcup}^{-}x_L \geq_N x_L$.

6.3. Converse Hedges

Two hedges h_1 and $h_2 \in H$ are converse to each other, if $\forall x \in Dom(x)$, $h_1 x \leq_N x$ is equivalent to $h_2 x \geq_N x$.

6.4. Compatible Hedges

Two hedges h_1 and $h_2 \in H$ are compatible, if $\forall x \in Dom(x)$, $h_1 x \leq_N x$ is equivalent to $h_2 x \leq_N x$.

6.5. Commutative Hedges

Two hedges h_1 and $h_2 \in H$ are commutative, if $\forall x \in Dom(x)$, $h_1 h_2 x = h_2 h_1 x$. Otherwise they are called non-commutative.

6.6. Cumulative Hedges

If $h_{1_{\sqcup}}^{+}$ and $h_{2_{\sqcup}}^{+} \in H^{+}$, then two neutrosophic edges can be cumulated into one:

$h_{1_{\sqcup}}^{+}\langle t_{h_{1_{\sqcup}}^{+}}, i_{h_{1_{\sqcup}}^{+}}, f_{h_{1_{\sqcup}}^{+}}\rangle\, h_{2_{\sqcup}}^{+}\langle t_{h_{2_{\sqcup}}^{+}}, i_{h_{2_{\sqcup}}^{+}}, f_{h_{2_{\sqcup}}^{+}}\rangle = h_{12_{\sqcup}}^{+}\langle t_{h_{1_{\sqcup}}^{+}}, i_{h_{1_{\sqcup}}^{+}}, f_{h_{1_{\sqcup}}^{+}}\rangle \boxed{\uparrow} \langle t_{h_{2_{\sqcup}}^{+}}, i_{h_{2_{\sqcup}}^{+}}, f_{h_{2_{\sqcup}}^{+}}\rangle$. (16)

Similarly, if $h_{1_{\sqcup}}^{-}$ and $h_{2_{\sqcup}}^{-} \in H^{-}$, then we can cumulate them into one:

$h_{1_{\sqcup}}^{-}\langle t_{h_{1_{\sqcup}}^{-}}, i_{h_{1_{\sqcup}}^{-}}, f_{h_{1_{\sqcup}}^{-}}\rangle\, h_{2_{\sqcup}}^{-}\langle t_{h_{2_{\sqcup}}^{-}}, i_{h_{2_{\sqcup}}^{-}}, f_{h_{2_{\sqcup}}^{-}}\rangle = h_{12_{\sqcup}}^{-}\langle t_{h_{1_{\sqcup}}^{-}}, i_{h_{1_{\sqcup}}^{-}}, f_{h_{1_{\sqcup}}^{-}}\rangle \boxed{\uparrow} \langle t_{h_{2_{\sqcup}}^{-}}, i_{h_{2_{\sqcup}}^{-}}, f_{h_{2_{\sqcup}}^{-}}\rangle$. (17)

Now, if the two hedges are converse, $h^+_{1_\sqcup}$ and $h^-_{1_\sqcup}$, but the neutrosophic components of the first (which is actually a neutrosophic number) are greater than the second, we cumulate them into one as follows:

$$h^+_{3_\sqcup} = \left(h^+_{1_\sqcup} h^-_{2_\sqcup}\right) \langle t_{h^+_{1_\sqcup}}, i_{h^+_{1_\sqcup}}, f_{h^+_{1_\sqcup}} \rangle \boxed{\downarrow} \langle t_{h^-_{2_\sqcup}}, i_{h^-_{2_\sqcup}}, f_{h^-_{2_\sqcup}} \rangle. \quad (18)$$

But, if the neutrosophic components of the second are greater, and the hedges are commutative, we cumulate them into one as follows:

$$h^+_{3_\sqcup} = \left(h^+_{1_\sqcup} h^-_{2_\sqcup}\right) \langle t_{h^-_{2_\sqcup}}, i_{h^-_{2_\sqcup}}, f_{h^-_{2_\sqcup}} \rangle \boxed{\downarrow} \langle t_{h^+_{1_\sqcup}}, i_{h^+_{1_\sqcup}}, f_{h^+_{1_\sqcup}} \rangle \quad (19)$$

7. Neutrosophic Hedge Algebra

$NHA = (x, G, C, H \cup I, \leq_N)$ constitutes an abstract algebra, called Neutrosophic Hedge Algebra.

7.1. Example of a Neutrosophic Hedge Algebra τ

Let $G = \{Small, Big\}$ the set of generators, repres-ented as neutrosophic generators as follows:

$Small_{\langle 0.3,0.6,0.7\rangle}, Big_{\langle 0.7,0.2,0.3\rangle}$.

Let $H = \{Very, Less\}$ the set of hedges, repres-ented as neutrosophic hedges as follows:

$Very_{\langle 0.1,0.1,0.1\rangle}, Less_{\langle 0.1,0.2,0.3\rangle}$,

where $Very \in H^+_\sqcup$ and $Less \in H^-_\sqcup$.

x is a neutrosophic linguistic variable whose domain is G at the beginning, but extended by generators.

The neutrosophic constants are

$C = \{0_{\langle 0,1,1\rangle}, Medium_{\langle 0.5,0.5,0.5\rangle}, 1_{\langle 1,0,0\rangle}\}$.

The neutrosophic identity is $I_{\langle 0,0,0\rangle}$.

We use the neutrosophic inequality $\leq_N$, and the neutrosophic increment / decrement operators previously defined.

Let's apply the neutrosophic hedges in order to generate new neutrosophic elements of the neutrosophic linguistic variable x.

$Very$ applied to Big [upper element] has a positive effect:

$Very_{\langle 0.1,0.1,0.1\rangle} Big_{\langle 0.7,0.2,0.3\rangle} = (Very\ Big)_{\langle 0.7+0.1,0.2-0.1,0.3-0.1\rangle} = (Very\ Big)_{\langle 0.8,0.1,0.2\rangle}$.

Then:

$Very_{\langle 0.1,0.1,0.1\rangle} (Very\ Big)_{\langle 0.9,0.1,0.2\rangle} = (Very\ Very\ Big)_{\langle 0.9,0,0.1\rangle}$.

$Very$ applied to $Small$ [lower element] has a negative effect:

$Very_{\langle 0.1,0.1,0.1\rangle} Small_{\langle 0.3,0.6,0.7\rangle} = (Very\ Small)_{\langle 0.3-0.1,0.6+0.1,0.7+0.1\rangle} =$
$(Very\ Small)_{\langle 0.2,0.7,0.8\rangle}$.

If we compute $(Very\ Very)$ first, which is a neutrosophic hedge-hedge operator:

$Very_{\langle 0.1,0.1,0.1\rangle} Very_{\langle 0.1,0.1,0.1\rangle} = (Very\ Very)_{\langle 0.1+0.1,0.1+0.1,0.1+0.1\rangle} =$
$(Very\ Very)_{\langle 0.2,0.2,0.2\rangle}$,

and we apply it to Big, we get:

$$(Very\ Very)_{\langle 0.2,0.2,0.2\rangle} Big_{\langle 0.7,0.2,0.3\rangle} = (Very\ Very\ Big)_{\langle 0.7+0.2,0.2-0.2,0.3-0.2\rangle}$$
$$= (Very\ Very\ Big)_{\langle 0.9,0,0.1\rangle},$$

so, we get the same result.

$Less$ applied to Big has a negative effect:

$Less_{\langle 0.1,0.2,0.3\rangle} Big_{\langle 0.7,0.2,0.3\rangle} = (Less\ Big)_{\langle 0.7-0.1,0.2+0.2,0.3\rangle} = (Less\ Big)_{\langle 0.6,0.4,0.6\rangle}$.

$Less$ applied to $Small$ has a positive effect:

$Less_{\langle 0.1,0.2,0.3\rangle} Small_{\langle 0.3,0.6,0.7\rangle} = (Less\ Small)_{\langle 0.1+0.3,0.6-0.2,0.7-0.3\rangle} = (Less\ Small)_{\langle 0.4,0.4,0.4\rangle}$.

The set of neutrosophic hedges H is enriched through the generation of new neutrosophic hedges by combining a hedge with another one using the neutrosophic hedge-hedge operators.

Further, the newly generated neutrosophic hedges are applied to the elements of the linguistic variable, and more new elements are generated.

Let's compute more neutrosophic elements:

$$VLB = Very_{\langle 0.1,0.1,0.1\rangle} Less_{\langle 0.1,0.2,0.3\rangle} Big_{\langle 0.7,0.2,0.3\rangle}$$
$$= (Very\ Less\ Big)_{\left[\langle 0.1,0.1,0.1\rangle \underset{h}{\boxed{\uparrow}} \langle 0.1,0.2,0.3\rangle\right] \boxed{\downarrow} \langle 0.7,0.2,0.3\rangle}$$
$$= (Very\ Less\ Big)_{\langle 0.1+0.1,0.1+0.2,0.1+0.3\rangle \boxed{\downarrow} \langle 0.7,0.2,0.3\rangle}$$
$$= (Very\ Less\ Big)_{\langle 0.7-0.2,0.2-0.3,0.3-0.4\rangle} = (Very\ Less\ Big)_{\langle 0.5,0,0\rangle}$$

$$VM = Very_{\langle 0.1,0.1,0.1\rangle} Medium_{\langle 0.5,0.5,0.5\rangle} = (Very\ Medium)_{\langle 0.1,0.1,0.1\rangle \boxed{\uparrow} \langle 0.5,0.5,0.5\rangle}$$
$$= (Very\ Medium)_{\langle 0.6,0.4,0.4\rangle}$$

$$LM = Less_{\langle 0.1,0.2,0.3\rangle} Medium_{\langle 0.5,0.5,0.5\rangle} = (Less\ Medium)_{\langle 0.1,0.2,0.3\rangle \boxed{\downarrow} \langle 0.5,0.5,0.5\rangle}$$
$$= (Less\ Medium)_{\langle 0.4,0.7,0.8\rangle}$$

$$VVS = Very_{\langle 0.1,0.1,0.1\rangle} Very_{\langle 0.1,0.1,0.1\rangle} Small_{\langle 0.3,0.6,0.7\rangle} = (Very\ Very)_{\langle 0.2,0.2,0.2\rangle} Small_{\langle 0.3,0.6,0.7\rangle}$$
$$= (Very\ Very\ Small)_{\langle 0.1,0.8,0.9\rangle}$$

$$VLS = Very_{\langle 0.1,0.1,0.1\rangle} Less_{\langle 0.1,0.2,0.3\rangle} Small_{\langle 0.3,0.6,0.7\rangle} = Very_{\langle 0.1,0.1,0.1\rangle} (Less\ Small)_{\langle 0.4,0.4,0.4\rangle}$$
$$= (Very\ Less\ Small)_{\langle 0.5,0.3,0.3\rangle}$$

$$LAMax = Less_{\langle 0.1,0.2,0.3\rangle} Absolute\ Maximum_{\langle 1,0,0\rangle}$$
$$= (Less\ Absolute\ Maximum)_{\langle 0.1,0.2,0.3\rangle \boxed{\downarrow} \langle 1,0,0\rangle}$$
$$= (Less\ Absolute\ Maximum)_{\langle 0.9,0.2,0.3\rangle}$$

$$LAMin = Less_{\langle 0.1,0.2,0.3\rangle} Absolute\ Minimum_{\langle 0,1,1\rangle} = (Less\ Absolute\ Minimum)_{\langle 0.1,0.2,0.3\rangle \boxed{\uparrow} \langle 0,1,1\rangle}$$
$$= (Less\ Absolute\ Maximum)_{\langle 0.1,0.8,0.7\rangle}$$

7.2. Theorem 4

Any increasing hedge $h_{\langle t,i,f\rangle}$ applied to the absolute maximum cannot overpass the absolute maximum.

Proof:

$$h_{\langle t,i,f\rangle} \boxed{\uparrow} 1_{\langle 1,0,0\rangle} = (h1)_{\langle 1+t,0-i,0-f\rangle}$$
$$= (h1)_{\langle 1,0,0\rangle} = 1_{\langle 1,0,0\rangle}.$$

7.3. Theorem 5

Any decreasing hedge $h_{\langle t,i,f\rangle}$ applied to the absolute minimum cannot pass below the absolute minimum.

Proof:

$$h_{\langle t,i,f\rangle} \boxed{\downarrow} 0_{\langle 0,1,1\rangle} = (ho)_{\langle 0-t,1+i,1+f\rangle}$$
$$= (ho)_{\langle 0,1,1\rangle} = 0_{\langle 0,1,1\rangle}.$$

8. Diagram of the Neutrosophic Hedge Algebra τ

$1_{\langle 1,0,0\rangle}$ ABSOLUTE MAXIMUM

$VVB_{\langle 0.9,0,0.1\rangle}$ Very Very Big

$LAM_{\langle 0.9,0.2,0.3\rangle}$ Less Absolute Maximum
$VB_{\langle 0.8,0.1,0.2\rangle}$ Very Big

$Big_{\langle 0.7,0.2,0.3\rangle}$
$VM_{\langle 0.6,0.4,0.4\rangle}$ Very Medium
$LV_{\langle 0.5,0.4,0.6\rangle}$ Less Big

$VLB_{\langle 0.5,0,0\rangle}$ Very Less Big
$VLS_{\langle 0.5,0.3,0.3\rangle}$ Very Less Small
$M_{\langle 0.5,0.5,0.5\rangle}$ MEDIUM
$LM_{\langle 0.4,0.7,0.8\rangle}$ Less Medium

$LS_{\langle 0.4,0.4,0.4\rangle}$ Less Small
$Small_{\langle 0.3,0.6,0.7\rangle}$
$VS_{\langle 0.2,0.7,0.8\rangle}$ Very Small
$LAMin_{\langle 0.1,0.8,0.7\rangle}$ Less Absolute Minimum
$VVS_{\langle 0.1,0.8,0.9\rangle}$ Very Very Small

$0_{\langle 0,1,1\rangle}$ ABSOLUTE MINIMUM

9. Conclusions

In this paper, the classical hedge algebras have been extended for the first time to neutrosophic hedge algebras. With respect to an attribute, we have inserted the neutrosophic degrees of membership / indeterminacy / nonmembership of each generator, hedge, and constant. More than in the classical hedge algebras, we have introduced several numerical hedge operators: for hedge applied to element, and for hedge combined with hedge. An extensive example of a neutrosophic hedge algebra is given, and important properties related to it are presented.

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Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras

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G. Muhiuddin, Florentin Smarandache, Young Bae Jun (2019). Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras. *Neutrosophic Sets and Systems* 25, 161-173

Abstract: In the present paper, we discuss the Neutrosophic quadruple q-ideals and (regular) neutrosophic quadruple ideals and investigate their related properties. Also, for any two nonempty subsets U and V of a BCI-algebra S, conditions for the set $NQ(U, V)$ to be a (regular) neutrosophic quadruple ideal and a neutrosophic quadruple q-ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$ are discussed. Furthermore, we prove that let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$ and $J \subseteq V$. If I and J are q-ideals of S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Keywords: neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-algebra, (regular) neutrosophic quadruple ideal, neutrosophic quadruple q-ideal.

1 Introduction

To deal with incomplete, inconsistent and indeterminate information, Smarandache introduced the notion of neutrosophic sets (see ([1], [2] and [3]). In fact, neutrosophic set is a useful mathematical tool which extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic set theory has useful applications in several branches (see for e.g., [4], [5], [6] and [7]).

In [8], Smarandache considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [9] and [10]. Recently, neutrosophic set theory has been applied to the BCK/BCI-algebras on various aspects (see for e.g., [11], [12] [13], [14], [15], [16], [17], [18], [19] and [20].) Using the notion of neutrosophic quadruple numbers based on a set, Jun et al. [21] constructed neutrosophic quadruple BCK/BCI-algebras. They investigated several properties, and considered ideal and positive implicative ideal in neutrosophic quadruple BCK-algebra, and closed ideal in neutrosophic quadruple BCI-algebra. Given subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, they considered sets $NQ(U,V)$ which consists of neutrosophic quadruple BCK/BCI-numbers with a condition. They provided conditions for the set $NQ(U,V)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra, and the set $NQ(U,V)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. They gave an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and then they considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra. Muhiuddin et al. [22] discussed several properties and (implicative) neutrosophic quadruple ideals in (implicative) neutrosophic quadruple BCK-algebras.

In this paper, we introduce the notions of (regular) neutrosophic quadruple ideal and neutrosophic quadruple q-ideal in neutrosophic quadruple BCI-algebras, and investigate related properties. Given nonempty subsets A and B of a BCI-algebra S, we consider conditions for the set $NQ(U,V)$ to be a (regular) neutrosophic quadruple ideal of $NQ(S)$ and a neutrosophic quadruple q-ideal of $NQ(S)$.

2 Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

A nonempty set S with a constant 0 and a binary operation $*$ is called a BCI-algebra if for all $x,y,z\in S$ the following conditions hold ([23] and [24]):

(I) $(((x*y)*(x*z))*(z*y)=0)$,

(II) $((x*(x*y))*y=0)$,

(III) $(x*x=0)$,

(IV) $(x*y=0,\, y*x=0 \;\Rightarrow\; x=y)$.

If a BCI-algebra S satisfies the following identity:

(V) $(\forall x\in S)\,(0*x=0)$,

then S is called a *BCK-algebra.* Define a binary relation $\leq$ on X by letting $x*y=0$ if and only if $x\leq y$. Then $(S,\leq)$ is a partially ordered set.

Theorem 2.1. *Let S be a BCK/BCI-algebra. Then following conditions are hold:*

$$(\forall x\in S)\,(x*0=x)\,, \tag{2.1}$$
$$(\forall x,y,z\in S)\,(x\leq y\;\Rightarrow\; x*z\leq y*z,\, z*y\leq z*x)\,, \tag{2.2}$$
$$(\forall x,y,z\in S)\,((x*y)*z=(x*z)*y)\,, \tag{2.3}$$
$$(\forall x,y,z\in S)\,((x*z)*(y*z)\leq x*y) \tag{2.4}$$

where $x\leq y$ if and only if $x*y=0$.

Any BCI-algebra S satisfies the following conditions (see [25]):

$$(\forall x,y\in S)(x*(x*(x*y))=x*y), \tag{2.5}$$
$$(\forall x,y\in S)(0*(x*y)=(0*x)*(0*y)), \tag{2.6}$$
$$(\forall x,y\in S)(0*(0*(x*y))=(0*y)*(0*x)). \tag{2.7}$$

A nonempty subset A of a BCK/BCI-algebra S is called a *subalgebra* of S if $x * y \in A$ for all $x, y \in A$. A subset I of a BCK/BCI-algebra S is called an *ideal* of S if it satisfies:

$$0 \in I, \tag{2.8}$$
$$(\forall x \in S)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.9}$$

An ideal I of a BCI-algebra S is said to be *regular* (see [26]) if it is also a subalgebra of S.

It is clear that every ideal of a BCK-algebra is regular (see [26]).

A subset I of a BCI-algebra S is called a *q-ideal* of S (see [27]) if it satisfies (2.8) and

$$(\forall x, y, z \in S)(x * (y * z) \in I,\, y \in I \;\Rightarrow\; x * z \in I). \tag{2.10}$$

We refer the reader to the books [25, 28] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Definition 2.2 ([21])**.** Let S be a set. A *neutrosophic quadruple S-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in S$ and T, I, F have their usual neutrosophic logic meanings.

The set of all neutrosophic quadruple S-numbers is denoted by $NQ(S)$, that is,

$$NQ(S) := \{(a, xT, yI, zF) \mid a, x, y, z \in S\},$$

and it is called the *neutrosophic quadruple set* based on S. If S is a BCK/BCI-algebra, a neutrosophic quadruple S-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $NQ(S)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let S be a BCK/BCI-algebra. We define a binary operation $\circledast$ on $NQ(S)$ by

$$(a, xT, yI, zF) \circledast (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in NQ(S)$. Given $a_1, a_2, a_3, a_4 \in S$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(S)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in NQ(S)$. It is easy to verify that "$\ll$" is an equivalence relation on $NQ(S)$.

Theorem 2.3 ([21])**.** *If S is a BCK/BCI-algebra, then $(NQ(S); \circledast, \tilde{0})$ is a BCK/BCI-algebra.*

We say that $(NQ(S); \circledast, \tilde{0})$ is a *neutrosophic quadruple BCK/BCI-algebra,* and it is simply denoted by $NQ(S)$.

Let S be a BCK/BCI-algebra. Given nonempty subsets A and B of S, consider the set

$$NQ(U, V) := \{(a, xT, yI, zF) \in NQ(S) \mid a, x \in U \;\&\; y, z \in V\},$$

which is called the *neutrosophic quadruple* (U, V)*-set*.

The set $NQ(U, U)$ is denoted by $NQ(U)$, and it is called the *neutrosophic quadruple* U*-set*.

3 (Regular) neutrosophic quadruple ideals

Definition 3.1. Given nonempty subsets U and V of a BCI-algebra S, if the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a (regular) ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$, we say $NQ(U, V)$ is a *(regular) neutrosophic quadruple ideal* of $NQ(S)$.

Question 1. *If U and V are subalgebras of a BCI-algebra S, then is the neutrosophic quadruple (U, V)-set $NQ(U, V)$ a neutrosophic quadruple ideal of $NQ(S)$?*

The answer to Question 1 is negative as seen in the following example.

Example 3.2. Consider a BCI-algebra $S = \{0, 1, a, b, c\}$ with the binary operation $*$, which is given in Table 1. Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 625 elements. Note that $U = \{0, a\}$ and $V = \{0, b\}$

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

are subalgebras of S. The neutrosophic quadruple (U, V)-set $NQ(U, V)$ consists of the following elements:

$$NQ(U, V) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, bF)$, $\tilde{2} = (0, 0T, bI, 0F)$,
$\tilde{3} = (0, 0T, bI, bF)$, $\tilde{4} = (0, aT, 0I, 0F)$, $\tilde{5} = (0, aT, 0I, bF)$,
$\tilde{6} = (0, aT, bI, 0F)$, $\tilde{7} = (0, aT, bI, bF)$, $\tilde{8} = (a, 0T, 0I, 0F)$,
$\tilde{9} = (a, 0T, 0I, bF)$, $\tilde{10} = (a, 0T, bI, 0F)$, $\tilde{11} = (a, 0T, bI, bF)$,
$\tilde{12} = (a, aT, 0I, 0F)$, $\tilde{13} = (a, aT, 0I, bF)$,
$\tilde{14} = (a, aT, bI, 0F)$, $\tilde{15} = (a, aT, bI, bF)$.

If we take $(1, aT, bI, 0F) \in NQ(S)$, then $(1, aT, bI, 0F) \notin NQ(U, V)$ and

$$(1, aT, bI, 0F) \circledast \tilde{9} = \tilde{15} \in NQ(U, V).$$

Hence the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is not a neutrosophic quadruple ideal of $NQ(S)$.

We consider conditions for the neutrosophic quadruple (U, V)-set $NQ(U, V)$ to be a regular neutrosophic quadruple ideal of $NQ(S)$.

Lemma 3.3 ([21]). *If U and V are subalgebras (resp., ideals) of a BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple subalgebra (resp., ideal) of $NQ(S)$.*

Theorem 3.4. *Let U and V be subalgebras of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \text{ (resp., } V), \ y \notin U \text{ (resp., } V) \ \Rightarrow \ y * x \notin U \text{ (resp., } V)). \tag{3.1}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.

Proof. By Lemma 3.3, $NQ(U, V)$ is a neutrosophic quadruple subalgebra of $NQ(S)$. Hence it is clear that $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(S)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F) \in NQ(S)$ be such that $\tilde{y} \circledast \tilde{x} \in NQ(U, V)$ and $\tilde{x} \in NQ(U, V)$. Then $x_i \in U$ and $x_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Also,

$$\begin{aligned}\tilde{y} \circledast \tilde{x} &= (y_1, y_2T, y_3I, y_4F) \circledast (x_1, x_2T, x_3I, x_4F) \\ &= (y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) \in NQ(U, V),\end{aligned}$$

and so $y_1 * x_1 \in U$, $y_2 * x_2 \in U$, $y_3 * x_3 \in V$ and $y_4 * x_4 \in V$. If $\tilde{y} \notin NQ(U, V)$, then $y_i \notin A$ or $y_j \notin B$ for some $i = 1, 2$ and $j = 3, 4$. It follows from (3.1) that $y_i * x_i \notin U$ or $y_j * x_j \notin V$ for some $i = 1, 2$ and $j = 3, 4$. This is a contradiction, and so $\tilde{y} \in NQ(U, V)$. Thus $NQ(U, V)$ is a neutrosophic quadruple ideal of $NQ(S)$, and therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$. □

Corollary 3.5. *Let U be a subalgebra of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U, \ y \notin U \ \Rightarrow \ y * x \notin U). \tag{3.2}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.

Theorem 3.6. *Let U and V be subsets of a BCI-algebra S. If any neutrosophic quadruple ideal $NQ(U, V)$ of $NQ(S)$ satisfies $\tilde{0} \circledast \tilde{x} \in NQ(U, V)$ for all $\tilde{x} \in NQ(U, V)$, then $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Proof. For any $\tilde{x}, \tilde{y} \in NQ(U, V)$, we have

$$(\tilde{x} \circledast \tilde{y}) \circledast \tilde{x} = (\tilde{x} \circledast \tilde{x}) \circledast \tilde{y} = \tilde{0} \circledast \tilde{y} \in NQ(U, V).$$

Since $NQ(U, V)$ is an ideal of $NQ(S)$, it follows that $\tilde{x} \circledast \tilde{y} \in NQ(U, V)$. Hence $NQ(U, V)$ is a neutrosophic quadruple subalgebra of $NQ(S)$, and therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$. □

Corollary 3.7. *Let U be a subset of a BCI-algebra S. If any neutrosophic quadruple ideal $NQ(U)$ of $NQ(S)$ satisfies $\tilde{0} \circledast \tilde{x} \in NQ(U)$ for all $\tilde{x} \in NQ(U)$, then $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Theorem 3.8. *If U and V are ideals of a finite BCI-algebra S, then the neutrosophic quadruple (U,V)-set $NQ(U,V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Proof. By Lemma 3.3, $NQ(U,V)$ is a neutrosophic quadruple ideal of $NQ(S)$. Since S is finite, $NQ(S)$ is also finite. Assume that $|NQ(S)| = n$. For any element $\tilde{x} \in NQ(U,V)$, consider the following $n+1$ elements:

$$\tilde{0}, \tilde{0} \circledast \tilde{x}, (\tilde{0} \circledast \tilde{x}) \circledast \tilde{x}, \cdots, (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{n\text{-times}}.$$

Then there exist natural numbers p and q with $p > q$ such that

$$(\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p\text{-times}} = (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q\text{-times}}.$$

Hence

$$\begin{aligned}
\tilde{0} &= ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p \text{ times}}) \circledast ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}}) \\
&= ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p-q \text{ times}}) \circledast ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}}) \\
&= (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast\tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p-q \text{ times}} \in NQ(U,V).
\end{aligned}$$

Since $NQ(U,V)$ is an ideal of $NQ(S)$, it follows that $\tilde{0} \circledast \tilde{x} \in NQ(U,V)$. Therefore $NQ(U,V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$ by Theorem 3.6. □

Corollary 3.9. *If U is an ideal of a finite BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

4 Neutrosophic quadruple q-ideals

Definition 4.1. Given nonempty subsets U and V of S, if the neutrosophic quadruple (U,V)-set $NQ(U,V)$ is a q-ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$, we say $NQ(U,V)$ is a *neutrosophic quadruple q-ideal* of $NQ(S)$.

Example 4.2. Consider a BCI-algebra $S = \{0,1,a\}$ with the binary operation $*$, which is given in Table 2. Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 81 elements. If we take $U = \{0,1\}$ and $V = \{0,1\}$, then

$$NQ(U,V) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

is a neutrosophic quadruple q-ideal of $NQ(S)$ where
$\tilde{0} = (0,0T,0I,0F)$, $\tilde{1} = (0,0T,0I,1F)$, $\tilde{2} = (0,0T,1I,0F)$,
$\tilde{3} = (0,0T,1I,1F)$, $\tilde{4} = (0,1T,0I,0F)$, $\tilde{5} = (0,1T,0I,1F)$,
$\tilde{6} = (0,1T,1I,0F)$, $\tilde{7} = (0,1T,1I,1F)$, $\tilde{8} = (1,0T,0I,0F)$,
$\tilde{9} = (1,0T,0I,1F)$, $\tilde{10} = (1,0T,1I,0F)$, $\tilde{11} = (1,0T,1I,1F)$,

Table 2: Cayley table for the binary operation “$*$”

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

$\tilde{12} = (1, 1T, 0I, 0F)$, $\tilde{13} = (1, 1T, 0I, 1F)$,
$\tilde{14} = (1, 1T, 1I, 0F)$, $\tilde{15} = (1, 1T, 1I, 1F)$.

Theorem 4.3. *For any nonempty subsets U and V of a BCI-algebra S, if the neutrosophic quadruple (U,V)-set $NQ(U,V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$, then it is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$.*

Proof. Assume that $NQ(U,V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. Since $\tilde{0} \in NQ(U,V)$, we have $0 \in U$ and $0 \in V$. Let $x, y, z \in S$ be such that $x * (y * z) \in U \cap V$ and $y \in U \cap V$. Then $(y, yT, yI, yF) \in NQ(U,V)$ and

$$\begin{aligned}
&(x, xT, xI, xF) \circledast ((y, yT, yI, yF) \circledast (z, zT, zI, zF)) \\
&= (x, xT, xI, xF) \circledast (y * z, (y * z)T, (y * z)I, (y * z)F) \\
&= (x * (y * z), (x * (y * z))T, (x * (y * z))I, (x * (y * z))F) \in NQ(U,V).
\end{aligned}$$

Since $NQ(U,V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$, it follows that

$$(x * z, (x * z)T, (x * z)I, (x * z)F) = (x, xT, xI, xF) \circledast (z, zT, zI, zF) \in NQ(U,V).$$

Hence $x * z \in U \cap V$, and therefore U and V are q-ideals of S. Since every q-ideal is both a subalgebra and an ideal, it follows from Lemma 3.3 that $NQ(U,V)$ is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$. $\square$

The converse of Theorem 4.3 is not true as seen in the following example.

Example 4.4. Consider a BCI-algebra $S = \{0, a, b, c\}$ with the binary operation $*$, which is given in Table 3.

Table 3: Cayley table for the binary operation “$*$”

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 256 elements. If we take $A = \{0\}$ and $B = \{0\}$, then $NQ(U, V) = \{\tilde{0}\}$ is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$. If we take $\tilde{x} := (c, bT, 0I, aF)$, $\tilde{z} := (a, bT, 0I, cF) \in NQ(S)$, then

$$\begin{aligned}\tilde{x} \circledast (\tilde{0} \circledast \tilde{z}) &= (c, bT, 0I, aF) \circledast (\tilde{0} \circledast (a, bT, 0I, cF)) \\ &= (c, bT, 0I, aF) \circledast (c, bT, 0I, aF) = \tilde{0} \in NQ(U, V).\end{aligned}$$

But

$$\begin{aligned}\tilde{x} \circledast \tilde{z} &= (c, bT, 0I, aF) \circledast (a, bT, 0I, cF) \\ &= (c * a, (b * b)T, (0 * 0)I, (a * c)F) \\ &= (b, 0T, 0I, bF) \notin NQ(U, V).\end{aligned}$$

Therefore $NQ(U, V)$ is not a neutrosophic quadruple q-ideal of $NQ(S)$.

We provide conditions for the neutrosophic quadruple (U, V)-set $NQ(U, V)$ to be a neutrosophic quadruple q-ideal.

Theorem 4.5. *If U and V are q-ideals of a BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Proof. Suppose that U and V are q-ideals of a BCI-algebra S. Obviously, $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(S)$ be such that $\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) \in NQ(U, V)$ and $\tilde{y} \in NQ(U, V)$. Then $y_i \in A$, $y_j \in B$ for $i = 1, 2$ and $j = 3, 4$, and

$$\begin{aligned}&\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) = (x_1, x_2T, x_3I, x_4F) \circledast ((y_1, y_2T, y_3I, y_4F) \circledast (z_1, z_2T, z_3I, z_4F)) \\ &= (x_1, x_2T, x_3I, x_4F) \circledast (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \\ &= (x_1 * (y_1 * z_1), (x_2 * (y_2 * z_2))T, (x_3 * (y_3 * z_3))I, (x_4 * (y_4 * z_4))F) \\ &\in NQ(U, V),\end{aligned}$$

that is, $x_i * (y_i * z_i) \in U$ and $x_j * (y_j * z_j) \in B$ for $i = 1, 2$ and $j = 3, 4$. It follows from (2.10) that $x_i * z_i \in U$ and $x_j * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{x} \circledast \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(U, V), \tag{4.1}$$

and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.6. *If A is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Corollary 4.7. *If $\{0\}$ is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ for any ideals U and V of S.*

Corollary 4.8. *If $\{0\}$ is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ for any ideal U of S.*

Theorem 4.9. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y, z \in S)(x * (y * z) \in U \cap V \;\Rightarrow\; (x * y) * z \in U \cap V). \tag{4.2}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. It is clear that $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(S)$ be such that $\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) \in NQ(U, V)$ and $\tilde{y} \in NQ(U, V)$. Then $y_1, y_2 \in U$, $y_3, y_4 \in V$ and

$$\begin{aligned}
\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) &= (x_1, x_2T, x_3I, x_4F) \circledast ((y_1, y_2T, y_3I, y_4F) \circledast (z_1, z_2T, z_3I, z_4F)) \\
&= (x_1, x_2T, x_3I, x_4F) \circledast (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \\
&= (x_1 * (y_1 * z_1), (x_2 * (y_2 * z_2))T, (x_3 * (y_3 * z_3))I, (x_4 * (y_4 * z_4))F) \\
&\in NQ(U, V),
\end{aligned}$$

that is, $x_i * (y_i * z_i) \in U$ and $x_j * (y_j * z_j) \in V$ for $i = 1, 2$ and $j = 3, 4$. It follows from (2.3) and (4.2) that $(x_i * z_i) * y_i = (x_i * y_i) * z_i \in U$ and $(x_j * z_j) * y_j = (x_j * y_j) * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Since U and V are ideals of S, we have $x_i * z_i \in U$ and $x_j * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{x} \circledast \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(U, V), \tag{4.3}$$

and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.10. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y, z \in S)(x * (y * z) \in U \;\Rightarrow\; (x * y) * z \in U). \tag{4.4}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.11. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y \in S)(x * (0 * y) \in U \cap V \;\Rightarrow\; x * y \in U \cap V). \tag{4.5}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x * (y * z) \in U \cap V$ for all $x, y, z \in S$. Note that

$$\begin{aligned}
((x * y)) * (0 * z)) * (x * (y * z)) &= ((x * y) * (x * (y * z))) * (0 * z) \\
&\leq ((y * z) * y) * (0 * z) \\
&= (0 * z) * (0 * z) = 0 \in U \cap V
\end{aligned}$$

Thus $(x * y) * (0 * z) \in U \cap V$ since U and V are ideals of S. It follows from (4.9) that $(x * y) * z \in U \cap V$. Using Theorem 4.9, $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.12. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y \in S)(x * (0 * y) \in U \;\Rightarrow\; x * y \in U). \tag{4.6}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.13. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \cap U \;\Rightarrow\; x * y \in U \cap V). \tag{4.7}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x * (y * z) \in U \cap V$ and $y \in U \cap V$ for all $x, y, z \in S$. Using (2.3) and (4.7), we get $(x * z) * (y * z) = (x * (y * z)) * z \in U \cap V$ and $y * z \in U \cap V$. Since U and V are ideals of S, it follows that $x * z \in U \cap V$. Hence U and V are q-ideals of S, and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.5. □

Corollary 4.14. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \;\Rightarrow\; x * y \in U). \tag{4.8}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.15. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$ and $J \subseteq V$. If I and J are q-ideals of S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Proof. Let $x, y, z \in S$ be such that $x * (0 * y) \in U \cap V$. Then

$$(x * (x * (0 * y))) * (0 * y) = (x * (0 * y)) * (x * (0 * y)) = 0 \in I \cap J$$

by (2.3) and (III). Since I and J are q-ideals of S, it follows from (2.3) and (2.10) that

$$(x * y) * (x * (0 * y)) = (x * (x * (0 * y))) * y \in I \cap J \subseteq U \cap V$$

Since U and V are ideals of S, we have $x * y \in U \cap V$. Therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.11. □

Corollary 4.16. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$. If I is a q-ideal of S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Theorem 4.17. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y, z \in S)(x * (y * z) \in I \cap J \;\Rightarrow\; (x * y) * z \in I \cap J). \tag{4.9}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Let $x, y, z \in S$ be such that $x * (y * z) \in I \cap J$ and $y \in I \cap J$. Then

$$(x * z) * y = (x * y) * z \in I \cap J$$

by (2.3) and (4.9). Since I and J are ideals of S, it follows that $x * z \in I \cap J$. This shows that I and J are q-ideals of S. Therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.15. □

Corollary 4.18. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y, z \in S)(x * (y * z) \in I \;\Rightarrow\; (x * y) * z \in I). \tag{4.10}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.19. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y \in S)(x \in I \cap J \ \Rightarrow \ x * y \in I \cap J). \tag{4.11}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. By the proof of Theorem 4.13, we know that I and J are q-ideals of S. Hence $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.15. □

Corollary 4.20. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y \in S)(x \in I \ \Rightarrow \ x * y \in I). \tag{4.12}$$

Then the neutrosophic quadruple A-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.21. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y \in S)(x * (0 * y) \in I \cap J \ \Rightarrow \ x * y \in I \cap J). \tag{4.13}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x*(y*z) \in I \cap J$ For all $x, y, z \in S$. Then $(x*y)*z \in I \cap J$ by the proof of Theorem 4.11. It follows from Theorem 4.17 that neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.22. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y \in S)(x * (0 * y) \in I \ \Rightarrow \ x * y \in I). \tag{4.14}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Future Work: Using the results of this paper, we will aply it to another algebraic structures, for example, MV-algebras, BL-algebras, MTL-algebras, R_0-algebras, hoops, (ordered) semigroups and (semi, near) rings etc.

Acknowledgements: We are very thankful to the reviewer(s) for careful detailed reading and helpful comments/suggestions that improve the overall presentation of this paper.

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Single–Valued Neutrosophic Filters in EQ–algebras

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Mohammad Hamidi, Arsham Borumand Saeid, Florentin Smarandache (2019). Single-valued neutrosophic filters in EQ-algebras. *Journal of Intelligent & Fuzzy Systems* 36(1), 805-818

Abstract. This paper introduces the concept of single–valued neutrosophic EQ–subalgebras, single–valued neutrosophic EQ–prefilters and single–valued neutrosophic EQ–filters. We study some properties of single–valued neutrosophic EQ–prefilters and show how to construct single–valued neutrosophic EQ–filters. Finally, the relationship between single–valued neutrosophic EQ–filters and EQ–filters are studied.

Keywords: (hyper)Single–valued neutrosophic EQ–algebras, Single–valued neutrosophic EQ–filters.

1. Introduction

EQ-algebra as an alternative to residuated lattices is a special algebra that was presented for the first time by V. Nov$\acute{a}$k [10,11]. Its original motivation comes from fuzzy type theory, in which the main connective is fuzzy equality and stems from the equational style of proof in logic [15]. EQ-algebras are intended to become algebras of truth values for fuzzy type theory (FTT) where the main connective is a fuzzy equality. Every EQ–algebra has three operations meet "$\wedge$", multiplication "$\otimes$", and fuzzy equality "$\sim$" and a unit element, while the implication "$\rightarrow$" is derived from fuzzy equality "$\sim$". This basic structure in fuzzy logic is ordering, represented by $\wedge$–semilattice, with maximal element "1". Further materials regarding EQ–algebras are available in the literature too [6,7,9,12]. Algebras including EQ-algebras have played an important role in recent years and have had its comprehensive applications in many aspects including dynamical systems and genetic code of biology [2]. From the point of view of logic, the main difference between residuated lattices and EQ–algebras lies in the way the implication operation is obtained. While in residuated lattices it is obtained from (strong) conjunction, in EQ–algebras it is obtained from equivalence. Consequently, the two kinds of algebras differ in several essential points despite their many similar or identical properties.

Filter theory plays an important role in studying various logical algebras. From logical point of view, filters correspond to sets of provable formulae. Filters are very important in the proof of the completeness of various logic algebras. Many researchers have studied the filter theory of various logical algebras [3,4,5].

Neutrosophy, as a newlyâĂŞborn science, is a branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. It can be defined as the incidence of the application of a law, an axiom, an idea, a conceptual accredited construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making it intelligible. Neutrosophic set and neutrosophic logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic) are tools for publications on advanced studies in neutrosophy. In neutrosophic logic, a proposition has a degree of truth (T), indeterminacy (I) and falsity (F), where T, I, F are standard or non–standard subsets of $]^{-}0, 1^{+}[$. In 1995, Smarandache talked for the first time about neutrosophy and in 1999 and 2005 [14] he initiated the theory of neutrosophic set as a new mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data. Alkhazaleh et al. generalized the concept of fuzzy soft set to neutrosophic soft set and they gave some applications of this concept in decision making and medical diagnosis [1].

Regarding these points, this paper aims to introduce the notation of single–valued neutrosophic EQ–subalgebras and single–valued neutrosophic EQ–filters. We investigate some properties of single–valued neutrosophic EQ–subalgebras and single–valued neutrosophic EQ–filters and prove them. Indeed show that how to construct single–valued neutrosophic EQ–subalgebras and single–valued neutrosophic EQ–filters. We applied the concept of homomorphisms in EQ–algebras and with this regard, new single–valued neutrosophic EQ–subalgebras and single–valued neutrosophic EQ–filters are generated.

2. Preliminaries

In this section, we recall some definitions and results are indispensable to our research paper.

Definition 2.1. [8] *An algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is called an EQ–algebra, if for all $x, y, z, t \in E$:*

- $(E1)$ $(E, \wedge, 1)$ *is a commutative idempotent monoid* (*i.e.* $\wedge$*–semilattice with top element* "1");
- $(E2)$ $(E, \otimes, 1)$ *is a monoid and* $\otimes$ *is isotone w.r.t.* "$\leq$" (*where* $x \leq y$ *is defined as* $x \wedge y = x$);
- $(E3)$ $x \sim x = 1$; (*reflexivity axiom*)
- $(E4)$ $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$; (*substitution axiom*)
- $(E5)$ $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$; (*congruence axiom*)
- $(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$; (*monotonicity axiom*)
- $(E7)$ $x \otimes y \leq x \sim y$, (*boundedness axiom*).

The binary operation "$\wedge$" is called meet (infimum), "$\otimes$" is called multiplication and "$\sim$" is called fuzzy equality. $(E, \wedge, \otimes, \sim, 1)$ is called a separated EQ–algebra if $1 = x \sim y$, implies that $x = y$.

Proposition 2.2. [8] Let $\mathcal{E}$ be an EQ–algebra, $x \rightarrow y := (x \wedge y) \sim x$ and $\tilde{x} = x \sim 1$. Then for all $x, y, z \in E$, the following properties hold:

- (i) $x \otimes y \leq x, y, \quad x \otimes y \leq x \wedge y$;
- (ii) $x \sim y = y \sim x$;
- (iii) $(x \wedge y) \sim x \leq (x \wedge y \wedge z) \sim (x \wedge z)$;
- (iv) $x \rightarrow x = 1$;
- (v) $(x \sim y) \otimes (y \sim z) \leq x \sim z$;
- (vi) $(x \rightarrow y) \otimes (y \rightarrow z) \leq x \rightarrow z$;
- (vii) $x \leq \tilde{x}, \quad \tilde{1} = 1$.

Proposition 2.3. [8] Let $\mathcal{E}$ be an EQ–algebra. Then for all $x, y, z \in E$, the following properties hold:

- (i) $x \otimes (x \sim y) \leq \overline{y}$;
- (ii) $(z \rightarrow (x \wedge y)) \otimes (x \sim t) \leq z \rightarrow (t \wedge y)$;
- (iii) $(y \rightarrow z) \otimes (x \rightarrow y) \leq x \rightarrow z$;
- (iv) $(x \rightarrow y) \otimes (y \rightarrow x) \leq x \sim y$;
- (v) if $x \leq y \rightarrow z$, then $x \otimes y \leq \overline{z}$;
- (vi) if $x \leq y \leq z$, then $z \sim x \leq z \sim y$ and $x \sim z \leq x \sim y$;
- (vi) $x \rightarrow (y \rightarrow x) = 1$.

Definition 2.4. [8] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be a separated EQ–algebra. A subset F of E is called an EQ–filter of E if for all $a, b, c \in E$ it holds that

- (i) $1 \in F$,
- (ii) if $a, a \rightarrow b \in F$, then $b \in F$,
- (iii) if $a \rightarrow b \in F$, then $a \otimes c \rightarrow b \otimes c \in F$ and $c \otimes a \rightarrow c \otimes b \in F$.

Theorem 2.5. [8] Let F be a prefilter of separated EQ–algebra $\mathcal{E}$. Then for all $a, b, c \in E$ it holds that

- (i) if $a \in F$ and $a \leq b$, then $b \in F$;
- (ii) if $a, a \sim b \in F$, then $b \in F$;
- (iii) If $a, b \in F$, then $a \wedge b \in F$;
- (iv) If $a \sim b \in F$ and $b \sim c \in F$ then $a \sim c \in F$.

Definition 2.6. [17] *Let $\mathcal{E}$ be an EQ–algebras. A fuzzy subset μ of E is called a fuzzy prefilter of $\mathcal{E}$, if for all $x, y, z \in E$:*

- $(FH1)$ $\nu(1) \geq \nu(x)$;
- $(FH2)$ $\nu(y) \geq \nu\big((x \wedge y) {\sim} y\big) \wedge \nu(x)$.

A fuzzy EQ–prefilter is called a fuzzy EQ–filter if it satisfies :

$(FH3)$ $\nu\big((x\wedge y)\sim y\big) \leq \nu\big(\big((x\otimes z)\wedge(y\otimes z)\big)\sim(y\otimes z)\big)$.

Definition 2.7. [16] *Let X be a set. A single valued neutrosophic set A in X (SVN–S A) is a function $A : X \to [0,1]\times[0,1]\times[0,1]$ with the form $A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$ where the functions T_A, I_A, F_A define respectively the truth–membership function, an indeterminacy–membership function, and a falsity–membership function of the element $x \in X$ to the set A such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. Moreover, $Supp(A) = \{x \mid T_A(x) \neq 0, I_A(x) \neq 0, F_A(x) \neq 0\}$ is a crisp set.*

3. Single–Valued Neutrosophic EQ–subalgebras

In this section, we introduce the concept of single–valued neutrosophic EQ–subalgebra and prove some their properties.

Definition 3.1. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra. A map A in E, is called a single–valued neutrosophic EQ–subalgebra of $\mathcal{E}$, if for all $x, y \in E$,*

(i) $T_A(x\wedge y) = T_A(x)\wedge T_A(y), I_A(x\wedge y) = I_A(x)\wedge I_A(y)$ and $F_A(x\wedge y) = F_A(x)\vee F_A(y)$,
(ii) $T_A(x\sim y) \geq T_A(x)\wedge T_A(y), I_A(x\sim y) \geq I_A(x)\wedge I_A(y)$ and $F_A(x\sim y) \leq F_A(x)\vee F_A(y)$.

From now on, when we say $(\mathcal{E}, A)$ is a single–valued neutrosophic EQ–subalgebra, means that $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ–algebra and A is a single–valued neutrosophic EQ–subalgebra of $\mathcal{E}$.

Theorem 3.2. *Let $(\mathcal{E}, A)$ be a single–valued neutrosophic EQ–subalgebra. Then for all $x, y \in H$,*

(i) if $x \leq y$, then $T_A(x) \leq T_A(y)$,
(ii) if $x \leq y$, then $I_A(x) \leq I_A(y)$,
(iii) if $x \leq y$, then $F_A(x) \geq F_A(y)$,
(iv) $T_A(x) \leq T_A(1), I_A(x) \leq I_A(1)$ and $F_A(x) \geq F_A(1)$,
(v) $T_A(x\otimes y) \leq T_A(x)\wedge T_A(y)$,
(vi) $I_A(x\otimes y) \leq I_A(x)\wedge T_A(y)$,
(vii) $F_A(x\otimes y) \geq F_A(x)\vee F_A(y)$,
$(viii)$ $T_A(x\to y) \geq T_A(x)\wedge T_A(y)$,
(ix) $I_A(x\to y) \geq I_A(x)\wedge I_A(y)$,
(x) $F_A(x\to y) \leq F_A(x)\vee F_A(y)$.

Proof. $(i), (ii), (iii), (iv)$ Let $x, y \in E$. Since $x \leq y$, we get that $x\wedge y = x$ and so $T_A(x)\wedge T_A(y) = T_A(x\wedge y) = T_A(x)$. It follows that $T_A(x) \leq T_A(y)$. In a similar way $I_A(x) \leq I_A(y)$ and $F_A(x) \geq F_A(y)$ are obtained.

$(v), (vi), (vii)$ By the previous items, for all $x, y \in E, x\otimes y \leq x\wedge y$ implies that $T_A(x\otimes y) \leq T_A(x)\wedge T_A(y)$, $I_A(x\otimes y) \leq I_A(x)\wedge I_A(y)$ and $F_A(x\otimes y) \geq F_A(x)\vee F_A(y)$.

$(viii), (ix), (x)$ Since $(x\sim y) \leq (x\to y)$, by the previous items we get that $T_A(x\to y) \geq T_A(x)\wedge T_A(y)$, $I_A(x\to y) \geq I_A(x)\wedge T_A(y)$ and $F_A(x\to y) \leq F_A(x)\vee F_A(y)$. $\square$

Example 3.3. *Let $E = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Define operations "$\otimes, \sim$" and "$\wedge$" on E as follows:*

$\wedge$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_2	a_2	a_2	a_2
a_3	a_1	a_2	a_3	a_3	a_3	a_3
a_4	a_1	a_2	a_3	a_4	a_4	a_4
a_5	a_1	a_2	a_3	a_4	a_5	a_5
a_6	a_1	a_2	a_3	a_4	a_5	a_6

,

$\otimes$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1	a_1	a_2
a_3	a_1	a_1	a_1	a_1	a_2	a_3
a_4	a_1	a_1	a_1	a_2	a_2	a_4
a_5	a_1	a_1	a_2	a_2	a_2	a_5
a_6	a_1	a_2	a_3	a_4	a_5	a_6

and

$\sim$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_6	a_4	a_3	a_2	a_1	a_1
a_2	a_4	a_6	a_3	a_2	a_2	a_2
a_3	a_3	a_3	a_6	a_3	a_3	a_3
a_4	a_2	a_2	a_3	a_6	a_4	a_4
a_5	a_1	a_2	a_3	a_4	a_6	a_5
a_6	a_1	a_2	a_3	a_4	a_5	a_6

.

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, a_6)$ is an EQ–algebra. Define a single valued neutrosophic set map A in E as follows:

T_A	a_1	a_2	a_3	a_4	a_5	a_6
	0.22	0.33	0.44	0.55	0.66	0.77

,

I_A	a_1	a_2	a_3	a_4	a_5	a_6
	0.21	0.31	0.41	0.51	0.61	0.71

and

F_A	a_1	a_2	a_3	a_4	a_5	a_6
	0.98	0.88	0.78	0.68	0.58	0.48

Hence $(A, \mathcal{E})$ is a single–valued neutrosophic EQ–subalgebra.

Corollary 3.4. *Let $(\mathcal{E}, A)$ be a single–valued neutrosophic EQ–subalgebra. Then for all $x, y \in H$,*

(i) if $x \leq y$, then $T_A(y\to x) = T_A(x\sim y)$,
(ii) if $x \leq y$, then $I_A(y\to x) = I_A(x\sim y)$,
(iii) if $x \leq y$, then $F_A(y\to x) = F_A(x\sim y)$.

3.1. Single–Valued Neutrosophic EQ–prefilters

In this section, we introduce the concept of single–valued neutrosophic EQ–prefilters and show how to construct of single–valued neutrosophic EQ–prefilters.

Definition 3.5. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra. A map A in E, is called a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$, if for all $x, y \in E$,*

$(SVNF1)$ $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$ *and* $F_A(x) \leq F_A(1)$,

$(SVNF2)$ $\wedge\{T_A(x), T_A(x \to y)\} \leq T_A(y)$, $\vee\{I_A(x), I_A(x \to y)\} \geq I_A(y)$ *and* $\wedge\{F_A(x), F_A(x \to y)\} \leq F_A(y)$.

In the following theorem, we will show that how to construct of single–valued neutrosophic EQ–prefilters in EQ–algebras.

Theorem 3.6. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y \in E$.*

(i) *If* $x \leq y$, *then* $\wedge\{T_A(x), T_A(x \to y)\} = T_A(x)$,

(ii) *If* $x \leq y$, *then* $\vee\{I_A(x), I_A(x \to y)\} = I_A(x)$,

(iii) *If* $x \leq y$, *then* $\wedge\{F_A(x), F_A(x \to y)\} = F_A(x)$,

(iv) *If* $x \leq y$, *then* $T_A(x) \leq T_A(y)$ *and* $F_A(x) \leq F_A(y)$,

(v) *If* $x \leq y$, *then* $I_A(y) \leq I_A(x)$.

Proof. $(i), (ii), (iii)$ Since $x \leq y$ we get that $x \to y = 1$, so by definition, $\wedge\{T_A(x), T_A(x \to y)\} = T_A(x)$, $\vee\{I_A(x), I_A(x \to y)\} = I_A(x)$ and $\wedge\{F_A(x), F_A(x \to y)\} = F_A(x)$.

(iv) Since $x \leq y$, by (i) we have $\wedge\{T_A(x), T_A(x \to y)\} = T_A(x)$. So by definition we get $T_A(x) = \wedge\{T_A(x), T_A(x \to y)\} \leq T_A(y)$. In a similar way $x \leq y$ implies that $F_A(x) \leq F_A(x)$.

(v) Since $x \leq y$, by (ii) we have $\vee\{I_A(x), I_A(x \to y)\} = I_A(x)$. Thus by definition we get $I_A(y) \leq \vee\{I_A(x), I_A(x \to y)\} = I_A(x)$ and it follows that $I_A(x) \geq I_A(y)$. □

Corollary 3.7. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $0 \in E$. If for every $y \in E, 0 \wedge y = 0$, then*

(i) $\wedge\{T_A(0), T_A(0 \to y)\} = T_A(0)$, $\vee\{I_A(0), I_A(0 \to y)\} = I_A(0)$,

(ii) $\wedge\{T_A(1), T_A(1 \to y)\} = T_A(\overline{y})$, $\vee\{I_A(1), I_A(1 \to y)\} = I_A(\overline{y})$,

(iii) $\wedge\{T_A(y), T_A(y \to 1)\} = T_A(y)$, $\vee\{I_A(y), I_A(y \to 1)\} = I_A(y)$,

(iv) $\wedge\{T_A(y), T_A(y \to y)\} = T_A(y)$, $\vee\{I_A(y), I_A(y \to y)\} = I_A(y)$,

(v) $T_A(0) \leq T_A(1)$ *and* $I_A(1) \leq I_A(0)$,

(vi) $T_A(x) \leq T_A(y \to x)$ *and* $I_A(x \to y) \geq I_A(y)$,

(vii) $T_A(x \otimes y) \leq T_A(y \sim x)$ *and* $I_A(x \otimes y) \geq I_A(y \sim x)$.

Example 3.8. *Let $E = \{a, b, c, d, 1\}$. Define operations "$\otimes, \sim$" and an operation "$\wedge$" on E as follows:*

$\wedge$	a	b	c	d	1
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	c	c
d	a	b	c	d	d
1	a	b	c	d	1

,

$\otimes$	a	b	c	d	1
a	a	a	a	a	a
b	a	a	a	a	b
c	a	a	a	c	c
d	a	a	a	d	d
1	a	b	c	d	1

and

$\sim$	a	b	c	d	1
a	1	b	a	a	a
b	b	1	b	b	b
c	a	b	1	c	c
d	a	b	c	1	d
1	a	b	c	d	1

.

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ–algebra and obtain the operation "$\to$" as follows: Define a single valued neutrosophic set map A in E as follows:

$\to$	a	b	c	d	1
a	1	1	1	1	1
b	b	1	1	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

.

T_A	a	b	c	d	1
	0.1	0.2	0.3	0.4	0.5

,

F_A	a	b	c	d	1
	0.55	0.45	0.35	0.25	0.15

and

I_A	a	b	c	d	1
	0.17	0.27	0.37	0.47	0.57

Hence A is a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$.

Theorem 3.9. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y \in E$. Then*

(i) $\wedge\{T_A(x), T_A(x \sim y)\} \leq T_A(y)$ *and* $(I_A(x) \vee I_A(x \sim y)) \geq I_A(y)$,

(ii) $\wedge\{T_A(x), T_A(x \otimes y)\} \leq T_A(y)$ *and* $(I_A(x) \vee I_A(x \otimes y)) \geq I_A(y)$,

(iii) $\wedge\{T_A(x), T_A(x \wedge y)\} \leq T_A(y)$ *and* $(I_A(x) \vee I_A(x \wedge y)) \geq I_A(y)$,
(iv) $T_A(x) \wedge T_A(y) \leq T_A(x) \wedge T_A(x \to y)$,
(v) $I_A(x) \vee I_A(x \to y) \leq I_A(x) \vee I_A(y)$,
(vi) $T_A(x \otimes y) \leq T_A(x) \wedge T_A(x)$,
(vii) $I_A(x \otimes y) \geq I_A(x) \vee I_A(x)$.

Proof. $(i), (ii), (iii)$ Let $x, y \in E$. Since $x \sim y \leq x \to y$ and T_A ia a monotone map, we get that $T_A(x \sim y) \leq T_A(x \to y)$. Hence

$$\wedge\{T_A(x), T_A(x \sim y)\} \leq \wedge\{T_A(x), T_A(x \to y)\} \leq T_A(y).$$

In addition, since I_A is an antimonotone map, $x \sim y \leq x \to y$ concludes that $I_A(x \sim y) \geq I_A(x \to y)$. Hence $\vee\{I_A(x), I_A(x \sim y)\} \geq \vee\{I_A(x), I_A(x \to y)\} \geq I_A(y)$. In a similar way $x \wedge y \leq y$ and $x \otimes y \leq x \to y$, imply that $\wedge\{T_A(x), T_A(x \otimes y)\} \leq T_A(y)$, $\wedge\{T_A(x), T_A(x \wedge y)\} \leq T_A(y)$, $(I_A(x) \vee I_A(x \otimes y)) \geq I_A(y)$ and $(I_A(x) \vee I_A(x \wedge y)) \geq I_A(y)$.

$(iv), (v)$ Let $x, y \in E$. Since $y \leq (x \to y)$, we get that

$$(T_A(x) \wedge T_A(y)) \leq (T_A(x) \wedge T_A(x \to y)) \leq T_A(y).$$

In a similar way we conclude that $I_A(y) \leq (I_A(x) \vee I_A(x \to y)) \leq I_A(x) \vee I_A(y)$.

$(vi), (vii)$ Since $x \otimes y \leq (x \wedge y)$ and T_A is a monotone map, then we get that $T_A(x \otimes y) \leq T_A(x \wedge y) \leq T_A(x) \wedge T_A(y)$. In a similar way since I_A is an antimonotone map, then we get that $I_A(x \otimes y) \geq T_A(x \wedge y) \geq I_A(x) \vee I_A(y)$. □

Corollary 3.10. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y \in E$. Then*

(i) $\wedge\{F_A(x), F_A(x \sim y)\} \leq F_A(y)$,
(ii) $\wedge\{F_A(x), F_A(x \otimes y)\} \leq F_A(y)$,
(iii) $\wedge\{F_A(x), F_A(x \wedge y)\} \leq F_A(y)$,
(iv) $F_A(x) \wedge F_A(y) \leq F_A(x) \wedge F_A(x \to y)$,
(v) $F_A(x \otimes y) \leq F_A(x) \wedge F_A(x)$.

Theorem 3.11. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y, z \in E$.*

(i) *If $x \leq y$, then $T_A(x) \wedge T_A(x \sim y) = T_A(x) \wedge T_A(y \to x)$,*
(ii) *If $x \leq y$, then $T_A(z) \wedge T_A(z \to x) \leq T_A(y)$,*
(iii) *If $x \leq y$, then $T_A(x) \wedge T_A(y \to z) = T_A(x) \wedge T_A(z)$,*
(iv) *If $x \leq y$, then $I_A(x) \vee I_A(x \sim y) = I_A(x) \vee I_A(y \to x)$,*
(v) *If $x \leq y$, then $I_A(z) \vee I_A(z \to x) = I_A(x) \vee I_A(z)$,*
(vi) *If $x \leq y$, then $I_A(x) \vee I_A(y \to z) = I_A(x) \vee I_A(z)$.*

Proof. (i) Let $x, y \in E$. Then $x \leq y$ follows that $x \sim y = y \to x$ and so $T_A(x) \wedge T_A(x \sim y) = T_A(x) \wedge T_A(y \to x)$.

(ii) Let $x, y, z \in E$. Since $z \to x \leq z \to y$, we get that $T_A(z \to x) \leq T_A(z \to y)$ and so $T_A(z) \wedge T_A(z \to x) \leq T_A(z) \wedge T_A(z \to y) \leq T_A(y)$.

(iii) Let $x, y, z \in E$. Since $y \to z \leq x \to z$, we get that $T_A(y \to z) \leq T_A(x \to z)$ and so $T_A(x) \wedge T_A(y \to z) \leq T_A(x) \wedge T_A(x \to z) \leq T_A(z)$. Moreover, $z \leq y \to z$ implies that $T_A(z) \leq T_A(y \to z)$, hence $T_A(z) \wedge T_A(x) \leq T_A(x) \wedge T_A(y \to z) \leq T_A(z) \wedge T_A(x)$ and so $T_A(x) \wedge T_A(y \to z) = T_A(z) \wedge T_A(x)$.

(v) Let $x, y, z \in E$. Since $z \to x \leq z \to y$, we get that $I_A(z \to y) \leq I_A(z \to x)$ and so $I_A(z) \vee I_A(z \to y) \leq I_A(z) \vee I_A(z \to x)$. Moreover, $x \leq y$ implies that $I_A(x) \vee I_A(y) = I_A(x)$, hence by Theorem 3.9, $I_A(z) \vee I_A(x) \vee I_A(y) \leq I_A(z) \vee I_A(z \to x) \leq T_A(x) \vee I_A(z)$ and so $T_A(z) \wedge I_A(z \to x) = I_A(z) \vee I_A(x)$.

(iv) and (vi) in a similar way are obtained. □

Corollary 3.12. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y, z \in E$.*

(i) *If $x \leq y$, then $F_A(x) \wedge F_A(x \sim y) = F_A(x) \wedge F_A(y \to x)$,*
(ii) *If $x \leq y$, then $F_A(z) \wedge F_A(z \to x) = F_A(x) \wedge F_A(z)$,*
(iii) *If $x \leq y$, then $F_A(x) \wedge F_A(y \to z) = F_A(x) \wedge F_A(z)$.*

Theorem 3.13. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y, z \in E$. Then*

(i) $T_A(x \wedge y) = T_A(x) \wedge T_A(y)$,
(ii) $T_A(x) \wedge T_A(x \sim y) \leq T_A(x) \wedge T_A(y)$,

Proof. (i) Since T_A is a monotone map, $x \wedge y \leq x$ and $x \wedge y \leq y$, we obtain $T_A(x \wedge y) \leq T_A(x) \wedge T_A(y)$. In addition from $y \leq x \to (x \wedge y)$ and Theorem 3.9, we conclude that $T_A(x) \wedge T_A(y) \leq (T_A(x) \wedge T_A(x \to (x \wedge y))) \leq T_A(x \wedge y)$. Hence $T_A(x \wedge y) = T_A(x) \wedge T_A(y)$.

(ii) Let $x, y \in E$. Then by Theorem 3.9, $T_A(x) \wedge T_A(x \sim y) \leq T_A(y)$. Since $x \sim y = y \sim x$, we obtain $T_A(x) \wedge T_A(x \sim y) = T_A(x) \wedge T_A(y \sim x) \leq T_A(x)$. So $T_A(x) \wedge T_A(x \sim y) \leq T_A(x) \wedge T_A(y)$.

□

Corollary 3.14. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y, z \in E$. Then*

(i) $F_A(x \wedge y) = F_A(x) \wedge F_A(y)$,
(ii) $F_A(x) \wedge F_A(x \sim y) \leq F_A(x) \wedge F_A(y)$,

Theorem 3.15. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y \in E$. Then*

(i) $I_A(x \wedge y) = I_A(x) \vee I_A(y)$,
(ii) $I_A(x) \vee I_A(x \sim y) \geq I_A(x \wedge y)$,

Proof. (i) Since I_A is an antimonotone map, $x \wedge y \leq x$ and $x \wedge y \leq y$, we obtain $I_A(x \wedge y) \geq I_A(x) \vee I_A(y)$. In addition from $y \leq x \to (x \wedge y)$, we conclude that

$$I_A(x) \vee I_A(y) \geq (I_A(x) \vee I_A(x \to (x \wedge y))) \geq I_A(x \wedge y).$$

Hence $I_A(x \wedge y) = I_A(x) \vee I_A(y)$.

(ii) Let $x, y \in E$. Then, $I_A(x) \vee I_A(x \sim y) \geq I_A(y)$. Since $x \sim y = y \sim x$, we obtain $I_A(x) \vee I_A(x \sim y) = I_A(x) \vee I_A(y \sim x) \geq I_A(x)$. So $I_A(x) \vee I_A(x \sim y) \geq I_A(x) \vee I_A(y)$.

□

Corollary 3.16. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $x, y \in E$. Then $x = y$, implies that $I_A(x) \vee I_A(x \sim y) = I_A(x \wedge y)$.*

In Example 3.8, for $x = a$ and $y = d$, we have $I_A(x) \vee I_A(x \sim y) = I_A(x \wedge y)$, while $x \neq y$.

4. Single–Valued Neutrosophic EQ–filters

In this section, we introduce the concept of single–valued neutrosophic EQ–filters as generalization of single–valued neutrosophic EQ–prefilters and prove some their properties.

Definition 4.1. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra. A map A in E, is called a single–valued neutrosophic EQ–filter of $\mathcal{E}$, if for all $x, y, z \in E$,*

$(SVNF1)$ $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$ *and* $F_A(x) \leq F_A(1)$,
$(SVNF2)$ $\wedge\{T_A(x), T_A(x \to y)\} \leq T_A(y)$, $\vee\{I_A(x), I_A(x \to y)\} \geq I_A(y)$ *and* $\wedge\{F_A(x), F_A(x \to y)\} \leq F_A(y)$,
$(SVNF3)$ $T_A(x \to y) \leq T_A((x \otimes z) \to (y \otimes z)), I_A(x \to y) \geq I_A((x \otimes z) \to (y \otimes z))$, *and* $F_A(x \to y) \leq F_A((x \otimes z) \to (y \otimes z))$.

In the following theorem, we will show that how to construct of single–valued neutrosophic EQ–prefilters in EQ–algebras.

Theorem 4.2. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–filter of $\mathcal{E}$ and $x, y \in E$.*

(i) *If $T_A(x \to y) = T_A(1)$, then for every $z \in E$, $T_A((x \otimes z) \to (y \otimes z)) = T_A(x \to y)$.*
(ii) *If $x \leq y$, then for every $z \in E$, $T_A((x \otimes z) \to (y \otimes z)) = T_A(x \to y)$.*
(iii) *If $T_A(x \to y) = T_A(0)$, then for every $z \in E$, $T_A((x \otimes z) \to (y \otimes z)) \geq T_A(x \to y)$.*
(iv) *If $I_A(x \to y) = I_A(1)$, then for every $z \in E$, $I_A((x \otimes z) \to (y \otimes z)) = I_A(x \to y)$.*
(v) *If $x \leq y$, then for every $z \in E$, $I_A((x \otimes z) \to (y \otimes z)) = I_A(x \to y)$.*
(vi) *If $I_A(x \to y) = I_A(0)$, then for every $z \in E$, $I_A((x \otimes z) \to (y \otimes z)) \leq I_A(x \to y)$.*

Proof. $(i), (iii), (iv)$ and (vi) by definition are obtained.

(ii) Since $x \leq y$ we get that $x \to y = 1$ and by definition $x \otimes z \leq y \otimes z$. Hence by item (i), we have $T_A((x \otimes z) \to (y \otimes z)) = T_A(x \to y)$.

(v) It is similar to the item (ii).

□

Corollary 4.3. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$ and $0, x, y, z \in E$. If for every $y \in E, 0 \wedge y = 0$, Then*

(i) $T_A(0 \to y) = T_A((x \otimes z) \to (y \otimes z))$,
(ii) $T_A(x \to x) = T_A((x \otimes z) \to (y \otimes z))$,
(iii) $T_A(x \to 1) = T_A((x \otimes z) \to (y \otimes z))$,
(iv) $I_A(0 \to y) = I_A((x \otimes z) \to (y \otimes z))$,
(v) $I_A(x \to x) = I_A((x \otimes z) \to (y \otimes z))$,
(vi) $I_A(x \to 1) = I_A((x \otimes z) \to (y \otimes z))$.

Corollary 4.4. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–filter of $\mathcal{E}$ and $x, y \in E$.*

(i) *If* $F_A(x \to y) = F_A(1)$*, then for every* $z \in E$*,* $F_A((x \otimes z) \to (y \otimes z)) = F_A(x \to y)$*,*
(ii) *If* $x \leq y$*, then for every* $z \in E$*,* $F_A((x \otimes z) \to (y \otimes z)) = F_A(x \to y)$*.*
(iii) *If* $F_A(x \to y) = F_A(0)$*, then for every* $z \in E$*,* $F_A((x \otimes z) \to (y \otimes z)) \geq F_A(x \to y)$*.*

Example 4.5. *Let* $E = \{0, a, b, c, 1\}$*. Define operations "*$\otimes, \sim$*" and an operation "*$\wedge$*" on* E *as follows:*

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	$-$	b
c	0	a	$-$	c	c
1	0	a	b	c	1

,

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

and

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	a	a	a
b	0	a	1	a	b
c	0	a	a	1	c
1	0	a	b	c	1

.

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *is an EQ–algebra, where* b *and* c *are non-comparable. Now, obtain the operation "*$\to$*" as follows:*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

.

Define a single valued neutrosophic set map A *in* E *as follows:*

T_A	0	a	b	c	1
	0.4	0.4	0.4	0.4	0.5

,

I_A	0	a	b	c	1
	0.62	0.62	0.62	0.62	0.11

and

F_A	0	a	b	c	1
	0.2	0.2	0.2	0.2	0.6

Hence A *is a single–valued neutrosophic EQ–filter of* $\mathcal{E}$*.*

Theorem 4.6. *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ–algebra,* A *be a single–valued neutrosophic EQ–filter of* $\mathcal{E}$ *and* $x, y \in E$*. Then*

(i) $T_A(x \otimes y) = T_A(x) \wedge T_A(y)$*,*
(ii) $I_A(x \otimes y) = I_A(x) \vee I_A(y)$*,*
(iii) $T_A(x \sim y) \leq T_A(y \to x)$*,*
(iv) $T_A(z) \wedge T_A(y) \leq T_A(x \to z)$*,*
(v) $T_A(x \sim y) \wedge T_A(y \sim z) \leq T_A(x \sim z)$*,*
(vi) $I_A(x \sim y) \geq I_A(y \to x)$*,*
(vii) $I_A(z) \vee I_A(y) \geq I_A(x \to z)$*,*
$(viii)$ $I_A(x \sim y) \vee I_A(y \sim z) \geq I_A(x \sim z)$*.*

Proof. (i) Let $x, y \in E$. Since A is a single–valued neutrosophic EQ–filter of $\mathcal{E}$, we get that

$$T_A(1 \to y) \leq T_A((1 \otimes x) \to (y \otimes x)) = T_A(x \to (y \otimes x)).$$

In addition by the item $(SVNF2)$, we have

$$T_A(x) \wedge T_A(x \to (y \otimes x)) \leq T_A(y \otimes x).$$

Hence

$$T_A(x) \wedge T_A(y) \leq T_A(x) \wedge T_A(1 \to y) \leq T_A(y \otimes x).$$

We apply Theorem 3.9 and obtain $T_A(x) \wedge T_A(y) = T_A(y \otimes x)$.

(ii) Let $x, y \in E$. By item $(SVNF2)$, we have

$$I_A(1 \to y) \geq I_A(1 \otimes x) \to (y \otimes x).$$

Then $I_A(x) \vee I_A(1 \to y) \geq I_A(x) \vee I_A(x \to (y \otimes x)) \geq I_A(y \otimes x)$. It follows that $I_A(x) \vee I_A(y) \geq I_A(x) \vee I_A(1 \to y) \geq I_A(y \otimes x)$. Therefore, Theorem 3.9 implies that $I_A(x) \vee I_A(y) = I_A(y \otimes x)$.

(iii) Let $x, y \in E$. Then $x \sim y \leq (x \to y) \wedge (y \to x)$ implies that $T_A(x \sim y) \leq T_A(y \to x)$.

(iv) Let $x, y, z \in E$. Since $(x \to y) \otimes (y \to z) \leq (x \to z)$, by item (i), we get that

$$T_A(y) \wedge T_A(z) \leq T_A(x \to y) \wedge T_A(y \to z) = T_A((x \to y) \otimes (y \to z)) \leq T_A(x \to z).$$

(v) Let $x, y, z \in E$. Since $(x \sim y) \otimes (y \sim z) \leq x \sim z$, we get that $T_A((x \sim y) \otimes (y \sim z)) \leq T_A(x \sim z)$. Now by item (i), we get that $T_A(x \sim y) \wedge T_A(y \sim z) = T_A((x \sim y) \otimes (y \sim z)) \leq T_A(x \sim z)$. (vi), (vii) and $(viii)$ in a similar way are obtained. □

Example 4.7. *Consider the EQ–algebra and the single–valued neutrosophic EQ–prefilter* A *of* $\mathcal{E}$ *which are defined in Example 3.8. Since* $0.1 = T_A(a) = T_A(d \otimes c) \neq 0.3 = 0.4 \wedge 0.3 = T_A(d) \wedge T_A(c)$*, we conclude that* A *is not a single–valued neutrosophic EQ–filter* A *of* $\mathcal{E}$*.*

Corollary 4.8. *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ–algebra,* A *be a single–valued neutrosophic EQ–filter of* $\mathcal{E}$ *and* $x, y, z \in E$*. Then*

(i) $F(x \otimes y) = F_A(x) \wedge F_A(y)$,
(ii) $F_A(x \sim y) \leq F_A(y \to x)$,
(iii) $F_A(z) \wedge F_A(y) \leq F_A(x \to z)$,
(iv) $F_A(x \sim y) \wedge F_A(y \sim z) \leq F_A(x \sim z)$.

4.1. Special single–valued neutrosophic EQ–filters

In this section, we apply the concept of homomorphisms and (α, β, γ)–level sets to construct of single–valued neutrosophic EQ–filters.

Theorem 4.9. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra and $\{A_i = (T_{A_i}, F_{A_i}, I_{A_i})\}_{i\in I}$ be a family of single–valued neutrosophic EQ–filters of $\mathcal{E}$. Then $\bigcap\limits_{i\in I} A_i$ is a single–valued neutrosophic EQ–filter of $\mathcal{E}$.*

Proof. Let $x \in E$, then for any $i \in I, T_{A_i}(x) \leq T_{A_i}(1), F_{A_i}(x) \leq F_{A_i}(1), I_{A_i}(x) \geq I_{A_i}(1)$ and so $(\bigcap\limits_{i\in I} T_{A_i})(x) = \bigwedge\limits_{i\in I} T_{A_i}(x) \leq T_{A_i}(1), (\bigcap\limits_{i\in I} F_{A_i})(x) = \bigwedge\limits_{i\in I} F_{A_i}(x) \leq F_{A_i}(1)$ and $(\bigcap\limits_{i\in I} I_{A_i})(x) = \bigwedge\limits_{i\in I} I_{A_i}(x) \geq I_{A_i}(1)$. Let $x, y \in E$. Then

$$(\bigcap_{i\in I} T_{A_i})(x) \wedge (\bigcap_{i\in I} T_{A_i})(x \to y)$$
$$= \bigwedge_{i\in I} T_{A_i}(x) \wedge \bigwedge_{i\in I} T_{A_i}(x \to y) \leq \bigwedge_{i\in I} T_{A_i}(y)$$
$$= \bigcap_{i\in I} T_{A_i}(y),$$
$$(\bigcap_{i\in I} F_{A_i})(x) \wedge (\bigcap_{i\in I} F_{A_i})(x \to y)$$
$$= \bigwedge_{i\in I} F_{A_i}(x) \wedge \bigwedge_{i\in I} F_{A_i}(x \to y) \leq \bigwedge_{i\in I} F_{A_i}(y)$$
$$= \bigcap_{i\in I} F_{A_i}(y) \text{ and}$$
$$(\bigcap_{i\in I} I_{A_i})(x) \vee (\bigcap_{i\in I} I_{A_i})(x \to y)$$
$$= \bigwedge_{i\in I} I_{A_i}(x) \vee \bigwedge_{i\in I} I_{A_i}(x \to y) \geq \bigwedge_{i\in I} I_{A_i}(y)$$
$$= \bigcap_{i\in I} I_{A_i}(y).$$

Let $x, y, z \in E$. Then

$$(\bigcap_{i\in I} T_{A_i})(x \to y) = \bigwedge_{i\in I} T_{A_i}(x \to y)$$
$$\leq \bigwedge_{i\in I} T_{A_i}(x \otimes z \to y \otimes z)$$
$$= \bigcap_{i\in I} T_{A_i}(x \otimes z \to y \otimes z),$$
$$(\bigcap_{i\in I} F_{A_i})(x \to y) = \bigwedge_{i\in I} F_{A_i}(x \to y)$$
$$\leq \bigwedge_{i\in I} F_{A_i}(x \otimes z \to y \otimes z)$$
$$= \bigcap_{i\in I} I_{A_i}(x \otimes z \to y \otimes z) \text{ and}$$
$$(\bigcap_{i\in I} I_{A_i})(x \to y) = \bigwedge_{i\in I} I_{A_i}(x \to y)$$
$$\leq \bigwedge_{i\in I} I_{A_i}(x \otimes z \to y \otimes z)$$
$$= \bigcap_{i\in I} I_{A_i}(x \otimes z \to y \otimes z).$$

Thus $\bigcap\limits_{i\in I} A_i$ is a single–valued neutrosophic EQ–filter of $\mathcal{E}$. □

Definition 4.10. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–filter of $\mathcal{E}$ and $\alpha, \beta, \gamma \in [0, 1]$. Consider $T^{\alpha}_A = \{x \in E \mid T_A(x) \geq \alpha\}, F^{\beta}_A = \{x \in E \mid F_A(x) \geq \beta\}, I^{\gamma}_A = \{x \in E \mid T_A(x) \leq \gamma\}$ and define $A^{(\alpha,\beta,\gamma)} = \{x \in E \mid T_A(x) \geq \alpha, F_A(x) \geq \beta, I_A(x) \leq \gamma\}$. For any $\alpha, \beta, \gamma \in [0, 1]$ the set $A^{(\alpha,\beta,\gamma)}$ is called an (α, β, γ)-level set.*

Example 4.11. *Consider the EQ–algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$, single–valued neutrosophic EQ–filter A of $\mathcal{E}$ which are defind in Example 4.5. If $\alpha = 0.3, \beta = 0.4$ and $\gamma = 0.5$, then $T^{\alpha}_A = E, F^{\beta}_A = \{1\}, I^{\gamma}_A = \{1\}$ and $A^{(\alpha,\beta,\gamma)} = \{1\}$.*

Theorem 4.12. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–filter of $\mathcal{E}$ and $\alpha, \beta, \gamma \in [0, 1]$. Then*

(i) *$A^{(\alpha,\beta,\gamma)} = T^{\alpha}_A \cap I^{\beta}_A \cap F^{\gamma}_A$,*
(ii) *if $\emptyset \neq A^{(\alpha,\beta,\gamma)}$, then $A^{(\alpha,\beta,\gamma)}$ is an EQ–filter of $\mathcal{E}$,*
(ii) *if $A^{(\alpha,\beta,\gamma)}$ is an EQ–filter of $\mathcal{E}$, then A is a single–valued neutrosophic EQ–filter in $\mathcal{E}$.*

Proof. (i) It is obtained by definition.

(ii) $\emptyset \neq A^{(\alpha,\beta,\gamma)}$, implies that there exists $x \in A^{(\alpha,\beta,\gamma)}$. By Theorem 3.6, we conclude that $\alpha \leq T_A(x) \leq T_A(1), \beta \leq F_A(x) \leq F_A(1)$ and $\gamma \geq I_A(x) \geq I_A(1)$. Therefore, $1 \in A^{(\alpha,\beta,\gamma)}$.

Let $x \in A^{(\alpha,\beta,\gamma)}$ and $x \leq y$. Since T_A and F_A are monotone maps and I_A is an antimonotone map, we get that $\alpha \leq T_A(x) \leq T_A(y), \beta \leq F_A(x) \leq F_A(y)$ and $\gamma \geq I_A(x) \geq I_A(y)$. Hence $y \in A^{(\alpha,\beta,\gamma)}$.

Let $x \in A^{(\alpha,\beta,\gamma)}$ and $x \to y \in A^{(\alpha,\beta,\gamma)}$. Since A is a single–valued neutrosophic EQ–filter of $\mathcal{E}$, by definition we get that $\alpha \leq T_A(x) \wedge T_A(x \to y) \leq T_A(y), \beta \leq F_A(x) \wedge F_A(x \to y) \leq F_A(y)$ and $\gamma \geq I_A(x) \vee I_A(x \to y) \geq I_A(y)$. So $y \in A^{(\alpha,\beta,\gamma)}$.

Let $x \to y \in A^{(\alpha,\beta,\gamma)}$ and $z \in E$. Since A is a single–valued neutrosophic EQ–filter of $\mathcal{E}$, by definition we get that $\alpha \leq T_A(x \to y) \leq T_A((x \otimes z) \to (y \otimes z)), \gamma \geq I_A(x \to y) \geq I_A((x \otimes z) \to (y \otimes z))$ and $\beta \leq F_A(x \to y) \leq F_A((x \otimes z) \to (y \otimes z))$. It follows that $(x \otimes z) \to (y \otimes z) \in A^{(\alpha,\beta,\gamma)}$ and so $A^{(\alpha,\beta,\gamma)}$ is an EQ–filter of $\mathcal{E}$.

(iii) Let $x, y, z \in E$. Consider $\alpha_x = T_A(x), \beta_x = F_A(x)$ and $\gamma_x = I_A(x)$. Since $A^{(\alpha,\beta,\gamma)}$ is an EQ–filter of $\mathcal{E}$, then $1 \in A^{(\alpha,\beta,\gamma)}$ implies that

$$T_A(1) \geq \alpha_x = T_A(x), F_A(1) \geq \beta_x = F_A(x),$$

$I_A(1) \leq \gamma_x = I_A(x)$.

Let $\alpha_{x\to y} = T_A(x \to y), \beta_{x\to y} = F_A(x \to y), \gamma_{x\to y} = I_A(x \to y), \alpha = \alpha_x \wedge \alpha_{x\to y}, \beta = \beta_x \wedge \beta_{x\to y}$ and $\gamma = \gamma_x \vee \gamma_{x\to y}$. We have $T_A(x) = \alpha_x \geq \alpha, T_A(x \to y) = \alpha_{x\to y} \geq \alpha, F_A(x) = \beta_x \geq \beta, F_A(x \to y) = \beta_{x\to y} \geq \beta$ and $I_A(x) = \gamma_x \leq \gamma, I_A(x \to y) = \gamma_{x\to y} \leq \gamma$, so $x, x \to y \in A^{(\alpha,\beta,\gamma)}$. Since $A^{(\alpha,\beta,\gamma)}$ is an EQ–filter of $\mathcal{E}$ we get $y \in A^{(\alpha,\beta,\gamma)}$. Thus we conclude that

$$T_A(y) \geq \alpha = \alpha_x \wedge \alpha_{x\to y} = T_A(x) \wedge T_A(x \to y),$$

$$F_A(y) \geq \beta = \beta_x \wedge \beta_{x\to y} = F_A(x) \wedge F_A(x \to y)$$

and $I_A(y) \leq \gamma = \gamma_x \vee \gamma_{x\to y} = I_A(x) \vee I_A(x \to y)$. We have $T_A(x \to y) = \alpha_{x\to y} \geq \alpha_{x\to y}, F_A(x \to y) = \beta_{x\to y} \geq \beta_{x\to y}$ and $I_A(x \to y) = \gamma_{x\to y} \leq \gamma_{x\to y}$, so $x \to y \in A^{(\alpha_{x\to y},\beta_{x\to y},\gamma_{x\to y})}$. Since $A^{(\alpha_{x\to y},\beta_{x\to y},\gamma_{x\to y})}$ is an EQ–filter of $\mathcal{E}$ we get $x \otimes z \to y \otimes z \in A^{(\alpha_{x\to y},\beta_{x\to y},\gamma_{x\to y})}$. Thus we conclude that

$$T_A((x \otimes z) \to (y \otimes z)) \geq \alpha_{x\to y} = T_A(x \to y),$$

$$F_A((x \otimes z) \to (y \otimes z)) \geq \beta_{x\to y} = F_A(x \to y)$$

and $I_A((x \otimes z) \to (y \otimes z)) \geq \gamma_{x\to y} = I_A(x \to y)$. It follows that A is a single–valued neutrosophic EQ–filter $\mathcal{E}$. □

Corollary 4.13. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–filter of $\mathcal{E}$, $\alpha, \beta, \gamma \in [0,1]$ and $\emptyset \neq A^{(\alpha,\beta,\gamma)}$.*

- (i) *$A^{(\alpha,\beta,\gamma)}$ is an EQ–filter of $\mathcal{E}$ if and only if A is a single–valued neutrosophic EQ–filter in $\mathcal{E}$.*
- (ii) *If $G_A = \{x \in E \mid T_A(1) = F_A(1) = 1, I_A(0) = 1\}$, then G_A is an EQ–filter in $\mathcal{E}$*

Let $A = (T_A, F_A, I_A)$ be a single–valued neutrosophic EQ–filter in $\mathcal{E}, \alpha, \alpha', \beta, \beta', \gamma, \gamma' \in [0,1]$ and $\emptyset \neq H \subseteq \mathcal{E}$. Consider

$$T^{[\alpha,\alpha']}_{A,H} = \begin{cases} \alpha & \text{if } x \in H, \\ \alpha' & \text{otherwise,} \end{cases} \quad F^{[\alpha,\alpha']}_{A,H} = \begin{cases} \beta & \text{if} x \in H \\ \beta' & \text{o.w,} \end{cases}$$

and $I^{[\alpha,\alpha']}_{A,H} = \begin{cases} \gamma & \text{if } x \in H, \\ \gamma' & \text{otherwise.} \end{cases}$ Then we have the following corollary.

Corollary 4.14. *Let $A = (T_A, F_A, I_A)$ be a single–valued neutrosophic EQ–filter in $\mathcal{E}$. Then*

- (i) *$T^{[\alpha,\alpha']}_{A,H}, F^{[\alpha,\alpha']}_{A,G}$ and $I^{[\alpha,\alpha']}_{A,G}$ are fuzzy subsets,*
- (ii) *$T^{[\alpha,\alpha']}_{A,H}$ is a fuzzy filter in E if and only if G is an EQ–filter of $\mathcal{E}$,*
- (iii) *$F^{[\alpha,\alpha']}_{A,H}$ is a fuzzy filter in E if and only if G is an EQ–filter of $\mathcal{E}$,*
- (iv) *$I^{[\alpha,\alpha']}_{A,H}$ is a fuzzy filter in E if and only if G is an EQ–filter of $\mathcal{E}$.*

Definition 4.15. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra, A be a single–valued neutrosophic EQ–filter of $\mathcal{E}$. Then A is said to be a normal single–valued neutrosophic EQ–filter of $\mathcal{E}$ if there exists $x, y, z \in E$ such that $T_A(x) = 1, I_A(y) = 1$ and $F_A(z) = 1$.*

Example 4.16. *Consider the EQ–algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$, which is defind in Example 4.5. If Define a single valued neutrosophic set map A in E as follows:*

T_A	0	a	b	c	1
	0.4	0.4	0.4	0.4	1

,

I_A	0	a	b	c	1
	1	1	1	1	0.11

and

F_A	0	a	b	c	1
	0.2	0.2	0.2	0.2	1

Hence A is a normal single–valued neutrosophic EQ–filter of $\mathcal{E}$.

Theorem 4.17. *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ–algebra and A be a single–valued neutrosophic EQ–filter of $\mathcal{E}$. Then A is a normal single–valued neutrosophic EQ–filter of $\mathcal{E}$ if and only if $T_A(1) = 1, F_A(1) = 1$ and $I_A(0) = 1$.*

Proof. By Corollary 3.7, it is straightforward. □

Corollary 4.18. *Let $A = (T_A, I_A, F_A)$ be a single–valued neutrosophic EQ–filter of $\mathcal{E}$. Then*

- (i) *A is a normal single–valued neutrosophic EQ–filter of $\mathcal{E}$ if and only if T_A, F_A and I_A are normal fuzzy subset.*
- (ii) *If there exists a sequence $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ of elements E in such a way that*

$$\{(T_A(x_n), I_A(y_n), F_A(z_n))\} \to (1, 1, 1),$$

then $A(1, 0, 1) = (1, 1, 1)$.

Corollary 4.19. *Let $\{A_i = (T_{A_i}, F_{A_i}, I_{A_i})\}_{i\in I}$ be a family of normal single–valued neutrosophic EQ–filters of $\mathcal{E}$. Then $\bigcap_{i\in I} A_i$ is a normal single–valued neutrosophic EQ–filter of $\mathcal{E}$.*

Let $A = (T_A, I_A, F_A)$ be a single–valued neutrosophic EQ–filter of $\mathcal{E}, x \in E$ and $p \in [1, +\infty)$. Consider $T_A^{+p}(x) = \dfrac{1}{p}(p + T_A(x) - T_A(1))$, $F_A^{+p}(x) = \dfrac{1}{p}(p + F_A(x) - F_A(1))$ and $I_A^{+p}(x) = \dfrac{1}{p}(p + I_A(x) - I_A(0))$.

Theorem 4.20. *Let $A = (T_A, I_A, F_A)$ be a single–valued neutrosophic EQ–filter of $\mathcal{E}$. Then*

- (i) *T_A^{+p} is a normal EQ–filter of $\mathcal{E}$,*
- (ii) *I_A^{+p} is a normal EQ–filter of $\mathcal{E}$,*
- (iii) *$(T_A^{+p})^{+p} = T_A^{+p}$ if and only if $p = 1$,*
- (iv) *$(I_A^{+p})^{+p} = I_A^{+p}$ if and only if $p = 1$,*
- (v) *$(T_A^{+p})^{+p} = T_A$ if and only if T_A is normal EQ–filter,*
- (vi) *$(I_A^{+p})^{+p} = I_A$ if and only if $I_A(0) = 1$.*

Proof. (i) Let $x \in E$. Because $T_A(x) \leq T_A(1)$, then we have $T_A^{+p}(x) = \dfrac{1}{p}(p + T_A(x) - T_A(1)) \leq 1$. Assume that $x, y \in E$. Using $(SVNF2)$, we get that

$$T_A^{+p}(x) \wedge T_A^{+p}(x \to y) = \frac{1}{p}(p + T_A(x) - T_A(1))$$
$$\wedge \frac{1}{p}(p + T_A(x \to y) - T_A(1))$$
$$= \frac{1}{p}\big[(p + T_A(x) - T_A(1))$$
$$\wedge (p + T_A(x \to y) - T_A(1))\big]$$
$$= \frac{1}{p}\big[((p \wedge p) + (T_A(x) \wedge T_A(x \to y))$$
$$- (T_A(1) \wedge T_A(1))\big]$$
$$\leq \frac{1}{p}(p + T_A(y) - T_A(1)) = T_A^{+p}(y).$$

Suppose that $x, y, z \in E$. Using $(SVNF3)$, we get that

$$\begin{aligned} T_A^{+p}(x \to y) &= \frac{1}{p}(p + T_A(x \to y) - T_A(1)) \\ &\leq \frac{1}{p}(p + T_A(x \otimes z \to y \otimes z) \\ &\quad - T_A(1)) = T_A^{+p}(x \otimes z \to y \otimes z). \end{aligned}$$

Thus T_A^{+p} is an EQ–filter of $\mathcal{E}$. In addition the equality $T_A^{+p}(1) = \dfrac{1}{p}(p + T_A(1) - T_A(1)) = 1$, implies that T_A^{+p} is a normal EQ–filter of $\mathcal{E}$.

(ii) Let $x \in E$. Since $I_A(x) \leq I_A(0)$ we get that $I_A^{+p}(x) = \dfrac{1}{p}(p + I_A(x) - I_A(0)) \leq 1$. Items $(SVNF2)$ and $(SVNF3)$ are obtained similar to the item (i).

(iii) Assume that $x \in E$. Then

$$\begin{aligned} (T_A^{+p})^{+p}(x) &= [\frac{1}{p}(p + T_A(x) - T_A(1))]^{+p} \\ &= \frac{1}{p}[p + \frac{1}{p}(p + T_A(x) - T_A(1)) \\ &\quad - \frac{1}{p}(p + T_A(1) - T_A(1))] \\ &= \frac{1}{p}(p + \frac{1}{p}(T_A(x) - T_A(1))). \end{aligned}$$

So

$$\begin{aligned}(T_A^{+p})^{+p}(x) &= T_A^{+p}(x)\\ &\Longleftrightarrow \frac{1}{p}(p+\frac{1}{p}(T_A(x)-T_A(1)))\\ &= \frac{1}{p}(p+T_A(x)-T_A(1))\\ &\Longleftrightarrow p=1.\end{aligned}$$

(iv) It is similar to (iii).
(v) Let $x \in E$. $(T_A^{+p})^{+p} = T_A$ if and only if

$$\frac{1}{p}(p+\frac{1}{p}(T_A(x)-T_A(1))) = T_A(x)$$

$$\Longleftrightarrow T_A(1) = (1-p^2)T_A(x)+p^2$$

$$\Longleftrightarrow p=1 \Longleftrightarrow T_A(1)=1.$$

(vi) It is similar to (v). □

Example 4.21. *Let $E = \{0,a,b,c,1\}$. Define operations "$\otimes, \sim$" and an operation "$\wedge$" on E as follows:*

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

,

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	a	b
c	0	0	0	a	c
1	0	a	b	c	1

and

$\sim$	0	a	b	c	1
0	1	c	b	a	0
a	c	1	c	b	a
b	b	c	1	c	b
c	a	b	c	1	c
1	0	a	b	c	1

.

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ–algebra, where b and c are non-comparable. Now, obtain the operation "$\rightarrow$" as follows:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

.

Define a single valued neutrosophic set map A in E as follows:

T_A	0	a	b	c	1
	0.41	0.42	0.43	0.44	0.5

,

I_A	0	a	b	c	1
	0.69	0.68	0.67	0.66	0.65

and

F_A	0	a	b	c	1
	0.21	0.22	0.23	0.24	0.25

Hence A is a single–valued neutrosophic EQ–prefilter of $\mathcal{E}$. Consider $p = 3$, then we obtain a single–valued neutrosophic EQ–prefilter A^{+3} in E as follows:

T_A^{+3}	0	a	b	c	1
	0.97	0.973	0.977	0.98	1

,

I_A^{+3}	0	a	b	c	1
	1	0.977	0.973	0.99	0.987

and

F_A^{+3}	0	a	b	c	1
	0.987	0.99	0.993	0.997	1

.

Corollary 4.22. *Let $A = (T_A, I_A, F_A)$ be a single–valued neutrosophic EQ–filter of $\mathcal{E}$. Then*

(i) F_A^{+p} is a normal EQ–filter of $\mathcal{E}$,
(ii) $(F_A^{+p})^{+p} = F_A^{+p}$ if and only if $p = 1$,
(iii) $(F_A^{+p})^{+p} = F_A$ if and only if F_A is normal EQ–filter.

Corollary 4.23. *Let $A = (T_A, I_A, F_A)$ be a single–valued neutrosophic EQ–filter of $\mathcal{E}$. Then*

(i) $A^{+p} = (T_A^{+p}, I_A^{+p}, F_A^{+p})$ is a normal single–valued neutrosophic EQ–filter of $\mathcal{E}$,
(ii) $(A^{+p})^{+p} = A^{+p}$ if and only if $p = 1$,
(ii) $(A^{+p})^{+p} = A$ if and only if A is a normal single–valued neutrosophic EQ–filter.

Proof. It is trivial by Theorem 4.20 and Corollary 4.22. □

Let $A = (T_A, I_A, F_A)$ be a single–valued neutrosophic EQ–filter of $\mathcal{E}$ and g be an endomorphism on $\mathcal{E}$. Now we define $A^g = (T_A^g, I_A^g, F_A^g)$ by $T_A^g(x) = T_A(g(x)), F_A^g(x) = F_A(g(x))$ and $I_A^g(x) = I_A(g(x))$.

Theorem 4.24. *Let $A = (T_A, I_A, F_A)$ be a single–valued neutrosophic EQ–filter of $\mathcal{E}$ and $x, y \in E$. Then*

(i) if $x \le y$, then $T_A^g(x) \le T_A^g(y), F_A^g(x) \le F_A^g(y)$ and $I_A^g(x) \ge I_A^g(y)$,
(ii) A^g is a single–valued neutrosophic EQ–filter of $\mathcal{E}$,
(iii) $T'_A(x) = \frac{1}{2}(T_A^g(x) + T_A(x))$ is a fuzzy filter in E,
(iv) $F'_A(x) = \frac{1}{2}(F_A^g(x) + F_A(x))$ is a fuzzy filter in E,

(v) $A'^g = (T'_A, I'_A, F'_A)$ *is a single–valued neutrosophic EQ–filter of* $\mathcal{E}$.

Proof. (i) Let $x, y \in E$. If $x \leq y$, then $g(x) \leq g(y)$. It follows that $T^g_A(x) = T_A(g(x)) \leq T_A(g(y)), F^g_A(x) = F_A(g(x)) \leq F_A(g(y))$ and $I^g_A(x) = I_A(g(x)) \geq I_A(g(y))$.

(ii) Since $g(x \to y) = g(x) \to g(y)$, we get that

$$\begin{aligned} T^g_A(x) \wedge T^g_A(x \to y) & \\ &= T_A(g(x)) \wedge T_A(g(x) \to g(y)) \\ &\leq T_A(g(y)) = T^g_A(y), F^g_A(x) \wedge F^g_A(x \to y) \\ &= F_A(g(x)) \wedge F_A(g(x) \to g(y)) \\ &\leq F_A(g(y)) = F^g_A(y) \end{aligned}$$

and $I^g_A(x) \vee I^g_A(x \to y) = I_A(g(x)) \vee I_A(g(x) \to g(y)) \leq I_A(g(y)) = I^g_A(y)$.

Let $z \in E$. Since $g(x \otimes z \to y \otimes z) = g(x \otimes z) \to g(y \otimes z)$, we get that

$$\begin{aligned} T^g_A(x \to y) &= T_A(g(x) \to g(y)) \\ &\leq T_A(g(x \otimes z \to y \otimes z)) \\ &= T_A(g(x \otimes z) \to (y \otimes z)) \\ &= T^g_A(x \otimes z \to y \otimes z), \\ F^g_A(x \to y) &= F_A(g(x) \to g(y)) \\ &\leq F_A(g(x \otimes z \to y \otimes z)) \\ &= F_A(g(x \otimes z) \to (y \otimes z)) \\ &= F^g_A(x \otimes z \to y \otimes z), \\ I^g_A(x \to y) &= I_A(g(x) \to g(y)) \\ &\geq I_A(g(x \otimes z \to y \otimes z)) \\ &= I_A(g(x \otimes z) \to (y \otimes z)) \\ &= I^g_A(x \otimes z \to y \otimes z). \end{aligned}$$

So by the item (i), A^g is a single–valued neutrosophic EQ–filter of $\mathcal{E}$.

$(iii), (iv)$ Let $x \in E$. Since $g(1) = 1$, so $T_A(x) + T_A(g(x)) \leq 2$ implies that $T'_A(x) = \frac{1}{2}(T^g_A(x) + T_A(x)) \leq T'_A(1)$. In a similar way $F'_A(x) \leq F'_A(1)$ and $I'_A(x) \geq I'_A(1)$ are obtained. Suppose that $x, y \in E$. Then we have

$$\begin{aligned} & T'_A(x) \wedge T'_A(x \to y) \\ &= \frac{1}{2}(T^g_A(x) + T_A(x)) \\ &\wedge \frac{1}{2}(T^g_A(x \to y) + T_A(x \to y)) \\ &= \frac{1}{2}(T^g_A(x) \wedge T^g_A(x \to y)) \\ &+ \frac{1}{2}(T_A(x) + T_A(x \to y)) \\ &\leq \frac{1}{2}(T^g_A(y) + T_A(y)) = T'_A(y). \end{aligned}$$

We can show that $F'_A(x) \wedge F'_A(x \to y) \leq F'_A(y)$ and $I'_A(x) \vee I'_A(x \to y) \geq I'_A(y)$. Let $x, y, z \in E$. Then

$$\begin{aligned} & T'_A(x \to y) = \frac{1}{2}(T^g_A(x \to y) + T_A(x \to y)) \\ &= \frac{1}{2}(T_A(g(x \to y)) + T_A(x \to y)) \\ &\leq \frac{1}{2}(T_A(g(x \otimes z \to y \otimes z)) + T_A(x \otimes z \to y \otimes z)) \\ &= \frac{1}{2}(T^g_A((x \otimes z \to y \otimes z)) + T_A(x \otimes z \to y \otimes z)) \\ &= T'_A(x \otimes z \to y \otimes z). \end{aligned}$$

In a similar way can see that $F'_A(x \to y) \leq F'_A(x \otimes z \to y \otimes z)$ and $I'_A(x \to y) \geq I'_A(x \otimes z \to y \otimes z)$.

(v) It is obtained from previous items. □

Example 4.25. *Let* $E = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. *Define operations "$\otimes, \sim$" and "$\wedge$" on E as follows:*

$\wedge$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_2	a_2	a_2	a_2
a_3	a_1	a_2	a_3	a_3	a_3	a_3
a_4	a_1	a_2	a_3	a_4	a_4	a_4
a_5	a_1	a_2	a_3	a_4	a_5	a_5
a_6	a_1	a_2	a_3	a_4	a_5	a_6

,

$\otimes$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1	a_1	a_1
a_3	a_1	a_1	a_1	a_1	a_1	a_1
a_4	a_1	a_1	a_1	a_1	a_1	a_1
a_5	a_1	a_1	a_1	a_1	a_5	a_5
a_6	a_1	a_2	a_3	a_4	a_5	a_6

and

$\sim$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_6	a_6	a_1	a_1	a_1	a_1
a_2	a_6	a_6	a_1	a_1	a_1	a_1
a_3	a_1	a_1	a_6	a_4	a_4	a_4
a_4	a_1	a_1	a_4	a_6	a_4	a_4
a_5	a_1	a_1	a_4	a_4	a_6	a_5
a_6	a_1	a_1	a_4	a_4	a_5	a_6

.

Now, we obtain the operation "$\to$" as follows:

$\to$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_6	a_6	a_6	a_6	a_6	a_6
a_2	a_6	a_6	a_6	a_6	a_6	a_6
a_3	a_1	a_1	a_6	a_6	a_6	a_6
a_4	a_1	a_1	a_4	a_6	a_6	a_6
a_5	a_1	a_1	a_4	a_4	a_6	a_6
a_6	a_1	a_1	a_4	a_4	a_5	a_6

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, a_6)$ is an EQ–algebra. Let $g \in End(E)$. Clearly $g(a_6) = a_6$. Since for any $1 \leq i \leq 4, 1 \leq j \leq 6, g(a_1) = g(a_i \otimes a_j) = g(a_i) \otimes g(a_j)$. So $a_1 = g(a_1) = g(a_5 \sim a_2) = g(a_5) \sim g(a_2) = g(a_5) \sim a_1 = a_1$ implies that $g(a_5) = a_1$. Hence define a single valued neutrosophic set map A in E and a map g on E as follows:

T_A	a_1	a_2	a_3	a_4	a_5	a_6
	0.01	0.02	0.03	0.04	0.05	0.06

,

F_A	a_1	a_2	a_3	a_4	a_5	a_6
	0.11	0.12	0.13	0.14	0.15	0.16

,

I_A	a_1	a_2	a_3	a_4	a_5	a_6
	0.61	0.52	0.43	0.34	0.25	0.16

and

g	a_1	a_2	a_3	a_4	a_5	a_6
	a_1	a_1	a_1	a_1	a_5	a_6

.

Hence $(A, \mathcal{E})$ is a single–valued neutrosophic EQ–prefilter. Now, we obtain a single valued neutrosophic EQ–prefilter A^g in E follows:

T_A^g	a_1	a_2	a_3	a_4	a_5	a_6
	0.01	0.01	0.01	0.01	0.05	0.06

,

F_A^g	a_1	a_2	a_3	a_4	a_5	a_6
	0.11	0.11	0.11	0.11	0.15	0.16

and

I_A^g	a_1	a_2	a_3	a_4	a_5	a_6
	0.61	0.61	0.61	0.61	0.25	0.16

.

and obtain a single valued neutrosophic EQ–prefilter A'^g in E follows:

$T_A'^g$	a_1	a_2	a_3	a_4	a_5	a_6
	0.01	0.015	0.02	0.025	0.05	0.06

,

$F_A'^g$	a_1	a_2	a_3	a_4	a_5	a_6
	0.11	0.115	0.12	0.125	0.15	0.16

and

$I_A'^g$	a_1	a_2	a_3	a_4	a_5	a_6
	0.61	0.565	0.52	0.475	0.25	0.16

.

5. Conclusion

The current paper considered the concept of single–valued neutrosophic EQ–algebras and introduce the concepts single–valued neutrosophic EQ–subalgebras, single–valued neutrosophic EQ–prefilters and single–valued neutrosophic EQ–filters.

(i) It is showed that single–valued neutrosophic EQ–subalgebras preserve some binary relation on EQ–algebras under some conditions.

(ii) Using the some properties of single–valued neutrosophic EQ–prefilters, we construct new single–valued neutrosophic EQ–prefilters.

(iii) We considered that single–valued neutrosophic EQ–filters as generalisation of single–valued neutrosophic EQ–prefilters and constructed them.

(iv) We connected the concept of EQ–prefilters to single–valued neutrosophic EQ–prefilters and the concept of EQ–filters to single–valued neutrosophic EQ–filters, so we obtained such structures from this connection.

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Semi-Idempotents in Neutrosophic Rings

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W.B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2019). Semi-Idempotents in Neutrosophic Rings. *Mathematics* 7, 507; DOI: 10.3390/math7060507

Abstract: In complex rings or complex fields, the notion of imaginary element i with $i^2 = -1$ or the complex number i is included, while, in the neutrosophic rings, the indeterminate element I where $I^2 = I$ is included. The neutrosophic ring $\langle R \cup I \rangle$ is also a ring generated by R and I under the operations of R. In this paper we obtain a characterization theorem for a semi-idempotent to be in $\langle Z_p \cup I \rangle$, the neutrosophic ring of modulo integers, where p a prime. Here, we discuss only about neutrosophic semi-idempotents in these neutrosophic rings. Several interesting properties about them are also derived and some open problems are suggested.

Keywords: semi-idempotent; neutrosophic rings; modulo neutrosophic rings; neutrosophic semi-idempotent

1. Introduction

According to Gray [1], an element $\alpha \neq 0$ of a ring R is called a semi-idempotent if and only if α is not in the proper two-sided ideal of R generated by $\alpha^2 - \alpha$, that is $\alpha \notin R(\alpha^2 - \alpha)R$ or $R = R(\alpha^2 - \alpha)R$. Here, 0 is a semi-idempotent, which we may term as trivial semi-idempotent. Semi-idempotents have been studied for group rings, semigroup rings and near rings [2–9].

An element I was defined by Smarandache [10] as an indeterminate element. Neutrosophic rings were defined by Vasantha and Smarandache [11]. The neutrosophic ring $\langle R \cup I \rangle$ is also a ring generated by R and the indeterminate element I ($I^2 = I$) under the operations of R [11]. The concept of neutrosophic rings is further developed and studied in [12–16]. As the newly introduced notions of neutrosophic triplet groups [17,18] and neutrosophic triplet rings [19], neutrosophic triplets in neutrosophic rings [20] and their relations to neutrosophic refined sets [21,22] depend on idempotents, thus the relative study of semi-idempotents will be an innovative research for any researcher interested in these fields. Finding idempotents is discussed in [18,23–25]. One can also characterize and study neutrosophic idempotents in these situations as basically neutrosophic idempotents are trivial neutrosophic semi-idempotents. A new angle to this research can be made by studying quaternion valued functions [26].

We call a semi-idempotents x in $\langle R \cup I \rangle$ as neutrosophic semi-idempotents if $x = a + bI$ and $b \neq 0; a, b \in \langle R \cup I \rangle$. Several interesting results about semi-idempotents are derived for neutrosophic rings in this paper. As the study pivots on idempotents it has much significance for the recent studies on neutrosophic triplets, duplets and refined sets.

Here, the notion of semi-idempotents in the case of neutrosophic rings is introduced and several interesting properties associated with them are analyzed. We discuss only about neutrosophic semi-idempotents in these neutrosophic rings. This paper is organized into three sections. Section 1 is introductory in nature. In Section 2, the notion of semi-idempotents in the case of

$$\langle Z_n \cup I\rangle = \{a + bI | a, b \in Z_n; n < \infty; I^2 = I\}$$

is considered. Section 3 gives conclusions and proposes some conjectures based on our study.

2. Semi-Idempotents in the Modulo Neutrosophic Rings $\langle Z_n \cup I\rangle$

Throughout this paper, $\langle Z_n \cup I\rangle = \{a + bI/a, b \in Z_n, 2 \leq n < \infty; I^2 = I\}$ denotes the neutrosophic ring of modulo integers. We illustrate some semi-idempotents of $\langle Z_n \cup I\rangle$ by examples and derive some interesting results related with them.

Example 1. *Let $S = \langle Z_2 \cup I\rangle = \{a + bI/a, b \in Z_2, I^2 = I\}$ be the neutrosophic ring of modulo integers. Clearly, $I^2 = I$ and $(1 + I)^2 = 1 + I$ are the two non-trivial idempotents of S. Here, 0 and 1 are trivial idempotents of S. Thus, S has no non-trivial semi-idempotents as all idempotents are trivial semi-idempotents of S.*

Example 2. *Let*

$$R = \langle Z_3 \cup I\rangle = \{a + bI|a, b \in Z^3, I^2 = I\} = \{0, 1, 2, I, 2I, 1 + I, 2 + I, 1 + 2I, 2 + 2I\}$$

be the neutrosophic ring of modulo integers. The trivial idempotents of R are 0 and 1. The non-trivial neutrosophic idempotents are I and $1 + 2I$. Thus, the idempotents I and $1 + 2I$ are trivial neutrosophic semi-idempotents of R. Clearly, 2 and $2 + 2I$ are units of R as 2×2 = 1(mod 3) and $2 + 2I \times 2 + 2I$ = 1(mod 3). $1 + I \in R$ is such that

$$(1 + I)^2 - (1 + I) = 1 + 2I + I - (1 + I) = 1 + 2 + 2I = 2I.$$

Thus, $1 + I$ is a semi-idempotent as the ideal generated by $1 + I$ is $\langle (1 + I)^2 - (1 + I)\rangle = \langle 2I\rangle$ is such that $1 + I \notin R$. However, it is important to note that $(1 + I) \in R$ is a unit as $(1 + I)^2 = 1 + 2I + I = 1$, hence $1 + I$ is a unit in R but it is also a non-trivial semi-idempotent of R. $2 + I$ is not a semi-idempotent as

$$(2 + I)^2 - (2 + I) = 1 + 4I + I - (2 + I) = 2 + I;$$

hence the claim. $2 + 2I \in R$ is a unit, now $(2 + 2I)^2 = 4 + 8I + 4I^2 = 1$, thus $2 + 2I$ is a unit. However, $(2 + 2I)^2 - (2 + 2I) = 1 + 1 + I = 2 + I$.

Now, the ideal generated by $\langle 2 + I\rangle$ does not contain $2 + 2I$ as $\langle 2 + I\rangle = \{0, 2 + I, 1 + 2I\}$, thus $2 + 2I$ is also a non-trivial semi-idempotent even though $2 + 2I$ is a unit of R. Thus, it is important to note that units in modulo neutrosophic rings contribute to non-trivial semi-idempotents. Let $P = \{0, 2 + 2I, 2 + I, 1 + 2I, I, 1 + I, 1\}$ be the collection of trivial and non-trivial semi-idempotents. $2I$ is not a semi-idempotent as $(2I)^2 - 2I = I + I = 2I$, hence the claim. Thus, P is not closed under sum or product.

Theorem 1. *Let $S = \{\langle Z_p \cup I\rangle, +, \times\}$ be the ring of neutrosophic modulo integers where p is a prime. x is semi-idempotent if and only if $x \in \langle Z_p \cup I\rangle \setminus \{Z_pI, 0, 1, a + bI$ with $a + b = 0\}$.*

Proof. The elements $x = a + bI \in S$ with $b = 0$ are such that $x^2 - x$ generates the ideal, which is S, thus x is a semi-idempotent. Let $y = a + bI$; if $a = 0$, the ideal generated by y is Z_pI, thus $y \in Z_pI \subset S$, hence $y \in Z_pI$, therefore y is not a semi-idempotent.

Consider $z = a + bI \in S$ with $a + b = 0(mod\ p)$; then, $z^2 - z$ generates an ideal M of S such that every element $x = d + cI$ in M is such that $d + c \equiv 0(mod\ p)$, thus z is not a semi-idempotent of S. Let $x = a + bI \in S(a \neq 0, b \neq 0$ and $a + b \neq 0)$.

$$x^2 - x = \begin{cases} m & m \in Z_p \text{ or} \\ nI & n \in Z_p \text{ or} \\ n + mI & m + n \neq 0 \end{cases}$$

If $x^2 - x = m$, then the ideal generated by $x^2 - x$ is S, thus x is a semi-idempotent. If $x^2 - x = nI$, then the ideal generated by nI is Z_pI, thus $x \notin Z_pI$, hence again x is a semi-idempotent. If $x^2 - x = n + mI(m + n \neq 0)$, then the ideal generated by $n + mI$ is S, thus x is a semi-idempotent by using properties of Z_p, p a prime. Hence, the theorem is proved. □

If we take $S = \{\langle Z_n \cup I\rangle, +, \times\}$ as a neutrosophic ring where n is not a prime, it is difficult to find all semi-idempotents.

Example 3. *Let $S = \{\langle Z_{15} \cup I\rangle, +, \times\}$ be the neutrosophic ring. How can the non-trivial semi-idempotents of S be found? Some of the neutrosophic idempotents of S are $\{1 + 9I,\ 6 + 4I,\ 1 + 5I,\ 1 + 14I, 6 + 5I, 6 + 9I, I, 6I, 10I, 10, 6, 6 + 10I, 10 + 11I, 10 + 6I, 10 + 5I\}$.*

The semi-idempotents are $\{1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 1 + 6I, 1 + 7I, 1 + 8I, 1 + 10I, 1 + 11I, 1 + 12I, 1 + 13I, 6 + I, 6 + 2I, 6 + 3I, 6 + 6I, 6 + 7I, 6 + 8I, 6 + 11I, 6 + 12I, 6 + 13I, 6 + 14I, 10 + I, 10 + 2I, 10 + 3I, 10 + 4I, 10 + 7I, 10 + 8I, 10 + 9I, 10 + 10I, 10 + 12I, 10 + 13I, 10 + 14I\}$.

Are there more non-trivial neutrosophic idempotents and semi-idempotents?

However, we are able to find all idempotents and semi-idempotents of S other than the once given. In view of all these, we have the following theorem.

Theorem 2. *Let $S = \{\langle Z_{pq} \cup I\rangle; \times, +\}$ where p and q are two distinct primes:*

1. *There are two idempotents in Z_{pq} say r and s.*
2. *$\{r, s, rI, sI, I, r + tI, s + tI | t \in \{Z_{pq} \setminus 0\}\}$ such that $r + t = s$, 1 or 0 and $s + t = 0, 1$ or r is the partial collection of idempotents and semi-idempotents of S.*

Proof. Given $S = \{\langle Z_{pq} \cup I\rangle, +, \times\}$ is a neutrosophic ring where p and q are primes, we know from [12,17,18,20,23–25] that Z_{pq} has two idempotents r and s to prove $A = \{r, s, rIsI, I, r + tI$ and $s + tI/t \in Z_{pq} \setminus \{0\}\}$ are idempotents or semi-idempotents of $S.\{r, s, rI, sI, I\}$ are non-trivial idempotents of S. Now, $r + tI \in A$ and $(r + tI)^2 - (r + tI) = mI$ as$r^2 = r$, thus the ideal generated by mI does not contain r_tI. Therefore, r_tI is a non-trivial semi-idempotent. Similarly, $s + tI$ is a non-trivial semi-idempotent. Hence, the theorem is proved. □

We in addition to this theorem propose the following problem.

Problem 1. *Let $S = \{\langle Z_{pq} \cup I\rangle, I, \times\}$, where p and q are two distinct primes, be the neutrosophic ring. Can S have non-trivial idempotents and non-trivial semi-idempotents other than the ones mentioned in (b) of the above theorem?*

Problem 2. *Can the collection of all trivial and non-trivial semi-idempotents have any algebraic structure defined on them?*

We give an example of Z_{pqr}, where p, q and r are three distinct primes, for which we find all the neutrosophic idempotents.

Example 4. *Let $S = \{\langle Z_{30} \cup I\rangle, +, \times\}$, be the neutrosophic ring. The idempotents of Z_{30} are 6, 10, 15, 16, 21 and 25. The non-trivial semi-idempotents of S are $\{1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 1 + 6I, 1 + 7I, 1 + 8I, 1 + 10I, 1 + 11I, 1 + 13I, 1 + 12I, 1 + 16I, 1 + 17I, 1 + 18I, 1 + 19I, 1 + 21I, 1 + 22I, 1 + 23I, 1 + 25I, 1 + 26I, 1 + 27I, 1 + 28I\}$.*

$P_1 = \{1+5I, 1+9I, 1+14I, 1+15I, 1+20I, 1+24I, 1+29I\}$ *are non-trivial idempotents of S.* $J_2 = \{6+I, 6+2I, 6+3I, 6+5I, 6+6I, 6+7I, 6+8I, 6+11I, 6+12I, 6+13I, 6+14I, 6+16I, 6+17I,$ *6 + 18I,* $6+20I, 6+21I, 6+22I, 6+23I, 6+26I, 6+27I, 6+28I, 6+29I\}$ *are non-trivial neutrosophic semi-idempotents of S.* $P_2 = \{6+4I, 6+9I, 6+10I, 6+15I, 6+24I, 6+19I, 6+25I\}$ *are non-idempotents of S.*

Now, we list the non-trivial semi-idempotents associated with 10 of Z_{30}. $J_3 = \{10+I,\ 10+2I,$ *10 + 3I,* $10+4I, 10+7I, 10+8I, 10+9I, 10+10I, 10+11I, 10+12I, 10+13I, 10+14I, 10+16I, 10+17I, 10+18I, 10+19I, 10+22I, 10+23I, 10+24I, 10+25I, 10+27I, 10+28I,$ *10 + 29I*$\}$

$P_3 = \{10+5, 10+6I, 10+15I, 10+20I, 10+21I, 10+26I, 10+11I\}$ *are the collection of non-trivial idempotent related with the idempotents. Now, we find the non-trivial idempotents associated with 15:* $J_4 = \{15+2I, 15+3I, 15+4I, 15+7I, 15+8I, 15+9I, 15+11I, 15+12I, 15+13I,$ *15 + 14I,* $15+17I, 15+18I, 15+19I, 15+20I, 15+22I, 15+23I, 15+24I, 15+25I, 15+26I, 15+27I,$ *15 + 28I, 15 + 29I*$\}$.

$P_4 = \{15+I, 15+5I, 15+6I, 15+10I, 15+15I, 15+16I, 15+21I\}$ *are the non-trivial idempotents associated with 15. The collection of non-trivial semi-idempotents associated with 16 are:* $J_5 = \{$*16 + I,* $16+2I, 16+3I, 16+4I, 16+6I, 16+7I, 16+8I, 16+10I, 16+19I, 16+27I, 16+21I, 16+22I, 16+23I, 16+25I, 16+11I, 16+12I, 16+13I, 16+17I, 16+18I,$ *16 + 28I.* $P_5 = \{16+14I, 16+15I, 16+20I, 16+24I, 16+29I, 16+5I, 16+9I\}$ *are the set of non-trivial idempotents related with the idempotent. We find the non-trivial semi-idempotents associated with the idempotent 21:* $J_6 = \{21+I, 21+2I, 21+3I, 21+5I, 21+6I, 21+7I, 21+8I, 21+12I, 21+11I, 21+13I, 21+14I,$ *21 + 16I,* $21+17I, 21+18I, 21+20I, 21+21I, 21+22I, 21+23I, 21+26I, 21+27I, 21+28I, 21+29I\}$. $P_6 = \{21+4I, 21+9I, 21+10I, 21+15I, 21+19I, 21+24I, 21+25I\}$ *is the collection of non-trivial idempotents related with the real idempotent 21. The collection of all non-trivial semi-idempotents associated with the idempotent 25.* $J_7 = \{25+I, 25+2I, 25+3I, 25+4I, 25+7I, 25+8I, 25+9I,$ *25 + 10I,* $25+12I, 25+13I, 25+14I, 25+16I, 25+24I, 25+17I, 25+18I, 25+19I, 25+22I,$ *25 + 23I,* $25+27I, 25+28I, 25+29I\}$ $P_7 = \{25+5I, 25+6I, 25+11I, 25+15I, 25+20I, 25+21I, 25+26I\}$ *are the non-trivial collection of neutrosophic semi-idempotents related with the idempotent 25.*

We tabulate the neutrosophic idempotents associated with the real idempotents in Table 1. Based on that table, we propose some open problems.

Table 1. Idempotents.

S.No	Real	Neutrosophic	Sum	Missing
1	1	$1+5I$ $1+9I$ $1+14I$ $1+15I$ $1+20I$ $1+24I$ $1+29I$	1 + 5 = 6 1 + 9 = 10 1 + 14 = 15 1 + 15 = 16 1 + 20 = 21 1 + 24 = 25 1 + 29 = 0	1
2	6	$6+4I$ $6+9I$ $6+10I$ $6+15I$ $6+24I$ $6+19I$ $6+25I$	6 + 4 = 10 6 + 9 = 15 6 + 10 = 16 6 + 15 = 1 6 + 24 = 0 6 + 19 = 25 $6 + 25 \equiv 1$	6

Table 1. *Cont.*

S.No	Real	Neutrosophic	Sum	Missing
3	10	$10+5I$ $10+6I$ $10+15I$ $10+20I$ $10+21I$ $10+26I$ $10+11I$	10 + 5 = 15 10 + 6 = 16 10 + 15 = 25 $10 + 20 \equiv 0$ $10 + 21 \equiv 1$ $10 + 26 \equiv 6$ 10 + 11 = 21	10
4	15	$15+I$ $15+5I$ $15+6I$ $15+10I$ $15+15I$ $15+16I$ $15+21I$	15 + 1 = 16 15 + 5 = 20 15 + 6 = 21 15 + 10 = 25 $15 + 15 \equiv 0$ $15 + 16 \equiv 1$ $15 + 21 \equiv 6$	15
5	16	$16+14I$ $16+15I$ $16+20I$ $16+24I$ $16+29I$ $16+5I$ $16+9I$	$16+14 \equiv 0$ $16+15 \equiv 1$ $16+20 \equiv 6$ $16+24 \equiv 10$ $16+29 \equiv 15$ 16 + 5 = 21 16 + 9 = 25	16
6	21	$21+4I$ $21+9I$ $21+10I$ $21+15I$ $21+19I$ $21+24I$ $21+25I$	21 + 4 = 25 $21 + 9 \equiv 0$ $21 + 10 \equiv 1$ $21 + 15\equiv 6$ $21 + 19 \equiv 10$ $21 + 24 \equiv 15$ $21 + 25 \equiv 16$	21
7	25	$25+I$ $25+5I$ $25+6I$ $25+10I$ $25+16I$ $25+21I$ $25+26I$	$25 + 5 \equiv 0$ $25 + 6 \equiv 1$ $25 + 11 \equiv 6$ $25 + 15 \equiv 10$ $25 + 20 \equiv 15$ $25 + 21 \equiv 16$ $25 + 26 \equiv 21$	25

We see there are eight idempotents including 0 and 1. It is obvious that using 0 we get only idempotents or trivial semi-idempotents.

In view of all these, we conjecture the following.

Conjecture 1. *Let $S = \{\langle Z_n \cup I\rangle, +, \times\}$ be the neutrosophic ring $n = pqr$, where p, q and r are three distinct primes.*

1. *$Z_n = Z_{pqr}$ has only six non-trivial idempotents associated with it.*
2. *If m_1, m_2, m_3, m_4, m_5 and m_6 are the idempotents, then, associated with each real idempotent m_i, we have seven non-trivial neutrosophic idempotents associated with it, i.e. $\{m_i + n_j I, j = 1, 2, \ldots, 7\}$, such that $m_i + n_j \equiv t$, where t_j takes the seven distinct values from the set $\{0, 1, m_k, k \neq i; k = 1, 2, 3, \ldots 6\}$. $i = 1, 2, \ldots, 6$.*

This has been verified for large values of p, q and r, where p, q and r are three distinct primes.

3. Conjectures, Discussion and Conclusions

We have characterized the neutrosophic semi-idempotents in $\langle Z_p \cup I \rangle$, with p a prime. However, it is interesting to find neutrosophic semi-idempotents of $\langle Z_n \cup I \rangle$, with n a non-prime composite number. Here, we propose a few new open conjectures about idempotents in Z_n and semi-idempotents in $\langle Z_n \cup I \rangle$.

Conjecture 2. *Given $\langle Z_n \cup I \rangle$, where $n = p_1, p_2, \ldots p_t; t > 2$ and p_is are all distinct primes, find:*

1. *the number of idempotents in Z_n;*
2. *the number of idempotents in $\langle Z_n \cup I \rangle \setminus Z_n$;*
3. *the number of non-trivial semi-idempotents in Z_n; and*
4. *the number of non-trivial semi-idempotents in $\langle Z_n \cup I \rangle \setminus Z_n$.*

Conjecture 3. *Prove if $\langle Z_n \cup I \rangle$ and $\langle Z_m \cup I \rangle$ are two neutrosophic rings where $n > m$ and $n = p^t q$ ($t > 2$, and p and q two distinct primes) and $m = p_1 p_2 \ldots p_s$ where p_is are distinct primes. $1 \leq i \leq s$, then*

1. *prove Z_n has more number of idempotents than Z_m; and*
2. *prove $\langle Z_m \cup I \rangle$ has more number of idempotents and semi-idempotents than $\langle Z_n \cup I \rangle$.*

Finding idempotents in the case of Z_n has been discussed and problems are proposed in [18,23,24]. Further, the neutrosophic triplets in Z_n are contributed by Z_n. In the case of neutrosophic duplets, we see units in Z_n contribute to them. Both units and idempotents contribute in general to semi-idempotents.

Author Contributions: The contributions of the authors are roughly equal.

Funding: This research received no external funding.

Acknowledgments: The authors would like to thank the reviewers for their reading of the manuscript and many insightful comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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Neutrosophic Nano ideal topological structure

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Abstract: This paper addressed the concept of Neutrosophic nano ideal topology which is induced by the two literature, they are nano topology and ideal topological spaces. We defined its local function, closed set and also defined and give new dimnesion to codense ideal by incorporating it to ideal topological structures. we investigate some properties of neutrosophic nano topology with ideal.

Keywords: neutrosophic nano ideal, neutrosophic nano local function, topological ideal, neutrosophic nano topological ideal.

1 Introduction and Preliminaries

In 1983, K. Atanassov [1] proposed the concept of IFS(intuitionstic fuzzy set) which is a generalization of FS(fuzzy set) [17], where each element has true and false membership degree. Smarandache [15] coined the concept of NS (neutrosophic set) which is new dimension to the sets. Neutrosophic set is classified into three independently related functions namely, membership, indeterminacy function and non-membership function. Lellis Thivagar [8], introduced the new notion of neutrosophic nano topology, which consist of upper, lower approximation and boundary region of a subset of a universal set using an equivalence class on it. There have been wide range of studies on neutrosophic sets, ideals and nano ideals [9, 10, 11,12,13,14]. Kuratowski [7] and Vaidyanathaswamy [16] introduced the new concept in topological spaces, called ideal topological spaces and also local function in ideal topological space was defined by them. Afterwards the properties of ideal topological spaces studied by Hamlett and Jankovic[5,6].

In this paper, we introduce the new concept of neutrosophic nano ideal topological structures, which is a generalized concept of neutrosophic nano and ideal topological structure. Also defined the codense ideal in neutrosophic nano topological structure.
We recall some relevant basic definitions which are useful for the sequel and in particular, the work of M. L. Thivagar [8], Parimala et al [9], F. Smarandache [15].

Definition 1.1. Let U be universe of discourse and R be an indiscernibility relation on U. Then U is divided into disjoint equivalence classes. The pair (U, R) is said to be the approximation space. Let F be a NS in U with the true μ_F, the indeterminancy σ_F and the false function ν_F. Then,

(i) The lower approximation of F with respect to equivalence class R is the set denoted by $\overline{N}(F)$ and defined as follows
$\overline{N}(F) = \left\{ \left\langle a, \mu_{\overline{R}(F)}(a), \sigma_{\overline{R}(F)}(a), \nu_{\overline{R}(F)}(a) \right\rangle |y \in [a]_R, a \in U \right\}$

(ii) The higher approximation of F with respect to equivalence class R is the set is denoted by $\underline{N}(F)$ and defined as follows, $\underline{N}(F) = \left\{ \left\langle a, \mu_{\underline{R}(F)}(a), \sigma_{\underline{R}(F)}(a), \nu_{\underline{R}(F)}(a) \right\rangle |y \in [a]_R, a \in U \right\}$

(iii) The boundary region of F with respect to equivalence class R is the set of all objects is denoted by $B(F)$ and defined by$B(F) = \overline{N}(F) - \underline{N}(F)$.

where,

$$\mu_{\overline{R}(F)}(a) = \bigcup_{y_1 \in [a]_R} \mu_F(y_1), \ \sigma_{\overline{R}(F)}(a) = \bigcup_{y_1 \in [a]_R} \sigma_F(y_1),$$
$$\nu_{\overline{R}(F)}(a) = \bigcap_{y_1 \in [a]_R} \nu_F(y_1). \ \mu_{\underline{R}(F)}(a) = \bigcap_{y_1 \in [a]_R} \mu_F(y_1),$$
$$\sigma_{\underline{R}(F)}(a) = \bigcap_{y_1 \in [a]_R} \sigma_F(y_1), \ \nu_{\underline{R}(F)}(a) = \bigcap_{y_1 \in [a]_R} \nu_F(y_1).$$

Definition 1.2. Let U be a nonempty set and the neutrosophic sets X and Y in the form $X = \{\langle a, \mu_X(a), \sigma_X(a), \nu_X(a) \rangle,$
and $Y = \{\langle a, \mu_Y(a), \sigma_Y(a), \nu_Y(a) \rangle, a \in U\}$. Then the following statements hold:

(i) $0_N = \{\langle a, 0, 0, 1 \rangle, a \in U\}$ and $1_N = \{\langle a, 1, 1, 0 \rangle, a \in U\}$.

(ii) $X \subseteq y$ if and only if $\mu_X(a) \leq \mu_Y(a), \sigma_X(a) \leq \sigma_Y(a), \nu_X(a) \geq \nu_Y(a)$ for all $a \in U$.

(iii) $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

(iv) $X^C = \{\langle a, \nu_X(a), 1 - \sigma_X(a), \mu_X(a) \rangle, a \in U\}$.

(v) $X \cap Y$ if and only if $\mu_X(a) \wedge \mu_X(a), \sigma_X(a) \wedge \sigma_Y(a), \nu_Y(a) \vee \nu_Y(a)$ for all $a \in U$.

(vi) $X \cup Y$ if and only if $\mu_Y(a) \vee \mu_Y(a), \sigma_X(a) \vee \sigma_Y(a), \nu_X(a) \wedge \nu_Y(a)$ for all $a \in U$.

(vii) $X - Y$ if and only if $\mu_X(a) \wedge \nu_Y(a), \sigma_X(a) \wedge 1 - \sigma_Y(a), \nu_X(a) \vee \mu_Y(a)$ for all $a \in U$.

Definition 1.3. Let X be a non-empty set and I is a neutrosophic ideal (NI for short) on X if

(i) $A_1 \in I$ and $B_1 \subseteq A_1 \Rightarrow B_1 \in I$ [heredity],

(ii) $A_1 \in I$ and $B_1 \in I \Rightarrow A_1 \cup B_1 \in I$ [finite additivity].

2 Neutrosophic nano ideal topological spaces

In this section we introduce a new type of local function in neutrosophic nano topological space. Before that we shall consider the following concepts.

Neutrosophic nano ideal topological space(in short NNI) is denoted by $(U, \tau_{\mathcal{N}}(F), I)$, where $(U, \tau_{\mathcal{N}}(F), I)$ is a neutrosophic nano topological space(in short NNT) $(U, \tau_{\mathcal{N}}(F))$ with an ideal I on U

Definition 2.1. Let $(U, \tau_{\mathcal{N}}(F), I)$ be a NNI with an ideal I on U and $(.)^*_{\mathcal{N}}$ be a set of operator from $P(U)$ to $P(U) \times P(U)$ ($P(U)$ is the set of all subsets of U). For a subset $X \subset U$, the neutrosophic nano local function $X^*_{\mathcal{N}}(I, \tau_{\mathcal{N}}(F))$ of X is the union of all neutrosophic nano points (NNP, for short) $C(\alpha, \beta, \gamma)$ such that $X^*_{\mathcal{N}}(I, \tau_{\mathcal{N}}(F)) = \vee\{C(\alpha, \beta, \gamma) \in U : X \cap G \notin I \text{for all } G \in N(C(\alpha, \beta, \gamma))\}$. We will simply write $X^*_{\mathcal{N}}$ for $X^*_{\mathcal{N}}(I, \tau_{\mathcal{N}}(F))$.

Example 2.2. Let $(U, \tau_{\mathcal{N}}(F))$ be a neutrosophic nano topological space with an ideal I on U and for every $X \subseteq U$.

(i) If $I = \{0_\sim\}$, then $X^*_{\mathcal{N}} = \mathcal{N}cl(X)$,

(ii) If $I = P(U)$, then $X^*_{\mathcal{N}} = 0_\sim$.

Theorem 2.3. Let $(U, \tau_{\mathcal{N}}(F))$ be a NNT with ideals I, I' on U and X, B be subsets of U. Then

(i) $X \subseteq B \Rightarrow X^*_{\mathcal{N}} \subseteq B^*_{\mathcal{N}}$,

(ii) $I \subseteq I' \Rightarrow X^*_{\mathcal{N}}(I') \subseteq X^*_{\mathcal{N}}(I)$,

(iii) $X^*_{\mathcal{N}} = \mathcal{N}cl(X^*_{\mathcal{N}}) \subseteq \mathcal{N}cl(X)$ ($X^*_{\mathcal{N}}$ is a neutrosophic nano closed subset of $\mathcal{N}cl(X)$),

(iv) $(X^*_{\mathcal{N}})^*_{\mathcal{N}} \subseteq X^*_{\mathcal{N}}$,

(v) $X^*_{\mathcal{N}} \cup B^*_{\mathcal{N}} = (X \cup B)^*_{\mathcal{N}}$,

(vi) $X^*_{\mathcal{N}} - B^*_{\mathcal{N}} = (X - B)^*_{\mathcal{N}} - B^*_{\mathcal{N}} \subseteq (X - B)^*_{\mathcal{N}}$,

(vii) $V \in \tau_{\mathcal{N}}(F) \Rightarrow V \cap X^*_{\mathcal{N}} = V \cap (V \cap X)^*_{\mathcal{N}} \subseteq (V \cap X)^*_{\mathcal{N}}$ and

(viii) $J \in I \Rightarrow (X \cup J)^*_{\mathcal{N}} = X^*_{\mathcal{N}} = (X - J)^*_{\mathcal{N}}$.

Proof. (i) Let $X \subset B$ and $a \in X^*_{\mathcal{N}}$. Assume that $a \notin B^*_{\mathcal{N}}$. We have $G_{\mathcal{N}} \cap B \in I$ for some $G_{\mathcal{N}} \in G_{\mathcal{N}}(a)$. Since $G_{\mathcal{N}} \cap X \subseteq G_{\mathcal{N}} \cap B$ and $G_{\mathcal{N}} \cap B \in I$, we obtain $G_{\mathcal{N}} \cap X \in I$ from the definition of ideal. Thus, we have $a \notin X^*_{\mathcal{N}}$. This is a contradiction. Clearly, $X^*_{\mathcal{N}} \subseteq B^*_{\mathcal{N}}$.
(ii) Let $I \subseteq I'$ and $a \in X^*_{\mathcal{N}}(I')$. Then we have $G_{\mathcal{N}} \cap X \notin I'$ for every $G_{\mathcal{N}} \in G_{\mathcal{N}}(a)$. By hypothesis, we obtain $G_{\mathcal{N}} \cap X \notin I$. So $a \in X^*_{\mathcal{N}}(I)$.
(iii) Let $a \in X^*_{\mathcal{N}}$. Then for every $G_{\mathcal{N}} \in G_{\mathcal{N}}(a), G_{\mathcal{N}} \cap X \notin I$. This implies that $G_{\mathcal{N}} \cap X \neq 0_\sim$. Hence

$a \in \mathcal{N}cl(X)$.
(iv) From (iii), $(X^*_{\mathcal{N}})^*_{\mathcal{N}} \subseteq \mathcal{N}cl(X^*_{\mathcal{N}}) = X^*_{\mathcal{N}}$, since $X^*_{\mathcal{N}}$ is a neutrosophic nano closed set.
The proofs of the other conditions are also obvious.
Theorem 2.4. If $(U, \tau_{\mathcal{N}}(F), I)$ is a NNT with an ideal I and $X \subseteq X^*_{\mathcal{N}}$, then $X^*_{\mathcal{N}} = \mathcal{N}cl(X^*_{\mathcal{N}}) = \mathcal{N}cl(X)$.
Proof. For every subset X of U, we have $X^*_{\mathcal{N}} = \mathcal{N}cl(X^*) \subseteq \mathcal{N}cl(X)$, by Theorem 2.3. (iii) $X \subseteq X^*_{\mathcal{N}}$ implies that $\mathcal{N}cl(X) \subseteq Ncl(X^*_{\mathcal{N}})$ and so $X^*_{\mathcal{N}} = \mathcal{N}cl(X^*_{\mathcal{N}}) = \mathcal{N}cl(X)$.

Definition 2.5. Let $(U, \tau_{\mathcal{N}}(F))$ be a NNT with an ideal I on U. The set operator $\mathcal{N}cl^*$ is called a neutrosophic nano$*$-closure and is defined as $\mathcal{N}cl^*(X) = X \cup X^*_{\mathcal{N}}$ for $X \subseteq a$.

Theorem 2.6. The set operator $\mathcal{N}cl^*$ satisfies the following conditions:

(i) $X \subseteq \mathcal{N}cl^*(X)$,

(ii) $\mathcal{N}cl^*(0_\sim) = 0_\sim$ and $\mathcal{N}cl^*(1_\sim) = 1_\sim$,

(iii) If $X \subset B$, then $\mathcal{N}cl^*(X) \subseteq \mathcal{N}cl^*(B)$,

(iv) $\mathcal{N}cl^*(X) \cup \mathcal{N}cl^*(B) = \mathcal{N}cl^*(X \cup B)$.

(v) $\mathcal{N}cl^*(\mathcal{N}cl^*(X)) = \mathcal{N}cl^*(X)$.

Proof. The proofs are clear from Theorem 2.3 and the definition of $\mathcal{N}cl^*$.

Now, $\tau_{\mathcal{N}}(F)^*(I, \tau_{\mathcal{N}}(F)) = \{V \subset U : \mathcal{N}cl^*(U - V) = U - V\}$. $\tau_{\mathcal{N}}(F)^*(I, \tau_{\mathcal{N}}(F))$ is called neutrosophic nano$*$-topology which is finer than $\tau_{\mathcal{N}}(F)$ (we simply write $\tau_{\mathcal{N}}(F)^*$ for $\tau_{\mathcal{N}}(F)^*(I, \tau_{\mathcal{N}}(F))$). The elements of $\tau_{\mathcal{N}}(F)^*(I, \tau_{\mathcal{N}}(F))$ are called neutrosophic nano$*$-open (briefly, $\mathcal{N}*$-open) and the complement of an $\mathcal{N}*$-open set is called neutrosophic nano$*$-closed (briefly, $\mathcal{N}*$-closed). Here $\mathcal{N}cl^*(X)$ and $\mathcal{N}int^*(X)$ will denote the closure and interior of X respectively in $(U, \tau_{\mathcal{N}}(F)^*)$.
Remark 2.7. (i) We know from Example 2.2 that if $I = \{0_\sim\}$ then $X^*_{\mathcal{N}} = \mathcal{N}cl(X)$. In this case, $\mathcal{N}cl^*(X) = \mathcal{N}cl(X)$.
(ii) If $(U, \tau_{\mathcal{N}}(F), I)$ is a NNI with $I = \{0_\sim\}$, then $\tau_{\mathcal{N}}(F)^* = \tau_{\mathcal{N}}(F)$.
Definition 2.8. A basis $\beta(I, \tau_{\mathcal{N}}(F))$ for $\tau_{\mathcal{N}}(F)^*$ can be described as follows:
$\beta(I, \tau_{\mathcal{N}}(F)) = \{X - B : X \in \tau_{\mathcal{N}}(F), B \in I\}$.
Theorem 2.9. Let $(U, \tau_{\mathcal{N}}(F))$ be a NNT and I be an ideal on U. Then $\beta(I, \tau_{\mathcal{N}}(F))$ is a basis for $\tau_{\mathcal{N}}(F)^*$.

Proof. We have to show that for a given space $(U, \tau_{\mathcal{N}}(F))$ and an ideal I on U, $\beta(I, \tau_{\mathcal{N}}(F))$ is a basis for $\tau_{\mathcal{N}}(F)^*$. If $\beta(I, \tau_{\mathcal{N}}(F))$ is itself a neutrosophic nano topology, then we have $\beta(I, \tau_{\mathcal{N}}(F)) = \tau_{\mathcal{N}}(F)^*$ and all the open sets of $\tau_{\mathcal{N}}(F)^*$ are of simple form $X - B$ where $X \in \tau_{\mathcal{N}}(F)$ and $B \in I$.
Theorem 2.10. Let $(U, \tau_{\mathcal{N}}(F), I)$ be a NNT with an ideal I on U and $X \subseteq U$. If $X \subseteq X^*_{\mathcal{N}}$, then

(i) $\mathcal{N}cl(X) = \mathcal{N}cl^*(X)$,

(ii) $\mathcal{N}int(U - X) = \mathcal{N}int^*(U - X)$.

Proof. (i) Follows immediately from Theorem 2.4.
(ii) If $X \subseteq X^*_{\mathcal{N}}$, then $\mathcal{N}cl(X) = \mathcal{N}cl^*(X)$ by (i) and so $U - \mathcal{N}cl(X) = U - \mathcal{N}cl^*(X)$. Therefore, $\mathcal{N}int(U - X) = \mathcal{N}int^*(U - X)$.

Theorem 2.11. Let $(U, \tau_{\mathcal{N}}(F), I)$ be a NNT with an ideal I on U and $X \subseteq X$. If $X \subseteq X^*_{\mathcal{N}}$, then $X^*_{\mathcal{N}} = \mathcal{N}cl(X^*_{\mathcal{N}}) = n\text{-}cl(X) = \mathcal{N}cl^*(X)$.

Definition 2.12. A subset A of a neutrosophic nano ideal topological space $(U, \tau_{\mathcal{N}}(F), I)$ is $\mathcal{N}*$-dense in itself (resp. $\mathcal{N}*$-perfect) if $X \subseteq X^*_{\mathcal{N}}$ (resp. $X = X^*_{\mathcal{N}}$).

Remark 2.13. A subset X of a neutrosophic nano ideal topological space $(U, \tau_{\mathcal{N}}(F), I)$ is $\mathcal{N}^*$-closed if and only if $X^*_{\mathcal{N}} \subseteq X$.

For the relationship related to several sets defined in this paper, we have the following implication:

$$\mathcal{N}*\text{-dense in itself} \Leftarrow \mathcal{N}*\text{-perfect} \Rightarrow \mathcal{N}^*\text{-closed}$$

The converse implication are not satisfied asthe following shows.

Example 2.14. Let U be the universe, $X = \{P_1, P_2, P_3, P_4, P_5\} \subset U$, $U/R = \{\{P_1, P_2\}, \{P_3\}, \{P_4, P_5\}\}$ and $\tau_{\mathcal{N}}(F) = \{1_\sim, 0_\sim, \overline{\mathcal{N}}, \underline{\mathcal{N}}, B\}$ and the ideal $I = 0_\sim, 1_\sim$. For $X = \{< P_1, (.5, .4, .7) >, < P_2, (.6, .4, .5) >, < P_3, (.4, .5, .4) >, < P_4, (.7, .3, .4) >, < P_5, (.8, .5, .2) >\}$, $\underline{N}(X) = \{\frac{P_1,P_2}{.5,.4,.7}, \frac{P_3}{.4,.5,.4}, \frac{P_4,P_5}{.7,.3,.4}\}$,
$\overline{N}(X) = \{\frac{P_1,P_2}{.6,.4,.5}, \frac{P_3}{.4,.5,.4}, \frac{P_4,P_5}{.8,.5,.2}\}$, $B(X) = \{\frac{P_1,P_2}{.6,.4,.5}, \frac{P_3}{.4,.5,.4}, \frac{P_4,P_5}{.4,.3,.7}\}$. If $I = 0_\sim$ then $X^*_{\mathcal{N}} = Ncl(a)$. Thus $X \subseteq X^*_{\mathcal{N}}$. Hence X is $\mathcal{N}^*$-dense but not $\mathcal{N}^*$-perfect.
If $I = 1_\sim$ then $X^*_{\mathcal{N}} = 0_\sim$. Thus $X \supseteq X^*_{\mathcal{N}}$. Hence $X^*_{\mathcal{N}}$ is $\mathcal{N}^*$-closed but not $\mathcal{N}^*$-perfect.

Lemma 2.15. Let $(U, \tau_{\mathcal{N}}(F), I)$ be a NNI and $X \subseteq U$. If X is $\mathcal{N}*$-dense in itself, then $X^*_{\mathcal{N}} = \mathcal{N}cl(X^*_{\mathcal{N}}) = \mathcal{N}cl(X) = \mathcal{N}cl^*(X)$.
Proof. Let X be $\mathcal{N}*$-dense in itself. Then we have $X \subseteq X^*_{\mathcal{N}}$ and using Theorem 2.11 we get$X^*_{\mathcal{N}} = \mathcal{N}cl(X^*_{\mathcal{N}}) = \mathcal{N}cl(X) = \mathcal{N}cl^*(X)$.

Lemma 2.16. If $(U, \tau_{\mathcal{N}}(F), I)$ is a NNT with an ideal I and $X \subseteq U$, then $X^*_{\mathcal{N}}(I, \tau_{\mathcal{N}}(F)) = X^*_{\mathcal{N}}(I, \tau_{\mathcal{N}}(F)^*)$ and hence $\tau_{\mathcal{N}}(F)^* = \tau_{\mathcal{N}}(F)^{**}$.

3 $\tau_{\mathcal{N}}(F)$-codence ideal

n this section we incorporated codence ideal [5] in ideal topological space and introduce similar concept in neutrosophic nano ideal topological spaces.

Definition 3.1. An ideal I in a space $(U, \tau_{\mathcal{N}}(F), I)$ is called $\tau_{\mathcal{N}}(F)$-codense ideal if $\tau_{\mathcal{N}}(F) \cap I = \{0_\sim\}$.
Following theorems are related to $\tau_{\mathcal{N}}(F)$-codense ideal.

Theorem 3.2. Let $(U, \tau_{\mathcal{N}}(F), I)$ be an NNI and I is $\tau_{\mathcal{N}}(F)$-codense with $\tau_{\mathcal{N}}(F)$. Then $U = U^*_{\mathcal{N}}$.
Proof. It is obvious that $U^*_{\mathcal{N}} \subseteq U$. For converse, suppose $a \in U$ but $a \notin U^*_{\mathcal{N}}$. Then there exists $G_x \in \tau_{\mathcal{N}}(F)(a)$ such that $G_x \cap U \in I$. That is $G_x \in I$, a contradiction to the fact that $\tau_{\mathcal{N}}(F) \cap I = \{0_\sim\}$. Hence $U = U^*_{\mathcal{N}}$.

Theorem 3.3. Let $(U, \tau_{\mathcal{N}}(F), I)$ be a NNI. Then the following conditions are equivalent:

(i) $U = U^*_{\mathcal{N}}$.

(ii) $\tau_{\mathcal{N}}(F) \cap I = \{0_{\sim}\}$.

(iii) If $J \in I$, then $\mathcal{N}int(J) = 0_{\sim}$.

(iv) For every $X \in \tau_{\mathcal{N}}(F)$, $X \subseteq X^*_{\mathcal{N}}$.

Proof. By Lemma 2.16, we may replace '$\tau_{\mathcal{N}}(F)$' by '$\tau_{\mathcal{N}}(F)^*$' in (ii), '$\mathcal{N}int(J) = 0_{\sim}$' by '$\mathcal{N}int^*(J) = 0_{\sim}$' in (iii) and '$X \in \tau_{\mathcal{N}}(F)$' by '$X \in \tau_{\mathcal{N}}(F)^*$' in (iv).

4 Conclusions

this paper, we introduced the notion of neutrosophic nano ideal topological structures and investigated some relations over neutrosophic nano topology and neutrosophic nano ideal topological structures and studied some of its basic properties. In future, it motivates to apply this concepts in graph structures.

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Separation Axioms in Neutrosophic Crisp Topological Spaces

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Ahmed B. Al-Nafee, Riad K. Al-Hamido, Florentin Smarandache (2019). Separation Axioms in Neutrosophic Crisp Topological Spaces. *Neutrosophic Sets and Systems* 25, 25-32

Abstract.The main idea of this research is to define a new neutrosophic crisp points in neutrosophic crisp topological space namely [NCP_N] ,the concept of neutrosophic crisp limit point was defind using [NCP_N],with some of its properties, the separation axioms [N-$\mathcal{T}_i$-space,i= 0,1,2] were constructed in neutrosophic crisp topological space using [NCP_N] and examine the relationship between them in details.

Keywords: Neutrosophic crisp topological spaces, neutrosophic crisp limit point, separation axioms.

Introduction

Smarandache [1,2,3] introduced the notions of neutrosophic theory and introduced the neutrosophic. components (T,I,F) which represent the membership , indeterminacy , and non membership values respectively, where $]-0,1^+[$ is a non standard unit interval . In [4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19 ,20] many scientists presented the concepts of the neutrosophic set theory in their works. Salama et al. [21,22] provided natural foundations to put mathematical treatments for the neutrosophic pervasively phenomena in our real world and for building new branches of neutrosophic mathematics .

Salama et al [23,24] put some basic concepts of the neutrosophic crisp set and their operations, and because of their wide applications and their grate flexibility to solve the problem , we used these concepts to define new types of neutrosophic points, that we called neutrosophic crisp points [NCP_N] .

Fainally ,we used these points [NCP_N] to define the concept of neutrosophic crisp limit point, with some of its properties and constructe the separation axioms[N-$\mathcal{T}_i$-space,i=0,1,2] in neutrosophic crisp topological and examine the relationship between them in details.

Throughout this paper,(NCTS) means a neutrosophic crisp topological space. Also, simply we denote neighborhood by (nhd).

1 Basic Concepts

1.1 Definition [25]

Let $\mathcal{X}$ be a non-empty fixed set . A neutrosophic crisp set [NCS for short] B is an object having the form $B = < B_1, B_2, B_3>$ where B_1, B_2 and B_3 are subsets of $\mathcal{X}$.

1.2 Definition [25]

The object having the form $B=<B_1 , B_2 , B_3 >$ is called :

1. A neutrosophic crisp set of Type1 [NCS/Type1] if satisfying
$B_1 \cap B_2 = \emptyset, B_1 \cap B_3 = \emptyset$ and $B_2 \cap B_3 = \emptyset$.
2. A neutrosophic crisp set of Type2 [NCS/Type2] if satisfying
$B_1 \cap B_2 = \emptyset, B_1 \cap B_3 = \emptyset$ and $B_2 \cap B_3 = \emptyset$, $B_1 \cup B_2 \cup B_3 = \mathcal{X}$.
3. A neutrosophic crisp set of Type3 [NCS/Type3] if satisfying
$B_1 \cap B_2 \cap B_3 = \emptyset$, $B_1 \cup B_2 \cup B_3 = \mathcal{X}$

1.3 Definition [25]

Types of NCSs $\emptyset_N$ & $\mathcal{X}_N$ in $\mathcal{X}$ as follows :

1 . $\emptyset_N$ may be defined in many ways as a NCS as follows :
   1. Type1 : $\emptyset_N = <\varphi , \varphi , \mathcal{X} >$
   2. Type2 : $\emptyset_N = <\varphi , \mathcal{X} , \mathcal{X} >$
   3. Type3 : $\emptyset_N = <\varphi , \mathcal{X}, \varphi >$
   4. Type4 : $\emptyset_N = <\varphi , \varphi , \varphi >$

2 . $\mathcal{X}_N$ may be defined in many ways as a NCS as follows :
   1. Type1: $\mathcal{X}_N = < \mathcal{X}, \varphi , \varphi >$

2. Type2: $\mathcal{X}_N = < \mathcal{X}, \mathcal{X}, \varphi >$
3. Type3: $\mathcal{X}_N = < \mathcal{X}, \varphi, \mathcal{X} >$
4. Type4: $\mathcal{X}_N = < \mathcal{X}, \mathcal{X}, \mathcal{X} >$

1.4 Definition [25]

Let $\mathcal{X}$ be a non-empty set and the NCSs C & D in the form $C = <C_1,C_2,C_3>$, $D = <D_1,D_2,D_3>$ then we may consider two possible definitions for subsets $C \subseteq D$, may be defined in two ways :

1. $C \subseteq D \Leftrightarrow C_1 \subseteq D_1$, $C_2 \subseteq D_2$ and $D_3 \subseteq C_3$
2. $C \subseteq D \Leftrightarrow C_1 \subseteq D_1$, $D_2 \subseteq C_2$ and $D_3 \subseteq C_3$

1.5 Definition [25]

Let $\mathcal{X}$ be a non-empty set and the NCSs C & D in the form $C = <C_1,C_2,C_3>$, $D = <D_1,D_2,D_3>$ then :

1. $C \cap D$ may be defined in two ways as a NCS as follows :
   - $C \cap D = [C_1 \cap D_1] , [C_2 \cup D_2] , [C_3 \cup D_3]$
   - $C \cap D = [C_1 \cap D_1] , [C_2 \cap D_2] , [C_3 \cup D_3]$
2. $C \cup D$ may be defined in two ways as a NCS as follows :
   - $C \cup D = [C_1 \cup D_1] , [C_2 \cup D_2] , [C_3 \cap D_3]$
   - $C \cup D = [C_1 \cup D_1] , [C_2 \cap D_2] , [C_3 \cap D_3]$

1.6 Definition [25]

A neutrosophic crisp topology (NCT) on a non-empty set χ is a family $\mathcal{T}$ of neutrosophic crisp subsets in $\mathcal{X}$ satisfying the following axioms :

1. $\emptyset_N , \mathcal{X}_N \in \mathcal{T}$
2. $C \cap D \in \mathcal{T}$, for any $C, D \in \mathcal{T}$
3. The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$

The pair $(\mathcal{X},\mathcal{T})$ is said to be a neutrosophic crisp topological space (NCTS) in $\mathcal{X}$. Moreover' The elements in $\mathcal{T}$ are said to be neutrosophic crisp open sets (NCOS), a neutrosophic crisp set F is closed (NCCS) iff its complement F^C is an open neutrosophic crisp set.

1.7 Definition [25]

Let $\mathcal{X}$ be a non-empty set and the NCS D in the form $D = <D_1,D_2,D_3>$ .Then D^c may be defined in three ways as a NCS as follows :

$D^c = < D_1^c, D_2^c, D_3^c>$, $D^c = < D_3,D_2,D_1>$ or $D^c = < D_3,D_2^c,D_1>$

1.8 Definition [25]

Let $(\mathcal{X},\mathcal{T})$ be neutrosophic crisp topological space (NCTS). A be neutrosophic crisp set then: The intersection of any neutrosophic crisp closed sets contained, A is called neutrosophic crisp closure of A (NC-Cl(A) for short) .

2 Neutrosophic crisp limit point :

In this section, we will introduce the neutrosophic crisp limit points with some of its properties. This work contains an adjustment for the above-mentioned definitions 1.4 & 1.5, this was necessary to homogeneous suitable results for the upgrade of this research.

2.1 Definition

Let $\mathcal{X}$ be a non-empty set and the NCSs C & D in the form $C = <C_1,C_2,C_3>$, $D = <D_1,D_2,D_3>$ then the additional new ways for the intersection , union and inclusion between C & D are

$C \cap D = [C_1 \cap D_1] , [C_2 \cap D_2] , [C_3 \cap D_3]$
$C \cup D = [C_1 \cup D_1] , [C_2 \cup D_2] , [C_3 \cup D_3]$
$C \subseteq D \Leftrightarrow C_1 \subseteq D_1$, $C_2 \subseteq D_2$ and $C_3 \subseteq D_3$

2.2 Definition

For all x, y, z belonging to a non-empty set $\mathcal{X}$. Then the neutrosophic crisp points related to x, y, z are defined as follows:

- $x_{N_1} = <\{x\}, \emptyset, \emptyset>$, is called a neutrosophic crisp point (NCP_{N_1}) in $\mathcal{X}$.
- $y_{N_2} = <\emptyset, \{y\}, \emptyset>$, is called a neutrosophic crisp point (NCP_{N_2}) in $\mathcal{X}$.
- $z_{N_3} = <\emptyset, \emptyset, \{z\}>$, is called a neutrosophic crisp point (NCP_{N_3}) in $\mathcal{X}$.

The set of all neutrosophic crisp points ($NCP_{N_1}, NCP_{N_2}, NCP_{N_3}$) is denoted by NCP_N.

2.3 Definition

Let $\mathcal{X}$ be to a non-empty set and x ,y, z $\in \mathcal{X}$. Then the neutrosophic crisp point:

- x_{N_1} is belonging to the neutrosophic crisp set B=<B_1,B_2,B_3>, denoted by $x_{N_1} \in B$, if $x \in B_1$,wherein x_{N_1} does not belong to the neutrosophic crisp set B denoted by $x_{N_1} \notin B$, if $x \notin B_1$.
- y_{N_2} is belonging to the neutrosophic crisp set B=<B_1,B_2,B_3>, denoted by $y_{N_2} \in B$, if $y \in B_2$. In contrast y_{N_2} does not belong to the neutrosophic crisp set B, denoted by $y_{N_2} \notin B$, if $y \notin B_2$.
- z_{N_3} is belonging to the neutrosophic crisp set B=<B_1,B_2,B_3>, denoted by $z_{N_3} \in B$, if $z \in B_3$. In contrast z_{N_3} does not belong to the neutrosophic crisp set B ,denoted by $z_{N_3} \notin B$, if $z \notin B_3$.

2.4 Remark

If B = <B_1,B_2,B_3> is a NCS in a non-empty set $\mathcal{X}$ then:

$B\backslash x_{N_1} =< B_1\backslash\{x\}, B_2, B_3 >$. $B\backslash x_{N_1}$ means that the component B doesn't contain x_{N_1}.

$B\backslash y_{N_2} =< B_1, B_2\backslash\{y\}, B_3 >$. $B\backslash y_{N_2}$ means that the component B doesn't contain y_{N_2}.

$B\backslash z_{N_3} =< B_1, B_2, B_3\backslash\{z\} >$. $B\backslash z_{N_3}$ means that the component B doesn't contain z_{N_3}.

2.5 Example

If B = $<\{a, b\}, \{c, b\}, \{c, a\}>$ is an NCS in $\mathcal{X} = \{a, b, c\}$, then:

$B\backslash a_{N_1} = <\{ \; b\}, \{c, b\}, \{c, a\}>$

$B\backslash b_{N_2} = <\{a, b\}, \{c\}, \{c, a\}>$

$B\backslash c_{N_3} = <\{a, b\}, \{c, b\}, \{ \; b\}>$

2.6 Remark

If B = <B_1,B_2,B_3> is a NCS in a non-empty set $\mathcal{X}$ then:

$B = (\cup\{x_{N_1} : x_{N_1} \in B\}) \cup (\cup\{y_{N_2} : y_{N_2} \in B\}) \cup (\cap\{z_{N_3} : z_{N_3} \in B\})$

$= (\cup\{<\{x\}, \emptyset, \emptyset> : x \in \mathcal{X}\}) \cup (\cup\{<\emptyset, \{y\}, \emptyset> : y \in \mathcal{X}\}) \cup (\cap\{<\emptyset, \emptyset, \{z\}> : z \in \mathcal{X}\})$

or $B = (\cup\{x_{N_1} : x_{N_1} \in B\}) \cup (\cup\{y_{N_2} : y_{N_2} \in B\}) \cup (\cup\{z_{N_3} : z_{N_3} \in B\})$

$= (\cup\{<\{x\}, \emptyset, \emptyset> : x \in \mathcal{X}\}) \cup (\cup\{<\emptyset, \{y\}, \emptyset> : y \in \mathcal{X}\}) \cup (\cup\{<\emptyset, \emptyset, \{z\}> : z \in \mathcal{X}\})$.

2.7 Definition

Let $(\mathcal{X}, \mathcal{T})$ be NCTS , $P \in NCP_N$ in $\mathcal{X}$, a neutrosophic crisp set B = <B_1,B_2,B_3> $\in \mathcal{T}$ is called neutrosophic crisp open nhd of P in $(\mathcal{X}, \mathcal{T})$ if $P \in B$.

2.8 Definition

Let $(\mathcal{X}, \mathcal{T})$ be NCTS , $P \in NCP_N$ in $\mathcal{X}$, a neutrosophic crisp set B = < B_1 , B_2 , B_3 > $\in \mathcal{T}$ is called neutrosophic crisp nhd of P in $(\mathcal{X}, \mathcal{T})$, if there is neutrosophic crisp open set A =< A_1, A_2, A_3 > containing P such that $A \subseteq B$.

2.9 Note

Every neutrosophic crisp open nhd of any point $P \in NCP_N$ in $\mathcal{X}$ is neutrosophic crisp nhd of P, but in general the inverse is not true , the following example illustrates this fact.

2.10 Example

If $\mathcal{X} = \{x, y, z\}$, $\mathcal{T} = \{\mathcal{X}_N, \emptyset_N, A, B, C\}$,

$A = <\{x\}, \emptyset, \emptyset>$, $B = <\{y\}, \emptyset, \emptyset>$, $G = <\{x, y\}, \emptyset, \emptyset>$

If we take $U = <\{x, y\}, \{z\}, \emptyset>$.

Then $G = <\{x, y\}, \emptyset, \emptyset>$ is an open set containing $P = x_{N_1} = <\{x\}, \emptyset, \emptyset>$ and $G \subseteq U$. That is U is a neutrosophic crisp nhd of P in $(\mathcal{X}, \mathcal{T})$, while it is not a neutrosophic crisp open nhd of P .

2.11 Definition

Let $(\mathcal{X},\mathcal{T})$ be NCTS and $B = < B_1, B_2, B_3 >$ be NCS of $\mathcal{X}$. A neutrosophic crisp point $P \in NCP_N$ in $\mathcal{X}$ is called a neutrosophic crisp limit point of $B = < B_1, B_2, B_3 >$ iff every neutrosophic crisp open set containing P must contains at least one neutrosophic crisp point of B different from P . It is easy to say that the point P is not neutrosophic crisp limit point of B if there is a neutrosophic crisp open set G of P and $B \cap (G\setminus P) = \emptyset_N$.

2.12 Definition

The set of all neutrosophic crisp limit points of a neutrosophic crisp set B is called neutrosophic crisp derived set of B , denoted by NCD(B) .

2.13 Example

If $\mathcal{X} = \{x, y, z\}$, $\mathcal{T} = \{\mathcal{X}_N, \emptyset_N, A, B, C\}$, $A = < \{x\}, \emptyset, \emptyset >$, $B = < \{y\}, \emptyset, \emptyset >$, $G = < \{x, y\}, \emptyset, \emptyset >$. If we take $D = < \{x, y\}, \emptyset, \emptyset >$, Then $P = Z_{N_1} = < \{Z\}, \emptyset, \emptyset >$ is the only neutrosophic crisp limit point of D. i.e. $NCD(D) = \{Z_{N_1}\}$

2.14 Remarks

- Let B be any neutrosophic crisp set of $\mathcal{X}$, If $P = < \{x\}, \emptyset, \emptyset > \in \mathcal{T}$ in any NCT space $(\mathcal{X},\mathcal{T})$, then $P \in NCD(B)$.
- Let B be any neutrosophic crisp set of $\mathcal{X}$, the following facts is true:
  $NCD(B) \not\subset B$, $B \not\subset NCD(B)$, and sometimes $NCD(B) \cap B = \emptyset_N$ or $NCD(B) \cap B \neq \emptyset_N$.
- In any NCT space $(\mathcal{X},\mathcal{T})$, we have $NCD(\emptyset) = \emptyset_N$.

2.15 Theorem

Let $(\mathcal{X},\mathcal{T})$ be NCTS and $B = < B_1, B_2, B_3 >$ be a neutrosophic crisp set of $\mathcal{X}$,then B is neutrosophic crisp closed set (NCCS for short) iff $NCD(B) \subseteq B$

Proof

Let B be NCCS, then $(\mathcal{X}\setminus B)$ is neutrosophic crisp open set (NCOS for short) this implies that for each neutrosophic crisp point $P \in NCP_N$ in $(\mathcal{X}\setminus B)$, $P \notin B$, there is a neutrosophic crisp open set G of P and $G \subseteq (\mathcal{X}\setminus B)$.

Since $B \cap (\mathcal{X}\setminus B) = \emptyset_N$, then P is not neutrosophic crisp limit point of B, thus $G \cap B = \emptyset_N$,which implies that $P \notin NCD(B)$.Hence $NCD(B) \subseteq B$

Conversely , assume that $P \notin NCD(B)$, implis that P is not neutrosophic crisp limit point of B, hence, there is a neutrosophic crisp open set G of P and $G \cap B = \emptyset_N$ which means that $G \subseteq (\mathcal{X}\setminus B)$ and since $(\mathcal{X}\setminus B)$ is a neutrosophic crisp open set . Hence B is neutrosophic crisp closed set . .

2.16 Theorem

Let $(\mathcal{X},\mathcal{T})$ be NCTS , B, G be a neutrosophic crisp sets of $\mathcal{X}$, then the following properties hold :

(1) $NCD(\emptyset_N) = \emptyset_N$

(2) If $B \subseteq G$, then $NCD(B) \subseteq NCD(G)$

(3) $NCD(B \cap G) \subseteq NCD(B) \cap NCD(G)$

(4) $NCD(B \cup G) = NCD(B) \cup NCD(G)$

Proof (1) the proof is, directly.

Proof (2)

Assume that NCD(B) be a neutrosophic crisp set containing a neutrosophic crisp point $P \in NCP_N$, then by definition 2.11, for each neutrosophic crisp open set V of P , we have $B \cap V\setminus P \neq \emptyset_N$,but $B \subseteq G$, hence $G \cap V\setminus P \neq \emptyset_N$, this means that $P \in NCD(G)$. Hence , $NCD(B) \subseteq NCD(G)$

Proof (3)

Since $B \cap G \subseteq B$, then by (2) $NCD(B \cap G) \subseteq NCD(B)$ (1)

$B \cap G \subseteq G$, implies $NCD(B \cap G) \subseteq NCD(G)$ (2)

From (1) & (2) $NCD(B \cap G) \subseteq NCD(B) \cap NCD(G)$

Proof (4)

Let $P \in NCP_N$ such that $P \notin NCD(B) \cup NCD(G)$, then either $P \notin NCD(B)$ and $P \notin NCD(G)$,then there is a neutrosophic crisp open set K of P and $B \cap K\setminus P = \emptyset_N$ and $G \cap K\setminus P = \emptyset_N$, this implies that $(B \cup G) \cap K\setminus P = \emptyset_N$, i.e $P \notin NCD(B \cup G)$, hence $NCD(B \cup G) \subseteq NCD(B) \cup NCD(G)$ (3)

Conversely, since $B \subseteq B \cup G$,$G \subseteq B \cup G$,then by property (2) $NCD(B) \subseteq NCD(B \cup G)$ and $NCD(G) \subseteq NCD(B \cup G)$, thus $NCD(B \cup G) \supseteq NCD(B) \cup NCD(G)$ (4)
from (3) and (4) we have $NCD(B \cup G) = NCD(B) \cup NCD(G)$.

2.17 Remark

In general, the inverse of property 2 & 3 in Th.(2.16) is not true. The following examples act as an evidence to this claim.

2.18 Example

If $\mathcal{X} = \{x, y, z\}$, $\mathcal{T} = \{\mathcal{X}_N, \emptyset_N, B\}$, $B = <\emptyset,\{x\},\emptyset>$. If we take $A = <\emptyset,\{x\},\emptyset>$, $G = <\emptyset,\{y\},\emptyset>$ Notes that; $NCD(A) = <\emptyset,\{y, z\},\emptyset>$, $NCD(G) = <\emptyset,\{y, z\},\emptyset>$ and $NCD(A) \subseteq NCD(G)$, but $A \not\subset G$.

2.19 Example

If $\mathcal{X} = \{x, y, z\}$, $\mathcal{T} = \{\mathcal{X}_N, \emptyset_N, B\}$, $B = <\emptyset,\{x\},\emptyset>$. If we take $A = <\emptyset,\{x\},\emptyset>$, $G = <\emptyset,\{y\},\emptyset>$.
Notes that; $NCD(B \cap G) \not\supset NCD(B) \cap NCD(G)$.

2.20 Theorem

For any neutrosophic crisp set B over the universe $\mathcal{X}$, then NC-Cl(B) = $B \cup NCD(B)$

Proof

Let us first prove that $B \cup NCD(B)$ is a neutrosophic crisp closed set that is
$\mathcal{X}_N \setminus (B \cup NCD(B)) = (\mathcal{X}_N \setminus B) \cap (\mathcal{X}_N \setminus NCD(B))$ is a neutrosophic crisp open set .
Now for a neutrosophic crisp point $P \in (\mathcal{X}_N \setminus (B)) \cap (\mathcal{X}_N \setminus NCD(B))$, then $P \in (\mathcal{X}_N \setminus (B))$ and $P \in \mathcal{X}_N \setminus NCD(B)$, thus $P \notin B$ and $P \notin NCD(B)$. So by definition 2.12, there is a neutrosophic crisp set R of P S.t $R \cap B = \emptyset_N$, hence $R \subseteq \mathcal{X}_N \setminus B$.
Now for each $P_1 \in R$, then $P_1 \notin NCD(B)$, then $R \cap NCD(B) = \emptyset_N$, this implies that $R \subseteq \mathcal{X}_N \setminus NCD(B)$ [i.e $R \subseteq (\mathcal{X}_N \setminus B) \cap (\mathcal{X}_N \setminus NCD(B))$] .Thus $(\mathcal{X}_N \setminus B) \cap (\mathcal{X}_N \setminus NCD(B))$ is a neutrosophic crisp nhd of all its elements and hence $(\mathcal{X}_N \setminus B) \cap (\mathcal{X}_N \setminus NCD(B))$ is a neutrosophic crisp open set and thus $B \cup NCD(B)$ is a neutrosophic crisp closed set containing B, therefore NC-Cl(B) $\subseteq B \cup NCD(B)$.
S ince NC-Cl(B) is a neutrosophic crisp closed set (see definition 2.12) and NC-Cl(B) contains all its neutrosophic crisp limits points .Thus $NCD(B) \subseteq$ NC-Cl(B) and $B \subseteq$ NC-Cl(B), hence NC-Cl(B) $= B \cup NCD(B)$.

3 Separation Axioms In a neutrosophic Crisp Topological Space

3.1 Definition

A neutrosophic crisp topological space $(\mathcal{X}, \mathcal{T})$ is called:

- N_1-$\mathcal{T}_o$-space if $\forall\, x_{N_1} \neq y_{N_1} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open set G in $\mathcal{X}$ containing one of them but not the other.
- N_2-$\mathcal{T}_o$-space if $\forall\, x_{N_2} \neq y_{N_2} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open set G in $\mathcal{X}$ containing one of them but not the other .
- N_3-$\mathcal{T}_o$-space if $\forall\, x_{N_3} \neq y_{N_3} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open set G in $\mathcal{X}$ containing one of them but not the other .
- N_1-$\mathcal{T}_1$-space if $\forall\, x_{N_1} \neq y_{N_1} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_1} \in G_1$, $y_{N_1} \notin G_1$ and $x_{N_1} \notin G_2$, $y_{N_1} \in G_2$
- N_2-$\mathcal{T}_1$-space if $\forall\, x_{N_2} \neq y_{N_2} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_2} \in G_1$, $y_{N_2} \notin G_1$ and $x_{N_2} \notin G_2$, $y_{N_2} \in G_2$
- N_3-$\mathcal{T}_1$-space if $\forall\, x_{N_3} \neq y_{N_3} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_3} \in G_1$, $y_{N_3} \notin G_1$ and $x_{N_3} \notin G_2$, $y_{N_3} \in G_2$
- N_1-$\mathcal{T}_2$-space if $\forall\, x_{N_1} \neq y_{N_1} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_1} \in G_1$, $y_{N_1} \notin G_1$ and $x_{N_1} \notin G_2$, $y_{N_1} \in G_2$ with $G_1 \cap G_2 = \emptyset$.
- N_2-$\mathcal{T}_2$-space if $\forall\, x_{N_2} \neq y_{N_2} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_2} \in G_1$, $y_{N_2} \notin G_1$ and $x_{N_2} \notin G_2$, $y_{N_2} \in G_2$ with $G_1 \cap G_2 = \emptyset$.

- N_3-$\mathcal{T}_2$-space if $\forall\, x_{N_3} \neq y_{N_3} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_3} \in G_1$, $y_{N_3} \notin G_1$ and $x_{N_3} \notin G_2$, $y_{N_3} \in G_2$ with $G_1 \cap G_2 = \emptyset$.

3.2 Definition

A neutrosophic crisp topological space ($\mathcal{X},\mathcal{T}$) is called:

- N- $\mathcal{T}_o$-space if $(\mathcal{X},\mathcal{T}\ )$ is N_1-$\mathcal{T}_o$-space , N_2-$\mathcal{T}_o$-space and N_3-$\mathcal{T}_o$-space
- N-$\mathcal{T}_1$-space if $(\mathcal{X},\mathcal{T})$ is N_1-$\mathcal{T}_1$-space , N_2-$\mathcal{T}_1$-space and N_3-$\mathcal{T}_1$-space
- N-$\mathcal{T}_2$-space if $(\mathcal{X},\mathcal{T})$ is N_1-$\mathcal{T}_2$-space , N_2-$\mathcal{T}_2$-space and N_3-$\mathcal{T}_2$-space

3.3 Remark

For a neutrosophic. crisp topological space $(\mathcal{X},\mathcal{T})$

- Every N- $\mathcal{T}_o$-space is N_1-$\mathcal{T}_o$-space
- Every N- $\mathcal{T}_0$- space is N_2-$\mathcal{T}_0$-space
- Every N- $\mathcal{T}_0$- space is N_3-$\mathcal{T}_0$-space

Proof the proof is directly from definition 3.2 .

The inverse of remark 3.3 is not true, the following example explain this state.

3.4 Example

If $\mathcal{X} = \{x,y\}$, $\mathcal{T}_1 = \{\mathcal{X}_N, \emptyset_N, A\}$,$\mathcal{T}_2 = \{\mathcal{X}_N, \emptyset_N, B\}$,$\mathcal{T}_3 = \{\mathcal{X}_N, \emptyset_N, G\}$, A =<{x},∅,∅>,B =<∅,{y},∅> , G =<∅,∅,{x}>, Then $(\mathcal{X},\mathcal{T}_1)$ is N_1-$\mathcal{T}_o$-space but it is not N-$\mathcal{T}_o$-space, $(\mathcal{X},\mathcal{T}_2)$ is N_2-$\mathcal{T}_o$-space but it is not N-$\mathcal{T}_o$-space, $(\mathcal{X},\mathcal{T}_3)$ is N_3-$\mathcal{T}_o$-space but it is not N-$\mathcal{T}_o$-space.

3.5 Remark

For a neutrosophic crisp topological space $(\mathcal{X},\mathcal{T})$

- Every N- $\mathcal{T}_1$-space is N_1-$\mathcal{T}_1$-space
- Every N- $\mathcal{T}_1$- space is N_2-$\mathcal{T}_1$-space
- Every N- $\mathcal{T}_1$- space is N_3-$\mathcal{T}_1$-space

Proof the proof is directly from definition 3.2 .

The inverse of remark (3.5) is not true as it is shown in the following example,

3.6 Example

If $\mathcal{X} = \{x,y\}$,$\mathcal{T}_1 = \{\mathcal{X}_N , \emptyset_N , A, B\}$, $\mathcal{T}_2 = \{\mathcal{X}_N, \emptyset_N, G , F\}$, A =<{x},{y},∅>, B =<{y},{x},∅>, G =<∅,∅,{x}>, F =<∅,∅,{y}>, Then $(\mathcal{X},\mathcal{T}_1)$ is N_1-$\mathcal{T}_1$-space but it is not N-$\mathcal{T}_1$-space. $(\mathcal{X},\mathcal{T}_1)$ is N_2-$\mathcal{T}_1$-space but it is not N-$\mathcal{T}_1$-space. $(\mathcal{X},\mathcal{T}_2)$ is N_3-$\mathcal{T}_1$-space but it is not N-$\mathcal{T}_1$-space

3.7 Remark

For a neutrosophic. crisp topological space $(\mathcal{X},\mathcal{T})$

- Every N-$\mathcal{T}_2$-space is N_1-$\mathcal{T}_2$-space
- Every N-$\mathcal{T}_2$-space is N_2-$\mathcal{T}_2$-space
- Every N-$\mathcal{T}_2$-space is N_3-$\mathcal{T}_2$-space

Proof the proof is directly from definition 3.2 .

The inverse of remark (3.7) is not true as it is shown in the example (3.6).

3.8 Remark

For a neutrosophic. crisp topological space $(\mathcal{X},\mathcal{T})$

- Every N-$\mathcal{T}_1$-space is N-$\mathcal{T}_0$-space
- Every N-$\mathcal{T}_2$-space is N-$\mathcal{T}_1$-space

Proof the proof is directly.

The inverse of remark (3.8) is not true as it is shown in the following example :

3.9 Example

If $= \{ x , y \}$, $\mathcal{T} = \{ \mathcal{X}_N , \emptyset_N , A, B, G\}$

A = < { x }, ∅ , ∅ > , B = < ∅, { y }, ∅ > , G = < ∅ , ∅ ,{ x } > ,

Then ($\mathcal{X},\mathcal{T}$) is N-$\mathcal{T}_o$-space but not N-$\mathcal{T}_1$-space

Conclusion

- We defined a new neutrosophic crisp points in neutrosophic crisp topological space
- We introduced the concept of neutrosophic crisp limit point, with some of its properties
- We constructed the separation axioms [N-$\mathcal{T}_i$-space , i= 0,1,2] in neutrosophic crisp topological and examine the relationship between them in details.

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Neutrosophic Soft Topological K-Algebras

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Muhammad Akram, Hina Gulzar, Florentin Smarandache (2019). Neutrosophic Soft Topological K-Algebras. *Neutrosophic Sets and Systems* 25, 104-124

Abstract: In this paper, we propose the notion of single-valued neutrosophic soft topological K-algebras. We discuss certain concepts, including interior, closure, C_5-connected, super connected, Compactness and Hausdorff in single-valued neutrosophic soft topological K-algebras. We illustrate these concepts with examples and investigate some of their related properties. We also study image and pre-image of single-valued neutrosophic soft topological K-algebras.

Keywords: K-algebras, Single-valued neutrosophic soft sets, Compactness, C_5-connectedness, Super connectedness, Hausdorff.

1 Introduction

A K-algebra $(G, \cdot, \odot, e)$ is a new class of logical algebra, introduced by Dar and Akram [1] in 2003. A K-algebra is constructed on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G and attached to an abstract K-algebra $(G, \cdot, \odot, e)$. This system is, in general, non-commutative and non-associative with a right identity e. If the given group G is not an elementary abelian 2-group, then the K-algebra is proper . Therefore, a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ is abelian and non-abelian, proper and improper purely depends upon the base group G. In 2004, a K-algebra renamed as $K(G)$-algebra due to its structural basis G and characterized by left and right mappings when the group G is abelian and non-abelian by Dar and Akram in [2, 3] . In 2007, Dar and Akram [4] investigated the K-homomorphisms of K-algebras.
Non-classical logic leads to classical logic due to various aspects of uncertainty. It has become a conventional tool for computer science and engineering to deal with fuzzy information and indeterminate data and executions. In our daily life, the most frequently encountered uncertainty is incomparability. Zadeh's fuzzy set theory [5] revolutionized the systems, accomplished with vagueness and uncertainty. A number of researchers extended the conception of Zadeh and presented different theories regarding uncertainty which includes intuitionistic fuzzy set theory, interval-valued intuitionistic fuzzy set theory [6] and so on. In addition, Smarandache [7] generalized intuitionistic fuzzy set by introducing the concept of neutrosophic set in 1998. It is such a branch of philosophy which studies the origin, nature, and scope of neutralities as well as their interactions with different ideational spectra. To have real life applications of neutrosophic sets such as in engineering and science, Wang et al. [8] introduced the single-valued neutrosophic set in 2010. In 1999, Molodtsov [9] introduced another mathematical approach to deal with ambiguous data, called soft set theory. Soft set theory gives a parameterized outlook to uncertainty. Maji [10] defined the notion of neutrosophic soft set by unifying

the fundamental theories of neutrosophic set and soft set to deal with inconsistent data in a much-unified mode. A large number of theories regarding uncertainty with their respective topological structures have been introduced. In 1968, Chang [11] introduced the concept of fuzzy topology. Chattopadhyay and Samanta [12], Pu and Liu [13] and Lowan [14] defined some certain notions related to fuzzy topology. Recently, Tahan et al. [15] presented the notion of topological hypergroupoids. Onasanya and Hoskova-Mayerova [16] discussed some topological and algebraic properties of $\alpha-$level subsets of fuzzy subsets. Coker [17] considered the notion of an intuitionistic fuzzy topology. Salama and Alblowi [18] studied the notion of neutrosophic topological spaces. In 2017, Bera and Mahapatra [19] described neutrosophic soft topological spaces. Akram and Dar [20, 21] considered fuzzy topological K-algebras and intuitionistic topological K-algebras. Recently, Akram et al. [22, 23, 24, 25] presented some notions, including single-valued neutrosophic K-algebras, single-valued neutrosophic topological K-algebras and single-valued neutrosophic Lie algebras. In this research article, In this paper, we propose the notion of single-valued neutrosophic soft topological K-algebras. We discuss certain concepts, including interior, closure, C_5-connected, super connected, Compactness and Hausdorff in single-valued neutrosophic soft topological K-algebras. We illustrate these concepts with examples and investigate some of their related properties. We also study image and pre-image of single-valued neutrosophic soft topological K-algebras.

The rest of the paper is organized as follows: In Section 2, we review some elementary concepts related to K-algebras, single-valued neutrosophic soft sets and their topological structures. In Section 3, we define the concept of single-valued neutrosophic soft topological K-algebras and discuss certain concepts with some numerical examples. In Section 4, we present concluding remarks.

2 Preliminaries

This section consists of some basic definitions and concepts, which will be used in the next sections.

Definition 2.1. [1] A K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ is an algebra of the type (2, 2, 0) defined on the group $(G, \cdot, e)$ in which each non-identity element is not of order 2 with the following $\odot-$ axioms:

(K1) $(x \odot y) \odot (x \odot z) = (x \odot (z^{-1} \odot y^{-1})) \odot x = (x \odot ((e \odot z) \odot (e \odot y))) \odot x,$

(K2) $x \odot (x \odot y) = (x \odot y^{-1}) \odot x = (x \odot (e \odot y)) \odot x,$

(K3) $(x \odot x) = e,$

(K4) $(x \odot e) = x,$

(K5) $(e \odot x) = x^{-1}$

for all x, y, $z \in G$.

Definition 2.2. [1] A nonempty set $\mathcal{S}$ in a K-algebra $\mathcal{K}$ is called a *subalgebra* of $\mathcal{K}$ if for all $x, y \in \mathcal{S}$, $x \odot y \in \mathcal{S}$.

Definition 2.3. [1] Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two K-algebras. A mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ is called a *homomorphism* if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}$.

Definition 2.4. [7] Let Z be a nonempty set of objects. A single-valued neutrosophic set H in Z is of the form $H = \{s \in Z : \mathcal{T}_H(s), \mathcal{I}_H(s), \mathcal{F}_H(s)\}$, where $\mathcal{T}, \mathcal{I}, \mathcal{F} : Z \to [0, 1]$ for all $s \in Z$ with $0 \leq \mathcal{T}_H(s) + \mathcal{I}_H(s) + \mathcal{F}_H(s) \leq 3$.

Definition 2.5. [22] Let $H = (\mathcal{T}_H, \mathcal{I}_H, \mathcal{F}_H)$ be a single-valued neutrosophic set in $\mathcal{K}$, then H is said to be a *single-valued neutrosophic K-subalgebra* of $\mathcal{K}$ if it possess the following properties:

(a) $\mathcal{T}_H(s \odot t) \geq \min\{\mathcal{T}_H(s), \mathcal{T}_H(t)\}$,

(b) $\mathcal{I}_H(s \odot t) \geq \min\{\mathcal{I}_H(s), \mathcal{I}_H(t)\}$,

(c) $\mathcal{F}_H(s \odot t) \leq \max\{\mathcal{F}_H(s), \mathcal{F}_H(t)\}$ for all $s, t \in \mathcal{K}$.

A K-subalgebra also satisfies the following conditions:
$\mathcal{T}_H(e) \geq \mathcal{T}_H(s), \mathcal{I}_H(e) \geq \mathcal{I}_H(s), \mathcal{F}_H(e) \leq \mathcal{F}_H(s)$ for all $s \neq e \in \mathcal{K}$.

Definition 2.6. [26] A *t-norm* is a two-valued function defined by a binary operation $*$, where $* : [0,1] \times [0,1] \to [0,1]$. A t-norm is an associative, monotonic and commutative function possess the following properties, for all $a, b, c, d \in [0,1]$,

(i) $*$ is a commutative binary operation.

(ii) $*$ is an associative binary operation.

(iii) $*(0,0) = 0$ and $*(a,1) = *(1,a) = a$.

(iv) If $a \leq c$ and $b \leq d$, then $*(a,b) \leq *(c,d)$.

Definition 2.7. [26] A *t-conorm (s-norm)* is a two-valued function defined by a binary operation $\circ$ such that $\circ : [0,1] \times [0,1] \to [0,1]$. A t-conorm is an associative, monotonic and commutative two-valued function, possess the following properties, for all $a, b, c, d \in [0,1]$,

(i) $\circ$ is a commutative binary operation.

(ii) $\circ$ is an associative binary operation.

(iii) $\circ(1,1) = 1$ and $\circ(a,0) = \circ(0,a) = a$.

(iv) If $a \leq c$ and $b \leq d$, then $\circ(a,b) \leq \circ(c,d)$.

Definition 2.8. [23] Let $\chi_{\mathcal{K}}$ be a single-valued neutrosophic topology over $\mathcal{K}$. Let H be a single-valued neutrosophic K-algebra of $\mathcal{K}$ and χ_H be a single-valued neutrosophic topology on H. Then H is called a *single-valued neutrosophic topological K-algebra* over $\mathcal{K}$ if the self map $\rho_a : (H, \chi_H) \to (H, \chi_H)$ for all $a \in \mathcal{K}$, defined as $\rho_a(s) = s \odot a$, is relatively single-valued neutrosophic continuous.

Definition 2.9. [9] Let Z be a universe of discourse and E be a universe of parameters. Let $P(Z)$ denotes the set of all subsets of Z and $A \subseteq E$. Then a *soft set* F_A over Z is represented by a set-valued function ζ_A, where $\zeta_A : E \to P(Z)$ such that $\zeta_A(\theta) = \emptyset$ if $\theta \in E - A$. In other words, F_A can be represented in the form of a collection of parameterized subsets of Z such as $F_A = \{(\theta, \zeta_A(\theta)) : \theta \in E, \zeta_A(\theta) = \emptyset \text{ if } \theta \in E - A\}$.

Definition 2.10. [27] Let Z be a universe of discourse and E be a universe of parameters. A *single-valued neutrosophic soft set* H in Z is defined by a set-valued function ζ_H, where $\zeta_H : E \to P(Z)$ and $P(Z)$ denotes the power set set of Z. In other words, a single-valued neutrosophic soft set is a parameterized family of single-valued neutrosophic sets in Z and therefore can be written as:
$H = \{(\theta, \langle u, \mathcal{T}_{\zeta_H(\theta)}(u), \mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{F}_{\zeta_H(\theta)}(u) \rangle : u \in Z) : \theta \in E\}$, where $\mathcal{T}_{\zeta_H(\theta)}, \mathcal{I}_{\zeta_H(\theta)}, \mathcal{F}_{\zeta_H(\theta)}$ are called truth , indeterminacy and falsity membership functions of $\zeta_H(\theta)$, respectively.

Definition 2.11. [27] Let H be a single-valued neutrosophic soft set. The *compliment* of H, denoted by H^c, is defined as follows:

$$H^c = \{(\theta,\ \langle u, \mathcal{F}_{\zeta_H(\theta)}(u), \mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{T}_{\zeta_H(\theta)}(u)\rangle\ : u \in Z) : \theta \in E\}.$$

Definition 2.12. [27] Let H and J be two single-valued neutrosophic soft sets over (Z, E). Then H is called a *neutrosophic soft subset* of J, denoted by $H \subseteq J$, if the following conditions hold:

(i) $\mathcal{T}_{\zeta_H(\theta)}(u) \leq \mathcal{T}_{\eta_J(\theta)}(u)$,

(ii) $\mathcal{I}_{\zeta_H(\theta)}(u) \leq \mathcal{I}_{\eta_J(\theta)}(u)$,

(iii) $\mathcal{F}_{\zeta_H(\theta)}(u) \geq \mathcal{F}_{\eta_J(\theta)}(u)$ for all $\theta \in E, u \in Z$.

Throughout this article, we take the t-norm $(*)$ as $\min(a, b)$ and t-conorm $(\circ)$ as $\max(a, b)$ for intersection of two single-valued neutrosophic soft sets and $(*)$ as $\max(a, b)$ and t-conorm $(\circ)$ as $\min(a, b)$ for union of two single-valued neutrosophic soft sets. The union and the intersection for two single-valued neutrosophic soft sets are defined as follows.

Definition 2.13. [27] Let H and J be two single-valued neutrosophic soft sets over (Z, E). Then the *union* of H and J is denoted by $H \cup J = L$ and defined as:

$$L = \Big\{\Big(\theta,\ \langle u, \mathcal{T}_{\vartheta_L(\theta)}(u), \mathcal{I}_{\vartheta_L(\theta)}(u), \mathcal{F}_{\vartheta_L(\theta)}(u)\rangle\ : u \in Z\Big) : \theta \in E\Big\},$$

where

$$\begin{aligned}
\mathcal{T}_{\vartheta_L(\theta)}(u) &= \{\mathcal{T}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \max\{\mathcal{T}_{\zeta_H(\theta)}(u), \mathcal{T}_{\eta_J(\theta)}(u)\},\\
\mathcal{I}_{\vartheta_L(\theta)}(u) &= \{\mathcal{I}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \max\{\mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{I}_{\eta_J(\theta)}(u)\},\\
\mathcal{F}_{\vartheta_L(\theta)}(u) &= \{\mathcal{F}_{\zeta_H(\theta)}(u) \circ \mathcal{F}_{\eta_J(\theta)}(u)\} = \min\{\mathcal{F}_{\zeta_H(\theta)}(u), \mathcal{F}_{\eta_J(\theta)}(u)\}.
\end{aligned}$$

Definition 2.14. [27] Let H and J be two single-valued neutrosophic soft sets over (Z, E). Then their *intersection* is denoted by $H \cap J = L$ and defined as:

$$L = \Big\{\Big(\theta,\ \langle u, \mathcal{T}_{\vartheta_L(\theta)}(u), \mathcal{I}_{\vartheta_L(\theta)}(u), \mathcal{F}_{\vartheta_L(\theta)}(u)\rangle\ : u \in Z\Big) : \theta \in E\Big\},$$

where

$$\begin{aligned}
\mathcal{T}_{\vartheta_L(\theta)}(u) &= \{\mathcal{T}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \min\{\mathcal{T}_{\zeta_H(\theta)}(u), \mathcal{T}_{\eta_J(\theta)}(u)\},\\
\mathcal{I}_{\vartheta_L(\theta)}(u) &= \{\mathcal{I}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \min\{\mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{I}_{\eta_J(\theta)}(u)\},\\
\mathcal{F}_{\vartheta_L(\theta)}(u) &= \{\mathcal{F}_{\zeta_H(\theta)}(u) \circ \mathcal{F}_{\eta_J(\theta)}(u)\} = \max\{\mathcal{F}_{\zeta_H(\theta)}(u), \mathcal{F}_{\eta_J(\theta)}(u)\}.
\end{aligned}$$

Definition 2.15. [27] A single-valued neutrosophic soft set H over the universe Z is termed to be an *empty or null single-valued neutrosophic soft set* with respect to the parametric set E if $\mathcal{T}_{\zeta_H(\theta)}(u) = 0, \mathcal{I}_{\zeta_H(\theta)}(u) = 0$, $\mathcal{F}_{\zeta_H(\theta)}(u) = 1$, for all $u \in Z$, $\theta \in E$, denoted by $\emptyset_E$ and can be written as:

$$\emptyset_E(u) = \{u \in Z : \mathcal{T}_{\zeta_H(\theta)}(u) = 0, \mathcal{I}_{\zeta_H(\theta)}(u) = 0, \mathcal{F}_{\zeta_H(\theta)}(u) = 1 : \theta \in E\}.$$

Definition 2.16. [27] A single-valued neutrosophic soft set H over the universe Z is called an *absolute or a whole single-valued neutrosophic soft set* if $\mathcal{T}_{\zeta_{H(\theta)}}(u) = 1$, $\mathcal{I}_{\zeta_{H(\theta)}}(u) = 1$, $\mathcal{F}_{\zeta_{H(\theta)}}(u) = 0$, for all $u \in Z$, $\theta \in E$, denoted by 1_E and can be written as:

$$1_E(u) = \{u \in Z : \mathcal{T}_{\zeta_{H(\theta)}}(u) = 1, \mathcal{I}_{\zeta_{H(\theta)}}(u) = 1, \mathcal{F}_{\zeta_{H(\theta)}}(u) = 0 : \theta \in E\}.$$

Definition 2.17. [10] Let (Z_1, E) and (Z_2, E) be two initial universes. Then a pair (φ, ρ) is called a *single-valued neutrosophic soft function* from (Z_1, E) into (Z_2, E), where $\varphi : Z_1 \to Z_2$ and $\rho : E \to E$, and E is a parametric set of Z_1 and Z_2.

Definition 2.18. [10] Let (H, E) and (J, E) be two single-valued neutrosophic soft sets over G_1 and G_2, respectively. If (φ, ρ) is a single-valued neutrosophic soft function from (G_1, E) into (G_2, E), then under this single-valued neutrosophic soft function (φ, ρ), *image* of (H, E) is a single-valued neutrosophic soft set on $\mathcal{K}_2$, denoted by $(\varphi, \rho)(H, E)$ and defined as follows:
for all $m \in \rho(E)$ and $y \in G_2$, $(\varphi, \rho)(H, E) = (\varphi(H), \rho(E))$, where

$$\mathcal{T}_{\varphi(\zeta)_m}(y) = \begin{cases} \bigvee_{\varphi(x)=y} \bigvee_{\rho(a)=m} \zeta_a(x) & \text{if } x \in \rho^{-1}(y), \\ 1, & \text{otherwise,} \end{cases}$$
$$\mathcal{I}_{\varphi(\zeta)_m}(y) = \begin{cases} \bigvee_{\varphi(x)=y} \bigvee_{\rho(a)=m} \zeta_a(x) & \text{if } x \in \rho^{-1}(y), \\ 1, & \text{otherwise,} \end{cases}$$
$$\mathcal{F}_{\varphi(\zeta)_m}(y) = \begin{cases} \bigwedge_{\varphi(x)=y} \bigwedge_{\rho(a)=m} \zeta_a(x) & \text{if } x \in \rho^{-1}(y), \\ 0, & \text{otherwise.} \end{cases}$$

The *preimage* of (J, E), denoted by $(\varphi, \rho)^{-1}(J, E)$, is defined as $\forall\, l \in \rho^{-1}(E)$ and for all $x \in G_1$, $(\varphi, \rho)^{-1}(J, E) = (\varphi^{-1}(J), \rho^{-1}(E))$, where

$$\mathcal{T}_{\varphi^{-1}(\eta)_l}(x) = \mathcal{T}_{\eta_{\rho(l)}}(\varphi(x)),$$
$$\mathcal{I}_{\varphi^{-1}(\eta)_l}(x) = \mathcal{I}_{\eta_{\rho(l)}}(\varphi(x)),$$
$$\mathcal{F}_{\varphi^{-1}(\eta)_l}(x) = \mathcal{F}_{\eta_{\rho(l)}}(\varphi(x)).$$

Proposition 2.19. Let Z_1 and Z_2 be two initial universes with parametric set E_1 and E_2, respectively. Let H, $(H_i, i \in I)$ be a single-valued neutrosophic soft set in Z_1 and J be a single-valued neutrosophic soft set in Z_2. Let $f : Z_1 \to Z_2$ be a function. Then

(i) $f(1_{E_1}) = 1_{E_2}$, if f is a surjective function.

(ii) $f(\emptyset_{E_1}) = \emptyset_{E_2}$.

(iii) $f^{-1}(1_{E_2}) = 1_{E_1}$.

(iv) $f^{-1}(\emptyset_{E_2}) = \emptyset_{E_1}$.

(v) $f^{-1}(\bigcup_{i=1}^{n} H_i) = \bigcup_{i=1}^{n} f^{-1}(H_i)$.

Through out this article, Z is considered as initial universe, E is a parametric set and $\theta \in E$ an arbitrary parameter.

3 Single-Valued Neutrosophic Soft Topological K-Algebras

Definition 3.1. Let Z be a nonempty set and E be a universe of parameters. A collection χ of single-valued neutrosophic soft sets is called a *single-valued neutrosophic soft topology* if the following properties hold:

(1) $\emptyset_E, 1_E \in \chi$.

(2) The intersection of any two single-valued neutrosophic soft sets of χ belongs to χ.

(3) The union of any collection of single-valued neutrosophic soft sets of χ belongs to χ.

The triplet (Z, E, χ) is called a single-valued neutrosophic soft topological space over (Z, E). Each element of χ is called a single-valued neutrosophic soft open set and compliment of each single-valued neutrosophic soft open set is a single-valued neutrosophic soft closed set in χ . A single-valued neutrosophic soft topology which contains all single-valued neutrosophic soft subsets of Z is called a discrete single-valued neutrosophic soft topology and indiscrete single-valued neutrosophic soft topology if it consists of $\emptyset_E$ and 1_E.

Definition 3.2. Let H be a single-valued neutrosophic soft set over a K-algebras $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft K-subalgebra* of $\mathcal{K}$ if the following conditions hold:

$$\begin{aligned}
&\text{(i)}\ \ \mathcal{T}_{\zeta_\theta}(s \odot t) \geq \min\{\mathcal{T}_{\zeta_\theta}(s), \mathcal{T}_{\zeta_\theta}(t)\},\\
&\text{(ii)}\ \ \mathcal{I}_{\zeta_\theta}(s \odot t) \geq \min\{\mathcal{I}_{\zeta_\theta}(s), \mathcal{I}_{\zeta_\theta}(t)\},\\
&\text{(iii)}\ \mathcal{F}_{\zeta_\theta}(s \odot t) \leq \max\{\mathcal{F}_{\zeta_\theta}(s), \mathcal{F}_{\zeta_\theta}(t)\} \text{ for all } s, t \in G \text{ and } \theta \in E.
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathcal{T}_{\zeta_\theta}(e) \geq \mathcal{T}_{\zeta_\theta}(s),\\
&\mathcal{I}_{\zeta_\theta}(e) \geq \mathcal{I}_{\zeta_\theta}(s),\\
&\mathcal{F}_{\zeta_\theta}(e) \leq \mathcal{F}_{\zeta_\theta}(s), \text{ for all } s \neq e \in G.
\end{aligned}$$

Example 3.3. Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ on a group $(G, \cdot)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$ is the cyclic group of order 8 and $\odot$ is given by the following Cayley's table as:

$\odot$	e	x	x^2	x^3	x^4	x^5	x^6	x^7
e	e	x^7	x^6	x^5	x^4	x^3	x^2	x
x	x	e	x^7	x^6	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^7	x^6	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^7	x^6	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^7	x^6	x^5
x^5	x^5	x^4	x^3	x^2	x	e	x^7	x^6
x^6	x^6	x^5	x^4	x^3	x^2	x	e	x^7
x^7	x^7	x^6	x^5	x^4	x^3	x^2	x	e

Let E be a set of parameters defined as $E = \{l_1, l_2\}$. We define single-valued neutrosophic soft sets H, J and L in $\mathcal{K}$ as:

$$\begin{aligned}
\zeta_H(l_1) &= \{(e, 0.8, 0.7, 0.2), (h, 0.6, 0.5, 0.4)\},\\
\zeta_H(l_2) &= \{(e, 0.7, 0.7, 0.2), (h, 0.6, 0.6, 0.5)\},
\end{aligned}$$

$$\zeta_J(l_1) = \{(e, 0.7, 0.7, 0.2), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_J(l_2) = \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.5, 0.7)\},$$

$$\zeta_L(l_1) = \{(e, 0.9, 0.8, 0.1), (h, 0.7, 0.6, 0.4)\},$$
$$\zeta_L(l_2) = \{(e, 0.9, 0.7, 0.1), (h, 0.7, 0.6, 0.4)\}$$

for all $h \neq e \in G$.
The collection $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J, L\}$ is a single-valued neutrosophic soft topology on $\mathcal{K}$ and the triplet $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a single-valued neutrosophic soft topological space over $\mathcal{K}$. It is interesting to note that corresponding to each parameter $\theta \in E$, we get a single-valued neutrosophic topology over $\mathcal{K}$ which means that a single-valued neutrosophic soft topological space gives a parameterized family of single-valued neutrosophic topological space on $\mathcal{K}$. Now, we define a single-valued neutrosophic soft set Q in $\mathcal{K}$ as:

$$\zeta_Q(l_1) = \{(e, 0.8, 0.5, 0.1), (h, 0.6, 0.4, 0.3)\},$$
$$\zeta_Q(l_2) = \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.6)\}.$$

Clearly, by Definition 3.2, Q is a single-valued neutrosophic soft K-subalgebra over $\mathcal{K}$.

Proposition 3.4. Let $(\mathcal{K}, E, \chi^{'}_{\mathcal{K}})$ and $(\mathcal{K}, E, \chi^{''}_{\mathcal{K}})$ be two single-valued neutrosophic topological spaces over $\mathcal{K}$. If $\chi^{'}_{\mathcal{K}} \cap \chi^{''}_{\mathcal{K}} = M^{'}$, where $M^{'}$ is a single-valued neutrosophic soft set from the set of all single-valued neutrosophic soft sets in $\mathcal{K}$, then $\chi^{'}_{\mathcal{K}} \cap \chi^{''}_{\mathcal{K}}$ is also a single-valued neutrosophic soft topology on $\mathcal{K}$.

Remark 3.5. The union of two single-valued neutrosophic soft topologies over $\mathcal{K}$ may not be a single-valued neutrosophic soft topology over $\mathcal{K}$.

Example 3.6. Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$ is the cyclic group of order 8 and Cayley's table for $\odot$ is given in Example 3.3. We take $E = \{l_1, l_2\}$ and two single-valued neutrosophic soft topological spaces $\chi^{'}_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J\}$, $\chi^{''}_{\mathcal{K}} = \{\emptyset_E, 1_E, R, S\}$ on $\mathcal{K}$, where $R = H$ and single-valued neutrosophic soft set S is defined as:

$$\zeta_S(l_1) = \{(e, 0.7, 0.6, 0.2), (h, 0.5, 0.5, 0.6)\},$$
$$\zeta_S(l_2) = \{(e, 0.9, 0.8, 0.2), (h, 0.7, 0.7, 0.3)\}.$$

Suppose that $\chi^{'''}_{\mathcal{K}} = \chi^{'}_{\mathcal{K}} \cup \chi^{''}_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J, S\}$. We see that $\chi^{'''}_{\mathcal{K}}$ is not a single-valued neutrosophic soft topology over $\mathcal{K}$ since $S \cap J \notin \chi^{'''}_{\mathcal{K}}$.

Definition 3.7. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$, where $\chi_{\mathcal{K}}$ is a single-valued neutrosophic soft topology over $\mathcal{K}$. Let F be a single-valued neutrosophic soft set in $\mathcal{K}$, then $\chi_F = \{F \cap H : H \in \chi_{\mathcal{K}}\}$ is called a single-valued neutrosophic soft topology on F and (F, E, χ_F) is called a *single-valued neutrosophic soft subspace* of $(\mathcal{K}, E, \chi_{\mathcal{K}})$.

Definition 3.8. Let $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are two K-algebras. Then, a mapping $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is called *single-valued neutrosophic soft continuous mapping* of single-valued neutrosophic soft topological spaces if it the following properties hold:

(i) For each single-valued neutrosophic soft set $H \in \chi_2$, $f^{-1}(H) \in \chi_1$.

(ii) For each single-valued neutrosophic soft K-subalgebra $H \in \chi_2$, $f^{-1}(H)$ is a single-valued neutrosophic soft K-subalgebra $\in \chi_1$.

Definition 3.9. Let H and J be two single-valued neutrosophic soft sets in a K-algebra $\mathcal{K}$ and $f : (H, E, \chi_H) \to (J, E, \chi_J)$. Then, f is called a *relatively single-valued neutrosophic soft open* function if for every single-valued neutrosophic soft open set V in χ_H, the image $f(V) \in \chi_J$.

Definition 3.10. If f is a mapping such that $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$. Then f is a mapping from (H, E, χ_H) into (J, E, χ_J) if $f(H) \subset J$, where (H, E, χ_H) and (J, E, χ_J) are two single-valued neutrosophic soft subspaces of $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$, respectively.

Definition 3.11. A mapping f such that $f : (H, E, \chi_H) \to (J, E, \chi_J)$ is called *relatively single-valued neutrosophic soft continuous* if for every single-valued neutrosophic soft open set $Y_J \in \chi_J$, $f^{-1}(y_J) \cap H \in \chi_H$.

Definition 3.12. Let $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces. Then, a function $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is called a *single-valued neutrosophic soft homomorphism* if it satisfies the following properties:

(i) f is a bijective function.

(ii) Both f and f^{-1} are single-valued neutrosophic soft continuous functions.

Proposition 3.13. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a single-valued neutrosophic soft continues mapping and (H, E, χ_H) and (J, E, χ_J) two single-valued neutrosophic soft topological subspaces of $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$, respectively. If $f(H) \subseteq J$, then f is a relatively single-valued neutrosophic soft continuous mapping from (H, E, χ_H) into (J, E, χ_J).

Proposition 3.14. Let $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces, where χ_1 is a single-valued neutrosophic soft topology on $\mathcal{K}_1$ and χ_2 is an indiscrete single-valued neutrosophic soft topology on $\mathcal{K}_2$. Then for each $\theta \in E$, every function $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft continues function.

Proof. Let χ_1 be a single-valued neutrosophic soft topology on $\mathcal{K}_1$ and χ_2 an indiscrete single-valued neutrosophic soft topology on $\mathcal{K}_2$ such that $\chi_2 = \{\emptyset_E, 1_E\}$. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be any function. Now, to prove that f is a single-valued neutrosophic soft continues function for each $\theta \in E$, we show that f satisfies both conditions of Definition 3.8. Clearly, every member of χ_2 is a single-valued neutrosophic soft K-subalgebra of $\mathcal{K}_2$ for each $\theta \in E$. Now, there is only need to show that for all $H \in \chi_2$ and for each $\theta \in E$, $f^{-1}(H) \in \chi_1$. For this purpose, let us assume that $\emptyset_\theta \in \chi_2$, for any $u \in \mathcal{K}_1$ and $\theta \in E$, we have $f^{-1}(\emptyset_\theta)(u) = \emptyset_\theta(f(u)) = \emptyset_\theta(u) \Rightarrow \emptyset_\theta \in \chi_1$. Similarly, $f^{-1}(1_\theta)(u) = 1_\theta(f(u)) = 1_\theta(u) \Rightarrow 1_\theta \in \chi_1$. For an arbitrary choice of θ, result holds for each $\theta \in E$. This shows that f is a single-valued neutrosophic soft continues function. □

Proposition 3.15. Let χ_1 and χ_2 be any two discrete single-valued neutrosophic soft topological spaces on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively and $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ two discrete single-valued neutrosophic soft topological spaces. Then for each $\theta \in E$, every homomorphism $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft continuous function.

Proof. Let $H = \{(\mathcal{T}_{\zeta_H(\theta)}, \mathcal{I}_{\zeta_H(\theta)}, \mathcal{F}_{\zeta_H(\theta)}) : \theta \in E\}$ be a single-valued neutrosophic soft set in $\mathcal{K}_2$ defined by a set-valued function ζ_H. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a homomorphism (not a usual inverse homomorphism). Since χ_1 and χ_2 be two discrete single-valued neutrosophic soft topologies, then for every $H \in \chi_2$, $f^{-1}(H) \in \chi_1$. Now, we show that for each $\theta \in E$, the mapping $f^{-1}(H)$ is a single-valued neutrosophic soft K-subalgebra of K-algebra $\mathcal{K}_1$. Then for any $s, t \in \mathcal{K}_1$ and $\theta \in E$, we have

$$\begin{aligned}
f^{-1}(\mathcal{T}_{\zeta_H(\theta)})(s \odot t) &= \mathcal{T}_{\zeta_H(\theta)}\big(f(s \odot t)\big) \\
&= \mathcal{T}_{\zeta_H(\theta)}(f(s) \odot f(t)) \\
&\geq \min\{\mathcal{T}_{\zeta_H(\theta)}(f(s)) \odot \mathcal{T}_{\zeta_H(\theta)}(f(t))\} \\
&= \min\{f^{-1}(\mathcal{T}_{\zeta_H(\theta)})(s), f^{-1}(\mathcal{T}_{\zeta_H(\theta)})(t)\},
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\mathcal{I}_{\zeta_H(\theta)})(s \odot t) &= \mathcal{I}_{\zeta_H(\theta)}(f(s \odot t)) \\
&= \mathcal{I}_{\zeta_H(\theta)}(f(s) \odot f(t)) \\
&\geq \min\{\mathcal{I}_{\zeta_H(\theta)}(f(s)) \odot \mathcal{I}_{\zeta_H(\theta)}(f(t))\} \\
&= \min\{f^{-1}(\mathcal{I}_{\zeta_H(\theta)})(s), f^{-1}(\mathcal{I}_{\zeta_H(\theta)})(t)\},
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\mathcal{F}_{\zeta_H(\theta)})(s \odot t) &= \mathcal{F}_{\zeta_H(\theta)}(f(s \odot t)) \\
&= \mathcal{F}_{\zeta_H(\theta)}(f(s) \odot f(t)) \\
&\geq \min\{\mathcal{F}_{\zeta_H(\theta)}(f(s)) \odot \mathcal{F}_{\zeta_H(\theta)}(f(t))\} \\
&= \min\{f^{-1}(\mathcal{F}_{\zeta_H(\theta)})(s), f^{-1}(\mathcal{F}_{\zeta_H(\theta)})(t)\},
\end{aligned}$$

Therefore, $f^{-1}(H)$ is single-valued neutrosophic soft K-subalgebra of $\mathcal{K}_1$. Hence $f^{-1}(H) \in \chi_2$ which shows that f is a single-valued neutrosophic soft continuous function from $(\mathcal{K}_1, E, \chi_1)$ into $(\mathcal{K}_2, E, \chi_2)$. $\square$

Proposition 3.16. Let χ_1 and χ_2 be any two single-valued neutrosophic soft topological spaces on $\mathcal{K}$ and $(\mathcal{K}, E, \chi_1)$ and $(\mathcal{K}, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces. Then for each $\theta \in E$, every homomorphism $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft continuous function.

Definition 3.17. Let χ be a single-valued neutrosophic soft topology on K-algebra $\mathcal{K}$. Let $H = (\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H})$ be a single-valued neutrosophic soft K-algebra (K-subalgebra) of $\mathcal{K}$ and χ_H a single-valued neutrosophic soft topology over H. Then H is called a *single-valued neutrosophic soft topological* K-algebra of $\mathcal{K}$ if the self mapping $\rho_a : (H, E, \chi_H) \to (H, E, \chi_H)$ defined as $\rho_a(u) = u \odot a$, $\forall\ a \in \mathcal{K}$, is a relatively single-valued neutrosophic soft continuous mapping.

Theorem 3.18. Let χ_1 and χ_2 be two single-valued neutrosophic soft topological spaces on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a homomorphism of K-algebras such that $f^{-1}(\chi_2) = \chi_1$. If for each $\theta \in E$, $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_2$, then for each $\theta \in E$, $f^{-1}(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_1$.

Proof. In order to prove that $f^{-1}(H)$ is a single-valued neutrosophic soft topological K-algebra of K-algebra $\mathcal{K}_1$. Firstly, we show that $f^{-1}(H)$ is a single-valued neutrosophic soft K-algebra of $\mathcal{K}_1$. One can easily show

that for all $s \neq e \in G$ and $\theta \in E$, $\mathcal{T}_{\zeta_\theta}(e) \geq \mathcal{T}_{\zeta_\theta}(s), \mathcal{I}_{\zeta_\theta}(e) \geq \mathcal{I}_{\zeta_\theta}(s), \mathcal{F}_{\zeta_\theta}(e) \leq \mathcal{F}_{\zeta_\theta}(s)$.
Let for any $s, t \in \mathcal{K}_1$ and $\theta \in E$,

$$\begin{aligned}\mathcal{T}_{f^{-1}(H)}(s \odot t) &= \mathcal{T}_H(f(s \odot t)) \\ &\geq \min\{\mathcal{T}_H(f(s)), \mathcal{T}_H(f(t))\} \\ &= \min\{\mathcal{T}_{f^{-1}(H)}(s), \mathcal{T}_{f^{-1}(H)}(t)\},\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{f^{-1}(H)}(s \odot t) &= \mathcal{I}_H(f(s \odot t)) \\ &\geq \min\{\mathcal{I}_H(f(s)), \mathcal{I}_H(f(t))\} \\ &= \min\{\mathcal{I}_{f^{-1}(H)}(s), \mathcal{I}_{f^{-1}(H)}(t)\},\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{f^{-1}(H)}(s \odot t) &= \mathcal{F}_H(f(s \odot t)) \\ &\geq \min\{\mathcal{F}_H(f(s)), \mathcal{F}_H(f(t))\} \\ &= \min\{\mathcal{F}_{f^{-1}(H)}(s), \mathcal{F}_{f^{-1}(H)}(t)\}.\end{aligned}$$

This shows that $f^{-1}(H)$ is a single-valued neutrosophic soft K-algebra of $\mathcal{K}_1$.
Since f is a homomorphism and also a single-valued neutrosophic soft continuous mapping, then clearly, f is relatively single-valued neutrosophic soft continuous mapping from (H, E, χ_H) into $(f^{-1}(H), E, \chi_{f^{-1}(H)})$ such that for a single-valued neutrosophic soft set V in χ_H, and a single-valued neutrosophic soft set U in $\chi_{(f^{-1}(H)}$,

$$f^{-1}(V) = U. \tag{1}$$

Now, we prove that the self mapping $\rho_a : (f^{-1}(H), E, \chi_{f^{-1}(H)}) \to (f^{-1}(H), E, \chi_{f^{-1}(H)})$ is relatively single-valued neutrosophic soft continuous mapping. Now, for any $a \in \mathcal{K}_1$ and $\theta \in E$, we have

$$\begin{aligned}\mathcal{T}_{\rho_a^{-1}(U)}(s) &= \mathcal{T}_{(U)}(\rho_a(s)) = \mathcal{T}_{(U)}(s \odot a) \\ &= \mathcal{T}_{f^{-1}(V)}(s \odot a) = \mathcal{T}_{(V)}(f(s \odot a)) \\ &= \mathcal{T}_{(V)}(f(s) \odot f(a)) = \mathcal{T}_{(V)}(\rho_{f(a)}(f(s))) \\ &= \mathcal{T}_{\rho^{-1}f(a)V}(f(s)) = \mathcal{T}_{f^{-1}}(\rho^{-1}_{f(a)}(V)(s)),\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{\rho_a^{-1}(U)}(s) &= \mathcal{I}_{(U)}(\rho_a(s)) = \mathcal{I}_{(U)}(s \odot a) \\ &= \mathcal{I}_{f^{-1}(V)}(s \odot a) = \mathcal{I}_{(V)}(f(s \odot a)) \\ &= \mathcal{I}_{(V)}(f(s) \odot f(a)) = \mathcal{I}_{(V)}(\rho_{f(a)}(f(s))) \\ &= \mathcal{I}_{\rho^{-1}f(a)V}(f(s)) = \mathcal{I}_{f^{-1}}(\rho^{-1}_{f(a)}(V)(s)),\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\rho_a^{-1}(U)}(s) = \mathcal{F}_{(U)}(\rho_a(s)) = \mathcal{F}_{(U)}(s \odot a)\\ = \mathcal{F}_{f^{-1}(V)}(s \odot a) = \mathcal{F}_{(V)}(f(s \odot a))\\ = \mathcal{F}_{(V)}(f(s) \odot f(a)) = \mathcal{F}_{(V)}(\rho_{f(a)}(f(s)))\\ = \mathcal{F}_{\rho^{-1} f(a) V}(f(s)) = \mathcal{F}_{f^{-1}}(\rho_{f(a)}^{-1}(V)(s)).\end{aligned}$$

This implies that $\rho_a^{-1}(U) = f^{-1}(\rho_{f(a)}^{-1}(V))$. Thus, $\rho_a^{-1}(U) \cap f^{-1}(H) = f^{-1}(\rho_{f(a)}^{-1}(V)) \cap f^{-1}(H)$ is a single-valued neutrosophic soft set in $f^{-1}(H)$ and a single-valued neutrosophic soft set in $\chi_{f^{-1}(H)}$. Hence $f^{-1}(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_1$. This completes the proof. □

Theorem 3.19. Let χ_1 and χ_2 be two single-valued neutrosophic soft topologies on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively and $f : \mathcal{K}_1 \to \mathcal{K}_2$ an isomorphism of K-algebras such that $f(\chi_1) = \chi_2$. If for each $\theta \in E$, $H = \{(\mathcal{T}_{\zeta_H(\theta)}, \mathcal{I}_{\zeta_H(\theta)}, \mathcal{F}_{\zeta_H(\theta)}) : \theta \in E\}$ is a single-valued neutrosophic soft topological K-algebra of K-algebra $\mathcal{K}_1$, then for each $\theta \in E$, $f(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_2$.

Proof. Let H be a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_1$. For $u, v \in \mathcal{K}_2$.

Let $t_o \in f^{-1}(u)$, $s_o \in f^{-1}(v)$ such that

$$\mathcal{T}_H(t_o) = \sup_{t \in f^{-1}(u)} \mathcal{T}_H(t), \mathcal{T}_H(y_o) = \sup_{t \in f^{-1}(v)} \mathcal{T}_H(t).$$

We now have,

$$\begin{aligned}\mathcal{T}_{f(H)}(u \odot v) = & \sup_{t \in f^{-1}(u \odot v)} \mathcal{T}_H(t)\\ \geq & \mathcal{T}_H(t_o, s_o)\\ \geq & \min\{\mathcal{T}_H(t_o), \mathcal{T}_H(s_o)\}\\ = & \min\{\sup_{t \in f^{-1}(u)} \mathcal{T}_H(t), \sup_{a \in f^{-1}(v)} \mathcal{T}_H(t)\}\\ = & \min\{\mathcal{T}_{f(H)}(u), \mathcal{T}_{f(H)}(v)\},\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{f(H)}(u \odot v) = & \sup_{t \in f^{-1}(u \odot v)} \mathcal{I}_H(t)\\ \geq & \mathcal{I}_H(t_o, s_o)\\ \geq & \min\{\mathcal{I}_H(t_o), \mathcal{I}_H(s_o)\}\\ = & \min\{\sup_{t \in f^{-1}(u)} \mathcal{I}_H(t), \sup_{t \in f^{-1}(v)} \mathcal{I}_H(t)\}\\ = & \min\{\mathcal{I}_{f(H)}(u), \mathcal{I}_{f(H)}(v)\},\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{f(H)}(u \odot v) = & \inf_{t \in f^{-1}(u \odot v)} \mathcal{F}_H(t) \\
\leq & \ \mathcal{F}_H(t_o, s_o) \\
\leq & \ \max\{\mathcal{F}_H(t_o), \mathcal{F}_H(s_o)\} \\
= & \ \max\{\inf_{t \in f^{-1}(u)} \mathcal{F}_H(t), \inf_{t \in f^{-1}(v)} \mathcal{F}_H(t)\} \\
= & \ \max\{\mathcal{F}_{f(H)}(u), \mathcal{F}_{f(H)}(v)\}.
\end{aligned}$$

Hence $f(H)$ is a single-valued neutrosophic soft K-subalgebra of $\mathcal{K}_2$. To show that $f(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_2$, i.e., the self map $\rho_b : (f(H), \chi_{f(H)}) \to (f(H), \chi_{f(H)})$, defined as $\rho_b(v) = v \odot b$, $\forall\, b \in \mathcal{K}_2$ is a relatively single-valued neutrosophic soft continuous mapping. Let Y_H be a single-valued neutrosophic soft set in χ_H, then there exists a single-valued neutrosophic soft set Y in χ_1 be such that $Y_H = Y \cap H$.

$$\rho^{-1}{}_b(Y_{f(H)}) \cap f(H) \in \chi_{f(H)}$$

Then $f(Y_H) = f(Y \cap H) = f(Y) \cap f(H)$ is a single-valued neutrosophic soft set in $\chi_{f(H)}$ since f is an injective function. Thus, f is relatively single-valued neutrosophic soft open. Since f is also an onto function, then for all $b \in \mathcal{K}_2$ and $a \in \mathcal{K}_1$, $a = f(b)$, we have

$$\begin{aligned}
\mathcal{T}_{f^{-1}(\rho^{-1}{}_b(Y_{f(H)}))}(u) &= \mathcal{T}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(H)}))}(u) \\
&= \mathcal{T}_{\rho^{-1}{}_{f(a)}(Y_{f(H)})}\big(f(u)\big) \\
&= \mathcal{T}_{(Y_{f(H)})}\big(\rho_{f(a)}(f(u))\big) \\
&= \mathcal{T}_{(Y_{f(H)})}\big(f(u) \odot f(a)\big) \\
&= \mathcal{T}_{f^{-1}(Y_{f(H)})}(u \odot a) \\
&= \mathcal{T}_{f^{-1}(Y_{f(H)})}(\rho_a(u)) \\
&= \mathcal{T}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(H)}))(u),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{f^{-1}(\rho^{-1}{}_b(Y_{f(H)}))}(u) &= \mathcal{I}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(H)}))}(u) \\
&= \mathcal{I}_{\rho^{-1}{}_{f(a)}(Y_{f(H)})}\big(f(u)\big) \\
&= \mathcal{I}_{(Y_{f(H)})}\big(\rho_{f(a)}(f(u))\big) \\
&= \mathcal{I}_{(Y_{f(H)})}\big(f(u) \odot f(a)\big) \\
&= \mathcal{I}_{f^{-1}(Y_{f(H)})}(u \odot a) \\
&= \mathcal{I}_{f^{-1}(Y_{f(H)})}(\rho_a(u)) \\
&= \mathcal{I}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(H)}))(u),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_{f^{-1}(\rho^{-1}{}_b(Y_{f(H)}))}(u) &= \mathcal{F}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(H)}))}(u)\\
&= \mathcal{F}_{\rho^{-1}{}_{f(a)}(Y_{f(H)})}\big(f(u)\big)\\
&= \mathcal{F}_{(Y_{f(H)})}\big(\rho_{f(a)}\big(f(u)\big)\big)\\
&= \mathcal{F}_{(Y_{f(H)})}\big(f(u)\odot f(a)\big)\\
&= \mathcal{F}_{f^{-1}(Y_{f(H)})}(u\odot a)\\
&= \mathcal{F}_{f^{-1}(Y_{f(H)})}(\rho_a(u))\\
&= \mathcal{F}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(H)}))(u).
\end{aligned}
$$

This shows that $f^{-1}(\rho_{(b)}^{-1}((Y_{f(H)}))) = \rho_{(a)}^{-1}(f^{-1}(Y_{(H)}))$. Since $\rho_a : (H, \chi_H) \to (H, \chi_H)$ is relatively single-valued neutrosophic soft continuous mapping and f is also relatively single-valued neutrosophic soft continues function. Therefore, $f^{-1}(\rho_{(b)}^{-1}((Y_{f(H)}))) \cap H = \rho_{(a)}^{-1}(f^{-1}(Y_{(H)})) \cap H$ is a single-valued neutrosophic soft set in χ_H. Thus, $f(f^{-1}(\rho_{(b)}((Y_{f(H)}))) \cap \mathcal{A}) = \rho_{(b)}^{-1}(Y_{f(\mathcal{A})}) \cap f(\mathcal{A})$ is a single-valued neutrosophic soft set in $\chi_{\mathcal{A}}$. $\square$

Example 3.20. Consider a K-algebra $\mathcal{K}$ on a cyclic group of order 8 and Cayley's table for $\odot$ is given Example 3.3, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. Consider a set of parameters $E = \{l_1, l_2\}$ and single-valued neutrosophic soft sets H, J, L defined as:

$$
\begin{aligned}
\zeta_H(l_1) &= \{(e, 0.8, 0.7, 0.2), (h, 0.6, 0.5, 0.4)\},\\
\zeta_H(l_2) &= \{(e, 0.7, 0.7, 0.2), (h, 0.6, 0.6, 0.5)\},
\end{aligned}
$$

$$
\begin{aligned}
\zeta_J(l_1) &= \{(e, 0.7, 0.7, 0.2), (h, 0.4, 0.1, 0.5)\},\\
\zeta_J(l_2) &= \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.5, 0.7)\},
\end{aligned}
$$

$$
\begin{aligned}
\zeta_L(l_1) &= \{(e, 0.9, 0.8, 0.1), (h, 0.7, 0.6, 0.4)\},\\
\zeta_L(l_2) &= \{(e, 0.9, 0.7, 0.1), (h, 0.7, 0.6, 0.4)\}
\end{aligned}
$$

for all $h \neq e \in G$. Then the family $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J, L\}$ is a single-valued neutrosophic soft topology on $\mathcal{K}$ and $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a single-valued neutrosophic soft topological space over $\mathcal{K}$. We define another single-valued neutrosophic soft set Q in $\mathcal{K}$ as:

$$
\begin{aligned}
\zeta_Q(l_1) &= \{(e, 0.8, 0.5, 0.1), (h, 0.6, 0.4, 0.3)\},\\
\zeta_Q(l_2) &= \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.6)\}.
\end{aligned}
$$

It is obvious that Q is a single-valued neutrosophic soft K-algebra of $\mathcal{K}$.
Now, we prove that the self map $\rho_a : (Q, E, \chi_Q) \to (Q, E, \chi_Q)$, defined as $\rho_a(s) = s \odot a$ for all $a \in \mathcal{K}$, is a relatively single-valued neutrosophic soft continuous mapping.
We get $Q \cap \emptyset_E = \emptyset_E, Q \cap 1_E = 1_E, Q \cap H = R_1, Q \cap J = R_2, Q \cap L = R_3$, where R_1, R_2, R_3 are as follows:

$$
\begin{aligned}
\zeta_{R_1}(l_1) &= \{(e, 0.8, 0.5, 0.2), (h, 0.6, 0.4, 0.4)\},\\
\zeta_{R_1}(l_2) &= \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.6)\},
\end{aligned}
$$

$$\zeta_{R_2}(l_1) = \{(e, 0.7, 0.5, 0.2), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_{R_2}(l_2) = \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.4, 0.7)\},$$

$$\zeta_{R_3}(l_1) = \{(e, 0.8, 0.5, 0.1), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_{R_3}(l_2) = \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.7)\}.$$

Thus, $\chi_Q = \{\emptyset_E, 1_E, R_1, R_2, R_3\}$ is a relatively topology of Q and (Q, E, χ_Q) is a single-valued neutrosophic soft subspace of $(\mathcal{K}, E, \chi_\mathcal{K})$. Since ρ_a is a homomorphism, then for a single-valued neutrosophic soft set $R \in \chi_Q$, $\rho_a^{-1}(R) \cap Q \in \chi_Q$. Which shows that $\rho_a : (Q, E, \chi_Q) \to (Q, E, \chi_Q)$ is relatively single-valued neutrosophic soft continuous mapping. Therefore, Q is a single-valued neutrosophic soft topological K-algebra.

4 Single-Valued Neutrosophic Soft C_5-connected K-Algebras

In this section, we discuss single-valued neutrosophic soft C_5-connected K-algebras.

Definition 4.1. Let $(\mathcal{K}, E, \chi_\mathcal{K})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. A *single-valued neutrosophic soft separation* of $(\mathcal{K}, E, \chi_\mathcal{K})$ is a pair of nonempty single-valued neutrosophic soft open sets H, J if the following conditions hold:

(i) $H \cup J = 1_E$.

(ii) $H \cap J = \emptyset_E$.

Definition 4.2. Let $(\mathcal{K}, E, \chi_\mathcal{K})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. Then $(\mathcal{K}, E, \chi_\mathcal{K})$ is called a *single-valued neutrosophic soft C_5-disconnected* if there exists a single-valued neutrosophic soft separation of $(\mathcal{K}, E, \chi_\mathcal{K})$, otherwise C_5-connected.

Definition 4.2 can be written as:

Definition 4.3. Let $(\mathcal{K}, E, \chi_\mathcal{K})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. If there exists a single-valued neutrosophic soft open set and single-valued neutrosophic soft closed set L such that $L \neq 1_E$ and $L \neq \emptyset_E$, then $(\mathcal{K}, E, \chi_\mathcal{K})$ is called a *single-valued neutrosophic soft C_5-disconnected*, otherwise $(\mathcal{K}, E, \chi_\mathcal{K})$ is called a single-valued neutrosophic soft C_5-connected.

Example 4.4. By considering Example 3.3, we consider a single-valued neutrosophic soft topological space $\chi_\mathcal{K} = \{\emptyset_E, 1_E, H, J, L\}$. Since $H \cap J \neq \emptyset_E$, $H \cap L \neq \emptyset_E$, $J \cap L \neq \emptyset_E$ and $H \cup J \neq 1_E$, $H \cup L \neq 1_E$, $J \cup L \neq 1_E$. Thus, $\chi_\mathcal{K}$ is a single-valued neutrosophic soft C_5-connected.

Example 4.5. Every indiscrete single-valued neutrosophic soft space is C_5-connected since the only single-valued neutrosophic soft sets in single-valued neutrosophic soft indiscrete space that are both single-valued neutrosophic soft open and single-valued neutrosophic soft closed are $\emptyset_E$ and 1_E.

Theorem 4.6. Let $(\mathcal{K}, E, \chi_\mathcal{K})$ be a single-valued neutrosophic soft topological space on K-algebra $\mathcal{K}$. Then $(\mathcal{K}, E, \chi_\mathcal{K})$ is a single-valued neutrosophic soft C_5-connected if and only if $\chi_\mathcal{K}$ contains only $\emptyset_E$ and 1_E which are both single-valued neutrosophic soft open and single-valued neutrosophic soft closed.

Proof. Straightforward. □

Proposition 4.7. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two K-algebras and $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$, $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ two single-valued neutrosophic soft topological spaces on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a single-valued neutrosophic soft continuous surjective function. If $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ is a single-valued neutrosophic soft C_5-connected space, then $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ is also single-valued neutrosophic soft C_5-connected.

Proof. Let $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ and $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ be two single-valued neutrosophic soft topological spaces and $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ be a single-valued neutrosophic soft C_5-connected space. We prove that $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ is also single-valued neutrosophic soft C_5-connected. Let us suppose on contrary that $(\mathcal{K}_2, \chi_2)$ be a single-valued neutrosophic soft C_5-disconnected space. According to Definition 4.3, we have both single-valued neutrosophic soft open set and single-valued neutrosophic soft closed set L such that $L \neq 1_{SN}$ and $L \neq \emptyset_{SN}$. Then $f^{-1}(L) = 1_{SN}$ or $f^{-1}(L) = \emptyset_{SN}$ since f is a single-valued neutrosophic soft continuous surjective mapping , where $f^{-1}(L)$ is both single-valued neutrosophic soft open set and single-valued neutrosophic soft closed set. Therefore, $L = f(f^{-1}(L)) = f(1_{SN}) = 1_{SN}$ and $L = f(f^{-1}(L)) = f(\emptyset_{SN}) = \emptyset_{SN}$, a contradiction. Hence $(\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft C_5-connected space. □

5 Single-Valued Neutrosophic Soft Super Connected K-Algebras

Definition 5.1. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$ and $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ a single-valued neutrosophic soft set in $\mathcal{K}$. Then the *interior* and *closure* of H in a K-algebra $\mathcal{K}$ is defines as:

$$H^{Int} = \bigcup\{O : O \text{ is a single-valued neutrosophic soft open set in } \mathcal{K} \text{ and } O \subseteq H\},$$
$$H^{Clo} = \bigcap\{C : C \text{ is a single-valued neutrosophic soft closed set in } \mathcal{K} \text{ and } H \subseteq C\}.$$

It is interesting to note that H^{Int}, being union of single-valued neutrosophic soft open sets is single-valued neutrosophic soft open and H^{Clo}, being intersection of single-valued neutrosophic soft closed set is single-valued neutrosophic soft closed.

Theorem 5.2. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space on $\mathcal{K}$. Let $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ be a single-valued neutrosophic soft set in $\chi_{\mathcal{K}}$. Then H^{Int} is the largest single-valued neutrosophic soft open set contained in H.

Proof. Obvious. □

Proposition 5.3. Let H be a single-valued neutrosophic soft set in $\mathcal{K}$. Then the following properties hold:

(i) $(1_E)^{Int} = 1_E$.

(ii) $(\emptyset_E)^{Clo} = \emptyset_E$.

(iii) $\overline{(H)}^{Int} = \overline{(H)^{Clo}}$.

(iv) $\overline{(H)}^{Clo} = \overline{(H)^{Int}}$.

Corollary 5.4. If H is a single-valued neutrosophic soft set in $\mathcal{K}$, then H is single-valued neutrosophic soft open if and only if $H^{Int} = H$ and H is a single-valued neutrosophic soft closed if and only if $H^{Clo} = H$.

Definition 5.5. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space on $\mathcal{K}$ and $\chi_{\mathcal{K}}$ be a single-valued neutrosophic soft topology on $\mathcal{K}$. Let $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ be a single-valued neutrosophic soft open set in $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft regular open* if

$$H = (H^{Clo})^{Int}.$$

Remark 5.6. (1) Every single-valued neutrosophic soft regular is single-valued neutrosophic soft open.
(2) Every single-valued neutrosophic soft clopen set is single-valued neutrosophic soft regular open.

Definition 5.7. Let $\chi_{\mathcal{K}}$ be a single-valued neutrosophic soft topology on $\mathcal{K}$. Then $\mathcal{K}$ is called a *single-valued neutrosophic soft super disconnected* if there exists a single-valued neutrosophic soft regular open set $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ such that $1_E \neq H$ and $\emptyset_E \neq H$. But if there does not exist such a single-valued neutrosophic soft regular open set H such that $1_E \neq H$ and $\emptyset_E \neq H$, then $\mathcal{K}$ is called *single-valued neutrosophic soft super connected.*

Example 5.8. Consider a K-algebra on a cyclic group of order 8 and Cayley's table for $\odot$ is given in Example 3.3, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. We have a single-valued neutrosophic soft topology $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J\}$, where H, J with a parametric set $E = \{l_1, l_2\}$ are given as:

$$\zeta_H(l_1) = \{(e, 0.8, 0.7, 0.2), (h, 0.6, 0.5, 0.4)\},$$
$$\zeta_H(l_2) = \{(e, 0.7, 0.7, 0.2), (h, 0.6, 0.6, 0.5)\},$$

$$\zeta_J(l_1) = \{(e, 0.7, 0.7, 0.2), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_J(l_2) = \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.5, 0.7)\},$$

for all $h \neq e \in G$.
Let L be a single-valued neutrosophic soft set in $\mathcal{K}$, defined by:

$$\zeta_L(l_1) = \{(e, 0.9, 0.8, 0.1), (h, 0.7, 0.6, 0.4)\},$$
$$\zeta_L(l_2) = \{(e, 0.9, 0.7, 0.1), (h, 0.7, 0.6, 0.4)\}.$$

Now, we have single-valued neutrosophic soft open sets : $\emptyset_E, 1_E, H, J$.
single-valued neutrosophic soft closed sets : $(\emptyset_E)^c = 1_E,\ (1_E)^c = \emptyset_E,\ (H)^c = H',\ (J)^c = J'$, where H', J' are obtained as:

$$\zeta_{H'}(l_1) = \{(e, 0.2, 0.7, 0.8), (h, 0.4, 0.5, 0.6)\},$$
$$\zeta_{H'}(l_2) = \{(e, 0.2, 0.7, 0.7), (h, 0.5, 0.6, 0.6)\},$$

$$\zeta_{J'}(l_1) = \{(e, 0.2, 0.7, 0.7), (h, 0.5, 0.1, 0.4)\},$$
$$\zeta_{J'}(l_2) = \{(e, 0.6, 0.6, 0.4), (h, 0.7, 0.5, 0.3)\},$$

for all $h \neq e \in G$. Then, interior and closure of a single-valued neutrosophic soft set L is obtained as:

$$L^{Int} = H,$$
$$L^{Clo} = 1_E.$$

For L to be a single-valued neutrosophic soft regular open, then $L = (L^{Clo})^{Int}$. But since $L = (1_E)^{Int} = 1_E \neq L$. This shows that $1_E \neq L \neq \emptyset_E$ is not a single-valued neutrosophic soft regular open set. By Definition 5.7, defined K-algebra is a single-valued neutrosophic soft super connected K-algebra.

6 Single-Valued Neutrosophic Soft Compactness K-Algebras

Definition 6.1. Let $\chi_{\mathcal{K}}$ be a single-valued neutrosophic soft topology on $\mathcal{K}$. Let H be a single-valued neutrosophic soft set in $\mathcal{K}$. A collection $\Omega = \{(\mathcal{T}_{\zeta_{H_i}}, \mathcal{I}_{\zeta_{H_i}}, \mathcal{F}_{\zeta_{H_i}}) : i \in I\}$ of single-valued neutrosophic soft sets in $\mathcal{K}$ is called a *single-valued neutrosophic soft open covering* of H if $H \subseteq \bigcup \Omega$. A finite sub-collection of Ω say $(\Omega^{'})$ is also a single-valued neutrosophic soft open covering of H, called a *finite subcovering* of H.

Definition 6.2. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space of $\mathcal{K}$. Let H be a single-valued neutrosophic soft set in $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft compact* if every single-valued neutrosophic soft open covering Ω of H has a finite sub-covering $(\Omega^{'})$.

Example 6.3. A single-valued neutrosophic soft topological space $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is single-valued neutrosophic soft compact if either $\mathcal{K}$ is finite or $\chi_{\mathcal{K}}$ is a finite single-valued neutrosophic soft topology on $\mathcal{K}$.

Proposition 6.4. Let $f : (\mathcal{K}_1, E, \chi_{\mathcal{K}_1}) \rightarrow (\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ be a single-valued neutrosophic soft continuous mapping, where $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ and $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ are two single-valued neutrosophic soft topological spaces of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. If H is a single-valued neutrosophic soft compact in $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$, then $f(H)$ is single-valued neutrosophic soft compact in $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$.

Proof. Let f be a single-valued neutrosophic soft continuous map from $\mathcal{K}_1$ into $\mathcal{K}_2$. Let $\Omega = \{f^{-1}(H_i : i \in I)\}$ be a single-valued neutrosophic soft open covering of H and $\Delta = \{H_i : i \in I\}$ a single-valued neutrosophic soft open covering of $f(H)$. Then there exists a single-valued neutrosophic soft finite sub-covering $\bigcup\limits_{I=1}^{n} f^{-1}(H_i)$ such that

$$H \subseteq \bigcup_{i=1}^{n} f^{-1}\big(H_i\big).$$

Thus,

$$f(H) \subseteq \bigcup_{i=1}^{n} \big(H_i\big)$$

$$
\begin{aligned}
H &\subseteq \bigcup_{i=1}^{n} f^{-1}(H_i) \\
f(H) &\subseteq f\Big(\bigcup_{i=1}^{n} f^{-1}(H_i)\Big) \\
f(H) &\subseteq \bigcup_{i=1}^{n} \big(f(f^{-1}(H_i))\big) \\
f(H) &\subseteq \bigcup_{i=1}^{n} (H_i).
\end{aligned}
$$

This shows that there exists a single-valued neutrosophic soft finite sub-covering of $f(H)$. Therefore, $f(H)$ is single-valued neutrosophic soft compact in $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$. □

7 Single-Valued Neutrosophic Soft Hausdorff K-Algebras

Definition 7.1. Let $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ be a single-valued neutrosophic soft set in a $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft point* if, for $\theta \in E$

$$\zeta_H(\theta) \neq \emptyset_E,$$

and

$$\zeta_H(\theta') = \emptyset_E,$$

for all $\theta' \in E - \{\theta\}$. A single-valued neutrosophic soft point in H is denoted by θ_H.

Definition 7.2. A single-valued neutrosophic soft point θ_H is said to *belong* to a single-valued neutrosophic soft set J, i.e., $\theta_H \in J$ if, for $\theta \in E$

$$\zeta_H(\theta) \leq \zeta_J(\theta).$$

Definition 7.3. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$ and θ_L, θ_Q be two single-valued neutrosophic soft points in $\mathcal{K}$. If for these two single-valued neutrosophic soft points, there exist two disjoint single-valued neutrosophic soft open sets H, J such that $\theta_L \in H$ and $\theta_Q \in J$. Then $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is called a *single-valued neutrosophic soft Hausdorff topological space* over $\mathcal{K}$ and $\mathcal{K}$ is called a single-valued neutrosophic soft Hausdorff K-algebra.

Example 7.4. Consider a K-algebra $\mathcal{K}$ on a cyclic group of order 8 and Cayley's table for $\odot$ is given in Example 3.3, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. Let $E = \{l\}$ and $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J\}$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. We define two single-valued neutrosophic soft points l_L, l_Q such that

$$
\begin{aligned}
l_L &= \{(e, 1, 0, 1), (h, 0, 0, 1)\}, \\
l_Q &= \{(e, 0, 0, 1), (h, 0, 1, 0)\}.
\end{aligned}
$$

Since for $l \in E, \zeta_L(l) \neq \emptyset_E, \zeta_Q(l) \neq \emptyset_E$, and $l_L \neq l_Q$, then clearly l_L and l_Q are two single-valued neutrosophic soft points. Now, consider two single-valued neutrosophic soft open sets H and J defined as:

$$\zeta_H(l) = \{(e,1,1,0),(h,0,0,1)\},$$
$$\zeta_J(l) = \{(e,0,0,1),(h,1,1,0)\},$$

for all $h \neq e \in G$. Since $\zeta_L(l) \leq \zeta_H(l)$ and $\zeta_Q(l) \leq \zeta_J(l)$, i.e., $l_L \in H$ and $l_Q \in J$ and $H \cap J = \emptyset_E$. Thus, $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a single-valued neutrosophic soft Hausdorff space and $\mathcal{K}$ is a single-valued neutrosophic soft Hausdorff K-algebra.

Theorem 7.5. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a single-valued neutrosophic soft homomorphism. Then $\mathcal{K}_1$ is a single-valued neutrosophic soft Hausdorff space if and only if $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra.

Proof. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a single-valued neutrosophic soft homomorphism and χ_1, χ_2 be two single-valued neutrosophic soft topologies on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Suppose that $\mathcal{K}_1$ is a single-valued neutrosophic soft Hausdorff space. To prove that $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra, Let for $l \in E$, l_L and l_Q be two single-valued neutrosophic soft points in χ_2 such that $l_L \neq l_Q$ with $u, v \in \mathcal{K}_1$, $u \neq v$. Then for these two distinct single-valued neutrosophic soft points, there exist two single-valued neutrosophic soft open sets H and J such that $l_L \in H$, $l_Q \in J$ with $H \bigcap J = \emptyset_E$. For $x \in \mathcal{K}_1$, we consider

$$(f^{-1}(l_L))(x) = l_L(f^{-1}(x)) = \begin{cases} s \in (0,1] & if x = f^{-1}(u), \\ 0 & otherwise. \end{cases}$$
$$= ((f^{-1}(l))_L(x))$$

Therefore, $f^{-1}(l_L) = (f^{-1}(l))_L$. Likewise, $f^{-1}(l_Q) = (f^{-1}(l))_Q$. Since f is a single-valued neutrosophic soft continuous function from $\mathcal{K}_1$ into $\mathcal{K}_2$ and also f^{-1} is a single-valued neutrosophic soft continuous function from $\mathcal{K}_2$ into $\mathcal{K}_1$, then there exist two disjoint single-valued neutrosophic soft open sets $f(H)$ and $f(J)$ of single-valued neutrosophic soft points l_L and l_Q, respectively be such that $f(H) \bigcap f(J) = f(\emptyset_E) = \emptyset_E$. This shows that $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra. The proof of converse part is straightforward. □

Theorem 7.6. let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a bijective single-valued neutrosophic soft continuous function, where $\mathcal{K}_1$ is a single-valued neutrosophic soft compact K-algebra and $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra. Then mapping f is a $\mathcal{K}_1$ is a single-valued neutrosophic soft homomorphism.

Proof. Let f be a bijective single-valued neutrosophic soft mapping from a single-valued neutrosophic soft compact K-algebra into a single-valued neutrosophic soft Hausdorff K-algebra. Then clearly, f is a single-valued neutrosophic soft homomorphism. We only prove that f is single-valued neutrosophic soft closed since f is a bijective mapping. Let a single-valued neutrosophic soft set $Q = \{\mathcal{T}_{\zeta_Q}, \mathcal{I}_{\zeta_Q}, \mathcal{F}_{\zeta_Q}\}$ be closed in K-algebra $\mathcal{K}_1$. Now if $Q = \emptyset_E$, then $f(Q) = \emptyset_E$ is single-valued neutrosophic soft closed in $\mathcal{K}_2$. But if $Q \neq \emptyset_E$, then being a subset of a single-valued neutrosophic soft compact K-algebra, Q is single-valued neutrosophic soft compact. Also $f(Q)$ is single-valued neutrosophic soft compact, being a single-valued neutrosophic soft continuous image of a single-valued neutrosophic soft compact K-algebra. Hence f is closed thus, f is a single-valued neutrosophic soft homomorphism. □

8 Conclusions

In 1998, Smarandache originally considered the concept of neutrosophic set from philosophical point of view. The notion of a single-valued neutrosophic set is a subclass of the neutrosophic set from a scientific and engineering point of view, and an extension of intuitionistic fuzzy sets [32]. In 1999, Molodtsov introduced the idea of soft set theory as another powerful mathematical tool to handle indeterminate and inconsistent data. This theory fixes the problem of establishing the membership function for each specific case by giving a parameterized outlook to indeterminacy. By using a hybrid model of these two mathematical techniques with a topological structure, we have developed the concept of single-valued neutrosophic soft topological K-algebras to analyze the element of indeterminacy in K-algebras. We have defined some certain concepts such as the interior, closure, C_5-connected, super connected, compactness and Hausdorff of single-valued neutrosophic soft topological K-algebras. In future, we aim to extend our notions to (1) Rough neutrosophic K-algebras, (2) Soft rough neutrosophic K-algebras, (3) Bipolar neutrosophic soft K-algebras, and (4) Rough neutrosophic K-algebras.

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Introduction to Non-Standard Neutrosophic Topology

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Mohammed A. Al Shumrani, Florentin Smarandache (2019). Introduction to Non-Standard Neutrosophic Topology. *Symmetry* 11, 0; DOI: 10.3390/sym11050000

Abstract: For the first time we introduce non-standard neutrosophic topology on the extended non-standard analysis space, called non-standard real monad space, which is closed under neutrosophic non-standard infimum and supremum. Many classical topological concepts are extended to the non-standard neutrosophic topology, several theorems and properties about them are proven, and many examples are presented.

Keywords: non-standard analysis; extended non-standard analysis; monad; binad; left monad closed to the right; right monad closed to the left; pierced binad; unpierced binad; non-standard neutrosophic mobinad set; neutrosophic topology; non-standard neutrosophic topology

1. Introduction to Non-Standard Analysis

The purpose of this study is to initiate for the first time a new field of research, called non-standard neutrosophic algebraic structures, and we start with non-standard neutrosophic topology (NNT) in this paper. Being constructed on the set of hyperreals, that includes the infinitesimals, NNT can further be utilized in neutrosophic calculus applications.

As a branch of mathematical logic, non-standard analysis [1] deals with *hyperreal numbers*, which include *infinitesimals* and *infinities*.

The introduction of infinitesimals in calculus has been debated philosophically in the history of mathematics since the time of G. W. Leibniz, with pros and cons. Many mathematicians prefer the *epsilon-delta* use in calculus concepts' definitions and theorems' proofs.

Besides calculus, non-standard analysis found applications in mathematical physics, mathematical economics, and in probability theory.

In 1998, Smarandache [3] used non-standard analysis in philosophy and in neutrosophic logic, in order to differentiate between *absolute truth* (which is truth in all possible worlds, according to Leibniz), and *relative truth* (which is, according to the same Leibniz, truth in at least one world). Let T represent the neutrosophic truth value, I the neutrosophic indeterminacy value, and F the neutrosophic falsehood value, with $T, I, F \in [^{-}0, 1^{+}]$. Then T (*absolute truth*) = $1^{+} = \mu\ (1^{+})$, while T (*relative truth*) = 1. This is analogously for *absolute falsehood* vs. *relative falsehood*, and *absolute indeterminacy* vs. *relative indeterminacy.*

Then he extended [3] the use of non-standard analysis to neutrosophic set (*absolute membership/indeterminacy/nonmembership* vs. *relative membership/indeterminacy/nonmembership* respectively) and to neutrosophic probability (*absolute occurrence/indeterminate occurrence/nonoccurence of an event* vs. *relative occurrence/indeterminate occurrence/nonoccurence of an event*, respectively).

We next recall several notions and results from classical non-standard analysis [2] that are needed to defining and developing the non-standard neutrosophic topology.

The set R^* of *nonstandard reals* (or *hyperreals*) is the generalization of the real numbers (R).

The *transfer principle* states that first-order statements that are valid in R are also valid in R^*.

$$R^* \text{ includes the infinites and the infinitesimals, which on the } \textit{hyperreal number line} \text{ may be represented as } 1/\varepsilon = \omega/1. \tag{1}$$

An *infinite* (or infinite number) (ω) is a number that is greater than anything:

$$1 + 1 + 1 + \ldots + 1 \text{ (for any number of finite terms)} \tag{2}$$

The infinitesimals are reciprocals of infinites.

An *infinitesimal* (or infinitesimal number) (ε) is a number ε such that $|\varepsilon| < 1/n$, for any non-null positive integer n.

An infinitesimal is so small that it cannot be measured, and it is very close to zero.

The infinitesimal in absolute value, is a number smaller than anything nonzero positive number.

In calculus one uses the infinitesimals.

$$\text{By } R_+{}^* \text{ we denote the set of positive non-zero hyperreal numbers.} \tag{3}$$

Left Monad {for simplicity, denoted [2] by (^-a) or only ^-a} was defined as:

$$\mu(^-a) = (^-a) = {}^-a = \overline{a} = \{a - x, x \in R_+{}^* | x \text{ is infinitesimal}\} \tag{4}$$

Right Monad {for simplicity, denoted [2] by (a^+) or only by a^+} was defined as:

$$\mu(a^+) = (a^+) = a^+ = \overset{+}{a} = \{a + x, x \in R_+{}^* | x \text{ is infinitesimal}\} \tag{5}$$

μ (a) is a *monad* (*halo*) of an element $a \in R^*$, which is formed by a subset of numbers infinitesimally close (to the left-hand side, or right-hand side) to a.

1.1. Non-Standard Analysis's First Extension

In 1998, Smarandache [3] introduced the pierced binad.

Pierced binad {for simplicity, denoted by $(^-a^+)$ or only $^-a^+$} was defined as:

$$\begin{aligned} &\mu(^-a^+) = (^-a^+) = {}^-a^+ = \overset{-+}{a} = \\ &= \{a - x, x \in R_+{}^* | x \text{ is infinitesimal}\} \cup \{a + x, x \in R_+{}^* | x \text{ is infinitesimal}\} \\ &= \{a \pm x, x \in R_+{}^* | x \text{ is infinitesimal}\} \end{aligned} \tag{6}$$

This extension was needed in order to be able to do union aggregations of non-standard neutrosophic sets, where a left monad μ ($^-$a) had to be united with a right monad μ (a$^+$), as such producing a pierced binad: $\mu(^-\mathrm{a}) \cup \mu(\mathrm{a}^+) =_N \mu(^-\mathrm{a}^+)$. Without this pierced binad we would not have been able to define the non-standard neutrosophic operators.

1.2. Non-Standard Analysis's Second Extension

Smarandache [4,5] introduced at the beginning of 2019 for the first time, the left monad closed to the right, the right monad closed to the left, and unpierced binad, defined as below:

Left Monad Closed to the Right

$$\begin{aligned} &\mu(\overset{-0}{a}) = (\overset{-0}{a}) = \overset{-0}{a} = \{a - x | x = 0, or\ x \in R_+{}^* \text{ and } x \text{ is infinitesimal}\} = \mu(^-a) \cup \{a\} = (^-a) \cup \{a\} \\ &= {}^-a \cup \{a\} \end{aligned} \tag{7}$$

Right Monad Closed to the Left

$$\mu(\overset{0+}{a}) = (\overset{0+}{a}) = \overset{0+}{a} = \{a + x | x = 0, or\ x \in R_+{}^*\ and\ x\ is\ infinitesimal\} = \mu(a^+) \cup \{a\} = (a^+) \cup \{a\} = a^+ \cup \{a\} \quad (8)$$

Unpierced Binad

$$\begin{aligned} \mu(\overset{-0+}{a}) = (\overset{-0+}{a}) = \overset{-0+}{a} &= \{a - x |\ x \in R_+{}^*\ and\ x\ is\ infinitesimal\} \cup \{a + x |\ x \in R_+{}^*\ and\ x\ is\ infinitesimal\} \cup \{a\} \\ &= \{a \pm x | x = 0, or\ x \in R_+{}^*\ and\ x\ is\ infinitesimal\} \\ &= \mu(^-a^+) \cup \{a\} = (^-a^+) \cup \{a\} = {}^-a^+ \cup \{a\} \end{aligned} \quad (9)$$

Therefore, as seen, the element $\{a\}$ has been included in both the left and right monads, and also in the pierced binad respectively.

All monads and binads are subsets of R^*.

This second extension was done in order to be able to compute the non-standard aggregation operators (negation, conjunction, disjunction, implication, equivalence) in non-standard neutrosophic logic, set, and probability, and now we need them in non-standard neutrosophic topology.

1.3. The Best Notations for Monads and Binads

For any standard real number $a \in R$, we employ the following notations for monads and binads:

$$\overset{m}{a} \in \{a, \overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a}\} \text{ and by convention } \overset{0}{a} = a; \quad (10)$$

where

$$m \in \{^-, {}^{-0}, {}^+, {}^{+0}, {}^{-+}, {}^{-0+}\} = \{^0, {}^-, {}^{-0}, {}^+, {}^{+0}, {}^{-+}, {}^{-0+}\}; \quad (11)$$

thus "m" written above the standard real number "a" means: a standard real number (^{0}, or nothing above), or a left monad ($^-$), or a left monad closed to the right ($^{-0}$), or a right monad ($^+$), or a right monad closed to the left ($^{0+}$), or a pierced binad ($^{-+}$), or a unpierced binad ($^{-0+}$) respectively.

Neutrosophic notations will have an index $_N$ associated to each symbol, for example: the classical symbol < (less than), becomes $<_N$ (neutrosophically less than, i.e., some indeterminacy is involved, especially with respect to infinitesimals, monads and binads).

Similarly for: $\cap$ and $\cap_N$, $\wedge$ and $\wedge_N$ etc.

1.4. Non-Standard Neutrosophic Inequalities

We have the following *neutrosophic non-standard inequalities* (taking into account the definitions of infinitesimals, monads and binads):

$$(^-a) <_N a <_N (a^+) \quad (12)$$

because

$$\forall x \in R^*_+, a - x < a < a + x \quad (13)$$

where x is a (nonzero) positive infinitesimal.

The converse also is true:

$$(a^+) >_N \text{a} >_N (-a) \quad (14)$$

Similarly:

$$(^-a) \leq_N (^-a^+) \leq_N (a^+) \quad (15)$$

To prove it, we rely on the fact that $(^-a^+) = (^-a) \cup (a^+)$ and the number a is in between the subsets (on the real number line) $^-a = (a - \varepsilon, a)$ and $a^+ = (a, a + \varepsilon)$, so:

$$(^-a) \leq_N (^-a) \cup (a^+) \geq_N (a^+) \quad (16)$$

Conversely, it is neutrosophically true too:

$$(a^{+}) \geq_N ({}^{-}a) \cup (a^{+}) \geq_N ({}^{-}a) \tag{17}$$

$$\text{Also, } \overset{-}{a} \leq_N \overset{-0}{a} \leq_N a \leq_N \overset{0+}{a} \leq_N \overset{+}{a} \text{ and } \overset{-}{a} \leq_N \overset{-+}{a} \leq_N \overset{-0+}{a} \leq_N \overset{+}{a} \tag{18}$$

Conversely, they are also neutrosophically true:

$$\overset{+}{a} \geq_N \overset{0+}{a} \geq_N a \geq_N \overset{-0}{a} \geq_N \overset{-}{a} \text{ and } \overset{+}{a} \geq_N \overset{-0+}{a} \geq_N \overset{-+}{a} \geq_N \overset{-}{a} \text{ respectively.} \tag{19}$$

Let a, b be two standard real numbers. If $a > b$, which is (standard) classical real inequality, then we have:

$$a >_N ({}^{-}b), a >_N (b^{+}), a >_N ({}^{-}b^{+}), a >_N \overset{-0}{b}, a >_N \overset{0+}{b}, a >_N \overset{-0+}{b}; \tag{20}$$

$$({}^{-}a) >_N b, ({}^{-}a) >_N ({}^{-}b), ({}^{-}a) >_N (b^{+}), ({}^{-}a) >_N ({}^{-}b^{+}), \overset{-}{a} >_N \overset{-0}{b}, \overset{-}{a} >_N \overset{0+}{b}, \overset{-}{a} >_N \overset{-0+}{b}; \tag{21}$$

$$(a^{+}) >_N b, (a^{+}) >_N b({}^{-}b), (a^{+}) >_N b(b^{+}), (a^{+}) >_N b({}^{-}b^{+}), \overset{+}{a} >_N \overset{-0}{b}, \overset{+}{a} >_N \overset{0+}{b}, \overset{+}{a} >_N \overset{-0+}{b}; \tag{22}$$

$$({}^{-}a^{+}) >_N b, ({}^{-}a^{+}) >_N ({}^{-}b), ({}^{-}a^{+}) >_N (b^{+}), ({}^{-}a^{+}) >_N ({}^{-}b^{+}), \text{etc.} \tag{23}$$

No non-standard order relationship between a and $({}^{-}a^{+})$,

$$\text{nor between } a \text{ and } ({}^{-0}a^{+}). \tag{24}$$

1.5. Neutrosophic Infimum and Neutrosophic Supremum

1.5.1. Neutrosophic Infimum

Let $(S, <_N)$ be a set, which is neutrosophically partially ordered, and let M be a subset of S.

The neutrosophic infimum of M, denoted by $inf_N(M)$, is the neutrosophically greatest element in S, which is neutrosophically less than or equal to all elements of M.

1.5.2. Neutrosophic Supremum

Let $(S, <_N)$ be a set, which is neutrosophically partially ordered, and let M be a subset of S.

The neutrosophic supremum of M, denoted by $sup_N(M)$, is the neutrosophically smallest element in S, which is neutrosophically greater than or equal to all elements of M.

The neutrosophic infimum and supremum are both extensions of the classical infimum and supremum respectively, using the *transfer principle* from the real set R to the neutrosophic real *MoBiNad* set NR_{MB} defined below.

1.5.3. Property

$$\begin{array}{l}\text{If } \overset{m_1}{a}, \overset{m_2}{b} \text{ are left monads, right monads, pierced binads, or unpierced monads,}\\ \text{then both } \inf_N\{\overset{m_1}{a}, \overset{m_2}{b}\} \text{ and } \sup_N\{\overset{m_1}{a}, \overset{m_2}{b}\} \text{ are left monads or right monads.}\end{array} \tag{25}$$

1.6. Non-Standard Real MoBiNad Set

MoBiNad [3] etymologically comes from **mo***nad* + **bi***nad*.

Let R and R^* be the set of standard real numbers, and respectively the set of hyper-reals (or non-standard reals) that contains the infinitesimals and infinites.

The Non-standard Real MoBiNad Set [2] is built as follows:

$$NR_{MB} = {}_N\left\{ \begin{array}{c} \varepsilon, \omega, a, (^-a), (^-a^0), (a^+), (^0a^+), (^-a^+), (^-a^{0+}) \text{ |where } \varepsilon \text{ are infinitesimals,} \\ \text{with } \varepsilon \in \mathbb{R}^*; \omega = 1/\varepsilon \text{ are infinites, with } \omega \in \mathbb{R}^*; \text{and } a \text{ are real numbers, with } a \in \mathbb{R} \end{array} \right\} \quad (26)$$

or,

$$NR_{MB} = {}_N\left\{ \varepsilon, \omega, \overset{m}{a} \,\middle|\, \begin{array}{c} \text{where } \varepsilon, \omega \in \mathbb{R}^*, \varepsilon \text{ are infinitesimals, } \omega = \frac{1}{\varepsilon} \text{ are infinitesimals;} \\ a \in \mathbb{R}; \text{ and } m \in \{,^-,^{-0},^+,^{+0},^{-+},^{-0+}\} \end{array} \right\} \quad (27)$$

As a set, NR_{MB} is closed under addition, subtraction, multiplication, division [except division by $\overset{m}{a}$, with $a = 0$ and $m \in \{,^-,^{-0},^+,^{0+},^{-+},^{-0+}\}$], and power

$$\left\{ \left(\overset{m_1}{a}\right)^{\left(\overset{m_2}{b}\right)} \text{ with : either } a > 0, \text{ or a } \le 0 \text{ but } b = \frac{p}{r} \text{ (irreducible fraction) and } p, r \text{ are} \right. \quad (28)$$
positive integers with r an odd number}.

1.7. Remark

The neutrosophic infimum and neutrosophic maximum are well-defined on the Non-standard Real MoBiNad Set NR_{MB}, in the sense that we can compute inf_N and sup_N of any subset of NR_{MB}.

1.8. Non-Standard Real Open Monad Unit Interval

Since there is no relationship of order between a and $^-a^+$, not between a and $(^{-0}a^+)$, and we need a total order relationship on the set of non-standard real numbers, we remove all binads and keep only the open left monads and open right monads [we also remove the monads closed to one side].

$$]^-0,\ 1^+[_M = \{a,\ \varepsilon,^- a, a^+ | a \in [0,1], \varepsilon \in R^*, \varepsilon > 0\}. \quad (29)$$

where a is subunitary real number, and ε is an infinitesimal number.

The non-standard neutrosophic unit interval $]^-0, 1^+[_M$ includes the previously defined $]^-0, 1^+[$ as follows:

$$]^-0, 1^+[=_N (^-0) \cup [0,1] \cup (1^+) \subset_N]^-0, 1^+[_M \quad (30)$$

where the index $_M$ means that the interval includes all open monads and infinitesimals between $^-0$ and 1^+.

2. General Monad Neutrosophic Set

Let U be a universe of discourse, and S ⊂ U be a subset. Then, a *Neutrosophic Set* is a set for which each element x from S has a degree of membership (T), a degree of indeterminacy (I), and a degree of non-membership (F), with T, I, F standard or non-standard real monad subsets or infinitesimals, neutrosophically included in or equal to the nonstandard real monad unit interval $]^-, ^+[_M$, or

$$T, I, F \subseteq_N]^-0, 1^+[_M \quad (31)$$

where

$$^-0 \le_N inf_N T + inf_N I + inf_N F \le_N sup_N T + sup_N I + sup_N F \le 3^+. \quad (32)$$

2.1. Non-Standard Neutrosophic Set

Let us consider the above general definition of general neutrosophic set, and assume that at least one of T, I, or F (the neutrosophic components) is a non-standard real monad subset or infinitesimal, neutrosophically included in or equal to $]^-0, 1^+[_M$, where

$$^-0 \le_N inf_N T + inf_N I + inf_N F \le_N sup_N T + sup_N I + sup_N F \le 3^+, \quad (33)$$

we have a non-standard neutrosophic set.

2.2. Non-Standard Fuzzy t-Norm and Fuzzy t-Conorm

Let T_1, and T_2, $\in]^{-}0, 1^{+}[_M$, be nonstandard real numbers (infinitesimals, or open monads), or standard (classical) real numbers, such that at least one of them is a non-standard real number. T_1 and T_2 are non-standard fuzzy degrees of membership. Then one has:

The non-standard fuzzy t-norms:

$$T_1 /\backslash_F\ T_2 = inf_N\ \{T_1, T_2\} \tag{34}$$

The non-standard fuzzy t-conorms:

$$T_1 \backslash/_F\ T_2 = sup_N\ \{T_1, T_2\} \tag{35}$$

2.3. Aggregation Operators on Non-Standard Neutrosophic Set

Let T_1, I_1, F_1 and $T_2, I_2, F_2 \in]^{-}0, 1^{+}[_{MB}$, be nonstandard real numbers (infinitesimals, or monads), or standard (classical) real numbers, such that at least one of them is a non-standard real number.

Non-Standard Neutrosophic Conjunction

$$\begin{gathered}(T_1, I_1, F_1) \wedge_N (T_2, I_2, \mathrm{F}_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2) = \\ (inf_N\ (T_1, T_2), sup_N\ (I_1, I_2), sup_N\ (F_1, F_2))\end{gathered} \tag{36}$$

Non-Standard Neutrosophic Disjunctions

$$\begin{gathered}(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) = \\ (sup_N\ (T_1, T_2), inf_N\ (I_1, I_2), inf_N\ (F_1, F_2))\end{gathered} \tag{37}$$

Non-Standard Neutrosophic Complement/Negation

We may use the notations $\mathrm{C_N}$ or $\neg_N$ for the neutrosophic complement.

$$\mathrm{C_N}(T_1, I_1, F_1) = {}_N \neg_N (T_1, I_1, F_1) =_N (F_1, I_1, T_1). \tag{38}$$

Non-Standard Neutrosophic Inclusion/Inequality

$$(T_1, I_1, F_1) \leq {}_N (T_2, I_2, F_2)\ iff\ T_1 \leq_N T_2, I_1 \geq_N I_2, F_1 \geq_N F_2. \tag{39}$$

Let $A, B \in P\ (X)$, if $A \subseteq_N B$ then B is called a *neutrosophic superset* of A.

Non-standard Neutrosophic Equality

$$(T_1, I_1, F_1) =_N (T_2, I_2, F_2)\ \mathit{iff}\ (T_1, I_1, F_1) \leq {}_N (T_2, I_2, F_2)\ \mathit{and}\ (T_2, I_2, F_2) \leq {}_N (T_1, I_1, F_1). \tag{40}$$

Non-Standard Monad Neutrosophic Universe of Discourse

We now introduce for the first time the non-standard neutrosophic universe.

Definition 1. *A general set U, defined such that each element $x \in U$ has neutrosophic coordinates of the form $x(T_x, I_x, F_x)$, such that T_x represents the degree of truth-membership of the element x with respect to set U, I_x represents the degree of indeterminate-membership of the element x with respect to the set U, and F_x represents the degree of false-membership of the element x with respect to the set U; where T_x, I_x, and F_x are non-standard or standard subsets of the neutrosophic real monad set NR_M, but at least one of all of them is non-standard (i.e., contains infinitesimals, or open monads).*

Single-Valued Non-Standard Neutrosophic Topology

Let U be a single-valued non-standard neutrosophic universe of discourse, i.e., for all $x \in U$, their neutrosophic components T_x, I_x, F_x are single-values (either real numbers, or infinitesimals, or open monads) belonging to $]^{-}0, 1^{+}[$

Definition 2. *Let X be a non-standard neutrosophic subset of U. The neutrosophic empty-set, denoted by* $\boldsymbol{0}_N = (^{-}0, 1^{+}, 1^{+})$, *is a set* $\Phi_N \subset X$ *whose all elements have the non-standard neutrosophic components equal to* $(^{-}0, 1^{+}, 1^{+})$. *The whole set, denoted by* $1_N = (1^{+}, ^{-}0, ^{-}0)$, *is a set* $W_N \subset X$ *whose all elements have the non-standard neutrosophic components equal to* $(1^{+}, ^{-}0, ^{-}0)$.

Definition 3. *Let X be a non-standard neutrosophic set. Let* $A = (T_1, I_1, F_1)$ *and* $B = (T_2, I_2, F_2)$ *be non-standard neutrosophic numbers. Then:*

$$\mathrm{A} \cap \mathrm{B} = (\inf_N (\mathrm{T}_1, \mathrm{T}_2), \sup_N (\mathrm{I}_1, \mathrm{I}_2), \sup_N (\mathrm{F}_1, \mathrm{F}_2)) \tag{41}$$

$$\mathrm{A} \cup \mathrm{B} = (\sup_N (\mathrm{T}_1, \mathrm{T}_2), \inf_N (\mathrm{I}_1, \mathrm{I}_2), \inf_N (\mathrm{F}_1, \mathrm{F}_2)) \tag{42}$$

$$\mathrm{C}_N\mathrm{A} = (\mathrm{F}_1, \mathrm{I}_1, \mathrm{T}_1) \tag{43}$$

Definition 4. *Let X be a non-standard neutrosophic set. Let A(X) be the family of all non-standard neutrosophic sets in X. Let* $\tau \subseteq A(X)$ *be a family of non-standard neutrosophic sets in X. Then* τ *is called a Non-standard Neutrosophic Topology on X, if it satisfies the following axioms:*

(i) $\boldsymbol{0}_N$ *and* $\boldsymbol{1}_N$ *are in* τ.
(ii) *The intersection of the elements of any finite subcollection of* τ *is in* τ.
(iii) *The union of the elements of any subcollection of* τ *is in* τ.

The pair (X, τ) *is called a non-standard neutrosophic topological space. All members of* τ *are called non-standard neutrosophic open sets in X.*

Example 1. *Let X be a non-standard neutrosophic set. Let* τ *be the set consisting of* $\boldsymbol{0}_N$ *and* $\boldsymbol{1}_N$. *Then* τ *is a topology on X. It is called the non-standard neutrosophic trivial topology.*

Example 2. *Let X be a non-standard neutrosophic set. Let A be a non-standard neutrosophic set in X. Let* $\tau = \{\boldsymbol{0}_N, \boldsymbol{1}_N, A\}$. *Then it can be easily shown that* τ *is a topology on X.*

Example 3. *Let X be a non-standard neutrosophic set. Let A and B be non-standard neutrosophic sets in X such that A is a neutrosophic superset of B. Let* $\tau = \{\boldsymbol{0}_N, \boldsymbol{1}_N, A, B\}$. *Then since* $A \cap B = B$ *and* $A \cup B = A$ *we deduce that* τ *is a topology on X.*

Example 4. *Let X be a non-standard neutrosophic set. Suppose we have a nested sequence*

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \subseteq A_{n-1} \subseteq A_n \subseteq \tag{44}$$

of non-standard neutrosophic sets in X such that each A_n *is a neutrosophic superset of* A_{n-1} *for each*

$$n \in \{1, 2, 3, \ldots\}.$$

Let $\tau = \{\boldsymbol{0}_N, \boldsymbol{1}_N, A_n \colon n \in N\}$. *Then since* $A_i \cap_N A_j = A_i$ *and* $A_i \cup_N A_j = A_j$ *for each i less than j, we deduce that* τ *is a topology on X.*

Example 5. *Let X be a non-standard neutrosophic infinite set:*

$$X = \left(x_{m,n,p}\left(\left(\overset{+}{0.7}\right)^m, (0.2)^n, \left(\overset{-}{0.6}\right)^p\right), x_{m,n,p} \in X; m, n, p \in \{1, 2, \ldots\}\right) \tag{45}$$

Let M_{100} be a family of subsets of X, such that each member $A_{m,n,p}$ of the family has:

$$m, n, p \in \{1, 2, \ldots, 100\}. \tag{46}$$

Then $\tau = \{\boldsymbol{0}_N, \boldsymbol{1}_N, M_{100}\}$ is a non-standard neutrosophic topology.

Proof. Any monad $(\overset{m}{a})$ raised to the integer power $k > 0$, is equal to the monad of a^k:

$$\left(\overset{m}{a}\right)^k = \left(\overset{m}{a^k}\right) \tag{47}$$

Let's consider two non-standard neutrosophic elements from X:

$$x_{m_1,n_1,p_1}\left(\left(\overset{+}{0.7}\right)^{m_1}, (0.2)^{n_1}, \left(\overset{-}{0.6}\right)^{p_1}\right) \text{ and } x_{m_2,n_2,p_2}\left(\left(\overset{+}{0.7}\right)^{m_2}, (0.2)^{n_2}, \left(\overset{-}{0.6}\right)^{p_2}\right) \tag{48}$$

where

$$m_1, n_1, p_1, m_2, n_2, p_2 \in \{1, 2, \ldots, 100\}. \tag{49}$$

It is sufficient to prove that their non-standard neutrosophic finite intersection and the random union of elements from M_{100} are in M_{100}.

$$\begin{aligned} & x_{m_1,n_1,p_1} \cap_N x_{m_2,n_2,p_2} =_N \left(\inf_N\{\left(\overset{+}{0.7}\right)^{m_1}, \left(\overset{+}{0.7}\right)^{m_2}\},\right. \\ & \mathrm{SUP}_N\{(0.2)^{n_1}, (0.2)^{n_2}\}, \mathrm{SUP}_N\{\left(\overset{-}{0.6}\right)^{p_1}, \left(\overset{-}{0.6}\right)^{p_2}\}) \\ & = \left(\left(\overset{+}{0.7}\right)^{\max\{m_1,m_2\}}, (0.2)^{\min\{n_1,n_2\}}, \left(\overset{-}{0.6}\right)^{\min\{p_1,p_2\}}\right) \in M_{100} \end{aligned} \tag{50}$$

because also $max\{m_1, m_2\}$, $min\{n_1, n_2\}$, $min\{p_1, p_2\} \in M_{100}$. (51)

$$\begin{aligned} & \bigcup_{m,n,p \in (\psi_1,\psi_2,\psi_3) \subseteq \{1,2,\ldots,100\}^3} \{x_{m,n,p}\left(\left(\overset{+}{0.7}\right)^{m}, (0.2)^{n}, \left(\overset{-}{0.6}\right)^{p}\right)\} \\ & = \left(\left(\overset{+}{0.7}\right)^{\min\{m, m \in \psi_1\}}, (0.2)^{\max\{n, n \in \psi_2\}}, \left(\overset{-}{0.6}\right)^{\max\{p, p \in \psi_3\}}\right) \in M_{100} \end{aligned} \tag{52}$$

□

Definition 5. *Let X be a non-standard neutrosophic set. Suppose that τ and τ′ are two topologies on X such that τ ⊂ τ ′. Then we say that τ′ is finer than τ.*

Example 6. *Let X be a non-standard neutrosophic set. Let A and B be non-standard neutrosophic sets in X such that A is a neutrosophic superset of B. Let $\tau = \{\boldsymbol{0}_N, \boldsymbol{1}_N, A\}$ and $\tau' = \{\boldsymbol{0}_N, \boldsymbol{1}_N, B\}$.*

Then τ′ is finer than τ.

Example 7. *Let's consider the above Example 5. In addition to M_{100}, let's define L_{100} as follows:*

$$L_{100} = \{x_{m,n,p}\left(\left(\overset{+}{0.7}\right)^{m}, (0.2)^{n}, \left(\overset{-}{0.6}\right)^{p}\right), x_{m,n,p} \in X; m, n, p \in \{2, 4, 6, \ldots, 100\}\} \tag{53}$$

The non-standard neutrosophic topology $\tau = \{\boldsymbol{0}_N, \boldsymbol{1}_N, M_{100}\}$ is a finer non-standard neutrosophic topology than the non-standard neutrosophic topology $\tau' = \{\boldsymbol{0}_N, \boldsymbol{1}_N, L_{100}\}$.

Definition 6. *The subset Z of a non-standard neutrosophic topological space X is called a non-standard neutrosophic closed set if its complement* C_N *(Z) is open in X.*

Example 8. *Let Y be a non-standard neutrosophic infinite set*

$$Y = \{y_{m,n}\left(\left(\overset{+}{0.5}\right)^m, \left(\overset{-}{0.1}\right)^n, \left(\overset{+}{0.5}\right)^m\right), y_{m,n} \in Y; m, n \in \{1, 2, \ldots\}\} \tag{54}$$

and P (Y) the power set of Y.

Let $\tau \subseteq P(Y)$ *be a non-standard neutrosophic topology.*

Each non-standard neutrosophic set $A \in \tau$ *is a non-standard neutrosophic open set and closed set in the same time, because its non-standard neutrosophic complement* $C_N(A) = A$.

Proof. For any $y_{m,n} \in Y$ one has:

$$C_N\left(y_{m,n}\right) = C_n\left(\left(\overset{+}{0.5}\right)^m, \left(\overset{-}{0.1}\right)^n, \left(\overset{+}{0.5}\right)^m\right) = \left(\left(\overset{+}{0.5}\right)^m, \left(\overset{-}{0.1}\right)^n, \left(\overset{+}{0.5}\right)^m\right) = y_{m,n} \tag{55}$$

□

Theorem 1. *Unlike in classical topology, the non-standard neutrosophic empty-set* $\mathbf{0}_N$ *and the non-standard neutrosophic whole set* $\mathbf{1}_N$ *are not necessarily closed, since they are not the non-standard neutrosophic complement of each other.*

Proof.

$$C_N\ ({}^-0, 1^+, 1^+) =_N (1^+, 1^+, {}^-0) \neq (1^+, {}^-0, {}^-0), \text{ and reciprocally:} \tag{56}$$

$$C_N\ (1^+, {}^-0, {}^-0) =_N ({}^-0, {}^-0, 1^+) \neq ({}^-0, 1^+, 1^+). \tag{57}$$

Theorem 2. *In a non-stardard neutrosophic topology there may be non-standard neutrosophic sets which are both open and closed set.*

Proof. See the above Example 8. □

Theorem 3. *Unlike in classical topology, the intersection of two non-standard neutrosophic closed sets is not necessarily a non-standard neutrosophic closed set. Moreover, the union of two non-standard neutrosophic closed sets is not necessarily a non-standard neutrosophic closed set.*

Proof. Consider Example 3 above.

Let $A = (T_2, I_2, F_2)$ and $B = (T_1, I_1, F_1)$. Note that $C_N A = (F_2, I_2, T_2)$ and $C_N B = (F_1, I_1, T_1)$. (58)

$$\text{Then } C_N A \cap_N C_N B = (F_2, I_1, T_2). \tag{59}$$

$$\text{Since } C_N\ (C_N A \cap_N C_N B) = (T_2, I_1, F_2) \tag{60}$$

is not non-standard neutrosophic open set in X, we have that $C_N A \cap_N C_N B$ is not a non-standard neutrosophic closed set in X. Also,

$$C_N A \cap_N C_N B = (F_1, I_2, T_1). \tag{61}$$

$$\text{Since } C_N\ (C_N A \cap_N C_N B) = (T_1, I_2, F_1) \tag{62}$$

is not non-standard neutrosophic open set in X, we have that $C_N A \cap_N C_N B$ is not a non-standard neutrosophic closed set in X. □

General Remark 1. *Since the non-standard neutrosophic aggregation operators (conjunction, disjunction, complement) needed in non-standard neutrosophic topology, are defined by* ***classes of operators*** *(not by exact unique operators) respectively, the classical topological space theorems and properties extended (by the transfer principle) to the non-standard neutrosophic topological space may be valid for some non-standard neutrosophic operators, but invalid for other classes of neutrosophic aggregation operators.*

Even worth, due to the fact that non-standard neutrosophic conjunction/disjunction/complement are, in addition, based on fuzzy t-norms and fuzzy t-conorms, which are not fixed either, but characterized by classes!

{Similarly for fuzzy and intuitionistic fuzzy aggregation operators.}

For example, the neutrosophic intersection/$\wedge_N$ can be defined in 2 ways:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) \tag{63}$$

And

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2). \tag{64}$$

In turn, the fuzzy t-norms ($\wedge_F$) and fuzzy t-conorm ($\vee_F$) are also defined in many ways; for example I know at least 3 types of fuzzy t-norms:

$$a \wedge_F b = min\,\{a, b\} \tag{65}$$

$$a \wedge_F b = ab \tag{66}$$

$$a \wedge_F b = max\,\{a + b - 1, 0\} \tag{67}$$

and 3 types of fuzzy t-conorms:

$$a \vee_F b = max\,\{a, b\} \tag{68}$$

$$a \wedge_F b = a + b - ab \tag{69}$$

$$a \wedge_F b = min\,\{a + b, 1\} \tag{70}$$

therefore there exist at least 2·3·3 = 18 possibilities to define the neutrosophic *t-norm* ($\wedge_N$).

There exist at least the same number 18 of possibilities of defining the neutrosophic *t-conorm* ($\vee_N$).

From these 18 possibilities of defining/$\wedge_N$ and $\vee_N$ for some of them the classical topological theorems extended to non-standard neutrosophic topology may be valid, for others invalid.

Definition 7. *Let (X, τ) be a nonstandard neutrosophic topological space. Let A be a non-standard neutrosophic set in X. Then the Non-standard Neutrosophic Closure of A is the intersection of all non-standard neutrosophic closed supersets of A, and we denote it by cl_N (A). The Non-standard Neutrosophic Closure of A is the smallest nonstandard neutrosophic closed set in X that neutrosophically includes A.*

Example 9. *Let X be a non-standard neutrosophic set:*

$$X = \{x_1(\overset{-}{0.4}, \overset{+}{0.1}, \overset{-}{0.5}), x_2(\overset{-}{0.5}, \overset{+}{0.1}, \overset{-}{0.4}), x_3(\overset{-}{0.5}, \overset{+}{0.1}, \overset{-}{0.5})\} \tag{71}$$

and the following non-standard neutrosophic topology:

$$\tau = \{\Phi_N, 1_N, A_1\{x_1(\overset{-}{0.4}, \overset{+}{0.1}, \overset{-}{0.5}), A_2\{x_2(\overset{-}{0.5}, \overset{+}{0.1}, \overset{-}{0.4}), A_3\{x_3(\overset{-}{0.5}, \overset{+}{0.1}, \overset{-}{0.5})\}\} \tag{72}$$

where

$$\Phi_N = \{x_1(\overset{-}{0}, \overset{+}{1}, \overset{+}{1}), x_2(\overset{-}{0}, \overset{+}{1}, \overset{+}{1}), x_3(\overset{-}{0}, \overset{+}{1}, \overset{+}{1}), 1_N = x_1(\overset{+}{1}, \overset{-}{0}, \overset{-}{0}), x_1(\overset{+}{1}, \overset{-}{0}, \overset{-}{0}), x_1(\overset{+}{1}, \overset{-}{0}, \overset{-}{0})\} \tag{73}$$

Proof. τ is a non-standard neutrosophic topology because:

$$A_1 \cap_N A_2 = A_1, A_1 \cap_N A_3 = A_1, A_2 \cap_N A_3 = A_3 \tag{74}$$

$$A_1 \cup_N A_2 = A_2, A_1 \cup_N A_3 = A_3, A_2 \cup_N A_3 = A_2, A_1 \cup_N A_2 \cup_N A_3 = A_2. \tag{75}$$

(X, τ) is a non-standard neutrosophic topological space.

The non-standard neutrosophic sets A_1, A_2, A_3 are open sets since they belong to τ.

A_2 is the non-standard neutrosophic complement of A_1, or C_N $(A_2) = A_1$, therefore A_2 is a non-standard neutrosophic closed set in X.

A_3 is the non-standard neutrosophic complement of A_3 (itself), or C_N $(A_3) = A_3$, therefore A_3 is also a non-standard neutrosophic closed set in X.

A_2 and A_3 are nonstandard neutrosophic supersets of A_1, since $A_1 \subset A_2$ and $A_1 \subset A_3$.

Whence, the *Non-standard Neutrosophic Closure* of A_1 is the intersection of its non-standard neutrosophic closed supersets A_2 and A_3, or

$$cl_N\ (A_1) =_N A_2 \cap_N A_3 =_N A_3 \tag{76}$$

□

Definition 8. *The Non-standard Neutrosophic Interior of A is the union of all non-standard neutrosophic open subsets of A that are contained in A, and we denote it by int_N (A).*

The Non-standard Neutrosophic Interior of A is the largest non-standard neutrosophic open set in X that is neutrosophically included into A.

Example 10. *Into the previous Example 9, let's compute int_N (A_2).*

$$A_1 \text{ and } A_3 \text{ are non-standard neutrosophic open sets in } X, \text{ with } A_1 \subset_N A_2 \text{ and } A_3 \subset_N A_2 \tag{77}$$

Whence

$$\mathrm{int}_N\ (A_2) = A_1 \cup_N A_3 = A_3. \tag{78}$$

Definition 9. *Let (X, τ) be a non-standard neutrosophic topological space, and let $Y \subseteq_N X$ be a non-standard neutrosophic subset of X. Then the collection $\tau_Y = \{O \cap_N Y, O \in \tau\}$ is a topology on Y. It is called the non-standard neutrosophic subspace topology and Y is called a non-standard neutrosophic subspace of X.*

Example 11. *In the same previous Example 9, let's take $Y = A_3 \subset X$, and the non-standard neutrosophic subspace topology*

$$\tau_Y = \{\Phi_N, 1_N, A_3, \{(\overset{-}{0.5}, \overset{+}{0.1}, \overset{-}{0.5})\}\} \tag{79}$$

Then Y is a non-standard neutrosophic topological subspace of X.

Definition 10. *Let X and Y be two non-standard neutrosophic topological spaces. A map f:*

$$X \to Y \tag{80}$$

is said to be non-standard neutrosophic continuous map if for each non-standard neutrosophic open set A in Y, the set f^{-1} (A) is a non-standard neutrosophic open set in X.

Example 12. *Let X be a non-standard neutrosophic space. Let Y be a non-standard neutrosophic subspace of X. Then the inclusion map i:* $Y \to X$ *is non-standard neutrosophic continuous.*

Example 13. *Let X be a non-standard neutrosophic set. Suppose that* τ *and* τ' *are two non-standard neutrosophic topologies on X such that* τ' *is finer than* τ*. Then the identity map id:* $(X, \tau') \to (X, \tau)$ *is obviously non-standard neutrosophic continuous.*

Definition 11. *Let* (X_1, τ_1) *and* (X_2, τ_2) *be two non-standard neutrosophic topological spaces. Then* $\tau_1 \times \tau_2 =_N \{U \times V : U \in \tau_1, V \in \tau_2\}$ *defines a topology on the product*

$$X_1 \times X_2 \tag{81}$$

The topology $\tau_1 \times \tau_2$ *is called non-standard neutrosophic product topology.*

3. Development of Neutrosophic Topologies

Since the first definition of neutrosophic topology and neutrosophic topological space [3] in 1998, the neutrosophic topology has been developed tremendously in multiple directions and has added new topological concepts such as: neutrosophic crisp topological [6–9], neutrosophic crisp α-topological spaces [10], neutrosophic soft topological k-algebras [11–13], neutrosophic nano ideal topological structure [14], neutrosophic soft cubic set in topological spaces [15], neutrosophic alpha m-closed sets [16], neutrosophic crisp bi-topological spaces [17], ordered neutrosophic bi-topological space [18], neutrosophic frontier and neutrosophic semi-frontier [19], neutrosophic topological functions [20], neutrosophic topological manifold [21], restricted interval valued neutrosophic topological spaces [22], smooth neutrosophic topological spaces [23], nω–closed sets in neutrosophic topological spaces [24], and other topological properties [25,26], arriving now to the neutrosophic topology extended to the non-standard analysis space.

4. Conclusions

We have introduced for the first time the non-standard neutrosophic topology, non-standard neutrosophic toplogical space and subspace constructed on the non-standard unit interval $]-0, 1+[_M$ that is formed by real numbers and positive infinitesimals and open monads, together with several concepts related to them, such as: non-standard neutrosophic open/closed sets, non-standard neutrosophic closure and interior of a given set, and non-standard neutrosophic product topology. Several theorems were proven and non-standard neutrosophic examples were presented.

Non-standard neutrosophic topology (NNT) is initiated now for the first time. It is a neutrosophic topology defined on the *set of hyperreals*, while the previous neutrosophic topologies were initiated and developed on the *set of reals*.

The novelty of NNT is its possibility to be used in calculus due to the involvement of infinitesimals, while the previous neutrosophic topologies could not be used due to lack of infinitesimals.

Thus, the paper has contributed to the foundation of a new field of study, called non-standard neutrosophic topology.

As future work, we intend to study more non-standard neutrosophic algebraic structures, such as: non-standard neutrosophic group, non-standard neutroosphic ring and field, non-standard neutrosophic vector space and so on.

Funding: This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant No. KEP-34-130-38. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

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Neutrosophic Quadruple Vector Spaces and Their Properties

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W.B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2019). Neutrosophic Quadruple Vector Spaces and Their Properties. *Mathematics* 7, 758; DOI: 10.3390/math7080758

Abstract: In this paper authors for the first time introduce the concept of Neutrosophic Quadruple (NQ) vector spaces and Neutrosophic Quadruple linear algebras and study their properties. Most of the properties of vector spaces are true in case of Neutrosophic Quadruple vector spaces. Two vital observations are, all quadruple vector spaces are of dimension four, be it defined over the field of reals R or the field of complex numbers C or the finite field of characteristic p, Z_p; p a prime. Secondly all of them are distinct and none of them satisfy the classical property of finite dimensional vector spaces. So this problem is proposed as a conjecture in the final section.

Keywords: Neutrosophic Quadruple (NQ); Neutrosophic Quadruple set; NQ vector spaces; NQ linear algebras; NQ basis; NQ vector spaces; orthogonal or dual NQ vector subspaces

1. Introduction

In this section we just give a brief literature survey of this new field of Neutrosophic Quadruples [1]. Neutrosophic triplet groups, modal logic Hedge algebras were introduced in [2,3]. Duplet semigroup, neutrosophic homomorphism theorem and triplet loops and strong AG(1, 1) loops are defined and described in [4–6]. Neutrosophic triplet neutrosophic rings application to mathematical modelling, classical group of neutrosophic triplets on $\{Z_{2p}, \times\}$ and neutrosophic duplets in neutrosophic rings are developed and analyzed in [7–11]. Study of Algebraic structures of neutrosophic triplets and duplets, quasi neutrosophic triplet loops, extended triplet groups, AG-groupoids, NT-subgroups are carried out in [6,12–17]. Refined neutrosophic sets were developed by [18–21]. Neutrosophic algebraic structures in general were studied in [22–25]. The new notion of Neutrosophic Quadruples which assigns a known part happens to be very interesting and innovative, and was introduced by Smarandache [1,26] in 2015. Several research papers on the algebraic structure of Neutrosophic Quadruples, such as groups, monoids, ideals, BCI-algebras, BCI-positive implicative ideals, hyperstructures, BCK/BCI algebras [27–32] have been recently studied and analyzed. However in this paper authors have defined the new notion of Neutrosophic Quadruple vector spaces (NQ vector spaces) and Neutrosophic Quadruple linear algebras (NQ linear algebras) and have studied a few related properties. This work can later be used to propose neutrosophic based dynamical systems in particular in the area of hyperchoaos from cellular neural networks [33].

This paper is organized into five sections. Basic concepts needed to make this paper a self contained one is given in Section 2. NQ vector spaces are introduced in Section 3, further NQ subspaces are introduced and the notion of direct sum and NQ bases are analysed. It is shown all NQ vector spaces are of dimension 4 be it defined over R or C or Z_p, p a prime. Section 4 defines and develops the properties of NQ linear algebras. The final section proposes a conjecture which is related with the finite dimensional vector spaces, which are always isomorphic to finite direct product of fields over which the vector space is defined. Finally we give the future direction of research on this topic.

2. Basic Concepts

In this section basic concepts on vector spaces and a few of its properties and some NQ algebraic structures and their properties needed for this paper are given.

Through out this paper R denotes the field of reals, C denotes the field of complex numbers and Z_p denotes the finite field of characteristic p, p a prime. $NQ = \{(a, bT, cI, dF)$ denotes the Neutrosophic Quadruple; with a, b, c, d in R or C or Z_p, where T, I and F has the usual neutrosophic logic meaning of Truth, Indeterminate and False respectively and a denotes the known part [26].

For basic properties of vector spaces and linear algebras please refer [22].

Definition 1 ([22])**.** *A vector space or a linear space V consists of the following;*

1. *A field of R or C or Z_p of scalars.*
2. *A set V of objects called vectors.*
3. *A rule (or operation) called vector addition; which associates with each pair of vectors x, y in V; $x + y$ is in V, called sum of the vectors x and y in such a way that ;*
   (a) *$x + y = y + x$ (addition is commutative).*
   (b) *$x + (y + z) = (x + y) + z$ (addition is associative).*
   (c) *There is a unique vector 0 in V such that $x + 0 = x$ for all $x \in V$.*
   (d) *For each vector $x \in V$ there is a unique vector $-x \in V$ such that $x + -x = 0$.*
   (e) *A rule or operation called scalar multiplication that associates with each scalar $c \in R$ or C or Z_p and for a vector $x \in V$, called product denoted by '.' of c and x in such a way that for $x \in V$ and $c.x \in V$ and ;*
      i. *$c.x = x.c$ for every $x \in V$.*
      ii. *$(c + d).x = c.x + d.x$*
      iii. *$c.(x + y) = c.x + c.y$*
      iv. *$c.(d.x) = (c.d)x$;*

      for all $x, y \in V$ and c, d in R or C or Z_p.

We can just say $(V, +)$ is a vector space over a field R or C or Z_p if $(V, +)$ is an additive abelian group and V is compatible with the product by the scalars. If on V is defined a product such that $(V, \times)$ is a monoid and $c(x \times y) = (cx) \times y$ then V is a linear algebra over R or C or Z_p [22].

Definition 2 ([22])**.** *Let V be a vector space over R (or C or Z_p). A subspace of V is a subset W of V which is itself a vector space over R (or C or Z_p) with the operations of addition and scalar multiplication as in V.*

Definition 3. *Let V be a vector space over R (or C or Z_p). A subset B of V is said to be linearly dependent or simply dependent if there exist distinct vectors, $x_1, x_2, x_3, \ldots, x_t \in B$ and scalars $a_1, a_2, a_3, \ldots, a_t \in R$ or C or Z_p not all of which are zero such that $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_tx_t = 0$. A set which is not linearly dependent is called independent or linearly independent. If B contains only finitely many vectors $x_1, x_2, x_3, \ldots, x_k$ we sometimes say $x_1, x_2, x_3, \ldots, x_k$ are dependent instead of saying B is dependent.*

The following facts are true [22].

1. A subset of a linearly independent set is linearly independent.
2. Any set which contains a linearly dependent subset is linearly dependent.
3. Any set which contains the zero vector (0 vector) is linearly dependent for 1.0 = 0.
4. A set B is linearly independent if and only if each finite subset of B is linearly independent; that is if and only if there exist distinct vectors $x_1, x_2, x_3, \ldots, x_k$ of B such that $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_kx_k = 0$ implies each $a_i = 0; i = 1, 2, \ldots, k$.

For a vector space V over a field R or C or Z_p , the basis for V is a linearly independent set of vectors in V which spans the space V. We say the vector space V over R or C or Z_p is a direct sum of subspaces $W_1, W_2, \ldots, W_t$ if and only if $V = W_1 + W_2 + \ldots + W_t$ and $W_i \cap W_j$ is the zero vector for $i \neq j$ and $1 \leq i, j \leq t$.

The other properties of vector spaces are given in book [22].

Now we proceed on to recall some essential definitions and properties of Neutrosophic Quadruples [26].

Definition 4 ([26]). *The quadruple (a, bT, cI, dF) where $a, b, c, d \in R$ or C or Z_p, with T, I, F as in classical Neutrosophic logic with a the known part and (bT, cI, dF) defined as the unknown part, denoted by $NQ = \{(a, bT, cI, dF) | a, b, c, d \in R$ or C or $Z_n\}$ in called the Neutrosophic set of quadruple numbers.*

The following operations are defined on NQ, for more refer [26].

For $x = (a, bT, cI, dF)$ and $y = (e, fT, gI, hF)$ in NQ [26] have defined

$$x + y = (a, bT, cI, dF) + (e, fT, gI, hF) = (a + e, (b + f)T, (c + g)I, (d + h)F)$$

$$\text{and} \quad x - y = (a - e, (b - f)T, (c - g)I, (d - h)F)$$

are in NQ. For $x = (a, bT, cI, dF)$ in NQ and s in R or C or Z_p where s is a scalar and x is a vector in V. $s.x = s.(a, bT, cI, dF) = (sa, sbT, scI, sdF) \in V$.

If $x = 0 = (0, 0, 0, 0)$ in V usually termed as zero Neutrosophic Quadruple vector and for any scalar s in R or C or Z_p we have $s.0 = 0$.

Further $(s + t)x = sx + tx, s(tx) = (st)x, s(x + y) = sx + sy$ for all $s, t \in R$ or C or Z_p and $x, y \in NQ$. $-x = (-a, -bT, -cI, -dF)$ which is in NQ.

The main results proved in [26] and which is used in this paper are mentioned below;

Theorem 1 ([26]). *$(NQ, +)$ is an abelian group.*

Theorem 2 ([26]). *$(NQ, .)$ is a monoid which is commutative.*

We mainly use only these two results in this paper, for more literature about Neutrosophic Quadruples refer [26].

3. Neutrosophic Quadruple Vector Spaces and Their Properties

In this section we proceed on to define for the first time the new notion of Neutrosophic Quadruple vector spaces (NQ -vector spaces) their NQ vector subspaces, NQ bases and direct sum of NQ vector subspaces. All these NQ vector spaces are defined over R, the field of reals or C, the field of complex numbers and finite field of characteristic p, Z_p, p a prime. All these three NQ vector spaces are different in their properties and we prove all three NQ vector spaces defined over R or C or Z_P are of dimension 4.

We mostly use the notations from [26]. They have proved $(NQ, +) = \{(a, bT, cI, dF) | a, b, c, d \in R$ or C or Z_p, p a prime; $+\}$ is an infinite abelian group under addition.

We prove the following theorem.

Theorem 3. *$(NQ, +) = \{(a, bT, cI, dF) | a, b, c, d \in R$ or C or Z_p; p a prime, $+\}$ be the Neutrosophic quadruple group. Then $V = (NQ, +, \circ)$ is a Neutrosophic Quadruple vector space (NQ-vector space) over R or C or Z_p, where '$\circ$' is the special type of operation between V and R (or C or Z_p) defined as scalar multiplication.*

Proof. To prove V is a Neutrosophic quadruple vector space over R (or C or Z_p, p is a prime), we have to show all the conditions given in Section two (Definition 1) of this paper is satisfied. In the first place we have R or C or Z_p are field of scalars, and elements of V we call as vectors. It has been proved by [26] that $V = (NQ, +)$ is an additive abelian group, which is the basic property on V to be a vector space. Further the quadruple is defined using R or C or Z_p, p a prime, or used in the mutually exclusive sense. Now we see if $x = (a, bT, cI, dF)$ is in V and $n \in R$ (or C or Z_p) then the scalar multiplication '$\circ$' which associates with each scalar $n \in R$ and the NQ vector $x \in V$,

$n \circ x = n \circ (a, bT, cI, dF) = (n \circ a, n \circ bT, n \circ cI, n \circ dF)$ which is in V, called the product of n with x in such a way that

1. $1 \circ x = x \circ 1 \quad \forall x \in V$
2. $(nm) \circ v = n \circ (mv)$
3. $n \circ (v + w) = n \circ v + n \circ w$
4. $(m + n) \circ v = m \circ v + n \circ v$

for all $m, n \in R$ or C or Z_p and $v, w \in V$.

$0 = (0, 0, 0, 0)$ is the zero vector of V and for 0 in R or C or Z_p; we have $0 \circ x = 0 \circ (a, bT, cI, dF) = (0, 0, 0, 0); \forall x \in V$.

Clearly $V = (NQ, +, \circ)$ is a vector space known as the NQ vector space over R or C or Z_p. $\square$

However we can as in case of vector spaces say in case of NQ-vector spaces also $(NQ, +)$ is a NQ vector space with special scalar multiplication $\circ$.

We now proceed on to define the concept of linear dependence, linear independence and basis of NQ vector spaces.

Definition 5. *Let $V = (NQ, +)$ be a NQ vector space over R (or C or Z_p). A subset L of V is said to be NQ linearly dependent or simply dependent, if there exists distinct vectors $a_1, a_2, \ldots, a_k \in L$ and scalars $d_1, d_2, \ldots, d_k \in R$ (or C or Z_p) not all zero such that $d_1 \circ a_1 + d_2 \circ a_2 + \ldots + d_k \circ a_k = 0$. We say the set of vectors $a_1, a_2, \ldots, a_k$ is NQ linearly independent if it is not NQ linearly dependent.*

We provide an example of this situation.

Example 4. *Let $V = (NQ, +)$ vector space over R. Let $x = (3, -4T, 5I, 2F), y = (-2, 3T, -2I, -2F)$ and $z = (-1, T, -3I, 0)$ be in V. We see $1 \circ x + 1 \circ y + 1 \circ z = (0, 0, 0, 0)$, so x, y and z are NQ linearly dependent. Let $x = (5, 0, 0, 2F)$ and $y = (0, 5T, -3I, 0)$ be in V. We cannot find a $a, b \in R$ such that $a \circ x + b \circ y = (0, 0, 0, 0)$. If possible $a \circ x + b \circ y = (0, 0, 0, 0)$; this implies $a \circ 5 + b \circ 0 = 0$, forcing $a = 0$; $a \circ 0 + b \circ 5 = 0$, forcing $b = 0$; $a \circ 0 + b \circ -3 = 0$, forcing $b = 0$ and $a \circ 2 + b \circ 0 = 0$ forcing $a = 0$. Thus the equations are consistent and $a = b = 0$. So x and y are NQ linearly independent over R.*

The following properties are true in case of all vector spaces hence true in case of NQ vector spaces also.

1. A subset of a NQ linearly independent set is NQ linearly independent.
2. A set L of vectors in NQ is linearly independent if and only if for any distinct vectors $a_1, a_2, \ldots, a_k$ of L; $d_1 \circ a_1 + d_2 \circ a_2 + \ldots + d_k \circ a_k = 0$ implies each $d_i = 0$, for $i = 1, 2, \ldots, k$.

We now proceed on to define Neutrosophic Quadruple basis (NQ basis) for $V = (NQ, +)$, Neutrosophic Quadruple vector space over R or C or Z_p (or used in the mutually exclusive sense).

Definition 6. *Let $V = (NQ, +)$ vector space over R (or C or Z_p). We say a subset L of V spans V if and only if every vector in V can be got as a linear combination of elements from L and scalars from R (or C or Z_p). That is if $a_1, a_2, \ldots, a_n$ are n elements in L; then $v = d_1 \circ a_1 + d_2 \circ a_2 + \ldots + d_n \circ a_n$, is the NQ linear combination of vectors of L; where $d_1, d_2, \ldots, d_n$ are in R or C or Z_p and not all these scalars are zero.*

The Neutrosophic Quadruple basis for $V = (NQ, +)$ is a set of vectors in V which spans V. We say a set of vectors B in V is a basis of V if B is a linearly independent set and spans V over R or C or Z_p.

We say V is finite dimensional if the number of elements in basic of V is a finite set; otherwise V is infinite dimensional.

Theorem 5. *Let $V = (NQ, +)$ be the Neutrosophic Quadruple vector space over R (or C or Z_p). V is a finite dimensional NQ vector space over R (or C or Z_p) and dimension of these NQ vector spaces over R(or C or Z_p) are always four.*

Proof. Let $V = (NQ, +) = \{(a, bT, cI, dF)|a, b, c, d \in R$ (or C or Z_p), $+\}$, be the collection of all neutrosophic quadruples of the Neutrosophic Quadruple vector space over R (or C or Z_p). To prove dimension of V over R is four it is sufficient to prove that V has four linearly independent vectors which can span V, which will prove the result. Take the set $B = \{(1, 0, 0, 0), (0, T, 0, 0), (0, 0, I, 0), (0, 0, 0, F)\}$ contained in V; to show B is independent and spans V it enough if we prove for any $v = (a, bT, cI, dF) \in V$, v can be represented uniquely as a linear combination of elements from B and scalars from R (or C or Z_p). Now $v = (a, bT, cI, dF) = a \circ (1, 0, 0, 0) + b \circ (0, T, 0, 0) + c \circ (0, 0, I, 0) + d \circ (0, 0, 0, F)$ for the scalars $a, b, c, d \in R$ (or C or Z_p). Hence we see the elements of V are uniquely represented as a linear combination of vectors using only B, further B is a set of linearly independent elements, hence B is a basis of V and B is finite, so V is finite dimensional over R (or C or Z_p). As order of B is four, dimension of all NQ vector spaces V over R (or C or Z_p) is four. Hence the theorem. □

We call the NQ basis B as the special standard NQ basis of V.

Definition 7. *Let $V = (NQ, +)$ be a NQ vector space over R (or C or Z_p). A subset W of V is said to be Neutrosophic Quadruple vector subspace of V if W itself is a Neutrosophic Quadruple vector space over R (or C or Z_p).*

We will illustrate this situation by examples.

Example 6. *Let $V = \{NQ, +\}$ be a NQ vector space over R. $W = \{(a, bT, 0, 0)|a, b \in R\}$ is a subset of V which is a NQ vector subspace of V over R. $U = \{(0, 0, cI, dF)|c, d \in R\}$ is again a vector subspace of V and is different from W.*

We observe that the only common element between W and U is the zero quadruple vector $(0, 0, 0, 0)$.

Further it is observed if we define the dot product or inner product on elements in V. For $x = (a, bT, cI, dF)$ and $y = (e, fT, gI, hF) \in V$, $x \bullet y$ denoted as $x \bullet y = (a \bullet e, bT \bullet fT, cI \bullet gI, dF \bullet hF)$; and $x \bullet y$ is in V. If $x \bullet y = (0, 0, 0, 0)$ for some $x, y \in V$ then we say x is orthogonal (or dual) with y and vice versa. In fact $x \bullet y = y \bullet x$; $\forall x, y \in V$. We say two NQ vector subspaces W and U are orthogonal (or dual subspaces) if for every $x \in W$ and for every $y \in U$; $x \bullet y = (0, 0, 0, 0)$, that is two NQ vector subspaces are orthogonal if and only if the dot product of every vector in W with every vector in U is the zero vector.

$\{(0, 0, 0, 0)\}$ is the zero vector subspace of V. Every NQ vector subspace of V trivial or nontrivial is orthogonal with the zero vector subspace $\{(0, 0, 0, 0)\}$ of V. V the NQ vector space is orthogonal with only the zero vector subspace of V, and with no other vector subspace of V. W orthogonal U = $W \bullet U$ = $\{w \bullet u|w \in W$ and $u \in U\}$ = $\{(0, 0, 0, 0)\}$; we call the pair of NQ subspaces as orthogonal or dual NQ subspaces of V.

Definition 8. *Let $V = (NQ, +)$ be a Neutrosophic Quadruple vector space over R (or C or Z_p); $W_1, W_2, \ldots, W_n$ be n distinct NQ vector subspaces of V. We say $V = W_1 \oplus W_2 \oplus \ldots \oplus W_n$ is a direct sum of NQ vector subspaces if and only if the following conditions are true;*

1. *Every vector $v \in V$ can be written in the form $v = d_1 \circ w_1 + d_2 \circ w_2 + \ldots + d_n \circ w_n$, where $d_1, d_2, \ldots, d_n$ are in R (or C or Z_p) not all zero with $w_i \in W_i, i = 1, 2, \ldots, n$.*
2. *$W_i \bullet W_j = \{(0, 0, 0, 0)\}$ for $i \neq j$ and true for all i, j varying in the set $\{1, 2, \ldots, n\}$.*

First we record that in case of all NQ vector spaces over R (or C or Z_p) we can have the value of n given in definition to be only four, we cannot have more than four as dimension of all NQ vector spaces are only four. Secondly the minimum of n can be two which is true in case of all vector spaces of any finite dimension. Finally we wish to prove not all NQ vector subspaces are orthogonal and there are only finitely many nontrivial NQ vector subspaces for any NQ vector space over R (or C or Z_p).

We prove as theorem a few of the properties.

Theorem 7. *Let $V = (NQ, +)$ be a NQ vector space over R (or C or Z_p). V has only finite number of NQ vector subspaces.*

Proof. We see in case of NQ vector spaces over R (or C or Z_p) the dimension is four and the special standard NQ basis for V is $B = \{(1,0,0,0),(0,T,0,0),(0,0,I,0),(0,0,0,F)\}$. So any non trivial subspace of V can be of dimension less than four; so it can be 1 or 2 or 3. Clearly there are some vector subspaces of dimension one given by, $W_1 = \langle(1,0,0,0)\rangle$, $W_2 = \langle(0,T,0,0)\rangle$, $W_3 = \langle(0,0,I,0)\rangle$, $W_4 = \langle(0,0,0,F)\rangle$, $W_5 = \langle(1,T,0,0)\rangle$, $W_6 = \langle(1,0,I,0)\rangle$, $W_7 = \langle(1,0,0,F)\rangle$, $W_8 = \langle(0,T,I,0)\rangle$, $W_9 = \langle(0,T,0,F)\rangle$, $W_{10} = \langle(0,0,I,F)\rangle$, $W_{11} = \langle(1,T,I,0)\rangle$, $W_{12} = \langle(1,T,0,F)\rangle$, $W_{13} = \langle(1,0,I,F)\rangle$, $W_{14} = \langle(0,T,I,F)\rangle$ and $W_{15} = \langle(1,T,I,F)\rangle$. Some the two dimensional vector spaces are $U_1 = \langle(1,0,0,0),(0,T,0,0)\rangle$, $U_2 = \langle(1,0,0,0),(0,0,I,0)\rangle, \ldots, U_{105} = \langle(0,T,I,F),(1,T,I,F)\rangle$;

in fact there are 105 NQ vector subspaces of dimension two. Further there are 1365 NQ vector subspaces of dimension three. Thus there are 1485 non trivial NQ vector subspaces in any NQ vector space $V = (NQ,+)$ over R (or C or Z_p). We have shown that there are four NQ vector subspaces of dimension three all of them are hyper subspaces of V, of course we are not enumerating other types of dimension three subspaces generated by vectors of the form $M_1 = \{\langle(1,T,0,0),(0,0,I,0),(0,0,0,F)\rangle\}$, or $M_2 = \{\langle(1,0,0,F),(0,0,I,0),(0,T,0,0)\rangle\}$ are spaces of dimension three which we do not take into account as hyper subspaces. □

We define the three dimensional NQ vector subspace generated only by $\{\langle(0,T,0,0),(0,0,I,0),(0,0,0,F)\rangle\}$ is defined as the special pseudo Singled Valued Neutrosophic hyper NQ vector subspace of V [22,24].

4. Neutrosophic Quadruple Linear Algebras over R or C or Z_p

In this section we take the basic concepts defined in [26] $(NQ,+)$ for the Neutrosophic Quadruple additive abelian group and $(NQ,.)$ as the commutative monoid with $(1,0,0,0)$ as the identity with respect to '.' and for any $(a,bT,cI,dF) = x$, and $y = (e,fT,gI,hF)$ in NQ [26] have defined $x.y = (ae,(af+be+bf)T,(ag+bg+ce+cf+cg)I,(ah+bh+ch+de+df+dg+dh)F)$.

Theorem 8. *$V = (NQ,+,.)$ is a Neutrosophic Quadruple linear algebra (NQ linear algebra) over R (or C or Z_p).*

Proof. To prove V is a NQ linear algebra we have to prove the following; $(NQ,+)$ is an abelian group under addition given in [26] and it is proved that $(NQ,+)$ is a vector space (Theorem 3). To prove V is a NQ linear algebra it is sufficient if we prove $(NQ,.)$ is a monoid under product '.' which is proved in [26], further $d\circ(x.y) = (d\circ x).y$ for $d \in R$ (or C or Z_p) and $x,y \in V$ which is true as $x.y$ is in V. Thus $(V,+,.)$ is a NQ linear algebra over R (or C or Z_p). □

Definition 9. *Let $V = (NQ,+,.)$ be a NQ linear algebra over R (or C or Z_p). Let W be a nonempty proper subset of V, we say W is a NQ sublinear algebra of V over R (or C or Z_p), if W itself is a linear algebra over R (or C or Z_p).*

We provide some examples of them.

Example 9. *Let $V = (NQ,+.)$ be a linear algebra over the field Z_7. $W = \{\langle(1,0,0,0)\rangle\}$ generated under +, . and '∘' multiplication by scalar from elements of Z_7 is a sublinear algebra and of order 7 and dimension of W over Z_7 is one. Similarly $U = \{\langle(1,t,0,0),(0,0,I,0)\rangle\}$ generated by these two vectors is a sublinear algebra of dimension two. Just we show how the product of $x = (3,4T,I,5F)$ and $y = (2,3T,4I,F)$ in V is carried out; $x.y = (6,2T,I,2F)$ which is in V.*

We can as in case of NQ vector spaces derive all properties of NQ linear algebras , further as in case of NQ vector spaces dimension of all these NQ-linear algebras is four.

We in the following section propose some open conjectures and the future work to be carried out in this direction.

5. Conclusions and Open Conjectures

In this paper for the first time we define the notion of NQ vector spaces and NQ linear algebras. All the three NQ vector spaces are of dimension four only. The NQ vector space V over R, is different from the NQ vector space W over C, and both has infinite number of vectors; but is of dimension four and U the NQ vector space over Z_p has only p^4 elements and is of dimension four.

We know the classical result on vector spaces states "A vector space V of say dimension n (n a finite integer) defined over the field F is isomorphic to $F \times F \times \ldots \times F$ n-times"; in view of this we propose the following conjectures:

1. Is the NQ vector space V defined over R isomorphic to $R \times R \times R \times R$?
2. Is the NQ vector space W defined over C isomorphic to $C \times C \times C \times C$?
3. Is the NQ vector space U defined over Z_p isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$?

Finally we would be developing the new notion of NQ algebraic codes and analyse them for future research. In our opinion a new type of NQ algebraic codes can certainly be defined with appropriate modifications. Also we would develop the notion of Neutrosophic quadruples in which the unknown part would be these neutrosophic triplets or modified form of neutrosophic duplets which would be taken for further study.

Abbreviations

The following abbreviations is used in this manuscript:

NQ Neutrosophic Quadruple

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New Results on Neutrosophic Extended Triplet Groups Equipped with a Partial Order

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Xin Zhou, Ping Li, Florentin Smarandache, Ahmed Mostafa Khalil (2019). New Results on Neutrosophic Extended Triplet Groups Equipped with a Partial Order. *Symmetry* 11, 1514; DOI: 10.3390/sym11121514

Abstract: Neutrosophic extended triplet group (NETG) is a novel algebra structure and it is different from the classical group. The major concern of this paper is to present the concept of a partially ordered neutrosophic extended triplet group (po-NETG), which is a NETG equipped with a partial order that relates to its multiplicative operation, and consider properties and structure features of po-NETGs. Firstly, in a po-NETG, we propose the concepts of the positive cone and negative cone, and investigate the structure features of them. Secondly, we study the specificity of the positive cone in a partially ordered weak commutative neutrosophic extended triplet group (po-WCNETG). Finally, we introduce the concept of a po-NETG homomorphism between two po-NETGs, construct a po-NETG on a quotient set by providing a multiplication and a partial order, then we discuss some fundamental properties of them.

Keywords: partially ordered neutrosophic extended triplet group; positive cone; homomorphism; quotient set

1. Introduction

Groups play a very important role in algebraic structures [1–3], and have been applied in many other areas such as chemistry, physics, biology, etc. The concept of neutrosophic set theory is proposed by Smarandache in [4], which is the generalization of classical sets [5], fuzzy sets [6], and intuitionistic fuzzy sets [5,7]. Neutrosophic sets have received wide attention both on practical applications [8–10] and on theory as well [11,12]. The main idea of the concept of a neutrosophic triplet group (NTG), is defined in [13,14]. For an NTG $(G, *)$, every element a in G has its own neutral element (denoted by $neut(a)$) satisfying $a * neut(a) = neut(a) * a = a$, and there exists at least one opposite element (denoted by $anti(a)$) in G relative to $neut(a)$ satisfying $a * anti(a) = anti(a) * a = neut(a)$. Here, $neut(a)$ is not allowed to be equal to the classical identity element as a special case. By removing this restriction, the concept of neutrosophic extended triplet group (NETG), is presented in [13]. Many significant results and several studies on NTGs and NETGs can be found in [15–20]. On the other hand, some algebraic structures are equipped with a partial order that relates to the algebraic operations, such as ordered groups, ordered semigroups, ordered rings and so on [21–28].

Regarding these developments, as the motivation of this article, we will consider what it is like to endow a NETG with a partial order and introduce the concepts of partially ordered NETGs and positive cones. Then we consider a question: is a subset P of a NETG G the positive cone relative to some compatible order on G if P satisfies some conditions? To solve this problem, we investigate structure features of partially ordered NETGs and try to characterize the positive cones. Finally, we study properties of homomorphisms and quotient sets in partially ordered NETGs, and discuss the relationships between homomorphisms and congruences. In particular, the quotient set equipped with a special multiplication and a partial order provides a way to obtain a partially ordered NETG. All these results lay the groundwork for investigation of category properties of partially ordered NETGs.

The rest of this paper is organized as follows. In Section 2, we review some basic concepts, such as a neutrosophic extended triplet set, a neutrosophic extended triplet group, a weak commutative neutrosophic extended triplet group and a completely regular semigroup, and several results were published in [16,19]. In Section 3, we define a partially ordered neutrosophic extended triplet group and partially ordered weak commutative neutrosophic extended triplet group. Several of their interesting properties of partially ordered neutrosophic extended triplet group and partially weak commutative neutrosophic extended triplet group are explained. The homomorphisms and quotient sets of partially ordered neutrosophic extended triplet group are shown in Section 4. Finally, conclusions are given in Section 5.

2. Preliminaries

In this section, we recall some basic notions and results which will be used in this paper as indicated below.

Definition 1. *([13]) Let G be a non-empty set together with a binary operation $*$. Then G is called a neutrosophic extended triplet set if for any $a \in G$, there exist a neutral of "a" (denoted by $neut(a)$) and an opposite of "a" (denoted by $anti(a)$), such that $neut(a) \in G$, $anti(a) \in G$, and*

$$a * neut(a) = neut(a) * a = a;$$

$$a * anti(a) = anti(a) * a = neut(a).$$

The triplet $(a, neut(a), anti(a))$ is called a neutrosophic extended triplet.

Definition 2. *([13]) Let $(G, *)$ be a neutrosophic extended triplet set. If $(G, *)$ is a semigroup, then G is called a neutrosophic extended triplet group (for short, NETG).*

Proposition 1. *([[16] Theorems 1 and 2]) Let $(G, *)$ be a NETG. The following properties hold: $\forall a \in G$*

(1) $neut(a)$ is unique;
(2) $neut(a) * neut(a) = neut(a)$;
(3) $neut(neut(a)) = neut(a)$.

Notice that $anti(a)$ may be not unique for every element a in a NETG $(G, *)$. To avoid confusion, we use the following notations:

$anti(a)$ denotes any certain one opposite of a and $\{anti(a)\}$ denotes the set of all opposites of a.

Proposition 2. *([[19], Theorem 1]) Let $(G, *)$ be a NETG. The following properties hold: $\forall\, a \in G$, $\forall\, p, q \in \{anti(a)\}$*

(1) $p * neut(a) \in \{anti(a)\}$;
(2) $p * neut(a) = q * neut(a) = neut(a) * q$;
(3) $neut(p * neut(a)) = neut(a)$;
(4) $a \in \{anti(p * neut(a))\}$;
(5) $anti(p * neut(a)) * neut(p * neut(a)) = a$.

Definition 3. *([16]) Let $(G, *)$ be a NETG. If $a * neut(b) = neut(b) * a$ $(\forall a \in G, \forall b \in G)$, then G is called a weak commutative neutrosophic extended triplet group (WCNETG).*

Proposition 3. *([[16], Theorem 2]) Let $(G, *)$ be a NETG. Then G is a WCNETG iff G satisfies the following conditions: $\forall a \in G, \forall b \in G$*

(1) $neut(a) * neut(b) = neut(b) * neut(a)$;
(2) $neut(a) * neut(b) * a = a * neut(b)$.

Proposition 4. *([[16], Theorem 3]) Let $(G, *)$ be a WCNETG. The following properties hold: $\forall a \in G, \forall b \in G$*

(1) $neut(a) * neut(b) = neut(b * a)$;
(2) $anti(a) * anti(b) \in \{anti(b * a)\}$.

Definition 4. *([29]) A semigroup $(S, *)$ will be called completely regular if there exists a unary operation $a \mapsto a^{-1}$ on S with the properties:*

$$(a^{-1})^{-1} = a, \ a * a^{-1} * a = a, \ a * a^{-1} = a^{-1} * a.$$

Proposition 5. *([[19], Theorem 2]) Let $(G, *)$ be a groupoid. Then G is a NETG iff it is a completely regular semigroup.*

Note 1. In semigroup theory, a^{-1} is called the inverse element of a and it is unique. However, in a NETG, $anti(a)$ is called an opposite element of a and it may not be unique. From Proposition 5, we get that for arbitrary element a of a NETG $(G, *)$, if we define a unary operation $a \mapsto a^{-1}$ by $a^{-1} = anti(a) * neut(a)$, then $(G, *)$ is a completely regular semigroup.

In the following, we will regard all NETGs as completely regular semigroups, in which $a^{-1} = anti(a) * neut(a)$ for arbitrary element a. Then by Proposition 2, we have in a NETG $(G, *)$, for each $a \in G$, $a^{-1} \in \{anti(a)\}$ and $a^{-1} * a = a * a^{-1} = neut(a)$.

3. Partially Ordered NETGs

An NETG is a special set endowed with a multiplicative operation. Assuming that we introduce a partial order which is compatible with multiplication in a NETG, we will get the definition of partially ordered NETGs as indicated below.

Definition 5. *Let $(G, *)$ be a NETG. If there exists a partial order relation $\leq$ on G such that $a \leq b$ implying $c * a \leq c * b$ and $a * c \leq b * c$ for all $a \in G, b \in G, c \in G$, then $\leq$ is called a compatible partial order on G, and $(G, *, \leq)$ is called a partially ordered NETG (for short, po-NETG).*

Similarly, if $(G, *)$ is a WCNETG and endowed with a compatible partial order, then $(G, *, \leq)$ is called a partially ordered WCNETG (po-WCNETG). Hence, po-WCNETGs must be po-NETGs.

Remark 1. *Obviously, the properties of NETGs and WCNETGs are holding in po-NETGs and po-WCNETGs, respectively.*

In the following, we give an example of a po-NETG.

Example 1. *Let $G = \{0, a, b, c, 1\}$ with the Hasse diagram as shown in Figure 1, in which 0 denotes the bottom element (mean the element is smallest element w.r.t. to partial order) and 1 denotes the top element (mean the element is largest element w.r.t. to partial order) of G. Then G is a partially ordered set.*

Define multiplication $*$ on G as shown in Table 1 , where a, b, c to label the elements in the po-NETG and the multiplication $*$ among these elements.

Table 1. Multiplication $*$ on G.

*	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	0	1	1	1	1

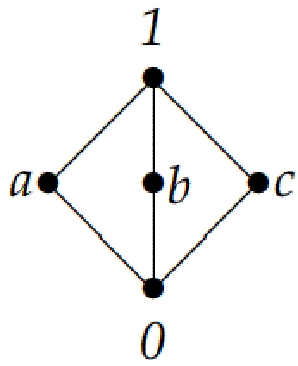

Figure 1. Hasse diagram.

We can verify that $(G,*)$ is a WCNETG. Moreover,

$$neut(0)=0,\ \{anti(0)\}=\{0,\ a,\ b,\ c,\ 1\},\ 0^{-1}=0;$$

$$neut(a)=c,\ \{anti(a)\}=\{b\},\ a^{-1}=b;$$

$$neut(b)=c,\ \{anti(b)\}=\{a\},\ b^{-1}=a;$$

$$neut(c)=c,\ \{anti(c)\}=\{c\},\ c^{-1}=c;$$

$$neut(1)=1,\ \{anti(1)\}=\{a,\ b,\ c,\ 1\},\ 1^{-1}=1.$$

It is easy to see that the partial order shown in Fig.1 is compatible with multiplication $*$. Hence, $(G,*,\leq)$ is a po-WCNETG.

Definition 6. *If $(G,*,\leq)$ is a po-NETG, then $a\in G$ is said to be a positive element if $neut(a)\leq a$; and a negative element if $a\leq neut(a)$. The subset P_G of all positive elements of G is called the positive cone of G, and the subset N_G of all negative elements the negative cone.*

Remark 2. *By Proposition 1, $\forall\ a\in G,\ neut(a)\in P_G\cap N_G$, so $P_G\cap N_G\neq\emptyset$.*

Lemma 1. *Let $(G,*)$ be an NETG. Then $\forall\ a\in G$,*

$$[neut(a)]^{-1}=neut(a)=neut(a^{-1}).$$

Proof. Let $a\in G$. Then

$$\begin{aligned}[neut(a)]^{-1}&=anti(neut(a))*neut(neut(a))\\&=anti(neut(a))*neut(a)\\&=neut(neut(a))\\&=neut(a).\end{aligned}$$

On the other hand, by Proposition 2(3), we have $neut(a^{-1}) = neut(anti(a) * neut(a)) = neut(a)$. $\square$

Remark 3. *If G is a po-NETG and $P \subseteq G$, we shall use the notation*

$$P^{-1} = \{a^{-1} : a \in P\}.$$

Proposition 6. *Let $(G, *, \leq)$ be a po-NETG. Then $P_G \cap P_G^{-1} = \{a \in G : a = neut(a) = a^{-1}\}$.*

Proof. ($\Longrightarrow$) Let $a \in G$. By Proposition 1 and Lemma 1, we have

$$neut(a) \in \{a \in G : a = neut(a) = a^{-1}\},$$

so $\{a \in G : a = neut(a) = a^{-1}\} \neq \emptyset$. By Lemma 1, it is clear that

$$\{a \in G : a = neut(a) = a^{-1}\} \subseteq P_G \cap P_G^{-1}.$$

($\Longleftarrow$) Let $b \in P_G \cap P_G^{-1}$, then $neut(b) \leq b$ and $\exists\, c \in P_G$ such that $b = c^{-1}$, so

$$b = c^{-1} = anti(c) * neut(c) \leq anti(c) * c = neut(c) = neut(b^{-1}) = neut(b),$$

that is, $b \leq neut(b)$, whence $b = neut(b)$. Hence,

$$c = b^{-1} = [neut(b)]^{-1} = neut(b) = b.$$

Then we can conclude that $b \in \{a \in G : a = neut(a) = a^{-1}\}$, and so

$$P_G \cap P_G^{-1} \subseteq \{a \in G : a = neut(a) = a^{-1}\}.$$

Thus, $P_G \cap P_G^{-1} = \{a \in G : a = neut(a) = a^{-1}\}$. $\square$

Remark 4. *If $(G, *, \leq)$ is a po-NETG and $P \subseteq G$, then we shall use the notation*

$$P^2 = \{a * b : a,\ b \in P\}.$$

Proposition 7. *(1) If $(G, *, \leq)$ is a po-NETG, then $P_G \subseteq P_G^2$.*
(2) If $(G, *, \leq)$ is a po-WCNETG, then $P_G = P_G^2$.

Proof. (1) If $(G, *, \leq)$ is a po-NETG, then $\forall\, a \in P_G$, by $neut(a) \in P_G$, we have $a = a * neut(a) \in P_G^2$, and so $P_G \subseteq P_G^2$.

(2) If $(G, *, \leq)$ is a po-WCNETG, then $\forall\, a \in P_G$, $\forall b \in P_G$, by Propositions 3 and 4, we have $neut(a * b) = neut(b) * neut(a) = neut(a) * neut(b) \leq a * b$, and so $a * b \in P_G$, thus $P_G^2 \subseteq P_G$. Consequently, $P_G = P_G^2$. $\square$

Proposition 8. *Let $(G, *, \leq)$ be a po-WCNETG. Then $\forall\, a \in G$, $aP_Ga^{-1} \subseteq P_G$.*

Proof. Let $a \in G$ and $b \in P_G$, then by Propositions 3 and 4, we have $neut(a * b * a^{-1}) = neut(a^{-1}) * neut(a * b) = neut(a * b) * neut(a^{-1}) = [neut(b) * neut(a)] * neut(a^{-1}) = neut(b) * [neut(a) * neut(a^{-1})] = neut(b) * neut(a^{-1} * a) = neut(b) * neut(neut(a)) = neut(b) * neut(a) = neut(b) * (a * a^{-1}) = [neut(b) * a] * a^{-1} = [a * neut(b)] * a^{-1} \leq a * b * a^{-1}$, thus $aba^{-1} \in P_G$. Therefore, $aP_Ga^{-1} \subseteq P_G$. $\square$

Lemma 2. *Let $(G, *)$ be a WCNETG. Then $\forall\, a \in G$, $\forall b \in G$, $(a * b)^{-1} = b^{-1} * a^{-1}$.*

Proof. We know $a * b$ is an element of $G\ \forall\ a \in G,\ \forall b \in G$ and by Proposition 4, we have $anti(b) * anti(a) \in \{anti(a * b)\}$. Then using Propositions 1, 5 and Note 1 we get the following identities:

$$\begin{aligned}
b^{-1} * a^{-1} &= [anti(b) * neut(b)] * [anti(a) * neut(a)] \\
&= anti(b) * [neut(b) * anti(a)] * neut(a) && \text{(Because the multiplication $*$ is associative)} \\
&= anti(b) * [anti(a) * neut(b)] * neut(a) && \text{(Because G is a WCNETG)} \\
&= [anti(b) * anti(a)] * [neut(b) * neut(a)] && \text{(Because the multiplication $*$ is associative)} \\
&= [anti(b) * anti(a)] * neut(a * b) && \text{(By Proposition 3)} \\
&= (a * b)^{-1}. \quad \square
\end{aligned}$$

Lemma 3. *Let $(G, *, \leq)$ be a po-NETG. Then $P_G = P_N^{-1}$ and $P_G^{-1} = P_N$.*

Proof. Let $a \in G$. If $a \in P_G$, then $neut(a) \leq a$, it follows by Lemma 1 that $a^{-1} = neut(a^{-1}) * a^{-1} = neut(a) * a^{-1} \leq a * a^{-1} = neut(a) = neut(a^{-1})$, and so $a^{-1} \in P_N$, whence $a = (a^{-1})^{-1} \in P_N^{-1}$. Hence, $P_G \subseteq P_N^{-1}$. Similarly, we can prove that if $a \in P_N$ then $a^{-1} \in P_G$, so $P_N^{-1} \subseteq P_G$. Consequently, $P_G = P_N^{-1}$. Similarly, $P_G^{-1} = P_N$. $\square$

Definition 7. *Let $(G, *)$ be a WCNETG. If $\forall a \in G,\ \forall b \in G,\ \forall c \in G,\ a * neut(c) = b * neut(c)$ implies $a = b$, then we say G satisfies neutrosophic cancellation law.*

Lemma 4. *Let $(G, *)$ be a WCNETG satisfying neutrosophic cancellation law and $P \subseteq G$ satisfy $\forall\ a \in P,\ a * a = a$. Then $\forall\ a \in G,\ \forall\ b \in G,\ a * neut(b) \in P$ implies $neut(a) = a = a^{-1}$.*

Proof. If $a * neut(b) \in P$, then $a * neut(b) = (a * neut(b)) * (a * neut(b)) = (a * a) * neut(b)$, and so $a * a = a$, whence $neut(a) = a\ \forall\ a \in G,\ \forall\ b \in G$. Then by Lemma 1, we get $a^{-1} = [neut(a)]^{-1} = neut(a) = a$. $\square$

Proposition 9. *Let $(G, *)$ be a WCNETG satisfying neutrosophic cancellation law and $P \subseteq G$ satisfy the following conditions:*

(1) $P^2 \subseteq P$;
(2) $P \cap P^{-1} = \{a \in G : neut(a) = a = a^{-1}\}$;
(3) $\forall\ a \in P,\ a * a = a$;
(4) $\forall\ a \in G,\ aPa^{-1} \subseteq P$,

then a compatible partial order on G exists such that P is the positive cone of G relative to it. Moreover, G is a chain with respect to this partial order if and only if $P \cup P^{-1} = G$.

Proof. Define the relation $\leq$ on G by

$$a \leq b \Leftrightarrow b * a^{-1} \in P.$$

By Proposition 1 and Lemma 1, we have $\forall\ a \in G,\ neut(a) \in P \cap P^{-1} \subseteq P$, and so $\leq$ is reflexive on G obviously.
If now $a \leq b$ and $b \leq a$, then $b * a^{-1} \in P$ and $a * b^{-1} \in P$. Since by Lemma 2 we know that

$$(a * b^{-1})^{-1} = (b^{-1})^{-1} * a^{-1} = b * a^{-1},$$

we conclude

$$b * a^{-1} \in P \cap P^{-1}.$$

It follows by (2) that $b * a^{-1} = neut(b * a^{-1})$. However, by Proposition 4 and Lemma 1,

$$neut(b * a^{-1}) = neut(a^{-1}) * neut(b) = neut(a) * neut(b),$$

thus

$b * neut(a) = b * a^{-1} * a = neut(b * a^{-1}) * a = [neut(a) * neut(b)] * a = neut(a) * [a * neut(b)] = [neut(a) * a] * neut(b) = a * neut(b)$, that is, $b * neut(a) = a * neut(b)$.

However, by Proposition 3, we have

$$b * neut(a) = neut(b) * neut(a) * b = neut(a * b) * b,$$

and similarly,

$$a * neut(b) = neut(a) * neut(b) * a = [neut(b) * neut(a)] * a = neut(a * b) * a,$$

therefore,

$$neut(a * b) * b = neut(a * b) * a,$$

and by neutrosophic cancellation law, consequently $a = b$. Hence, $\leq$ is anti-symmetric.

To prove that $\leq$ is transitive, let $a \leq b$ and $b \leq c$. Then

$$b * a^{-1} \in P \text{ and } c * b^{-1} \in P.$$

It follows by (1) that

$$P \supseteq P^2 \ni (c * b^{-1}) * (b * a^{-1}) = c * (b^{-1} * b) * a^{-1} = c * neut(b) * a^{-1} = (c * a^{-1}) * neut(b).$$

By (3) and Lemma 4, we have

$$neut(c * a^{-1}) = c * a^{-1} = (c * a^{-1})^{-1},$$

and so

$$c * a^{-1} \in P \cap P^{-1} \subseteq P,$$

that is, $c * a^{-1} \in P$. Thus, $a \leq c$. Therefore, $\leq$ is a partial order on G.

To see that it is compatible, let $x \leq y$. Then $y * x^{-1} \in P$ and it follows by (1) and (4) that, for every $a \in G$,

$$(a * y) * (a * x)^{-1} = (a * y) * (x^{-1} * a^{-1}) = a * (y * x^{-1}) * a^{-1} \in P,$$

$$(y * a) * (x * a)^{-1} = y * (a * a^{-1}) * x^{-1} = y * neut(a) * x^{-1} = (y * x^{-1}) * neut(a) \in P^2 \subseteq P,$$

which shows that

$$a * x \leq a * y \text{ and } x * a \leq y * a.$$

It follows that $\leq$ is compatible.

Finally, note that $\forall\, a \in G$,

$$neut(a) \leq a \Leftrightarrow a * [neut(a)]^{-1} \in P \Leftrightarrow a * neut(a) \in P \Leftrightarrow a \in P,$$

so P is the associated positive cone. Suppose now that $(G, \leq)$ is a chain, then for every $a \in G$, we have either

$$neut(a) \leq a \text{ or } a \leq neut(a).$$

It follows by Lemma 3 that

$$a \in P \text{ or } a \in P^{-1}.$$

Thus $G = P \cup P^{-1}$. Conversely, if $G = P \cup P^{-1}$, then for all $a,\ b \in G$, we have

$$a * b^{-1} \in P \text{ or } a * b^{-1} \in P^{-1},$$

that is,

$$a * b^{-1} \in P \text{ or } b * a^{-1} = (a * b^{-1})^{-1} \in P.$$

Hence, we have either $b \leq a$ or $a \leq b$. Therefore, $(G, \leq)$ is a chain. □

By the following example, we clarify the above proposition as:

Example 2. *Let $G = \{a,\ b,\ c\}$. Define multiplication $*$ on G as shown in Table 2, where a, b, c to label the elements in the po-NETG and the multiplication $*$ among these elements.*

Table 2. Multiplication $*$ on G.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

It is easy to verify that $(G, *)$ is a WCNETG and $(G, *)$ satisfies neutrosophic cancellation law, in which

$$neut(a) = neut(b) = neut(c) = a,$$

$$\{anti(a)\} = \{a\},\ a^{-1} = a;$$

$$\{anti(b)\} = \{c\},\ b^{-1} = c;$$

$$\{anti(c)\} = \{b\},\ c^{-1} = b.$$

Let $P = \{a\}$, then P satisfies all conditions mentioned in Proposition 9. Define the relation $\leq$ on G by $x \leq y \Leftrightarrow y * x^{-1} \in P$, then $\leq$ is a partial order on G and $(G, \leq)$ is a antichain. Obviously, P is the positive cone of G with respect to this partial order $\leq$.

Proposition 10. *Let $(G, *)$ be a po-WCNETG. Then $\forall\ x \in G$, $\forall y \in G$, $x \leq y$ implies $y * x^{-1} \in P_G$.*

Proof. Let $\forall\ x \in G$, $\forall y \in G$. If $x \leq y$, then $neut(x) = x * x^{-1} \leq y * x^{-1}$, hence, by Proposition 4 and Lemma 1, we have $neut(y * x^{-1}) = neut(x^{-1}) * neut(y) = neut(x) * neut(y) \leq (y * x^{-1}) * neut(y) = neut(y) * (y * x^{-1}) = (neut(y) * y) * x^{-1} = y * x^{-1}$. Thus, $y * x^{-1} \in P_G$. □

4. Homomorphisms and Quotient Sets of po-NETGs

Definition 8. *Let $(G, *, \leq_1)$ and $(T, \cdot, \leq_2)$ be two po-NETGs. The map $f : G \to T$ is called a po-NETG homomorphism of po-NETGs, if f satisfies: $\forall a \in G$, $\forall b \in G$*

(1) $f(a * b) = f(a) \cdot f(b)$;
(2) $a \leq_1 b$ *implies* $f(a) \leq_2 f(b)$.

Proposition 11. *Let $(G, *, \leq_1)$ and $(T, \cdot, \leq_2)$ be two po-NETGs, and let $f : G \to T$ be a po-NETG homomorphism of po-NETGs. The following properties hold:*

(1) $\forall a \in G$, $f(neut(a)) = neut(f(a))$;
(2) $\forall a \in G$, $\{f(b) : b \in \{anti(a)\}\} \subseteq \{anti(f(a))\}$, *and if f is bijective, then* $\{f(b) : b \in \{anti(a)\}\} = \{anti(f(a))\}$;
(3) $\forall a \in G$, $[f(a)]^{-1} = f(a^{-1})$;
(4) $\forall a \in P_G$, $f(a) \in P_T$;
(5) $\forall a \in N_G$, $f(a) \in N_T$.

Proof.

(1) $\forall\, a \in G$, $\forall\, b \in \{anti(a)\}$, since

$$f(a) \cdot f(neut(a)) = f(a * neut(a)) = f(a) = f(neut(a) * a) = f(neut(a)) \cdot f(a),$$

$$f(a) \cdot f(b) = f(a * b) = f(neut(a)) = f(b * a) = f(b) \cdot f(a),$$

then we obtain $f(neut(a)) = neut(f(a))$.

(2) From the proof of (1), we can get that

$$\forall\, a \in G,\ \forall\, b \in \{anti(a)\},\ f(b) \in \{anti(f(a))\},$$

and so

$$\{f(b) : b \in \{anti(a)\}\} \subseteq \{anti(f(a))\}.$$

If f is bijective, then $\forall\, d \in \{anti(f(a))\}$, $\exists\, c \in G$ such that $f(c) = d$. Since

$$f(c * a) = f(c) \cdot f(a) = d \cdot f(a) = neut(f(a)) = f(neut(a)),$$

we have $c * a = neut(a)$. Similarly, we can get $a * c = neut(a)$. Thus, $c \in anti(a)$ and so

$$d = f(c) \in \{f(b) : b \in \{anti(a)\}\}.$$

By the arbitrariness of d, we have

$$\{anti(f(a))\} \subseteq \{f(b) : b \in \{anti(a)\}\}.$$

Then,

$$\{f(b) : b \in \{anti(a)\}\} = \{anti(f(a))\}.$$

(3) Let $a \in G$ and $b \in \{anti(a)\}$. By (2), $f(b) \in \{anti(f(a))\}$. Then by (1), we have

$$[f(a)]^{-1} = anti(f(a)) \cdot neut(f(a)) = f(b) \cdot f(neut(a)) = f(b * neut(a)) = f(a^{-1}).$$

(4) Since $\forall\, a \in P_G,\ neut(a) \leq_1 a$, we have $neut(f(a)) = f(neut(a)) \leq_2 f(a)$, and so $f(a) \in P_T$.

(5) It is similar to (4).

□

Definition 9. *Let $(G, *, \leq)$ be a po-NETG and θ be an equivalence relation on G. If θ satisfies*

$$\forall a \in G,\ \forall b \in G,\ \forall c \in G,\ \forall d \in G,\ (a, b) \in \theta \,\&\, (c, d) \in \theta \Rightarrow (a * c, b * d) \in \theta,$$

then θ is called a congruence on G.

Obviously, $\theta_1 = \{(a, a) : a \in G\}$ and $\theta_2 = \{(a, b) : \forall\, a,\ b \in G\}$ are both congruences on G, and they are called identity congruence on G and pure congruence on G, respectively.

Definition 10. *Let $(G, *, \leq)$ be a po-NETG and θ be a congruence on G. A multiplication $\circ$ on the quotient set $G/\theta = \{[a]_\theta : a \in G\}$ is defined by*

$$[a]_\theta \circ [b]_\theta = [a * b]_\theta.$$

Proposition 12. *Let a relation $\preceq$ on $(G/\theta, \circ)$ be defined by*

$$\forall\ [a]_\theta \in G/\theta,\ \forall [b]_\theta \in G/\theta,\ [a]_\theta \preceq [b]_\theta \Leftrightarrow a \leq b.$$

Then, $(G/\theta, \circ, \preceq)$ is a po-NETG.

Proof. We can verify that $\circ$ is associative. Let $[a]_\theta \in G/\theta$ (see Definition 10), since

$$[neut(a)]_\theta \circ [a]_\theta = [neut(a) * a]_\theta = [a]_\theta = [a * neut(a)]_\theta = [a]_\theta \circ [neut(a)]_\theta,$$

and

$$[anti(a)]_\theta \circ [a]_\theta = [anti(a) * a]_\theta = [neut(a)]_\theta = [a * anti(a)]_\theta = [a]_\theta \circ [anti(a)]_\theta,$$

we conclude that $(G/\theta, \circ)$ is a NETG, in which $\forall\ [a]_\theta \in G/\theta,\ neut([a]_\theta) = [neut(a)]_\theta$ and $[anti(a)]_\theta \in \{anti([a]_\theta)\}$. Then it is easy to see that $\preceq$ is a partial order on $(G/\theta, \circ)$. Moreover, $\forall\ [a]_\theta \in G/\theta,\ \forall [b]_\theta \in G/\theta,\ \forall [c]_\theta \in G/\theta$, if $[a]_\theta \preceq [b]_\theta$, then $a \leq b$, so we have $a * c \leq b * c$, and $c * a \leq c * b$. Thus,

$$[a]_\theta \circ [c]_\theta = [a * c]_\theta \preceq [b * c]_\theta = [b]_\theta \circ [c]_\theta$$

and

$$[c]_\theta \circ [a]_\theta = [c * a]_\theta \preceq [c * b]_\theta = [c]_\theta \circ [b]_\theta.$$

Thus, $(G/\theta, \circ, \preceq)$ is a po-NETG. □

In the following, we give an example to illustrate Proposition 12.

Example 3. *Consider the po-NETG $(G, *, \leq)$ is given in Example 1. Now we define a relation θ on G by*

$$\theta = \{(0,0),\ (a,a),\ (b,b),\ (c,c),\ (1,1),\ (a,b),\ (b,a),\ (a,c),\ (c,a),\ (b,c),\ (c,b)\}.$$

Then we can verify that θ is a congruence on G with the following blocks:

$$[0]_\theta = \{0\},\ [a]_\theta = \{a,\ b,\ c\},\ [1]_\theta = \{1\}.$$

So the quotient set $G/\theta = \{[0]_\theta,\ [a]_\theta,\ [1]_\theta\}$. By Proposition 12, we know $(G/\theta, \circ, \preceq)$ is a po-NETG, in which $neut([0]_\theta) = [0]_\theta$, $neut([a]_\theta) = [c]_\theta = [a]_\theta$, $neut([1]_\theta) = [1]_\theta$, $\{anti([0]_\theta)\} = \{[0]_\theta,\ [a]_\theta,\ [1]_\theta\}$, $\{anti([a]_\theta)\} = \{[a]_\theta\}$, $\{anti([1]_\theta)\} = \{[a]_\theta,\ [1]_\theta\}$, and then G/θ is a chain, because $[0]_\theta \preceq [a]_\theta \preceq [1]_\theta$.

Proposition 13. *Let $(G, *, \leq)$ be a po-NETG and θ be a congruence on G. Then the natural mapping $\natural_\theta : (G, *, \leq) \to (G/_\theta, \circ, \preceq)$ given by $\natural_\theta(a) = [a]_\theta$ is a po-NETG homomorphism of po-NETGs.*

Proof. As $\natural_\theta(a * b) = [a * b]_\theta = [a]_\theta \circ [b]_\theta = \natural_\theta(a) \circ \natural_\theta(b)\ \forall\ a \in G,\ \forall b \in G$. If $a \leq b$, then $[a]_\theta \preceq [b]_\theta$ which implies $\natural_\theta(a) \preceq \natural_\theta(b)$. Thus, the natural mapping $\natural_\theta : (G, *, \leq) \to (G/_\theta, \circ, \preceq)$ is a po-NETG homomorphism of po-NETGs. □

Next, we give an example to explain Proposition 13.

Example 4. *From Example 3, we consider the natural mapping $\natural_\theta : (G, *, \leq) \to (G/_\theta, \circ, \preceq)$. Thus, $\natural_\theta(0) = [0]_\theta$, $\natural_\theta(a) = \natural_\theta(b) = \natural_\theta(c) = [a]_\theta$, $\natural_\theta(1) = [1]_\theta$. It is easy to verify that $\natural_\theta$ is a po-NETG homomorphism of po-NETGs.*

Proposition 14. *Let $(G, *, \leq_1)$ and $(T, \cdot, \leq_2)$be two po-NETGs and $f : (G, *, \leq_1) \to (T, \cdot, \leq_2)$ be a po-NETG homomorphism of po-NETGs. We shall use the notation*

$$Kerf = \{(a, b) \in G \times G : f(a) = f(b)\},$$

then we can get the following properties:

(1) $Kerf$ is a congruence on G;
(2) f is a injective po-NETG homomorphism of po-NETGs if and only if $kerf$ is an identity congruence on G;
(3) There exists an injective po-NETG homomorphism of po-NETGs $g : (G/Kerf, \circ, \preceq) \to (T, \cdot, \leq_2)$ such that $f = g \circ \natural_{Kerf}$.

Proof.

(1) Obviously, $Kerf$ is an equivalence relation on G. Let $\forall\ a \in G, \forall\ b \in G, \forall\ c \in G, \forall\ d \in G$, if $(a, b) \in Kerf$ and $(c, d) \in Kerf$, then $f(a) = f(b)$ and $f(c) = f(d)$. Since f is a po-NETG homomorphism of po-NETGs, we have $f(a * c) = f(a) \cdot f(c) = f(b) \cdot f(d) = f(b * d)$, and so $(a * c, b * d) \in Kerf$. Thus, $Kerf$ is a congruence on G.

(2) If f is an injective po-NETG homomorphism of po-NETGs and if $(a, b) \in kerf$ then $f(a) = f(b)$. Therefore, we get $a = b$. Hence, by the arbitrariness of (a, b), we obtain $kerf$ is an identity congruence on G.

Conversely, suppose that $kerf$ is an identity congruence on G. $\forall\ a \in G, \forall\ b \in G$, if $f(a) = f(b)$, then $(a, b) \in kerf$, so $a = b$. Therefore, f is an injective po-NETG homomorphism of po-NETGs.

(3) We define a map $g : G/Kerf \to T$ by $\forall\ [a]_{Kerf} \in G/Kerf$, $g([a]_{Kerf}) = f(a)$, then g is injective. $\forall\ [a]_{Kerf},\ [b]_{Kerf} \in G/Kerf$, we have $g([a]_{Kerf} \circ [b]_{Kerf}) = g([a * b]_{Kerf}) = f(a * b) = f(a) \cdot f(b) = g([a]_{Kerf}) \cdot g([b]_{Kerf})$, and if $[a]_{Kerf} \preceq [b]_{Kerf}$, then $a \leq_1 b$, thus, $f(a) \leq_2 f(b)$, that is, $g([a]_{Kerf}) \leq_2 g([b]_{Kerf})$. Hence, g is an injective po-NETG homomorphism of po-NETGs.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\natural_{Kerf}"'] & & T \\ & G/Kerf \arrow[ur, "g"'] & \end{tikzcd}$$

$\forall\ a \in G$, $(g \circ \natural_{Kerf})(a) = g(\natural_{Kerf}(a)) = g([a]_{Kerf}) = f(a)$, that is, $f = g \circ \natural_{Kerf}$.
□

In the following, we present an example to illustrate Proposition 14.

Example 5. *Consider $(G, *, \leq_1)$ be the po-NETG is given in Example 1, in which the partial order $\leq_1$ is the same as the partial order $\leq$ in Example 1. Assume that $T = \{m, n, p, q, r\}$ be a bounded lattice with a partial order $\leq_2$ with the Hasse diagram shown as in Figure 2 whose multiplication $\cdot$ is defined as $\wedge$.*

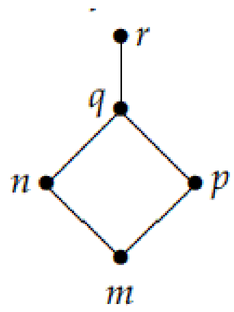

Figure 2. Hasse diagram.

We can verify that $(T, \cdot, \leq_2)$ is a po-NETG, in which $\forall\ x \in T$, $neut(x) = x$, $\{anti(m)\} = \{m,\ n,\ p,\ q,\ r\}$, $\{anti(n)\} = \{n,\ q,\ r\}$, $\{anti(p)\} = \{p,\ q,\ r\}$, $\{anti(q)\} = \{q,\ r\}$, $\{anti(r)\} = \{r\}$. Now, we define a map $f : G \to T$ by $f(0) = m$, $f(a) = f(b) = f(c) = f(1) = r$, then f is a po-NETG homomorphism of po-NETGs, and $Kerf = \{(0,0), (a,a), (b,b), (c,c), (1,1), (a,b), (a,c), (a,1), (b,a), (b,c), (b,1), (c,a), (c,b), (c,1), (1,a), (1,b), (1,c)\}$. Obviously, $Kerf$ is a congruence on G. f is not injective, and of course, $kerf$ is not an identity congruence on G.

5. Conclusions

In this paper, inspired by the research work in algebraic structures equipped with a partial order, we proposed the concepts of po-NETGs, deeply studied the relationships between po-NETGs and their positive cones, and characterized the positive cone of a WCNETG after defining a partial order relation on it. Moreover, we found that the quotient set of a po-NETG can construct another po-NETG by defining a special multiplication and a partial order on the quotient set, and we also achieved the interrelation of homomorphisms and congruences of po-NETGs. All these results are useful for exploring the structure characterization (for example, category properties) of po-NETGs. As a direction of future research, we will consider the application of the fuzzy set theory and the rough set theory to the research of algebraic structure of po-NETGs. Furthermore, we will discuss the relation between the homomorphisms and congruences of po-NETG and the morphisms of ordered lattice ringoids [30]. Finally, in the next paper, we will study sub-structures of po-NETGs and we give some examples using constructions such as central extensions or direct products related to sub-structures of po-NETGs.

Funding: This work was supported by the National Natural Science Foundation of China (Grant no. 11501435) and supported by the PhD research start-up fund (Grant no. BS1529).

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On neutrosophic extended triplet groups (loops) and Abel-Grassmann's groupoids (AG-groupoids)

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Xiaohong Zhang, Xiaoying Wu, Xiaoyan Mao, Florentin Smarandache, Choonkil Park (2019). On neutrosophic extended triplet groups (loops) and Abel-Grassmann's groupoids (AG-groupoids). *Journal of Intelligent & Fuzzy Systems* 37, 5743–5753; DOI:10.3233/JIFS-181742

Abstract. From the perspective of semigroup theory, the characterizations of a neutrosophic extended triplet group (NETG) and AG-NET-loop (which is both an Abel-Grassmann groupoid and a neutrosophic extended triplet loop) are systematically analyzed and some important results are obtained. In particular, the following conclusions are strictly proved: (1) an algebraic system is neutrosophic extended triplet group if and only if it is a completely regular semigroup; (2) an algebraic system is weak commutative neutrosophic extended triplet group if and only if it is a Clifford semigroup; (3) for any element in an AG-NET-loop, its neutral element is unique and idempotent; (4) every AG-NET-loop is a completely regular and fully regular Abel-Grassmann groupoid (AG-groupoid), but the inverse is not true. Moreover, the constructing methods of NETGs (completely regular semigroups) are investigated, and the lists of some finite NETGs and AG-NET-loops are given.

Keywords: Semigroup, neutrosophic extended triplet group (NETG), completely regular semigroup, Clifford semigroup, Abel-Grassmann's groupoid (AG-groupoid)

1. Introduction

Smarandache proposed the new concept of neutrosophic set, which is an extension of fuzzy set and intuitionistic fuzzy set [1]. Until now, neutrosophic sets have been applied to many fields [2–4], and some new theoretical studies are developed [5, 6].

As an application of the basic idea of neutrosophic sets (more general, neutrosophy), the new notion of neutrosophic triplet group (NTG) is introduced by Smarandache and Ali in [7, 8]. As a new algebraic structure, NTG is a generalization of classical group, but it has different properties from classical group. For NTG, the neutral element is relative and local, that is, for a neutrosophic triplet group $(N,^*)$, every element a in N has its own neutral element (denote by $neut(a)$) satisfying condition $a * neut(a) = neut(a)^* a = a$, and there exits at least one opposite element (denote by $anti(a)$) in N relative to $neut(a)$ such condition $a^* anti(a) = anti(a)^* a = neut(a)$. In the original definition of NTG in [8], $neut(a)$ is different from the traditional unit element. Later, the concept of neutrosophic extended triplet group (NETG) was introduced (see [7]), in which the neutral element may be traditional unit element, it is just a special case.

For the structure of NETG, some exploratory research papers are published and a series of results are got [9–12]. Recently, we have analyzed these new results and studied them from the perspective of semi-group theory. Miraculously, we have obtained some unexpected results: every NETG is a completely regular semigroup, and the inverse is true. In fact, the research of completely regular semigroups originated from the study of Clifford [13], and have been greatly developed [14–16], and have been extended to a wide range of algebraic systems [17–20]. This paper will focus on the latest results of the authors, mainly discuss the relationships between neutrosophic extended triplet groups and completely regular semigroups.

Moreover, this paper also investigates the relationships between neutrosophic extended triplet loops and Abel-Grassmann's groupoids (AG-groupoids). The concept of an Abel-Grassmann's groupoid was first given by Kazim and Naseeruddin [21] in 1972 and they have called it a left almost semigroup (LA-semigroup). In [22], the same structure is called a left invertive groupoid. In [23–29], some properties and different classes of an AG-groupoid are investigated. In this paper, we combine the notions of neutrosophic extended triplet loop and AG-groupoid, introduce the new concept of Abel-Grassmann's neutrosophic extended triplet loop (AG-NET-loop), that is, AG-NET-loop is both AG-groupoid and neutrosophic extended triplet loop (NET-loop). We deeply analyze the internal connecting link between AG-NET-loop and completely regular AG-groupoid and obtain some important and interesting results.

2. Preliminaries

Definition 1. [7, 8] Let N be a non-empty set together with a binary operation*. Then, N is called a neutrosophic extended triplet set if for any$a \in N$, there exist a neutral of "a" (denote by $neut(a)$), and an opposite of "a" (denote by $anti(a)$), such that $neut(a) \in N$, $anti(a) \in N$ and:

$$a * neut(a) = neut(a)^* a = a;$$

$$a^* anti(a) = anti(a)^* a = neut(a).$$

The triplet $(a, neut(a), anti(a))$ is called a neutrosophic extended triplet.

Note that, for a neutrosophic triplet set $(N,^*)$, $a \in N$, $neut(a)$ and $anti(a)$ may not be unique. In order not to cause ambiguity, we use the following notations to distinguish:

$neut(a)$: denote any certain one of neutral of a;
$\{neut(a)\}$: denote the set of all neutral of a.
$anti(a)$: denote any certain one of opposite of a;
$\{anti(a)\}$: denote the set of all opposite of a.

Definition 2. [7, 8] Let $(N,^*)$ be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet group (NETG), if the following conditions are satisfied:

(1) $(N,^*)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$.
(2) $(N,^*)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

N is called a commutative neutrosophic extended triplet group if for all $a, b \in N$, $a * b = b * a$.

Proposition 1. [11] *Let $(N,^*)$ be a NETG. Then*

(1) *$neut(a)$ is unique for any a in N.*
(2) *$neut(a) * neut(a) = neut(a)$ for any a in N.*
(3) *$neut(neut(a)) = neut(a)$ for any a in N.*

Definition 3. [11] Let $(N,^*)$ be a NETG. Then N is called a weak commutative neutrosophic extended triplet group (briefly, WCNETG) if $a * neut(b) = neut(b) * a$ *for all* $a, b \in N$.

Proposition 2. [11] *Let $(N,^*)$ be a NETG. Then $(N,^*)$ is weak commutative if and only if N satisfies the following conditions:*

(1) *$neut(a) * neut(b) = neut(b) * neut(a)$ for all $a, b \in N$.*
(2) *$neut(a) * neut(b)^* a = a * neut(b)$ for all $a, b \in N$.*

Proposition 3. [11] *Let $(N,^*)$ be a weak commutative NETG. Then (for all $a, b \in N$)*

(1) *$neut(a) * neut(b) = neut(b^* a)$;*
(2) *$anti(a)^* anti(b) \in \{anti(b^* a)\}$.*

Definition 4. [14] A semigroup $(S, *)$ will be called completely regular if there exists a unary operation $a_- \mapsto a^{-1}$ on S with the properties

$$(a^{-1})^{-1} = a, a^* a^{-1*} a = a, a^* a^{-1} = a^{-1*} a.$$

Proposition 4. [14] *Let $(S,^*)$ be a semigroup. Then the following statements are equivalent:*

(1) *S is completely regular;*
(2) *every element of S lies in a subgroup of S;*
(3) *every H-class in S is a group.*

Here, recall some basic concepts in semigroup theory. A non-empty subset A of a semigroup $(S,^*)$ is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$, and an ideal if it both a left and a right ideal. Evidently, every ideal (whether one- or two-sided) is a subsemigroup. If a is an element of a semigroup $(S,^*)$, the smallest left ideal containing a is $Sa \cup \{a\}$, which we may conveniently write as S^1a, and which we shall call the principle left ideal generated by a.

An equivalent relation L on S is defined by the rule that aLb if and only if $S^1a = S^1b$; an equivalent relation R on S is defined by the rule that aLb if and only if $aS^1 = bS^1$; denote $H = L \wedge R$, $D = L \vee R$, that is, aHb if and only if $S^1a = S^1b$ and $aS^1 = bS^1$; aDb if and only if $S^1a = S^1b$ or $aS^1 = bS^1$. An equivalent relation J on S is defined by the rule that aJb if and only if $S^1aS^1 = S^1bS^1$, where

$$S^1aS^1 = SaS \cup aS \cup Sa \cup \{a\}$$

That is, aJb if and only if there exists x, y, u, $v \in S^1$ for which $x^*a^*y = b$, $u^*b^*v = a$. The L-class (R-class, H-class, D-class, J-class) containing the element a will be written L_a (R_a, H_a, D_a, J_a).

Definition 5. [14] A semigroup $(S,^*)$ will be called Clifford semigroup, if it is completely regular and in which, for all x, y in S,

$$(x^*x^{-1})^*(y^*y^{-1}) = (y^*y^{-1})^*(x^*x^{-1}).$$

In an arbitrary semigroup S, we say that an element c is central if $c^*s = s^*c$ for every s in S. The set of central elements forms a subsemigroup of S, called the center of S.

Proposition 5. [14] *Let $(S,^*)$ be a semigroup. Then the following statements are equivalent:*

(1) *S is Clifford semigroup;*
(2) *S is a semilattice of groups;*
(3) *S is regular, and the idempotents of S are central.*

Abel-Grassmann's groupoid (AG-groupoid) [21, 22], is a groupoid $(S,^*)$ holding left invertive law, that is, for all $a, b, c \in S$, $(a^*b)^*\, c = (c^*b)^*\, a$. In an AG-groupoid the medial law holds, for all $a, b, c, d \in S$, $(a^*b)^*\, (c^*d) = (a^*c)^*\, (b^*d)$.

There can be a unique left identity in an AG-groupoid. In an AG-groupoid S with left identity the paramedial laws hold for all $a, b, c, d \in S$, $(a^*b)^*\, (c^*d) = (d^*c)^*\, (b^*a)$. Further if an AG-groupoid contain a left identity, hen he following law holds: for all $a, b, c \in S$, $a^*\, (b^*c) = b^*\, (a^*c)$.

An AG-groupoid is a non-associative algebraic structure midway between a groupoid and a commutative semigroup, because if an AG-groupoid contains right identity then it becomes a commutative semigroup.

Definition 6. [25] (1) An element a of an AG-groupoid $(S,^*)$ is called a regular if there exists $x \in S$ such that $a = (a^*x^*)^*a$ and S is called regular if all elements of S are regular.

(2) An element a of an AG-groupoid $(S,^*)$ is called a weakly regular if there exists $x, y \in S$ such that $a = (a * x) * (a * y)$ and S is called weakly regular if all elements of S are weakly regular.
(3) An element a of an AG-groupoid $(S,^*)$ is called an intra-regular if there exists $x, y \in S$ such that $a = \left(x^*a^2\right) * y$ and S is called an intra-regular if all elements of S are intra-regular.
(4) An element a of an AG-groupoid $(S,^*)$ is called a right regular if there exists $x \in S$ such that $a = a^2 * x = (a^*a) * x$ and S is called a right regular if all elements of S are right regular.
(5) An element a of an AG-groupoid $(S,^*)$ is called a left regular if there exists $x \in S$ such that $a = x^*a^2 = x * (a^*a)$ and S is called left regular if all elements of S are left regular.
(6) An element a of an AG-groupoid $(S,^*)$ is called a left quasi regular if there exists $x, y \in S$ such that $a = (x^*a) * (y^*a)$ and S is called left quasi regular if all elements of S are left quasi regular.
(7) An element a of an AG-groupoid $(S,^*)$ is called a completely regular if a is regular and left (right) regular. S is called completely regular if it is regular, left and right regular.

Proposition 6. [25] *If $(S,^*)$ is regular (weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular) AG-groupoid, then $S = S^2$*

Proposition 7. [25] *In an AG-groupoid $(S,^*)$ with left identity, the following are equivalent:*

(i) *S is weakly regular.*
(ii) *S is an intra-regular.*
(iii) *S is right regular.*
(iv) *S is left regular.*
(v) *S is left quasi regular.*
(vi) *S is completely regular.*

Definition 7. [26] An element a of an AG-groupoid $(S,^*)$ is called a fully regular element of S if there exist some $p, q, r, s, t, u, v, w, x, y, z \in S$ (p, q,..., z may be repeated) such that

$$\begin{aligned} a &= \left(p^* a^2\right) * q = \left(r^* a\right) * (a * s) \\ &= (a * t) * (a * u) = \left(a^* a\right) * v \\ &= w * \left(a^* a\right) = \left(x^* a\right) * \left(y^* a\right) \\ &= \left(a^2 * z\right)^* a^2. \end{aligned}$$

An AG-groupoid $(S,^*)$ is called fully regular if all elements of S are fully regular.

A non-empty subset A of an AG-groupoid $(S,^*)$ called left (right) ideal of S if and only if $SA \subseteq A (AS \subseteq A)$ and is called two-sided ideal or ideal of S if and only if it is both left and right ideal of S.

Definition 8. [26] A non-empty subset A of an AG-groupoid $(S,^*)$ called semiprime if and only if

$$a^2 \in A \Rightarrow a \in A.$$

Definition 9. [26] An AG-groupoid is called left (right) simple if and only if it has no proper left (right) ideal and is called simple if and only if it has no proper two-sided ideal.

Proposition 8. [26] *The following conditions are equivalent for an AG-groupoid $(S,^*)$ with left identity:*

(i) $aS = S$, *for some* $a \in S$.
(ii) $Sa = S$, *for some* $a \in S$.
(iii) S *is simple*.
(iv) $AS = S = SA$, *where A two-sided ideal of S*.
(v) S *is fully regular*.

3. NETG and completely regular semigroup

Theorem 1. *Let $(N,^*)$ be a NETG. Then for all* $a \in N$,

(1) $p * neut(a) \in \{anti(a)\}$, *for any* $p \in \{anti(a)\}$;
(2) $p * neut(a) = q * neut(a) = neut(a) * q$, *for any* $p, q \in \{anti(a)\}$;
(3) $neut(p * neut(a)) = neut(a)$, *for any* $p \in \{anti(a)\}$;
(4) $a \in \{anti(p * neut(a))\}$, *for any* $p \in \{anti(a)\}$;
(5) $anti(p * neut(a)) * neut(p * neut(a)) = a$, *for any* $p \in \{anti(a)\}$.

Proof. (1) Suppose $p \in \{anti(a)\}$, then $p^* a = a * p = neut(a)$.

From this, and applying Proposition 1, we get$(p * neut(a))^* a = p * \left(neut(a)^* a\right) = p^* a = neut(a)$, $a * (p * neut(a)) = (a * p) * neut(a) = neut(a) * neut(a) = neut(a)$.

It follows that $p * neut(a) \in \{anti(a)\}$.

(2) Suppose $p, q \in \{anti(a)\}$, then
$p^* a = a * p = neut(a)$; $q^* a = a * q = neut(a)$.
Thus,
$p * neut(a) = p * (a * q) = (p^* a) * q = neut(a) * q$
$= (q^* a) * q = q * (a * q) = q * neut(a)$.
That is, $p * neut(a) = neut(a) * q = q * neut(a)$.

(3) For any $p \in \{anti(a)\}$, by Proposition 1 and (2), we have
$(p * neut(a)) * neut(a) = p * (neut(a) * neut(a)) = p * neut(a)$,
$neut(a) * (p * neut(a)) = (neut(a) * p) * neut(a) = (p * neut(a)) * neut(a)$
$= p * (neut(a) * neut(a)) = neut(a)$.
Moreover, using Proposition 1,
$(p * neut(a))^* a = p * \left(neut(a)^* a\right)$
$= p^* a = neut(a)$, $a * (p * neut(a)) = (a * p) * neut(a) = neut(a) * neut(a) = neut(a)$.
Applying Definition 1, $neut(a) = neut(p * neut(a))$.

(4) For any $p \in \{anti(a)\}$, by Proposition 1, we have
$a * (p * neut(a)) = (a * p) * neut(a)$
$= neut(a) * neut(a) = neut(a)$,
$(p * neut(a))^* a = p * (a * neut(a))$
$= p^* a = neut(a)$.
By Definition 1 we know that $a \in \{anti(p * neut(a))\}$.

(5) Assume $p \in \{anti(a)\}$. For all $anti(p * neut(a)) \in \{anti(p * neut(a))\}$, by (2) we know that $anti(p * neut(a)) * neut(p * neut(a))$ is unique. Applying (4), $a \in \{anti(p * neut(a))\}$, it follows that

$$\begin{aligned} & anti(p * neut(a)) * neut(p * neut(a)) \\ & = a * neut(p * neut(a)). \end{aligned}$$

Using (3), $neut(p * neut(a)) = neut(a)$. *Therefore,*

$$\begin{aligned} & anti(p * neut(a)) * neut(p * neut(a)) \\ & = a * neut(p * neut(a)) \\ & = a * neut(a) = a. \end{aligned}$$
□

Theorem 2. *Let $(N, *)$ be a groupoid. Then N is a NETG if and only if it is a completely regular semigroup.*

Proof. Assume that N is a NETG. By Theorem 1, we define a unary operation $a_\shortmid \to a^{-1}$ on N as follows:
$a^{-1} = anti\,(a) * neut\,(a)$, for any a in N.

By Theorem 1 (2), a^{-1} is unique. Applying Theorem 1 (5) we get

$$(a^{-1})^{-1} = anti(anti\,(a) * neut\,(a))$$
$$*neut(anti\,(a) * neut\,(a)) = a.$$

Moreover, by Proposition 1,

$$\begin{aligned} a^*a^{-1*}a &= a^*anti\,(a) * neut\,(a)^*\, a = a, \\ a^*a^{-1} &= a^*anti\,(a) * neut\,(a) \\ &= neut\,(a)^*\, anti\,(a) = neut\,(a) \\ &= anti\,(a)^*\, a = anti\,(a) * neut\,(a) \\ &\quad ^*a = a^{-1*}a. \end{aligned}$$

Thus, by Definition 4, N is a completely regular semigroup.

Conversely, suppose that N is a completely regular semigroup. For any a in N, denote $neut\,(a) = a^*a^{-1}$, then

$$neut\,(a)^*\, a = a^*a^{-1*}a = a,$$

$$a * neut\,(a) = a^*a^*a^{-1} = a^*a^{-1*}a = a.$$

Moreover,

$$a^{-1*}a = a^*a^{-1} = neut\,(a)\,.$$

By Definition 1, we know that N is a NETG, and$a^{-1} \in \{anti\,(a)\}$. □

Note that, in semigroup theory, a^{-1} is called inverse element, it is unique; in NETG, $anti\,(a)$ is called opposite element, it may be not unique, please see the following example.

Example 1. Let $N = \{a, b, c, d, e\}$, define operations $*$ on N as following Table 1. Then, $(N,^*)$ is a NETG and a completely regular semigroup. We can get that

$$a^{-1} = a; a^{-1*}a = a^*a^{-1} = a.$$

$$neut\,(a) = a, \{anti\,(a)\} = \{a, c, d, e\}\,.$$

Table 1
The operation * on N

*	a	b	c	d	e
a	a	b	a	a	a
b	b	a	b	b	b
c	a	b	d	c	a
d	a	b	c	d	a
e	a	b	a	a	e

4. Weak commutative NETG and Clifford semigroup

Applying Theorem 2 and Definition 5, we can get the following result (the proof is omitted).

Proposition 9. Let $(N,^*)$ *be a completely regular semigroup. Then N is a Clifford semigroup, if and only if it satisfies:*

$neut\,(a) * neut\,(b) = neut\,(b) * neut\,(a)\,,$
for all $a, b \in N$.

Theorem 3. *Let $(N,^*)$ be a groupoid. Then N is a weak commutative neutrosophic extended triplet group (NETG) if and only if it is a Clifford semigroup.*

Proof. Suppose that N is a weak commutative NETG. By Theorem 2, we know that N is a completely regular semigroup. Using Proposition 2, for any $a, b \in N$, $neut\,(a) * neut\,(b) = neut\,(b) * neut\,(a)$. Then, by Proposition 9 we know that N is a Clifford semigroup.

Conversely, assume that N is a Clifford semigroup. Applying Theorem 2 and Proposition 1, $neut\,(a) * neut\,(a) =$, for any a in . That is, $neut\,(a)$ is idempotent. Thus, by Proposition 3, $neut\,(a)$ is central. Therefore, for any b in N,

$$neut\,(a) * b = b * neut\,(a)\,.$$

This means that N is a weak commutative NETG, by Definition 3. □

Applying Theorem 3 and Proposition 2, we can get the following result (the proof is omitted).

Proposition 10. *Let $(N, *)$ be a NETG. Then N is weak commutative, if and only if it satisfies:*

$$neut\,(a) * neut\,(b) = neut\,(b) * neut\,(a)\,,$$
$$\textit{for all } a,\ b \in N.$$

In other words, in NETG, the following conditions are equivalent:

(1) $a * neut\,(b) = neut\,(b)^*\, a$, *for all* $a,\ b \in N$;
(2) $neut\,(a) * neut\,(b) = neut\,(b) * neut\,(a)$, *for all* $a,\ b \in N$

Now, we discuss the method of establishing Clifford semigroup (that is, weak commutative NETG) by two given groups.

Theorem 4. *Let $(G_1, *_1)$ and $(G_2, *_2)$ be two groups, e_1 and e_2 identity elements of $(G_1, *_1)$ and $(G_2, *_2)$, $G_1 \cap G_2 = \emptyset$. Denote $N = G_1 \cup G_2$, and define the operation $*$ in N as follows:*

(1) *if $a, b \in G_1$, then $a * b = a *_1 b$;*
(2) *if $a, b \in G_2$, then $a * b = a *_2 b$;*
(3) *if $a \in G_1, b \in G_2$, then $a * b = a$;*
(4) *if $a \in G_2, b \in G_1$, then $a * b = b$.*

Then $(N,^*)$ is a Clifford semigroup (weak commutative NETG).

Proof. *It is only necessary to prove that the associative law hold in $(N,^*)$, that is, $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$. We will discuss the following situations separately.*

Case 1: $a, b, c \in G_1$, *or* $a, b, c \in G_2$. Since G_1 and G_2 are groups, so $(a * b) * c = *(b * c)$.

Case 2: $a \in G_1, b \in G_2$, *and* $c \in G_1$. Then, by the definition of $*$, we have $(a * b) * c = a * c = a * (b * c)$.

Case 3: $a \in G_1, b \in G_2$, *and* $c \in G_2$. Then, by the definition of $*$, we have $(a * b) * c = a * c = a = a * (b * c)$.

Case 4: $a \in G_2, b \in G_1$, *and* $c \in G_1$. Then, $(a * b) * c = b * c = a * (b * c)$.

Case 5: $a \in G_2, b \in G_1$, *and* $c \in G_2$. Then, $(a * b) * c = b * c = b = a * b = a * (b * c)$.

Case 6: $a \in G_1, b \in G_1$, *and* $c \in G_2$. From the definition of operation $*$ we have $(a * b) * c = a * b = a * (b * c)$.

Case 7: $a \in G_2, b \in G_2$, *and* $c \in G_1$. From the definition of operation $*$ we have $(a * b) * c = c = a * c = a * (b * c)$.

Therefore, $(N,^*)$ is a semigroup. Moreover, for any $a \in N$,

if $a \in G_1$, then $a * e_1 = e_1^* a = a$, and $a * (a^{-1}) = (a^{-1})^* a = e_1$, where a^{-1} is the inverse of a in group $(G_1, *_1)$;

if $a \in G_2$, then $a * e_2 = e_2^* a = a$, and $a * (a^{-1}) = (a^{-1})^* a = e_2$, where a^{-1} is the inverse of a in group $(G_2, *_2)$.

This means that $(N,^*)$ is a NETG by Definition 1. Moreover, by the definition of operation *, we have $x * e_1 = e_1 * x$, $x * e_2 = e_2 * x$, for any x in N. Hence, $(N,^*)$ is a weak commutative NETG by Definition 3. Using Theorem 3 we know that $(N,^*)$ is a Clifford semigroup. □

Similarly, we can get the following result.

Theorem 5. *Let $(G_1, *_1)$ and $(G_2, *_2)$ be two groups, e_1 and e_2 identity elements of $(G_1, *_1)$ and $(G_2, *_2)$, $G_1 \cap G_2 = \emptyset$. Denote $N = G_1 \cup G_2$, and define the operation * in N as follows:*

(1) *if $a, b \in G_1$, then $a * b = a *_1 b$;*
(2) *if $a, b \in G_2$, then $a * b = a *_2 b$;*
(3) *if $a \in G_1, b \in G_2$, then $a * b = b$;*
(4) *if $a \in G_2, b \in G_1$, then $a * b = a$.*

Then $(N,^*)$ is a Clifford semigroup (weak commutative NETG).

Example 2. Let $G_1 = \{e, a, b, c\}$ and $G_2 = \{1, 2, 3, 4, 5, 6\}$. efine operations $*_1$ and $*_2$ on G_1, G_2 following Tables 2 and 3. Then, $N = G_1 \cup G_2 = \{e, a, b, c, 1, 2, 3, 4, 5, 6\}$ is $(N,^*)$ is a weak commutative NETG with the operation * in Table 4.

Moreover, according the method in Theorem 5, we can get another weak commutative NETG (Clifford semigroup) $(N,^{*'})$, in which the peration$*$' is defined as Table 5.

Table 2
Commutative group $(G_1, *_1)$

$*_1$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table 3
Non-commutative group $(G_2, *_2)$

$*_2$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	4	3
3	3	5	1	6	2	4
4	4	6	5	1	3	2
5	5	3	4	2	6	1
6	6	4	2	3	1	5

Table 4
First weak commutative NETG (Clifford semigroup) $(N,^*)$

*	e	a	b	c	1	2	3	4	5	6
e	e	a	b	c	e	e	e	e	e	e
a	a	e	c	b	a	a	a	a	a	a
b	b	c	e	a	b	b	b	b	b	b
c	c	b	a	e	c	c	c	c	c	c
1	e	a	b	c	1	2	3	4	5	6
2	e	a	b	c	2	1	6	5	4	3
3	e	a	b	c	3	5	1	6	2	4
4	e	a	b	c	4	6	5	1	3	2
5	e	a	b	c	5	3	4	2	6	1
6	e	a	b	c	6	4	2	3	1	5

Table 5
Second weak commutative NETG (Clifford semigroup) (N,* ')

*'	e	a	b	c	1	2	3	4	5	6
e	e	a	b	c	1	2	3	4	5	6
a	a	e	c	b	1	2	3	4	5	6
b	b	c	e	a	1	2	3	4	5	6
c	c	b	a	e	1	2	3	4	5	6
1	1	1	1	1	1	2	3	4	5	6
2	2	2	2	2	2	1	6	5	4	3
3	3	3	3	3	3	5	1	6	2	4
4	4	4	4	4	4	6	5	1	3	2
5	5	5	5	5	5	3	4	2	6	1
6	6	6	6	6	6	4	2	3	1	5

5. AG-NET-loops and completely regular AG-groupoids

Definition 10. Let $(N,^*)$ be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet loop (NET-loop), if $(N,^*)$ is ell-defined, i.e., for any$a, b \in N$, one has$a * b \in N$.

Remark 1. In [10, 12], the name of neutrosophic triplet loop is used. In order to be more rigorous and echoed with neutrosophic extended triplet group (NETG), the name of neutrosophic extended triplet loop (NET-loop) is used in this paper.

Definition 11. Let $(N,^*)$ be a neutrosophic extended triplet loop (NET-loop). Then, N is called an AG-NET-loop, if $(N,^*)$ is an AG-groupoid.

Theorem 6. *Assume that* $(N,^*)$ *is an AG-NET-loop. Then*

(1) *for all a in N, neut* (a) *is unique*
(2) *for all a in N,* $neut\,(a) * neut\,(a) = neut\,(a)$.

Proof. *Suppose that there exists* $x,\ y \in \{neut\,(a)\}$. *By Definition 1 and 10,* $a * x = x^*a = a$, $a * y = y^*a = a$, *and there exists* $u, v \in N$ *which satisfy* $a * u = u^*a = x$, $a * v = v^*a = y$. *Applying the invertive law, we have*

(i) $y * u = (v^*a) * u = (u^*a) * v = x * v$.
(ii) $x * y = (a * u) * y = (y * u)^*a = (x * v)^*\, a = (a * v) * x = y * x$. (by the invertive law and (i))
(iii) $x = a * u = (y^*a) * u = (u^*a) * y = x * y$.
(iv) $y = a * v = (x^*a) * v = (v^*a) * x = y * x$.
(v) $(x = x * y = y * x = y$. (by iii), (ii) and (iv))

Therefore, $neut\,(a)$ is unique. Moreover, by (v) and (iii) we get that $x = x * x$, that is, $neut\,(a) * neut\,(a) = neut\,(a)$. □

Theorem 7. *Let* $(N,^*)$ *be an AG-NET-loop. Then*

(1) *for any* $x, y \in \{anti(a)\}$, $neut(a) * x = neut(a) * y$, that is, $|neut(a) * \{anti(a)\}| = 1$;
(2) *for all a in N* $neut(neut(a)) * neut(a) = neut(a) = neut(a) * neut(neut(a))$;
(3) *for all a in N* $neut(neut(a)) = neut(a)$;
(4) *for any a in N and* $p \in anti(neut(a))$, $a * p = a$;
(5) *for any a in N* $q \in \{anti(a)\}$, $neut(a) * neut(q) = neut(a)$ and $neut(a) * q = q * neut(a)$;
(6) *for any a in N and any* $q \in \{anti(a)\}$, $neut(a)^*anti(q) = neut(q)^*a$;
(7) *for any a in N and for any* $q \in \{anti(a)\}$, $(q * neut(a))^*a = (neut(a) * q)^*a = neut(a)$;
(8) *for any a in N and for any* $q \in \{anti(a)\}$, $a * (q * neut(a)) = a * (neut(a) * q) = neut(a)$;
(9) *for any a in N and for any* $q \in \{anti(a)\}$, $q * neut(a) \in \{anti(a)\}$ and $neut(a) * q \in \{anti\,(a)\}$;
(10) *for any a in N* $q \in \{anti(a)\}$, $neut(q) * neut(a) = neut(a)$;
(11) *for any a in N* $q \in \{anti(a)\}$, $a * neut(q) = a$;
(12) *for any a in N* $q \in \{anti(a)\}$, $q * (a^*a) = a$;
(13) *for all a in N* $a * neut(a^*a) = a$.

Proof. (1) Assume $x,\ y \in \{anti\,(a)\}$, by Definition 1 and 10,

$$x^*a = a * x = neut\,(a)\,,\ y^*a = a * y = neut\,(a)\,.$$

Using the invertive law, we have
$neut\,(a) * x = (y^*a) * x = (x^*a) * y$
$= neut\,(a) * y$.

(2) Since $neut(neut\,(a))$ is the neutral element of$neut\,(a)$, by Theorem 6 (1), Definition 1 and 10, we have$neut(neut\,(a)) * neut\,(a) = neut\,(a) = neut\,(a) * neut\,(neut\,(a))$.
(3) Let $p \in \{anti\,(neut\,(a))\}$, then
$neut\,(a) * p = neut\,(a)^*\, anti\,(neut\,(a)) = neut(neut\,(a))$.
$p * neut\,(a) = anti\,(neut\,(a)) * neut\,(a) = neut(neut\,(a))$.
By the invertive law,
$(p * x)^*a = (a * x) * p = neut\,(a) * p = neut(neut\,(a))$.
On the other hand, by the medial law and (2) we have
$(p * x)^*a = (p * x) * (neut\,(a)^*\, a) = (p * neut\,(a)) * (x^*a) = neut(neut\,(a)) * neut\,(a) = neut\,(a)$.

Therefore, $neut(neut(a)) = (p * x)^* a = neut(a)$.

(4) Let $p \in \{anti(neut(a))\}$, applying the invertive law and (3) we get
$a * p = (a * neut(a)) * p = (p * neut(a))^* a$
$= (anti(neut(a)) * neut(a))^* a = neut(neut(a))^* a$
$= neut(a)^* a = a$.

(5) Assume $q \in \{anti(a)\}$, then $a * q = q^* a = neut(a)$.
Applying the invertive law,
$neut(a) * neut(q) = (a * q) * neut(q)$
$= (neut(q) * q)^* a = q^* a = neut(a)$.
Moreover,
$neut(a) * q = (neut(a) * neut(q)) * q$
$= (q * neut(q)) * neut(a) = q * neut(a)$

(6) Assume $q \in \{anti(a)\}$, then $a * q = q^* a = neut(a)$, $q^* anti(q) = anti(q) * q = neut(q)$. Applying the invertive law and (5),
$neut(q)^* a = (anti(q) * q)^* a$
$= (a * q)^* anti(q) = neut(a)^* anti(q)$.

(7) Suppose $q \in \{anti(a)\}$, then
$(q * neut(a))^* a = (a * neut(a)) * q = a * q = neut(a)$.
And, applying (5), $(neut(a) * q)^* a = (q * neut(a))^* a = neut(a)$.

(8) Suppose $q \in \{anti(a)\}$, using the invertive law and (7) we have
$a * (q * neut(a)) = (a * neut(a)) * (q * neut(a))$
$= ((q * neut(a)) * neut(a))^* a$
$= ((neut(a) * neut(a)) * q)^* a$
$= (neut(a) * q)^* a$
$= neut(a)$.
Also, applying (5), $a * (neut(a) * q) = a * (q * neut(a)) = neut(a)$.

(9) If $q \in \{anti(a)\}$, by (7) and (8), we get that $q * neut(a) \in \{anti(a)\}$ and $neut(a) * q \in \{anti(a)\}$.

(10) If $q \in \{anti(a)\}$, then
$neut(q) * neut(a) = (q^* anti(q)) * neut(a)$
$= (neut(a)^* anti(q)) * q$..........
$= (neut(q)^* a) * q$...................... (by (6))
$= (q^* a) * neut(q)$
$= neut(a) * neut(q)$ ....(by $q \in \{anti(a)\}$)
$= neut(a)$ .... (using (5))

(11) Assume $q \in \{anti(a)\}$, then (applying (10))
$a * neut(q) = (a * neut(a)) * neut(q) = (neut(q) * neut(a))^* a = neut(a)^* a = a$.

Table 6
Non-Commutive AG-NET-loop

$*$	a	b	c	d	e
a	a	a	e	c	d
b	a	b	e	c	d
c	d	d	c	e	a
d	e	e	a	d	c
e	c	c	d	a	e

(12) Assume $q \in \{anti(a)\}$, then (applying (10))
$q * (a^* a) = (q * neut(q)) * (a^* a)$
$= (q^* a) * (neut(q)^* a)$ (applying the medial law)
$= (q^* a) * (a * neut(q))$......................... (by (5))
$= (q^* a) * (neut(a)^* anti(q))$.............. (by (6))
$= (q * neut(a)) * (a^* anti(q))$.... (by the medial law)
$= (neut(a) * q) * (a^* anti(q))$.............. (by (5))
$= (neut(a)^* a) * (q^* anti(q))$.... (by the medial law)
$= a * neut(q))$
$= a$................................ (by (11))

(13) For all a in N, there exists $q \in \{anti(a)\}$, then $a * neut(a^* a)$
$= (q * (a^* a)) * neut(a^* a)$........... (using (12))
$= (neut(a^* a) * (a^* a)) * q$... (by the invertive law)
$= (a^* a) * q$
$= (q^* a)^* a$......... (applying the invertive law)
$= neut(a)^* a$
$= a$.

The proof complete. ☐

Example 3. Let $N = \{a, b, c, d, e\}$. Define operation $*$ on N as following Table 6. Then, $(N,^*)$ is a non-commutative AG-NET-loop. And,

$$neut(a) = a, \{anti(a)\} = \{a, b\};$$

$$neut(b) = b, \{anti(b)\} = \{b\};$$

$$neut(c) = c, \{anti(c)\} = \{c\}; neut(d) = d,$$

$$\{anti(d)\} = \{d\}; neut(e) = e, \{anti(e)\} = \{e\}.$$

Theorem 8. *Let $(N,^*)$ be an AG-NET-loop. Then N is a completely regular AG-groupoid.*

Table 7
Non-commutative completely regular AG-groupoid

$*_1$	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	4	2	3
4	1	3	4	2

Proof. For any a in N, by Definition 1 and 11 we have

$$(a^* anti\,(a))^* a = neut\,(a)^*\, a = a.$$

From this and Definition 6 (1), we know that N is a regular AG-groupoid.

Moreover, assume $a \in N$, we have

$$(a^* a)\ \ ^* anti\,(a) = (anti\,(a)^*\, a)$$
$$^* a = neut\,(a)^*\, a = a.$$

From this and Definition 6 (4), N is a right regular AG-groupoid.

For all $a \in N$, there exists $q \in \{anti\,(a)\}$, $a * q = q^* a = neut\,(a)$. Denote $x = q * neut\,(a)$, then (using the medial law)

$$\begin{aligned} x * (a^* a) &= (q * neut\,(a)) * (a^* a) \\ &= (q^* a) * (neut\,(a)^*\, a) \\ &= (q^* a)^* a = neut\,(a)^*\, a = a. \end{aligned}$$

From this and Definition 6 (5), N is a left regular AG-groupoid.

Therefore, by Definition 6 (7) we know that N is a completely regular AG-groupoid. □

The following example shows that a completely regular AG-groupoid may be not an AG- NET-loop.

Example 4. Let $N = \{1, 2, 3, 4\}$. Define operations $*$ on N as following Table 7. Then, $(N,^*)$ is a non-commutative completely regular AG-groupoid, but it is not an AG-NET-loop, since there is no $a \in N$ such that $a * 4 = 4^* a = 4$.

Theorem 9. *Let $(N,^*)$ be an AG-NET-loop. Then N is a fully regular AG-groupoid.*

Proof. Suppose $a \in N$. Then there exists $m \in \{anti\,(a)\}$, $a * m = m^* a = neut\,(a)$. Denote $p = m * neut\,(a)$, $q = neut\,(a)$; $r = m$, $s = neut\,(a)$; $t = m, u = neut\,(a)$; $v = m$; $w = m * neut\,(a)$; $x = m$, $y = neut\,(a)$. Then

$\left(p^* a^2\right) * q = ((m * neut\,(a))^* a^2)) * neut\,(a)$
$= ((a^{2*} neut\,(a)) * m)) * neut\,(a)$
$= (((a^* a) * neut\,(a)) * m)) * neut\,(a)$
$= (((neut\,(a)^*\, a)^* a) * m)) * neut\,(a)$
$= ((a^* a) * m)) * neut\,(a)$
$= ((w^* a)^* a)) * neut\,(a)$
$= (neut\,(a)^*\, a)) * neut\,(a)$
$= a * neut\,(a) = a.$

$(r^* a) * (a * s) = (m^* a) * (a * neut\,(a)) = neut\,(a)^*\, a = a$

$(a * t) * (a * u) = (a * m) * (a * neut\,(a)) = neut\,(a)^*\, a = a$

$(a^* a) * v = (a^* a) * m = (m^* a)^*\, a = neut\,(a)^*\, a = a$

$w * (a^* a) = (m * neut\,(a)) * (a^* a)$
$= ((a^* a) * neut\,(a)) * (m * neut\,(a))$
$= \left(\left(neut\,(a)^*\, a\right)^* a\right) * (m * neut\,(a))$
$= (a^* a) * (m * neut\,(a))$
$= \left((m * neut\,(a))^*\, a\right)^* a$
$= ((a * neut\,(a)) * m)^*\, a$
$= (a * m)^*\, a$
$= neut\,(a)^*\, a = a$

$(x^* a) * (y^* a) = (m^* a) * (neut\,(a)^*\, a) = neut\,(a)^*\, a = a.$

Moreover, for $a^2 \in N$, there exists $n \in \{anti(a^2)\}$. Denote $z = n * m$, then

$\left(a^2 * z\right)^* a^2 = ((a^* a) * z)^*\, a^2.$
$= \left((z^* a)^*\, a\right)^* a^2$..... (applying the invertive law)
$= \left(a^{2*} a\right) * (z^* a)$..... (applying the invertive law)
$= \left(a^{2*} a\right) * ((n * m)^* a)$
$= \left(a^{2*} a\right) * ((a * m)^* n)$.......(*by the invertive law*)
$= \left(a^{2*} a\right) * (neut\,(a)^*\, n)(by\ m \in \{anti\,(a)\})$
$= ((a^* a) * (neut\,(a)^*\, a)) * (neut\,(a)^*\, n)$
$= ((a * neut\,(a)) * (a^* a)) * (neut\,(a)^*\, n)$...(applying the medial law)
$= \left(a^* a^2\right) * (neut\,(a)^*\, n)$........(by the medial law)
$= (a * neut\,(a)) * \left(a^{2*} n\right)$..(applying the medial law)
$= a * neut(a^2)$ (by the definition of $n \in \{anti(a^2)\}$)
$= a$......................... (by Theorem 7 (13))

Therefore, combing above results, by Definition 7, we know that N is a fully regular AG- groupoid. □

The following example shows that a fully regular AG-groupoid may be not an AG-NET-loop.

Example 5. Let $N = \{1, 2, 3, 4, 5, 6, 7\}$. Define operations $*$ on N as following Table 8. Then, $(N,^*)$ is a non- commutative fully regular AG-groupoid (see [26]), but it is not an AG-NET-loop, since there is no $x \in N$ such that $x * 3 = 3 * x = 3$.

6. On finite NETGs and finite AG-NET-loops

The instances with finite order and their constructions are of great significance for exploring structural

Table 8
Non-commutative fully regular AG-groupoid

*	1	2	3	4	5	6	7
1	2	4	6	1	3	5	7
2	5	7	2	4	6	1	3
3	1	3	5	7	2	4	6
4	4	6	1	3	5	7	2
5	7	2	4	6	1	3	5
6	3	5	7	2	4	6	1
7	6	1	3	5	7	2	4

features of abstract algebraic systems. By designing the MATLAB program, we have found all NTEGs of order 3, 4 and 5, which have 13, 67 and 353 respectively and they are not isomorphic to each other. Moreover, we obtained all AG-NET-loops of order 3, 4 and 5, which have 5, 17 and 54 respectively and they are not isomorphic to each other. In this section, we present our results in the form of theorems for the sake of further study. For NETGs with order 5, we only list all of commutative NETGs, a total of 51.

Theorem 10. *Let $(N, *)$ be a NETG with order 3 and denote $N = \{1, 2, 3\}$. Then N must be isomorphic to one of the NETGs represented by the following tables, and these NETGs are not mutually isomorphic:*

(1) $T3_1 = \{\{1, 1, 1\}, \{2, 2, 2\}, \{3, 3, 3\}\}$;
(2) $T3_2 = \{\{1, 2, 3\}, \{2, 2, 2\}, \{3, 3, 3\}\}$;
(3) $T3_3 = \{\{1, 3, 3\}, \{2, 2, 2\}, \{3, 3, 3\}\}$;
(4) $T3_4 = \{\{3, 2, 1\}, \{2, 2, 2\}, \{1, 2, 3\}\}$;
(5) $T3_5 = \{\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}\}$;
(6) $T3_6 = \{\{1, 2, 3\}, \{2, 2, 3\}, \{3, 2, 3\}\}$;
(7) $T3_7 = \{\{1, 3, 3\}, \{3, 2, 3\}, \{3, 3, 3\}\}$;
(8) $T3_8 = \{\{1, 2, 1\}, \{2, 2, 2\}, \{3, 2, 3\}\}$;
(9) $T3_9 = \{\{1, 2, 3\}, \{2, 2, 3\}, \{3, 3, 3\}\}$;
(10) $T3_{10} = \{\{3, 1, 1\}, \{1, 2, 3\}, \{1, 3, 3\}\}$;
(11) $T3_{11} = \{\{1, 2, 3\}, \{2, 2, 2\}, \{1, 2, 3\}\}$;
(12) $T3_{12} = \{\{1, 3, 3\}, \{1, 2, 3\}, \{1, 3, 3\}\}$;
(13) $T3_{13} = \{\{3, 1, 2\}, \{1, 2, 3\}, \{2, 3, 1\}\}$.

Theorem 11. *Let $(N, *)$ be a NETG with order 4 and denote $N = \{1, 2, 3, 4\}$. Then N must be isomorphic to one of the NETGs represented by the following 67 tables, and these NETGs are not mutually isomorphic: (the tables are omitted).*

Theorem 12. *Let $(N, *)$ be a commutative NETG with order 5 and denote $N = \{1, 2, 3, 4, 5\}$. Then N must be isomorphic to one of the NETGs represented by the following 51 tables, and these NETGs are not mutually isomorphic: (the tables are omitted).*

Theorem 13. *Let $(N, *)$ be an AG-NET-loop with order 3 and denote $N = \{1, 2, 3\}$. Then N must be isomorphic to one of the AG-NET-loops represented by the following tables, and these AG-NET-loops are not mutually isomorphic:*

(1) $L3_1 = \{\{1, 1, 1\}, \{1, 2, 1\}, \{1, 1, 3\}\}$;
(2) $L3_2 = \{\{1, 1, 1\}, \{1, 2, 2\}, \{1, 2, 3\}\}$;
(3) $L3_3 = \{\{1, 1, 1\}, \{1, 2, 3\}, \{1, 3, 2\}\}$;
(4) $L3_4 = \{\{1, 1, 3\}, \{1, 2, 3\}, \{3, 3, 1\}\}$;
(5) $L3_5 = \{\{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\}\}$.

Theorem 14. *Let $(N, *)$ be an AG-NET-loop order 4 and denote $N = \{1, 2, 3, 4\}$. Then N must be isomorphic to one of the AG-NET-loops represented by the following 17 tables, and these AG-NET-loops are not mutually isomorphic: (the tables are omitted).*

Theorem 15. *Let $(N, *)$ be an AG-NET-loop order 5 and denote $N = \{1, 2, 3, 4, 5\}$. Then N must be isomorphic to one of the AG-NET-loops represented by the following 54 tables, and these AG-NET-loops are not mutually isomorphic: (the tables are omitted).*

Table 9
Finite NETGs and AG-NET-loops

Order	NETGs	AG-NET-loops
3	13	5
4	67	17
5	353	54

7. Conclusions

In the paper, from the perspective of semigroup theory, we studied neutrosophic extended triplet group (NETG) and AG-NET-loop which is both an AG-groupoid and a neutrosophic extended triplet loop, and obtained some important results. We proved that the notion of NETG is equal to the notion of completely regular semi group, and the notion of weak commutative NETG is equal to the notion of Clifford semigroup. Moreover, we investigated the relationships among AG-NET-loops, and completely regular AG-groupoids and fully regular AG-groupoids, we proved that every AG-NET-loop is a completely regular and fully regular AG-groupoid, but the inverse is not true by constructing some counter examples. We also give some construction methods and low order instances of finite NETGs and AG-NET-loops (the order ≤ 5), see Table 9. These results are interesting for exploring the structure characterizations of NETGs and AG-NET-loops.

As a direction of future research, we will discuss the integration of the related topics, such as the combination of neutrosophic set, fuzzy set, soft set and algebra systems (see [30–34]).

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Neutrosophic Triplets in Neutrosophic Rings

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W.B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2019). Neutrosophic Triplets in Neutrosophic Rings. *Mathematics* 7, 563; DOI: 10.3390/math7060563

Abstract: The neutrosophic triplets in neutrosophic rings $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$ are investigated in this paper. However, non-trivial neutrosophic triplets are not found in $\langle Z \cup I \rangle$. In the neutrosophic ring of integers $Z \setminus \{0, 1\}$, no element has inverse in Z. It is proved that these rings can contain only three types of neutrosophic triplets, these collections are distinct, and these collections form a torsion free abelian group as triplets under component wise product. However, these collections are not even closed under component wise addition.

Keywords: neutrosophic ring; neutrosophic triplets; idemponents; special neutrosophic triplets

1. Introduction

Handling of indeterminacy present in real world data is introduced in [1,2] as neutrosophy. Neutralities and indeterminacies represented by Neutrosophic logic has been used in analysis of real world and engineering problems [3–5].

Neutrosophic algebraic structures such as neutrosophic rings, groups and semigroups are presented and analyzed and their application to fuzzy and neutrosophic models are developed in [6]. Subsequently, researchers have been studying in this direction by defining neutrosophic rings of Types I and II and generalization of neutrosophic rings and fields [7–12]. Neutrosophic rings [9] and other neutrosophic algebraic structures are elaborately studied in [6–8,10,13–17]. Related theories of neutrosophic triplet, duplet, and duplet set were developed by Smarandache [18]. Neutrosophic duplets and triplets have fascinated several researchers who have developed concepts such as neutrosophic triplet normed space, fields, rings and their applications; triplets cosets; quotient groups and their application to mathematical modeling; triplet groups; singleton neutrosophic triplet group and generalization; and so on [19–36]. Computational and combinatorial aspects of algebraic structures are analyzed in [37].

Neutrosophic duplet semigroup [23], classical group of neutrosophic triplet groups [27], the neutrosophic triplet group [12], and neutrosophic duplets of $\{Z_{pn}, \times\}$ and $\{Z_{pq}, \times\}$ have been analyzed [28]. Thus, Neutrosophic triplets in case of the modulo integers $Z_n (2 < n < \infty)$ have been extensively researched [27].

Neutrosophic duplets in neutrosophic rings are characterized in [29]. However, neutrosophic triplets in the case of neutrosophic rings have not yet been researched. In this paper, we for the first time completely characterize neutrosophic triplets in neutrosophic rings. In fact, we prove this collection of neutrosophic triplets using neutrosophic rings are not even closed under addition. We also prove that they form a torsion free abelian group under component wise multiplication.

2. Basic Concepts

In this section, we recall some of the basic concepts and properties associated with both neutrosophic rings and neutrosophic triplets in neutrosophic rings. We first give the following notations: I denotes the indeterminate and it is such that $I \times I = I = I^2$. I is called as the neutrosophic value. Z, Q and R denote the ring of integers, field of rationals and field of reals, respectively. $\langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}$ is the neutrosophic ring of integers, $\langle Q \cup I \rangle = \{a + bI | a, b \in Q, I^2 = I\}$ is the neutrosophic ring of rationals and $\langle R \cup I \rangle = \{a + bI | a, b \in R, I^2 = I\}$ is the neutrosophic ring of reals with usual addition and multiplication in all the three rings.

3. Neutrosophic Triplets in $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$

In this section, we prove that the neutrosophic rings $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$ have infinite collection of neutrosophic triplets of three types. Both collections enjoy strong algebraic structures. We explore the algebraic structures enjoyed by these collections of neutrosophic triplets. Further, the neutrosophic ring of integers $\langle Z \cup I \rangle$ has no nontrivial neutrosophic triplets. An example of neutrosophic triplets in $\langle Q \cup I \rangle$ is provided before proving the related results.

Example 1. *Let $S = \langle Q \cup I \rangle, +, \times$ (or $\langle R \cup I \rangle, +, \times$) be the neutrosophic ring. If $x = a - aI \in S (a \neq 0)$, then*

$$y = \frac{1}{a} - \frac{I}{a} \in S$$

is such that

$$x \times y = (a - aI) \times \left(\frac{1}{a} - \frac{I}{a}\right) = 1 - I - I + I = 1 - I.$$

Thus, for every $x = a - aI$, of this form in S we have a unique y of the form

$$\frac{1}{a} - \frac{I}{a}$$

such that $x \times y = 1 - I$. Further, $1 - I \in S$ is such that $1 - I \times 1 - I = 1 - I + I - I = 1 - I \in S$. Thus, these triplets

$$\left\{a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right\} \text{ and } \left\{\frac{1}{a} - \frac{I}{a}, 1 - I, a - aI\right\}$$

form neutrosophic triplets with $1 - I$ as a neutral element.

Similarly, for $aI \in S (a \neq 0)$, we have a unique

$$\frac{I}{a} \in S \text{ such that } aI \times \frac{I}{a} = I$$

and $I \times I = I$ is an idempotent. Thus,

$$\left\{aI, I, \frac{I}{a}\right\} \text{ and } \left\{\frac{I}{a}, I, aI\right\}$$

are neutrosophic triplets with I as the neutral element.

First, we prove $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$ have only I and $1 - I$ as nontrivial idempotents as invariably one idempotents serve as neutrals of neutrosophic triplets.

Theorem 1. *Let $S = \langle Q \cup I \rangle, +, \times$ (or $\{\langle R \cup I \rangle, +, \times\}$) be a neutrosophic ring. The only non-trivial idempotents in S are I and $1 - I$.*

Proof. We call 0 and $1 \in S$ as trivial idempotents. Suppose $x \in S$ is a non-trivial idempotent, then $x = aI$ or $x = a + bI \in S (a \neq 0, b \neq 0)$. Now, $x \times x = aI \times aI = a^2 I$ (as $I^2 = I$); if x is to be an idempotent, we must have $aI = a^2 I$; that is, $(a - a^2)I = 0 (I \neq 0)$, thus $a^2 = a$. However, in Q or R,

$a^2 = a$ implies $a = 0$ or $a = 1$; as $a \neq 0$, we have $a = 1$; thus, $x = I$ and x is a nontrivial idempotent in S. Now, let $y = a + bI; a \neq 0$ and $b \neq 0$ for $a = 0$ will reduce to case $y = I$ is an idempotent.

$$y^2 = (a + bI) \times (a + bI) = a^2 + b^2 I + 2abI$$

That is, $y^2 = a + bI \times a - bI = a^2 + abI + abI + b^2 I = a + bI$, equating the real and neutrosophic parts.

$$a^2 = a \text{ i.e., } a(a-1) = 0 \Rightarrow a = 1 \text{ as } a \neq 0 \text{ and } 2ab + b^2 - b = 0$$

$b(2a + b - 1) = 0; b \neq 0$, thus $2a + b - 1 = 0$; further, $a \neq 0$ as $a = 0$ will reduce to the case $I^2 = I$, thus $a = 1$. Hence, $2 + b - 1 = 0$, thus $b = -1$. Hence, $a = 1$ and $b = -1$ leading to $y = 1 - I$. Thus, only the non-trivial idempotents of S are I and $1 - I$. □

We next find the form of the triplets in S.

Theorem 2. *Let $S = \{\langle Q \cup I \rangle, +, \times\}$ (or $\langle R \cup I \rangle, +, \times$) be the neutrosophic ring. The neutrosophic triplets in S are only of the following form for $a, b \in Q$ or R.*

(i)
$$\left(a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right) \text{ and } \left(\frac{1}{a} - \frac{I}{a}, 1 - I, a - aI\right); a \neq 0.$$

(ii)
$$\left(bI, I, \frac{I}{b}\right) \text{ and } \left(\frac{I}{b}, I, b\right); b \neq 0.$$

(iii)
$$\left(a + bI, 1, \frac{1}{a} - \frac{bI}{a(a+b)}\right); a + b \neq 0 \text{ and } \left(\frac{1}{a} - \frac{bI}{a(a+b)}, 1, a + bI\right).$$

Proof. Let S be the neutrosophic ring. Let $x = \{a + bI, e + fI, c + dI\}$ be a neutrosophic triplet in $S; a, b, c, d, e, f \in Q$ or R. We prove the neutrosophic triplets of S are in one of the forms. If x is a neutrosophic triplet, then we have

$$a + bI \times e + fI = a + bI \tag{1}$$

$$e + fI \times c + dI = c + dI \tag{2}$$

and

$$a + bI \times c + dI = e + fI \tag{3}$$

Now, solving Equation (1), we get

$$ae + (bfI + beI + afI) = a + bI$$

Equating the real and neutrosophic parts, we get

$$ae = a \tag{4}$$

$$bf + be + af = b \tag{5}$$

Expanding Equation (2), we get

$$ce + fcI + deI + fdI = c + dI.$$

Equating the real and neutrosophic parts, we get

$$ce = c \tag{6}$$

$$fc + de + fd = d. \tag{7}$$

Solving Equation (3), we get

$$ac + bcI + bdI + adI = e + fI$$

Equating the real and neutrosophic parts, we get

$$ac = e \tag{8}$$

$$bc + bd + ad = f \tag{9}$$

We find conditions so that Equations (4) and (5) are true.

Now, $ae = a$ and $bf + be + af = b$; $ae = a$ gives $a(e-1) = 0$ if $a = 0$ and $e \neq 1$ using in Equation (4), thus if $a = 0$, we get $e = 0$ and using $e = 0$ in Equation (6), we get $c = 0$. Thus, $a = c = e = 0$. This forces $b \neq 0$, $d \neq 0$ and $f \neq 0$. We solve for b, d and f using Equations (5), (7) and (9). Equations (5) and (7) gives $bf = b$ as $b \neq 0, f = 1$. Now, $fd = d$ as $f = 1; d = d$. Equation (9) gives $bd = f$ or $bd = 1$, thus

$$d = \frac{1}{b}(b \neq 0).$$

Thus, we get

$$\left(bI, I, \frac{I}{b}\right)$$

to be neutrosophic triplet then

$$\left(\frac{I}{b}, I, bI\right)$$

is also a neutrosophic triplet. Thus, we have proved (ii) of the theorem.

Assume in Equation (4) $ae = a; a \neq 0$, which forces $e = 1$. Now, using Equation (8), we get $ac = 1$, thus

$$c = \frac{1}{a}.$$

Using Equation (5), we get $bf + b + af = b$, thus $(a+b)f = 0$. If $f = 0$, then we have

$$\left(a + bI, 1, \frac{1}{a} + dI\right)$$

should be a neutrosophic triplet. That is,

$$(a + bI) \times \left(\frac{1}{a} + dI\right) = 1$$

$$1 + \frac{b}{a}I + daI + dbI = 1$$

$$\frac{b}{a} + da + db = 0$$

$$b + a^2d + abd = 0$$

$$b(ad + 1) + a^2d = 0$$

$$d(a^2 + ab) = -b.$$

$$d = \frac{-b}{a^2 + ab} = \frac{-b}{a(a+b)}$$

$a \neq 0$ and $a + b \neq 0$. $a + b \neq 0$ for if $a + b = 0$, then $b = 0$ we get $d = 0$. Thus, the trivial triplet

$$(a, 1, \frac{1}{a})$$

will be obtained. Thus, $a + b \neq 0$ and

$$\left(a + bI, 1, \frac{1}{a} - \frac{bI}{a(a+b)}\right) \text{ and } \left(\frac{1}{a} - \frac{bI}{a(a+b)}, 1, a + bI\right)$$

are neutrosophic triplets so that Condition (iii) of theorem is proved.

Now, let $f \neq 0$, thus $a + b = 0$ and $c + d = 0$. We get $a = -b$ or $b = -a$ and $d = -c$. We have already proved $c = \frac{1}{a}$. Using Equations (8) and (9) and conditions $a = -b$ and $c = -d$, we get $f = -1$.

Hence, the neutrosophic triplets are

$$\left(a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right) \text{ and } \left(\frac{1}{a} - \frac{I}{a}, 1 - I, a - aI\right)$$

which is Condition (i) of the theorem. $\square$

Theorem 3. *Let $S = \{\langle Q \cup I \rangle, +, \times\}$ (or $\langle R \cup I \rangle, +, \times\}$) be the neutrosophic ring.*

$$M = \left\{\left(a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right) | a \in Q \backslash \{0\}\right\}$$

be the collection of neutrosophic triplets of S with neutral $1 - I$ is commutative group of infinite order with $(1 - I, 1 - I, 1 - I)$ as the multiplicative identity.

Proof. To prove M is a group of infinite order, we have to prove M is closed under component-wise product and has an identity with respect to which every element has an inverse.

Let

$$x = \left(a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right) \text{ and } y = \left(c - cI, 1 - I, \frac{1}{c} - \frac{I}{c}\right) \in M$$

$$x \times y = \left(a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right) \times \left(c - cI, 1 - I, \frac{1}{c} - \frac{I}{c}\right)$$

$$= \left(ac - acI - acI + acI, 1 - 2I + I, \frac{1}{ac} - \frac{I}{ac} - \frac{I}{ac} + \frac{I}{ac}\right)$$

$$= \left(ac - acI, 1 - I, \frac{1}{ac} - \frac{I}{ac}\right) \in M.$$

Thus, M is closed under component wise product.

We see that, when $a = 1$, we get $e = (1 - I, 1 - I, 1 - I) \in M$ is the identity of M under component wise multiplication. Clearly, $e \times x = x \times e = x$ for all $x \in M$, thus e is the identity of M. For every

$$x = \left(a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right),$$

we have a unique

$$x^{-1} = \left(\frac{1}{a} - \frac{I}{a}, 1 - I, a - aI\right) \in M$$

such that

$$x \times x^{-1} = x^{-1} \times x = e = (1 - I, 1 - I, 1 - I)$$

$$\begin{aligned} x \times x^{-1} &= \left(a - aI, 1 - I, \frac{1}{a} - \frac{I}{a}\right) \times \left(\frac{1}{a} - \frac{I}{a}\right) - \left(\frac{1}{a} - \frac{I}{a}, 1 - I, a - aI\right) \\ &= \left(\frac{a}{a} - \frac{aI}{a} - \frac{aI}{a} + \frac{aI}{a}, 1 - 2I + I, \frac{a}{a} - \frac{aI}{a} - \frac{aI}{a} + \frac{aI}{a}\right) \\ &= (1 - I, 1 - I, 1 - I) \end{aligned}$$

as $a \neq 0$. Thus, $(M, \times)$ is a group under component wise product, which is known as the neutrosophic triplet group. □

Theorem 4. *Let $S = \{\langle Q \cup I\rangle, +, \times\}$ (or $\{\langle R \cup I\rangle, +, \times\}$) be the neutrosophic ring. The collection of neutrosophic triplets*

$$N = \left\{ \left(aI, I, \frac{I}{a}\right) | a \in Q \backslash \{0\} \right\}$$

(or $R\backslash\{0\}$) forms a commutative group of infinite order under component wise multiplication with (I, I, I) as the multiplicative identity.

Proof. Let

$$N = \left\{ \left(aI, I, \frac{I}{a}\right) | a \neq 0 \in Q \text{ or } R \right\}$$

be a collection of neutrosophic triplets. To prove N is commutative group under component wise product, let

$$x = \left(aI, I, \frac{I}{a}\right)$$

and

$$y = \left(bI, I, \frac{I}{b}\right) \in M.$$

To show $x \times y \in N$.

$$x \times y = \left(aI, I, \frac{I}{a}\right) \times \left(bI, I, \frac{I}{b}\right) = \left(abI, I, \frac{I}{ab}\right),$$

using the fact $I^2 = I$. Hence, $(N, \times)$ is a semigroup under product.

Considering $e = (I, I, I) \in N$, we see that $e \times e = x \times e = x$ for all $x \in N$.

$$e \times x = (I, I, I) \times \left(aI, I, \frac{I}{a}\right) = \left(aI, I, \frac{I}{a}\right) = x(\text{ using } I^2 = I).$$

Thus, (I, I, I) is the identity element of $(N, \times)$. For every

$$x = \left(aI, I, \frac{I}{a}\right),$$

we have a unique

$$x^{-1} = \left(\frac{I}{a}, I, a\right) \in N$$

is such that

$$x \times x^{-1} = \left(aI, I, \frac{I}{a}\right) = (I, I, I)$$

as $a \neq 0$ and $I^2 = I$.

Thus, $\{N, \times\}$ is a commutative group of infinite order.

It is interesting to note both the sets M and N are not even closed under addition.
Next, let

$$P = \left\{ a + bI, 1, \frac{1}{a} - \frac{bI}{a(a+b)}; a \neq b; a + b \neq 0, a \neq 0. \right\}$$

We get

$$a + bI \times \frac{1}{a} - \frac{bI}{a(a+b)} = 1.$$

□

We call these neutrosophic triplets as special neutrosophic triplets contributed by the unity 1 of the ring which is the trivial idempotent of S; however, where it is mandatory, x and $anti(x)$ are nontrivial neutrosophic numbers with $neut(x) = 1$.

Theorem 5. *Let $S = \langle Q \cup I \rangle, +, \times$ (or $\langle R \cup I \rangle, +, \times$) be the neutrosophic ring. Let*

$$P = \left\{ (a + bI, 1, \frac{1}{a} - \frac{bI}{a(a+b)}; a \neq b, \text{ where } a, b \in Q \backslash \{0\} (\text{ or } R \backslash 0) \text{ and } a + b \neq 0 \right\}$$

be the collection of special neutrosophic triplets with 1 as the neutral. P is a torsion free abelian group of infinite order with $(1, 1, 1)$ as its identity under component wise product.

Proof. It is easily verified P is closed under the component wise product and (1, 1, 1) acts as the identity for component wise product. For every

$$x = \left(a - bI, 1, \frac{1}{a} + \frac{bI}{a(a-b)} \right) \in P,$$

we have a unique

$$y = \left(\frac{1}{a} + \frac{bI}{a(a-b)}, 1, a - bI \right) \in P$$

such that $x \times y = (1, 1, 1)$. We also see $x^n \neq (1, 1, 1)$ for any $x \in P$ and $n \neq 0 (n > 0)$; $x \neq (1, 1, 1)$, hence P is a torsion free abelian group. □

4. Discussion and Conclusions

We show that, in the case of neutrosophic duplets in $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ or $\langle R \cup I \rangle$, the collection of duplets $\{a - aI\}$ forms a neutrosophic subring. However, in the case of neutrosophic triplets, we show that $\langle Z \cup I \rangle$ has no nontrivial triplets and we have shown there are three distinct collection of neutrosophic triplets in $\langle R \cup I \rangle$ and $\langle Q \cup I \rangle$. We have proved there are only three types of neutrosophic triplets in these neutrosophic rings and all three of them form abelian groups that are torsion free under component wise product. For future research, we would apply these neutrosophic triplets to concepts akin to SVNS and obtain some mathematical models.

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Refined Neutrosophy and Lattices vs. Pair Structures and YinYang Bipolar Fuzzy Set

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Florentin Smarandache (2019). Refined Neutrosophy and Lattices vs. Pair Structures and YinYang Bipolar Fuzzy Set. *Mathematics* 7, 353; DOI: 10.3390/math7040353

Abstract: In this paper, we present the lattice structures of neutrosophic theories. We prove that Zhang-Zhang's YinYang bipolar fuzzy set is a subclass of the Single-Valued bipolar neutrosophic set. Then we show that the pair structure is a particular case of refined neutrosophy, and the number of types of neutralities (sub-indeterminacies) may be any finite or infinite number.

Keywords: neutrosophic set; Zhang-Zhang's YinYang bipolar fuzzy set; single-valued bipolar neutrosophic set; bipolar fuzzy set; YinYang bipolar fuzzy set

1. Introduction

First, we prove that Klement Dand Mesiar's lattices [1] do not fit the general definition of neutrosophic set, and we construct the appropriate nonstandard neutrosophic lattices of the first type (as neutrosophically ordered set) [2], and of the second type (as neutrosophic algebraic structure, endowed with two binary neutrosophic laws, $\inf_N$ and $\sup_N$) [2].

We also present the novelties that neutrosophy, neutrosophic logic, set, and probability and statistics, with respect to the previous classical and multi-valued logics and sets, and with the classical and imprecise probability and statistics, respectively.

Second, we prove that Zhang-Zhang's YinYang bipolar fuzzy set [3,4] is not equivalent with but a subclass of the Single-Valued bipolar neutrosophic set.

Third, we show that Montero, Bustince, Franco, Rodríguez, Gómez, Pagola, Fernández, and Barrenechea's paired structure of the knowledge representation model [5] is a particular case of Refined Neutrosophy (a branch of philosophy that generalized dialectics) and of the Refined Neutrosophic Set [6]. We disprove again the claim that the bipolar fuzzy set (renamed as YinYang bipolar fuzzy set) is the same of neutrosophic set as asserted by Montero et al [5].

About the three types of neutralities presented by Montero et al., we show, by examples and formally, that there may be any finite number or an infinite number of types of neutralities n, or that indeterminacy (I), as neutrosophic component, can be refined (split) into $1 \leq n \leq \infty$ number of sub-indeterminacies (not only 3 as Montero et al. said) as needed to each application to solve.

Also, we show, besides numerous neutrosophic applications, many innovatory contributions to science were brought on by the neutrosophic theories, such as: generalization of Yin Yang Chinese philosophy and dialectics to neutrosophy [7], a new branch of philosophy that is based on the dynamics of opposites and their neutralities, the sum of the neutrosophic components T, I, F up to 3, the degrees of dependence/independence between the neutrosophic components [8,9]; the distinction between absolute truth and relative truth in the neutrosophic logic [10], the introduction of nonstandard neutrosophic logic, set, and probability after we have extended the nonstandard analysis [11,12], the refinement of neutrosophic components into subcomponents [6]; the ability to express incomplete information, complete information, paraconsistent (conflicting) information [13,14]; and the extension of the middle principle to the multiple-included middle principle [15], introduction of neutrosophic crisp set and topology [16], and so on.

2. Answers to Erich Peter Klement and Radko Mesiar

2.1. Oversimplification of the Neutrosophic Set

At [1], page 10 (Section 3.3) in their paper, related to neutrosophic sets, they wrote:

"As a straightforward generalization of the product lattice $\left(\mathbb{I}\times\mathbb{I}, \leq_{comp}\right)$, for each $n \in N$, the n-dimensional unit cube $\left(\mathbb{I}^n, \leq_{comp}\right)$, i.e., the n-dimensional product of the lattice $(\mathbb{I}, \leq_{comp})$, can be defined by means of (1) and (2).

The so-called "neutrosophic" sets introduced by F. Smarandache [93] (see also [94–97], which are based on the bounded lattices $\left(\mathbb{I}^3, \leq_{\mathbb{I}^3}\right)$ and $\left(\mathbb{I}^3, \leq^{\mathbb{I}^3}\right)$, where the orders $\leq_{I^3}$ and $\leq_{I^3}$ on the unit cube I^3 are defined by the Equations below.

$$(x_1, x_2, x_3) \leq_{I^3} (y_1, y_2, y_3) \Leftrightarrow x_1 \leq y_1 \ AND \ x_2 \leq y_2 \ AND \ x_3 \geq y_3 \quad (\text{-}13)$$

$$(x_1, x_2, x_3) \leq^{I^3} (y_1, y_2, y_3) \Leftrightarrow x_1 \leq y_1 \ AND \ x_2 \geq y_2 \ AND \ x_3 \geq y_3 \quad (\text{-}14)$$

The authors have defined Equations *(1)* and *(2)* as follows:

$$\left(\prod_{i=1}^{n} L_i, \leq_{comp}\right), where \left(L_i, \leq_{L_i}\right) are\ fuzzy\ lattices,\ for\ all\ 1 \leq i \leq n \quad (1)$$

$$(x_1, x_2, \ldots, x_n) \leq_{comp} (y_1, y_2, \ldots, y_n) \Leftrightarrow x_1 \leq y_1 \ AND \ x_2 \leq y_2 \ AND \ \ldots \ AND \ x_n \leq y_n \quad (2)$$

The authors did not specify what type of lattices they employ: of the first type (lattice, as a partially ordered set), or the second type (lattice, as an algebraic structure). Since their lattices are endowed with some inequality (referring to the neutrosophic case), we assume it is as the first type.

The authors have used the notations:

$\mathbb{I} = [0,\ 1]$,
$\mathbb{I}^2 = [0,\ 1]^2$,
$\mathbb{I}^3 = [0,\ 1]^3$.

The order relationship $\leq_{\text{comp}}$ on $\mathbb{I}^3$ can be defined as:

$$(x_1, x_2, x_3) \leq_{\text{comp}} (y_1, y_2, y_3) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2 \text{ and } x_3 \leq y_3$$

The three lattices they constructed are denoted by KL_1, KL_2, KL_3, respectively.

$$KL_1 = (\mathbb{I}^3, \leq_{\text{comp}}),\ KL_2 = (\mathbb{I}^3, \leq_{I^3}),\ KL_3 = (\mathbb{I}^3, \leq^{\mathbb{I}^3})$$

Contain only the very particular case of standard single-valued neutrosophic set, i.e., when the neutrosophic components T (truth-membership), I (indeterminacy-membership), and F (false-membership) of the generic element $x(T,\ I,\ F)$, of a neutrosophic set N are single-valued (crisp) numbers from the unit interval $[0, 1]$.

The authors have *oversimplified* the neutrosophic set. Neutrosophic is much more complex. Their lattices do not characterize the *initial definition of the neutrosophic set* ([10], 1998): a set whose elements have the degrees of appurtenance T, I, F, where T, I, F are standard or nonstandard subsets of the nonstandard unit interval: $]^-0,\ 1^+[$, where $]^-0,\ 1^+[$ overpasses the classical real unit interval $[0,\ 1]$ to the left and to the right.

2.2. *Neutrosophic Cube vs. Unit Cube*

Clearly, their $\mathbb{I}^3 = [0,1]^3 \subsetneq\ ]^-0,\ 1^+[^3$ that is our neutrosophic cube (Figure 1), where $]^-0 = \mu(^-0)$ is the left nonstandard monad of number 0, and $1^+ = \mu(1^+)$ is the right nonstandard monad of number 1.

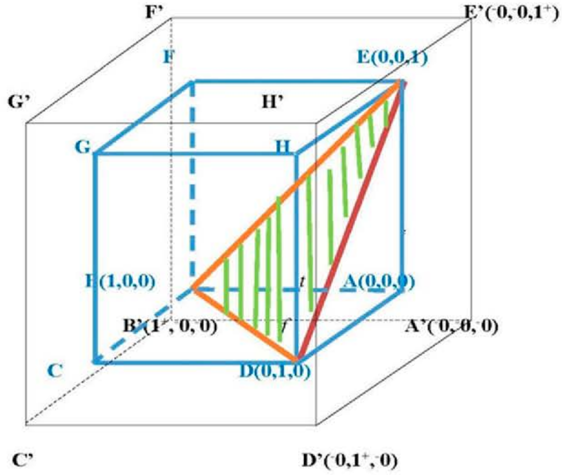

Figure 1. Neutrosophic cube.

The unit cube $\mathbb{I}^3$ used by the authors does not equal the above neutrosophic cube. The neutrosophic cube $A'B'C'D'E'F'G'H'$ was introduced by Dezert [17] in 2002.

2.3. *The Most General Neutrosophic Lattices*

The authors' lattices are far from catching the *most general definition of the neutrosophic set*.

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set. Then an element $x(T(x), I(x), F(x)) \in M$, where $T(x), I(x), F(x)$ are standard or nonstandard subsets of nonstandard interval: $]^-\Omega,\ \Psi^+[$, where $\Omega \leq 0 < 1 \leq \Psi$, with $\Omega,\ \Psi \in \mathbb{R}$, whose values Ω and Ψ depend on each application, and

$$]^-\Omega,\ \Psi^+[=_N \left\{\varepsilon, a, a^-, a^{-0},\ a^+,\ a^{+0},\ a^{\mp},\ a^{-0+} \middle|\ \varepsilon, a \in [\Omega,\ \Psi],\ \varepsilon \text{ is infinitesimal}\right\},$$

where $\overset{m}{a}$, $m \in \left\{^-, ^{-0}, ^+, ^{+0}, ^{-+}, ^{-0+}\right\}$ are monads or binads [12].

It follows that the nonstandard neutrosophic mobinad real offsets lattices $\left(]^-\Omega,\ \Psi^+[,\ \leq_N^{nonS}\right)$ and $\left(]^-\Omega,\ \Psi^+[,\ \inf_N,\ \sup_N, ^-\Omega,\ \Psi^+\right)$ of the first type and, respectively, of the second type are the most general (non-refined) neutrosophic lattices.

While the most general refined neutrosophic lattices of the first type is: $\left(]^-\Omega,\ \Psi^+[,\ \leq_{nN}^{nonS}\right)$, where $\leq_{nN}^{nonS}$ is the n-tuple nonstandard neutrosophic inequality dealing with nonstandard subsets, defined as:

$$(T_1(x),\ T_2(x),\ \ldots,\ T_p(x);\ I_1(x),\ I_2(x),\ \ldots,\ I_r(x);\ F_1(x),\ F_2(x),\ \ldots,\ F_s(x)\ ) \leq_{nN}^{nonS} (T_1(y),\ T_2(y),\ \ldots,\ T_p(y);\ I_1(y),\ I_2(y),\ \ldots,\ I_r(y);\ F_1(y),\ F_2(y),\ \ldots,\ F_s(y)) \text{ iff}$$

$$T_1(x) \leq_{nN}^{nonS} T_1(y),\ T_2(x) \leq_{nN}^{nonS} T_2(y),\ \ldots,\ T_p(x) \leq_{nN}^{nonS} T_p(y)$$

$$I_1(x) \geq_{nN}^{nonS} I_1(y),\ I_2(x) \geq_{nN}^{nonS} I_2(y),\ \ldots,\ I_r(x) \geq_{nN}^{nonS} I_r(y)$$

$$F_1(x) \geq_{nN}^{nonS} F_1(y),\ F_2(x) \geq_{nN}^{nonS} F_2(y),\ \ldots,\ F_s(x) \geq_{nN}^{nonS} F_s(y)$$

2.4. Distinction between Absolute Truth and Relative Truth

The authors' lattices are incapable of making distinctions between absolute truth (when $T = 1^+ >_N 1$) and relative truth (when $T = 1$) in the sense of Leibniz, which is the essence of nonstandard neutrosophic logic.

2.5. Neutrosophic Standard Subset Lattices

Their three lattices are not even able to deal with *standard subsets* [including intervals [8], and hesitant (discrete finite) subsets] $T, I, F \subseteq [0, 1]$, since they have defined the 3D-inequalities with respect to single-valued (crisp) numbers: $x_1, x_2, x_3 \in [0, 1]$ and $y_1, y_2, y_3 \in [0, 1]$.

In order to deal with standard subsets, they should use *inf/sup*, i.e.,

$(T_1, I_1, F_1) \leq (T_2, I_2, F_2) \Leftrightarrow$
$\inf T_1 \leq \inf T_2$ and $\sup T_1 \leq \sup T_2$,
$\inf I_1 \geq \inf I_2$ and $\sup I_1 \geq \sup I_2$,
and $\inf F_1 \geq \inf F_2$ and $\sup F_1 \geq \sup F_2$

[I have displayed the most used 3D-inequality by the neutrosophic community.]

2.6. Nonstandard and Standard Refined Neutrosophic Lattices

The *Nonstandard Refined Neutrosophic Set* [2,6,12], defined on $]^-0, 1^+[^n$, strictly includes their n-dimensional unit cube ($\mathbb{I}^n$), and we use a nonstandard neutrosophic inequality, not the classical inequalities, to deal with inequalities of monads and binads, such as $\leq_{nN}^{nonS}$ and $\leq_N^{nonS}$.

Not even the Standard Refined Single-Valued Neutrosophic Set [6] (2013) may be characterized with KL_1, KL_2, and KL_3 nor with $\left(\mathbb{I}^n, \leq_{\text{comp}}\right)$, since the n-D neutrosophic inequality is different from n-D $\leq_{\text{comp}}$, and from n-D extensions of $\leq_{I_3}$ or $\leq^{I_3}$ respectively, as follows:

Let T be refined into $T_1, T_2, \ldots, T_p$;

I be refined into $I_1, I_2, \ldots, I_r$;

and F be refined into $F_1, F_2, \ldots, F_s$;

with $p, r, s \geq 1$ are integers, and $p + r + s = n \geq 4$, produced the following n-D neutrosophic inequality.

Let $x\left(T_1^x, T_2^x, \ldots, T_p^x;\ I_1^x, I_2^x, \ldots, I_r^x;\ F_1^x, F_2^x, \ldots, F_s^x\right)$, and $y\left(T_1^y, T_2^y, \ldots, T_p^y;\ I_1^y, I_2^y, \ldots, I_r^y;\ F_1^y, F_2^y, \ldots, F_s^y\right)$.

Then:

$$x \leq_N y \Leftrightarrow \begin{pmatrix} T_1^x \leq T_1^y, T_2^x \leq T_2^y, \ldots, T_p^x \leq T_p^y; \\ I_1^x \geq I_1^y, I_2^x \geq I_2^y, \ldots, I_r^x \geq I_r^y; \\ F_1^x \geq F_1^y, F_2^x \geq F, \ldots, F_s^x \geq F_s^y. \end{pmatrix}$$

2.7. Neutrosophic Standard Overset/Underset/Offset Lattice

Their three lattices KL_1, KL_2 and KL_3 are no match for neutrosophic overset (when the neutrosophic components $T, I, F > 1$), nor for neutrosophic underset (when the neutrosophic components $T, I, F < 0$), and, in general, no match for the neutrosophic offset (when the neutrosophic components T, I, F take values outside the unit interval $[0, 1]$ as needed in real life applications [13,14,18–20] (2006–2018): $[\Omega, \Psi]$ with $\Omega \leq 0 < 1 \leq \Psi$.)

Therefore, a lattice may similarly be built on the *non-unitary neutrosophic cube* $[\varphi, \psi]^3$.

2.8. Sum of Neutrosophic Components up to 3

The authors do not mention the novelty of neutrosophic theories regarding the sum of single-valued neutrosophic components $T + I + F \leq 3$, extended up to 3, and, similarly, the corresponding inequality when T, I, F are subsets of $[0, 1]$: $\sup T + \sup I + \sup F \leq 3$, for neutrosophic set, neutrosophic logic, and neutrosophic probability never done before in the previous classic logic and multiple-valued logics and set theories, nor in the classical or imprecise probabilities.

This makes a big difference, since, for a single-valued neutrosophic set S, all unit cubes $[0,\ 1]^3$ are fulfilled with points, each point $P(a,\ b,\ c)$ into the unit cube may represent the neutrosophic coordinates $(a,\ b,\ c)$ of an element $x(a,\ b,\ c) \in S$, which was not the case for previous logics, sets, and probabilities.

This is not the case for the Picture Fuzzy Set (Cuong [21], 2013) whose domain is $\frac{1}{6}$ of the unit cube (a cube corner):

$$\mathbb{D}^* = \left\{(x_1, x_2, x_3) \in \mathbb{I}^3 \middle| x_1 + x_2 + x_3 \leq 1\right\}$$

For Intuitionistic Fuzzy Set (Atanassov [22], 1986), the following is true.

$$\mathbb{D}_A = \left\{(x_1, x_2, x_3) \in \mathbb{I}^3 \middle| x_1 + x_2 + x_3 = 1\right\}$$

where x_1 = membership degree, x_2 = hesitant degree, and x_3 = nonmembership degree, whose domain is the main cubic diagonal triangle that connects the vertices: $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, and $(0,\ 0, 1)$, i.e., triangle BDE (its sides and its interior) in Figure 1.

2.9. Etymology of Neutrosophy and Neutrosophic

The authors [1] write ironically twice, in between quotations, "neutrosophic" because they did not read the etymology [10] of the word published into my first book (1998), etymology, which also appears into Denis Howe's 1999 *The Free Online Dictionary of Computing* [23], and, afterwards, repeated by many researchers from the neutrosophic community in their published papers:

Neutrosophy [23]: *<philosophy> (From Latin "neuter"—neutral, Greek "sophia"—skill/wisdom). A branch of philosophy, introduced by Florentin Smarandache in 1980, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity, "A"in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics.*

While **neutrosophic** means what is derived/resulted from *neutrosophy*.

Unlike the "intuitionistic|" and "picture fuzzy" notions, the notion of *neutrosophic* was carefully and meaningfully chosen, coming from **neutral** (or indeterminate, denoted by <neutA>) between two opposites, $\langle A \rangle$ and $\langle \text{anti}A \rangle$, which made the main distinction between neutrosophic logic/set/probability, and the previous fuzzy, intuitionistic fuzzy logics and sets, i.e.,

- For neutrosophic logic neither true nor false, but neutral (or indeterminate) in between them;
- Similarly for neutrosophic set: neither membership nor non-membership, but in between (neutral, or indeterminate);
- And analogously for neutrosophic probability: chance that an event E occurs, chance that the event E does not occur, and indeterminate (neutral) chance of the event E of occurring or not occuring.

Their irony is malicious and ungrounded.

2.10. Neutrosophy as Extension of Dialectics

Let $\langle A \rangle$ be a concept, notion, idea, or theory.

Then $\langle \text{anti}A \rangle$ is the opposite of $\langle A \rangle$, while $\langle \text{neut}A \rangle$ is the neutral (or indeterminate) part between them.

While in philosophy, Dialectics is the dynamics of opposites ($\langle A \rangle$ and $\langle \text{anti}A \rangle$), Neutrosophy is an extension of dialectics. In other words, neutrosophy is the dynamics of opposites and their neutrals ($\langle A \rangle$, $\langle \text{anti}A \rangle$, $\langle \text{neut}A \rangle$), because the neutrals play an important role in our world, interfering in one side or the other of the opposites.

Refined Neutrosophy is an extension of Neutrosophy, and it is the dynamics of the refined-items $<A_1>$, $<A_2>$, ... , $<A_n>$, their refined-opposites $<antiA_1>$, $<antiA_2>$, ... , $<antiA_n>$, and their refined-neutrals $<neutA_1>$, $<neutA_2>$, ... , $<neutA_n>$.

As an extension of Refined Neutrosophy one has the Plithogeny [24–27].

2.11. Refined Neutrosophic Set and Lattice

At page 11, Klement and Mesiar ([1], 2018) assert that: Considering, for $n > 3$, lattices which are isomorphic to $\left(L_n(\mathbb{I}),\ \leq_{comp}\right)$, further generalizations of "neutrosophic" sets can be introduced.

The authors are uninformed so that a generalization was done in 2013 when we have published a paper [6] that introduced, for the first time, the refined neutrosophic set/logic/probability, where T, I, F were refined into n neutrosophic subcomponents:

$T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s,$

With $p,\ r,\ s \geq 1$ are integers and $p + r + s = n \geq 4$.

But in our lattice $(\mathbb{I}^n,\ \leq_{nN})$, the neutrosophic inequality is adjusted to the categories of sub-truths, sub-indeterminacies, and sub-falsehood, respectively.

$$(T_1(x),\ T_2(x),\ \ldots,\ T_p(x);\ I_1(x),\ I_2(x),\ \ldots,\ I_r(x);\ F_1(x),\ F_2(x),\ \ldots,\ F_s(x)) \leq_{nN} (T_1(y),\ T_2(y),\ \ldots,\ T_p(y);\ I_1(y),\ I_2(y),\ \ldots,\ I_r(y);\ F_1(y),\ F_2(y),\ \ldots,\ F_s(y)) \text{ if and only if}$$

$$T_1(x) \leq T_1(y),\ T_2(x) \leq T_2(y),\ \ldots,\ T_p(x) \leq T_p(y)$$

$$I_1(x) \geq I_1(y),\ I_2(x) \geq I_2(y),\ \ldots,\ I_r(x) \geq I_r(y)$$

$$F_1(x) \geq F_1(y),\ F_2(x) \geq F_2(y),\ \ldots,\ F_s(x) \geq F_s(y)$$

Therefore, $\leq_{nN}$ is different from the n-D inequalities $\leq_{\text{comp}}$, and from $\leq_{\mathbb{I}^n}$ and $\leq^{\mathbb{I}^n}$ (extending from authors inequalities $\leq_{\mathbb{I}^3}$ and $\leq^{\mathbb{I}^3}$, respectively).

2.12. Nonstandard Refined Neutrosophic Set and Lattice

Even more, Nonstandard Refined Neutrosophic Set/Logic/Probability (which include infinitesimals, monads, and closed monads, binads and closed binads) has no connection and no isomorphism whatsoever with any of the authors' lattices or extensions of their lattices for *2D* and *3D* to *nD*.

2.13. Nonstandard Neutrosophic Mobinad Real Lattice

We have built ([2], 2018) a more complex Nonstandard Neutrosophic Mobinad Real Lattice, on the nonstandard mobinad unit interval $]^-0,\ 1^+[$ defined as:

$$]^-0,\ 1^+[= \left\{\varepsilon, a, a^-, a^{-0},\ a^+,\ a^{+0},\ a^{-+},\ a^{-0+} \middle| \text{ with } 0 \leq a \leq 1,\ a \in \mathbb{R}, \text{ and } \varepsilon > 0,\ \varepsilon \text{ infinitesimal},\ \varepsilon \in \mathbb{R}^*\right\}$$

which is both **nonstandard neutrosophic lattice of the first type** (as partially ordered set, under neutrosophic inequality $\leq_N$) and lattice of the second type (as algebraic structure, endowed with two binary nonstandard neutrosophic laws: $\inf_N$ and $\sup_N$).

Now, $]^-0,\ 1^+[^3$ is a nonstandard unit cube, with much higher density than $[0,\ 1]^3$ and which comprise not only real numbers $a \in [0,\ 1]$ but also infinitesimals $\varepsilon > 0$ and monads and binads neutrosophically included in $]^-0,\ 1^+[$.

2.14. New Ideas Brought by the Neutrosophic Theories and Never Done Before

- The sum of the neutrosophic components is up to 3 (previously the sum was up to 1);
- Degree of independence and dependence between the neutrosophic components T, I, F, making their sum $T + I + F$ vary between 0 and 3.

For example, when T, I, and F are totally dependent with each other, then $T + I + F \leq 1$. Therefore, we obtain the particular cases of intuitionistic fuzzy set (when $T + I + F = 1$) and picture set when $T + I + F \leq 1$.

- Nonstandard analysis used in order to distinguish between absolute and relative (truth, membership, chance).

— Refinement of the components into sub-components:

$$\left(T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s\right)$$

with the newly introduced Refined Neutrosophic Logic/Set/Probability.

— Ability to express *incomplete information* ($T + I + F < 1$) and paraconsistent (conflicting) and subjective information ($T + I + F > 1$).

— Law of Included Middle explicitly/independently expressed as $\langle$neut$A\rangle$ (indeterminacy, neutral).

— Law of Included Middle expanded to the Law of Included Multiple-Middles within the refined neutrosophic set as well as logic and probability.

— A large array of applications [28–30] in a variety of fields, after two decades from their foundation ([10], 1998), such as: Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Communication, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences, and more [30].

Klement's and Mesiar's claim that the neutrosophic set (I do not talk herein about intuitionistic fuzzy set, picture fuzzy set, and Pythagorean fuzzy set that they criticized) is not a new result is far from the truth.

3. Neutrosophy vs. Yin Yang Philosophy

Ying Han, Zhengu Lu, Zhenguang Du, Gi Luo, and Sheng Chen [3] have defined the "YinYang bipolar fuzzy set" (2018).

However, the "YinYang bipolar" is already a pleonasm, because, in Taoist Chinese philosophy, from the 6th century BC, Yin and Yang was already a bipolarity, between negative (Yin)/positive (Yang), or feminine (Yin)/masculine (Yang).

Dialectics was derived, much later in time, from Yin Yang.

Neutrosophy, as the dynamicity and harmony between opposites (Yin <A> and Yang (antiA>) together with their neutralities (things which are neither Yin nor Yang, or things which are blends of both: <neutA>) is an extension of Yin Yang Chinese philosophy. Neutrosophy came naturally since, into the dynamicity, conflict, cooperation, and even ignorance between opposites, the neutrals are attracted and play an important role.

3.1. YinYang Bipolar Fuzzy Set Is the Bipolar Fuzzy Set

The authors sincerely recognize that: "*In the existing papers, YinYang bipolar fuzzy set also was called bipolar fuzzy set [5] and bipolar-valued fuzzy set [13,16].*"

These papers are cited as References [31–33].

We prove that the YinYang bipolar fuzzy set is not equivalent with the neutrosophic set, but a particular case of the bipolar neutrosophic set.

The authors [3] say that: "Denote $I^P = [0, 1]$ and $I^N = [-1, 0]$, and $L = \left\{\tilde{\alpha} = (\tilde{\alpha}^P, \tilde{\alpha}^N) \middle| \tilde{\alpha}^P \in I^P, \tilde{\alpha}^N \in I^N\right\}$, then $\tilde{\alpha}$ is called the YinYang bipolar fuzzy number. (YinYang bipolar fuzzy set) $X = \{x_1, \cdots, x_n\}$ represents the finite discourse. YinYang bipolar fuzzy set in X is defined by the mapping below.

$$\tilde{A} : X \to L, x \to \left(\tilde{A}^P(x), \tilde{A}^N(x)\right), \forall x \in X.$$

where the functions $\overset{\sim P}{A} : X \to I^P, x \to \overset{\sim P}{A}(x) \in I^P$ and $\overset{\sim N}{A} : X \to I^N, x \to \overset{\sim N}{A}(x) \in I^N$ define the satisfaction degree of the element $x \in X$ to the property, and the implicit counter-property to the YinYang bipolar fuzzy set $\tilde{A}$ in X, respectively (see [3], page 2).

With simpler notations, the above set L is equivalent to:

$L = \{(a, b),\ with\ a \in [0, 1],\ b \in [-1, 0]\}$, and the authors denote (a, b) as the YinYang bipolar fuzzy number.

Further on, again with simpler notations, the so-called *YinYang bipolar fuzzy set* in

$X = \{x_1, \ldots, x_n\}$ is equivalent to:

$X = \{x_1(a_1, b_1), \ldots, x_n(a_n, b_n)\}$, where all $a_1, \ldots, a_n \in [0, 1]$, and all $b_1, \ldots, b_n \in [-1, 0]\}$. Clearly, this is the bipolar fuzzy set and there is no need to call it the "YinYang bipolar fuzzy set." The authors added that: "Montero et al. pointed out that the neutrosophic set is equivalent to the YinYang bipolar fuzzy set in syntax." However, the bipolar fuzzy set is not equivalent to the neutrosophic set at all. The bipolar fuzzy set is actually a particular case of the bipolar neutrosophic set, defined as (keeping the previous notations):

$$X = \{x_1(\ (a_1, b_1), (c_1, d_1), (e_1, f_1)\ ), \ldots, x_n(\ (a_n, b_n), (c_n, d_n), (e_n, f_n)\ )\}$$

where

all $a_1, \ldots, a_n, c_1, \ldots, c_n, e_1, \ldots, e_n \in [0, 1]$, and all $b_1, \ldots, b_n, d_1, \ldots, d_n, f_1, \ldots, f_n \in [-1, 0]\}$;

for a generic $x_j((a_j, b_j), (c_j, d_j), (e_j, f_j)) \in X$, $1 \le j \le n$,

a_i = *positive membership degree of* x_i, and b_i = *negative membership degree of* x_i;

c_i = *positive indeterminate-membership degree of* x_i, and d_i = *negative indeterminate membership degree of* x_i;

e_i = *positive non-membership degree of* x_i, and f_i = *negative non-membership degree of* x_i.

Using notations adequate to the neutrosophic environment, one found the following.

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set. M is a **single-valued bipolar fuzzy set** (that authors call *YinYang bipolar fuzzy set*) if, for any element, $x(T^+_{(x)}, T^-_{(x)}) \in M$, $T^+_{(x)} \in [0, 1]$, and $T^-_{(x)} \in [-1, 0]$, where $T^+_{(x)}$ is the positive membership of x, and $T^-_{(x)}$ is the negative membership of x. (BFS).

The authors write that: "*Montero et al. pointed that the neutrosophic set [22] is equivalent to the YinYang bipolar fuzzy set in syntax [17]*".

Montero et al.'s paper is cited below as Reference [5].

If somebody says something, it does not mean it is true. They have to verify. Actually, it is *untrue*, since the neutrosophic set is totally different from the so-called YinYang bipolar fuzzy set.

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set, if for any element.

$$x(T(x), I(x), F(x)) \in M$$

$T(x), I(x), F(x)$ are standard or nonstandard real subsets of the nonstandard real subsets of the nonstandard real unit interval $]^-0,\ 1^+[$. (NS).

Clearly, the definitions (BFS) and (NS) are totally different. In the so-called YinYang bipolar fuzzy set, there is no indeterminacy $I(x)$, no nonstandard analysis involved, and the neutrosophic components may be subsets as well.

3.2. Single-Valued Bipolar Fuzzy Set as a Particular Case of the Single-Valued Bipolar Neutrosophic Set

The Single-Valued bipolar fuzzy set (alias YinYang bipolar fuzzy set) is a particular case of the Single-Valued bipolar neutrosophic set, employed by the neutrosophic community, and defined as follows:

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set. M is a single-valued bipolar neutrosophic set, if for any element:

$$x(T^+_{(x)}, T^-_{(x)}; I^+_{(x)}, I^-_{(x)}; F^+_{(x)}, F^-_{(x)}) \in M$$

$$T^+_{(x)},\ I^+_{(x)},\ F^+_{(x)} \in [0,\ 1]$$

$$T^-_{(x)},\ I^-_{(x)},\ F^-_{(x)} \in [-1,\ 0]$$

3.3. Dependent Indeterminacy vs. Independent Indeterminacy

The authors say: "*Attanassov's intuitionistic fuzzy set [4] perfectly reflects indeterminacy but not bipolarity.*"

We disagree, since Atanassov's intuitionistic fuzzy set [22] perfectly reflects **hesitancy** between membership and non-membership not **indeterminacy**, since **hesitancy is dependent** on membership and non-membership: $H = 1 - T - F$, where H = hesitancy, T = membership, and F = non-membership.

It is the single-valued neutrosophic set that "perfectly reflects indeterminacy" since indeterminacy (I) in the neutrosophic set is **independent** from membership (T) and from nonmembership (F).

On the other hand, the neutrosophic set perfectly reflects the bipolarity membership/non-membership as well, since the membership (T) and nonmembership (F) are independent of each other.

3.4. Dependent Bipolarity vs. Independent Bipolarity

The *bipolarity* in the single-valued fuzzy set and intuitionistic fuzzy set is **dependent** (restrictive) in the sense that, if the truth-membership is T, then it involves the falsehood-nonmembership $F \le 1 - T$ while the *bipolarity* in a single-valued neutrosophic set is independent (nonrestrictive): if the truth-membership $T \in [0,\ 1]$, the falsehood-nonmebership is not influenced at all, then $F \in [0,\ 1]$.

3.5. Equilibriums and Neutralities

Again: "While, in semantics, the YinYang bipolar fuzzy set suggests *equilibrium*, and neutrosophic set suggests a general *neutrality*. While the neutrosophic set has been successfully applied to a medical diagnosis [9,27], from the above analysis and the conclusion in [31], we see that the YinYang bipolar fuzzy set is clearly the suitable model to a bipolar disorder diagnosis and will be adopted in this paper."

I'd like to add that the single-valued bipolar neutrosophic set suggests:

— three types of equilibrium, between: $T^+_{(x)}$ and $T^-_{(x)}$, $I^+_{(x)}$ and $I^-_{(x)}$, and $F^+_{(x)}$ and $F^-_{(x)}$;
— and two types of neutralities (indeterminacies) between $T^+_{(x)}$ and $F^+_{(x)}$, and between $T^-_{(x)}$ and $F^-_{(x)}$.

Therefore, the single-valued bipolar neutrosophic set is $3 \times 2 = 6$ times more complex and more flexible than the YinYang bipolar fuzzy set. Due to higher complexity, flexibility, and capability of catching more details (such as falsehood-nonmembership, and indeterminacy), the single-valued bipolar neutrosophic set is more suitable than the YinYang bipolar fuzzy set to be used in a bipolar disorder diagnosis.

3.6. Zhang-Zhang's Bipolar Model is not Equivalent with the Neutrosophic Set

Montero et al. [5] wrote: "*Zhang-Zhang's bipolar model is, therefore, equivalent to the neutrosophic sets proposed by Smarandache [70]" (p. 56).*

This sentence is false and we proved previously that what Zhang & Zhang proposed in 2004 is a subclass of the single-valued bipolar neutrosophic set.

3.7. Tripolar and Multipolar Neutrosophic Sets

Not talking about the fact that, in 2016, we have extended our bipolar neutrosophic set to tripolar and even multipolar neutrosophic sets [18], the sets have become more general than the bipolar fuzzy model.

3.8. Neutrosophic Overset/Underset/Offset

Not talking that the unit interval $[0, 1]$ was extended in 2006 below 0 and above 1 into the neutrosophic overset/underset/offset: $[\Omega, \Psi]$ with $\Omega \le 0 < 1 \le \Psi$ (as explained above).

3.9. Neutrosophic Algebraic Structures

The Montero et al. [5] continue: "*Notice that none of these two equivalent models include any formal structure, as claimed in [48]*".

First, we have proved that these two models (Zhang-Zhang's bipolar fuzzy set, and neutrosophic logic) are not equivalent at all. Zhang-Zhang's bipolar fuzzy set is a subclass of a particular type of neutrosophic set, called the single-valued bipolar neutrosophic set.

Second, since 2013, Kandasamy and Smarandache have developed various algebraic structures (such as neutrosophic semigroup, neutrosophic group, neutrosophic ring, neutrosophic field, neutrosophic vector space, etc.) [28] on the set of neutrosophic numbers:

$S_R = \{a + bI|, \text{ where } a,\ b \in \mathbb{R}, \text{ and } I = \text{indeterminacy},\ I^2 = I\}$, where $\mathbb{R}$ is the set of real numbers.

And extended on:

$S_C = \{a + bI|, \text{ where } a,\ b \in C, \text{ and } I = \text{indeterminacy},\ I^2 = I\}$, where C is the set of complex numbers.

However, until 2016 [year of Montero et al.'s published paper], I did not develop a formal structure on the neutrosophic set. Montero et al. are right.

Yet, in 2018, and, consequently at the beginning of 2019, we [2] developed, then generalized, and proved that the neutrosophic set has a structure of the lattice of the first type (as the neutrosophically partially ordered set): $\left(\,]^{-}0,\ 1^{+}[, \le_N\right)$, where $]^{-}0,\ 1^{+}[$ is the nonstandard neutrosophic mobinad (monads and binads) real unit interval, and $\le_N$ is the nonstandard neutrosophic inequality. Moreover, $\left(\,]^{-}0,\ 1^{+}[, \inf_N, \sup_N, {}^{-}0,\ 1^{+}\right)$ has the structure of the bound lattice of the second type (as algebraic structure), under two binary laws $\inf_N$ (nonstandard neutrosophic infimum) and $\sup_N$ (nontandard neutrosophic supremum).

3.10. Neutrality (<neutA>)

Montero et al. [5] continue: " ... *the selected denominations within each model might suggest different underlying structures: while the model proposed by Zhang and Zhang suggests conflict between categories (a specific type of neutrality different from Atanassov's indeterminacy), Smarandache suggests a general neutrality that should, perhaps jointly, cover some of the specific types of neutrality considered in our paired approach.*"

In neutrosophy and neutrosophic set/logic/probability, the neutrality <neutA> means everything in between <A> and <antiA>, everything which is neither <A> nor <antiA>, or everything which is a blending of <A> and <antiA>.

Further on, in Refined Neutrosophy and Refined Neutrosophic Set/Logic/Probability [9], the neutrality <neutA> was split (refined) in 2013 into sub-neutralities (or sub-indeterminacies), such as: <neutA$_1$>, <neutA$_2$>, ... , <neutA$_n$> whose number could be finite or infinite depending on each application that needs to be solved.

Thus, the paired structure becomes a particular case of refined neutrosophy (see next).

4. The Pair Structure as a Particular Case of Refined Neutrosophy

Montero et al. [5] in 2016 have defined a **paired structure**: "*composed by a pair of opposite concepts and three types of neutrality as primary valuations: L = {concept, opposite, indeterminacy, ambivalence, conflict}.*"

Therefore, each element $x \in X$, where X is a universe of discourse, is characterized by a degree function, with respect to each attribute value from L:

$$\mu : X \to [0,\ 1]^5$$

$$\mu(x) = (\mu_1(x), \mu_2(x), \mu_3(x), \mu_4(x), \mu_5(x))$$

where $\mu_1(x)$ represents the degree of x with respect to the concept;
$\mu_2(x)$ represents the degree of x with respect to the opposite (of the concept);
$\mu_3(x)$ represents the degree of x with respect to 'indeterminacy';
$\mu_4(x)$ represents the degree of x with respect to 'ambivalence';
$\mu_5(x)$ represents the degree of x with respect to 'conflict'.
However, this paired structure is a particular case of Refined Neutrosophy.

4.1. Antonym vs. Negation

First, Dialectics is the dynamics of opposites. Denote them by $\langle A\rangle$ and $\langle \text{anti}A\rangle$, where $\langle A\rangle$ may be an item, a concept, attribute, idea, theory, and so on while $\langle \text{anti}A\rangle$ is the opposite of $\langle A\rangle$.

Secondly, Neutrosophy ([10], 1998), as a generalization of Dialectics, and a new branch of philosophy, is the dynamics of opposites and their neutralities (denoted by $\langle \text{neut}A\rangle$). Therefore, Neutrosophy is the dynamics of $\langle A\rangle$, $\langle \text{anti}A\rangle$, and $\langle \text{neut}A\rangle$.

$\langle \text{neut}A\rangle$ means everything, which is neither $\langle A\rangle$ nor $\langle \text{anti}A\rangle$, or which is a mixture of them, or which is indeterminate, vague, or unknown.

The **antonym** of $\langle A\rangle$ is $\langle \text{anti}A\rangle$.

The **negation** of $\langle A\rangle$ (which we denote by $\langle \text{non}A\rangle$) is what is not $\langle A\rangle$, therefore:

$$\neg_N\langle A\rangle = \langle nonA\rangle =_N \langle neutA\rangle \cup_N \langle antiA\rangle$$

We preferred to use the lower index $_N$ (neutrosophic) because we deal with items, concepts, attributes, ideas, and theories such as $\langle A\rangle$ and, in consequence, its derivates $\langle \text{anti}A\rangle$, $\langle \text{neut}A\rangle$, and $\langle \text{non}A\rangle$, whose borders are ambiguous, vague, and not clearly delimited.

4.2. Refined Neutrosophy as an Extension of Neutrosophy

Thirdly, Refined Neutrosophy ([6], 2013), as an extension of Neutrosophy, and a refined branch of philosophy, is the dynamics of refined opposites: $\langle A_1\rangle, \langle A_2\rangle, \ldots, \langle A_p\rangle$ with $\langle \text{anti}A_1\rangle, \langle \text{anti}A_2\rangle, \ldots, \langle \text{anti}A_s\rangle$, and their refined neutralities: $\langle \text{neut}A_1\rangle, \langle \text{neut}A_2\rangle, \ldots, \langle \text{neut}A_r\rangle$, for integers p, r, $s \geq 1$, and $p + r + s = n \geq 4$. Therefore, the item $\langle A\rangle$ has been split into sub-items $\langle A_j\rangle$, $1 \leq j \leq p$, the $\langle \text{anti}A\rangle$ into sub-(anti-items) $\langle \text{anti}A_k\rangle$, $1 \leq l \leq s$, and the $\langle \text{neut}A\rangle$ into sub-(neutral-items) $\langle \text{neut}A_l\rangle$, $1 \leq k \leq r$.

4.3. Qualitative Scale as a Particular Case of Refined Neutrosophy

Montero et al.'s qualitative scale [5] is a particular case of Refined Neutrosophy where the neutralities are split into three parts.

L = {concept, opposite, indeterminacy, ambivalence, conflict} = {<A>, <antiA>, <neutA$_1$>, <neutA$_2$>, <neutA$_3$>}

where: *<A> = concept*, *<antiA> = opposite*, *<neutA1> = indeterminacy*, *<neutA2> = ambivalence*, *<neutA3> = conflict*.

Yin Yang, Dialectics, Neutrosophy, and Refined Neutrosophy (the last one having only $\langle \text{neut}A\rangle$ as refined component), are bipolar: $\langle A\rangle$ and $\langle \text{anti}A\rangle$ are the poles.

Montero et al.'s qualitative scale is bipolar ('concept', and its 'opposite').

4.4. Multi-Subpolar Refined Neutrosophy

However, the Refined Neutrosophy, whose at least one of $\langle A\rangle$ or $\langle \text{anti}A\rangle$ is refined, is *multi-subpolar*.

4.5. Multidimensional Fuzzy Set as a Particular Case of the Refined Neutrosophic Set

Montero et al. [5] defined the *Multidimensional Fuzzy Set* A_L as: $A_t = \left\{< x; (\mu_s(x))_{s\in L} > \middle| x \in X\right\}$, where X is the universe of discourse, L = the previous qualitative scale, and $\mu_s(x) \in S$, where S is a valuation scale (in most cases $S = [0, 1]$), $\mu_s(x)$ is the degree of x with respect to $s \in L$.

A *Single-Valued Neutrosophic Set* is defined as follows. Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ a set. For each element $x(T(x), I(x), F(x)) \in M$, $T(x) \in [0,\ 1]$ is the degree of truth-membership of element x with respect to the set M, $I(x) \in [0,\ 1]$ is the degree of indeterminacy-membership of element x with respect to the set M, and $F(x) \in [0,\ 1]$ is the degree of falsehood-nonmembership of element x with respect to the set M.

Let's refine $I(x)$ as $I_1(x)$, $I_2(x)$, and $I_3(x) \in [0,\ 1]$ sub-indeterminacies. Then we get a single-valued refined neutrosophic set.

$\mu_{concept}(x) = T(x)$ (truth-membership);
$\mu_{opposite}(x) = F(x)$ (falsehood-non-membership);
$\mu_{\text{indeterminacy}}(x) = I_1(x)$ (first sub-indeterminacy);
$\mu_{\text{ambivalence}}(x) = I_2(x)$ (second sub-indeterminacy);
$\mu_{\text{conflict}}(x) = I_3(x)$ (third sub-indeterminacy).

The *Single-Valued Refined Neutrosophic Set* is defined as follows. Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ a set. For each element:

$$x\Big(T_1(x), T_2(x), \ldots, T_p(x);\ I_1(x), I_2(x), \ldots, I_r(x);\ F_1(x), F_2(x), \ldots, F_s(x)\Big) \in M$$

$T_j(x)$, $1 \leq j \leq p$, are degrees of subtruth-submembership of element x with respect to the set M.

$I_k(x)$, $1 \leq k \leq r$, are degrees of subindeterminacy-membership of element x with respect to the set M.

Lastly, $F_l(x)$, $1 \leq l \leq s$, are degrees of sub-falsehood-sub-non-membership of element x with respect to the set M, where integers p, r, $s \geq 1$, and $p + r + s = n \geq 4$.

Therefore, Montero et al.'s **multidimensional fuzzy set** is a particular case of the **refined neutrosophic set**, when $p = 1$, $r = 3$, and $s = 1$, where $n = 1 + 3 + 1 = 5$.

4.6. Plithogeny and Plithogenic Set

Fourthly, in 2017 and in 2018 [24–27], the Neutrosophy was extended to Plithogeny, which is multipolar, being the dynamics and hermeneutics [methodological study and interpretation] of many opposites and/or their neutrals, together with non-opposites.

$\langle A\rangle$, $\langle \text{neut}A\rangle$, $\langle \text{anti}A\rangle$;
$\langle B\rangle$, $\langle \text{neut}B\rangle$, $\langle \text{anti}B\rangle$; etc.
$\langle C\rangle$, $\langle D\rangle$, etc.

In addition, the **Plithogenic Set** was introduced, as a generalization of **Crisp**, **Fuzzy**, **Intuitionistic Fuzzy**, and **Neutrosophic Sets**.

Unlike previous sets defined, whose elements were characterized by the attribute 'appurtenance' (to the set), which has only one (membership), or two (membership, nonmembership), or three (membership, nonmembership, indeterminacy) attribute values, respectively. For the Plithogenic Set, each element may be characterized by a multi-attribute, with any number of attribute values.

4.7. Refined Neutrosophic Set as a Unifying View of Opposite Concepts

Montero et al.'s statement [5] from their paper Abstract: *"we propose a consistent and unifying view to all those basic knowledge representation models that are based on the existence of two somehow opposite fuzzy concepts."*

With respect to the "unifying" claim, their statement is not true, since, as we proved before, their **paired structure** together with three types on neutralities (**indeterminacy**, **ambivalence**, and **conflict**) is a simple, particular case of the refined neutrosophic set.

The real unifying view currently is the **Refined Neutrosophic Set**.

{I was notified about this paired structure article [5] by Dr. Said Broumi, who forwarded it to me.}

4.8. Counter-Example to the Paired Structure

As a counter example to the paired structure [5], it cannot catch a simple voting scenario.

The election for the United States President from 2016: Donald Trump vs. Hillary Clinton. USA has 50 states and since, in the country, there is an **Electoral vote**, not a **Popular vote**, it is required to know the winner of each state.

There were two opposite candidates.

The candidate that receives more votes than the other candidate in a state gets all the points of that state.

As in the neutrosophic set, there are three possibilities:

T = percentage of USA people voting for Mr. Trump;

I = percentage of USA people not voting, or voting but giving either a blank vote (not selecting any candidate) or a black vote (cutting all candidates);

F = percentage of USA people voting against Mr. Trump.

The opposite concepts, using Montero et al.'s knowledge representation, are T (voting for, or truth-membership) and F (voting against, or false-membership). However, $T > F$, or $T = F$, or $T < F$, that the Paired Structure can catch, mean only the Popular vote, which does not count in the United States.

Actually, it happened that $T < F$ in the US 2016 presidential election, or Mr. Trump lost the Popular vote, but he won the Presidency using the Electoral vote.

The paired structure is not capable of refining the opposite concepts (T and F), while the indeterminate (I) could be refined by the paired structure only in three parts.

Therefore, the paired structure is not a unifying view of all basic knowledge that uses opposite fuzzy concepts. However, the refined neutrosophic set/logic/probability do.

Using the refined neutrosophic set and logic, and splits (refines) T, I, and F as:

T_j = percentage of American state S_j people voting for Mr. Trump;

I_j = percentage of American state S_j people not voting, or casting a blank vote or a black vote;

F_j = percentage of American state S_j people voting against Mr. Trump, with $T_j, I_j, F_j \in [0, 1]$ and $T_j + I_j + F_j = 1$, for all $j \in \{1, 2, \ldots, 50\}$.

Therefore, one has:

$(T_1, T_2, \ldots, T_{50};\ I_1, I_2, \ldots, I_{50};\ F_1, F_2, \ldots, F_{50})$.

On the other hand, due to the fact that the sub-indeterminacies $I_1, I_2, \ldots, I_{50}$ did not count towards the winner or looser (only for indeterminate voting statistics), it is not mandatory to refine I. We could simply refine it as:

$(T_1, T_2, \ldots, T_{50};\ I;\ F_1, F_2, \ldots, F_{50})$.

4.9. Finite Number and Infinite Number of Neutralities

Montero et al. [5]: *"(...) we emphasize the key role of certain neutralities in our knowledge representation models, as pointed out by Atanassov [4], Smarandache [70], and others. However, we notice that our notion of neutrality should not be confused with the neutral value in a traditional sense (see [22–24,36,54], among others).*

Instead, we will stress the existence of different kinds of neutrality that emerge (in the sense of Reference [11]) from the semantic relation between two opposite concepts (and notice that we refer to a neutral category that does not entail linearity between opposites)."

In neutrosophy, and, consequently, in the neutrosophic set, logic, and probability, between the opposite items (concepts, attributes, ideas, etc.) $\langle A \rangle$ and $\langle \text{anti}A \rangle$, there may be a large number of neutralities/indeterminacies (all together denoted by $\langle \text{neut}A \rangle$ even an infinite spectrum—depending on the application to solve.

We agree with different kinds of neutralities and indeterminacies (vague, ambiguous, unknown, incomplete, contradictory, linear and non-linear information, and so on), but the authors display only three neutralities.

In our everyday life and in practical applications, there are more neutralities and indeterminacies.

In another example (besides the previous one about Electoral voting), there may be any number of sub indeterminacies/sub neutralities.

The opposite concepts attributes are: $\langle A \rangle$ = white, $\langle \text{anti}A \rangle$ = black, while neutral concepts in between may be: $\langle \text{neut}A_1 \rangle$ = yellow, $\langle \text{neut}A_2 \rangle$ = orange, $\langle \text{neut}A_3 \rangle$ = red, $\langle \text{neut}A_4 \rangle$ = violet, $\langle \text{neut}A_5 \rangle$ = green, and $\langle \text{neut}A_6 \rangle$ = blue. Therefore, we have six neutralities. Example with infinitely many neutralities:

— The opposite concepts: $\langle A \rangle$ = white, $\langle \text{anti}A \rangle$ = black;
— The neutralities: $\langle \text{neut}A_{1,\,2,\,\ldots,\,\infty} \rangle$ = the whole light spectrum between white and black, measured in nanometers (*nn*) [a nanometer is a billionth part of a meter].

5. Conclusions

The neutrosophic community thank the authors for their criticism and interest in the neutrosophic environment, and we wait for new comments and criticism, since, as Winston Churchill had said, *the eagles fly higher against the wind*.

Notations

$\leq_{nN}^{nonS}$	means nonstandard n-tuple neutrosophic inequality;
$\leq_{nN}$	means standard (real) n-tuple inequality;
$\leq_{N}^{nonS}$	means nonstandard unary neutrosophic inequality;
$\leq_{N}$	mean standard (real) unary neutrosophic inequality;
$=_{N}$	means neutrosophic equality;
$\neg_{N}$	means neutrosophic negation;
$\cup_{N}$	means neutrosophic union;
$=$	means classical equality;
$<, >, \leq, \geq$	mean classical inequalities.

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Symmetry in Hyperstructure: Neutrosophic Extended Triplet Semihypergroups and Regular Hypergroups

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Xiaohong Zhang, Florentin Smarandache, Yingcang Ma (2019). Symmetry in Hyperstructure: Neutrosophic Extended Triplet Semihypergroups and Regular Hypergroups. *Symmetry*, 11, 1217; DOI: 10.3390/sym11101217

Abstract: The symmetry of hyperoperation is expressed by hypergroup, more extensive hyperalgebraic structures than hypergroups are studied in this paper. The new concepts of neutrosophic extended triplet semihypergroup (NET- semihypergroup) and neutrosophic extended triplet hypergroup (NET-hypergroup) are firstly introduced, some basic properties are obtained, and the relationships among NET- semihypergroups, regular semihypergroups, NET-hypergroups and regular hypergroups are systematically are investigated. Moreover, pure NET-semihypergroup and pure NET-hypergroup are investigated, and a strucuture theorem of commutative pure NET-semihypergroup is established. Finally, a new notion of weak commutative NET-semihypergroup is proposed, some important examples are obtained by software MATLAB, and the following important result is proved: every pure and weak commutative NET-semihypergroup is a disjoint union of some regular hypergroups which are its subhypergroups.

Keywords: hypergroup; semihypergroup; neutrosophic extended triplet group; neutrosophic extended triplet semihypergroup (NET-semihypergroup); NET-hypergroup

1. Introduction and Preliminaries

As a generalization of traditional algebraic structures, hyper algebraic structures (or hypercompositional structures) have been extensively studied and applied [1–7]. Especially, hypergroups and semihypergroups are basic hyper structures which are extensions of groups and semigroups [8]. In fact, hypergroups characterize the symmetry of hyperoperations.

On the other hand, as an extension of fuzzy set and intuitionistic fuzzy set, the concept of neutrosophic set firstly proposed by F. Smarandache in [9], has been applied to many fields [10–12]. Moreover, as an application of the ideal of neutrosphic sets, a new notion of neutrosophic triplet group (NTG) was proposed by F. Smarandache and Ali in [13], while the new notion of neutrosophic extended group (NETG) was proposed by Smarandache in [14]. Furthermore, the basic properties and structural characteristics of neutrosophic extended groups (NETGs) are studied in [15,16]; the closed connection between between NETG and regular semigroup investigated, and the new notion of neutrosophic extended triplet Abel-Grassmann's Groupoid is proposed in [17]; the decomposition theorem of NETG is poved in [18]; the generalized neutrosophic extended groups are presented in [19]; the relationship and difference between NETGs and generalized groups are systematically studied in [20]. From these research results, we know that NETG is a typical algebraic system with important research value.

In this paper, we combine the two directions mentioned above to study the hyperalgebraic structures related to neutrosophic extended triplet groups (NETGs), which can be regarded as a further development of the research ideas in [21].

At first, we recall some concepts and results on hypergroups, semigroups and NETGs.

Let H be a non-empty set and $P^*(H)$ the set of all non-empty subsets of H. A map $\circ$: $H \times H \to P^*(H)$ is called (binary) hyperoperation (or hypercomposition), and $(H, \circ)$ is called a hypergroupoid. If A, $B \in P^*(H)$, $x \in H$, then

$$A \circ B = \bigcup_{a\in A, b\in B} (a \circ b),\ A \circ x = A \circ \{x\},\ x \circ B = \{x\} \circ B.$$

Definition 1. *([1–4]) Let $(H, \circ)$ be a hypergroupoid. If $(\forall x,y,z \in H)$ $(x\circ y)\circ z = x\circ(y\circ z)$, then $(H, \circ)$ is called a semihypergroup. That is,*

$$\bigcup_{u\in x\circ y} (u \circ z) = \bigcup_{v\in y\circ z} (x \circ v).$$

For a semihypergroup $(H, \circ)$, if $(\forall x,y \in H)$ $x\circ y = y\circ x$, then we call that H is commutative.

Note that, if $(H, \circ)$ is a semihypergroup, then $(A\circ B)\circ C = A\circ(B\circ C)$, $\forall A, B, C \in P^*(H)$.

Definition 2. *([1–4]) Assume that $(H, \circ)$ is a semihypergroup. (1) If $a\in H$ satisfies $(\forall x\in H)$ $|a\circ x| = |x\circ a| = 1$, then a is called to be scalar. (2) If $e\in H$ satisfies $(\forall x\in H)$ $x\circ e = e\circ x = \{x\}$, then e is called scalar identity. (3) If $e\in H$ satisfies $(\forall x\in H)$ $x\in(e\circ x)\cap(x\circ e)$, then e is called identity. (4) Let a, $b\in H$. If there exists an identity $e\in H$ satisfies $e\in (a\circ b)\cap(b\circ a)$, then b is called an inverse of a. (5) If $0\in H$ satisfies $(\forall x\in H)$ $x\circ 0 = 0\circ x = \{0\}$, then 0 is called zero element.*

Definition 3. *([1–4]) Let $(H, \circ)$ be a semihypergroup. (1) If $(\forall x\in H)$ $a\circ H = H\circ a = H$ (reproductive axiom), then $(H, \circ)$ is called a hypergroup. (2) If $(H, \circ)$ is a hypergroup and $(H, \circ)$ has at least one identity and each element has at least one inverse, then $(H, \circ)$ is called to be regular.*

Definition 4. *([1–4]) Let $(H, \circ)$ be a semihypergroup. If $x\in H$ satisfies $x\in x\circ H\circ x$, i.e., there exists an element $y\in H$, $x\in x\circ y\circ x$, then x is said to be regular. If $(\forall x\in H)$ x is regular, then $(H, \circ)$ is called to be regular.*

Note that, Every regular semigroup is a regular semihypergroup, and every hypergroup is a regular semihypergroup.

Definition 5. *([14]) Let N be a non-empty set, and * a binary operation on N. If $(\forall a\in N)$ there exist neut(a)$\in$N, anti(a)$\in$N satisfy*

*neut(a)*a = a*neut(a) = a, and*
*anti(a)*a = a*anti(a) = neut(a).*

Then N is called a neutrosophic extended triplet set (NETS). Moreover, for $a\in N$, (a, neut(a), anti(a)) is called a neutrosophic extend triplet, neut(a) is called an extend neutral of "a", and anti(a) is called an opposite of "a".

For a neutrosophic extended triplet set N, $a\in N$, the set of *neut*(a) is denoted by {*neut*(a)}, and the set of *anti*(a) is denoted by {*anti*(a)}.

Definition 6. *([13,14]) Let (N, *) be a NETS. If (N,*) is a semigroup, then (N, *) is called to be a neutrosophic extended triplet group (NETG).*

About some basic properties of neutrosophic extended triplet groups, plesse see [15,17,20].

2. Neutrosophic Extended Triplet Semihypergroups (NET-Semihypergroups) and Neutrosophic Extended Triplet Hypergroups (NET-Hypergroups)

In this section, we propose the new concepts of neutrosophic extended triplet semihypergroup (NET-semihypergroup) and neutrosophic extended triplet hypergroup (NET-hypergroup), and give some typical examples to illustrate their wide representativeness.

Definition 7. *Let (H,*) be a semihypergroup (i.e., * be a binary hyperoperation on nonempty set H such that (x*y)*z = x*(y*z), for all x, y, z∈H). (H,*) is called a neutrosophic extended triplet semihypergroup (shortened form, NET-semihypergroup), if for every x∈H, there exist neut(x) and anti(x) such that*

$$x \in (neut(x)*x)\cap(x*neut(x)), \text{ and}$$

$$neut(x) \in (anti(x)*x)\cap(x*anti(x)).$$

Here, we call that (x, neut(x), anti(x)) to be a hyper-neutrosophic-triplet.

Example 1. *Denote H = {a, b, c}, define hyperoperations * on H as shown in Table 1. We can verify that (H, *) is semihypergroup by software MATLAB (see Figular 1).*

Table 1. The hyperoperation * on *H*.

*	a	b	c
a	a	{a, b}	{a, b, c}
b	a	{a, b}	{a, b, c}
c	a	{a, b}	c

Moreover,

$$a \in (a*a)\cap(a*a);$$
$$b \in (b*b)\cap(b*b);$$
$$c \in (c*c)\cap(c*c).$$

*This means that (H, *) is neutrosophic extended triplet semihypergroup (NET-semihypergroup) and (a, a, a), (b, b, b), (c, c, c) are hyper-neutrosophic-triplets.*

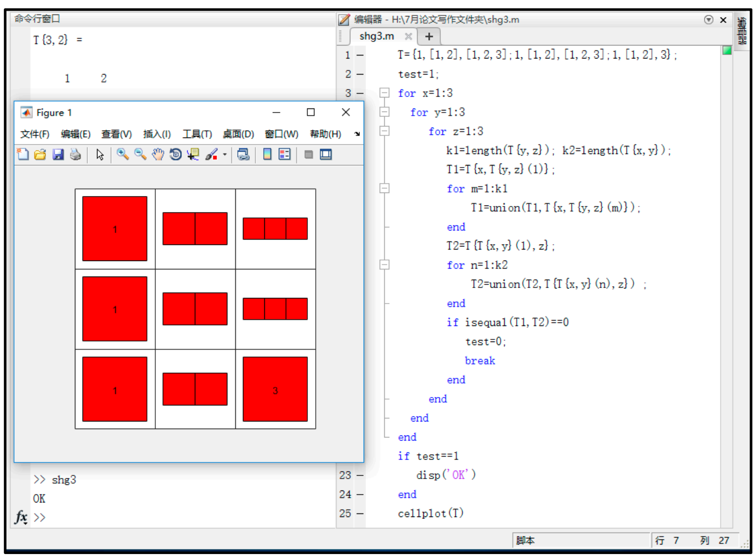

Figure 1. A program by Matlab to verify hyperoperation.

Example 2. *Denote H = {a, b, c, d}, define hyperoperations * on H as shown in Table 2. We can verify that (H, *) is semihypergroup by software MATLAB (see Figular 2).*

Table 2. The hyperoperation * on *H*.

*	a	b	c	d
a	{a, b}	{a, b}	{c, d}	{c, d}
b	{a, b}	{a, b}	{c, d}	{c, d}
c	{c, d}	{c, d}	a	b
d	{c, d}	{c, d}	b	a

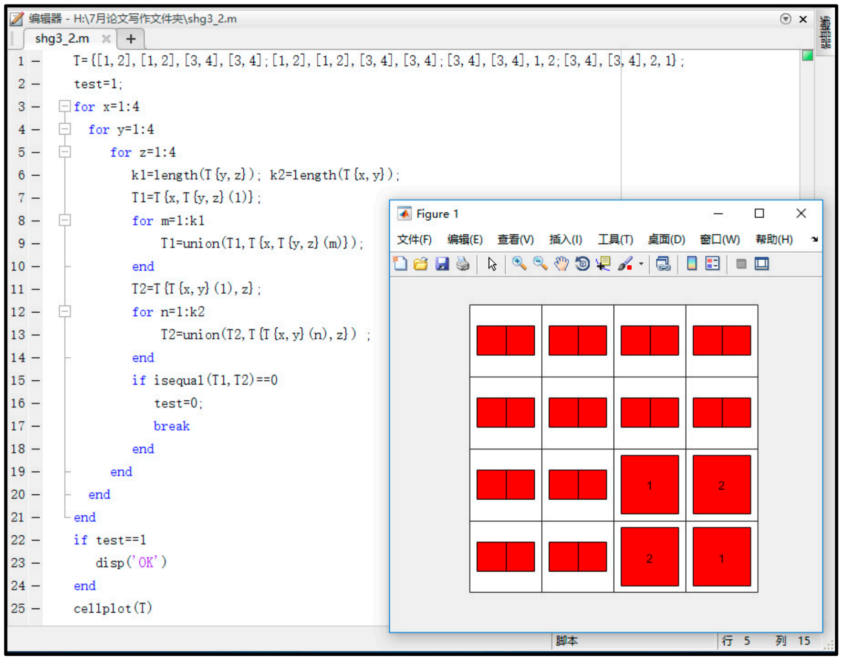

Figure 2. Verify hyperoperation by Matlab.

Moreover,

$$a \in (a*a) \cap (a*a);\ a \in (b*a) \cap (a*b),\ b \in (b*a) \cap (a*b).$$

$$b \in (b*b) \cap (b*b).$$

$$c \in (a*c) \cap (c*a),\ a \in (c*c) \cap (c*c);\ c \in (b*c) \cap (c*b),\ b \in (d*c) \cap (c*d).$$

$$d \in (a*d) \cap (d*a),\ a \in (d*d) \cap (d*d);\ d \in (b*d) \cap (d*b),\ b \in (c*d) \cap (d*c).$$

*This means that (H, *) is neutrosophic extended triplet semihypergroup (NET-semihypergroup) and (a, a, a), (a, b, b), (b, b, b), (c, a, c), (c, b, d), (d, a, d), (d, b, c) are hyper-neutrosophic-triplets.*

Remark 1. From Example 2 we know that *neut*(*x*) may be not unique for an element *x* in a neutrosophic extended triplet semihypergroup (NET-semihypergroup). In fact, in Example 2, we have

$$\{neut(a)\} = \{a, b\},\ neut(b) = b,\ \{neut(c)\} = \{a, b\},\ \{neut(d)\} = \{a, b\}.$$

Example 3. *Let H be the set of all nonnegative integers, and define a hyperoperation * on H as following:*

$$x*y = \{z \in H \mid z \geq max\{x,y\}\}.$$

For examples,

$$3*5 = \{5, 6, 7, 8, \ldots\};\ 9*9 = \{9, 10, 11, 12, \ldots\};\quad 2019*0 = \{2019, 2020, 2021, 2022, \ldots\}.$$

*Then (H, *) is a commutative semihypergroup. Moreove, for any $x \in H$, we have*

$$x \in (x*x) \cap (x*x);\ x \in (x*x) \cap (x*x).$$

*This means that (H, *) is a neutrosophic extended triplet semihypergroup (NET-semihypergroup). In fact, we have*

$$neut(0)=0;\ \{neut(1)\}=\{0,1\};\ \{neut(2)\}=\{0, 1, 2\};\ \{neut(3)\}=\{0, 1, 2, 3\}\ldots$$

Example 4. *Let R be the set of all real numbers, and Z the set of integers. We use the modulo of real numbers (that we denote by mod$_R$) in the following way:*

$\forall a, b \in R$, *then* $a = b$ (mod$_R$ 6), *if and only if* $a - b = 6n$, *where n is an integer.*

For examples, $14.73 = 2.73$ (mod$_R$ 6), *since* $14.73 - 2.73 = 12 = 6 \times 2$; *but* $18 \neq 15$ (mod$_R$ 6), *since* $18 - 15 = 3 \neq 6n$ *with n integer. Now, we define a hyperoperation # on R as following:*

$$a\#b = \{x \in R \mid x = 4ab \ (\mathrm{mod}_R\ 6)\}.$$

Then (R, #) is a commutative semihypergroup, since $a\#b = b\#a = 4ab$ (mod$_R$ 6), *and associative because:*

$$(a\#b)\#c = (4ab)\#c = 4(4ab)c = 16abc \ (\mathrm{mod}_R\ 6), \text{ and}$$

$$a\#(b\#c) = a\#(4bc) = 4a(4bc) = 16abc \ (\mathrm{mod}_R\ 6).$$

Moreove, for any $a \in R$, *we have*

(1) *when* $a=0$, $(a, 6m, r)$ *are hyper-triplets for any integer number m and real number r;*

(2) *when* $a \neq 0$, $\left(a, \frac{1}{4} + \frac{3m}{2a}, \frac{1}{16a} + \frac{3m}{8a} + \frac{3n}{2a}\right)$ *are hyper-neutrosophic-triplets for any integer numbers m, n.*

This means that (R, #) is a neutrosophic extended triplet semihypergroup (NET-semihypergroup), and infinitely many neut(a) and infinitely many anti(a) for any element a in R.

Remark 2. The following example shows that a sub-semihypergroup of a NET-semihypergroup may be not a NET-semihypergroup.

Example 5. *Denote H = {a, b, c, d, e}, define hyperoperations * on H as shown in Table 3. We can verify that (H, *) is semihypergroup by software MATLAB (see Figular 3).*

Table 3. The hyperoperation * on *H*.

*	a	b	c	d	e
a	a	a	a	d	{a, b, c, d, e}
b	a	{a, b}	{a, c}	d	{a, b, c, d, e}
c	a	a	a	d	{a, b, c, d, e}
d	d	d	d	d	{a, b, c, d, e}
e	{a, b, c, d, e}	{a, b, c, d, e}	{a, b, c, d, e}	{a, b, c, d, e}	{a, b, c, d, e}

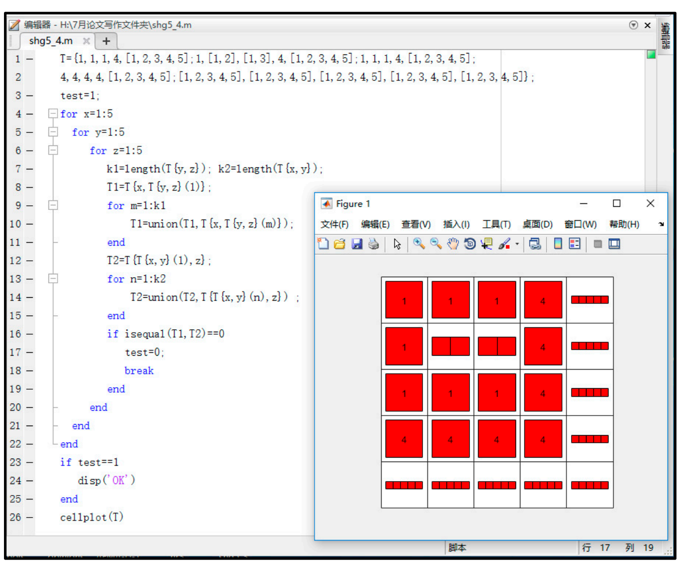

Figure 3. Verify the hyperoperation by Matlab.

*Moreover, (a, a, a), (a, e, e), (b, b, b), (b, e, e), (c, e, e), (d, d, d), (d, e, e), (e, e, e), (e, a, e), (e, b, e), (e, c, e), (e, d, e) are hyper-neutrosophic-triplets.This means that (H, *) is a NET-semihypergroup. For S={a, b, c }⊆H, (S, *) is sub-semihypergroup of (H, *). But, (S, *) is not a NET-semihypergroup.*

Remark 3. For the traditional algebraic structures, we have the conclusion that any group must be a neutrosophic extended triplet group (NETG). For hyper algebraic structures, we know from Example 1 that a NET-semihypergroup is not necessarily a hypergroup (since $a*H \neq H$ in Example 1). Moreover, the following example shows that a hypergroup may be not a NET-semihypergroup. Therefore, hypergroup and NET-semihypergroup are two non-inclusion hyperalgebraic systems.

Example 6. *Denote H = {1, 2, 3}, define hyperoperations * on H as shown in Table 4. We can verify that (H, *) is semihypergroup by software MATLAB.*

Table 4. The hyperoperation * on *H*.

*	1	2	3
1	2	2	{1, 3}
2	{1, 2, 3}	{2, 3}	{1, 2, 3}
3	2	{1, 2, 3}	{1, 3}

Moreover,

$$1*H = H*1 = H,\ 2*H = H*2 = H,\ 3*H = H*3 = H.$$

*This means that (H, *) is a hypergroup. But, for* $1 \in H$, *we cannot find* $x, y \in H$ *such that* $1 \in (x*1) \cap (1*x)$, *and* $x \in (y*1) \cap (1*y)$. *That is, (H, *) is not a NET- semihypergroup.*

Definition 8. *Let (H,*) be a semihypergroup. (H,*) is called a neutrosophic extended triplet hypergroup (shortened form, NET-hypergroup), if (H,*) is both a NET-semihypergroup and a hypergroup.*

Obviously, the NET-semihypergroups in Example 2 and Example 5 are all NET-hypergroups. And, the following propostion is true (the proof is omitted).

Proposition 1. *Every regular hypergroup is a NET-hypergroup.*

The NET-hypergroup in Example 2 is not a regular hypergroup, it shows that the inverse of Proposition 1 is not true.

Proposition 2. *Let (H,*) be a NET-semihypergroup (or a NET-hypergroup). Then (H,*) is a regular semihypergroup.*

Proof. Assume that $(H,*)$ is a NET-semihypergroup. For any $x \in H$, by Definition 7 we get that there exist $neut(x)$ and $anti(x)$ such that

$$x \in (neut(x)*x) \cap (x*neut(x)),\ \text{and}\ neut(x) \in (anti(x)*x) \cap (x*anti(x)).$$

Then,

$$x \in neut(x)*x \subseteq (x*anti(x))*x.$$

That is, $x \in x*anti(x)*x$. From this, by Definition 4, we know that $(H,*)$ is a regular semihypergroup.

If $(H,*)$ is a NET-hypergroup, by Definition 8, it follows that $(H,*)$ is a NET-semihypergroup. Then, by the proof above, $(H,*)$ is a regular semihypergroup. □

The following example shows that the inverse of Proposition 2 is not true. Moreover, it also shows that a regular semihypergroup may be not a hypergroup.

Example 7. *Denote H = {a, b, c}, define hyperoperations * on H as shown in Table 5. We can verify that (H, *) is semihypergroup.*

Table 5. The hyperoperation * on *H*.

*	*a*	*b*	*c*
a	*a*	*a*	*a*
b	{*a*, *b*, *c*}	{*a*, *b*, *c*}	{*a*, *b*, *c*}
c	{*a*, *b*, *c*}	{*a*, *b*}	{*a*, *b*}

Moreover, $a \in a*a*a$; $b \in b*b*b$; $c \in c*a*c$. *This means that (H, *) is a regular semihypergroup. But it is not a NET-semihypergroup, since there is not any* $x \in H$ *such that* $c \in x*c$ *and* $c \in c*x$. *Obviously, (H, *) is not a hypergroup.*

Therefore, the relationships among semihypergroup, NET-semihypergroup, NET-hypergroup, (regular) hypergroup and regular semihypergroup can be expressed by Figure 4.

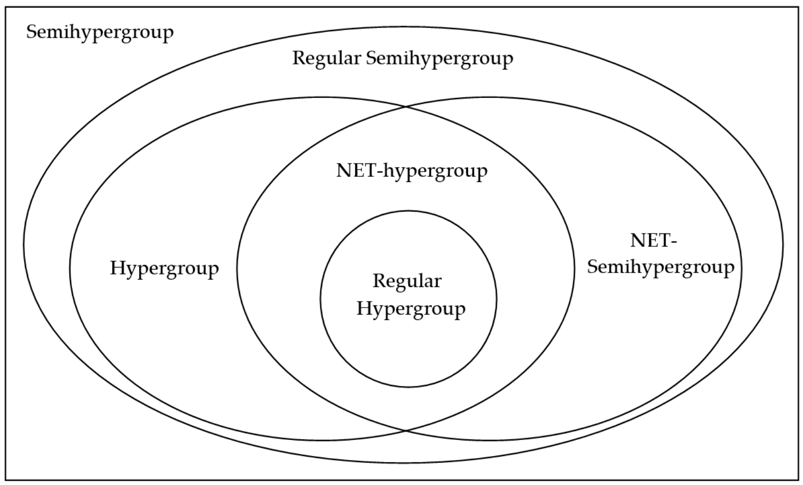

Figure 4. The relationships among some kinds of semihypergroups.

For basic properties of NET-semihypergroups and NET-hypergroups, we can get following results.

Theorem 1. *Let (H,*) be a semihypergroup. Then*

(1) if (H,) is commutative NET-semihypergroup, then for any* $x \in H$ *and hyper-neutrosophic-triplet (x, neut(x), anti(x)), there exists* $p \in neut(x)*neut(x)$ *and* $q \in anti(x)*neut(x)$ *such that (x, p, q) is also a hyper-neutrosophic-triplet.*

(2) if (H,) is commutative NET-semihypergroup, then for any* $x \in H$ *and* $neut(x) \in \{neut(x)\}$*, there exists* $p \in neut(x)*neut(x)$ *such that* $p \in \{neut(x)\}$.

(3) if (H,) is NET-semihypergroup and* $x \in H$ *is scalar, then* $|\{neut(x)\}|=1$*, that is, the neutral element of x is unique; Moreover, if x is scalar, then* $neut(x)*neut(x)=neut(x)$.

(4) if (H,) is commutative hypergroup, then (H,*) is NET-hypergroup.*

Proof. (1) Assume that $x \in H$ and $(x, neut(x), anti(x))$ is a hyper-neutrosophic-triplet. By Definition 7:

$$x \in (neut(x)*x) \cap (x*neut(x)), \text{ and } neut(x) \in (anti(x)*x) \cap (x*anti(x)).$$

Since $(H, *)$ is commutative, then:

$$x \in neut(x)*x \subseteq neut(x)*(neut(x)*x) = (neut(x)*neut(x))*x = x*(neut(x)*neut(x)).$$

This means that there exists $p \in neut(x)*neut(x)$ such that $x \in p*x = x*p$. Moreover:

$$p \in neut(x)*neut(x) \subseteq (x*anti(x))*neut(x) = x*(anti(x)*neut(x)) = (anti(x)*neut(x))*x.$$

It follows that there exists $q \in anti(x)*neut(x)$ such that $p \in q*x = x*q$. By Definition 7 we know that (x, p, q) is also a hyper-neutrosophic-triplet.

(2) It follows from (1).

(3) Suppose that $x\in H$ and x is scalar. Using Definition 2, $|x*a|=|a*x|=1$ for any $a\in H$. From this, for a hyper-neutrosophic-triplet (x, $neut(x)$, $anti(x)$), applying Definition 7, we have:

$$x = neut(x)*x = x*neut(x), \text{ and } neut(x) = anti(x)*x = x*anti(x).$$

Assume $p_1, p_2\in \{neut(x)\}$, then there exists $q_1, q_2\in H$ such that:

$$x = p_1*x = x*p_1,\ p_1 = q_1*x = x*q_1;\ x = p_2*x = x*p_2,\ p_2 = q_2*x = x*q_2.$$

Then:

$$p_1 = q_1*x = q_1*(x*p_2) = (q_1*x)*p_2 = p_1*p_2;$$

$$p_2 = x*q_2 = (x*p_1)*q_2 = (x*(q_1*x))*q_2 = (x*q_1)*(x*q_2) = p_1*p_2.$$

It follows that $p_1 = p_2$ and $p_1 = p_1*p_1$. That is, $|\{neut(x)\}|=1$ and $neut(x)*neut(x) = neut(x)$.

(4) Let (H, $*$) be a commutative hypergroup. By Definition 3, for any $x\in H$, $x*H = H*x = H$. Then, for any $x\in H$, there exists $h\in H$ such that $x = h*x = x*h$. Moreover, for $h\in H$, there exists $u\in H$ such that $h = u*x = x*u$. Thus, (x, h, u) is a hyper-neutrosophic-triplet, and it means that (H, $*$) is a NET-semihypergroup by Definition 7. On the other hand, since (H, $*$) is a hypergroup, so (H, $*$) is a NET-hypergroup by Definition 8.

3. Pure NET-semihypergroups and Regular hypergroups

In this section, we discuss some properties of NET-semihypergroups. We'll propose the new notion of pure NET-semihypergroup, investigate the structure of pure NET-semihypergroups.

Definition 9. *Let $(H,*)$ be a NET-semihypergroup. $(H,*)$ is called a pure NET-semihypergroup, if for every $x\in H$, there exist $neut(x)$ and $anti(x)$ such that*

$$x = (neut(x)*x)\cap(x*neut(x)), \text{ and } neut(x) = (anti(x)*x)\cap(x*anti(x)).$$

Obviously, the following proposition is true and the proof is omitted.

Proposition 3. *(1) Every neutrosophic extended triplet group (NETG) is pure NET-semihypergroup. (2) If $(H,*)$ is a pure NET-semihypergroup and the hyper operation $*$ is commutative, then for every $x\in H$, there exists $y, z\in H$ such that*

$$x = y*x = x*y, \text{ and } y = z*x = x*z.$$

Example 8. *Denote $H = \{a, b, c\}$, define hyperoperations $*$ on H as shown in Table 6. We can verify that (H, $*$) is semihypergroup.*

Table 6. The hyperoperation * on *H*.

*	a	b	c
a	a	{a, b, c}	{a, b, c}
b	{a, b, c}	c	b
c	{a, b, c}	b	c

Moreover,

$$a=(a*a)\cap(a*a);\ b=(c*b)\cap(b*c),\ c=(b*b)\cap(b*b);\ c=(c*c)\cap(c*c).$$

This means that (H, $*$) is a pure NET-semihypergroup.

Example 9. *Denote $H = \{a, b, c, d, e\}$, define hyperoperations $*$ on H as shown in Table 7. We can verify that (H, $*$) is semihypergroup.*

Table 7. The hyperoperation * on *H*.

*	a	b	c	d	e
a	a	{a, b, c}	{a, b, c}	d	a

b	{a, b, c}	b	c	d	b
c	{a, b, c}	c	b	d	c
d	d	d	d	d	d
e	a	b	c	d	e

Moreover:

$$a=(a*a)\cap(a*a);\ b=(b*b)\cap(b*b);\ c=(b*c)\cap(c*b),\ b=(c*c)\cap(c*c);\ d=(d*d)\cap(d*d);\ e=(e*e)\cap(e*e).$$

*This means that (H, *) is a pure NET-semihypergroup.*

Remark 4. From Example 8 and Example 9, we have:

$$a=(a*a)\cap(a*a);$$

$$a\in(b*a)\cap(a*b),\ b\in(b*a)\cap(a*b);\ a\in(c*a)\cap(a*c),\ c\in(c*a)\cap(a*c).$$

This means that $\{neut(a)\} = \{a, b, c\}$. But, $b\in\{neut(a)\}$ and $c\in\{neut(a)\}$ are different to $a\in\{neut(a)\}$, since one is "$\in$" and the other is "=". In order to clearly express the difference between the two kinds of neutral elements, we introduce a new concept: pure neutral element.

Definition 10. *Let (H,*) be a NET-semihypergroup and $x\in H$. An element $y\in H$ is called a pure neutral element of the element x, if there exist $z\in H$ such that:*

$$x = y*x = x*y,\text{ and } y = z*x = x*z.$$

Here, we denote y by pneut(x).

Proposition 4. *Let (H,*) be a NET-semihypergroup and $x\in H$. If there exists a pure neutral element of x, then the pure neutral element of x, that is, pneut(x), is unique.*

Proof. Assume that there exists two pure neutral elements y_1, y_2 for $x\in H$. Then there exists $z_1, z_2 \in H$ such that:

$$x = y_1*x = x*y_1,\text{ and } y_1 = z_1*x = x*z_1;$$

$$x = y_2*x = x*y_2,\text{ and } y_2 = z_2*x = x*z_2.$$

Therefore,

$$y_1 = z_1*x = z_1*(x*y_2) = (z_1*x)*y_2 = y_1*y_2;$$

$$y_2 = x*z_2 = (x*y_1)*z_2 = (x*(z_1*x))*z_2 = (x*z_1)*(x*z_2) = y_1*y_2.$$

Hence, $y_1 = y_2$. That is, $pneut(x)$ is unique. □

By the proof of Proposition 4, we know that $y_1 = y_2 = y_1*y_2$, it follows that $y_1 = y_1*y_1$. Therefore, we have the following corollary.

Corollary 1. *Let (H,*) be a NET-semihypergroup and $x\in H$. If there exists a pure neutral element of x, then the pure neutral element of x is idempotent, that is, pneut(x)*pneut(x)= pneut(x).*

Remark 5. From Proposition 4, we know that the pure neutral element of an elemetn x is unique when there exists one pure neutral element of x. Particularly, for commutative pure NET-semihypergroups, applying Proposition 3 (2), we get following Proposition 5 (the proof is omitted).

Proposition 5. *Let (H,*) be a commutative pure NET-semihypergroup. Then for any $x\in H$, pneut(x) is unique.*

Proposition 6. *Let (H,*) be a commutative pure NET-semihypergroup. Then for any $x, y\in H$, $pneut(x*y)= pneut(x)*pneut(y)$ when $|x*y|=1$. Moreover, if $pneut(x) = z_1*x = x*z_1$ and $pneut(y) = z_2*y = y*z_2$, $z_1, z_2\in H$, then:*

$$pneut(x*y) = (z_1*z_2)*(x*y) = (x*y)*(z_1*z_2).$$

Proof. Assume that $x, y\in H$ and $|x*y|=1$. Since $(H, *)$ be a commutative pure NET-semihypergroup, then:

$(x^*y)^*(pneut(x)^*pneut(y)) = (x^*y)^*(pneut(y)^*pneut(x))$

$= x^*(y^*pneut(y))^*pneut(x)$

$= x^*y^*pneut(x)$

$= (x^*pneut(x))^*y$

$= x^*y;$

$(pneut(x)^*pneut(y))^*(x^*y) = (pneut(y)^*pneut(x))^*(x^*y)$

$= pneut(y)^*(pneut(x)^*x)^* y$

$= pneut(y)^*x^*y$

$= x^*(pneut(y)^*y)$

$= x^*y.$

On the other hand, assume that (x, $pneut(x)$, $anti(x)$) and (y, $pneut(y)$, $anti(y)$) are hyper-neutrosophic-triplets, then:

$(x^*y)^*(anti(x)^*anti(y)) = (x^*y)^*(anti(y)^*anti(x))$

$= x^*(y^*anti(y))^*anti(x)$

$= x^*pneut(y)^*anti(x)$

$= (x^*anti(x))^*pneut(y)$

$= pneut(x)^*pneut(y);$

$(anti(x)^*anti(y))^*(x^*y) = (anti(x)^*anti(y))^*(y^*x)$

$= anti(x)^*(anti(y)^*y)^*x$

$= anti(x)^*pneut(y)^*x$

$= (anti(x)^*x)^*pneut(y)$

$= pneut(x)^*pneut(y).$

Applying Proposition 5 we get that $pneut(x^*y)= pneut(x)^*pneut(y)$.

Moroeover, assume $pneut(x) = z_1^*x = x^*z_1$, $pneut(y) = z_2^*y = y^*z_2$. Then, by commutativity of the hyper operation *:

$(z_1^*z_2)^*(x^*y) = (z_1^*x)^*(z_2^*y)$

$= pneut(x)^*pneut(y)$

$= pneut(x^*y);$

$(x^*y)^*(z_1^*z_2) =(x^*z_1)^*(y^*z_2)$

$= pneut(x)^*pneut(y)$

$= pneut(x^*y).$

Therefore, the proof is completed.

Theorem 2. *Let (H,*) be a commutative pure NET-semihypergroup and H satisfies:*

$$\forall x,\, y\in H,\ pneut(x)=pneut(y)\Rightarrow |\, x^*y\,|=1. \qquad \text{(C1)}$$

Define a binary relation $\approx$ *on H as following:*

$$\forall x,\, y\in H,\ x\approx y \text{ if and only if } pneut(x)=pneut(y).$$

Then:

(1) The binary relation is a equivalent relation on H;
(2) For any $x\in H$, $[x]\approx$ is a sub-NET-semihypergroup of H, where $[x]\approx$ is the equivalent class of x based on equivalent relation $\approx$;
(3) For any $x\in H$, $[x]\approx$ is a regular hypergroupe.

Proof. (1) It is obviously.

(2) Assume a, $b\in[x]_\approx$, then $pneut(a)= pneut(b)=pneut(x)$. Applying Proposition 6 and Corollary 1, we have

$$pneut(a*b) = pneut(a)*pneut(b) = pneut(x)*pneut(x) = pneut(x).$$

This means that $[x]\approx$ is closed on the hyper operation *.

Moreover, by Corollary 1, we have $pneut(x)*pneut(x) = pneut(x)$. From this and using Proposition 5, we get that $pneut(pneut(x)) = pneut(x)$. It follows that $pneut(a) \in [x]\approx$ for any $a\in[x]\approx$. Moreover, assume that $a\in[x]\approx$, by the definition of commutative pure NET-semihypergroup, there exists $r\in H$ such that:

$$pneut(a) = r*a = a*r.$$

It follows that:

$$pneut(a) = (r*pneut(a))*a = a*(r*pneut(a)). \quad \text{(C2)}$$

Applying Proposition 6 and Corollary 1:

$$pneut(r*pneut(a)) = pneut(r)* pneut(pneut(a)) = pneut(r)* pneut(a) = pneut(r*a) = pneut(pneut(a)) = pneut(a).$$

That is, $pneut(r*pneut(a)) = pneut(a) = pneut(x)$. This means that $r*pneut(a) \in [x]\approx$. Therefore, by (C2), there exists $anti(a)$ (see Definition 7), it is in $[x]\approx$. This means that $[x]\approx$ is a sub-NET-semihypergroup of *H.*

(3) For any $x\in H$, from (2) we know that $[x]\approx$ is a sub-NET-semihypergroup of *H.* By the definition of $\approx$, for any $a\in[x]\approx$, $pneut(a) = pneut(x)$. Then, $a*[x]\approx*a =[x]\approx$, and $pneut(x)$ is a (local) identity in $[x]\approx$. By Definition 3, we get that $[x]\approx$ is a regular hypergroup. □

From Theorem 2 we know that for a commutative pure NET-semihypergroup (it satisfies the condition in Theorem 2), it is a union of some regular hypergroups. The following picture (Figure 5) shows this special structure.

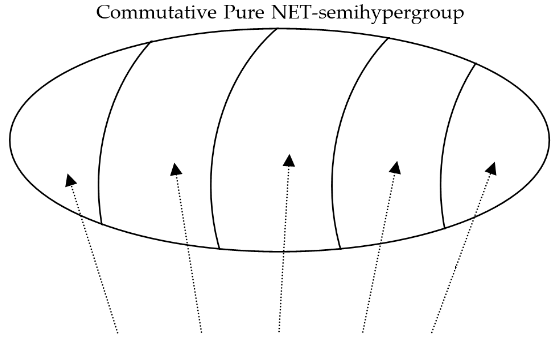

Figure 5. The structure of a commutative pure NET-semihypergroups.

Example 10. *Denote H = {a, b, c, d, e}, define hyperoperations * on H as shown in Table 8. We can verify that (H, *) is commutative pure NET-semihypergroup.*

Table 8. The hyperoperation * on *H.*

*	a	b	c	d	e
a	a	{a, b, c}	{a, b, c}	d	{a, d, e}
b	{a, b, c}	b	c	d	{b, c, d, e}
c	{a, b, c}	c	b	d	{b, c, d, e}
d	d	d	d	d	d
e	{a, d, e}	{b, c, d, e}	{b, c, d, e}	d	e

Moreover:

$$H_1=\{a\}=[a]_{\approx};$$
$$H_2=\{b, c\}=[b]_{\approx}=[c]_{\approx};$$
$$H_3=\{d\}=[d]_{\approx};$$
$$H_4=\{e\}=[e]_{\approx};$$

and $H= H_1\cup H_2\cup H_3\cup H_4$, *where,* H_i *(i=1, 2, 3, 4) are regular hypergroups.*

Remark 6. The above example shows that a commutative pure NET-semihypergroup may be not a hypergroup (since $d*H\neq H$ in Example 10).

4. Weak Commutative NET-Semihypergroups and Their Structures

In this section, we discuss generalized commutativity in NET-semihypergroups. We propose a new notion of weak commutative NET-semihypergroup, and prove the structure theorem of weak commutative pure NET-semihypergroup (WCP-NET-semihypergroup), which can be regarded as a generalization of Cliffod Theorem in semigroup theory.

Definition 11. *Let (H,*) be a NET-semihypergroup. (H,*) is called a weak commutative NET-semihypergroup, if for every x∈H, every hyper-neutrosophic-triplet (x, neut(x), anti(x)), the following conditions are satisfied:*

(H,) is called a weak commutative pure NET-semihypergroup (shortly, WCP-NET-semihypergroup), if it both weak commutative and pure.*

Obviously, the following proposition is true and the proof is omitted.

Proposition 7. *Every commutative NET-semihypergroup is weak commutative.*

The following examples show that there exists some weak commutative NET-semihypergroups which are not commutative.

Example 11. *Denote H = {1, 2, 3, 4, 5, 6, 7, 8}, define hyperoperations * on H as shown in Table 9. We can verify that (H, *) is NET-semihypergroup.*

Table 9. The hyperoperation * on *H.*

*	1	2	3	4	5	6	7	8
1	1	{1, 2}	1	1	1	1	1	1
2	{1, 2}	2	1	1	1	1	1	1
3	1	1	3	4	5	6	7	8
4	1	1	4	3	8	7	6	5
5	1	1	5	7	3	8	4	6
6	1	1	6	8	7	3	5	4
7	1	1	7	5	6	4	8	3
8	1	1	8	6	4	5	3	7

Moreover, (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 3, 4), (5, 3, 5), (6, 3, 6), (7, 3, 8) and (8, 3, 7) are hyper-neutrosophic-triplets, and ($\forall x\in H$*) 1*x = x*1, 2*x = x*2 and 3*x = x*3, 7*8 = 8*7. This means that (H, *) is a weak commutative NET-semihypergroup. Since 4*5 ≠ 5*4, (H, *) is not commutative.*

Remark 7. The above example shows that there exists WCP-NET-semihypergroup (by Definition 9, we know that the NET-semihypergroup in Example 11 is pure).

Example 12. *Denote H = {1, 2, 3, 4, 5, 6, 7, 8, 9}, define hyperoperations * on H as shown in Table 10. We can verify that (H, *) is NET-semihypergroup.*

Table 10. The hyperoperation * on *H*.

*	1	2	3	4	5	6	7	8	9
1	2	{1, 3}	3	1	1	1	1	1	1
2	{1, 3}	2	{1, 3}	{1, 2, 3}	{1, 2, 3}	{1, 2, 3}	{1, 2, 3}	{1, 2, 3}	{1, 2, 3}
3	3	{1, 3}	1	{1, 3}	{1, 3}	{1, 3}	{1, 3}	{1, 3}	{1, 3}
4	1	{1, 2, 3}	{1, 3}	4	5	6	7	8	9
5	1	{1, 2, 3}	{1, 3}	5	4	9	8	7	6
6	1	{1, 2, 3}	{1, 3}	6	8	4	9	5	7
7	1	{1, 2, 3}	{1, 3}	7	9	8	4	6	5
8	1	{1, 2, 3}	{1, 3}	8	6	7	5	9	4
9	1	{1, 2, 3}	{1, 3}	9	7	5	6	4	8

Moreover, (1, 2, 1), (2, 2, 2), (3, 1, 3), (4, 4, 4), (5, 4, 5), (6, 4, 6), (7, 4, 7), (8, 4, 9) and (9, 4, 8) are hyper-neutrosophic-triplets, and ($\forall x \in H$*) 2*x = x*2, 1*x = x*1 and 4*x = x*4, 8*9 = 9*8. This means that (H, *) is a weak commutative NET-semihypergroup. Since 5*6 ≠ 6*5, (H, *) is not commutative.*

Proposition 8. *Let (H,*) be a weak commutative pure NET-semihypergroup (WCP-NET-semihypergroup). Then for any x∈H, there exists a pure neutral element of x, and pneut(x) is unique, pneut(x)*pneut(x)= pneut(x).*

Proof. For any $x \in H$. Since (*H*, *) is pure, by Definition 9, there exists hyper-neutrosophic-triplet (*x*, *neut*(*x*), *anti*(*x*)) such that

$$x = (neut(x)*x) \cap (x*neut(x)), \text{ and } neut(x) = (anti(x)*x) \cap (x*anti(x)).$$

Moreover, since (*H*, *) is weak commutative, by Definition 11, $neut(x)*x = x*neut(x)$, and $anti(x)*x = x*anti(x)$. Thus

$$x = neut(x)*x = x*neut(x), \text{ and } neut(x) = anti(x)*x = x*anti(x).$$

Therefore, by Definition 10, *neut*(*x*) is a pure neutral element of *x*. Applying Proposition 4 we know that pure neutral element of *x* is unique. Moreover, using Corollary 1, $pneut(x)*pneut(x)= pneut(x)$.□

Proposition 9. *Let (H,*) be a weak commutative pure NET-semihypergroup (WCP-NET-semihypergroup). Then for any x, y∈H, pneut(x*y)= pneut(x)*pneut(y) when* $|x*y|=1$*. Moreover, if* $pneut(x) = z_1*x = x*z_1$ *and* $pneut(y) = z_2*y = y*z_2$*,* $z_1, z_2 \in H$*, then*

$$pneut(x*y) = (z_2*z_1)*(x*y) = (x*y)*(z_2*z_1).$$

Proof. Since (*H*, *) be a WCP-NET-semihypergroup, then for any *x*, $y \in H$ and $|x*y|=1$, $pneut(x)*y = y*pneut(x)$ by Definition 11. Then

$$x*y)*(pneut(x)*pneut(y)) = (x*y)*(pneut(y)*pneut(x)) = x*y*pneut(x) = (x*pneut(x))*y = x*y;$$

$$(pneut(x)*pneut(y))*(x*y) = (pneut(y)*pneut(x))*(x*y) = pneut(y)*x*y = x*(pneut(y)*y) = x*y.$$

On the other hand, let (*x*, *pneut*(*x*), *anti*(*x*)) and (*y*, *pneut*(*y*), *anti*(*y*)) are hyper-neutrosophic-triplets, then

$$xx*y)*(anti(y)*anti(x))$$
$$= x*(y*anti(y))*anti(x)$$
$$= x*pneut(y)*anti(x)$$

$$= pneut(y)*x*anti(x)$$

$$= pneut(y)*pneut(x)$$

$$= pneut(x)*pneut(y);$$

$$(anti(y)*anti(x))*(x*y) = anti(y)*(anti(x)*x)*y = anti(y)*pneut(x)*y = pneut(x)*anti(y)*y = pneut(x)*pneut(y).$$

Thus, $pneut(x)*pneut(y)$ is a pure neutral element of $x*y$ by Definition 7 and Definition 10. Applying Proposition 8 we get that $pneut(x*y)= pneut(x)*pneut(y)$.

Moroeover, assume $pneut(x) = z_1*x = x*z_1$, $pneut(y) = z_2*y = y*z_2$. Then, by weak commutativity (Definition 11) we have

$$(z_2*z_1)*(x*y) = z_2*(z_1*x)*y = z_2*pneut(x)*y = pneut(x)*(z_2*y) = pneut(x)*pneut(y) = pneut(x*y);$$

$$(x*y)*(z_2*z_1) = x*(y*z_2)*z_1 = x*pneut(y)*z_1 = (x*z_1)*pneut(y) = pneut(x)*pneut(y) = pneut(x*y).$$

Therefore, the proof is completed. □

Theorem 3. *Let (H,*) be a WCP-NET-semihypergroup and H satisfies*

$$(\forall x, y\in H, pneut(x)=pneut(y) \Rightarrow |x*y|=1. \qquad \text{(C1)}$$

Define a binary relation ≈ on H as following:

$$\forall x, y\in H,\ x\approx y \text{ if and only if } pneut(x)=pneut(y).$$

Then

(1) The binary relation ≈ is a equivalent relation on H;

(2) For any $x\in H$, $[x]_\approx$ is a sub-NET-semihypergroup of H, where $[x]_\approx$ is the equivalent class of x based on equivalent relation ≈;

(3) For any $x\in H$, $[x]_\approx$ is a regular hypergroupe.

Proof. (1) From the definition of ≈, by Proposition 8 and Proposition 9, we know that the binary relation ≈ is a equivalent relation.

(2) Suppose a, $b\in[x]_\approx$. By the definition of ≈, $pneut(a) = pneut(b) = pneut(x)$. Using Proposition 8 and Proposition 9, we have

$$pneut(a*b) = pneut(a)*pneut(b) = pneut(x)*pneut(x) = pneut(x).$$

It follows that $[x]_\approx$ is closed on the hyper operation *.

And, applying Proposition 8, we have $pneut(x)*pneut(x) = pneut(x)$. From this and using Proposition 8, we get that $pneut(pneut(x)) = pneut(x)$. It follows that $pneut(a)\in[x]_\approx$ for any $a\in[x]_\approx$. Moreover, assume that $a\in[x]_\approx$, by the definition of WCP-NET-semihypergroup, there exists $r\in H$ such that $pneut(a) = r*a = a*r$. Thus (by Proposition 9)

$$pneut(a) = (r*pneut(a))*a = a*(r*pneut(a))$$
$$\Rightarrow r*pneut(a) \in \{anti(a)\}.$$

$$pneut(r*pneut(a))$$

$$= pneut(r)* pneut(pneut(a))$$

$$= pneut(r)* pneut(a)$$

$$= pneut(r*a)$$

$$= pneut(pneut(a))$$

$$= pneut(a).$$

That is, $pneut(r*pneut(a)) = pneut(a) = pneut(x)$. This means that $r*pneut(a) \in [x]_{\approx}$. Combining this and $r*pneut(a) \in \{anti(a)\}$, we know that there exists $anti(a)$ which is in $[x]_{\approx}$. This means that $[x]_{\approx}$ is a sub-NET- semihypergroup of *H.*

(3) Assume $x \in H$, from (2) we know that $[x]_{\approx}$ is a sub-NET-semihypergroup of *H.* By the definition of ≈, for any $a \in [x]_{\approx}$, $pneut(a) = pneut(x)$. From the proof of (2), there exists $anti(a) \in \{anti(a)\}$ and $anti(a) \in [x]_{\approx}$. Then, $[x]_{\approx} \subseteq a*[x]_{\approx}*a$. Obviously, $a*[x]_{\approx}*a \subseteq [x]_{\approx}$. Thus, $a*[x]_{\approx}*a=[x]_{\approx}$.

On the other hand, $pneut(x)$ is a (local) identity in $[x]_{\approx}$. Therefore, by Definition 3, we get that $[x]_{\approx}$ is a regular hypergroup. □

Example 13. *Denote H = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, define hyperoperations * on H as shown in Table 11. We can verify that (H, *) is WCP-NET-semihypergroup, and not commutative.*

Table 11. The hyperoperation * on *H.*

*	1	2	3	4	5	6	7	8	9	10	11
1	1	{1,2,3}	{1,2,3}	1	1	1	1	1	1	1	1
2	{1,2,3}	3	2	3	2	{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}
3	{1,2,3}	2	3	2	3	{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}
4	1	3	2	5	4	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}
5	1	2	3	4	5	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}
6	1	{1,2,3}	{1,2,3}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	6	7	8	9	10	11
7	1	{1,2,3}	{1,2,3}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	7	6	11	10	9	8
8	1	{1,2,3}	{1,2,3}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	8	10	6	11	7	9
9	1	{1,2,3}	{1,2,3}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	9	11	10	6	8	7
10	1	{1,2,3}	{1,2,3}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	10	8	9	7	11	6
11	1	{1,2,3}	{1,2,3}	{6,7,8, 9,10,11}	{6,7,8, 9,10,11}	11	9	7	8	6	10

Moreover,

$$H_1 = \{1\} = [1]_{\approx};$$
$$H_2 = \{2, 3\} = [2]_{\approx} = [3]_{\approx};$$
$$H_3 = \{4, 5\} = [4]_{\approx} = [5]_{\approx};$$
$$H_4 = \{6, 7, 8, 9, 10, 11\} = [6]_{\approx} = [7]_{\approx} = [8]_{\approx} = [9]_{\approx} = [10]_{\approx} = [11]_{\approx};$$

and $H = H_1 \cup H_2 \cup H_3 \cup H_4$, *where,* H_i *(i=1, 2, 3, 4) are regular hypergroups.*

5. Conclusions

In this paper, we propose some new notions of neutrosophic extended triplet semihypergroup (NET-semihypergroup), neutrosophic extended triplet hypergroup (NET-hypergroup), pure NET-semihypergroup and weak commutative NET-semihypergroup, investigate some basic properties and the relationships among them (see Figure 6), study their close connections with regular hypergroups and regular semihypergroups. Particularly, we prove two structure theorems of commutative pure NET-semihypergroup (CP-NET-semihypergroup) and weak commutative pure NET-semihypergroup (WCP-NET-semihypergroup) under the condition (C1) (see Theorem 2 and Theorem 3). From these results, we know that NET-semihypergroup is a hyperalgebraic structure independent of hypergroup, and NET-semihypergroup is also a generalization of group concept in hyperstructures. The research results in this paper show that NET-semihypergroups and NET-hypergroups have important theoretical research value, which greatly enriches the traditional theory of hyperalgebraic structures.

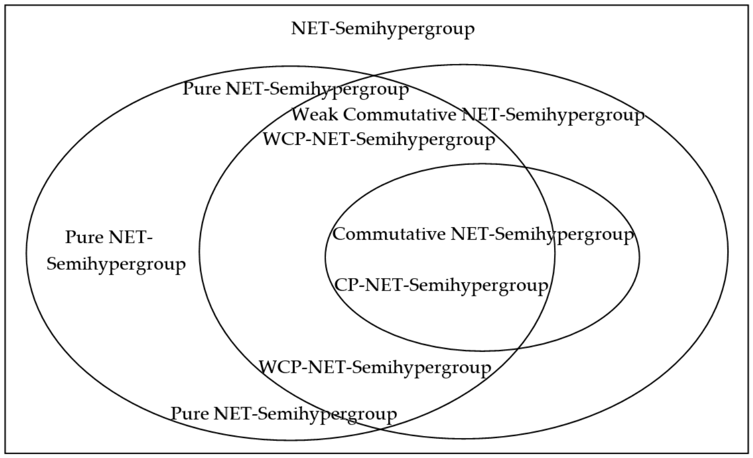

Figure 6. The relationships among some kinds of NET-semihypergroups.

In the future, we will investigate the combinations of NET-semihypergroups and related algebraic systems ([22–24]).

Funding: This research was funded by National Natural Science Foundation of China (Grant Nos. 61573240 and 61976130).

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The Structure of Idempotents in Neutrosophic Rings and Neutrosophic Quadruple Rings

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Yingcang Ma, Xiaohong Zhang, Florentin Smarandache, Juanjuan Zhang (2019). The Structure of Idempotents in Neutrosophic Rings and Neutrosophic Quadruple Rings. *Symmetry*, 11, 1254; DOI: 10.3390/sym11101254

Abstract: This paper aims to reveal the structure of idempotents in neutrosophic rings and neutrosophic quadruple rings. First, all idempotents in neutrosophic rings $\langle R \cup I \rangle$ are given when R is $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ or $\mathbb{Z}_n$. Secondly, the neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is introduced and all idempotents in neutrosophic quadruple rings $\langle \mathbb{C} \cup T \cup I \cup F \rangle, \langle \mathbb{R} \cup T \cup I \cup F \rangle$, $\langle \mathbb{Q} \cup T \cup I \cup F \rangle, \langle \mathbb{Z} \cup T \cup I \cup F \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ are also given. Furthermore, the algorithms for solving the idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ for each nonnegative integer n are provided. Lastly, as a general result, if all idempotents in any ring R are known, then the structure of idempotents in neutrosophic ring $\langle R \cup I \rangle$ and neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ can be determined.

Keywords: neutrosophic rings; neutrosophic quadruple rings; idempotents; neutrosophic extended triplet group; neutrosophic set

1. Introduction

The notions of neutrosophic set and neutrosophic logic were proposed by Smarandache [1]. In neutrosophic logic, every proposition is considered by the truth degree T, the indeterminacy degree I, and the falsity degree F, where T, I and F are subsets of the nonstandard unit interval $]0^-, 1^+[= 0^- \cup [0, 1] \cup 1^+$.

Using the idea of neutrosophic set, some related algebraic structures have been studied in recent years. Among these algebraic structures, by extending classical groups, the neutrosophic triplet group (NTG) and the neutrosophic extended triplet group (NETG) have been introduced in refs. [2–4]. As an example, paper [5] shows that $(\mathbb{Z}_{p_1 p_2 \cdots p_t}, \cdot)$ is not only a semigroup, but also a NETG, where $\cdot$ the classical mod multiplication and $p_1, p_2, \cdots, p_t$ are distinct primes. After the notions were put forward, NTG and NETG have been carried out in-depth research. For example, the inclusion relations of neutrosophic sets [6], neutrosophic triplet coset [7], neutrosophic duplet semi-groups [8], AG-neutrosophic extended triplet loops [9,10], the neutrosophic set theory to pseudo-BCI algebras [11], neutrosophic triplet ring and a neutrosophic triplet field [12,13], neutrosophic triplet normed space [14], neutrosophic soft sets [15], neutrosophic vector spaces [16], and so on.

In contrast to the neutrosophic triplet ring, the neutrosophic ring $\langle R \cup I \rangle$, which is a ring generated by the ring R and the indeterminate element I ($I^2 = I$), was proposed by Vasantha and Smarandache in [17]. The concept of neutrosophic ring was further developed and studied in [18–20].

As a special kind of element in an algebraic system, the idempotent element plays a major role in describing the structure and properties of the algebra. For example, Boolean rings refer to rings in which all elements are idempotent, clean rings [21] refer to rings in which each element is clean (an element in a ring is clean, if it can be written as the sum of an idempotent element and an invertible element), and Albel ring is a ring if each element in the ring is central. From these we can see that some rings can be characterized by idempotents. Thus, it is also quite meaningful to find all idempotents in a ring. In this paper, the idempotents in neutrosophic rings and neutrosophic quadruple rings will be studied in depth, and all idempotents in them can be obtained if the idempotents in R are known. In addition, the relationship between idempotents and neutral elements will be given. The elements of each NETG can be partitioned by neutrals [10]. Therefore, as an application, if $R = \mathbb{F}$, where $\mathbb{F}$ is any field, we can divide the elements of $\langle R \cup I \rangle$ (or $\langle R \cup T \cup I \cup F \rangle$) by idempotents. As another application, in paper [22], the authors explore the idempotents and semi-idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$ and some open problems and conjectures are given. In this paper, we will answer partial open problems and conjectures in paper [22] and some further studies are discussed.

The outline of this paper is organized as follows. Section 2 gives the basic concepts. In Section 3, the idempotents in neutrosophic ring $\langle R \cup I \rangle$ will be explored. For neutrosophic rings $\langle \mathbb{Z}_n \cup I \rangle$, $\langle \mathbb{C} \cup I \rangle$, $\langle \mathbb{R} \cup I \rangle$, $\langle \mathbb{Q} \cup I \rangle$ and $\langle \mathbb{Z} \cup I \rangle$, all idempotents will be given. Moreover, the open problem and conjectures proposed in paper [22] about idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$ will be solved. In Section 4, the neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is introduced and all idempotents in neutrosophic quadruple rings $\langle \mathbb{C} \cup T \cup I \cup F \rangle$, $\langle \mathbb{R} \cup T \cup I \cup F \rangle$, $\langle \mathbb{Q} \cup T \cup I \cup F \rangle$, $\langle \mathbb{Z} \cup T \cup I \cup F \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ will be given. Finally, the summary and future work is presented in Section 5.

2. Basic Concepts

In this section, the related basic definitions and properties of neutrosophic ring $\langle R \cup I \rangle$ and NETG are provided, the details can be seen in [3,4,17,18].

Definition 1. *([17,18]) Let $(R, +, \cdot)$ be any ring. The set*

$$\langle R \cup I \rangle = \{a + bI : a, b \in R\}$$

is called a neutrosophic ring generated by R and I. Let $a_1 + b_1 I, a_2 + b_2 I \in \langle R \cup I \rangle$, The operators $\oplus$ and $\otimes$ on $\langle R \cup I \rangle$ are defined as follows:

$$(a_1 + b_1 I) \oplus (a_2 + b_2 I) = (a_1 + a_2) + (b_1 + b_2)I,$$

$$(a_1 + b_1 I) \otimes (a_2 + b_2 I) = (a_1 \cdot a_2) + (a_1 \cdot b_2 + b_1 \cdot a_2 + b_1 \cdot b_2)I.$$

Remark 1. *It is easy to verify that $(\langle R \cup I \rangle, \oplus, \otimes)$ is a ring, so $\langle R \cup I \rangle$ is named by a neutrosophic ring is reasonable.*

Remark 2. *It should be noted that the operators $+, \cdot$ are defined on ring R and $\oplus, \otimes$ are defined on neutrosophic ring $\langle R \cup I \rangle$. For simplicity of notation, we also use $+, \cdot$ to replace $\oplus, \otimes$ on ring $\langle R \cup I \rangle$. That is $a + b$ also means $a \oplus b$ if $a, b \in \langle R \cup I \rangle$. $a \cdot b$ also means $a \otimes b$ if $a, b \in \langle R \cup I \rangle$. For short $a \cdot b$ denoted by ab and $a \cdot a$ denoted by a^2.*

Example 1. *$\langle \mathbb{Z} \cup I \rangle$, $\langle \mathbb{Q} \cup I \rangle$, $\langle \mathbb{R} \cup I \rangle$ and $\langle \mathbb{C} \cup I \rangle$ are neutrosophic rings of integer, rational, real and complex numbers, respectively. $\langle \mathbb{Z}_n \cup I \rangle$ is neutrosophic ring of modulo integers. Of course, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Z}_n$ are neutrosophic rings when $b = 0$.*

Definition 2. *([17,18]) Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is said to be commutative if*

$$ab = ba, \forall a, b \in \langle R \cup I \rangle.$$

In addition, if there exists $1 \in \langle R \cup I \rangle$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \langle R \cup I \rangle$ then we call $\langle R \cup I \rangle$ a commutative neutrosophic ring with unity.

Definition 3. *([17,18]) An element a in a neutrosophic ring $\langle R \cup I \rangle$ is called an idempotent element if $a^2 = a$.*

Definition 4. *([3,4]) Let N be a non-empty set together with a binary operation $*$. Then, N is called a neutrosophic extended triplet set if for any $a \in N$, there exists a neutral of "a" (denote by $neut(a)$), and an opposite of "a"(denote by $anti(a)$), such that $neut(a) \in N$, $anti(a) \in N$ and:*

$$a * neut(a) = neut(a) * a = a, \;\; a * anti(a) = anti(a) * a = neut(a).$$

The triplet $(a, neut(a), anti(a))$ is called a neutrosophic extended triplet.

Definition 5. *([3,4]) Let $(N, *)$ be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet group (NETG), if the following conditions are satisfied:*
(1) $(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$.
(2) $(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.
A NETG N is called a commutative NETG if for all $a, b \in N, a * b = b * a$.

Proposition 1. *([4]) $(N, *)$ be a NETG. We have:*
(1) $neut(a)$ is unique for any $a \in N$.
(2) $neut(a) * neut(a) = neut(a)$ for any $a \in N$.
(3) $neut(neut(a)) = neut(a)$ for any $a \in N$.

Proposition 2. *([10]) Let $(N, *)$ is a NETG, denote the set of all different neutral element in N by $E(N)$. For any $e \in E(N)$, denote $N(e) = \{x | neut(x) = e, x \in N\}$. Then:*
(1) $N(e)$ is a classical group, and the unit element is e.
(2) For any $e_1, e_2 \in E(N), e_1 \neq e_2 \Rightarrow N(e_1) \cap N(e_2) = \emptyset$.
(3) $N = \bigcup_{e \in E(N)} N(e)$. i.e., $\bigcup_{e \in E(N)} N(e)$ is a partition of N.

3. The Idempotents in Neutrosophic Rings

In this section, we will explore the idempotents in neutrosophic rings $\langle R \cup I \rangle$. If R is $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{Z}_n$, all idempotents in neutrosophic rings $\langle \mathbb{Z}_n \cup I \rangle, \langle \mathbb{C} \cup I \rangle, \langle \mathbb{R} \cup I \rangle, \langle \mathbb{Q} \cup I \rangle$ or $\langle \mathbb{Z} \cup I \rangle$ will be given. Moreover, we can also obtain all idempotents in neutrosophic ring $\langle R \cup I \rangle$ if all idempotents in any ring R are known. As an application, the open problem and conjectures about the idempotents of neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$ in paper [22] will be solved. Moreover, an example is given to show how to use the idempotents to get a partition for a neutrosophic ring. The following proposition reveal the relation of a neutral element and an idempotent element.

Proposition 3. *Let G be a non-empty set, $*$ is a binary operation on G. For each $a \in G$, a is idempotent iff it is a neutral element.*

Proof. Necessity: If a is idempotent, i.e., $a * a = a$, from Definition 4, which shows that a has neutral element a and opposite element a, so a is a neutral element.

Sufficiency: If a is a neutral element, from Proposition 1(2), we have $a * a = a$, thus a is idempotent. □

Theorem 1. *The set of all idempotents in neutrosophic ring $\langle \mathbb{C} \cup I \rangle, \langle \mathbb{R} \cup I \rangle, \langle \mathbb{Q} \cup I \rangle$ or $\langle \mathbb{Z} \cup I \rangle$ is $\{0, 1, I, 1 - I\}$.*

Proof. We just give the proof for $\langle \mathbb{R} \cup I \rangle$, and the same result can be obtained for $\langle \mathbb{C} \cup I \rangle, \langle \mathbb{Q} \cup I \rangle$ or $\langle \mathbb{Z} \cup I \rangle$.

Let $a + bI \in \langle \mathbb{R} \cup I \rangle$. If $a + bI$ is idempotent, so $(a + bI)^2 = a + bI$, which means

$$\begin{cases} a^2 = a \\ 2ab + b^2 = b \end{cases} \tag{1}$$

From $a^2 = a$, we can get $a = 0$ or $a = 1$. When $a = 0$, from $2ab + b^2 = b$, we can get $b = 0$ or $b = 1$. That is 0 and I are idempotents. When $a = 1$, from $2ab + b^2 = b$, we can get $b = 0$ or $b = -1$. That is 1 and $1 - I$ are idempotents. Thus, the set of all idempotents of neutrosophic ring $\langle \mathbb{R} \cup I \rangle$ is $\{0, 1, I, 1 - I\}$. $\square$

The above theorem reveals that the set of all idempotents in neutrosophic ring $\langle R \cup I \rangle$ is $\{0, 1, I, 1 - I\}$ when R is $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$. For any ring R, we have the following results.

Proposition 4. *If a is idempotent in any ring R, then aI is also idempotent in neutrosophic ring $\langle R \cup I \rangle$.*

Proof. If $a \in R$ is idempotent, i.e., $a^2 = a$, so $(aI)^2 = (0 + aI)(0 + aI) = a^2 I = aI$, thus, aI is also idempotent in neutrosophic ring $\langle R \cup I \rangle$. $\square$

Proposition 5. *In neutrosophic ring $\langle R \cup I \rangle$, then $a - aI$ is idempotent iff a is idempotent.*

Proof. Necessity: If $a - aI$ is idempotent, i.e., $(a - aI)^2 = a - aI$, so $(a - aI)^2 = (a - aI)(a - aI) = a^2 - 2aI + a^2 I = a^2 + (a^2 - 2a)I = a - aI$, which means $a^2 = a$ and $a^2 - 2a = -a$. Thus, we have $a^2 = a$, so a is idempotent.

Sufficiency: If a is idempotent, so $(a - aI)^2 = a^2 + (a^2 - 2a)I = a - aI$, thus $a - aI$ is idempotent. $\square$

Theorem 2. *In neutrosophic ring $\langle R \cup I \rangle$, let $a + bI \in \langle R \cup I \rangle$, then $a + bI$ is idempotent iff a is idempotent in R and $b = c - a$, where c is any idempotent element in R.*

Proof. Necessity: If $a + bI$ is idempotent, i.e., $(a + bI)^2 = a + bI$, so $(a + bI)^2 = a^2 + (2ab + b^2) = a + bI$, which means $a^2 = a$ and $2ab + b^2 = b$. From $a^2 = a$, we can get a is idempotent. From $2ab + b^2 = b$ and $a^2 = a$, we can get $(b + a)^2 = b^2 + 2ab + a^2 = b + a$, so $b + a$ is also idempotent in R, denoted by c, so $b = c - a$.

Sufficiency: If a and c are any idempotents in R, let $b = c - a$, so $(a + bI)^2 = (a + (c - a)I)^2 = a^2 + (2a(c - a) + (c - a)^2)I = a^2 + (2ac - 2a^2 + c^2 - 2ac + a^2) = a + (c - a)I = a + bI$, thus $a + bI$ is idempotent. $\square$

Theorem 3. *If the number of different idempotents in ring R is t, then the number of different idempotents in the neutrosophic ring $\langle R \cup I \rangle$ is t^2.*

Proof. If the number of idempotents in R is t and let $a + bI \in \langle R \cup I \rangle$ is idempotent, so from Theorem 2, we can infer that a is idempotent in R, i.e., a has t different selections. When a is fixed, set $b = c - a$, where c is any idempotent in R and c also has t different selections, which means b has t different selections. Thus, $a + bI$ has $t \cdot t = t^2$ different selections, i.e., the number of all idempotents in $\langle R \cup I \rangle$ is t^2. $\square$

From the above analysis, for any ring R, all idempotents in $\langle R \cup I \rangle$ can be determined if all idempotents in R are known. In the following, we will explore all idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$, i.e., when $R = \mathbb{Z}_n$.

Theorem 4. *([5]) In the algebra system $(\mathbb{Z}_n, \cdot)$ (see Appendix A), $\cdot$ is the classical mod multiplication, for each $a \in \mathbb{Z}_n$, a has $neut(a)$ and $anti(a)$ iff* $\gcd(\gcd(a, n), n/\gcd(a, n)) = 1$.

Theorem 5. *([5]) For an algebra system $(\mathbb{Z}_n, \cdot)$ and $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$, where each $p_i (i = 1, 2, \cdots, t)$ is a prime, then the number of different neutral elements in $\mathbb{Z}_n$ is 2^t.*

Remark 3. *From Proposition 3 and Theorem 5, we can infer that the number of all idempotents in $\mathbb{Z}_{p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}}$ is also 2^t.*

Example 2. *For $(\mathbb{Z}_{36}, \cdot)$, $n = 36 = 2^2 3^2$. From Theorem 5, the number of different neutral elements in $\mathbb{Z}_{36}$ is $2^2 = 4$. They are:*

(1) $[0]$ *has the neutral element* $[0]$.
(2) $[1], [5], [7], [11], [13], [17], [19], [23], [25], [29], [31]$ *and* $[35]$ *have the same neutral element* $[1]$.
(3) $[9]$ *and* $[27]$ *have the same neutral element* $[9]$ *being* $\gcd(9, 36) = \gcd(27, 36) = 9$.
(4) $[4]$ *and* $[8]$ *have the same neutral element being* $\gcd(4, 36) = \gcd(8, 36) = 4$. *In fact,* $[4], [8], [16], [20], [28]$ *and* $[32]$ *have the same neutral element, which is* $[28]$.

From Remark 3, the number of idempotents in $\mathbb{Z}_{36}$ is also 4, which are $[0], [1], [9]$ and $[28]$.

From Theorems 2 and 3 and Remark 3, it follows easily that:

Corollary 1. *In neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$, let $a + bI \in \langle \mathbb{Z}_n \cup I \rangle$, then $a + bI$ is idempotent iff $a^2 = a$ and $b = c - a$, where c is any idempotent element in $\mathbb{Z}_n$.*

Corollary 2. *For an algebra system $(\mathbb{Z}_n, \cdot)$ and $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$, where each $p_1, p_2, \cdots$, and p_k are distinct primes. Then the number of different idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ is 2^{2t}.*

The solving process for $\langle \mathbb{Z}_n \cup I \rangle$ is given by Algorithm 1. Just only input n, then we can get all idempotents in $\langle \mathbb{Z}_n \cup I \rangle$. The MATLAB code is provided in the Appendix B.

Example 3. *Solve all idempotents in $\langle \mathbb{Z}_{600} \cup I \rangle$.*

Since $n = 600 = 2^3 \cdot 3 \cdot 5^2$, from Theorem 5, we can get the different neutral elements in $\mathbb{Z}_{600}$ are $neut(1), neut(2^3), neut(3), neut(5^2), neut(2^3 \cdot 3), neut(2^3 \cdot 5^2), neut(3 \cdot 5^2)$ and $neut(0)$, i.e., the different idempotents in $\mathbb{Z}_{600}$ are 1, 376, 201, 25, 576, 400, 225, 0. From Corollary 2, the number of different idempotents in neutrosophic ring $\langle \mathbb{Z}_{600} \cup I \rangle$ is $2^{2 \cdot 3} = 64$.

From Algorithm 1, the set of all 64 idempotents in $\langle \mathbb{Z}_{600} \cup I \rangle$ is: $\{0, I, 25I, 201I, 225I, 376I, 400I, 576I, 1 + 599I, 1, 1 + 24I, 1 + 200I, 1 + 224I, 1 + 375I, 1 + 399I, 1 + 575I, 25 + 575I, 25 + 576I, 25, 25 + 176I, 25 + 200I, 25 + 351I, 25 + 375I, 25 + 551I, 201 + 399I, 201 + 400I, 201 + 424I, 201, 201 + 24I, 201 + 175I, 201 + 199I, 201 + 375I, 225 + 375I, 225 + 376I, 225 + 400I, 225 + 576I, 225, 225 + 151I, 225 + 175I, 225 + 351I, 376 + 224I, 376 + 225I, 376 + 249I, 376 + 425I, 376 + 449I, 376, 376 + 24I, 376 + 200I, 400 + 200I, 400 + 201I, 400 + 225I, 400 + 401I, 400 + 425I, 400 + 576I, 400, 400 + 176I, 576 + 24I, 576 + 25I, 576 + 49I, 576 + 225I, 576 + 249I, 576 + 400I, 576 + 424I, 576\}$.

Algorithm 1: Solving the different idempotents in $\langle \mathbb{Z}_n \cup I \rangle$

Input: n
1: Factorization of integer n, we can get $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$.
2: Computing the neutral element of $1, p_1^{k_1}, p_2^{k_2}, \cdots, p_t^{k_t}, p_1^{k_1} p_2^{k_2}, \cdots p_1^{k_1} p_t^{k_t}, \cdots, p_2^{k_2} p_3^{k_3} \cdots p_t^{k_t}$ and $p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$. So, we can get all idempotents in $\mathbb{Z}_n$, denoted by $a_1, a_2, \cdots, a_{2^t}$.
3: Let ID=[];
4: for $i = 1 : 2^t$
5: $\quad a = a_i$
6: $\quad$ for $j = 1 : 2^t$
7: $\quad\quad b = \mathrm{mod}(a_j - a, n)$;
8: $\quad\quad ID = [ID;\ [a, b]]$;
9: $\quad$ end
10: end
Output: ID: all the idempotents in $\langle \mathbb{Z}_n \cup I \rangle$

In paper [22], the authors studied the idempotents and semi-idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ and proposed some open problems and conjectures. We list partial open problems and conjectures about idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ as follows and answer them.

Problem 1. *([22]) Let $S = \langle \mathbb{Z}_{pq}, +, \cdot \rangle$, where p and q are two distinct primes, be the neutrosophic ring. Can S have non-trivial idempotents other than the ones mentioned in (b) of the Theorem 6?*

Conjecture 1. *([22]) Let $S = \langle \mathbb{Z}_n, +, \cdot \rangle$ be the neutrosophic ring $n = pqr$, where p, q and r are three distinct primes.*

1. *$\mathbb{Z}_n = \mathbb{Z}_{pqr}$ has only six non-trivial idempotents associated with it.*
2. *If m_1, m_2, m_3, m_4, m_5 and m_6 are the idempotents, then, associated with each real idempotent m_i, we have seven non-trivial neutrosophic idempotents associated with it, i.e., $\{m_i + n_j I, j = 1, 2, \cdots, 7\}$, such that $m_i + n_j \equiv t$, where t_j takes the seven distinct values from the set $\{0, 1, m_k, k \neq i; k = 1, 2, 3, \cdots, 6\}.i = 1, 2, \cdots, 6$.*

Conjecture 2. *([22]) Given $\langle \mathbb{Z}_n \cup I \rangle$, where $n = p_1 p_2 \cdot p_t; t > 2$ and p_is are all distinct primes, find:*

1. *the number of idempotents in $\mathbb{Z}_n$;*
2. *the number of idempotents in $\langle \mathbb{Z}_n \cup I \rangle \backslash \mathbb{Z}_n$;*

Conjecture 3. *([22]) Prove if $\langle \mathbb{Z}_n \cup I \rangle$ and $\langle \mathbb{Z}_m \cup I \rangle$ are two neutrosophic rings where $n > m$ and $n = p^t q$ ($t > 2$, and p and q two distinct primes) and $m = p_1 p_2 \cdots p_s$ where p_is are distinct primes. $1 \leq i \leq s$, then*

1. *prove $\mathbb{Z}_n$ has a greater number of idempotents than $\mathbb{Z}_m$; and*
2. *prove $\langle \mathbb{Z}_n \cup I \rangle$ has a greater number of idempotents than $\langle \mathbb{Z}_n \cup I \rangle$.*

Theorem 6. *([22]) Let $S = \langle \mathbb{Z}_{pq}, +, \cdot \rangle$ where p and q are two distinct primes:*

(a) There are two idempotents in $\mathbb{Z}_{pq}$ say r and s.
(b) $\{r, s, rI, sI, I, r + tI, s + tI | t \in \{\mathbb{Z}_{pq} \backslash 0\}\}$ such that $r + t = s, 1$ or 0 and $s + t = 0, 1$ or r is the partial collection of idempotents of S.

For Problem 1, from Remark 3, there are four idempotents in $\mathbb{Z}_{pq}$, which are $\{1, neut(p), neut(q), neut(pq) = 0\}$. Let $r = neut(p), s = neut(q)$, so there are two non-trivial idempotents r, s in $\mathbb{Z}_{pq}$. From Corollary 1 and 2, the number of all idempotents in $\langle \mathbb{Z}_{pq} \cup I \rangle$ is $2^4 = 16$, they are $\{0 + (0 - 0)I = 0, 0 + (1 - 0)I = I, 0 + (r - 0)I = rI, 0 + (s - 0)I = sI, 1 + (0 - 1)I = 1 + (n - 1)I, 1 + (1 - 1)I = 1, 1 + (r - 1)I, 1 + (s - 1)I, r + (0 - r)I = r + (n - r)I, r + (1 - r)I = r +$

$(n+1-r)I, r+(r-r)I = r, r+(s-r)I, s+(0-s)I = s+(n-r)s, s+(1-s)I = s+(n+1-s)I, s+(r-s)I, s+(s-s)I = s\}$. So there are 14 non-trivial idempotents in $\langle \mathbb{Z}_{pq} \cup I \rangle$, but there are only include 11 non-trivial idempotents in (b) of the Theorem 6, missing $\{1+(n-1)I, 1+(r-1)I, 1+(s-1)I\}$.

For Conjecture 1, from Corollary 1 and 2, there are eight idempotents in $\mathbb{Z}_{pqr}$, which are $\{1 = m_0, neut(p) = m_1, neut(q) = m_2, neut(r) = m_3, neut(pq) = m_4, neut(pr) = m_5, neut(qr) = m_6, neut(pqr) = 0 = m_7\}$. There are six non-trivial idempotents in $\mathbb{Z}_{pqr}$. In $\langle \mathbb{Z}_n \cup I \rangle$, all idempotents are $\{m_i + (m_j - m_i)I | i, j = 0, 1, 2, \cdots, 7\}$.

For Conjecture 2, from Remark 3, the number of idempotents in $\mathbb{Z}_{p_1 p_2 \cdots p_t}$ is 2^t, and the number of idempotents in $\langle \mathbb{Z}_{p_1 p_2 \cdots p_t} \cup I \rangle \backslash \mathbb{Z}_{p_1 p_2 \cdots p_t}$ is $2^{2t} - 2^t$.

For Conjecture 3, from Remark 3, the number of idempotents in $\mathbb{Z}_n$ is 2^2, and the number of idempotents in $\mathbb{Z}_m$ is 2^s, where $n = p^t q, m = p_1 p_2 \cdot p_s$. So, if $s > 2$, $\mathbb{Z}_m$ is characterized by a larger number of idempotents than $\mathbb{Z}_n$. In similarly way, the number of idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ is 2^4, and the number of idempotents in $\langle \mathbb{Z}_m \cup I \rangle$ is 2^{2s}. So, if $s > 2$, we can infer that $\langle \mathbb{Z}_m \cup I \rangle$ is characterized by a larger number of idempotents than $\langle \mathbb{Z}_n \cup I \rangle$.

As another application, we will use the idempotents to divide the elements of the neutrosophic rings $\langle R \cup I \rangle$ when $R = \mathbb{F}$.

For each NETG $(N, *)$, $a \in N$, from Proposition 1, the neutral element of a is uniquely determined. From Proposition 2, $\bigcup_{e \in E(N)} N(e)$ is a partition of N. Since the idempotents and neutral elements are same, we can use the idempotents to get a partition of N. Let us illustrate these with the following example.

Example 4. *Let $R = \mathbb{Z}_3$, which is a field. Since $n = 3$, from Theorem 5, we can get the different neutral elements in $\mathbb{Z}_3$ are $neut(1)$ and $neut(0)$, i.e., the different idempotents in $\mathbb{Z}_3$ are $1, 0$. From Corollary 2, the number of different idempotents in neutrosophic ring $\langle \mathbb{Z}_3 \cup I \rangle$ is $2^{2 \cdot 1} = 4$.*

From Algorithm 1, the set of all 4 idempotents in $\langle \mathbb{Z}_3 \cup I \rangle$ is: $\{0, 1, I, 1+2I\}$. We have $E(0) = \{0\}, E(1) = \{1, 2, 1+I, 2+2I\}$, $E(I) = \{I, 2I\}, E(1+2I) = \{1+2I, 2+I\}$. So $\langle \mathbb{Z}_3 \cup I \rangle = E(0) \cup E(1) \cup E(I) \cup E(1+2I)$.

4. The Idempotents in Neutrosophic Quadruple Rings

In the above section, we explored the idempotents in $\langle R \cup I \rangle$. In neutrosophic logic, each proposition is approximated to represent respectively the truth (T), the falsehood (F), and the indeterminacy (I). In this section, according the idea of neutrosophic ring $\langle R \cup I \rangle$, the neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is proposed and the idempotents are given in this section.

Definition 6. *Let $(R, +, \cdot)$ be any ring. The set*

$$\langle R \cup T \cup I \cup F \rangle = \{a_1 + a_2 T + a_3 I + a_4 F : a_1, a_2, a_3, a_4 \in R\} \tag{2}$$

is called a neutrosophic quadruple ring generated by R and T, I, F. Consider the order $T \prec I \prec F$. Let $a = a_1 + a_2 T + a_3 I + a_4 F, b = b_1 + b_2 T + b_3 I + b_4 F \in \langle R \cup T \cup I \cup F \rangle$, the operators $\oplus, \otimes$ on $\langle R \cup T \cup I \cup F \rangle$ are defined as follows:

$$\begin{aligned} a \oplus b &= (a_1 + a_2 T + a_3 I + a_4 F) \oplus (b_1 + b_2 T + b_3 I + b_4 F) \\ &= a_1 + b_1 + (a_2 + b_2) T + (a_3 + b_3) I + (a_4 + b_4) F. \end{aligned} \tag{3}$$

$$\begin{aligned} a * b = \; & (a_1 + a_2 T + a_3 I + a_4 F) * (b_1, b_2 T, b_3 I, b_4 F) \\ & = a_1 b_1 + (a_1 b_2 + a_2 b_1 + a_2 b_2) T + (a_1 b_3 + a_2 b_3 + a_3 b_1 + a_3 b_2 + a_3 b_3) I \\ & + (a_1 b_4 + a_2 b_4 + a_3 b_4 + a_4 b_1 + a_4 b_2 + a_4 b_3 + a_4 b_4) F. \end{aligned} \tag{4}$$

Remark 4. *It is easy to verify that* $(\langle R \cup T \cup I \cup F\rangle, \oplus, *)$ *is a ring, moreover, it also has the same algebra structure with neutrosophic quadruple numbers (see [23–25]), so the we call* $\langle R \cup T \cup I \cup F\rangle$ *is a neutrosophic quadruple ring is reasonable.*

Remark 5. *Similarly with Remark 2, for simplicity of notation, we use* $+, \cdot$ *to replace* $\oplus, *$ *on neutrosophic quadruple ring* $\langle R \cup T \cup I \cup F\rangle$. *That is* $a + b$ *also means* $a \oplus b$ *if* $a, b \in \langle R \cup T \cup I \cup F\rangle$. *and* $a \cdot b$ *also means* $a * b$ *if* $a, b \in \langle R \cup T \cup I \cup F\rangle$. *For short* $a \cdot b$ *denoted by* ab *and* $a \cdot a$ *denoted by* a^2.

Example 5. $\langle \mathbb{Z} \cup T \cup I \cup F\rangle, \langle \mathbb{Q} \cup T \cup I \cup F\rangle, \langle \mathbb{R} \cup T \cup I \cup F\rangle$ *and* $\langle \mathbb{C} \cup T \cup I \cup F\rangle$ *are neutrosophic quadruple rings of integer, rational, real and complex numbers, respectively.* $\langle \mathbb{Z}_n \cup T \cup I \cup F\rangle$ *is neutrosophic quadruple ring of modulo integers. Of course,* $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ *and* $\mathbb{Z}_n$ *are neutrosophic quadruple rings when coefficients of* T, I *and* F *equal zero.*

Definition 7. *Let* $\langle R \cup T \cup I \cup F\rangle$ *be a neutrosophic quadruple ring.* $\langle R \cup T \cup I \cup F\rangle$ *is commutative if*

$$ab = ba, \forall a, b \in \langle R \cup T \cup I \cup F\rangle.$$

In addition, if there exists $1 \in \langle R \cup T \cup I \cup F\rangle$, *such that* $1 \cdot a = a \cdot 1 = a$ *for all* $a \in \langle R \cup T \cup I \cup F\rangle$, *then* $\langle R \cup T \cup I \cup F\rangle$ *is called a commutative neutrosophic quadruple ring with unity.*

Definition 8. *An element* a *in a neutrosophic quadruple ring* $\langle R \cup T \cup I \cup F\rangle$ *is called an idempotent element if* $a^2 = a$.

Theorem 7. *The set of all idempotents of neutrosophic quadruple rings* $\langle \mathbb{C} \cup T \cup I \cup F\rangle, \langle \mathbb{R} \cup T \cup I \cup F\rangle, \langle \mathbb{Q} \cup T \cup I \cup F\rangle$ *and* $\langle \mathbb{Z} \cup T \cup I \cup F\rangle$ *is*

$$\{(1,0,0,0),(0,0,0,F),(0,0,I,-F),(0,0,I,0),(0,T,-I,0),(0,T,-I,F),(0,T,0,-F),(0,T,0,0),$$

$$(1,-T,0,0),(1,-T,0,F),(1,-T,I,-F),(1,-T,I,0),(1,0,-I,0),(1,0,-I,F),(1,0,0,-F),(1,0,0,0)\}.$$

Proof. We only give the proof for $\langle \mathbb{R} \cup T \cup I \cup F\rangle$, and the same result can be obtained for $\langle \mathbb{C} \cup T \cup I \cup F\rangle, \langle \mathbb{Q} \cup T \cup I \cup F\rangle$ or $\langle \mathbb{Z} \cup T \cup I \cup F\rangle$.

Let $a = a_1 + a_2T + a_3I + a_4F$, if a is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F\rangle$, so $a^2 = a$, i.e., $(a_1 + a_2T + a_3I + a_4F)^2 = (a_1 + a_2T + a_3I + a_4F)$, which means

$$\begin{cases} a_1^2 = a_1, \\ 2a_1a_2 + a_2^2 = a_2, \\ 2(a_1 + a_2)a_3 + a_3^2 = a_3, \\ 2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4. \end{cases}$$

Since $a_1 \in \mathbb{R}$, so from $a_1^2 = a_1$, we can get $a_1 = 0$ or $a_1 = 1$.

Case A: if $a_1 = 0$, then from $2a_1a_2 + a_2^2 = a_2$, we can infer $a_2^2 = a_2$, so $a_2 = 0$ or $a_2 = 1$.

Case A1: if $a_1 = 0$ and $a_2 = 0$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = 1$.

Case A11: if $a_1 = 0$, $a_2 = 0$ and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.

Case A111: if $a_1 = a_2 = a_3 = a_4 = 0$, i.e., $(0,0,0,0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F\rangle$.

Case A112: if $a_1 = a_2 = a_3 = 0$ and $a_4 = 1$, i.e., $(0,0,0,F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F\rangle$.

Case A12: if $a_1 = a_2 = 0$ and $a_3 = 1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.

Case A121: if $a_1 = a_2 = 0$, $a_3 = 1$ and $a_4 = 0$, i.e., $(0, 0, I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case A122: if $a_1 = a_2 = 0$, $a_3 = 1$ and $a_4 = -1$, i.e., $(0, 0, I, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case A2: if $a_1 = 0$ and $a_2 = 1$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $2a_3 + a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = -1$.

Case A21: if $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.

Case A121: if $a_1 = 0$, $a_2 = 1$, $a_3 = 0$ and $a_4 = 0$, i.e., $(0, T, 0, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case A112: if $a_1 = 0$, $a_2 = 1$, $a_3 = 0$ and $a_4 = -1$, i.e., $(0, T, 0, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case A22: if $a_1 = 0$, $a_2 = 1$ and $a_3 = -1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.

Case A121: if $a_1 = 0$, $a_2 = 1$, $a_3 = -1$ and $a_4 = 0$, i.e., $(0, T, -I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case A112: if $a_1 = 0$, $a_2 = 1$, $a_3 = -1$ and $a_4 = 1$, i.e., $(0, T, -I, F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B: if $a_1 = 1$, then from $2a_1a_2 + a_2^2 = a_2$, we can infer $2a_2 + a_2^2 = a_2$, so $a_2 = 0$ or $a_2 = -1$.

Case B1: if $a_1 = 1$ and $a_2 = 0$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $2a_3 + a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = -1$.

Case B11: if $a_1 = 1$, $a_2 = 0$ and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.

Case B111: if $a_1 = 1$, $a_2 = 0$, $a_3 = 0$ and $a_4 = 0$, i.e., $(1, 0, 0, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B112: if $a_1 = 1$, $a_2 = 0$, $a_3 = 0$ and $a_4 = -1$, i.e., $(1, 0, 0, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B12: if $a_1 = 1$, $a_2 = 0$ and $a_3 = -1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.

Case B121: if $a_1 = 1$, $a_2 = 0$, $a_3 = -1$ and $a_4 = 0$, i.e., $(1, 0, -I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B122: if $a_1 = 1$, $a_2 = 0$, $a_3 = -1$ and $a_4 = 1$, i.e., $(1, 0, -I, F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B2: if $a_1 = 1$ and $a_2 = -1$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = 1$.

Case B21: if $a_1 = 1$, $a_2 = -1$, and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.

Case B121: if $a_1 = 1$, $a_2 = -1$, $a_3 = 0$ and $a_4 = 0$, i.e., $(1, -T, 0, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B112: if $a_1 = 1$, $a_2 = -1$, $a_3 = 0$ and $a_4 = 1$, i.e., $(1, -T, 0, F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B22: if $a_1 = 1$, $a_2 = -1$ and $a_3 = 1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.

Case B121: if $a_1 = 1$, $a_2 = -1$, $a_3 = 1$ and $a_4 = 0$, i.e., $(1, -T, I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case B112: if $a_1 = 1$, $a_2 = -1$, $a_3 = 1$ and $a_4 = -1$, i.e., $(1, -T, I, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

From the above analysis, we can get the set of all idempotents in neutrosophic quadruple ring $\langle \mathbb{R} \cup T \cup I \cup F \rangle$ are $\{(1, 0, 0, 0), (0, 0, 0, F), (0, 0, I, -F), (0, 0, I, 0), (0, T, -I, 0), (0, T, -I, F), (0, T, 0, -F), (0, T, 0, 0), (1, -T, 0, 0), (1, -T, 0, F), (1, -T, I, -F), (1, -T, I, 0), (1, 0, -I, 0), (1, 0, -I, F), (1, 0, 0, -F), (1, 0, 0, 0)\}$. □

The above theorem reveals that the idempotents in neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is fixed when R is $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$. For any ring R, we have the following results.

Theorem 8. *For neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$, $a = a_1 + a_2T + a_3I + a_4F$ is idempotent in neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ iff a_1 is idempotent in R, $a_2 = c - a_1$, $a_3 = d - (a_1 + a_2)$ and $a_4 = e - (a_1 + a_2 + a_3)$, where c, d and e are any idempotents in R.*

Proof. Necessity: If $a = a_1 + a_2T + a_3I + a_4F$ is idempotent, i.e., $(a_1 + a_2T + a_3I + a_4F)^2 = a_1 + a_2T + a_3I + a_4F$, which means

$$\begin{cases} a_1^2 = a_1, \\ 2a_1a_2 + a_2^2 = a_2, \\ 2(a_1 + a_2)a_3 + a_3^2 = a_3, \\ 2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4. \end{cases}$$

Since $a_1 \in R$, from $a_1^2 = a_1$, we can get a_1 is idempotent in R.

From $2a_1a_2 + a_2^2 = a_2$ and $a_1^2 = a_1$, we can get $(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2 = a_1 + a_2$, so $a_1 + a_2$ is also idempotent in R, denoted by c, so $a_2 = c - a_1$.

From $2(a_1 + a_2)a_3 + a_3^2 = a_3$, and $(a_1 + a_2)^2 = a_1 + a_2$, we can get $(a_1 + a_2 + a_3)^2 = (a_1 + a_2)^2 + 2(a_1 + a_2)a_3 + a_3^2 = a_1 + a_2 + a_3$, so $a_1 + a_2 + a_3$ is also idempotent in R, denoted by d, so $a_3 = d - a_1 - a_2$.

From $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, and $(a_1 + a_2 + a_3)^2 = a_1 + a_2 + a_3$, we can get $(a_1 + a_2 + a_3 + a_4)^2 = (a_1 + a_2 + a_3)^2 + 2(a_1 + a_2 + a_3)a_3 + a_4^2 = a_1 + a_2 + a_3 + a_4$, so $a_1 + a_2 + a_3 + a_4$ is also idempotent in R, denoted by e, so $a_4 = e - a_1 - a_2 - a_3$.

Sufficiency: If a_1, c, d and e are arbitrary idempotents in R, let $a_2 = c - a_1$, $a_3 = d - (a_1 + a_2)$ and $a_4 = e - (a_1 + a_2 + a_3)$. so $(a_1 + a_2T + a_3I + a_4F)^2 = (a_1 + (c - a_1)T + (d - a_1 - a_2)I + (e - a_1 - a_2 - a_3)F)^2 = a_1^2 + (2(c - a_1)a_1 + (c - a_1)^2)T + (2c(d - a_1 - a_2) + (d - a_1 - a_2)^2)I + (2d(e - a_1 - a_2 - a_3) + (e - a_1 - a_2 - a_3)^2)F = a_1 + (c - a_1)T + (d - a_1 - a_2)I + (e - a_1 - a_2 - a_3)F$. Thus, $a = a_1 + a_2T + a_3I + a_4F$ is idempotent. □

Theorem 9. *If the number of different idempotents in R is t, then the number of different idempotents in neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is t^4.*

Proof. If the number of different idempotents in R is t, let $a_1 + a_2T + a_3I + a_4F \in \langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ is idempotent, so a_1 is idempotent in R, i.e., a_1 has t different selections. When a_1 is selected, $a_2 = c - a_1$, where c is idempotent, which also has t different selections. When a_1, a_2 are selected, $a_3 = d - a_1 - a_2$, where d is idempotent, which also has t different selections. When a_1, a_2, a_3 is selected, $a_4 = e - a_1 - a_2 - a_3$, where e is idempotent, which also has t different selections. Thus, the number of all selections is $t \cdot t \cdot t \cdot t = t^4$, i.e., the number of different idempotents in $\langle R \cup T \cup I \cup F \rangle$ is t^4. □

From Theorems 8 and 9 and Remark 3, it follows easily that:

Corollary 3. *In neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$, $a = a_1 + a_2T + a_3I + a_4F$ is idempotent in neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ iff a_1 is idempotent in $\mathbb{Z}_n$, $a_2 = c - a_1$, $a_3 = d - (a_1 + a_2)$ and $a_4 = e - (a_1 + a_2 + a_3)$, where c, d and e are any idempotents in $\mathbb{Z}_n$.*

Corollary 4. *The number of different idempotents in neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ is 2^{4t}.*

The solving process for neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ is given by Algorithm 2. Just only input n, we can get all idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$. The MATLAB code is provided in the Appendix C.

Algorithm 2: Solving the different idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$

Input: n

1: Factorization of integer n, we can get $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$.

2: Computing the neutral element of $1, p_1^{k_1}, p_2^{k_2}, \cdots, p_t^{k_t}, p_1^{k_1} p_2^{k_2}, \cdots p_1^{k_1} p_t^{k_t}, \cdots, p_2^{k_2} p_3^{k_3} \cdots p_t^{k_t}$ and $p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$. So, we can get all idempotents in $\mathbb{Z}_n$, denoted by $c_1, c_2, \cdots, c_{2^t}$.

3: Let ID=[];

4: for $i = 1 : 2^t$

5: $\quad a_1 = c_i$

6: $\quad$ for $j = 1 : 2^t$

7: $\qquad a_2 = \mathrm{mod}(c_j - a_1, n);$

8: $\qquad$ for $m = 1 : 2^t$

9: $\qquad\quad a_3 = \mathrm{mod}(c_m - a_1 - a_2, n);$

10: $\qquad\quad$ for $q = 1 : 2^t$

11: $\qquad\qquad a_4 = \mathrm{mod}(c_q - a_1 - a_2 - a_3, n);$

12: $\qquad\qquad ID = [ID; \ [a_1, a_2, a_3, a_4]];$

13: $\qquad\quad$ end

14: $\qquad$ end

15: $\quad$ end

16: end

Output: ID: all the idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$

Example 6. *Solve all idempotents in* $\langle \mathbb{Z}_{12} \cup T \cup I \cup F \rangle$.

Since $n = 12 = 2^2 \cdot 3$, *from Theorems 4 and 5, we can get the different neutral elements in* $\mathbb{Z}_{12}$ *are* $neut(1), neut(2^2), neut(3), neut(2^3 \cdot 3)$ *and* $neut(0)$, *i.e., the different idempotents in* $\mathbb{Z}_{12}$ *are* $1, 4, 9, 0$. *From Corollary 4, the number of different idempotents in neutrosophic quadruple ring* $\langle \mathbb{Z}_{12} \cup T \cup I \cup F \rangle$ *is* $2^{4 \cdot 2} = 256$.

From Algorithm 2, the set of all 256 idempotents in $\langle \mathbb{Z}_{12} \cup T \cup I \cup F \rangle$ *is:* $\{0, 1F, 4F, 9F, I + 11F, I, I + 3F, I + 8F, 4I + 8F, 4I + 9F, 4I, 4I + 5F, 9I + 3F, 9I + 4F, 9I + 7F, 9I, T + 11I, T + 11I + F, T + 11I + 4F, T + 11I + 9F, T + 11F, T, T + 3F, T + 8F, T + 3I + 8F, T + 3I + 9F, T + 3I, T + 3I + 5F, T + 8I + 3F, T + 8I + 4F, T + 8I + 7F, T + 8, 4T + 8I, 4T + 8I + F, 4T + 8I + 4F, 4T + 8I + 9F, 4T + 9I + 11F, 4T + 9I, 4T + 9I + 3F, 4T + 9I + 8F, 4T + 8F, 4T + 9F, 4T, 4T + 5F, 4T + 5I + 3F, 4T + 5I + 4F, 4T + 5I + 7F, 4T + 5I, 9T + 3I, 9T + 3I + F, 9T + 3I + 4F, 9T + 3I + 9F, 9T + 4I + 11F, 9T + 4I, 9T + 4I + 3F, 9T + 4I + 8F, 9T + 7I + 8F, 9T + 7I + 9F, 9T + 7I, 9T + 7I + 5F, 9T + 3F, 9T + 4F, 9T + 7F, 9T, 1 + 11T, 1 + 11T + F, 1 + 11T + 4F, 1 + 11T + 9F, 1 + 11T + I + 11F, 1 + 11T + I, 1 + 11T + I + 3F, 1 + 11T + I + 8F, 1 + 11T + 4I + 8F, 1 + 11T + 4I + 9F, 1 + 11T + 4I, 1 + 11T + 4I + 5F, 1 + 11T + 9I + 3F, 1 + 11T + 9I + 4F, 1 + 11T + 9I + 7F, 1 + 11T + 9I, 1 + 11I, 1 + 11I + F, 1 + 11I + 4F, 1 + 11I + 9F, 1 + 11F, 1, 1 + 3F, 1 + 8F, 1 + 3I + 8F, 1 + 3I + 9F, 1 + 3I, 1 + 3I + 5F, 1 + 8I + 3F, 1 + 8I + 4F, 1 + 8I + 7F, 1 + 8I, 1 + 3T + 8I, 1 + 3T + 8I + F, 1 + 3T + 8I + 4F, 1 + 3T + 8I + 9F, 1 + 3T + 9I + 11F, 1 + 3T + 9I, 1 + 3T + 9I + 3F, 1 + 3T + 9I + 8F, 1 + 3T + 8F, 1 + 3T + 9F, 1 + 3T, 1 + 3T + 5F, 1 + 3T + 5I + 3F, 1 + 3T + 5I + 4F, 1 + 3T + 5I + 7F, 1 + 3T + 5I, 1 + 8T + 3I, 1 + 8T + 3I + F, 1 + 8T + 3I + 4F, 1 + 8T + 3I + 9F, 1 + 8T + 4I + 11F, 1 + 8T + 4I, 1 + 8T + 4I + 3F, 1 + 8T + 4I + 8F, 1 + 8T + 7I + 8F, 1 + 8T + 7I + 9F, 1 + 8T + 7I, 1 + 8T + 7I + 5F, 1 + 8T + 3F, 1 + 8T + 4F, 1 + 8T + 7F, 1 + 8T, 4 + 8T, 4 + 8T + F, 4 + 8T + 4F, 4 + 8T + 9F, 4 + 8T + I + 11F, 4 + 8T + I, 4 + 8T + I + 3F, 4 + 8T + I + 8F, 4 + 8T + 4I + 8F, 4 + 8T + 4I + 9F, 4 + 8T + 4I, 4 + 8T + 4I + 5F, 4 + 8T + 9I + 3F, 4 + 8T + 9I + 4F, 4 + 8T + 9I + 7F, 4 + 8T + 9I, 4 + 9T + 11I, 4 + 9T + 11I + F, 4 + 9T + 11I + 4F, 4 + 9T + 11I + 9F, 4 + 9T + 11F, 4 + 9T, 4 + 9T + 3F, 4 + 9T + 8F, 4 + 9T + 3I + 8F, 4 + 9T + 3I + 9F, 4 + 9T + 3I, 4 + 9T + 3I + 5F, 4 + 9T + 8I + 3F, 4 + 9T + 8I + 4F, 4 + 9T + 8I + 7F, 4 + 9T + 8I, 4 + 8I, 4 + 8I + F, 4 + 8I + 4F, 4 + 8I + 9F, 4 + 9I + 11F, 4 + 9I, 4 + 9I + 3F, 4 + 9I + 8F, 4 + 8F, 4 + 9F, 4, 4 + 5F, 4 + 5I + 3F, 4 + 5I + 4F, 4 + 5I + 7F, 4 + 5I, 4 + 5T + 3I, 4 + 5T + 3I + F, 4 + 5T + 3I + 4F, 4 + 5T + 3I + 9F, 4 + 5T + 4I + 11F, 4 + 5T + 4I, 4 + 5T + 4I + 3F, 4 + 5T + 4I + 8F, 4 + 5T + 7I + 8F, 4 + 5T + 7I + 9F, 4 + 5T + 7I, 4 + 5T + 7I + 5F, 4 + 5T + 3F, 4 +$

$5T+4F, 4+5T+7F, 4+5T, 9+3T, 9+3T+F, 9+3T+4F, 9+3T+9F, 9+3T+I+11F, 9+3T+I, 9+3T+I+3F, 9+3T+I+8F, 9+3T+4I+8F, 9+3T+4I+9F, 9+3T+4I, 9+3T+4I+5F, 9+3T+9I+3F, 9+3T+9I+4F, 9+3T+9I+7F, 9+3T+9I, 9+4T+11I, 9+4T+11I+F, 9+4T+11I+4F, 9+4T+11I+9F, 9+4T+11F, 9+4T, 9+4T+3F, 9+4T+8F, 9+4T+3I+8F, 9+4T+3I+9F, 9+4T+3I, 9+4T+3I+5F, 9+4T+8I+3F, 9+4T+8I+4F, 9+4T+8I+7F, 9+4T+8I, 9+7T+8I, 9+7T+8I+F, 9+7T+8I+4F, 9+7T+8I+9F, 9+7T+9I+11F, 9+7T+9I, 9+7T+9I+3F, 9+7T+9I+8F, 9+7T+8F, 9+7T+9F, 9+7T, 9+7T+5F, 9+7T+5I+3F, 9+7T+5I+4F, 9+7T+5I+7F, 9+7T+5I, 9+3I, 9+3I+F, 9+3I+4F, 9+3I+9F, 9+4I+11F, 9+4I, 9+4I+3F, 9+4I+8F, 9+7I+8F, 9+7I+9F, 9+7I, 9+7I+5F, 9+3F, 9+4F, 9+7F, 9.\}$

Similarly, we will use the idempotents to divide the elements of the neutrosophic rings $\langle R \cup T \cup I \cup F \rangle$ when $R = \mathbb{F}$. Let us illustrate these with the following example.

Example 7. *Let $R = \mathbb{Z}_3$, which is a field. From Example 4, the different idempotents in $\mathbb{Z}_3$ are $1, 0$. From Corollary 4, the number of different idempotents in neutrosophic quadruple ring $\langle \mathbb{Z}_3 \cup T \cup I \cup F \rangle$ is $2^{4\cdot} = 16$.*

From Algorithm 2, the set of all 16 idempotents in $\langle \mathbb{Z}_3 \cup I \rangle$ is: $E = \{0, F, I+2F, I, T+2I, T+2I+F, T+2F, T, 1+2T, 1+2T+F, 1+2T+I+2F, 1+2T+I, 1+2I, 1+2I+F, 1+2F, 1\}$. We have $E(0) = \{0\}, E(F) = \{F, 2F\}, E(I+2F) = \{I+2F, 2I+F\}$, $E(I) = \{I, I+F, 2I, 2I+2F\}$, $E(T+2I) = \{T+2I, 2T+I\}$, $E(T+2I+F) = \{T+2I+F, T+2I+2F, 2T+I+F, 2T+I+2F\}$, $E(T+2F) = \{T+2F, T+I+F, 2T+F, 2T+2I+2F\}$, $E(T) = \{T+F, T, T+I, T+I+2F, 2T, 2T+2F, 2T+2I, 2T+2I+F\}$, $E(1+2T) = \{1+2T, 2+T\}$, $E(1+2T+F) = \{1+2T+F, 1+2T+2F, 2+T+F, 2+T+2F\}$, $E(1+2T+I+2F) = \{1+2T+I+2F, 1+2T+2I+F, 2+T+I+2F, 2+T+2I+F\}$, $E(1+2T+I) = \{1+2T+I, 1+2T+I+F, 1+2T+2I, 1+2T+2I+2F, 2+T+I, 2+T+I+F, 2+T+2I, 2+T+2I+2F\}$, $E(1+2I) = \{1+2I, 1+T+I, 2+I, 2+2T+2I\}$, $E(1+2I+F) = \{1+2I+F, 1+2I+2F, 1+T+I+F, 1+T+I+2F, 2+I+F, 2+I+2F, 2+2T+2I+F, 2+2T+2I+2F\}$, $E(1+2F) = \{1+2F, 1+I+F, 1+T+F, 1+T+2I+2F, 2+F, 2+2I+2F, 2+2T+2F, 2+2T+I+F\}$, $E(1) = \{1, 1+F, 1+I, 1+I+2F, 1+T, 1+T+2F, 1+T+2I, 1+T+2I+F, 2, 2+2F, 2+2I, 2+2I+F, 2+2T, 2+2T+F, 2+2T+I, 2+2T+I+2F\}$. So $\langle \mathbb{Z}_3 \cup T \cup I \cup F \rangle = \bigcup_{e \in E} E(e)$.

5. Conclusions

In this paper, we study the idempotents in neutrosophic ring $\langle R \cup I \rangle$ and neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$. We not only solve the open problem and conjectures in paper [22] about idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$, but also give algorithms to obtain all idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ for each n. Furthermore, if $R = \mathbb{F}$, then the neutrosophic rings (neutrosophic quadruple rings) can be viewed as a partition divided by the idempotents. As a general result, if all idempotents in ring R are known, then all idempotents in $\langle R \cup I \rangle$ and $\langle R \cup T \cup I \cup F \rangle$ can be obtained too. Moreover, if the number of all idempotents in ring R is t, then the numbers of all idempotents in $\langle R \cup I \rangle$ and $\langle R \cup T \cup I \cup F \rangle$ are t^2 and t^4 respectively. In the following, on the one hand, we will explore semi-idempotents in neutrosophic rings, on the other hand, we will study the algebra properties of neutrosophic rings and neutrosophic quadruple rings.

Funding: This research was funded by National Natural Science Foundation of China (Grant No. 61976130), Discipline Construction Funding of Xi'an Polytechnic University, Instructional Science and Technology Plan Projects of China National Textile and Apparel Council (No. 2016073) and Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 18JS042).

Appendix A. The MATLAB code for solving the idempotents in $(\mathbb{Z}_n, \cdot)$

```
function neut = solve_neut(n)

% n: nonnegative integer
% neut: all idempotents in Z_n

B=[];
digits(32);
for i=1:n
    for j=1:n
        A1(i,j)=mod((i-1)*(j-1),n);
    end
end
a1=factor(n);
a2=unique(a1);
for i=1:length(a2)
    b=length(find(a1==a2(i)));
    B(i)=a2(i)^b;
end
D=[1];
for i=1:length(a2)
    C=combnk(B,i);
    A=prod(C,2);
    D=[D;A];
end
D=mod(D,n);
for i=1:length(D)
    if D(i)==1
        neut(i)=1;
    elseif D(i)==0
        neut(i)=0;
    else
        for j=1:n
            if mod(D(i)*j,n)==D(i)
                for k=1:n
                    if mod(D(i)*k,n)==j
                        neut(i)=j;
                        break
                    end
                end
            end
        end
    end
end
neut=sort(neut);
```

Appendix B. The MATLAB code for solving the idempotents in $\langle\mathbb{Z}_n \cup I\rangle$

```
function ID = Idempotents_ZR(n)
% n: nonnegative integer
```

```
% ID: all idempotents in in neutrosophic  ring <Z_n \cup I>

neut = solve_neut(n);

neutall=[];
for i=1:length(neut)
    for j=1:length(neut)
        c1=mod(neut(j)-neut(i),n);
        neutall=[neutall;  [neut(i), c1]];
    end
end

ID=sortrows(neutall',1)';
```

Appendix C. The MATLAB code for solving the idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$

```
function ID = Idempotents_ZRTIF(n)
% n: nonnegative integer
% ID: all idempotents in in neutrosophic quadruple ring <Z_n\cup T\cup I\cup F>


neut = solve_neut(n);
neutall=[];
for i=1:length(neut)
    a1=neut(i);
    for j=1:length(neut)
        a2=mod(neut(j)-a1,n);
        for m=1:length(neut)
            a3=mod(neut(m)-a1-a2,n);
            for q=1:length(neut)
               a4=mod(neut(q)-a1-a2-a3,n);
               neutall=[neutall;  [a1 a2 a3 a4]];
            end
        end
    end
end

ID=sortrows(neutall',1)';
```

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NeutroAlgebra is a Generalization of Partial Algebra

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Florentin Smarandache (2020). NeutroAlgebra is a Generalization of Partial Algebra. *International Journal of Neutrosophic Science* 2(1), 8-17

Abstract

In this paper we recall, improve, and extend several definitions, properties and applications of our previous 2019 research referred to NeutroAlgebras and AntiAlgebras (also called NeutroAlgebraic Structures and respectively AntiAlgebraic Structures).

Let <A> be an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, we split the nonempty space we work on into three regions {two opposite ones corresponding to <A> and <antiA>, and one corresponding to neutral (indeterminate) <neutA> (also denoted <neutroA>) between the opposites}, which may or may not be disjoint – depending on the application, but they are exhaustive (their union equals the whole space).

A NeutroAlgebra is an algebra which has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements).

A Partial Algebra is an algebra that has at least one Partial Operation, and all its Axioms are classical (i.e. axioms true for all elements).

Through a theorem we prove that NeutroAlgebra is a generalization of Partial Algebra, and we give examples of NeutroAlgebras that are not Partial Algebras. We also introduce the NeutroFunction (and NeutroOperation).

Keywords: neutrosophy, algebra, neutroalgebra, neutroFunction, neutroOperation, neutroAxiom

1. Neutrosophication by Tri-Sectioning the Space

Let X be a given nonempty space (or simply set) included into a universe of discourse U.

Let <*A*> be an item (concept, attribute, idea, proposition, theory, etc.) defined on the set X. Through the process of neutrosphication, we split the set X into three regions [two opposite ones <*A*> and <*antiA*>, and one neutral (indeterminate) <*neutroA*> between them], regions which may or may not be disjoint – depending on the application, but they are exhaustive (their union equals the whole space).

The region denoted just by <*A*> is formed by all set's elements where <*A*> is true {degree of truth (T)}, the region denoted by <*antiA*> is formed by all set's elements where <*A*> is false {degree of falsehood (F)}, and the region denoted by <*neutroA*> is formed by all set's elements where <*A*> is indeterminate (neither true nor false) {degree of indeterminacy (I)}.

We further on work with the following <*A*> concepts: Function, Operation, Axiom, and Algebra.

Therefore, by tri-sectioning the set X with respect to each such <*A*> concept, we get the following neutrosophic triplets corresponding to *(<A>, <NeutroA>, <AntiA>)*:

<Function, NeutroFunction, AntiFunction>,

<Operation, NeutroOperation, AntiOperation>,

<Axiom, NeutroAxiom, AntiAxiom>,

<Algebra, NeutroAlgebra, AntiAlgebra>.

A NeutroAlgebra is an algebra which has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements).

We have proposed for the first time the NeutroAlgebraic Structures (or NeutroAlgebras), and in general the NeutroStructures, in 2019 [1] and further on in 2020 [2].

The NeutroAlgebra is a generalization of Partial Algebra, which is an algebra that has at least one Partial Operation, while all its Axioms are totally true (classical axioms).

We recall the Boole's Partial Algebras and the Effect Algebras as particular cases of Partial Algebras, and by consequence as particular cases of NeutroAlgebras.

In comparison between the Partial Algebra and the NeutroAlgebra:

i) When the NeutroAlgebra has no NeutroAxiom, it coincides with the Partial Algebra.
ii) There are NeutroAlgebras that have no NeutroOperations, but have NeutroAxioms. These are different from Partial Algebras.
iii) And NeutroAlgebras that have both, NeutroOperations and NeutroAxioms.
These are different from Partial Algebras too.

All the above will be proved in the following.

2-4. Partially Inner-Defined, Partially Outer-Defined, or Partially Indeterminate

Let U be a nonempty universe of discourse, and X and Y be two nonempty subsets of U.

Let's consider a function:

$$\boldsymbol{f}\mathbf{:}\ X \rightarrow Y.$$

Let $a \in X$ be an element. Then, there are three possibilities:

i) [Inner-defined, or *Well-defined*; corresponding in neutrosophy to Truth (T)*]*
$f(a) \in Y;$
ii) [Outer-defined; corresponding in neutrosophy to Falsehood (F)*]*
$f(a) \in U\text{-}Y;$
iii) *[Indeterminacy*; corresponding in neutrosophy to Indeterminate (I)*]*
α) f(a) = indeterminacy;
{i.e. the value of *f(a)* does exist, but we do not know it exactly;
for example, $f(a) = c$ or d, we know that $f(a)$ may be equal to c or d (but we are not sure to which one);
or, another example, we only know that $f(a) \neq e$, where the previous $c, d, e \in U$};
β) f(a) = undefined (i.e. the value of *f(a)* is not defined, or it does not exist – as in Partial Function); *undefined* is considered part of *indeterminacy*;
δ) f(indeterminacy) $\in U$, but we either do not know the indeterminacy at all, or we only partially know some information about it
{for example we know that $f(a$ or b or $c) \in U$, where $a, b, c \in X$, but we are not sure if the argument is either a, or b, or c};
δ) more general: *f(indeterminacy1) = indeterminacy2*, where *indeterminacy1* is a vaguely known value in X and *indeterminacy2* is a vaguely known value in U;
ε) By the way, there are many types of indeterminacies, we only gave above some elementary examples.
Consequently we have:

5-7. Definitions of Total InnerFunction, Total OuterFunction, Total IndeterminateFunction, and Total UndefinedFunction

i) If for any $x \in X$ one has $f(x) \in Y$ (*inner-ness,* or *well-defined*), then f is called a ***Total InnerFunction*** (or classical **Total Function,** or in general **Function**).

ii) If for any $x \in X$ one has $f(x) \in U\text{-}Y$ (*outer-ness,* or *outer-defined*)*,* then f is called a ***Total OuterFunction*** *(or **AntiFunction**).*

iii) If for any $x \in X$ one has either *f(x) = indeterminacy,* or *f(indeterminacy)* $\in U$ *,* or *f(indeterminacy1) = indeterminacy2,* then f is called a ***Total IndeterminateFunction***.

{ As a particular case of the Total IndeterminateFunction there is the ***Total UndefinedFunction****:* when for any $x \in X$ one has *f(x) = undefined. }*

8. Definition of Partial Function

In the previous literature {[3], [4]}, the Partial Function was defined as follows:

A function $f: X \rightarrow Y$ is called a **Partial Function** if it is well-defined for some elements in X, and undefined for all the other elements in X. Therefore, there exist some elements $a \in$ X such that $f(a) \in Y$ (*well-defined*), and for all other element $b \in X$ one has $f(b)$ = (is) *undefined*.

*

We extend the partial function to NeutroFunction in order to comprise all previous *i) – iii)* situations.

9. Definition of NeutroFunction

A function $f: X \rightarrow Y$ is called a **NeutroFunction** if it has elements in X for which the function is well-defined {degree of truth (T)}, elements in X for which the function is indeterminate {degree of indeterminacy (I)}, and elements in X for which the function is outer-defined {degree of falsehood (F)}, where $T, I, F \in [0, 1]$, with (T, I, F) ≠ (1, 0, 0) that represents the (Total) Function, and (T, I, F) ≠ (0, 0, 1) that represents the AntiFunction.

In this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, unknown-ness, or indeterminacy in general.

A NeutroFunction is more general, and it may include all three previous situations: elements in X for which the function f is well-defined, elements in X for which function f is indeterminate (including function's undefined values), and elements in X for which function f is outer-defined.

We have formed the following neutrosophic triplet:

*<Function, NeutroFunction (*that includes the *Partial Function), AntiFunction>*.

Therefore, according to the above definitions, we have the following:

10. Classification of Functions

i) (Classical) **Function**, which is a function well-defined for all the elements in its domain of definition.

ii) **NeutroFunction**, which is a function partially well-defined, partially indeterminate, and partially outer-defined on its domain of definition.

iii) **AntiFunction**, which is a function outer-defined for all the elements in its domain of definition.

11. Example of NeutroFunction

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ be a universe of discourse, and two of its nonempty subsets $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{7, 8, 9, 10, 11, 12\}$, and the function f constructed as follows:

$f: X \rightarrow Y$ such that

$f(1) = 7 \in Y$ (well-defined);

$f(2) = 8 \in U\text{-}Y$ (outer-defined);

f(3) = undefined (doesn't exist);

f(4) = 9 or *10* or *11* (it does exist, but we do not know it exactly), therefore f(4) = *indeterminate*;

f(some number greater ≥ 5) = 12, {i.e. it can be *f(5) = 12* or *f(6) = 12,* we are not sure about}, therefore *f(indeterminate) = 12*.

*

Similarly we defined the NeutroOperation.

12. Definition of NeutroOperation

An *n-ary* (for integer $n \geq 1$) operation $\omega: X^n \rightarrow Y$ is called a **NeutroOperation** if it is has *n-plets* in X^n for which the operation is well-defined {degree of truth (T)}, *n-plets* in X^n for which the operation is indeterminate {degree of indeterminacy (I)}, and *n-plets* in X^n for which the operation is outer-defined {degree of falsehood (F)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (Total) Operation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary AntiOperation.

Again, in this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general.

A NeutroOperation is more general, and it may include all previous situations: elements in X^n for which the operation ω is well-defined, elements for which operation ω is outer-defined, and elements for which operation ω is indeterminate (including undefined).

13. Definition of AntiOperation

An *n-ary* (for integer $n \geq 1$) operation $\omega: X^n \rightarrow Y$ is called **AntiOperation** if for all *n-plets* $(x_1, x_2, \ldots, x_n) \in X^n$ one has $\omega(x_1, x_2, \ldots, x_n) \in U\text{-}Y$.

We have formed the neutrosophic triplet: *<Operation, NeutroOperation, AntiOperation>*.

Therefore, according to the above definitions, we have the following:

14. Classification of Operations

On a given set:

i) (Classical) **Operation** is an operation well-defined for all the set's elements.
ii) **NeutroOperation** is an operation partially well-defined, partially indeterminate, and partially outer-defined on the given set.
iii) **AntiFunction** is an operation outer-defined for all the set's elements.

*

Further, we define the NeutroHyperOperation.

15. Definition of NeutroHyperOperation

Similarly, an *n-ary* (for integer $n \geq 1$) hyperoperation $\omega: X^n \rightarrow P(Y)$ is called a **NeutroHyperOperation** if it is has *n-plets* in X^n for which the operation is well-defined $\omega(a_1, a_2, \ldots, a_n) \in P(Y)$ {degree of truth (T)}, *n-plets* in X^n for which the operation is indeterminate {degree of indeterminacy (I)}, and *n-plets* in X^n for which the operation is outer-defined $\omega(a_1, a_2, \ldots, a_n) \notin P(Y)$ {degree of falsehood (F)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (Total) HyperOperation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary AntiHyperOperation.

Again, in this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general.

A NeutroOperation is more general, and it may include all previous situations: elements in X^n for which the operation ω is well-defined, elements for which operation ω is outer-defined, and elements for which operation ω is indeterminate (including undefined).

16. Definition of AntiHyperOperation

An *n-ary* (for integer $n \geq 1$) operation $\omega: X^n \rightarrow P(Y)$ is called **AntiHyperOperation** if it is outer-defined for all the *n-plets* in X^n. Or, for any *n*-plet $(x_1, x_2, \ldots, x_n) \in X^n$ one has $\omega(x_1, x_2, \ldots, x_n) \notin P(Y)$.

Again, we have formed a neutrosophic triplet:

<HyperOperation, NeutroHyperOperation, AntiHyperOperation>.

Similarly, according to the above definitions, we have the following:

17. Classification of HyperOperations

On a given set:

i) (Classical) **HyperOperation** is a hyper-operation well-defined for all the set's elements.

ii) **NeutroHyperOperation** is a hyper-operation partially well-defined, partially indeterminate, and partially outer-defined on the given set.

iii) **AntiHyperFunction** is a hyper-operation outer-defined for all the set's elements.

*

18. Definition of Universal Algebra

In the previous literature there exist the following.

The (classical) **Universal Algebra** (or **General Algebra**) is a branch of mathematics that studies classes of (classical) algebraic structures.

19. Definition of Algebraic Structure

A (classical) **Algebraic Structure** (or **Algebra**) is a nonempty set A endowed with some (totally well-defined) operations (functions) on A, and satisfying some (classical) axioms (totally true) - according to the Universal Algebra.

20. Definition of Partial Algebra

A (classical) **Partial Algebra** is an algebra defined on a nonempty set PA that is endowed with some partial operations (or partial functions: partially well-defined, and partially undefined). While the axioms (laws) defined on a Partial Algebra are all totally (100%) true.

21. Definition of Effect Algebra

A set L that contains two special elements $0, 1 \in L$, and endowed with a partially defined binary operation $\oplus$ that satisfies the following conditions (Foulis and Bennett [4]).

For all $p, q, r \in L$ one has:

i) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$ *[Commutativity].*

ii) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$ *[Associativity].*

iii) For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$ (*Orthosupplementation*).

iv) If $1 \oplus p$ is defined, then $p = 0$ (*Zero-One Law*).

Clearly, the Effect Algebra is a particular case of Partial Algebra, since it has a partial operation $\oplus$, and all its (Commutative, Associative, Orthosupplementation, and Zero-One) Laws are totally true.

22. Definition of Boole's Partial Algebras

Let U be a universe of discourse, Su(U) the collection of subsets of U, and S(U) the partial algebra *(Su(U), +, ·, -, 0, 1).* Two partial operations (+ and –) were defined by George Boole (Burris and Sankappanavar [5]):

$A + B := A \cup B$, provided $A \cap B = \phi$, otherwise *undefined*;

and

$A - B := A \setminus B$, provided $B \subseteq A$, otherwise *undefined*;

one total operation:

$A \cdot B = A \cap B$;

and two constants:

$1 := U$,

$0 := \phi$.

Obviously, Boole's Partial Algebras are partial algebras since they have at least one partial operation, while its axioms are totally true.

*

Now we extend the Partial Algebra to NeutroAlgebra, but first we recall the below.

23. Classification of Axioms:

i) A (classical) **Axiom** defined on a nonempty set is an axiom that is totally true (i.e. true for all set's elements).

ii) A **NeutroAxiom** (or **Neutrosophic Axiom**) defined on a nonempty set is an axiom that is true for some set's elements {degree of truth (*T*)}, indeterminate for other set's elements {degree of indeterminacy (*I*)}, or false for the other set's elements {degree of falsehood (*F*)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the (classical) Axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.

iii) An **AntiAxiom** defined on a nonempty set is an axiom that is false for all set's elements.

Therefore, we have formed the **neutrosophic triplet**: *<Axiom, NeutroAxiom, AntiAxiom>*.

24. Classification of Algebras

i) A (classical) **Algebra** is a nonempty set CA that is endowed with total operations (or total functions, i.e. true for all set's elements) and (classical) **Axioms** (also true for all set's elements).

ii) A **NeutroAlgebra** (or **NeutroAlgebraic Structure**) is a nonempty set NA that is endowed with: at least one NeutroOperation (or NeutroFunction), or one NeutroAxiom that is referred to the set's (partial-, neutro-, or total-) operations.

iii) An **AntiAlgebra** (or **AntiAlgebraic Structure**) is a nonempty set AA that is endowed with at least one AntiOperation (or AntiFunction) or at least one AntiAxiom.

Therefore, we have formed the neutrosophic triplet:

*<Algebra, NeutroAlgebra (*which includes the *Partial Algebra), AntiAlgebra>*.

25. Definition of Universal NeutroAlgebra

The **Universal NeutroAlgebra** (or **General NeutroAlgebra**) is a branch of neutrosophic mathematics that studies classes of NeutroAlgebras and AntiAlgebras.

26. Applications of NeutroFunctions and NeutroAlgebras

Applicability of Partial Functions, when the domain is not well-known, are in computer science, computability theory, programming language, real analysis, complex analysis, charts in the atlases, recursion theory, category theory, etc.

NeutroFunctions (NeutroOperations), when the domain and/or range are/is not well-known, have a larger applicable field since, besides Partial Functions' undefined values, NeutroFunctions include functions' outer-defined and/or indeterminate values referred not only to the functions' not-well-known domain but to the functions' not-well-known range too.

NeutroAlgebras, in addition to NeutroFunctions, is equipped with NeutroAxioms that better reflect our reality where not all individuals totally agree or totally disagree with some regulation (law, rule, action, organization, idea, etc.), but each individual expresses partial degree of approval, partial degree of ignorance, and partial degree of disapproval of the regulation. NeutroAxioms are true for some elements, indeterminate for others, and false for other elements.

27. NeutroAxioms in our World

Unlike the idealistic or abstract algebraic structures, from pure mathematics, constructed on a given perfect space (set), where the axioms (laws, rules, theorems, results etc.) are totally (100%) true for all space's elements, our World and Reality consist of approximations, imperfections, vagueness, and partialities.

Most of mathematical models are too rigid to completely describe the imperfect reality. Many axioms are actually NeutroAxioms (i.e. axioms that are true for some space's elements, indeterminate for other space's elements, and false for other space's elements). See below several examples.

In Soft Sciences [2] the laws are interpreted and re-interpreted; in social, political, religious legislation the laws are flexible; the same law may be true from a point of view, and indeterminate or false from another point of view. Thus the law is partially true and partially indeterminate (neutral) or false (it is a neutrosophic law, or NeutroLaw). Many interpretations have a *degree of objectivity,* a *degree of neutrality (indeterminacy),* and a *degree of subjectivity*. The cultural, tradition, religious, and psychological factors play important roles in interpretations and actions for or against some regulations.

a) For example, "gun control". There are people supporting it because of too many crimes and violence (and they are right), and people that oppose it because they want to be able to defend themselves and their houses (and they are right too); there also are ignorant people who do not care (so, they do not manifest for or against it).

Besides ignorant (neutral) people, we see two opposite propositions, both of them true, but from different points of view (from different criteria/parameters; plithogenic logic may better be used herein, since the truth-value of a proposition is calculated from various points of view – obtaining different results). How to solve this? Going to the middle, in between opposites (as in neutrosophy): <u>allow</u> military, police, security, registered hunters to bear arms; <u>prohibit</u> mentally ill, sociopaths, criminals, violent people from bearing arms; and <u>background check</u> on everybody that buys arms, etc.

b) Similarly for "abortion". Some people argue that by abortion one kills a life (which is true), others support the idea of the woman to be master of her body (which is true as well), and again the category of ignorants.
c) A law applying for a category of people (degree of truth), but not applying for another category of people (degree of falsehood).

For example, in India a Hindi man is allowed to marry only one wife, while a Muslim man is allowed to marry up to four wives.

d) *Double Standard*: a rule applying for some people, but not applying for other people that for example may have a higher social rank.
e) *Hypocrisy*: criticizing your enemies (but not your friends!) for what your friends do too!

Or praising your friends (but not your enemies!) for what your enemies do too!

That's why the NeutroAlgebras better model our imprecise reality and they are needed to be studied, since they are equipped with NeutroOperations (partially true, partially indeterminate, and partially false operations) and NeutroAxioms (partially true, partially indeterminate, and partially false axioms), all designed on a not-well-known space.

28. Theorem 1

The NeutroAlgebra is a generalization of Partial Algebra.

As a consequence, NeutroAlgebra is a generalization of Effect Algebra and of Boole's Partial Algebras.

Proof.

Since the Partial Algebra is equipped with partially defined operations, they are NeutroAlgebras according with the above definition of NeutroAlgebras. But the converse is not true.

Further on, the Effect Algebra and the Boole's Partial Algebras are particular cases of Partial Algebra, therefore particular cases of NeutroAlgebra.

29. Example of NeutroAlgebra that is not a Partial Algebra

Let the set $S = (0, \infty)$, endowed with the real division $\div$ of numbers. $(S, \div)$ is well defined, since

$\div$ is a total operation (there is no division by zero).

S is NeutroAssociative, because, from $x, y, z \in S$ such that

$x \div (y \div z) = (x \div y) \div z$

one gets $\frac{xz}{y} = \frac{x}{yz}$ or $xyz^2 = xy$ or $z^2 = 1$ (since both $x, y \neq 0$), whence $z = 1$ (the other solution

$z = -1$ does not belong to S).

Therefore, $(S, \div)$ is: associative for the triplets of the form $\{(x, y, 1), x, y \in S\}$, while for other triplets $\{(x, y, z), x, y, z \in S, \text{ and } z \neq 1\}$ it is not associative. So, S is partially associative and partially nonassociative (that we call NeutroAssociative).

Thus $(S, \div)$ is a classical groupoid, it is neither a partial algebra nor an effect algebra since its operation $\div$ is not a partial operation (but a total operation), and it is a NeutroSemigroup (since it is well-defined and neutroassociative) which means part of the general NeutroAlgebra.

Thus we proved that there are NeutroAlgebras that are different from Partial Algebras.

30. Other Examples of NeutroAlgebras vs. Partial Algebras

Let $U = \{a, b, c\}$ be a universe of discourse and $S = \{a, b\}$ one of its nonempty subsets.

i) Structure $S_1 = (S, *_1)$, constructed as below using Cayley Table:

$*_1$	a	b
a	b	a
b	a	*undefined*

$*_1$ is a partially defined operation since $b *_1 b = undefined$, but for all $x \neq b$ or $y \neq b$, $x *_1 y$ is defined.
The axiom of commutativity is totally true, since $a *_1 b$ and $b *_1 a$ are defined, and they are equal: $a *_1 b = a = b *_1 a$.
Therefore, S_1 equipped with the axiom of commutativity is a partial algebra.
But S_1 equipped with the axiom of associativity is not a partial algebra, since the associativity is partially true and partially indeterminate or partially false (i.e. NeutroAssociativity):

$a*_1(b*_1a) = a*_1 a = b$, and $(a*_1b)*_1a = a*_1a = b$ *(degree of truth);*

but $a*_1(a*_1b) = a*_1 a = b$ while $(a*_1a)*_1b = b*_1b = undefined \neq b$ *(degree of falsehood).*

Therefore, S_1 equipped with the axiom of associativity is a Neutro-algebra.

ii) Structure $S_2 = (S, *_2)$, constructed as below using Cayley Table:

$*_2$	a	b
a	b	a
b	a	$c \notin S$

$*_2$ is an outer-operation since $b*_2b = c \in U - S$ is *outer-defined*, but for all $x \neq b$ *or* $y \neq b$, $x*_2y$ is inner-defined. Because $*_2$ is not partially defined (since $b*_2b \neq undefined$), S_2 cannot be a partial algebra.

Similarly, the axiom of commutativity is totally true, since $a*_2b$ and $b*_2a$ are defined, and they are equal: $a*_2b = a = b*_2a$.

Therefore, S_2 equipped with the axiom of commutativity is an outer-algebra (which is a particular case of NeutroAlgebra).

But S_2 equipped with the axiom of associativity is not an outer-algebra, since the associativity is partially true and partially indeterminate or partially false (i.e. NeutroAssociativity):

$a*_2(b*_2a) = a*_2 a = b$, and $(a*_2b)*_2a = a*_2a = b$ *(degree of truth);*

but $a*_2(a*_2b) = a*_2 a = b$ while $(a*_2a)*_2b = b*_2b = c \neq b$ *(degree of falsehood).*

Therefore, S_2 equipped with the axiom of associativity is a Neutro-algebra.

iii) Structure $S_3 = (S, *_3)$, constructed as below using Cayley Table:

$*_2$	a	b
a	b	a
b	a	a or b

$*_3$ is an indeterminate-operation since $b*_3b = a$ *or* b *(indeterminate)*, but for all $x \neq b$ *or* $y \neq b$, $x*_3y$ is well-defined. The same, because $*_3$ is not partially defined (since $b*_3b \neq undefined$), S_3 cannot be a partial algebra.

Similarly, the axiom of commutativity is totally true, since $a*_3b$ and $b*_3a$ are defined, and they are equal: $a*_3b = a = b*_3a$.

Therefore, S_3 equipped with the axiom of commutativity is an indeterminate-algebra (a particular case of NeutroAlgebra).

But S_3 equipped with the axiom of associativity is not an indeterminate-algebra, since the associativity is partially true and partially indeterminate or partially false (i.e. NeutroAssociativity):

$a*_3(b*_3a) = a*_3 a = b$, and $(a*_3b)*_3a = a*_3a = b$ *(degree of truth);*

but $a*_3(a*_3b) = a*_3 a = b$ while $(a*_3a)*_3b = b*_3b = (a \text{ or } b) \neq b$ *(degree of falsehood).*

Therefore, S_3 equipped with the axiom of associativity is a NeutroAlgebra.

31. The main distinction between Partial Algebra vs. NeutroAlgebra

A Partial Algebra has at least one Partial Operation, while all Axioms involving its partial and total operations (Associativity, Commutativity, etc.) are 100% true.
Whilst a NeutroAlgebra has at least one NeutroOperation (which is an extension of Partial Operation) or one NeutroAxiom:

i) When the NeutroAlgebra has no NeutroAxiom, it coincides with the Partial Algebra.

ii) There are NeutroAlgebras that have no NeutroOperations, but have NeutroAxioms. These are different from Partial Algebras.

iii) And NeutroAlgebras that have both, NeutroOperations and NeutroAxioms.
Also, these are different from Partial Algebras.

32. Remark 1

For the study of NeutroAlgebras the names of axioms (to be taken into consideration if they are partially true, partially indeterminate, partially false) and similarly for the study of AntiAlgebras the names of axioms (to be taken into consideration if they are totally false) should from the beginning be specified - since many axioms may fall in such categories.

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Refined Neutrosophic Rings I

E.O. Adeleke, A.A.A. Agboola, Florentin Smarandache

E.O. Adeleke, A.A.A. Agboola, Florentin Smarandache (2020). Refined Neutrosophic Rings I. *International Journal of Neutrosophic Science* 2(2), 77-81. DOI: 10.5281/zenodo.3728222

Abstract

The study of refined neutrosophic rings is the objective of this paper. Substructures of refined neutrosophic rings and their elementary properties are presented. It is shown that every refined neutrosophic ring is a ring.

Keywords: Neutrosophy, refined neutrosophic set, refined neutrosophic group, refined neutrosophic ring.

1 Introduction

The notion of neutrosophic ring $R(I)$ generated by the ring R and the indeterminacy component I was introduced for the first time in the literature by Vasantha Kandasamy and Smarandache in.[12] Since then, further studies have been carried out on neutrosophic ring, neutrosophic nearring and neutrosophic hyperring see.[1,3,4,6–8] Recently, Smarandache[10] introduced the notion of refined neutrosophic logic and neutrosophic set with the splitting of the neutrosophic components $< T, I, F >$ into the form $< T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s >$ where T_i, I_i, F_i can be made to represent different logical notions and concepts. In,[11] Smarandache introduced refined neutrosophic numbers in the form $(a, b_1I_1, b_2I_2, \ldots, b_nI_n)$ where $a, b_1, b_2, \ldots, b_n \in \mathbb{R}$or $\mathbb{C}$. The concept of refined neutrosophic algebraic structures was Antroduced by Agboola An[5] and An particular, refined neutrosophic Aroups and their substructures were studied. The present paper As devoted to the study of refined neutrosophic rings and their substructures. At As shown that every refined neutrosophic ring As a ring.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [**contradiction** (true (T) and false (F))] and I_2 [**ignorance** (true (T) or false (F))]. It then follows logically that:

$$\begin{aligned} I_1I_1 &= I_1^2 = I_1, & (1)\\ I_2I_2 &= I_2^2 = I_2, \text{and} & (2)\\ I_1I_2 &= I_2I_1 = I_1. & (3)\end{aligned}$$

If X is any nonempty set, then the set

$$X(I_1, I_2) =< X, I_1, I_2 >= \{(x, yI_1, zI_2) : x, y, z \in X\} \quad (4)$$

is called a refined neutrosophic set generated by X, I_1 and I_2. For $x, y, z \in X$, any element of $X(I_1, I_2)$ is of the form (x, yI_1, zI_2) and it is called a refined neutrosophic element.

If $+$ and . are the usual addition and multiplication of numbers, then I_k with $k = 1, 2$ have the following properties:

(1) $I_k + I_k + \cdots + I_k = nI_k$.

(2) $I_k + (-I_k) = 0$.

(3) $I_k.I_k.\cdots.I_k = I_k^n = I_k$ for all positive integer $n > 1$.

(4) $0.I_k = 0$.

(5) I_k^{-1} is undefined with respect to multiplication and therefore does not exist.

For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$\begin{aligned}(a, bI_1, cI_2) + (d, eI_1, fI_2) &= (a + d, (b + e)I_1, (c + f)I_2), \quad (5)\\ (a, bI_1, cI_2).(d, eI_1, fI_2) &= (ad, (ae + bd + be + bf + ce)I_1,\\ &\quad (af + cd + cf)I_2). \quad (6)\end{aligned}$$

For any algebraic structure $(X, *)$, the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$. For instance, if $(X, *)$ is a group, then $(X(I_1, I_2), *)$ is called a refined neutrosophic group generated by X, I_1, I_2.

Given any two refined neutrosophic algebraic structures $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$, the mapping $\phi : (X(I_1, I_2), *) \to (Y(I_1, I_2), *')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2)) \quad \forall (a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$.

(2) $\phi(I_k) = I_k$ for $k = 1, 2$.

Example 1.1. [5] Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 1.2. [5] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

For more details about refined neutrosophic sets, refined neutrosophic numbers and refined neutrosophic groups, we refer to.[5,10,11]

2 Main Results

Definition 2.1. Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring generated by R, I_1, I_2.

The abstract system $(R(I_1, I_2), +, .)$ is called a commutative refined neutrosophic ring if for all $x, y \in R(I_1, I_2)$, we have $xy = yx$. If there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ for all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ is a refined neutrosophic ring with unity.

Definition 2.2. Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If for the least positive integer n such that $nx = 0$ for all $x \in R(I_1, I_2)$, then we call $(R(I_1, I_2), +, .)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1, I_2), +, .)$.

(ii) $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1, I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 2.3. (i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 2.4. Let $M^{\mathbb{R}}_{n\times n}(I_1,I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1,I_2) \right\}$ be a refined neutrosophic set of all $n \times n$ matrix. Then $(M^{\mathbb{R}}_{n\times n}(I_1,I_2),+,.)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

Theorem 2.5. *Let $(R(I_1,I_2),+,.)$ be any refined neutrosophic ring. Then $(R(I_1,I_2),+,.)$ is a ring.*

Proof. It is clear that $(R(I_1,I_2),+)$ is an abelian group and and that $(R(I_1,I_2),.)$ is a semigroup. It remains to show that the distributive laws hold. To this end, let $x = (a_1,a_2I_1,a_3I_2), y = (b_1,b_2I_1,b_3I_2), z = (c_1,c_2I_1,c_3I_2)$ be any arbitrary elements of $R(I_1,I_2)$. Then

$$\begin{aligned} x(y+z) &= (a_1,a_2I_1,a_3I_2)((b_1,b_2I_1,b_3I_2)+(c_1,c_2I_1,c_3I_2)) \\ &= (a_1,a_2I_1,a_3I_2)(b_1+c_1,(b_2+c_2)I_1,b_3+c_3)I_2) \\ &= (a_1(b_1+c_1),a_1(b_2+c_2)+a_2(b_1+c_1)+a_2(b_2+c_2)+a_2(b_3+c_3)+a_3(b_2+c_2))I_1, \\ &\quad (a_1(b_3+c_3)+a_3(b_1+c_1)+a_3(b_3+c_3))I_2) \\ &= (a_1b_1+a_1c_1,(a_1b_2+a_1c_2+a_2b_1+a_2c_1+a_2b_2+a_2c_2+a_2b_3+a_2c_3+a_3b_2+a_3c_2)I_1, \\ &\quad (a_1b_3+a_1c_3+a_3b_1+a_3c_1+a_3b_3+a_3c_3)I_2). \end{aligned}$$

Also,

$$\begin{aligned} xy+xz &= ((a_1,a_2I_1,a_3I_2))((b_1,b_2I_1,b_3I_2))+((a_1,a_2I_1,a_3I_2))((c_1,c_2I_1,c_3I_2)) \\ &= (a_1b_1,(a_1b_2+a_2b_1+a_2b_2+a_2b_3+a_3b_2)I_1, \\ &\quad (a_1b_3+a_3b_1+a_3b_3)I_2)+(a_1c_1,(a_1c_2+a_2c_1+a_2c_2+a_2c_3+a_3c_2)I_1, \\ &\quad (a_1c_3+a_3c_1+a_3c_3)I_2) \\ &= (a_1b_1+a_1c_1,(a_1b_2+a_2b_1+a_2b_2+a_2b_3+a_3b_2+a_1c_2+a_2c_1+a_2c_2+a_2c_3+a_3c_2)I_1, \\ &\quad (a_1b_3+a_3b_1+a_3b_1+a_3b_3+a_1c_3+a_3c_1+a_3c_3)I_2). \end{aligned}$$

These show that $x(y+z) = xy+xz$. Similarly, it can be shown that $(y+z)x = yx+zx$. Hence $(R(I_1,I_2),+,.)$ is a ring. □

Definition 2.6. Let $(R(I_1,I_2),+,.)$ be a refined neutrosophic ring and let $J(I_1,I_2)$ be a nonempty subset of $R(I_1,I_2)$. $J(I_1,I_2)$ is called a refined neutrosophic subring of $R(I_1,I_2)$ if $(J(I_1,I_2),+,.)$ is itself a refined neutrosophic ring.

It is essential that $J(I_1,I_2)$ contains a proper subset which is a ring. Otherwise, $J(I_1,I_2)$ will be called a pseudo refined neutrosophic subring of $R(I_1,I_2)$.

Example 2.7. Let $(R(I_1,I_2),+,.) = (\mathbb{Z}(I_1,I_2),+)$ be the refined neutrosophic ring of integers. The set $J(I_1,I_2) = n\mathbb{Z}(I_1,I_2))$ for all positive integer n is a refined neutrosophic subring of $R(I_1,I_2)$.

Example 2.8. Let $(R(I_1,I_2),+,.) = (\mathbb{Z}_6(I_1,I_2),+)$ be the refined neutrosophic ring of integers modulo 6. The set

$$\begin{aligned} J(I_1,I_2) &= \{(0,0,0),(0,I_1,0),(0,0,I_2),(0,I_1,I_2), \\ &\quad (0,2I_1,0),(0,0,2I_2),(0,2I_1,2I_2), \\ &\quad (0,3I_1,0),(0,0,3I_2),(0,3I_1,3I_2), \\ &\quad (0,4I_1,0),(0,0,4I_2),(0,4I_1,4I_2), \\ &\quad (0,5I_1,0),(0,0,5I_2),(0,5I_1,5I_2)\}. \end{aligned}$$

is a refined neutrosophic subring of $R(I_1,I_2)$.

Theorem 2.9. *Let $\{J_k(I_1,I_2)\}_1^n$ be a family of all refined neutrosophic subrings (pseudo refined neutrosophic subrings) of a refined neutrosophic ring $(R(I_1,I_2),+,.)$. Then $\bigcap_1^n J_k(I_1,I_2)\}$ is a refined neutrosophic subring (pseudo refined neutrosophic subring) of $R(I_1,I_2)$.*

Definition 2.10. Let $A(I_1, I_2)$ and $B(I_1, I_2)$ be any two refined neutrosophic subrings (pseudo refined neutrosophic subrings) of a refined neutrosophic ring $(R(I_1, I_2), +)$. We define the sum $A(I_1, I_2) \oplus B(I_1, I_2)$ by the set

$$A(I_1, I_2) \oplus B(I_1, I_2) = \{a + b : a \in A(I_1, I_2), b \in B(I_1, I_2)\} \tag{7}$$

which is a refined neutrosophic subring (pseudo refined neutrosophic subring) of $R(I_1, I_2)$

Theorem 2.11. *Let $A(I_1, I_2)$ be any refined neutrosophic subring of a refined neutrosophic ring $(R(I_1, I_2), +)$ and let $B(I_1, I_2)$ be any pseudo refined neutrosophic subring of $(R(I_1, I_2), +)$. Then:*

(i) $A(I_1, I_2) \oplus A(I_1, I_2) = A(I_1, I_2)$.

(ii) $B(I_1, I_2) \oplus B(I_1, I_2) = B(I_1, I_2)$.

(iii) $A(I_1, I_2) \oplus B(I_1, I_2)$ *is a refined neutrosophic subring of* $R(I_1, I_2)$.

Definition 2.12. Let R be a non-empty set and let $+$ and $.$ be two binary operations on R such that:

(i) $(R, +)$ is an abelian group.

(ii) $(R, .)$ is a semigroup.

(iii) There exists $x, y, z \in R$ such that

$$x(y + z) = xy + xz, (y + z)x = yx + zx.$$

(iv) R contains elements of the form (x, yI_1, zI_2) with $x, y, z \in R$ such that $y, z \neq 0$ for at least one value.

Then $(R, +, .)$ is called a pseudo refined n eutrosophic ring.

Example 2.13. Let R be a set given by

$$R = \{(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 6I_1, 0), (0, 0, 6I_2)\}.$$

Then $(R, +, .)$ is a pseudo refined n eutrosophic ring which is also a refined neutrosophic ring where $+$ and $.$ are addition and multiplication modulo 8.

Example 2.14. Let $R(I_1, I_2) = \mathbb{Z}_{12}(I_1, I_2)$ be a refined n eutrosophic ring of integers modulo 12 and let T be a subset of $\mathbb{Z}_{12}(I_1, I_2)$ given by

$$\begin{aligned} T \ = \ & \{(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 0), (0, 0, 4I_2), \\ & (0, 6I_1, 0), (0, 0, 6I_2)(0, 8I_1, 0), (0, 0, 8I_2), (0, 10I_1, 0), (0, 0, 10I_2)\}. \end{aligned}$$

It is clear that $(T, +, .)$ is a pseudo refined n eutrosophic ring.

Since $T \subset R(I_1, I_2)$, it follows that $T \cup R(I_1, I_2) \subseteq R(I_1, I_2)$ and consequently, $(T \cup R(I_1, I_2), +, .)$ is a refined n eutrosophic ring.

Theorem 2.15. *Let $(R(I_1, I_2), +, .)$ be any refined n eutrosophic ring and let $(T, +, .)$ be any pseudo refined neutrosophic ring. Then $(T \cup R(I_1, I_2), +, .)$ is a refined n eutrosophic ring if and only if $T \subset R(I_1, I_2)$.*

Theorem 2.16. *Let $(R(I_1, I_2), +, .)$ be any refined n eutrosophic ring and let $(T, +, .)$ be any pseudo refined neutrosophic ring. Then $(T \oplus R(I_1, I_2), +, .)$ is a refined neutrosophic ring.*

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Refined Neutrosophic Rings II

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E.O. Adeleke, A.A.A. Agboola, Florentin Smarandache (2020). Refined Neutrosophic Rings II. *International Journal of Neutrosophic Science* 2(2), 89-94. DOI: 10.5281/zenodo.3728235

Abstract

This paper is the continuation of the work started in the paper titled "Refined Neutrosophic Rings I". In the present paper, we study refined neutrosophic ideals and refined neutrosophic homomorphisms along their elementary properties. It is shown that if $R = \mathbb{Z}(I_1, I_2)$ is a refined neutrosophic ring of integers and $J = n\mathbb{Z}(I_1, I_2)$ is a refined neutrosophic ideal of R, then $R/J \cong \mathbb{Z}_n(I_1, I_2)$.

Keywords: Neutrosophy, refined neutrosophic ring, refined neutrosophic ideal, refined neutrosophic ring homomorphism.

1 Preliminaries

In this section, we only state some useful definitions, examples and results. For full details about refined neutrosophic rings, the readers should see.[9]

Definition 1.1 ([9])**.** Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ As called a refined neutrosophic ring Aenerated by R, I_1, I_2. $(R(I_1, I_2), +, .)$ As called a commutative refined neutrosophic ring Af for all $x, y \in R(I_1, I_2)$, we Aave $xy = yx$. Af there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ Aor all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ As a refined neutrosophic ring with unity.

Definition 1.2 ([9])**.** Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If for the least positive integer n such that $nx = 0$ for all $x \in R(I_1, I_2)$, we call $(R(I_1, I_2), +, .)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1, I_2), +, .)$.

(ii) $(R(I_1, I_2), +, .)$ is call a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1, I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 1.3 ([9])**.** (i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2),$ $(0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 1.4 ([9])**.** Let $M^{\mathbb{R}}_{n\times n}(I_1, I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1, I_2) \right\}$ be a refined neutrosophic set of all $n \times n$ matrix. Then $(M^{\mathbb{R}}_{n\times n}(I_1, I_2), +, .)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

Theorem 1.5 ([9]). *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring. Then $(R(I_1, I_2), +, .)$ is a ring.*

Definition 1.6 ([9]). Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $J(I_1, I_2)$ be a nonempty subset of $R(I_1, I_2)$. $J(I_1, I_2)$ is called a refined neutrosophic subring of $R(I_1, I_2)$ if $(J(I_1, I_2), +, .)$ is itself a refined neutrosophic ring. It is essential that $J(I_1, I_2)$ contains a proper subset which is a ring. Otherwise, $J(I_1, I_2)$ will be called a pseudo refined neutrosophic subring of $R(I_1, I_2)$.

Example 1.7 ([9]). Let $(R(I_1, I_2), +, .) = (\mathbb{Z}(I_1, I_2), +)$ be the refined neutrosophic ring of integers. The set $J(I_1, I_2) = n\mathbb{Z}(I_1, I_2))$ for all positive integer n is a refined neutrosophic subring of $R(I_1, I_2)$.

Example 1.8 ([9]). Let $(R(I_1, I_2), +, .) = (\mathbb{Z}_6(I_1, I_2), +)$ be the refined neutrosophic ring of integers modulo 6. The set

$$\begin{aligned} J(I_1, I_2) &= \{(0,0,0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), \\ &\quad (0, 2I_1, 0), (0, 0, 2I_2), (0, 2I_1, 2I_2), \\ &\quad (0, 3I_1, 0), (0, 0, 3I_2), (0, 3I_1, 3I_2), \\ &\quad (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 4I_2), \\ &\quad (0, 5I_1, 0), (0, 0, 5I_2), (0, 5I_1, 5I_2)\}. \end{aligned}$$

is a refined neutrosophic subring of $R(I_1, I_2)$.

Definition 1.9 ([9]). Let R be a non-empty set and let $+$ and . be two binary operations on R such that:

(i) $(R, +)$ is an abelian group.

(ii) $(R, .)$ is a semigroup.

(iii) There exists $x, y, z \in R$ such that

$$x(y + z) = xy + xz, (y + z)x = yx + zx.$$

(iv) R contains elements of the form (x, yI_1, zI_2) with $x, y, z \in \mathbb{R}$ such that $y, z \neq 0$ for at least one value.

Then $(R, +, .)$ is called a pseudo refined neutrosophic ring.

Example 1.10 ([9]). Let R be a set given by

$$R = \{(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 6I_1, 0), (0, 0, 6I_2)\}.$$

Then $(R, +, .)$ is a pseudo refined neutrosophic ring where $+$ and . are addition and multiplication modulo 8.

Example 1.11 ([9]). Let $R(I_1, I_2) = \mathbb{Z}_{12}(I_1, I_2)$ be a refined neutrosophic ring of integers modulo 12 and let T be a subset of $\mathbb{Z}_{12}(I_1, I_2)$ given by

$$\begin{aligned} T &= \{(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 0), (0, 0, 4I_2), \\ &\quad (0, 6I_1, 0), (0, 0, 6I_2)(0, 8I_1, 0), (0, 0, 8I_2), (0, 10I_1, 0), (0, 0, 10I_2)\}. \end{aligned}$$

It is clear that $(T, +, .)$ is a pseudo refined neutrosophic ring.

2 Main Results

In this section except if otherwise stated, all refined neutrosophic rings $R(I_1, I_2)$ will be assumed to be commutative refined neutrosophic rings with unity.

Definition 2.1. Let $R(I_1, I_2)$ be a refined neutrosophic ring.

(i) An element $x \in R(I_1, I_2)$ is called an idempotent element if $x^2 = x$.

(ii) A nonzero element $x \in R(I_1, I_2)$ is called a zero divisor if there exists a nonzero element $y \in R(I_1, I_2)$ such that $xy = 0$.

(ii) A nonzero element $x \in R(I_1, I_2)$ is said to be invertible if there exists an element $y \in R(I_1, I_2)$ such that $xy = 1$.

Example 2.2. Consider the refined neutrosophic rings $\mathbb{Z}_2(I_1, I_2)$ and $\mathbb{Z}_3(I_1, I_2)$ of integers modulo 2 and 3 respectively. The element $x = (1, I_1, I_2)$ is idempotent in $\mathbb{Z}_2(I_1, I_2)$ and the element $x = (1, 0, I_2)$ is invertible in $\mathbb{Z}_3(I_1, I_2)$. The elements $x = (0, I_1, 0)$ and $y = (1, I_1, 0)$ are zero divisors in $\mathbb{Z}_2(I_1, I_2)$ because $xy = (0, 0, 0)$.

Definition 2.3. Let $R(I_1, I_2)$ be a refined neutrosophic ring. Then $R(I_1, I_2)$ is called a refined neutrosophic integral domain if it has no zero divisors.

Theorem 2.4. *$\mathbb{Z}_n(I_1, I_2)$ is not a refined neutrosophic integral domain for all n.*

Proof. For nonzero integers α, β, let $x = (0, \alpha I_1, 0)$ and $y = (0, \beta(1 - I_1), 0)$ be arbitrary elements in $\mathbb{Z}_n(I_1, I_2)$. It is clear that x and y are zero divisors since $xy = (0, 0, 0)\ \forall\ \alpha, \beta \in \mathbb{Z}^+$ and therefore, $\mathbb{Z}_n(I_1, I_2)$ is not a refined neutrosophic integral domain for all n. $\square$

Corollary 2.5. *Let $R(I_1, I_2)$ be a refined neutrosophic ring where R is an integral domain. Then $R(I_1, I_2)$ is not necessarily a refined neutrosophic integral domain.*

Theorem 2.6. *If $R = \mathbb{Z}_n$ is a ring of integers modulo n, then $R(I_1, I_2)$ is a finite refined neutrosophic ring of order n^3.*

Definition 2.7. Let F be a field. A refined neutrosophic field is a set $F(I_1, I_2)$ generated by F, I_1, I_2 defined by

$$F(I_1, I_2) = \{(x, yI_1, zI_2) : x, y, z \in F\}.$$

Example 2.8.
(i) $\mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2)$ and $\mathbb{C}(I_1, I_2)$ of rational, real and complex numbers are examples of refined neutrosophic fields.

(ii) $\mathbb{Z}_p(I_1, I_2)$ for a prime p is a refined neutrosophic field.

It is worthy of noting that refined neutrosophic fields are not fields in the classical sense since not every element of refined neutrosophic fields is invertible.

Definition 2.9. Let $R(I_1, I_2)$ be a refined neutrosophic ring and let J be a nonempty subset of $R(I_1, I_2)$. Then J is called a refined neutrosophic ideal of $R(I_1, I_2)$ if the following conditions hold:

(i) J is a refined neutrosophic subring of $R(I_1, I_2)$.

(ii) For every $x \in J$ and $r \in R(I_1, I_2)$, we have $xr \in J$.

If J is a pseudo refined neutrosophic subring of $R(I_1, I_2)$, and, for every $x \in J$ and $r \in R(I_1, I_2)$, we have $xr \in J$, then J is called a pseudo refined neutrosophic ideal of $R(I_1, I_2)$.

Example 2.10. In the refined neutrosophic ring $\mathbb{Z}_4(I_1, I_2)$ of integers modulo 4, the set
$J = \{(0, 0, 0), (2, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (2, 2I_1, 2I_2)\}$ is a refined neutrosophic ideal.

Example 2.11. Consider

$$\begin{aligned}
\mathbb{Z}_3(I_1, I_2) \quad = \quad & \{(0,0,0), (1,0,0), (2,0,0), (0,0,I_2), (0,0,2I_2), (0,I_1,0), \\
& (0,I_1,I_2), (0,I_1,2I_2), (0,2I_2,0), (0,2I_1,I_1), (0,2I_1,2I_2), \\
& (1,0,I_2), (1,0,2I_2), (1,I_1,0), (1,I_1,I_2), (1,I_1,2I_2), (1,2I_2,0), \\
& (1,2I_1,I_1), (1,2I_1,2I_2), (2,0,I_2), (2,0,2I_2), (2,I_1,0), \\
& (2,I_1,I_2), (2,I_1,2I_2), (2,2I_2,0), (2,2I_1,I_1), (2,2I_1,2I_2)\}
\end{aligned}$$

the refined neutrosophic ring of integers modulo 3. The set

$$J = \{(0, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, 2I_1, 0), (0, 0, 2I_2)\}$$

is a pseudo refined neutrosophic ideal. Consider the set

$$K = \{(0, 0, 0), (2, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (2, 2I_1, 2I_2)\}.$$

It can easily be shown that K is not a refined neutrosophic ideal of $\mathbb{Z}_3(I_1, I_2)$ and J is the only pseudo refined neutrosophic ideal.

Theorem 2.12. *Let $\{J_k(I_1, I_2)\}_1^n$ be a family of refined neutrosophic ideals (pseudo refined neutrosophic ideals) of a refined neutrosophic ring $R(I_1, I_2)$. Then $\bigcap_1^n J_k(I_1, I_2)\}$ is a refined neutrosophic ideal (pseudo refined neutrosophic ideal) of $R(I_1, I_2)$.*

Definition 2.13. Let $J(I_1, I_2)$ and $K(I_1, I_2)$ be any two refined neutrosophic ideals (pseudo refined neutrosophic ideals) of a refined neutrosophic ring $R(I_1, I_2)$. We define the sum $J(I_1, I_2) \oplus K(I_1, I_2)$ by the set

$$J(I_1, I_2) \oplus K(I_1, I_2) = \{x + y : x \in J(I_1, I_2), y \in K(I_1, I_2)\}$$

which can easily be shown to be a refined neutrosophic ideal (pseudo refined neutrosophic ideal) of $R(I_1, I_2)$

Theorem 2.14. *Let $J(I_1, I_2)$ be any refined neutrosophic ideal of a refined neutrosophic ring $R(I_1, I_2)$ and let $K(I_1, I_2)$ be any pseudo refined neutrosophic ideal of $R(I_1, I_2)$. Then:*

(i) $J(I_1, I_2) \oplus J(I_1, I_2) = J(I_1, I_2)$.

(ii) $K(I_1, I_2) \oplus K(I_1, I_2) = K(I_1, I_2)$.

(iii) $J(I_1, I_2) \oplus K(I_1, I_2)$ *is a pseudo refined neutrosophic ideal of* $R(I_1, I_2)$.

(iv) $x + J = J \ \forall\, x \in J$.

Definition 2.15. Let J be a refined neutrosophic ideal of the refined neutrosophic ring $R(I_1, I_2)$. The set $R(I_1, I_2)/J$ is defined by

$$R(I_1, I_2)/J = \{r + J : r \in R(I_1, I_2)\}.$$

If $\bar{x} = r_1 + J$ and $\bar{y} = r_2 + J$ are two arbitrary elements of $R(I_1, I_2)/J$ and $\oplus, \odot$ are two binary operations on $R(I_1, I_2)/J$ defined by

$$\begin{aligned} \bar{x} \oplus \bar{y} &= (x + y) + J, \\ \bar{x} \odot \bar{y} &= (xy) + J. \end{aligned}$$

It can be shown that $(R(I_1, I_2)/J, \oplus, \odot)$ is a refined neutrosophic ring with the additive identity J. $(R(I_1, I_2)/J, \oplus, \odot)$ is called a refined quotient neutrosophic ring.

Example 2.16. (i) Let $R = \mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ring of integers and let $J = 2\mathbb{Z}(I_1, I_2)$. It is clear that J is a refined neutrosophic ideal of R. Now, R/J is obtained as follows:

$$\begin{aligned} R/J = \ &\{J, (1, 0, 0) + J, (0, I_1, 0) + J, (0, 0, I_2) + J, (0, I_1, I_2) + J, \\ &(1, I_1, 0) + J, (1, 0, I_2) + J, (1, I_1, I_2) + J\} \end{aligned}$$

which is a refined neutrosophic ring of order 8.

(ii) Let $S = \mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ring of integers and let $K = 3\mathbb{Z}(I_1, I_2)$. It is also clear that K is a refined neutrosophic ideal of S. Now, S/K is obtained as follows:

$$\begin{aligned} S/K = \ &\{K, (1, 0, 0) + K, (2, 0, 0) + K, (0, I_1, 0) + K, (0, 2I_1, 0) + K, (0, 0, I_2) + K, \\ &(0, 0, 2I_2) + K, (0, 2I_1, I_2) + K, (0, 2I_1, 2I_2) + K, (0, I_1, I_2) + K, (0, I_1, 2_2) + K, \\ &(1, 0, I_2) + K, (1, I_1, 0) + K, (1, I_1, I_2) + K, (1, 2I_1, 0) + K, (1, 0, 2I_2) + K, \\ &(1, 2I_1, 2I_2) + K, (1, 2I_1, I_2) + K, (1, I_1, 2_2) + K, (2, 0, I_2) + K, \\ &(2, 0, 2I_2) + K, (2, I_1, 0) + K, (2, I_1, I_2) + K, (2, I_1, 2I_2) + K, (2, 2I_1, 0) + K, \\ &(2, 2I_1, I_2) + K, (2, 2I_2, 2I_2) + K\} \end{aligned}$$

which is a refined neutrosophic ring of order 27.

These two examples lead to the following general result:

Theorem 2.17. *Let $R = \mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ring of integers and let $J = n\mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ideal of R. Then*

$$R/J \cong \mathbb{Z}_n(I_1, I_2).$$

Definition 2.18. Let $(R(I_1, I_2), +, .)$ and and $(S(I_1, I_2), +, .)$ be two refined neutrosophic rings. The mapping $\phi : (R(I_1, I_2), +, .) \to (S(I_1, I_2), +, .)$ is called a refined neutrosophic ring homomorphism if the following conditions hold:

(i) $\phi(x+y) = \phi(x) + \phi(y)$.

(ii) $\phi(x.y) = \phi(x).\phi(y)$.

(iii) $\phi(I_k) = I_k \;\; \forall x, y \in R(I_1, I_2)$ and $k = 1, 2$.

The image of ϕ denoted by $Im\phi$ is defined by the set

$$Im\phi = \{y \in S(I_1, I_2) : y = \phi(x) \text{ for some } x \in R(I_1, I_2)\}.$$

The kernel of ϕ denoted by $Ker\phi$ is defined by the set

$$Ker\phi = \{x \in R(I_1, I_2) : \phi(x) = (0, 0, 0)\}.$$

Epimorphism, monomorphism, isomorphism, endomorphism and automorphism of ϕ are similarly defined as in the classical cases.

Example 2.19. Let $R_1(I_1, I_2)$ and $R_2(I_1, I_2)$ be two refined neutrosophic rings. Let $\phi : R_1(I_1, I_2) \times R_2(I_1, I_2) \to R_1(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : R_1(I_1, I_2) \times R_2(I_1, I_2) \to R_2(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$ for all $(x, y) \in R_1(I_1, I_2) \times R_2(I_1, I_2)$. Then ϕ and ψ are refined neutrosophic ring homomorphisms.

Example 2.20. Let $\phi : \mathbb{Z}_2(I_1, I_2) \times \mathbb{Z}_2(I_1, I_2) \to \mathbb{Z}_2(I_1, I_2)$ be a refined neutrosophic ring homomorphism defined by $\phi(x, y) = x$ for all $x, y \in \mathbb{Z}_2(I_1, I_2)$. Then

(i)

$$\begin{aligned} Im\phi \;\; = \;\; &\{(0,0,0), (1,0,0), (0,I_1,0), (0,0,I_2), \\ &(0,I_1,I_2), (1,I_1,0), (1,0,I_2), (1,I_1,I_2)\} \end{aligned}$$

which is a refined neutrosophic subring.

(ii) Also,

$$\begin{aligned} Ker\phi \;\; = \;\; &\{((0,0,0),(0,0,0)), ((0,0,0),(1,0,0)), ((0,0,0),(0,I_1,0)), \\ &((0,0,0),(0,I_1,I_2)), ((0,0,0),(0,0,I_2)), ((0,0,0),(1,I_1,0)), \\ &((0,0,0),(1,0,I_2)), ((0,0,0),(1,I_1,I_2))\} \end{aligned}$$

which is just a subring, not a refined neutrosophic subring and equally not a refined neutrosophic ideal.

This example leads to the following general results:

Theorem 2.21. *Let $\phi : R_1(I_1, I_2) \to R_2(I_1, I_2)$ be a refined neutrosophic ring homomorphism. Then*

(i) $Im\phi$ is a refined neutrosophic subring $R_2(I_1, I_2)$.

(ii) $Ker\phi$ is a subring of R_1.

(iii) $Ker\phi$ is not a refined neutrosophic subring of R_1.

(iv) $Ker\phi$ is not a refined neutrosophic ideal of R_1.

Theorem 2.22. *Let $R = R(I_1, I_2)$ be a refined n eutrosophic r ings a nd l et $J = J(I_1, I_2)$ b e a refined neutrosophic ideal. Then the mapping $\phi : R \to R/J$ defined by $\phi(r) = r + J \;\forall\, r \in R$ is not a refined neutrosophic ring homomorphism.*

Proof. It is clear that $\phi(r+s) = (r+s)+J = (r+J)+(s+J) = \phi(r)+\phi(s)$ and $\phi(rs) = (rs)+J = (r+J)(s+J) = \phi(r)\phi(s)$. But then, $\phi(I_k) \neq I_k$ for $k = 1, 2$ and so, ϕ is not a refined n eutrosophic ring homomorphism. □

This is different from what is obtainable in the classical rings and consequently, classical isomorphism theorems cannot hold in refined neutrosophic rings.

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n-Refined Neutrosophic Rings

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Florentin Smarandache, Mohammad Abobala (2020). n-Refined Neutrosophic Rings. *International Journal of Neutrosophic Science* 3(2), 83-90. DOI:10.5281/zenodo.3828996

Abstract

The aim of this paper is to introduce the concept of n-refined neutrosophic ring as a generalization of refined neutrosophic ring. Also, wepresent concept of n-refined polynomial ring. We study some basic concepts related to these rings such as AH-subrings, AH-ideals, AH-factors, and AH-homomorphisms.

Keywords: n-Refined neutrosophic ring, AH-ideal, AHS-ideal, AH-homomorphism, n-Refined neutrosophic polynomial ring.

1.Introduction

Neutrosophy as a new branch of philosophy founded by F.Smarandache became a useful tool in algebraic studies.Many neutrosophic algebraic structures were defined and studied such as neutrosophic groups, neutrosophic rings, and neutrosophic vector spaces. (See [1,2,3,4,5,6]).Refined neutrosophic theory was introduced by Smarandache in 2013 when he extended the neutrosophic set / logic / probability to refined [n-valued] neutrosophic set / logic / probability respectively, i.e. the truth value T is refined/split into types of sub-truths such as (T_1, T_2, …), similarly indeterminacy I is refined/split into types of sub-indeterminacies (I_1, I_2, …) and the falsehood F is refined/split into sub-falsehood (F_1, F_2,..)[10]. In [9], Smarandache proposed a way to split the Indeterminacy element I into n sub-indeterminacies $I_1, I_2, \dots, I_n$. This idea is very interesting and helps to define new generalizations of refined neutrosophic algebraic structures.

For our purpose we define multiplication operation between indeterminacies $I_1, I_2, \dots, I_n$ as follows:

$I_m I_s = I_{\min(m,s)}$For examples if $n = 4$ we get

$I_4 I_2 = I_2, I_1 I_2 = I_1, I_2 I_3 = I_2$If $n = 6$ we get $I_2 I_4 = I_2, I_1 I_4 = I_1, I_4 I_5 = I_4$If $n = 2$ we get $I_1 I_2 = I_1$ (2-refined neutrosophic ring).

AH-subtructures were firstly defined in [1]. AH-ideal in a neutrosophic ring R(I) has the form P+QI, where P,Q are ideals in the ring R. We can understand these substructures as two sections, each one is ideal (in rings). These ideals are interesting since they have properties which are similar to classical ideals and they lead us to study the concept of AHS-homomorphisms which are ring homomorphisms but not neutrosophic homomorphisms. In this article we aim to define these ideals in n-refined neutrosophic rings too.

2. Preliminaries

Definition 2.1: [7]

Let (R,+,×) be a ring, $R(I) = \{a + bI : a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Remark 2.2: [4]

The element I can be split into two indeterminacies I_1, I_2 with conditions:

$I_1^2 = I_1, I_2^2 = I_2, I_1 I_2 = I_2 I_1 = I_1$.

Definition 2.3: [4]

If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2) : a, b, c \in X\}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 2.4: [4]

Let (R,+,×) be a ring, $(R(I_1, I_2)+,\times)$ is called a 2-refined neutrosophic ring generated by R ,I_1, I_2.

Theorem 2.5: [4]

Let $(R(I_1, I_2)+,\times)$ be a 2-refined neutrosophic ring then it is a ring.

In the following we remind the reader about some AH-substructures.

Definition 2.6: [2]

Let $(R(I_1, I_2)+,\cdot)$ be a refined neutrosophic ring and P_0, P_1, P_2 be ideals in the ring R then the set $P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2) : a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

Definition 2.7: [1]

Let R be a ring and R(I) be the related neutrosophic ring and

$P = P_0 + P_1 I = \{a_0 + a_1 I : a_0 \in P_0, a_1 \in P_1\}$; P_0, P_1 are two subsets of R.

(a) We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R.

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Definition 2.8: [2]

(a) Let $f: R(I_1, I_2) \to T(I_1, I_2)$ be an AHS-homomorphism we define AH-Kernel of f by : $AH - Kerf = \{(a, bI_1, cI_2) : a, b, c \in Kerf_R\} = (Kerf_R, Kerf_R I_1, Kerf_R I_2)$

(b) let $S=(S_0, S_1 I_1, S_2 I_2)$ be a subset of $R(I_1, I_2)$, then : $f(S) = (f_R(S_0), f_R(S_1)I_1, f_R(S_2)I_2) = \{(f_R(a_0), f_R(a_1)I_1, f_R(a_2)I_2) : a_i \in S_i\}$.

(c) let $S=(S_0, S_1 I_1, S_2 I_2)$ be a subset of $T(I_1, I_2)$. Then

$f^{-1}(S) = (f_T^{-1}(S_0), f_T^{-1}(S_1)I_1, f_T^{-1}(S_2)I_2)$.

Definition 2.9: [2]

Let f:R $(I_1, I_2) \to$ T(I_1, I_2) be an AHS-homomorphism we say that f is an AHS-isomorphism if it is a bijective map and R(I_1, I_2), T (I_1, I_2) are called AHS-isomorphic refined neutrosophic rings.

It is easy to see that f_R will be an isomorphism between R , T .

Theorem 2.10 :

Let f:R $(I_1, I_2) \to$ T(I_1, I_2) be an AHS-homomorphism then we have :

(a) AH-Kerf is an AHS-ideal of R(I_1, I_2).

(b) If P is a refined neutrosophic AH-ideal of R(I_1, I_2) , f(P) is a refined neutrosophic AH-ideal of T(I_1, I_2).

(c) If P is a refined neutrosophic AHS-ideal of R(I_1, I_2) , f(P) is a refined neutrosophic AHS-ideal of T(I_1, I_2).

3. n-Refined neutrosophic rings

Definition 1.3:

Let (R,+,×) be a ring and I_k;$1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring. If n=2 we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$.

Additionand multiplication on R_n(I)are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i$, $\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R.

It is easy to see that R_n(I) is a ring in the classical concept and contains a proper ring R.

Definition 2.3:

Let R_n(I) be an n-refined neutrosophic ring, it is said to be commutative if $xy = yx$ for each x , y ∈ R_n(I), if there is 1∈ R_n(I) such $1x = x1 = x$, then it is called an n-refined neutrosophic ring with unity.

Theorem 3.3:

Let R_n(I) be an n-refinedneutrosophic ring. Then

(a) R is commutative if and only if R_n(I) is commutative,

(b) R has unity if and only if R_n(I) has unity,

(c) R_n(I) = $\sum_{i=0}^{n} R I_i$ = {$\sum_{i=0}^{n} x_i I_i$; $x_i \in R$}.

Proof:

(a) Holds directly from the definition of multiplication on R_n(I).

(b) If 1 is a unity of R then for each $a_0 + a_1 I + \cdots + a_n I^n \in R_n$(I) we have

1·$(a_0 + a_1 I + \cdots + a_n I_n) = (a_0 + a_1 I + \cdots + a_n I_n)$·1=$a_0 + a_1 I + \cdots + a_n I_n$ so 1 is the unity of R_n(I).

(c) It is obvious that $\sum_{i=0}^{n} RI_i \leq R(I)$. Conversely assume that $a_0 + a_1 I + \cdots + a_n I_n \in R_n$(I) then by the definition we have that $a_0 + a_1 I + \cdots + a_n I_n \in \sum_{i=0}^{n} RI_i$.Thus the proof is complete.

Definition 4.3:

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n; a_i \in P_i\}$ where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i. AHS-subring is defined by the condition $P_i = P_j$ for all i, j.

(b)P is an AH-ideal if P_i is an two sides ideal of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i, j.

(c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i.

Theorem 5.3:

Let R_n(I) be an n-refined neutrosophic ring and P is an AH-ideal,(P,+) is an abelian neutrosophic group with $k \leq n$ and r.p$\in P$ for all p $\in P$ and r $\in R$.

Proof :

Since P_i is abelian subgroup of $(R, +)$ and $rx \in P_i$ for all $r \in R, x \in P_i$, the proof holds.

Remark 6.3:

We can define the right AH-ideal by the condition that P_i is a right ideal of R, the left AH-ideal can be defined as the same.

Definition 7.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$, Q = $\sum_{i=0}^{n} Q_i I_i$ be two AH-ideals then we define:

P+Q = $\sum_{i=0}^{n}(P_i + Q_i)I_i$, P∩Q = $\sum_{i=0}^{n}(P_i \cap Q_i)I_i$.

Theorem 8.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$, Q = $\sum_{i=0}^{n} Q_i I_i$ be two AH- ideals then P+Q, P∩Q are AH-ideals. If P, Q are AHS-ideals then P+Q, P∩Q are AHS-ideals.

Proof :

Since $P_i + Q_i, P_i \cap Q_i$ are ideals of R then P+Q, P∩Q are AH-ideals of R_n(I).

Definition 9.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ be an AH-ideal then the AH- radical of P can be defined as $H - rad(P) = \sum_{i=0}^{n}(\sqrt{P_i})I_i$

Theorem 10.3:

The AH-radical of an AH-ideal is an AH-ideal.

Proof :

Since $\sqrt{P_i}$ is an ideal of R then $AH - Rad(P)$ is an AH-ideal of R_n(I).

Definition 11.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ be an AH-ideal, we define AH-factor R(I)/P = $\sum_{i=0}^{n} (R/P_i) I_i = \sum_{i=0}^{n} (x_i + P_i) I_i ; x_i \in R$.

Theorem 12.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ be an AH-ideal:

R_n(I)/P is aring with the following two binary operations

$\sum_{i=0}^{n}(x_i + P_i)I_i + \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i + y_i + P_i)I_i$,

$\sum_{i=0}^{n}(x_i + P_i)I_i \times \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i \times y_i + P_i)I_i$.

Proof :

Proof is similar tothat of Theorem 3.9 in [1].

Definition 13.3:

(a) Let R_n(I), T_n(I) be two n-refined neutrosophic rings respectively, and $f_R : R \to T$ be a ring homomorphism. We define n-refined neutrosophic AHS-homomorphism as :

$f : R_n$(I)→ T_n(I); $f(\sum_{i=0}^{n} x_i I_i) = \sum_{i=0}^{n} f_R(x_i) I_i$.

(b) f is an n-refined neutrosophic AHS-isomorphism if it is a bijective n-refined neutrosophic AHS-homomorphism.

(c) AH-Ker f = $\sum_{i=0}^{n} Ker(f_R) I_i$= $\{\sum_{i=0}^{n} x_i I_i ; x_i \in Ker f_R\}$.

Theorem 14.3:

Let R_n(I), T_n(I) be two n-refined neutrosophic rings respectively and f be an n-refined neutrosophic AHS-homomorphism f: R_n(I)→ T_n(I). Then

(a) If P = $\sum_{i=0}^{n} P_i I_i$ is an AH- subring of R_n(I) then f(P) is an AH- subring of T_n(I),

(b) If P = $\sum_{i=0}^{n} P_i I_i$ is an AHS- subring of R_n(I) then f(P) is an AHS- subring of T_n(I),

(c) If P = $\sum_{i=0}^{n} P_i I_i$ is an AH-ideal of R_n(I) then f(P) is an AH-ideal of f(R_n(I)),

(d)P = $\sum_{i=0}^{n} P_i I_i$ is an AHS-ideal of R_n(I) then f(P) is an AHS-ideal of f(R_n(I)),

(e) $R_n(I) / AH - Ker(f)$ is $AHS - isomorphic$ to $f(R_n(I))$,

(f) Inverse image of an AH-ideal P in T_n(I) is an AH-ideal in R(I).

Proof :

(a) Since $f(P_i)$ is a subring of T then f(P) is an AH- subring of T_n(I).

(b) Holds by a similar way to (a).

(c) Since $f(P_i)$ is an ideal of f(R) then f(P) is an AH- ideal of f(R(I)).

(d) It is similar to (c).

(e) We have $R/Ker(f_R) \cong f(R)$, by definition of AH-factor and $AH - Ker(f)$ we find that $R(I)/P \cong f(R(I))$.

(f) It is similar to the classical case.

Definition15.3:

(a) Let R(I) be a commutative n-refined neutrosophic ring, and P = $\sum_{i=0}^{n} P_i I_i$ be an AH- ideal, we define P to be a weak prime AH-ideal if P_i is a prime ideal of R for all i

(b) P is called a weak maximal AH-ideal if P_i is a maximal ideal of R for all i

(c) P is called a weak principal AH-ideal if P_i is a principal ideal of R for all i

Theorem16.3:

Let R_n(I), T_n(I) be two commutative n-refined neutrosophic rings with an n-refined neutrosophic AHS-homomorphism $f: R_n(I) \to T_n(I)$:

(a) If P = $\sum_{i=0}^{n} P_i I_i$ is an AHS- ideal of R_n(I) and Ker(f_R)$\leq P_i \neq R_n$(I):

(a) P is a weak prime AHS-ideal if and only if f(P) is a weak prime AHS-ideal in f(R_n(I)).

(b) P is a weak maximal AHS-ideal if and only if f(P) is a weak maximal AHS-ideal in f(R_n(I)).

(c) If Q = $\sum_{i=0}^{n} Q_i I_i$ is an AHS-ideal of T_n(I)then it is a weak prime AHS-ideal if and only if $f^{-1}(Q)$ is a weak prime inR_n(I).

(d)if Q = $\sum_{i=0}^{n} Q_i I_i$ is an AHS-ideal of T_n(I) then it is a weak maximal AHS-ideal if and only if $f^{-1}(Q)$ is a weak maximal in R_n(I).

Proof :

Proof is similar to that of Theorem 3.8 in [1].

Example 17.3:

Let $R = Z$ be the ring of integers, $T = Z_6$ be thering of integers modulo 6 with multiplication and addition modulo 6, we have:

(a)$f_R: R \to T; f(x) = x mod 6$ is a ring homomorphism, $ker(f_R) = 6Z$, the corresponding AHS-homomorphism between R_4(I), T_4(I) is:

$f: R_4(I) \to T_4(I); f(a + bI_1 + cI_2 + dI_3 + eI_4) = (a mod 6) + (b mod 6)I_1 + (c mod 6)I_2 + (d mod 6)I_3 + (e mod 6) I_4$; $a, b, c, d, e \in Z$.

(b) $P = <2>, Q = <3>$ are two prime and maximal and principal ideals in R,

M= $P + PI_1 + QI_2 + QI_3 + PI_4 = \{2a + 2bI_1 + 3cI_2 + 3dI_3 + 2eI_4; a, b, c, d, e \in Z\}$ is a weak prime/ maximal AH-ideal of R_4(I).

(c) $Ker(f_R) = 6Z \leq P, Q, f_R(P) = \{0,2,4\}, f_R(Q) = \{0,3\}$,

$f(M) = f(P) + f(P)I_1 + f(Q)I_2 + f(Q)I_3 + f(P)I_4$ which is a weak maximal/ prime/principal AH-ideal of T_4(I).

(d) $AH - Ker(f) = 6Z + 6ZI_1 + 6ZI_2 + 6ZI_3 + 6ZI_4$ which is an AHS-ideal of $R_4(I)$.

(e) $R_4(I)/AH - Kerf = R/6Z + R/6ZI_1 + R/6ZI_2 + R/6ZI_3 + R/6ZI_4$ which is AHS-isomorphic to $f(R_4(I)) = T_4(I)$, since $R/6Z \cong T$.

Example 18.3:

Let $R = Z_8$ be a ring with addition and multiplication modulo 8.

(a) 3-refined neutrosophic ring related with R is Z_{8_3}(I)={a+$bI_1 + cI_2 + dI_3; a, b, c, d \in Z_8$}

(b) P={0,4} is an ideal of R, $\sqrt{P}$={0,2,4,6}, $M = P + PI_1 + PI_2 + PI_3$ is an AHS-ideal of Z_{8_3}(I),

$AH - Rad(M) = \sqrt{P} + \sqrt{P}I_1 + \sqrt{P}I_2 + \sqrt{P}I_3$ which is an AHS-ideal of Z_{8_3}(I).

Example 19.3:

Let R=Z_2 the ring of integers modulo 2, let $n = 3$. The corresponding 3-refined neutrosophic ring is

Z_{2_3}(I)={$0, 1, I_1, I_2, I_3, 1 + I_1, 1 + I_2, 1 + I_3, I_1 + I_2, I_1 + I_3, I_1 + I_2 + I_3, I_2 + I_3, 1 + I_1 + I_2 + I_3, 1 + I_2 + I_3, 1 + I_1 + I_3, 1 + I_1 + I_2$}.

4. n-Refined neutrosophic polynomial rings

Definition1.4:

Let R_n(I) be a commutative n-refined neutrosophic ring and $P: R_n(I) \to R_n$(I) is a function defined as $P(x) = \sum_{i=0}^{m} a_i x^i$ such $a_i \in R_n$(I), we call P a neutrosophic polynomial on R_n(I).

We denote by R_n(I)[x] to the ring of neutrosophic polynomials over R_n(I).

Since R_n(I) is a classical ring then R_n(I)[x] is a classical ring.

Theorem 2.4:

Let R(I) be a commutative n-refined neutrosophic ring. Then R_n(I)[x] = $\sum_{i=0}^{n}$ R$[x]I_i$.

Proof :

Let P(x) = $\sum_{i=0}^{n} P_i(x) I^i \in \sum_{i=0}^{n}$ R$[x]I^i$,by rearranging the previous sum we can write it as P(x) = $\sum_{i=0}^{m} a_i x^i \in R_n(I)[x]$.

Conversely, if P(x) = $\sum_{i=0}^{n} a_i x^i \in R_n(I)[x]$, then we can write it as

P(x) = $\sum_{i=0}^{n} P_i(x) I_i \in \sum_{i=0}^{n} R[x]I_i$, by the previous argument we find the proof.

Example 3.4:

Let $Z_3(I)$ be a 3-refined neutrosophic ring and P(x) = I_1 + (2+I_1) x + (I_1+I_3)x^2 a polynomial over $Z_{3_n}(I)$, then we can write P(x) = 2x + I_1 (1 + x + x^2) + I_2x^2.

It is obvious that R_n(I)≤ $R_n(I)[x]$.

Definition4.4:

Let P(x) = $\sum_{i=0}^{n} P_i(x)I^{\ i}$ a neutrosophic polynomial over R_n(I) we define the degree of P by deg P = max(deg P_i).

5. Conclusion

In this paper we have defined the n-refined neutrosophic ring and n-refined neutrosophic polynomial ring, we have introduced and studied AH-structures such as:

AH-ideal, AHS-ideal, AH-weak principal ideal, AH-weak prime ideal. Authors hope that other n-refined neutrosophic algebraic structures will be defined in future research.

Funding: “This research received no external funding”

Conflicts of Interest: “The authors declare no conflict of interest.”

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On Neutro-BE-algebras and Anti-BE-algebras (revisited)

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Akbar Rezaei, Florentin Smarandache (2020). On Neutro-BE-algebras and Anti-BE-algebras (revisited). *International Journal of Neutrosophic Science* 4(1), 8-15.
DOI: 10.5281/zenodo.3751862

Abstract

In this paper, the concepts of Neutro-BE-algebra and Anti-BE-algebra are introduced, and some related properties and four theorems are investigated. We show that the classes of Neutro-BE-algebra and Anti-BE-algebras are alternatives of the class of BE-algebras.

Keywords: BE-algebra; Neutro-sophication; Neutro-BE-algebra; Anti-sophication; Anti-BE-algebra.

1. Introduction

Neutrosophy, introduced by F. Smarandache in 1998, is a new branch of philosophy that generalized the dialectics and took into consideration not only the dynamics of opposites, but the dynamics of opposites and their neutrals [8]. Neutrosophic Logic / Set / Probability / Statistics / Measure / Algebraic Structures etc. are all based on it. One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, vague set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic vague set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been a very important tool in all various areas of data mining, decision making, e-learning, engineering, computer science, graph theory, medical diagnosis, probability theory, topology, social science, etc [9-13].

A classical Algebra may be transformed into a NeutroAlgebra by a process called neutro-sophication, and into an AntiAlgebra by a process called anti-sophication.

In [2], H.S. Kim et al. introduced the notion of a BE-algebra as a generalization of a BCK-algebra. S.S. Ahn et al. introduced the notion of ideals in BE-algebras, and they stated and proved several properties of such ideals [1]. A. Borumand Saeid et al defined some filters in BE-algebras and investigated relation between them [3]. A. Rezaei et al. investigated the relationship between Hilbert algebras and BE-algebras and showed that commutative self-distributive BE-algebras and Hilbert algebras are equivalent [4]. In this paper, the concepts of a Neutro-BE-algebra and Anti-BE-algebra are introduced, and some related properties are investigated. We show that the class of Neutro-BE-algebra is an alternative of the class of BE-algebras.

2. NeutroLaw, NeutroOperation, NeutroAxiom, and NeutroAlgebra

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

The Neutrosophy's Triplet is (<A>, <neutroA>, <antiA>), where <A> may be an item (concept, idea, proposition, theory, structure, algebra, etc.), <antiA> the opposite of <A>, while <neutroA> {also the notation <neutA> was employed before} the neutral between these opposites. Based on the above triplet the following Neutrosophic Principle one has: a law of composition defined on a given set may be true (T) for some set elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroLaw. A law of composition defined on a given sets, such that the law is false (F) for all set's elements is called AntiLaw. Similarly, an operation defined on a given set may be well-defined for some set elements, indeterminate for other set's elements, and undefined for the remainder of the set's elements; we call it NeutroOperation. While, an operation defined on a given set that is undefined for all set's elements is called AntiOperation.

In classical algebraic structures, the laws of compositions or operations defined on a given set are automatically well-defined [i.e. true (T) for all set's elements], but this is idealistic. Consequently, an axiom (let's say Commutativity, or Associativity, etc.) defined on a given set, may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroAxiom. In classical algebraic structures, similarly an axiom defined on a given set is automatically true (T) for all set's elements, but this is idealistic too. A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or some NeutroAxiom. The NeutroLaw, NeutroOperation, NeutroAxiom, NeutroAlgebra and respectively AntiLaw, AntiOperation, AntiAxiom and AntiAlgebra were introduced by Smarandache in 2019 [6] and afterwards he recalled, improved and extended them in 2020 [7]. Recently, the concept of a Neutrosophic Triplet of BI-algebra was defined [5].

3. Neutro-BE-algebras, Anti-BE-Algebras

Definition 3.1. (Definition of classical BE-algebras [1])
An algebra $(X,*,0)$ of type $(2,0)$ (i.e. X is a nonempty set, $*$ is a binary operation and 0 is a constant element of X) is said to be a *BE-algebra* if:

(L) **The law $*$ is well-defined**, i.e. $(\forall x, y \in X)(x * y \in X)$.

And the following axioms are totally true on X:

$(BE1)$ $(\forall x \in X)(x * x = 0)$,

$(BE2)$ $(\forall x \in X)(0 * x = x)$,

$(BE3)$ $(\forall x \in X)(x * 0 = 0)$,

$(BE4)$ $(\forall x, y, z \in X, with\ x \neq y)(x * (y * z) = y * (x * z))$.

Example 3.2.

(i) Let $\mathbb{N}$ be the set of all natural numbers and $*$ be the binary operation on $\mathbb{N}$ defined by

$$x * y = \begin{cases} y & if\ x = 1; \\ 1 & if\ x \neq 1. \end{cases}$$

Then $(\mathbb{N},*,1)$ is a BE-algebra.

(ii) Let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and let $*$ be the binary operation on $\mathbb{N}_0$ defined by

$$x * y = \begin{cases} 0 & if\ x \geq y; \\ y - x & otherwise. \end{cases}$$

Then $(\mathbb{N}_0, *, 0)$ is a BE-algebra.

Definition 3.3. (Neutro-sophications)

The Neutro-sophication of the Law (degree of well-defined, degree of indeterminacy, degree of outer-defined)

(NL) $(\exists x, y \in X)(x * y \in X)$ and $(\exists x, y \in X)(x * y =\ indeterminate$ or $x * y \notin X)$,

The Neutro-sophication of the Axioms (degree of truth, degree of indeterminacy, degree of falsehood)

$(NBE1)$ $(\exists x \in X)(x * x = 0)$ and $(\exists x \in X)(x * x = indeterminate$ or $x * x \neq 0)$,

$(NBE2)$ $(\exists x \in X)(0 * x = x)$ and $(\exists x \in X)(0 * x = indeterminate$ or $0 * x \neq x)$,

$(NBE3)$ $(\exists x \in X)(x * 0 = 0)$ and $(\exists x \in X)(x * 0 =\ indeterminate$ or $x * 0 \neq 0)$,

$(NBE4)$ $(\exists x, y, z \in X, with\ x \neq y)(x * (y * z) = y * (x * z))$ and

$(\exists x, y, z \in X, with\ x \neq y)\big(x * (y * z) = indeterminate$ or $x * (y * z) \neq y * (x * z)\big)$.

Definition 3.4. (Anti-sophications)

The Anti-sophication of the Law (totally outer-defined)

(AL) $(\forall x, y \in X)(x * y \notin X)$.

The Anti-sophication of the Axioms (totally false)

$(ABE1)$ $(\forall x \in X)(x * x \neq 0)$,

$(ABE2)$ $(\forall x \in X)(0 * x \neq x)$,

$(ABE3)$ $(\forall x \in X)(x * 0 \neq 0)$,

$(ABE4)$ $(\forall x, y, z \in X$, with $x \neq y)(x * (y * z) \neq y * (x * z))$.

Definition 3.5. (Neutro-BE-algebras)

A Neutro-BE-algebra is an alternative of BE-algebra that has at least a (NL) or at least one $(NBEi)$, $i \in \{1, 2, 3, 4\}$, with no anti-law and no anti-axiom.

Example 3.6.

(i) Let $\mathbb{N}$ be the set of all natural numbers and $*$ be the Neutro-sophication of the Law $*$ on $\mathbb{N}$ from Example 2.2. (i) defined by

$$x * y = \begin{cases} y & if\ x = 1; \\ \frac{1}{2} & if\ x \in \{3,5,7\}; \\ 1 & otherwise. \end{cases}$$

Then $(\mathbb{N},*,1)$ is a Neutro-BE-algebra. Since

(NL) if $x \in \{3,5,7\}$, then $x * y = \frac{1}{2} \notin \mathbb{N}$, for all $y \in \mathbb{N}$, while if $x \notin \{3,5,7\}$ and $x \in \mathbb{N}$, then $x * y \in \{1, y\} \subseteq \mathbb{N}$, for all $y \in \mathbb{N}$.

(NBE1) $1 * 1 = 1 \in \mathbb{N}$ and $3 * 3 = \frac{1}{2} \notin \mathbb{N}$,

(BE2) holds always since $1 * x = x$, for all $x \in \mathbb{N}$.

(NBE3) $5 * 1 = \frac{1}{2} \neq 1$ and if $x \notin \{3,5,7\}$, then $x * 1 = 1$,

(NBE4) $5 * (3 * 4) = 5 * \frac{1}{2} = ?\,(indeterminate)$ and $3 * (5 * 4) = 3 * \frac{1}{2} = ?\,(indeterminate)$

Also, $2 * (3 * 4) = 2 * \frac{1}{2} = \ ?\,(indeterminate)$, but $3 * (2 * 4) = 3 * 1 = \frac{1}{2}$.

Further, $4 * (8 * 2) = 4 * 1 = 1 = 8 * (4 * 2)$.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S), \cap, \emptyset)$ is a Neutro-BE-algebra.

$\cap$ is the binary set intersection operation, but

(NBE1) is valid, since $\emptyset \cap \emptyset = \emptyset$ and for all $\emptyset \neq A \in \mathcal{P}(S)$, $A \cap A = A \neq \emptyset$.

(NBE2) $\emptyset \cap \emptyset = \emptyset$ and if $\emptyset \neq A$, then $\emptyset \cap A = \emptyset \neq A$,

(BE3) holds, since $A \cap \emptyset = \emptyset$,

(BE4) holds, since $A \cap (B \cap C) = B \cap (A \cap C)$.

(iii) Similarly, $(\mathcal{P}(S), \cup, \emptyset)$, $(\mathcal{P}(S), \cap, S)$, $(\mathcal{P}(S), \cup, S)$, where $\cup$ is the binary set union operation, are Neutro-BE-algebras.

(iv) Let $X := \{0, a, b, c, d\}$ be a set with the following table.

Table 1

*	0	a	b	c	d
0	c	a	b	c	a
a	b	0	b	c	d
b	0	a	0	c	c
c	?	0	b	0	b
d	0	0	0	0	?

Then $(X,*,0)$ is a Neutro-BE-algebra.

(NL) $c * 0 = ?\,(indeterminate)$, and $d * d = ?\,(indeterminate)$, and for all $x, y \in \{0, a, b\}$, then $x * y \in X$.

(NBE1) $a * a = 0$ and $0 * 0 = c \neq 0$ or $d * d = ?\,(indeterminate)$.

(NBE2) holds since $0 * b = b$, and $0 * d = a \neq d$.

(NBE3) $c * 0 = ?\,(indeterminate) \neq 0$ and if $x \in \{b, d\}$, then $x * 0 = 0$,

(NBE4) $d * (c * b) = d * b = 0 \neq c * (d * b) = c * 0 = ?\,(indeterminate)$ and

$a * (b * c) = a * c = c = b * (a * c)$.

(v) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S), -, \emptyset)$ is an Anti-BE-algebra, where $-$ is the binary operation of set subtraction, because:

(BE1) is valid, since $A - A = \emptyset$,

(NBE2) holds, since $\emptyset - A = \emptyset \neq A$ and $\emptyset - \emptyset = \emptyset$,

(NBE3) holds, since $A - \emptyset = A \neq \emptyset$ and $\emptyset - \emptyset = \emptyset$

(ABE4) is valid, since for A ≠ B, one has $A - (B - C) \neq B - (A - C)$, because:

x ∈ $A - (B - C)$ means (x ∈ A and x ∉ B-C), or {x ∈ A and (x ∉ B or x ∈ C) }, or {(x ∈ A and x ∉ B) or (x ∈ A and x ∈ C)}; while x ∈ $B - (A - C)$ means {(x ∈ B and x ∉ A) or (x ∈ B and x ∈ C)}.

(vi) Let $\mathbb{R}$ be the set of all real numbers and $*$ be a binary operation on $\mathbb{R}$ defined by $x * y = |x - y|$. Then ($\mathbb{R}$,* ,0) is a Neutro-BE-algebra.

(BE1) holds, since $x * x = |x - x| = 0$, for all $x \in \mathbb{R}$.

(NBE2) is valid, since if $x \geq 0$, then $x * 0 = |x - 0| = |x| = x$, and if $x < 0$, then $x * 0 = |x - 0| = |x| = -x \neq x$.

(NBE3) is valid, since if $x \neq 0$, then $0 * x = |0 - x| = |-x| \neq 0$, and if $x = 0$, then $0 * 0 = 0$.

(NBE4) holds, if x = 2, y = 3, z = 4 we get |2-|3-4|| = |2 – 1| = 1 and |3-|2-4|| = |3-2| = 1;

while for x = 4, y =8, z =3 we get |4 -|8-3|| = |4-5| = 1 and |8-|4-3|| = |8-1| = 7 ≠ 1.

Theorem 3.7.

The total number of Neutro-BE-algebras is 31.

Proof.

The classical BE-algebra has: 1 classical Law and 4 classical Axioms:

$1 + 4 = 5$ classical mathematical propositions.

Let C_n^m mean combinations of n elements taken by m, where n, m are positive integers, n ≥ m ≥ 0.

We transform (neutro-sophicate) the classical BE-algebra, by neutro-sophicating some of the 5 classical mathematical propositions, while the others remain classical (unchanged) mathematical propositions:

either only 1 of the 5 classical mathematical propositions (hence we have $C_5^1 = 5$ possibilities) – so 4 classical mathematical propositions remain unchanged,

or only 2 of the 5 classical mathematical propositions (hence we have $C_5^2 = 10$ possibilities) – so 3 classical mathematical propositions remain unchanged, or only 3 of the 5 classical mathematical propositions (hence we have $C_5^3 = 10$ possibilities) – so 2 classical mathematical propositions remain unchanged,

or only 4 of the 5 classical mathematical propositions (hence we have $C_5^4 = 5$ possibilities) – so 1 classical mathematical proposition remainsnchanged,

or all 5 of the 5 classical mathematical propositions (hence we have $C_5^1 = 1$ possibilities).

Whence the total number of possibilities will be:

$$C_5^1 + C_5^2 + C_5^3 + C_5^4 + C_5^5 = (1+1)^5 - C_5^0 = 2^5 - 1 = 31.$$

Definition 3.8. (Anti-$\boldsymbol{BE}$-algebras)

An Anti-BE-algebra is an alternative of BE-algebra that has at least an (AL) or at least one $(ABEi), i \in \{1,2,3,4\}$.

Example 3.9.

(i) Let $\mathbb{N}$ be the natural number set and $X := \mathbb{N} \cup \{0\}$. Define a binary operation $*$ on X by $x *_A y = x^2 + y^2 + 1$. Then $(X,*,0)$ is not a BE-algebra, nor a Neutro-BE-algebra, but an Anti-BE-algebra.

Since $x *_A x = x^2 + x^2 + 1 \neq 0$, for all $x \in X$, and so $(ABE1)$ holds.

For all $x \in \mathbb{N}$, we have $x * 0 = x^2 + 1 \neq 0$, so $(ABE2)$ is valid. By a similar argument $(ABE3)$ is valid.

Since for $x \neq y$, one has $x *_A (y *_A z) = x^2 + (y^2 + z^2 + 1)^2 + 1 \neq y *_A (x *_A z) = y^2 + (x^2 + z^2 + 1)^2 + 1$,

thus $(ABE4)$ is valid.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Define the binary operation Δ (i.e. symmetric difference) by $A\Delta B = (A \cup B) - (A \cap B)$ for every $A, B \in \mathcal{P}(S)$. Then $(\mathcal{P}(S), \Delta, S)$ is not a BE-algebra, nor Neutro-BE-algebra, but it is an Anti-BE-algebra.

Since $A\Delta A = \emptyset \neq S$ for every $A \in \mathcal{P}(S)$ we get $(ABE1)$ holds, and so $(BE1)$ and $(NBE1)$ are not valid.

Also, for all $A, B, C \in \mathcal{P}(S)$ one has $A\Delta(B\Delta C) = B\Delta(A\Delta C)$. Thus, $(BE4)$ is valid.

Since there is at least one anti-axiom (ABE1), then $(\mathcal{P}(S), \Delta, S)$ is an Anti-BE-algebra.

(iii) Let $\mathcal{U} = \{0, a, b, c, d\}$ be a universe of discourse, and a subset $S = \{0, c\}$, and the below binary well-defined Law * with the following Cayley table.

Table 2

*	0	c
0	c	0
c	c	c

Then $(S,*,0)$ is an Anti-BE-algebra, since (ABE1) is valid, because: 0*0 = c ≠ 0 and c*c = c ≠ 0, and it is sufficient to have a single anti-axiom.

Theorem 3.10.

The total number of Anti-BE-algebras is 211.

Proof.

The classical BE-algebra has: 1 classical Law and 4 classical Axioms:

$1 + 4 = 5$ classical mathematical propositions.

Let C_n^m mean combinations of n elements taken by m, where n, m are positive integers, n ≥ m ≥ 0.

We transform (anti-sophicate) the classical BE-algebra, by anti-sophicating some of the 5 classical mathematical propositions, while the others remain classical (unchanged) or neutro-mathematical propositions:

either only 1 of the 5 classical mathematical propositions (hence we have $C_5^1 = 5$ subpossibilities) – so 4 classical mathematical propositions remain some unchanged others neutro-sophicated or $2^4 = 16$ subpossibilities; hence total number of possibilities in this case is: $5 \cdot 16 = 80$;

or 2 of the 5 classical mathematical propositions (hence we have $C_5^2 = 10$ subpossibilities) – so 3 classical mathematical propositions remain some unchanged other neutro-sophicated or $2^3 = 8$ subpossibilities; hence total number of possibilities in this case is: $10 \cdot 8 = 80$;

or 3 of the 5 classical mathematical propositions (hence we have $C_5^3 = 10$ subpossibilities) – so 2 classical mathematical propositions remain some unchanged other neutro-sophicated or $2^2 = 4$ subpossibilities; hence total number of possibilities in this case is: $10 \cdot 4 = 40$;

or 4 of the 5 classical mathematical propositions (hence we have $C_5^4 = 5$ subpossibilities) – so 1 classical mathematical propositions remain either unchanged other neutro-sophicated or $2^1 = 2$ subpossibilities; hence total number of possibilities in this case is: $5 \cdot 2 = 10$;

or all 5 of the 5 classical mathematical propositions (hence we have $C_5^5 = 1$ subpossibility) – so no classical mathematical propositions remain.

Hence, the total number of Anti-BE-algebras is:

$$C_5^1 . 2^{5-1} + C_5^2 . 2^{5-2} + C_5^3 . 2^{5-3} + C_5^4 . 2^{5-4} + C_5^5 . 2^{5-5} = 5 \cdot 16 + 10 \cdot 8 + 10 \cdot 4 + 5 \cdot 2 + 1 \cdot 1 = 211.$$

Theorem 3.11.

As a particular case, for BE-algebras, we have:

1 (classical) BE-algebra + 31 Neutro-BE-algebras + 211 Anti-BE-algebras = $243 = 3^5$ algebras.

Where, $31 = 2^5 - 1$, and $211 = 3^5 - 2^5$.

Proof.

It results from the previous Theorem 3.10 and 3.11.

Theorem 3.12.

Let U be a nonempty finite or infinite universe of discourse, and S a nonempty finite or infinite subset of U. A classical Algebra is defined on S.

In general, for a given classical Algebra, having n operations (laws) and axioms altogether, for integer $n \geq 1$, there are 3^n total number of Algebra / NeutroAlgebras / AntiAlgebras as below:

1 (classical) Algebra, $(2^n - 1)$ Neutro-Algebras, and $(3^n - 2^n)$ Anti-Algebras.

The finite or infinite cardinal of set the classical algebra is defined upon, does not influence the numbers of Neutro-BE-algebras and Anti-BE-algebras.

Proof.

It is similar to Theorem 3.11, and based on Theorems 3.10 and 3.11.

Where 5 (total number of classical laws and axioms altogether) is extended/replaced by n.

5. Conclusion.

We have studied and presented the neutrosophic triplet (BE-algebra, Neutro-BE-algebra, Anti-BE-algebra) together with many examples, several properties and four theorems.

Funding: “The first author has been supported by Payame Noor University (Grant No. /47416/7).”

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A New Trend to Extensions of CI-algebras

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Florentin Smarandache, Akbar Rezaei, Hee Sik Kim (2020). A New Trend to Extensions of CI-algebras. International Journal of Neutrosophic Science 5(1), 8-15. DOI: 10.5281/zenodo.3788124

Abstract

In this paper, as an extension of CI-algebras, we discuss the new notions of Neutro-CI-algebras and Anti-CI-algebras. First, some examples are given to show that these definitions are different. We prove that any proper CI-algebra is a Neutro-BE-algebra or Anti-BE-algebra. Also, we show that any NeutroSelf-distributive and AntiCommutative CI-algebras are not BE-algebras.

Keywords: CI-algebra, Neutro-CI-algebra, Anti-CI-algebra, Self-distributive, NeutroSelf-distributive, AntiSelf-distributive, Commutative, NeutroCommuative, AntiCommutative.

1. Introduction

H.S. Kim et al. introduced the notion of BE-algebras as a generalization of dual BCK-algebras [1]. A. Walendziak defined the notion of commutative BE-algebras and discussed some of their properties [11]. A. Rezaei et al. investigated the relationship between Hilbert algebras and BE-algebras [5]. B.L. Meng introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCI/BCK-algebras, and studied some relations with BE-algebras [2]. Then he defined the notion of atoms in CI-algebras and singular CI-algebras and investigated their properties [3]. Filters and upper sets were studied in detail by B. Piekart et al. [4].

Recently, in 2019-2020 F. Smarandache [8, 9, 10] constructed for the first time the neutrosophic triple corresponding to the Algebraic Structures as (Algebraic Structure, NeutroAlgebraic Structure, AntiAlgebraic Structure), where a (classical) Algebraic Structure is an algebraic structure dealing only with (classical) Operations) (that are totally well-defined) and (classical) Axioms (that are totally true). A NeutroAlgebraic Structure is an algebraic structure that has at least one NeutroOperation or NeutroAxiom, and no AntiOperation and no AntiAxiom, while an AntiAlgebraic Structure is an algebraic structure that has at least one AntiOperation or one AntiAxiom. Moreover, some left (right)-quasi neutrosophic triplet structures in BE-algebras were studied by X. Zhang et al. [12].

The aim of this paper is to characterize these definitions to CI-algebras. Also, the notions of NeutroSelf-distributive / AntiSelf-distributive and NeutroCommutative / AntiCommutative in CI-algebras are studied. Finally, as an alternative to the definition of CI-algebra, Neutro-CI-algebra and Anti-CI-algebra are defined.

2. Preliminaries

In this section we recall some basic notions and results regarding CI-algebras and BE-algebras. CI-algebras were introduced in [2] as a generalization of BE-algebras (see [1]) and properties of them have recently been studied in [3] and [4].

Definition 2.1. ([2]) A *CI-algebra* is an algebra $(X,\to,1)$ of type $(2,0)$ (i.e. X is a non-empty set, $\to$ is a binary operation and 1 is a constant element) satisfying the following axioms, for all $x,y,z\in X$:

(CI1) $(\forall x\in X)(x\to x=1)$;

(CI2) $(\forall x\in X)(1\to x=x)$;

(CI3) $(\forall x,y,z\in X, with\; x\neq y)(x\to(y\to z)=y\to(x\to z))$.

We introduce a binary relation $\leq$ on X by $x\leq y$ if and only if $x\to y=1$. A CI-algebra $(X,\to,1)$ is said to be a *BE-algebra* ([1]) if

(BE) $(\forall x\in X)(x\to 1=1)$.

By (CI1) $\leq$ is only reflexive.

In what follows, let X be a CI-algebra unless otherwise specified. A CI-algebra X is proper if it is not a BE-algebra.

For example, the set $X=\{1,a\}$, with the following Cayley table is a proper CI-algebra, since $a\to 1=a\neq 1$.

Table 1

$\to$	1	a
1	1	a
a	a	1

Theorem 2.2. Let $(X,\to,1)$ be a CI-algebra. The binary operation $\to$ is associative if and only if $x\to 1=x$, for all $x\in X$.

Proof. Assume that $\to$ is associative. Using (CI2) and associativity, we have

$$x=1\to x=(x\to x)\to x=x\to(x\to x)=x\to 1.$$

Conversely, suppose that $x\to 1=x$, for all $x\in X$. Let $x,y,z\in X$, then by applying assumption and three times (CI3), we get

$(x\to y)\to z=(x\to y)\to(z\to 1)=z\to((x\to y)\to 1)=z\to(x\to y)=z\to(x\to(y\to 1))=x\to(\to(y\to 1))=x\to(y\to(z\to 1))=x\to(y\to z)$.

Thus, $(x\to y)\to z=x\to(y\to z)$.

Also, if $\to$ is associative relation, then CI-algebra $(X,\to,1)$ is an Abelian group with identity 1, since

$$x \to y = x \to (y \to 1) = y \to (x \to 1) = y \to x.$$

Example 2.3. (i) Let $\mathbb{R}$ be the set of all real numbers and $\to$ be the binary operation on $\mathbb{R}$ defined by $x \to y = y \div x$, where $\div$ is the binary operation of division. Then $(\mathbb{R} - \{0\}, \to, 1)$ is a CI-algebra, but it is not a BE-algebra.

(CI1) holds, since for every $0 \neq x \in \mathbb{R}$, $x \to x = x \div x = 1$;

(CI2) valid, since for all $x \in X$, $1 \to x = x$;

(CI3) holds. Let $x, y, z \in X$. Then we have

$x \to (y \to z) = x \to (z \div y) = (z \div y) \div x = (z \div x) \div y = y \to (z \div x) = y \to (x \to z)$.

(BE) is not valid, since $5 \to 1 = 1 \div 5 = \frac{1}{5} \neq 1$. Thus, $(\mathbb{R} - \{0\}, \to, 1)$ is a proper CI-algebra.

(ii) Consider the real interval [0,1] and let $\to$ be the binary operation on [0,1] defined by $x \to y = 1 - x + xy$. Then $([0,1], \to, 1)$ is not a CI-algebra (so, is not a BE-algebra), since (CI1) and (CI3) are not valid. Note that (BE) holds, since $x \to 1 = 1 - x + x.1 = 1 - x + x = 1$.

Proposition 2.4. ([2]) Let $\boldsymbol{X}$ be a CI-algebra. Then for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$,

(i) $\boldsymbol{y} \to ((\boldsymbol{y} \to \boldsymbol{x}) \to \boldsymbol{x}) = \mathbf{1}$;

(ii) $(\boldsymbol{x} \to \mathbf{1}) \to (\boldsymbol{y} \to \mathbf{1}) = (\boldsymbol{x} \to \boldsymbol{y}) \to \mathbf{1}$.

Definition 2.5. ([1, 2]) A CI/BE-algebra $\boldsymbol{X}$ is said to be *self-distributive* if for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{X}$,

$$\boldsymbol{x} \to (\boldsymbol{y} \to \boldsymbol{z}) = (\boldsymbol{x} \to \boldsymbol{y}) \to (\boldsymbol{x} \to \boldsymbol{z}).$$

Example 2.6. Consider the CI-algebra given in Example 2.3 (i). It is not self-distributive. Let $\boldsymbol{x} := \mathbf{5}, \boldsymbol{y} := \mathbf{4}$ and $\boldsymbol{z} := \mathbf{7}$. Then we have $\mathbf{5} \to (\mathbf{4} \to \mathbf{7}) = \mathbf{5} \to \frac{\mathbf{7}}{\mathbf{4}} = \frac{\mathbf{7}}{\mathbf{20}} \neq (\mathbf{5} \to \mathbf{4}) \to (\mathbf{5} \to \mathbf{7}) = \frac{\mathbf{4}}{\mathbf{5}} \to \frac{\mathbf{7}}{\mathbf{5}} = \frac{\mathbf{7}}{\mathbf{4}}$.

Proposition 2.7. ([2]) Every self-distributive CI-algebra $\boldsymbol{X}$ is a BE-algebra.

Note that if $\boldsymbol{X}$ is self-distributive, then $\leq$ is transitive ([6]).

Definition 2.8. ([2, 5, 11]) A CI/BE-algebra X is said to be commutative if for any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$,

$$\boldsymbol{x} \to (\boldsymbol{x} \to \boldsymbol{y}) = \boldsymbol{y} \to (\boldsymbol{y} \to \boldsymbol{x}).$$

Example 2.9. ([6]) (i) Let $\mathbb{N}$ be the set of all natural numbers and $\to$ be the binary operation on $\mathbb{N}$ defined by

$$\boldsymbol{x} \to \boldsymbol{y} = \begin{cases} \boldsymbol{y} & \boldsymbol{if\ x = 1}; \\ \mathbf{1} & \boldsymbol{otherwise}. \end{cases}$$

Then $(\mathbb{N}, \to, \mathbf{1})$ is a non-commutative BE-algebra.

(ii) Let $\mathbb{N}_{\mathbf{0}} = \mathbb{N} \cup \{\mathbf{0}\}$ and let $\to$ be the binary operation on $\mathbb{N}_{\mathbf{0}}$ defined by

$$\boldsymbol{x} \to \boldsymbol{y} = \begin{cases} \mathbf{0} & \boldsymbol{if\ x \geq y}; \\ \boldsymbol{y - x} & \boldsymbol{otherwise}. \end{cases}$$

Then $(\mathbb{N}_0, \rightarrow, 0)$ is a commutative BE-algebra ([6]), but it is not self-distributive, since

$$5 \rightarrow (6 \rightarrow 7) = 5 \rightarrow 1 = 0 \neq (5 \rightarrow 6) \rightarrow (5 \rightarrow 7) = 1 \rightarrow 2 = 1.$$

Proposition 2.10. ([2]) Every commutative CI-algebra X is a BE-algebra.

Note that if X is commutative, then $\leq$ is anti-symmetric ([6]). Hence, if X is a commutative and self-distributive CI-algebra, then $\leq$ is a partially ordered set ([6]).

3. On NeutroSelf-distributive and AntiSelf-distributive CI-algebras

Definition 3.1. A CI-algebra X is said to be *NeutroSelf-distributive* if

$$(\exists x, y, z \in X)(x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)) \text{ and } (\exists x, y, z \in X)(x \rightarrow (y \rightarrow z) \neq (x \rightarrow y) \rightarrow (x \rightarrow z)).$$

Example 3.2. Consider the non-self-distributive CI-algebra given in Example 2.3 (i). If $x := 1$, then for all $y, z \in \mathbb{R} - \{0\}$, we have $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$. If $x \neq 1$, then for all $y, z \in \mathbb{R} - \{0\}$, we have $x \rightarrow (y \rightarrow z) \neq (x \rightarrow y) \rightarrow (x \rightarrow z)$. Hence $(\mathbb{R} - \{0\}, \rightarrow, 1)$ is a NeutroSelf-distributive CI-algebra.

Definition 3.3. A CI-algebra X is said to be *AntiSelf-distributive* if

$$(\forall x, y, z \in X, \textit{with } x \neq 1)(x \rightarrow (y \rightarrow z) \neq (x \rightarrow y) \rightarrow (x \rightarrow z)).$$

Example 3.4. Consider the CI-algebra given in Example 2.3 (i). Then it is an AntiSelf-distributive CI-algebra, since for all $x, y, z \in \mathbb{R} - \{0\}$ and $x \neq 1$, we can see that

$$x \rightarrow (y \rightarrow z) = (z \div y) \div x = \frac{z}{yx} \neq (x \rightarrow y) \rightarrow (x \rightarrow z) = (z \div x) \div (y \div x) = \frac{z}{y}.$$

Theorem 3.5. Let X be an AntiSelf-distributive CI-algebra. Then X is not a BE-algebra.

Proof. Assume that X is an AntiSelf-distributive CI-algebra and $1 \neq x \in X$. Take $y = z = 1$ and using AntiSelf-distributivity and applying (CI1) two times, we have

$$x \rightarrow 1 = x \rightarrow (1 \rightarrow 1) \neq (x \rightarrow 1) \rightarrow (x \rightarrow 1) = 1.$$

Thus, $(\forall x \in X, \textit{with } x \neq 1)(x \rightarrow 1 \neq 1)$, and so X is not a BE-algebra.

Corollary 3.6. There is no AntiSelf-distributive BE-algebra.

Proposition 3.7. Let X be an AntiSelf-distributive CI-algebra. Then

$$(\forall x, y \in X, \textit{with } x \neq 1)(x \rightarrow (x \rightarrow y) \neq x \rightarrow y)$$

Proof. Let X be a CI-algebra and $x, y \in X$. Using AntiSelf-distributivity and (CI2), we get

$$x \rightarrow (x \rightarrow y) \neq (x \rightarrow x) \rightarrow (x \rightarrow y) = 1 \rightarrow (x \rightarrow y) = x \rightarrow y.$$

Thus, $x \rightarrow (x \rightarrow y) \neq x \rightarrow y$.

Proposition 3.8. Let $\boldsymbol{X}$ be an AntiSelf-distributive CI-algebra, and $\boldsymbol{x} \leq \boldsymbol{y}$. Then $\boldsymbol{z} \rightarrow \boldsymbol{x} \nleq \boldsymbol{z} \rightarrow \boldsymbol{y}$, for all $\mathbf{1} \neq \boldsymbol{z} \in \boldsymbol{X}$.

Proof. Suppose that $\boldsymbol{X}$ is an AntiSelf-distributive CI-algebra, $\boldsymbol{x} \leq \boldsymbol{y}$ and $\mathbf{1} \neq \boldsymbol{z} \in \boldsymbol{X}$. Then $\boldsymbol{x} \rightarrow \boldsymbol{y} = \mathbf{1}$. Applying AntiSelf-distributivity and (BE), we get

$$(\boldsymbol{z} \rightarrow \boldsymbol{x}) \rightarrow (\boldsymbol{z} \rightarrow \boldsymbol{y}) \neq \boldsymbol{z} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) = \boldsymbol{z} \rightarrow \mathbf{1} \neq \mathbf{1}.$$

Thus, $\boldsymbol{z} \rightarrow \boldsymbol{x} \nleq \boldsymbol{z} \rightarrow \boldsymbol{y}$, for all $\mathbf{1} \neq \boldsymbol{z} \in \boldsymbol{X}$.

Proposition 3.9. Let $\boldsymbol{X}$ be an AntiSelf-distributive CI-algebra. Then $\leq$ is not transitive.

Proof. Suppose that $\boldsymbol{X}$ is an AntiSelf-distributive CI-algebra, $\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{y} \leq \boldsymbol{z}$. Then $\boldsymbol{x} \rightarrow \boldsymbol{y} = \mathbf{1}$ and $\boldsymbol{y} \rightarrow \boldsymbol{z} = \mathbf{1}$. Using (CI2) and AntiSelf-distributivity, we have

$$\boldsymbol{x} \rightarrow \boldsymbol{z} = \mathbf{1} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{z}) = (\boldsymbol{x} \rightarrow \boldsymbol{y}) \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{z}) \neq \boldsymbol{x} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{z}) = \boldsymbol{x} \rightarrow \mathbf{1} \neq \mathbf{1}.$$

Thus, $\boldsymbol{x} \nleq \boldsymbol{z}$.

4. On NeutroCommutative and AntiCommutative CI-algebras

Definition 4.1. A CI/BE-algebra $\boldsymbol{X}$ is said to be *NeutroCommutative* if

$$(\exists \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X} \text{ with } \boldsymbol{x} \neq \boldsymbol{y})(\boldsymbol{x} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) = \boldsymbol{y} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x})) \text{ and } (\exists \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) \neq \boldsymbol{y} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x})).$$

Example 4.2. (i) Consider the non-commutative BE-algebra given in Example 2.9 (i). If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N} - \{\mathbf{1}\}$, then $\boldsymbol{x} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) = \boldsymbol{y} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x})$. If $\boldsymbol{x} = \mathbf{1}$ and $\boldsymbol{y} \neq \mathbf{1}$, then $\boldsymbol{x} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) = \boldsymbol{y} \neq \boldsymbol{y} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x}) = \mathbf{1}$.

(ii) Consider the CI-algebra given in Example 2.3 (i). Then it is not a NeutroCommutative CI-algebra, since, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R} - \{\mathbf{0}\}$, we have $\boldsymbol{x} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) \neq \boldsymbol{y} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x})$, only if $\boldsymbol{x} = \boldsymbol{y} = \mathbf{1}$, then $\boldsymbol{x} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) = \boldsymbol{y} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x})$. Thus, there is not $\boldsymbol{x} \neq \boldsymbol{y}$ such that $\boldsymbol{x} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{y}) = \boldsymbol{y} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x})$.

Definition 4.3. A CI/BE-algebra $\boldsymbol{X}$ is said to be *AntiCommutative* if

$$(\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X} \text{ with } \boldsymbol{x} \neq \boldsymbol{y})((\boldsymbol{x} \rightarrow \boldsymbol{y}) \rightarrow \boldsymbol{y} \neq (\boldsymbol{y} \rightarrow \boldsymbol{x}) \rightarrow \boldsymbol{x}).$$

Example 4.4. Consider the CI-algebra given in Example 2.3 (i). Then it is an AntiCommutative CI-algebra.

Proposition 4.5. Let $\boldsymbol{X}$ be an AntiCommutative CI-algebra. Then $\boldsymbol{X}$ is not a BE-algebra.

Proof. By contrary, let $\boldsymbol{X}$ be a BE-algebra. Then for all $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x} \rightarrow \mathbf{1} = \mathbf{1}$. Hence $(\boldsymbol{x} \rightarrow \mathbf{1}) \rightarrow \mathbf{1} = \mathbf{1} \rightarrow \mathbf{1} = \mathbf{1}$, by assumption and (CI1). Now, applying AntiCommutaitivity and (CI2) we get

$$\mathbf{1} = (\boldsymbol{x} \rightarrow \mathbf{1}) \rightarrow \mathbf{1} \neq (\mathbf{1} \rightarrow \boldsymbol{x}) \rightarrow \boldsymbol{x} = \boldsymbol{x} \rightarrow \boldsymbol{x} = \mathbf{1}.$$

Thus, $\mathbf{1} \neq \mathbf{1}$ which is a contradiction.

Corollary 4.6. There is no AntiCommutative BE-algebra.

Proposition 4.7. Let $\boldsymbol{X}$ be an AntiCommutative CI-algebra. If $\boldsymbol{x} \leq \boldsymbol{y}$, then $\boldsymbol{y} \neq (\boldsymbol{y} \rightarrow \boldsymbol{x}) \rightarrow \boldsymbol{x}$.

Proof. Assume that $\boldsymbol{X}$ be an AntiCommutative CI-algebra and $\boldsymbol{x} \leq \boldsymbol{y}$. Then $\boldsymbol{x} \rightarrow \boldsymbol{y} = \mathbf{1}$. Using (CI2) and AntiCommutaitivity, we have

$$\boldsymbol{y} = \mathbf{1} \rightarrow \boldsymbol{y} = (\boldsymbol{x} \rightarrow \boldsymbol{y}) \rightarrow \boldsymbol{y} \neq (\boldsymbol{y} \rightarrow \boldsymbol{x}) \rightarrow \boldsymbol{x}.$$

Proposition 4.8. Let $\boldsymbol{X}$ be an AntiCommutative CI-algebra. Then $\leq$ is not an anti-symmetric relation on $\boldsymbol{X}$.

Proof. Assume that $\boldsymbol{X}$ be an AntiCommutative CI-algebra. Let $\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{y} \leq \boldsymbol{x}$. Then $\boldsymbol{x} \rightarrow \boldsymbol{y} = \mathbf{1}$ and $\boldsymbol{y} \rightarrow \boldsymbol{x} = \mathbf{1}$. Applying (CI2) and AntiCommutativity, we get

$$\boldsymbol{x} = \mathbf{1} \rightarrow \boldsymbol{x} = (\boldsymbol{y} \rightarrow \boldsymbol{x}) \rightarrow \boldsymbol{x} \neq (\boldsymbol{x} \rightarrow \boldsymbol{y}) \rightarrow \boldsymbol{y} = \mathbf{1} \rightarrow \boldsymbol{y} = \boldsymbol{y}.$$

Corollary 4.9. If $\boldsymbol{X}$ is an AntiSelf-distributive or an AntiCommutative CI-algebra, then X endowed with the induced relation $\leq$ is not a partially ordered set.

Proof. By Propositions 3.9 and 4.8, we get the desired result.

If $\boldsymbol{X}$ is not a partially ordered set, then $\boldsymbol{X}$ is either totally ordered set, or totally unordered set (i.e. for any two distinct elements $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, neither $\boldsymbol{x} \leq \boldsymbol{y}$ nor $\boldsymbol{y} \leq \boldsymbol{x}$).

We have the neutrosophic triplet for the order relationship $\leq$ in a similar way as for CI-algebras:

(totally ordered, partially ordered and partially unordered, totally unordered) or (Ordered, NeutroOrdered, AntiOrdered).

Corollary 4.10. If $\boldsymbol{X}$ is an AntiSelf-distributive or an AntiCommutative CI-algebra, then $\boldsymbol{x} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x}) \neq \mathbf{1}$, for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$.

Proof. Using Corollaries 3.6 and 4.5, $\boldsymbol{X}$ is not a BE-algebra, and so applying (CI3) and (CI1) we get, for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$

$$\boldsymbol{x} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{x}) = \boldsymbol{y} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{x}) = \boldsymbol{y} \rightarrow \mathbf{1} \neq \mathbf{1}.$$

5. On Neutro-CI-algebras and Anti-CI-algebras

The Neutro-BE-algebra and the Anti-BE-algebra as an alternative of a BE-algebra was defined in 2020 by A. Rezaei and F. Smarandache. Now, we can define Neutro-CI-algebra and Anti-CI-algebra (for detail see [7]).

Definition 5.1. (Neutro-sophications)

The Neutro-sophication of the Law (degree of well-defined, degree of indeterminacy, degree of outer-defined)

(NL) $(\exists \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \boldsymbol{y} \in \boldsymbol{X})$ and $(\exists \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \boldsymbol{y} = \boldsymbol{indeterminate}$ or $\boldsymbol{x} \rightarrow \boldsymbol{y} \notin \boldsymbol{X})$.

The Neutro-sophication of the Axioms (degree of truth, degree of indeterminacy, degree of falsehood)

(NCI1) $(\exists \boldsymbol{x} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \boldsymbol{x} = \mathbf{1})$ and $(\exists \boldsymbol{x} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \boldsymbol{x}=$ *indeterminate* or $\boldsymbol{x} \rightarrow \boldsymbol{x} \neq \mathbf{1})$;

(NCI2) $(\exists \boldsymbol{x} \in \boldsymbol{X})(\mathbf{1} \rightarrow \boldsymbol{x} = \boldsymbol{x})$ **and** $(\exists \boldsymbol{x} \in \boldsymbol{X})(\mathbf{1} \rightarrow \boldsymbol{x}=$ *indeterminate* or $1 \rightarrow \boldsymbol{x} \neq \boldsymbol{x})$;

(NCI3) $(\exists \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{X}, \boldsymbol{with}\ \boldsymbol{x} \neq \boldsymbol{y})(\boldsymbol{x} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{z}) = \boldsymbol{y} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{z}))$ and $(\exists \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{X}, \boldsymbol{with}\ \boldsymbol{x} \neq \boldsymbol{y})(\boldsymbol{x} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{z}) = \boldsymbol{indeterminate}\ \boldsymbol{or}\ \boldsymbol{x} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{z}) \neq \boldsymbol{y} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{z}))$.

Definition 5.2. (Anti-sophications)

The Anti-sophication of the Law (totally outer-defined)

(AL) $(\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \boldsymbol{y} \notin \boldsymbol{X})$.

The Anti-sophication of the Axioms (totally false)

(ACI1) $(\forall \boldsymbol{x} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \boldsymbol{x} \neq \boldsymbol{1})$;

(ACI2) $(\forall \boldsymbol{x} \in \boldsymbol{X})(\boldsymbol{1} \rightarrow \boldsymbol{x} \neq \boldsymbol{x})$;

(ACI3) $(\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{X}, \mathbf{with} \boldsymbol{x} \neq \boldsymbol{y})(\boldsymbol{x} \rightarrow (\boldsymbol{y} \rightarrow \boldsymbol{z}) \neq \boldsymbol{y} \rightarrow (\boldsymbol{x} \rightarrow \boldsymbol{z}))$.

Definition 5.3. A *Neutro-CI-algebra* is an alternative of CI-algebra that has at least a (NL) or at least one (NCIt), $\boldsymbol{t} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, with no anti-law and no anti-axiom.

Definition 5.4. An *Anti-CI-algebra* is an alternative of CI-algebra that has at least an (AL) or at least one (NCIt), $\boldsymbol{t} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$.

A Neutro-BE-algebra ([7]) is a Neutro-CI-algebra or has (NBE), where

(NBE) $(\exists \boldsymbol{x} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \mathbf{1} = \mathbf{1})$ and $(\exists \boldsymbol{x} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \mathbf{1} = \boldsymbol{indeterminate}$ or $\boldsymbol{x} \rightarrow \mathbf{1} \neq \mathbf{1})$.

An Anti-BE-algebra ([7]) is an Anti-CI-algebra or has (ABE), where

(ABE) $(\forall \boldsymbol{x} \in \boldsymbol{X})(\boldsymbol{x} \rightarrow \mathbf{1} \neq \mathbf{1})$.

Note that any proper CI-algebra may be a Neutro-BE-algebra or Anti-BE-algebra.

Proposition 5.5. Every NeutroSelf-distributive CI-algebra is a Neutro-CI-algebra.

Proposition 5.6. Every AntiSelf-distributive CI-algebra is an Anti-BE-algebra.

Proposition 5.7. Every AntiCommutative CI-algebra is an Anti-BE-algebra.

6. Conclusions

In this paper, Neutro-CI-algebras and Anti-CI-algebras are introduced and discussed based on the definition of CI-algebras. By some examples we showed that these notions were different. Some of their properties were provided. We proved that any proper CI-algebra is a Neutro-BE-algebra or Anti-BE-algebra. Further, for every classical CI-algebra, it was shown that, if it is AntiSelf-distributive or AntiCommutative, then it is an Anti-BE-algebra.

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Neutro-BCK-Algebra

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Mohammad Hamidi, Florentin Smarandache (2020). Neutro-BCK-Algebra. *International Journal of Neutrosophic Science* 8(2), 110-117. DOI: 10.5281/zenodo.3902754

Abstract

This paper introduces the novel concept of Neutro-BCK-algebra. In Neutro-BCK-algebra, the outcome of any given two elements under an underlying operation (neutro-sophication procedure) has three cases, such as: appurtenance, non-appurtenance, or indeterminate. While for an axiom: equal, non-equal, or indeterminate. This study investigates the Neutro-BCK-algebra and shows that Neutro-BCK-algebra are different from BCK-algebra. The notation of Neutro-BCK-algebra generates a new concept of NeutroPoset and Neutro-Hass-diagram for NeutroPosets. Finally, we consider an instance of applications of the Neutro-BCK-algebra.

Keywords: Neutro-BCK-algebra, NeutroPoset, Neutro-Hass diagram.

1 Introduction

Neutrosophy, as a newly-born science, is a branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. It can be defined as the incidence of the application of a law, an operation, an axiom, an idea, a conceptual accredited construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making it intelligible. Neutrosophic Sets and Systems international journal (which is in Scopus and Web of Science) is a tool for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics, neutrosophic measure, neutrosophic integral, and so on, studies that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc. Recently, Florentin Smarandache [2019] generalized the classical Algebraic Structures to NeutroAlgebraic Structures NeutroAlgebras) and AntiAlgebraic Structures (AntiAlgebras) and he proved that the NeutroAlgebra is a gen-eralization of Partial Algebra.[7] He considered $< A >$ as an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, he split the nonempty space and worked onto three regions two opposite ones corresponding to $< A >$ and $< antiA >$, and one corresponding to neutral (indeterminate) $< neutA >$ (also denoted $< neutroA >$) between the opposites, regions that may or may not be disjoint -depending on the application, but they are exhaustive (their union equals the whole space). A NeutroAlgebra is an algebra which has at least one NeutroOperation operation that is well-defined (also called inner-defined) for some elements, indeterminate for others, and outer-defined for the others or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A Partial Alge-bra is an algebra that has at least one partial operation (well-defined for some elements, and indeterminate for other elements), and all its axioms are classical (i.e., the axioms are true for all elements). Through a theorem he proved that NeutroAlgebra is a generalization of Partial Algebra, and examples of NeutroAlgebras that are not partial algebras were given. Also, the NeutroFunction and NeutroOperation were introduced.[7]

Regarding these points, we now introduce the concept of Neutro-BCK-algebras based on axioms of BCK-algebras, but having a different outcome. In the system of BCK-algebras, the operation is totally well-defined for any two given elements, but in Neutro-BCK-algebras its outcome may be well-defined, outer-defined, or indeterminate. Any BCK-algebra is a system which considers that all its axioms are true; but we weaken the conditions that the axioms are not necessarily totally true, but also partially false, and partially indeterminate. So, one of our main motivation is a weak coverage of the classical axioms of BCK-algebras. This causes new partially ordered relations on a non-empty set, such as NeutroPosets and Neutro-Hass Diagrams. Indeed Neutro-Hass Diagrams of NeutroPosets contain relations between elements in the set that are true, false, or indeterminate.

2 Preliminaries

In this section, we recall some definitions and results from,[7] which are needed throughout the paper.

Let $n \in \mathbb{N}$. Then an n-ary operation $\circ : X^n \to Y$ is called a NeutroOperation if it has $x \in X^n$ for which $\circ(x)$ is well-defined (degree of truth (T)), $x \in X^n$ for which $\circ(x)$ is indeterminate (degree of indeterminacy (I)), and $x \in X^n$ for which $\circ(x)$ is outer-defined (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (total, or classical) Operation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary AntiOperation. Again, in this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general. In this regards, for any given set X, we classify n-ary operation on X^n by (i); (**classical**) **Operation** is an operation well-defined for all set's elements, (ii); **NeutroOperation** is an operation partially well-defined, partially indeterminate, and partially outer-defined on the given set and (iii); **AntiOperation** is an operation outer-defined for all set's elements.

Moreover, we have (i); a (**classical**) **Axiom** defined on a non-empty set is an axiom that is totally true (i.e. true for all set's elements), (ii); **NeutroAxiom (or neutrosophic axiom)** defined on a non-empty set is an axiom that is true for some elements (degree of true = T), indeterminate for other elements (degree of indeterminacy = I), and false for the other elements (degree of falsehood = F), where T, I, F are in $[0, 1]$ and (T, I, F) is different from $(1, 0, 0)$ i.e., different from totally true axiom, or classical Axiom and (T, I, F) is different from $(0, 0, 1)$ i.e., different from totally false axiom, or AntiAxiom. (iii); an **AntiAxiom** of type $\mathcal{C}$ defined on a non-empty set is an axiom that is false for all set's elements.

Based on the above definitions, there is a classification of algebras as follows. Let X be a non-empty set and $\mathcal{O}$ be a family of binary operations on X. Then $(A, \mathcal{O})$ is called

(i) a (**classical**) **Algebra** of type $\mathcal{C}$, if $\mathcal{O}$ is the set of all total Operations (i.e. well-defined for all set's elements) and $(A, \mathcal{O})$ is satisfied by (classical) Axioms of type $\mathcal{C}$(true for all set's elements).

(ii) a **NeutroAlgebra (or neutro-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one NeutroOperation (or NeutroFunction), or $(A, \mathcal{O})$ is satisfied by at least one NeutroAxiom of type $\mathcal{C}$ that is referred to the set's (partial-, neutro-, or total-) operations or axioms;

(iii) an **AntiAlgebra (or anti-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one AntiOperation or $(A, \mathcal{O})$ is satisfied by at least one AntiAxiom of type $\mathcal{C}$.

3 Neutro-BCK-algebra

3.1 Concept of Neutro-BCK-algebra

In this section, we introduce several concepts suc has: Neutro-BCK-algebra, Neutro-BCK-algebra of type 5, NeutroPoset and Neutro-Hass Diagram and investigate the properties of these concepts.

Definition 3.1. [2] Let X be a non-empty set with a binary operation "$*$" and a constant "0". Then, $(X, *, 0)$ is called a BCK*-algebra* if it satisfies the following conditions:

$(BCI\text{-}1)$ $((x * y) * (x * z)) * (z * y) = 0$,

$(BCI\text{-}2)$ $(x * (x * y)) * y = 0$,

$(BCI\text{-}3)$ $x * x = 0$,

$(BCI\text{-}4)$ $x * y = 0$ and $y * x = 0$ imply $x = y$,

$(BCK\text{-}5)$ $0 * x = 0$.

Now, we define Neutro-BCK-algebras as follows.

Definition 3.2. Let X be a non-empty set, $0 \in X$ be a constant and "$*$" be a binary operation on X. An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a Neutro-BCK-algebra, if it satisfies at least one of the following NeutroAxioms (while the others are classical BCK-axioms):

$(NBCI\text{-}1)$ $\big(\exists\, x, y, z \in X$ such that $((x * y) * (x * z)) * (z * y) = 0)\big)$ and $\big(\exists\, x, y, z \in X$ such that $((x * y) * (x * z)) * (z * y) \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}2)$ $\big(\exists\, x, y \in X$ such that $(x * (x * y)) * y = 0\big)$ and $\big(\exists\, x, y \in X$ such that $(x * (x * y)) * y \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}3)$ $\big(\exists\, x \in X$ such that $x * x = 0\big)$ and $\big(\exists\, x \in X$ such that $x * x \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}4)$ $\big(\exists\, x, y \in X$, such that if $x*y = y*x = 0$, we have $x = y\big)$ and $\big(\exists\, x, y \in X$, such that if $x*y = y*x = 0$, we have $x \neq y$ $\big)$;

$(NBCK\text{-}5)$ $\big(\exists\, x \in X$ such that $0 * x = 0\big)$ and $\big(\exists\, x \in X$ such that $0 * x \neq 0$ or indeterminate $\big)$. Each above NeutroAxiom has a degree of equality (T), degree of non-equality (F), and degree of indeterminacy (I), where $(T, I, F) \notin (1, 0, 0), (0, 0, 1)$.

If $(X, *, 0)$ is a NeutroAlgebra and satisfies the conditions $(NBCI\text{-}1)$ to $(NBCI\text{-}4)$ and $(NBCK\text{-}5)$, then we will call it is a Neutro-BCK-algebra of type 5 (i.e. it satisfies 5 NeutroAxioms).

Example 3.3. Let $X = \mathbb{Z}$. Then

(i) $(X, *, 0)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x * y = x - y + xy$.

(ii) $(X, *, 1)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x * y = xy$.

(iii) $(X, *, 1)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x * y = \begin{cases} 1 & \text{if } x \text{ an even} \\ xy & \text{if } x \text{ an odd} \end{cases}$.

Let $X \neq \emptyset$ be a finite set. We denote $\mathcal{N}_{BCK}(X)$ and $\mathcal{N}_{NBCK}(X)$ by the set of all Neutro-BCK-algebras and Neutro-BCK-algebras of type 5 that are constructed on X, respectively.

Theorem 3.4. *Let $(X, *, 0)$ be a Neutro BCK-algebra. Then*

(i) *If $|X| = 1$, then $(X, *, 0)$ is a trivial BCK-algebra.*

(ii) *If $|X| = 2$, then $|\mathcal{N}_{BCK}(X)| = 2$ and $|\mathcal{N}_{NBCK}(X)| = \infty$.*

(iii) *If $|X| = 3$, then there exists $\emptyset \neq Y \subseteq X$, such that $(Y, *', 0)$ is a nontrivial or trivial BCK-algebra.*

Proof. We consider only the cases $(ii), (iii)$, because the others are immediate.

(ii) Let $X = \{0, x\}$. Then we have 2 trivial Neutro-BCK-algebras $(X, *_1), (X, *_2)$ and an infinite number of trivial Neutro-BCK-algebras of type 5 $(X, *, 0)$ in Table 1, where $w \notin X$.

(iii) Let $X = \{0, x, y\}$. Now consider $Y = \{0, x\}$ and define a Neutro-BCK-algebra $(X, *', 0)$ in Table 1. Clearly $(Y, *', 0)$ is a non-trivial BCK-algebra. If $Y = \{0\}$, it is a trivial BCK-algebra. $\square$

Table 1: Neutro-BCK-algebras of order 2

$*_1$	0	x
0	0	x
x	0	x

,

$*_2$	0	x
0	x	0
x	x	0

,

$*$	0	x
0	x	0
x	w	0

and

$*'$	0	x	y
0	0	0	y
x	x	0	0
y	0	y	x

.

Theorem 3.5. *Every BCK-algebra, can be extended to a Neutro-BCK-algebra.*

Proof. Let $(X, *, 0)$ be a BCK-algebra and $\alpha \notin X$, and U be the universe of discourse that strictly includes $X \cup \alpha$. For all $x, y \in X \cup \{\alpha\}$, define $*_\alpha$ on $X \cup \{\alpha\}$ by $x *_\alpha y = x * y$ where, $x, y \in X$ and if $\alpha \in \{x, y\}$, define $x *_\alpha y$ as indeterminate or $x *_\alpha y \notin X \cup \alpha$. Then $(X \cup \{\alpha\}, *_\alpha, 0)$ is a Neutro-BCK-algebra. $\square$

Example 3.6. Let $X = \{0, 1, 2, 3, 4, 5\}$. Consider Table 3.
Then

(i) If $a = 0$, then $(X, *_1, 0)$ is a Neutro-BCK-algebra and if $a = 1$, then $(X \setminus \{3, 4, 5\}, *_1, 0)$ is a BCK-algebra.

(ii) $(X, *_2, 0)$ is a Neutro-BCK-algebra and $(X \setminus \{4, 5\}, *_2, 0)$ is a BCK-algebra.

(iii) If $s = t = y = z = 0, w = 3$, then $(X, *_3, 0)$ is a Neutro-BCK-algebra and for $s = t = 1, y = 2, z = 3$, $(X \setminus \{5\}, *_3, 0)$ is a BCK-algebra. If $s = t = y = z = 0, w = \sqrt{2}$, then $(X, *_3, 0)$ is a Neutro-BCK-algebra of type 5 where $s, t \in \{0, 1\}, x \in \{4, 5\}, y \in \{2, 0\}, z \in \{3, 0\}$ and $w \in \{3, \sqrt{2}\}$.

Table 2: Neutro-BCK-algebras and Neutro-BCK-algebra of type 5

$*_1$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	a	2	0	3
2	2	2	0	0	2	0
3	3	0	1	2	0	5
4	0	4	0	1	4	0
5	4	0	1	0	2	3

,

$*_2$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	0	0	0	5
2	2	1	0	0	5	0
3	3	2	1	0	0	2
4	0	1	0	4	1	2
5	5	0	4	0	0	x

and

$*_3$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	t	0	s	0
2	2	2	0	y	0	3
3	3	1	3	0	z	0
4	4	4	4	4	0	1
5	0	2	0	2	0	w

,

Remark 3.7. In Neutro-BCK-algebra $(X, *_3, 0)$, which is defined as in Example 3.6, we have $(1, 5) \in \leq$ and $(5, 0) \in \leq$, but $(1, 0) \notin \leq$, where $(x, y) \in \leq$ means $x *_3 y = 0$. Thus $\leq$, necessarily, is not a transitive relation. So we have the following definition of neutro-partially ordered relation on Neutro-BCK-algebra.

Definition 3.8. Let X be a non-empty set and R be a binary relation on X. Then R is called a

(i) neutro-reflexive, if $\exists\, x \in X$ such that $(x, x) \in R$ (degree of truth T), and $\exists\, x \in X$ such that (x, x) is indeterminate (degree of indeterminacy I), and $\exists\, x \in X$ such that $(x, x) \notin R$ (degree of falsehood F);

(ii) neutro-antisymmetric, if $\exists\, x, y \in X$ such that $(x, y) \in R$ and $(x, y) \in R$ imply that $x = y$ (degree of truth T), and $\exists\, x, y \in X$ such that (x, y) or (y, x) are indeterminate in R (degree of indeterminacy I), and $\exists\, x, y \in X$ such that $(x, y) \in R$ and $(y, x) \in R$ imply that $x \neq y$ (degree of falsehood F);

(iii) neutro-transitive, if $\exists\, x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$ imply that $(x, z) \in R$ (degree of truth T),and $\exists\, x, y, z \in X$ such that (x, y) or (y, z) are indeterminate in R (degree of indeterminacy I), and $\exists\, x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$, but $(x, z) \notin R$ (degree of falsehood F). In all above neutro-axioms $(T, I, F) \notin (1, 0, 0), (0, 0, 1)$.

(iv) neutro-partially ordered binary relation, if the relation satisfies at least one of the above neutro-axioms neutro-reflexivity, neutro-antisymmetry, neutro-transitivity, while the others (if any) are among the classical axioms reflexivity, antisymmetry, transitivity.

If R is a neutro-partially ordered relation on X, we will call (X, R) by neutro-poset. We will denote, the related diagram with to neutro-poset (X, R) by neutro-Hass diagram.

We define binary relations " $\leq_1, \leq_2$ " on X by $\big(x \leq_1 y$ if or only if $x * y = 0$ or $x \leq_1 x\ \big)$ and $\big(x \leq_2 y$ if and only if ($x * y \neq 0$ or indeterminate) or $x \leq_2 x\ \big)$. So we have the following theorem.

Theorem 3.9. *An algebra $(X, *, 0)$ is a Neutro-BCK-algebra if and only if it satisfies the following conditions:*

$(NBCI\text{-}1')$ $\big(\exists\, x, y \in X$ *such that* $((x*y)*(x*z)) \leq_1 (z*y)\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $((x*y)*(x*z)) \leq_2 (z*y)\big)$,

$(NBCI\text{-}2')$ $\big(\exists\, x, y \in X$ *such that* $(x * (x * y)) \leq_1 y\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $(x * (x * y)) \leq_2 y\big)$,

$(NBCI\text{-}3')$ $\big(\exists\, x, y \in X$ *such that* $x \leq_1 x\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $x \leq_2 x\big)$,

$(NBCI\text{-}4')$ $\big(\forall\, x, y \in X$, *if* $x \leq_1 y$ *and* $y \leq_1 x$, *we get* $x = y\big)$ *and* $\big(\forall\, x, y \in X$, *if* $x \leq_2 y$ *and* $y \leq_2 x$, *we get* $x = y\big)$,

$(NBCK\text{-}5')$ $\big(\exists\, x, y \in X$ *such that* $0 \leq_1 x\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $0 \leq_2 x\big)$.

Proof. Let $(X, *, 0)$ be a Neutro-BCK-algebra. We prove only the item $(NBCI\text{-}1')$, other items are similar to. Since $(X, *, 0)$ be a Neutro-BCK-algebra, $\big(\exists\, x, y \in X$ such that $(x*(x*y))*y = 0\big)$ and $\big(\exists\, x, y \in X$ such that $(x*(x*y))*y \neq 0$ or indeterminate$\big)$. By definition, $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_1 (z*y)\big)$ and $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_2 (z*y)\big)$. Conversely, let the items $(NBCI\text{-}1')$ to $(NBCI\text{-}4')$ and $(NBCK\text{-}4')$. Just prove $(NBCI\text{-}1)$ and other items are similar to. Since $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_1 (z*y)\big)$ and $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_2 (z*y)\big)$, we get that $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z))*(z*y) = 0)\big)$ and $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z))*(z*y) \neq 0$ or indeterminate $\big)$. $\square$

Let $(X,*,0)$ be a Neutro-BCK algebra. Define binary relation $\leq$ on X, by $x \leq y$ if and only $x \leq_1 y$ and $y \leq_2 x$. So we have the following results.

Theorem 3.10. *Let $(X,*,0)$ be a Neutro-BCK algebra and $x,y,z \in X$. Then*

(i) *if $x \neq y$ and $x \leq y$, then $y \leq x$;*

(ii) *$\leq$ is a reflexive and symmetric relation on X;*

(iii) *$\leq$ is a neutro-transitive algebra relation on X.*

Proof. (i) Let $x \neq y \in X$ and $x \leq y$. If $y \leq x$, by definition we obtain $(x*y = y*x = 0)$ and $(x*y = y*x \neq 0)$ and so $x = y$.

$(ii), (iii)$ It is clear by item (i) and Remark 3.7.

(iii) It is obtained by (ii). □

Corollary 3.11. *Let $(X,*,0)$ be a Neutro-BCK algebra. Then $(X,*,0,\leq_1), (X,*,0,\leq_2)$ and $(X,*,0,\leq)$ are neutro-posets.*

Let $(X_1,*_1,0_1)$ and $(X_2,*_2,0_2)$ be BCK-algebras, where $X_1 \cap X_2 = \emptyset$. For some $x,y \in X$, define an operation $*$ as follows: $x*y = \begin{cases} x *_1 y & \text{if if } x,y \in X_1 \setminus X_2 \\ x *_2 y & \text{if if } x,y \in X_2 \setminus X_1 \\ 0_1 & \text{if if } x \in X_1, y \in X_2 \\ 0_2 & \text{if if } x \in X_2, y \in X_1 \end{cases}$, where $0_1 * 0_2 = 0_2$ and $0_2 * 0_1 = 0_1$.

Theorem 3.12. *Let $(X_1,*_1,0_1)$ and $(X_2,*_2,0_2)$ be BCK-algebras, where $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. Then*

(i) *$(X,*,0_1)$ is a Neutro-BCK-algebra;*

(ii) *$(X,*,0_2)$ is a Neutro-BCK-algebra;*

Proof. (i) We only prove $(NBCI\text{-}4)$. Let $x*y = 0_1$. It follows that $x \in X_1$ and $y \in X_2$ or $x,y \in X_1$. If $x,y \in X_1$, because $(X_1,*_1,0_1)$ is a BCK-algebra, $y*x = 0_1$ implies that $x = y$. But for $x \in X_1$ and $y \in X_2$, we have $y*x \neq 0_1$ so $(NBCI\text{-}4)$ is valid in any cases. Other items are clear.

(ii) It is similar to item (i). □

Example 3.13. Let $X_1 = \{a,b\}$ and $X_2 = \{w,x,y,z\}$. Then $(X_1,*,a)$ and $(X_2,*,w)$ are BCK-algebras. So by Theorem 3.12, $(X_1 \cup X_1,*,a)$ and $(X_1 \cup X_1,*,w)$ are Neutro-BCK-neutralgebras in Table 3.

Table 3: BCK-algebras and Neutro-BCK-algebra

$*$	a	b	w	x	y	z
a	a	a	w	a	a	a
b	b	a	a	a	a	a
w	a	w	w	w	w	w
x	w	w	x	w	w	w
y	w	w	y	x	w	w
z	w	w	z	x	x	w

.

Corollary 3.14. *Let $(X_1,*_1,0_1)$ and $(X_2,*_2,0_2)$ be BCK-algebras. Then*

(i) *$(X,*,0_1,\leq_1), (X,*,0_2,\leq_2)$ and $(X,*,0_2,\leq_2)$ are posets.*

(ii) *$(X,*,0_1,\leq_2), (X,*,0_2,\leq_1)$ are neutro-posets.*

Example 3.15. Consider the Neutro-BCK-algebra in Example 3.13. Then we have neutro-posets $(X,*,w,\leq_1), (X,*,a,\leq_2)$ and $(X,*,0_2,\leq)$ in Table 4, where $-$ means that elements are not comparable and I means that are indeterminates.

Definition 3.16. Let $(X,*,0)$ be a Neutro-BCK-algebra, $\theta \in X$ and $Y \subseteq X$. Then

Table 4: neutro-posets

$\leq_1$	a	b	w	x	y	z
a	a	$-$	a	x	y	z
b	$-$	b	w	x	y	z
w	a	w	w	w	w	w
x	x	x	w	x	x	x
y	y	y	w	x	y	y
z	z	z	w	x	y	z

$\leq_2$	a	b	w	x	y	z
a	a	b	a	x	y	z
b	b	b	w	x	y	z
w	a	w	w	I	I	I
x	x	x	I	x	I	I
y	y	y	I	I	y	I
z	z	z	I	I	I	z

$\leq$	a	b	w	x	y	z
a	a	a	w	a	a	a
b	a	b	b	b	b	b
w	w	b	w	$-$	$-$	$-$
x	a	b	$-$	x	$-$	$-$
y	a	b	$-$	$-$	y	$-$
z	a	b	$-$	$-$	$-$	z

(*i*) Y is called a Neutro-BCK-subalgebra, if (1) $0 \in Y$, (2) for all $x, y \in Y$, we have $x * y \in Y$, (3) satisfies in conditions ($NBCI$-3), ($NBCI$-4) and ($NBCK$-5).

(*ii*) $\theta \in X$ is called a source element, if it is a minimum or maximum element in neutro-Hass diagram of $(X, *, 0)$.

Theorem 3.17. *Let $(X, *, 0)$ be a Neutro-BCK-algebra and $Y \subseteq X$. If Y is a Neutro-BCK-subalgebra of X, then*

(*i*) *$(Y, *, 0)$ is a Neutro-BCK-algebra.*

(*ii*) *X is a Neutro-BCK-subalgebra of X.*

Proof. They are clear. □

Corollary 3.18. *Let $(X, *, 0)$ be a Neutro-BCK-algebra and $|X| = n$. Then there exist $m \leq n$ and $x_1, x_2, \ldots, x_m \in X$ such that $(\{0, x_1, x_2, \ldots, x_m\}, *, 0)$ is a Neutro-BCK-algebra of X.*

Theorem 3.19. *Let X be a non-empty set. Then there exists a binary operation "$\bullet$" on X and $0 \in X$ such that*

(*i*) *$(X, \bullet, x_0)$ is a Neutro-BCK-algebra.*

(*ii*) *For all $\emptyset \neq Y \subseteq X$, $Y \cup \{x_0\}$ is a Neutro-BCK-subalgebra of X.*

(*iii*) *If X is a countable set, then in neutro-Hass diagram $(X, \bullet, x_0)$, we have $|Maximal(X)| = 1$ and $Minimal(X) = |X| - 1(|X|$ is cardinal of $X)$.*

(*iv*) *neutro-Hass diagram $(X, \bullet, x_0)$ has a source element.*

Proof. Let $x, y \in X$. Fixed $x_0 \in X$ and define $x * y = y$.
(*i*) Some modulations show that $(X, *, x_0)$ is a Neutro-BCK-algebra.
(*ii*) By Theorem 3.4 and definition, it is clear.
(*iii*) Let $X = \{x_0, x_1, x_2, x_3, \ldots\}$. Then by Corollary 3.11, $(X, \leq, x_0)$ is a neutro-poset and so has a neutro-Hass diagram as Figure 1. □

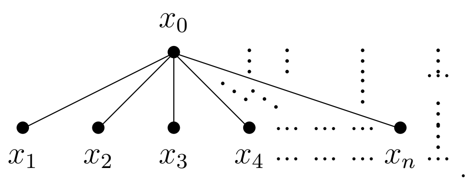

Figure 1: neutro-Hass diagram $(X, \leq, x_0)$ with source x_0.

Theorem 3.20. *Let $(X, \leq_X)$ be a chain. Then*

(*i*) *there exists $*_X$ on X and $0 \in X$ such that $(X, *_X, 0)$ is a Neutro-BCK-algebra.*

(*ii*) *for all $x, y \in X$, we have $x \leq y$ if and only if $y \leq_X x$.*

(*iii*) *In neutro-Hass diagram $(X, \bullet, x_0)$, 0 is source element.*

there exists $*_X$ on X and $0 \in X$ such that $(X, *_X, 0)$ is a Neutro-BCK-algebra.

Proof. Let $0, x, y \in X$, where $0 = Min(X)$.

(i) Define $x *_X y = \begin{cases} x \vee y & \text{if } x \leq_X y \\ x \wedge y & \text{otherwise} \end{cases}$. Some modulations show that $(X, *_X, 0)$ is a Neutro-BCK-algebra.

(ii) Let $x, y \in X$. Clearly $x * x = x$, then by definition $x \leq y$ if and only if $x * y = 0$ and $y * x \neq 0$ if and only if $y = 0$ if and only if $y \leq_X x$.

(iii) By item (ii), we get the neutro-Hass diagram $(X, \leq_X, 0)$ in Figure 1, so 0 is source element. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras, where $X_1 \cap X_2 = \emptyset$. Define $*$ on $X_1 \cup X_2$, by $x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1 \setminus X_2 \\ x *_2 y & \text{if } x, y \in X_2 \setminus X_1 \\ y & \text{otherwise} \end{cases}$.

Theorem 3.21. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Then*

(i) $(X_1 \cup X_2, *, 0_1)$ *is a Neutro-BCK-algebra.*

(ii) $(X_1 \cup X_2, *, 0_2)$ *is a Neutro-BCK-algebra.*

Proof. It is obvious. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Define $*$ on $X_1 \times X_2$, by $(x, y) * (x', y') = (x *_1 x', y *_2 y')$, where $(x, y), (x', y') \in X_1 \times X_2$.

Theorem 3.22. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Then $(X_1 \times X_2, *, (0_1, 0_2))$ is a Neutro-BCK-algebra.*

Proof. We prove only the item $(NBCI$-4$)$. Let $(x, y), (x', y') \in X_1 \times X_2$. If $(x, y) * (x', y') = (x', y') * (x, y) = (0_1, 0_2)$, then $(x *_1 x', y *_2 y') = (0_1, 0_2)$ and $(x' *_1 x, y' *_2 y) = (0_2, 0_1)$. It follows that $(x, y) = (x', y')$. In a similar way, $(x, y) * (x', y') = (x', y') * (x, y) \neq (0_1, 0_2)$, we get that $(x, y) = (x', y')$. Thus, $(X_1 \times X_2, *, (0_1, 0_2))$ is a Neutro-BCK-algebra. □

3.2 Application of Neutro-BCK-algebra

In this subsection, we describe some applications of Neutro-BCK-algebra.

In the following example, we describe some applications of Neutro-BCK-algebra. We discuss applications of Neutro-BCK-algebra for studying the competition along with algorithms. The Neutro-BCK-algebra has many utilizations in different areas, where we connect Neutro-BCK-algebra to other sciences such as economics, computer sciences and other engineering sciences. We present an example of application of Neutro-BCK-algebra in COVID-19.

Example 3.23. (COVID-19) Let $X = \{a = China, b = Italy, c = USA, d = Spain, e = Germany, f = Iran\}$ be a set of top six COVID-19 affected countries. There are many relations between the countries of the world. Suppose $*$ is one of relations on X which is described in Table 5. This relation can be economic impact, political influence, scientific impact or other chasses. For example $x * y = z$, means that the country z influences the relationship $*$ from country x to country y. Clearly $(X, *, China)$ is a Neutro-BCK-algebra.

Table 5: Neutro-BCK-algebra

$*$	$China$	$Italy$	USA	$Spain$	$Germany$	$Iran$
$China$	$China$	$Iran$	$Spain$	$Germany$	$Italy$	USA
$Italy$	$China$	$Italy$	$Iran$	$Germany$	$Spain$	$Germany$
USA	$China$	$Italy$	USA	USA	$Iran$	$Iran$
$Spain$	$China$	$China$	$China$	$Spain$	USA	$Italy$
$Germany$	$China$	$Germany$	$Italy$	$Spain$	$Germany$	$Italy$
$Iran$	$China$	$Spain$	USA	USA	$China$	$Iran$

And so we obtain neutro-Hass diagram as Figure 2. Applying Figure 2, we obtain that China is main source of COVID-19 to top five affected countries and Iran, Spain, Italy are indeterminated countries in COVID-19 affection together, USA effects Spain and Germany effects Iran.

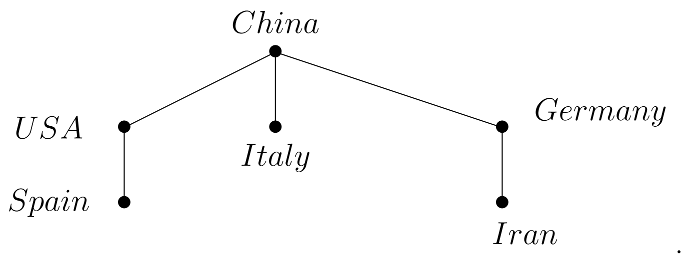

Figure 2: neutro-Hass diagram $(X, *, China)$ associated to infected COVID-19 .

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Generalizations and Alternatives of Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures

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Florentin Smarandache (2020). Generalizations and Alternatives of Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures. J*ournal of Fuzzy Extension & Applications* 1(2), 85-87. DOI: 10.22105/jfea.2020.248816.1008

1. From Paradoxism to Neutrosophy

1.1. Paradoxism

Paradoxism is an international movement in science and culture, founded and developed by Smarandache in 1980s, based on excessive use of antitheses, oxymoron, contradictions, and paradoxes in science, literature, and arts. During three decades (1980-2020) hundreds of authors from tens of countries around the globe contributed papers in various languages to 15 international paradoxist anthologies.

1.2. Neutrosophy

In 1995, the author extended the paradoxism (based on opposites) to a new branch of philosophy called neutrosophy (based on opposites and their neutral) that gave birth to many scientific branches, such as neutrosophic logic, neutrosophic set, neutrosophic probability and statistics, neutrosophic algebraic structures, and so on with multiple applications in engineering, computer science, administrative work, medical research, biology, psychology, social sciences etc.

1.3. Extensions

Neutrosophy is also an extension of Dialectics (characterized by the dynamics of opposites in philosophy), and of Yin-Yang Ancient Chinese philosophy (based also on opposites: male/female, good/bad, sky/earth, etc.) that was founded and studied two and half millennia ahead of Hegel's and Marx's Dialectics.

2. From Classical Algebras to NeutroAlgebras and AntiAlgebras

2.1. Operation, NeutroOperation, and AntiOperation

When we define an operation on a given set, it does not automatically mean that the operation is well-defined. There are three possibilities:

- The operation is well-defined (or inner-defined) for all set's elements (as in classical algebraic structures; this is classical *Operation*).
- The operation if well-defined for some elements, indeterminate for other elements, and outer-defined for others elements (this is *NeutroOperation*).
- The operation is outer-defined for all set's elements (this is *AntiOperation*).

2.2. Axiom, NeutroAxiom, and AntiAxiom

Similarly for an axiom defined on a given set endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set's elements. We have three possibilities:

- The axiom is true for all set's elements [totally true] (as in classical algebraic structures; this is classical *Axiom*).
- The axiom if true for some elements, indeterminate for other elements, and false for other elements (this is *NeutroAxiom*).
- The axiom is false for all set's elements (this is *AntiAxiom*).

Similarly for any statement, theorem, lemma, algorithm, property, etc. For example: Classical *Theorem* (which is true for all space's elements), *NeutroTheorem* (which is partially true, partially indeterminate, and partially false), and *AntiTheorem* (which is false for all space's elements).

2.3. Algebra, NeutroAlgebra, and AntiAlgebra

An algebraic structure who's all operations are well-defined and all axioms are totally true is called Classical Algebraic Structure (or Algebra). An algebraic structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called NeutroAlgebraic Structure (or NeutroAlgebra).

An algebraic structure that has at least one AntiOperation or AntiAxiom is called AntiAlgebraic Structure (or AntiAlgebra). Therefore, a neutrosophic triplet structure is formed:
<Algebra, NeutroAlgebra, AntiAlgebra>.

"Algebra" can be any classical algebraic structure, such as: groupoid, semigroup, monoid, group, commutative group, ring, field, vector space, BCK-Algebra, BCI-Algebra, etc.

3. Foundation of NeutroAlgebra and AntiAlgebra

The classical algebraic structures were generalized in 2019 and improved and extended in 2020 by Smarandache [1, 2, 3] to NeutroAlgebraic Structures (or NeutroAlgebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of partial algebra, and to AntiAlgebraic Structures (or AntiAlgebras) whose operations and axioms are totally false.

4. Foundation of NeutroStructures and AntiStructures

And in general, we extended any classical Structure, which is a space characterized by some properties, ideas, laws, shapes, hierarchy, etc., in no matter what field of knowledge, to a NeutroStructure and an AntiStructure. So, we formed a general neutrosophic triplet: Structure, NeutroStructure, and AntiStructure.

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Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited)

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Florentin Smarandache (2020). Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited). *Neutrosophic Sets and Systems*, 31, 2-16

Abstract: In all classical algebraic structures, the *Laws of Compositions* on a given set are well-defined. But this is a restrictive case, because there are many more situations in science and in any domain of knowledge when a law of composition defined on a set may be only partially-defined (or partially true) and partially-undefined (or partially false), that we call *NeutroDefined*, or totally undefined (totally false) that we call *AntiDefined*.

Again, in all classical algebraic structures, the *Axioms* (Associativity, Commutativity, etc.) defined on a set are totally true, but it is again a restrictive case, because similarly there are numerous situations in science and in any domain of knowledge when an Axiom defined on a set may be only

partially-true (and partially-false), that we call *NeutroAxiom*, or totally false that we call *AntiAxiom*. Therefore, we open for the first time in 2019 new fields of research called *NeutroStructures* and *AntiStructures* respectively.

Keywords: Neutrosophic Triplets, (Axiom, NeutroAxiom, AntiAxiom), (Law, NeutroLaw, AntiLaw), (Associativity, NeutroAssociaticity, AntiAssociativity), (Commutativity, NeutroCommutativity, AntiCommutativity), (WellDefined, NeutroDefined, AntiDefined), (Semigroup, NeutroSemigroup, AntiSemigroup), (Group, NeutroGroup, AntiGroup), (Ring, NeutroRing, AntiRing), (Algebraic Structures, NeutroAlgebraic Structures, AntiAlgebraic Structures), (Structure, NeutroStructure, AntiStructure), (Theory, NeutroTheory, AntiTheory), S-denying an Axiom, S-geometries, Multispace with Multistructure.

1. Introduction

For the necessity to more accurately reflect our reality, Smarandache [1] introduced for the first time in 2019 the NeutroDefined and AntiDefined Laws, as well as the NeutroAxiom and AntiAxiom, inspired from Neutrosophy ([2], 1995), giving birth to new fields of research called NeutroStructures and AntiStructures.

Let's consider a given classical algebraic Axiom. We defined for the first time the neutrosophic triplet corresponding to this Axiom, which is the following: *(Axiom, NeutroAxiom, AntiAxiom)*; while the classical Axiom is 100% or totally true, the NeutroAxiom is partially true and partially false (the degrees of truth and falsehood are both > 0), while the AntiAxiom is 100% or totally false [1].

For the classical algebraic structures, on a non-empty set endowed with well-defined binary laws, we have properties (axioms) such as: associativity & non-associativity, commutativity & non-commutativity, distributivity & non-distributivity; the set may contain a neutral element with respect to a given law, or may not; and so on; each set element may have an inverse, or some set elements may not have an inverse; and so on.

Consequently, we constructed for the first time the neutrosophic triplet corresponding to the Algebraic Structures [1], which is this: *(Algebraic Structure, NeutroAlgebraic Structure, AntiAlbegraic Structure).*

Therefore, we had introduced for the first time [1] the *NeutroAlgebraic Structures* & the *AntiAlgebraic Structures*. A (classical) Algebraic Structure is an algebraic structure dealing only with (classical) Axioms (which are totally true). Then a NeutroAlgebraic Structure is an algebraic structure that has at least one NeutroAxiom, and no AntiAxioms.

While an AntiAlgebraic Structure is an algebraic structure that has at least one AntiAxiom.
These definitions can straightforwardly be extended from Axiom/NeutroAxiom/AntiAxiom to any Property/NeutroProperty/AntiProperty, Proposition/NeutroProposition/AntiProposition, Theorem/NeutroTheorem/AntiTheorem, Theory/NeutroTheory/AntiTheory, etc. and from Algebraic Structures to other Structures in any field of knowledge.

2. Neutrosophy

We recall that in neutrosophy we have for an item <A>, its opposite <antiA>, and in between them their neutral <neutA>.

We denoted by <nonA> = <neutA> ∪ <antiA>, where ∪ means union, and <nonA> means what is not <A>.

Or <nonA> is refined/split into two parts: <neutA> and <antiA>.

The neutrosophic triplet of <A> is: $(\langle A\rangle, \langle neutA\rangle, \langle antiA\rangle)$, *with* $\langle neutA\rangle \cup \langle antiA\rangle = \langle nonA\rangle$.

3. Definition of Neutrosophic Triplet Axioms

Let $\mathcal{U}$ be a universe of discourse, endowed with some well-defined laws, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and an Axiom α, defined on S, using these laws. Then:

1) If all elements of $\mathcal{S}$ verify the axiom α, we have a *Classical Axiom*, or simply we say *Axiom*.
2) If some elements of $\mathcal{S}$ verify the axiom α and others do not, we have a *NeutroAxiom* (which is also called *NeutAxiom*).
3) If no elements of $\mathcal{S}$ verify the axiom α, then we have an *AntiAxiom*.

The Neutrosophic Triplet Axioms are:
(Axiom, NeutroAxiom, AntiAxiom) with

NeutroAxiom ∪ *AntiAxiom* = *NonAxiom*,
and *NeutroAxiom* ∩ *AntiAxiom* = φ (empty set),
where ∩ means intersection.

Theorem 1: The Axiom is 100% true, the NeutroAxiom is partially true (its truth degree > 0) and partially false (its falsehood degree > 0), and the AntiAxiom is 100% false.

Proof is obvious.

Theorem 2: Let *d: {Axiom, NeutroAxiom, AntiAxiom}* → *[0 ,1]* represent the degree of negation function.

The NeutroAxiom represents a degree of partial negation {$d \in (0, 1)$} of the Axiom, while the AntiAxiom represents a degree of total negation {$d = 1$} of the Axiom.
Proof is also evident.

4. Neutrosophic Representation

We have: $\langle A \rangle$ = *Axiom;*

$\langle neutA \rangle$ = *NeutroAxiom (or NeutAxiom);*

$\langle antiA \rangle$ = *AntiAxiom;* and $\langle nonA \rangle$ = *NonAxiom.*

Similarly, as in *Neutrosophy*, *NonAxiom* is refined/split into two parts: *NeutroAxiom* and *AntiAxiom*.

5. Application of NeutroLaws in Soft Science

In *soft sciences* the laws are interpreted and re-interpreted; in social and political legislation the laws are flexible; the same law may be true from a point of view, and false from another point of view. Thus, the law is partially true and partially false (it is a *Neutrosophic Law*).
For example, "gun control". There are people supporting it because of too many crimes and violence (and they are right), and people that oppose it because they want to be able to defend themselves and their houses (and they are right too).
We see two opposite propositions, both of them true, but from different points of view (from different criteria/parameters; plithogenic logic may better be used herein). How to solve this? Going to the middle, in between opposites (as in neutrosophy): allow military, police, security, registered hunters to bear arms; prohibit mentally ill, sociopaths, criminals, violent people from bearing arms; and background check on everybody that buys arms, etc.

6. Definition of Classical Associativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is associative on the set $\mathcal{S}$, iff $\forall\, a, b, c \in \mathcal{S}$, $a * (b * c) = (a * b) * c$.

7. Definition of Classical NonAssociativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is non-associative on the set $\mathcal{S}$, iff $\exists a, b, c \in \mathcal{S}$, such that $a * (b * c) \neq (a * b) * c$.
So, it is sufficient to get a single triplet a, b, c (where a, b, c may even be all three equal, or only two of them equal) that doesn't satisfy the associativity axiom.
Yet, there may also exist some triplet $d, e, f \in \mathcal{S}$ that satisfies the associativity axiom: $d * (e * f) = (d * e) * f$.
The classical definition of NonAssociativity does not make a distinction between a set $(\mathcal{S}_1, *)$ whose all triplets $a, b, c \in \mathcal{S}_1$ verify the non-associativity inequality, and a set $(\mathcal{S}_2, *)$ whose some triplets verify the non-associativity inequality, while others don't.

8. NeutroAssociativity & AntiAssociativity

If ⟨A⟩ = (classical) Associativity, then ⟨nonA⟩ = (classical) NonAssociativity.

But we refine/split ⟨nonA⟩ into two parts, as above:

⟨*neutA*⟩ = *NeutroAssociativity;*

⟨*antiA*⟩ = *AntiAssociativity.*

Therefore, *NonAssociativity* = *NeutroAssociativity* ∪ *AntiAssociativity*.

The Associativity's neutrosophic triplet is: <*Associativity*, *NeutroAssociativity*, *AntiAssociativity*>.

9. Definition of NeutroAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $(\mathcal{S},*)$ is NeutroAssociative if and only if:

there exists at least one triplet $a_1, b_1, c_1 \in \mathcal{S}$ such that: $a_1 * (b_1 * c_1) = (a_1 * b_1) * c_1$; and there exists at least one triplet $a_2, b_2, c_2 \in \mathcal{S}$ such that: $a_2 * (b_2 * c_2) \neq (a_2 * b_2) * c_2$. Therefore, some triplets verify the associativity axiom, and others do not.

10. Definition of AntiAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $(\mathcal{S},*)$ is AntiAssociative if and only if: for any triplet $a, b, c \in \mathcal{S}$ one has $a * (b * c) \neq (a * b) * c$. Therefore, none of the triplets verify the associativity axiom.

11. Example of Associativity

Let *N* = *{0, 1, 2, …, ∞}*, the set of natural numbers, be the universe of discourse, and the set

$\mathcal{S} = \{0, 1, 2, \dots, 9\} \subset$ N, also the binary law $*$ be the classical addition modulo 10 defined on N.

Clearly the law * is well-defined on S, and associative since:

$a + (b + c) = (a + b) + c$ (mod 10), for all $a, b, c \in \mathcal{S}$.

The degree of negation is 0%.

12. Example of NeutroAssociativity

$\mathcal{S} = \{0, 1, 2, \dots, 9\}$, and the well-defined binary law $*$ constructed as below:

$a * b = 2a + b$ (mod 10).

Let's check the associativity: $a * (b * c) = 2a + (b * c) = 2a + 2b + c$

$(a * b) * c = 2(a * b) + c = 2(2a + b) + c = 4a + 2b + c$

The triplets that verify the associativity result from the below equality: $2a + 2b + c = 4a + 2b + c$ or $2a = 4a$ (mod 10) or $0 = 2a$ (mod 10), whence $a \in \{0, 5\}$.

Hence, two general triplets of the form: $\{(0, b, c), (5, b, c), \text{where } b, c \in \mathcal{S}\}$ verify the associativity.

The degree of associativity is $\frac{2}{10} = 20\%$, corresponding to the two numbers $\{0, 5\}$ out of ten.

While the other general triplet: $\{(a, b, c), \text{where } a \in \mathcal{S} \setminus \{0, 5\}, \text{while } b, c \in \mathcal{S}\}$

do not verify the associativity.

The degree of negation of associativity is $\frac{8}{10} = 80\%$.

13. Example of AntiAssociativity

$\mathcal{S} = \{a, b\}$, and the binary law $*$ well-defined as in the below Cayley Table:

*	a	b
a	b	b
b	a	a

Theorem 3. For any $x, y, z \in \mathcal{S}$, $x * (y * z) \neq (x * y) * z$.

Proof. We have $2^3 = 8$ possible triplets on $\mathcal{S}$:

1) (a, a, a)

$a * (a * a) = a * b = b$

while $(a * a) * a = b * a = a \neq b$.

2) (a, a, b)

$a * (a * b) = a * b = b$

$(a * a) * b = b * b = a \neq b$.

3) (a, b, a)

$a * (b * a) = a * a = b$

$(a * b) * a = b * a = a \neq b$.

4) (b, a, a)

$b * (a * a) = b * b = a$

$(b * a) * a = a * a = b \neq a$.

5) (a, b, b)

$a * (b * b) = a * a = b$

$(a * b) * b = b * b = a \neq b$.

6) (b, a, b)

$b * (a * b) = b * b = a$

$(b * a) * b = a * b = b \neq a$.

7) (b, b, a)

$b * (b * a) = b * a = a$

$(b * b) * a = a * a = b \neq a$.

8) (b, b, b)

$b * (b * b) = b * a = a$

$(b * b) * b = a * b = b \neq a$.

Therefore, there is no possible triplet on $\mathcal{S}$ to satisfy the associativity. Whence the law is AntiAssociative. The degree of negation of associativity is $\frac{8}{8} = 100\%$.

14. Definition of Classical Commutativity

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is Commutative on the set $\mathcal{S}$, iff $\forall\, a, b \in \mathcal{S},\ a * b = b * a$.

15. Definition of Classical NonCommutativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is NonCommutative on the set $\mathcal{S}$, iff $\exists a, b \in \mathcal{S}$, such that $a * b \neq b * a$.
So, it is sufficient to get a single duplet $a, b \in \mathcal{S}$ that doesn't satisfy the commutativity axiom.
However, there may exist some duplet $c, d \in \mathcal{S}$ that satisfies the commutativity axiom: $c * d = d * c$.

The classical definition of NonCommutativity does not make a distinction between a set $(\mathcal{S}_1,*)$ whose all duplets $a, b \in \mathcal{S}_1$ verify the NonCommutativity inequality, and a set $(\mathcal{S}_2,*)$ whose some duplets verify the NonCommutativity inequality, while others don't.
That's why we refine/split the NonCommutativity into NeutroCommutativity and AntiCommutativity.

16. NeutroCommutativity & AntiCommutativity

Similarly to Associativity we do for the Commutativity:
If $\langle A \rangle$ = (classical) *Commutativity*, then $\langle \text{nonA} \rangle$ = (classical) *NonCommutativity*.
But we refine/split $\langle \text{nonA} \rangle$ into two parts, as above:

$\langle neutA \rangle$ = *NeutroCommutativity;*

$\langle antiA \rangle$ = *AntiCommutativity.*

Therefore, *NonCommutativity* = *NeutroCommutativity* $\cup$ *AntiCommutativity*.
The Commutativity's neutrosophic triplet is:

<Commutativity, NeutroCommutativity, AntiCommutativity>.

In the same way, Commutativity means all elements of the set commute with respect to a given binary law, NeutroCommutativity means that some elements commute while others do not, while AntiCommutativity means that no elements commute.

17. Example of NeutroCommutativity

$\mathcal{S} = \{a, b, c\}$, and the well-defined binary law $*$.

$*$	a	b	c
a	b	c	c
b	c	b	a
c	b	b	c

$a * b = b * a = c$ (commutative);

$\begin{cases} a * c = c \\ c * a = b \neq c \end{cases}$ (not commutative);

$\begin{cases} b * c = a \\ c * b = b \neq a \end{cases}$ (not commutative).

We conclude that $(\mathcal{S},*)$ is $\frac{1 \text{ pair}}{3 \text{ pairs}} \approx 33\%$ commutative, and $\frac{2 \text{ pair}}{3 \text{ pairs}} \approx 67\%$ not commutative.

Therefore, the degree of negation of the commutativity of $(\mathcal{S},*)$ is 67%.

18. Example of AntiCommutativity

$\mathcal{S} = \{a, b\}$, and the below binary well-defined law $*$.

*	a	b
a	b	b
b	a	a

where $a * b = b$, $b * a = a \neq b$ (not commutative)

Other pair of different element does not exist, since we cannot take $a * a$ nor $b * b$. The degree of negation of commutativity of this $(\mathcal{S},*)$ is 100%.

19. Definition of Classical Unit-Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $\mathcal{S}$ has a classical unit element $e \in \mathcal{S}$, iff e is unique, and for any $x \in \mathcal{S}$ one has $x * e = e * x = x$.

20. Partially Negating the Definition of Classical Unit-Element

It occurs when at least one of the below statements occurs:

1) There exists at least one element $a \in \mathcal{S}$ that has no unit-element.

2) There exists at least one element $b \in \mathcal{S}$that has at least two distinct unit-elements e_1 $e_2 \in \mathcal{S}$, $e_1 \neq e_2$, such that:

$b * e_1 = e_1 * b = b$,

$b * e_2 = e_2 * b = b$.

3) There exists at least two different elements $c, d \in \mathcal{S}$, $c \neq d$, such that they have different unit-elements $e_c, e_d \in \mathcal{S}$, $e_c \neq e_d$, with $c * e_c = e_c * c = c$, and $d * e_d = e_d * d = d$.

21. Totally Negating the Definition of Classical Unit-Element

The set $(\mathcal{S},*)$ has *AntiUnitElements*, if:

Each element $x \in S$ has either no unit-element, or two or more unit-elements (unicity of unit-element is negated).

22. Definition of NeutroUnitElements

The set $(S,*)$ has *NeutroUnit Elements*, if:

1) [Degree of Truth] There exist at least one element $a \in S$ that has a single unit-element.

2) [Degree of Falsehood] There exist at least one element $b \in S$ that has either no unit-element, or at least two distinct unit-elements.

23. Definition of AntiUnit Elements

The set $(S,*)$ has *AntiUnit Elements*, if:

Each element $x \in S$ has either no unit-element, or two or more distinct unit-elements.

24. Example of NeutroUnit Elements

$S = \{a, b, c\}$, and the well-defined binary law $*$:

*	a	b	c
a	b	b	a
b	b	b	a
c	a	b	c

Since,

$a * c = c * a = a$

$c * c = c$

the common unit element of a and c is c (two distinct elements $a \neq c$ have the same unit element c).

From $b * a = a * b = b$

$b * b = b$

we see that the element b has two distinct unit elements a and b.

Since only one element b does not verify the classical unit axiom (i.e. to have a unique unit), out of 3 elements, the degree of negation of unit element axiom is $\frac{1}{3} \approx 33\%$, while $\frac{2}{3} \approx 67\%$ is the degree of truth (validation) of the unit element axiom.

25. Example of AntiUnit Elements

$S = \{a, b, c\}$, endowed with the well-defined binary law $*$ as follows:

$*$	a	b	c
a	a	a	a
b	a	c	b
c	a	c	b

Element a has 3 unit-elements: a, b, c, because:

$a * a = a$

$a * b = b * a = a$

and $a * c = c * a = a$.

Element b has no u-it element, since:

$b * a = a \neq b$

$b * b = c \neq b$

and $b * c = b$, but $c * b \neq b$.

Element c has no unit-element, since:

$c * a = a \neq c$

$c * b = c$, but $b * c = b \neq c$,

and $c * c = b \neq c$.

The degree of negation of the unit-element axiom is $\frac{3}{3} = 100\%$.

26. Definition of Classical Inverse Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non $-$ empty set $\mathcal{S} \subseteq \mathcal{U}$.

Let $e \in \mathcal{S}$ be the classical unit element, which is unique.

For any element $x \in \mathcal{S}$, there exists a unique element, named the inverse of x, denoted by x^{-1}, such that:

$x * x^{-1} = x^{-1} * x = e$.

27. Partially Negating the Definition of Classical Inverse Element

It occurs when at least one statement from below occurs:

1) There exists at least one element $a \in \mathcal{S}$ that has no inverse with respect to no ad-hoc unit-element;

or

2) There exists at least one element $b \in \mathcal{S}$ that has two or more inverses with respect to some ad-hoc unit-elements.

28. Totally Negating the Definition of Classical Inverse Element

Each element has either no inverse, or two or more inverses with respect to some ad-hoc unit-elements respectively.

29. Definition of NeutroInverse Elements

The set $(\mathcal{S},*)$ has NeutroInverse Elements if:

1) [Degree of Truth] There exist at least one element that has a unique inverse with respect to some ad-hoc unit-element.

2) [Degree of Falsehood] There exists at least one element $c \in \mathcal{S}$ that does not have any inverse with respect to no ad-hoc unit element, or has at least two distinct inverses with respect to some ad-hoc unit-elements.

30. Definition of AntiInverse Elements

The set $(\mathcal{S},*)$ has AntiInverse Elements, if: each element has either no inverse with respect to no ad-hoc unit-element, or two or more distinct inverses with respect to some ad-hoc unit-elements.

31. Example of NeutroInverse Elements

$S = \{a, b, c\}$, endowed with the binary well-defined law * as below:

*	a	b	c
a	a	b	c
b	b	a	a
c	b	b	b

Because $a * a = a$, hence its ad-hoc unit/neutral element $neut(a) = a$ and correspondingly its inverse element is $inv(a) = a$.

Because $b * a = a * b = b$, hence its ad-hoc inverse/neutral element $neut(b) = a$; from $b * b = a$, we get $inv(b) = b$.

No $neut(c)$, hence no $inv(c)$.

Hence *a* and *b* have ad-hoc inverses, but *c* doesn't.

32. Example of AntiInverse Elements

Similarly, $S = \{a, b, c\}$, endowed with the binary well-defined law * as below:

*	a	b	c
a	b	b	c
b	a	a	a
c	c	a	a

There is no *neut(a)* and no *neut(b)*, hence: no *inv(a)* and *no inv(b)*.

$c * a = a * c = c$, hence: $neut(c) = a$.

$c * b = b * c = a$, hence: $inv(c) = b$;

$c * c = c * c = a,$ hence: $inv(c) = c$; whence we get two inverses of c.

33. Cases When Partial Negation (NeutroAxiom) Does Not Exist

Let's consider the classical geometric *Axiom*:

On a plane, through a point exterior to a given line it's possible to draw a single parallel to that line.

The *total negation* is the following *AntiAxiom*:

On a plane, through a point exterior to a given line it's possible to draw either no parallel, or two or more parallels to that line.

The *NeutroAxiom* does not exist since it is not possible to partially deny and partially approve this axiom.

34. Connections between the neutrosophic triplet *(Axiom, NeutroAxiom, AntiAxiom)* and the *S-denying an Axiom*

The *S-denying of an Axiom* was first defined by Smarandache [3, 4] in 1969 when he constructed *hybrid geometries* (or *S-geometries*) [5 – 18].

35. Definition of S-denying an Axiom

An *Axiom* is said *S-denied* [3, 4] if in the same space the axiom behaves differently (i.e., validated and invalided; or only invalidated but in at least two distinct ways). Therefore, we say that an axiom is partially or totally negated { or there is a degree of negation in (0, 1] of this axiom }: http://fs.unm.edu/Geometries.htm.

36. Definition of S-geometries

A geometry is called *S-geometry* [5] if it has at least one *S*-denied axiom.

Therefore, the Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries were united altogether for the first time, into the same space, by some *S*-geometries. These *S*-geometries could be partially Euclidean and partially Non-Euclidean, or only Non-Euclidean but in multiple ways.

The most important contribution of the *S*-geometries was the introduction of the *degree of negation of an axiom* (and more general the degree of negation of any theorem, lemma, scientific or humanistic proposition, theory, etc.).

Many geometries, such as pseudo-manifold geometries, Finsler geometry, combinatorial Finsler geometries, Riemann geometry, combinatorial Riemannian geometries, Weyl geometry, Kahler geometry are particular cases of *S*-geometries. (Linfan Mao).

37. Connection between *S-denying an Axiom* and *NeutroAxiom / AntiAxiom*

"Validated and invalidated" Axiom is equivalent to NeutroAxiom. While "only invalidated but in at least two distinct ways" Axiom is part of the AntiAxiom (depending on the application).

"Partially negated" (or $0 < d < 1$, where d is the degree of negation) is referred to NeutroAxiom. While "there is a degree of negation of an axiom" is referred to both NeutroAxiom (when $0 < d < 1$) and AntiAxiom (when $d = 1$).

38. Connection between NeutroAxiom and MultiSpace

In any domain of knowledge, a *S-multispace with its multistructure* is a finite or infinite (countable or uncountable) union of many spaces that have various structures (Smarandache, 1969, [19]). The multi-spaces with their multi-structures [20, 21] may be non-disjoint. The multispace with multistructure form together a *Theory of Everything*. It can be used, for example, in the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak, and strong interactions in physics.

Therefore, a NeutroAxiom splits a set M, which it is defined upon, into two subspaces: one where the Axiom is true and another where the Axiom is false. Whence M becomes a BiSpace with BiStructure (which is a particular case of MultiSpace with MultiStructure).

39. (Classical) WellDefined Binary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and a binary law $*$ defined on U. For any $x, y \in \mathcal{S}$, one has $x * y \in \mathcal{S}$.

40. NeutroDefined Binary Law

There exist at least two elements (that could be equal) $a, b \in \mathcal{S}$ such that $a * b \in \mathcal{S}$. And there exist at least other two elements (that could be equal too) $c, d \in \mathcal{S}$ such that $c*d \notin S$.

41. Example of NeutroDefined Binary Law

Let $U = \{a, b, c\}$ be a universe of discourse, and a subset $\mathcal{S} = \{a, b\}$, endowed with the below NeutroDefined Binary Law $*$:

$*$	a	b
a	b	b
b	a	c

We see that: $a * b = b \in S$, $b * a = a \in S$, but $b * b = c \notin S$.

42. AntiDefined Binary Law

For any $x, y \in \mathcal{S}$ one has $x * y \notin \mathcal{S}$.

43. Example of AntiDefined Binary Law

Let $U = \{a, b, c, d\}$ a universe of discourse, and a subset $\mathcal{S} = \{a, b\}$, and the below binary well-defined law $*$.

$*$	a	b
a	c	d
b	d	c

where all combinations between *a* and *b* using the law * give as output *c* or *d* who do not belong to S.

44. Theorem 4 (The Degenerate Case)

If a set is endowed with AntiDefined Laws, all its algebraic structures based on them will be AntiStructures.

45. WellDefined n-ary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and a n-ary law, for n integer, $n \geq 1$, defined on $\mathcal{U}$.

$L: \mathcal{U}^n \to \mathcal{U}$.

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $L(x_1, x_2, \dots, x_n) \in \mathcal{S}$.

46. NeutroDefined n-ary Law

There exists at least a n-plet $a_1, a_2, \dots, a_n \in \mathcal{S}$ such that $L(a_1, a_2, \dots, a_n) \in S$. The elements $a_1, a_2, \dots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \dots, b_n \in \mathcal{S}$ such that $L(a_1, a_2, \dots, a_n) \notin S$. The elements $b_1, b_2, \dots, b_n$ may be equal or not among themselves.

47. AntiDefined n-ary Law

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $L(x_1, x_2, \dots, x_n) \notin \mathcal{S}$.

48. WellDefined n-ary HyperLaw

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subset_{\neq} \mathcal{U}$, and a n-ary hyperlaw, for n integer, $n \geq 1$:

$H: \mathcal{U}^n \to \mathcal{P}(\mathcal{U})$, where $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$.

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \dots, x_n) \in \mathcal{P}(\mathcal{S})$.

49. NeutroDefined n-ary HyperLaw

There exists at least a n-plet $a_1, a_2, \dots, a_n \in \mathcal{S}$ such that $H(a_1, a_2, \dots, a_n) \in \mathcal{P}(\mathcal{S})$. The elements $a_1, a_2, \dots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \dots, b_n \in \mathcal{S}$ such that $H(b_1, b_2, \dots, b_n) \notin \mathcal{P}(\mathcal{S})$. The elements $b_1, b_2, \dots, b_n$ may be equal or not among themselves.

50. AntiDefined n-ary HyperLaw

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \dots, x_n) \notin \mathcal{P}(\mathcal{S})$.

*

The most interesting are the cases when the composition law(s) are well-defined (classical way) and neutro-defined (neutrosophic way).

51. WellDefined NeutroStructures

Are structures whose laws of compositions are well-defined, and at least one axiom is NeutroAxiom, while not having any AntiAxiom.

52. NeutroDefined NeutroStructures

Are structures whose at least one law of composition is NeutroDefined, and all other axioms are NeutroAxioms or Axioms.

53. Example of NeutroDefined NeutroGroup

Let U = {a, b, c, d} be a universe of discourse, and the subset

$S = \{a, b, c\}$, endowed with the binary law $*$:

*	a	b	c
a	a	c	c
b	a	a	a
c	c	a	d

NeutroDefined Law of Composition:
Because, for example: $a*b = c \in S$, but $c*c = d \notin S$.
NeutroAssociativity:
Because, for example: $a*(a*c) = a*c = c$ and $(a*a)*c = a*c = c$;
while, for example: $a*(b*c) = a*a = a$ and $(a*b)*c = c*c = d \neq a$.
NeutroCommutativity:
Because, for example: $a*c = c*a = c$, but $a*b = c$ while $b*a = a \neq c$.
NeutroUnit Element:
There exists the same unit-element a for a and c, or $neut(a) = neut(c) = a$, since $a*a = a$ and $c*a = a*c = c$.
But there is no unit element for b, because $b*x = a$, not b, for any $x \in S$ (see the above Cayley Table).
NeutroInverse Element:
With respect to the same unit element a, there exists an inverse element for a, which is a, or $inv(a) = a$, because $a*a = a$, and an inverse element for c, which is b, or $inv(c) = b$, because $c*b = b*c = a$.
But there is no inverse element for b, since b has no unit element.
Therefore $(S, *)$ is a NeutroDefined NeutroCommutative NeutroGroup.

54. WellDefined AntiStructures

Are structures whose laws of compositions are well-defined, and have at least one AntiAxiom.

55. NeutroDefined AntiStructures

Are structures whose at least one law of composition is NeutroDefined and no law of composition is AntiDefined, and has at least one AntiAxiom.

56. AntiDefined AntiStructures

Are structures whose at least one law of composition is AntiDefined, and has at least one AntiAxiom.

57. Conclusion

The neutrosophic triplet (<A>, <neutA>, <antiA>), where <A> may be an "Axiom", a "Structure", a "Theory" and so on, <antiA> the opposite of <A>, while <neutA> (or <neutroA>) their neutral in between, are studied in this paper.

The NeutroAlgebraic Structures and AntiAlgebraic Structures are introduced now for the first time, because they have been ignored by the classical algebraic structures. Since, in science and technology and mostly in applications of our everyday life, the laws that characterize them are not necessarily well-defined or well-known, and the axioms / properties / theories etc. that govern their spaces may be only partially true and partially false (as *<neutA>* in neutrosophy, which may be a blending of truth and falsehood).

Mostly in idealistic or imaginary or abstract or perfect spaces we have rigid laws and rigid axioms that totally apply (that are 100% true). But the laws and the axioms should be more flexible in order to comply with our imperfect world.

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Length Neutrosophic Subalgebras of BCK/BCI-Algebras

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Young Bae Jun, Madad Khan, Florentin Smarandache, Seok-Zun Song (2020). Length Neutrosophic Subalgebras of BCK/BCI-Algebras. *Bulletin of the Section of Logic*, 49(4), 377-400. DOI: 10.18778/0138-0680.2020.21

Abstract

Given $i, j, k \in \{1, 2, 3, 4\}$, the notion of (i, j, k)-length neutrosophic subalgebras in BCK/BCI-algebras is introduced, and their properties are investigated. Characterizations of length neutrosophic subalgebras are discussed by using level sets of interval neutrosophic sets. Conditions for level sets of interval neutrosophic sets to be subalgebras are provided.

Keywords: Interval neutrosophic set, interval neutrosophic length, length neutrosophic subalgebra.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

1. Introduction

The intuitionistic fuzzy set, which has been introduced by Atanassov [1], consider both truth-membership and falsity membership. The neutrosophic set developed by Smarandache [6, 7, 8] is a formal framework which generalizes the concept of the classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, interval valued intuitionistic fuzzy set and paraconsistent set etc. Neutrosophic set theory is applied to various part, includ-ing algebra, topology, control theory, decision making problems, medicines and in many real life problems. Wang et al. [9, 11] presented the con-cept of interval neutrosophic sets, which is more precise and more flex-ible than the single-valued neutrosophic set. An interval-valued neutro-sophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$. The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent in-formation which exists in real world. Jun et al. [4] discussed interval neutrosophic sets in BCK/BCI-algebras. They introduced the notion of $(T\,(i, j), I(k, l), F\,(m, n))$-interval neutrosophic subalgebras in BCK/BCI-algebras for $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, and investigated several properties and relations. They also introduced the notion of interval neutrosophic length of an interval neutrosophic set, and investigated related properties.

In this paper, we introduce the notion of (i, j, k)-length neutrosophic subalgebras in BCK/BCI-algebras for $i, j, k \in \{1, 2, 3, 4\}$, and investigate several properties. We consider relations of (i, j, k)-length neutrosophic subalgebras, and discuss characterizations of (i, j, k)-length neutrosophic subalgebras. Using subalgebras of a BCK-algebra, we construct (i, j, k)-length neutrosophic subalgebras for $i, j, k \in \{1, 4\}$. We consider conditions for level sets of interval neutrosophic set to be subalgebras of a BCK/BCI-algebra.

2. Preliminaries

By a BCI-*algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [2] and [5] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Definition 2.1 ([3]). For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- *fuzzy subalgebra* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{2.1}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{2.2}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{2.3}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{2.4}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (see [7]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T\colon X \to [0, 1]$ is a truth membership function, $A_I\colon X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function.

An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ (see [11, 10]).

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ and $\mathcal{P}^*([0, 1])$ be the family of all subintervals of $[0, 1]$ unless otherwise specified.

Definition 2.2 ([11, 10]). An *interval neutrosophic set* in a nonempty set X is a structure of the form:

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}$$

where

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval truth-membership function*,

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval indeterminacy-membership function*, and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval falsity-membership function.*

For the sake of simplicity, we will use the notation $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ for the interval neutrosophic set

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we consider the following functions (see [4]):

$$\mathcal{I}[T]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[F](x)\}$$

and

$$\mathcal{I}[T]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[F](x)\}.$$

Definition 2.3 ([4]). Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we define the *interval neutrosophic length* of $\mathcal{I}$ as an ordered triple $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ where

$$\mathcal{I}[T]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x),$$

$$\mathcal{I}[I]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x),$$

and

$$\mathcal{I}[F]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x),$$

which are called *interval neutrosophic T-length*, *interval neutrosophic I-length* and *interval neutrosophic F-length* of $\mathcal{I}$, respectively.

3. Length neutrosophic subalgebras

Definition 3.1. Given $i, j, k \in \{1, 2, 3, 4\}$, an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is called an *(i, j, k)-length neutrosophic subalgebra* of $(X, *, 0)$ if the interval neutrosophic T-length of $\mathcal{I}$ is an i-fuzzy subalgebra of $(X, *, 0)$, the interval neutrosophic I-length of $\mathcal{I}$ is a j-fuzzy subalgebra of $(X, *, 0)$, and the interval neutrosophic F-length of $\mathcal{I}$ is a k-fuzzy subalgebra of $(X, *, 0)$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1 (see [5]).
Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ where $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.1, 0.8) & \text{if } x = 0, \\ (0.3, 0.7] & \text{if } x = 1, \\ [0.0, 0.6] & \text{if } x = 2, \\ [0.4, 0.8] & \text{if } x = 3, \\ [0.2, 0.5] & \text{if } x = 4, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.2, 0.8) & \text{if } x = 0, \\ (0.4, 0.8] & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ [0.6, 0.9] & \text{if } x = 3, \\ [0.3, 0.5] & \text{if } x = 4, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.1, 0.4) & \text{if } x = 0, \\ (0.4, 0.8] & \text{if } x = 1, \\ [0.1, 0.5] & \text{if } x = 2, \\ [0.2, 0.7) & \text{if } x = 3, \\ [0.3, 0.9] & \text{if } x = 4. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 2.
It is routine to verify that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,4)$-length neutrosophic subalgebra of $(X, *, 0)$.

Table 2. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.7	0.6	0.3
1	0.4	0.4	0.4
2	0.6	0.5	0.4
3	0.4	0.3	0.5
4	0.3	0.2	0.6

PROPOSITION 3.3. Given an (i,j,k)-length neutrosophic subalgebra $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ of $(X,*,0)$, we have the following assertions.

(1) If $i,j,k \in \{1,3\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \geq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \geq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \geq \mathcal{I}[F]_\ell(x)). \tag{3.1}$$

(2) If $i,j,k \in \{2,4\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \leq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \leq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \leq \mathcal{I}[F]_\ell(x)). \tag{3.2}$$

(3) If $i,j \in \{1,3\}$ and $k \in \{2,4\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \geq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \geq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \leq \mathcal{I}[F]_\ell(x)). \tag{3.3}$$

(4) If $i,j \in \{2,4\}$ and $k \in \{1,3\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \leq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \leq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \geq \mathcal{I}[F]_\ell(x)). \tag{3.4}$$

PROOF: Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an (i,j,k)-length neutrosophic subalgebra of $(X,*,0)$. If $(i,j,k) = (1,3,1)$, then

$$\mathcal{I}[T]_\ell(0) = \mathcal{I}[T]_\ell(x*x) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(x)\} = \mathcal{I}[T]_\ell(x)$$

$$\mathcal{I}[I]_\ell(0) = \mathcal{I}[I]_\ell(x * x) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(x)\} = \mathcal{I}[I]_\ell(x)$$

$$\mathcal{I}[F]_\ell(0) = \mathcal{I}[F]_\ell(x * x) \geq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(x)\} = \mathcal{I}[F]_\ell(x)$$

for all $x \in X$. Similarly, we can verify that (3.1) is true for other cases of (i, j, k). Using the similar way to the proof of (1), we can prove that (2), (3) and (4) hold. □

THEOREM 3.4. *Given a subalgebra S of $(X, *, 0)$ and $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathcal{P}^*([0,1])$, let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by*

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} A_2 & \text{if } x \in S, \\ A_1 & \text{otherwise,} \end{cases} \tag{3.5}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{otherwise,} \end{cases} \tag{3.6}$$

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} C_2 & \text{if } x \in S, \\ C_1 & \text{otherwise.} \end{cases} \tag{3.7}$$

(1) *If $A_1 \subsetneq A_2$, $B_1 \subsetneq B_2$ and $C_1 \subsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(2) *If $A_1 \supsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \supsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,4,4)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(3) *If $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,4,1)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(4) *If $A_1 \supsetneq A_2$, $B_1 \subsetneq B_2$ and $C_1 \supsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,1,4)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(5) *If $A_1 \subsetneq A_2$, $B_1 \subsetneq B_2$ and $C_1 \supsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,4)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(6) *If $A_1 \supsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,4,1)$-length neutrosophic subalgebra of $(X, *, 0)$.*

PROOF: We will prove (3) only, and others can be obtained by the similar way. Assume that $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$. If $x \in S$, then $\mathcal{I}[T](x) = A_2$, $\mathcal{I}[I](x) = B_2$ and $\mathcal{I}[F](x) = C_2$. Hence

$$\mathcal{I}[T]_\ell(x) = \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \sup\{A_2\} - \inf\{A_2\},$$
$$\mathcal{I}[I]_\ell(x) = \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \sup\{B_2\} - \inf\{B_2\},$$
$$\mathcal{I}[F]_\ell(x) = \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \sup\{C_2\} - \inf\{C_2\}.$$

If $x \notin S$, then $\mathcal{I}[T](x) = A_1$, $\mathcal{I}[I](x) = B_1$ and $\mathcal{I}[F](x) = C_1$, and so

$$\mathcal{I}[T]_\ell(x) = \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \sup\{A_1\} - \inf\{A_1\},$$
$$\mathcal{I}[I]_\ell(x) = \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \sup\{B_1\} - \inf\{B_1\},$$
$$\mathcal{I}[F]_\ell(x) = \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \sup\{C_1\} - \inf\{C_1\}.$$

Since $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$, we have

$$\sup\{A_2\} - \inf\{A_2\} \geq \sup\{A_1\} - \inf\{A_1\},$$
$$\sup\{B_2\} - \inf\{B_2\} \leq \sup\{B_1\} - \inf\{B_1\},$$
$$\sup\{C_2\} - \inf\{C_2\} \geq \sup\{C_1\} - \inf\{C_1\}.$$

Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so

$$\mathcal{I}[T]_\ell(x * y) = \sup\{A_2\} - \inf\{A_2\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},$$
$$\mathcal{I}[I]_\ell(x * y) = \sup\{B_2\} - \inf\{B_2\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},$$
$$\mathcal{I}[F]_\ell(x * y) = \sup\{C_2\} - \inf\{C_2\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.$$

If $x, y \notin S$, then

$$\mathcal{I}[T]_\ell(x * y) \geq \sup\{A_1\} - \inf\{A_1\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},$$
$$\mathcal{I}[I]_\ell(x * y) \leq \sup\{B_1\} - \inf\{B_1\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},$$
$$\mathcal{I}[F]_\ell(x * y) \geq \sup\{C_1\} - \inf\{C_1\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.$$

Assume that $x \in S$ and $y \notin S$ (or, $x \notin S$ and $y \in S$). Then

$$\mathcal{I}[T]_\ell(x * y) \geq \sup\{A_1\} - \inf\{A_1\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},$$
$$\mathcal{I}[I]_\ell(x * y) \leq \sup\{B_1\} - \inf\{B_1\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},$$
$$\mathcal{I}[F]_\ell(x * y) \geq \sup\{C_1\} - \inf\{C_1\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.$$

Therefore $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$. $\square$

Remark 3.5. We have the following relations.

(1) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{1, 3\}$ is a $(1, 1, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

(2) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{2, 4\}$ is a $(4, 4, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(3) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j \in \{1, 3\}$ and $k \in \{2, 4\}$ is a $(1, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(4) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j \in \{2, 4\}$ and $k \in \{1, 3\}$ is a $(4, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

(5) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{2, 4\}$ and $j \in \{1, 3\}$ is a $(4, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(6) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{1, 3\}$ and $j \in \{2, 4\}$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

The following example shows that the converse in Remark 3.5 is not true in general. We consider the cases (5) and (6) only in Remark 3.5.

Example 3.6. Consider the BCK-algebra $(X, *, 0)$ in Example 3.2. Given a subalgebra $S = \{0, 1, 2\}$ of $(X, *, 0)$, let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.7) & \text{if } x \in S, \\ (0.1, 0.8] & \text{otherwise,} \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.9) & \text{if } x \in S, \\ (0.3, 0.7] & \text{otherwise,} \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x \in S, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

Then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$ by Theorem 3.4(4). Since

$$\mathcal{I}[I]_\ell(2) = \mathcal{I}[I]_{\sup}(2) - \mathcal{I}[I]_{\inf}(2) = 0.9 - 0.2 = 0.7$$

and

$$\mathcal{I}[I]_\ell(3 * 2) = \mathcal{I}[I]_\ell(3) = \mathcal{I}[I]_{\sup}(3) - \mathcal{I}[I]_{\inf}(3) = 0.7 - 0.3 = 0.4,$$

we have $\mathcal{I}[I]_\ell(3 * 2) = 0.4 < 0.7 = \max\{\mathcal{I}[I]_\ell(3), \mathcal{I}[I]_\ell(2)\}$. Hence $\mathcal{I} := (\mathcal{I}[T],\ \mathcal{I}[I],\ \mathcal{I}[F])$ is not an $(i, 3, k)$-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{2, 4\}$. Given a subalgebra $S = \{0, 1, 2, 3\}$ of $(X, *, 0)$, let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.7) & \text{if } x \in S, \\ (0.3, 0.5] & \text{otherwise,} \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.4, 0.6) & \text{if } x \in S, \\ (0.3, 0.8] & \text{otherwise,} \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.8) & \text{if } x \in S, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

Then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$ by Theorem 3.4(3). But it is not an $(i, 2, k)$-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{1, 3\}$ since

$$\mathcal{I}[I]_\ell(4 * 2) = \mathcal{I}[I]_\ell(4) = 0.5 > 0.2 = \min\{\mathcal{I}[I]_\ell(4), \mathcal{I}[I]_\ell(2)\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T],\ \mathcal{I}[I],\ \mathcal{I}[F])$ in $(X, *, 0)$, we consider the following level sets:

$$\begin{aligned} U_\ell(\mathcal{I}[T]; \alpha_T) &:= \{x \in X \mid \mathcal{I}[T]_\ell(x) \geq \alpha_T\}, \\ U_\ell(\mathcal{I}[I]; \alpha_I) &:= \{x \in X \mid \mathcal{I}[I]_\ell(x) \geq \alpha_I\}, \\ U_\ell(\mathcal{I}[F]; \alpha_F) &:= \{x \in X \mid \mathcal{I}[F]_\ell(x) \geq \alpha_F\}, \end{aligned}$$

and

$$\begin{aligned} L_\ell(\mathcal{I}[T]; \beta_T) &:= \{x \in X \mid \mathcal{I}[T]_\ell(x) \leq \beta_T\}, \\ L_\ell(\mathcal{I}[I]; \beta_I) &:= \{x \in X \mid \mathcal{I}[I]_\ell(x) \leq \beta_I\}, \\ L_\ell(\mathcal{I}[F]; \beta_F) &:= \{x \in X \mid \mathcal{I}[F]_\ell(x) \leq \beta_F\}. \end{aligned}$$

THEOREM 3.7. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$ *and for any* α_T , α_I , $\alpha_F \in [0, 1]$, *the following assertions are equivalent.*

(1) $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(1,1,1)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(2) $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

PROOF: Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$ and let $\alpha_T, \alpha_I, \alpha_F \in [0,1]$ be such that $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are nonempty. If $x, y \in U_\ell(\mathcal{I}[T]; \alpha_T)$, then $\mathcal{I}[T]_\ell(x) \geq \alpha_T$ and $\mathcal{I}[T]_\ell(y) \geq \alpha_T$. Hence

$$\mathcal{I}[T]_\ell(x * y) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} \geq \alpha_T,$$

that is, $x*y \in U_\ell(\mathcal{I}[T]; \alpha_T)$. Similarly, we can see that if $x, y \in U_\ell(\mathcal{I}[I]; \alpha_I)$, then $x * y \in U_\ell(\mathcal{I}[I]; \alpha_I)$, and if $x, y \in U_\ell(\mathcal{I}[F]; \alpha_F)$, then $x * y \in U_\ell(\mathcal{I}[F]; \alpha_F)$. Therefore $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are subalgebras of $(X, *, 0)$.

Conversely, suppose that (2) is valid. If there exist $a, b \in X$ such that

$$\mathcal{I}[T]_\ell(a * b) < \min\{\mathcal{I}[T]_\ell(a), \mathcal{I}[T]_\ell(b)\},$$

then $a, b \in U_\ell(\mathcal{I}[T]; \alpha_T)$ by taking $\alpha_T = \min\{\mathcal{I}[T]_\ell(a), \mathcal{I}[T]_\ell(b)\}$, and so $a * b \in U_\ell(\mathcal{I}[T]; \alpha_T)$. It follows that $\mathcal{I}[T]_\ell(a * b) \geq \alpha_T$, a contradiction. Hence

$$\mathcal{I}[T]_\ell(x * y) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$$

for all $x, y \in X$. Similarly, we can check that

$$\mathcal{I}[I]_\ell(x * y) \geq \min\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\}$$

and

$$\mathcal{I}[F]_\ell(x * y) \geq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}$$

for all $x, y \in X$. Thus $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$. $\square$

COROLLARY 3.8. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{1, 3\}$, then $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are subalgebras of $(X, *, 0)$ whenever they are nonempty for all α_T, α_I, $\alpha_F \in [0, 1]$.

The following example shows that the converse of Corollary 3.8 is not true.

Example 3.9. Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the binary operation $*$ which is given in Table 3 (see [5]).

Table 3. Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.3, 0.9) & \text{if } x = 0, \\ (0.5, 0.7] & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ [0.4, 0.7] & \text{if } x = a, \\ (0.3, 0.5] & \text{if } x = b, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.2, 0.9) & \text{if } x = 0, \\ (0.1, 0.8] & \text{if } x = 1, \\ [0.5, 0.9] & \text{if } x = 2, \\ [0.4, 0.7] & \text{if } x = a, \\ (0.4, 0.7] & \text{if } x = b, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.1, 0.6) & \text{if } x = 0, \\ (0.6, 0.9) & \text{if } x = 1, \\ (0.4, 0.8] & \text{if } x = 2, \\ [0.5, 0.7] & \text{if } x = a, \\ (0.5, 0.7] & \text{if } x = b. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 4.

Table 4. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.6	0.7	0.5
1	0.2	0.7	0.3
2	0.5	0.4	0.4
a	0.3	0.3	0.2
b	0.2	0.3	0.2

Hence we have

$$U_\ell(\mathcal{I}[T]; \alpha_T) = \begin{cases} \emptyset & \text{if } \alpha_T \in (0.6, 1], \\ \{0\} & \text{if } \alpha_T \in (0.5, 0.6], \\ \{0, 2\} & \text{if } \alpha_T \in (0.3, 0.5], \\ \{0, 2, a\} & \text{if } \alpha_T \in (0.2, 0.3], \\ X & \text{if } \alpha_T \in [0, 0.2], \end{cases}$$

$$U_\ell(\mathcal{I}[I]; \alpha_I) = \begin{cases} \emptyset & \text{if } \alpha_I \in (0.7, 1], \\ \{0, 1\} & \text{if } \alpha_I \in (0.4, 0.7], \\ \{0, 1, 2\} & \text{if } \alpha_I \in (0.3, 0.4], \\ X & \text{if } \alpha_I \in [0, 0.3], \end{cases}$$

and

$$U_\ell(\mathcal{I}[F];\alpha_F) = \begin{cases} \emptyset & \text{if } \alpha_F \in (0.5,1], \\ \{0\} & \text{if } \alpha_F \in (0.4,0.5], \\ \{0,2\} & \text{if } \alpha_F \in (0.3,0.4], \\ \{0,1,2\} & \text{if } \alpha_F \in (0.2,0.3], \\ X & \text{if } \alpha_F \in [0,0.2], \end{cases}$$

and so $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ for all α_T, α_I, $\alpha_F \in [0,1]$ such that $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are nonempty. But $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not an (i,j,k)-length neutrosophic subalgebra of $(X,*,0)$ for $i,j,k \in \{1,3\}$ with $(i,j,k) \neq (1,1,1)$ since

$$\mathcal{I}[T]_\ell(b*2) = \mathcal{I}[T]_\ell(b) = 0.2 \ngeq 0.5 = \max\{\mathcal{I}[T]_\ell(b), \mathcal{I}[T]_\ell(2)\},$$

$$\mathcal{I}[I]_\ell(a*1) = \mathcal{I}[I]_\ell(a) = 0.3 \ngeq 0.7 = \max\{\mathcal{I}[I]_\ell(a), \mathcal{I}[I]_\ell(1)\},$$

and/or

$$\mathcal{I}[F]_\ell(b*1) = \mathcal{I}[F]_\ell(a) = 0.2 \ngeq 0.3 = \max\{\mathcal{I}[F]_\ell(b), \mathcal{I}[F]_\ell(1)\}.$$

THEOREM 3.10. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$ *and for any* β_T, β_I, $\beta_F \in [0,1]$, *the following assertions are equivalent.*

(1) $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(4,4,4)$*-length neutrosophic subalgebra of* $(X,*,0)$.

(2) $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ *and* $L_\ell(\mathcal{I}[F];\beta_F)$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty.*

PROOF: Suppose that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,4,4)$-length neutrosophic subalgebra of $(X,*,0)$ and let β_T, β_I, $\beta_F \in [0,1]$ be such that $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are nonempty. For any $x,y \in X$, if $x,y \in L_\ell(\mathcal{I}[T];\beta_T)$, then $\mathcal{I}[T]_\ell(x) \leq \beta_T$ and $\mathcal{I}[T]_\ell(y) \leq \beta_T$. It follows that

$$\mathcal{I}[T]_\ell(x*y) \leq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} \leq \beta_T$$

and so that $x*y \in L_\ell(\mathcal{I}[T];\beta_T)$. Similarly, if $x,y \in L_\ell(\mathcal{I}[I];\beta_I)$, then $x*y \in L_\ell(\mathcal{I}[I];\beta_I)$, and if $x,y \in L_\ell(\mathcal{I}[F];\beta_F)$, then $x*y \in L_\ell(\mathcal{I}[F];\beta_F)$.

Therefore (2) is valid.

Conversely, assume that $L_\ell(\mathcal{I}[T]; \beta_T)$, $L_\ell(\mathcal{I}[I]; \beta_I)$ and $L_\ell(\mathcal{I}[F]; \beta_F)$ are subalgebras of $(X, *, 0)$ whenever they are nonempty for all $\beta_T, \beta_I, \beta_F \in [0, 1]$. If there are $a, b \in X$ such that

$$\mathcal{I}[F]_\ell(a * b) > \max\{\mathcal{I}[F]_\ell(a), \mathcal{I}[F]_\ell(b)\},$$

then $a, b \in L_\ell(\mathcal{I}[F]; \beta_F)$ by taking $\beta_F = \max\{\mathcal{I}[F]_\ell(a), \mathcal{I}[F]_\ell(b)\}$. Thus $a * b \in L_\ell(\mathcal{I}[F]; \beta_F)$, which implies that $\mathcal{I}[F]_\ell(a * b) \leq \beta_F$. This is a contradiction, and so

$$\mathcal{I}[F]_\ell(x * y) \leq \max\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}$$

for all $x, y \in X$. Similarly, we get

$$\mathcal{I}[T]_\ell(x * y) \leq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$$

and

$$\mathcal{I}[I]_\ell(x * y) \leq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\}$$

for all $x, y \in X$. Consequently, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a (4, 4, 4)-length neutrosophic subalgebra of $(X, *, 0)$.

Corollary 3.11. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{2, 4\}$, then $L_\ell(\mathcal{I}[T]; \beta_T)$, $L_\ell(\mathcal{I}[I]; \beta_I)$ and $L_\ell(\mathcal{I}[F]; \beta_F)$ are subalgebras of $(X, *, 0)$ whenever they are nonempty for all $\beta_T, \beta_I, \beta_F \in [0, 1]$. □

The following example shows that the converse of Corollary 3.11 is not true.

Example 3.12. Consider the BCI-algebra $X = \{0, 1, 2, a, b\}$ in Example 3.9 and let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.5, 0.7) & \text{if } x = 0, \\ (0.2, 0.6] & \text{if } x = 1, \\ [0.3, 0.6] & \text{if } x = 2, \\ [0.1, 0.7] & \text{if } x = a, \\ (0.2, 0.8] & \text{if } x = b, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.66, 0.99) & \text{if } x = 0, \\ (0.15, 0.59] & \text{if } x = 1, \\ [0.22, 0.88) & \text{if } x = 2, \\ (0.35, 0.90] & \text{if } x = a, \\ (0.20, 0.75) & \text{if } x = b, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.75, 0.90) & \text{if } x = 0, \\ (0.45, 0.90) & \text{if } x = 1, \\ (0.25, 0.50] & \text{if } x = 2, \\ [0.50, 0.85] & \text{if } x = a, \\ (0.15, 0.60] & \text{if } x = b. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 5.

Table 5. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.2	0.33	0.15
1	0.4	0.44	0.45
2	0.3	0.66	0.25
a	0.6	0.55	0.35
b	0.6	0.55	0.45

Hence we have

$$L_\ell(\mathcal{I}[T]; \beta_T) = \begin{cases} \emptyset & \text{if } \beta_T \in [0, 0.2), \\ \{0\} & \text{if } \beta_T \in [0.2, 0.3), \\ \{0, 2\} & \text{if } \beta_T \in [0.3, 0.4), \\ \{0, 1, 2\} & \text{if } \beta_T \in [0.4, 0.6), \\ X & \text{if } \beta_T \in [0.6, 1], \end{cases}$$

$$L_\ell(\mathcal{I}[I]; \beta_I) = \begin{cases} \emptyset & \text{if } \beta_I \in [0, 0.33), \\ \{0\} & \text{if } \beta_I \in [0.33, 0.44), \\ \{0, 1\} & \text{if } \beta_I \in [0.44, 0.55), \\ \{0, 1, a, b\} & \text{if } \beta_I \in [0.55, 0.66), \\ X & \text{if } \beta_I \in [0.66, 1], \end{cases}$$

and

$$L_\ell(\mathcal{I}[F];\beta_F)=\begin{cases}\emptyset & \text{if } \beta_F\in[0,0.15),\\ \{0\} & \text{if } \beta_F\in[0.15,0.25),\\ \{0,2\} & \text{if } \beta_F\in[0.25,0.35),\\ \{0,2,a\} & \text{if } \beta_F\in[0.35,0.45),\\ X & \text{if } \beta_F\in[0.45,1],\end{cases}$$

which are subalgebras of $(X,*,0)$ for all β_T, β_I, $\beta_F\in[0,1]$ such that $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are nonempty. But $\mathcal{I}:=(\mathcal{I}[T],$ $\mathcal{I}[I]$, $\mathcal{I}[F])$ is not an (i,j,k)-length neutrosophic subalgebra of $(X,*,0)$ for $i,j,k\in\{2,4\}$ with $(i,j,k)\neq(4,4,4)$ since

$$\mathcal{I}[T]_\ell(a*1)=0.6\not\leq 0.4=\min\{\mathcal{I}[T]_\ell(a),\mathcal{I}[T]_\ell(1)\},$$

$$\mathcal{I}[I]_\ell(a*0)=0.55\not\leq 0.33=\min\{\mathcal{I}[I]_\ell(a),\mathcal{I}[I]_\ell(0)\},$$

and/or

$$\mathcal{I}[F]_\ell(2*a)=0.35\not\leq 0.25=\min\{\mathcal{I}[F]_\ell(2),\mathcal{I}[F]_\ell(a)\}.$$

Using the similar way to the proofs of Theorems 3.7 and 3.10, we have the following theorem.

THEOREM 3.13. *Given an (i,j,k)-length neutrosophic subalgebra $\mathcal{I}:=(\mathcal{I}[T],$ $\mathcal{I}[I]$, $\mathcal{I}[F])$ of $(X,*,0)$ for $i,j,k\in\{1,2,3,4\}$, the following assertions are valid.*

(1) *If $i,j\in\{1,3\}$ and $k\in\{2,4\}$, then $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(2) *If $i,k\in\{1,3\}$ and $j\in\{2,4\}$, then $U_\ell(\mathcal{I}[T];\alpha_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(3) *If $i\in\{2,4\}$ and $j,k\in\{1,3\}$, then $L_\ell(\mathcal{I}[T];\beta_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(4) *If $i,j\in\{2,4\}$ and $k\in\{1,3\}$, then $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(5) *If* $i, k \in \{2,4\}$ *and* $j \in \{1,3\}$, *then* $L_\ell(\mathcal{I}[T];\beta_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ *and* $L_\ell(\mathcal{I}[F];\beta_F)$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty.*

(6) *If* $i \in \{1,3\}$ *and* $j, k \in \{2,4\}$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ *and* $L_\ell(\mathcal{I}[F];\beta_F)$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty.*

THEOREM 3.14. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(2,3,2)$*-length neutrosophic subalgebra of* $(X,*,0)$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ *and* $U_\ell(\mathcal{I}[F];\alpha_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* α_T, β_I, $\alpha_F \in [0,1]$.

PROOF: Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(2,3,2)$-length neutrosophic subalgebra of $(X,*,0)$. Let α_T, β_I, $\alpha_F \in [0,1]$ be such that $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ and $U_\ell(\mathcal{I}[F];\alpha_F)^c$ are nonempty. If $x, y \in U_\ell(\mathcal{I}[T];\alpha_T)^c$, then $\mathcal{I}[T]_\ell(x) < \alpha_T$ and $\mathcal{I}[T]_\ell(y) < \alpha_T$. Hence

$$\mathcal{I}[T]_\ell(x*y) \leq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} < \alpha_T,$$

and so $x*y \in U_\ell(\mathcal{I}[T];\alpha_T)^c$. If $x, y \in L_\ell(\mathcal{I}[I];\beta_I)^c$, then $\mathcal{I}[I]_\ell(x) > \beta_I$ and $\mathcal{I}[I]_\ell(y) > \beta_I$. Thus

$$\mathcal{I}[I]_\ell(x*y) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\} > \beta_I,$$

which implies that $x*y \in L_\ell(\mathcal{I}[I];\beta_I)^c$. Let $x, y \in U_\ell(\mathcal{I}[F];\alpha_F)^c$. Then $\mathcal{I}[F]_\ell(x) < \alpha_F$ and $\mathcal{I}[F]_\ell(y) < \alpha_F$. Hence

$$\mathcal{I}[F]_\ell(x*y) \leq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\} < \alpha_F,$$

and so $x*y \in U_\ell(\mathcal{I}[F];\alpha_F)^c$. Therefore $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ and $U_\ell(\mathcal{I}[F];\alpha_F)^c$ are subalgebras of $(X,*,0)$ for all α_T, β_I, $\alpha_F \in [0,1]$. □

The converse of Theorem 3.14 is not true in general as seen in the following example.

Example 3.15. Consider a BCI-algebra $X = \{0,1,a,b,c\}$ with the binary operation $*$ which is given in Table 6 (see [5]).
Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X,*,0)$ given by

Table 6. Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.50, 0.75) & \text{if } x = 0, \\ (0.25, 0.70] & \text{if } x = 1, \\ [0.10, 0.65] & \text{if } x = a, \\ [0.05, 0.70) & \text{if } x = b, \\ (0.10, 0.75] & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.05, 0.80] & \text{if } x = 0, \\ (0.10, 0.80) & \text{if } x = 1, \\ [0.26, 0.89] & \text{if } x = a, \\ (0.16, 0.79) & \text{if } x = b, \\ (0.07, 0.75] & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.23, 0.67) & \text{if } x = 0, \\ (0.03, 0.58] & \text{if } x = 1, \\ (0.18, 0.73) & \text{if } x = a, \\ [0.14, 0.80] & \text{if } x = b, \\ (0.07, 0.73] & \text{if } x = c. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 7.
Then

$$U_\ell(\mathcal{I}[T]; \alpha_T)^c = \begin{cases} \emptyset & \text{if } \alpha_T \in [0, 0.25], \\ \{0\} & \text{if } \alpha_T \in (0.25, 0.45], \\ \{0, 1\} & \text{if } \alpha_T \in (0.45, 0.55], \\ \{0, 1, a\} & \text{if } \alpha_T \in (0.55, 0.65], \\ X & \text{if } \alpha_T \in (0.65, 1], \end{cases}$$

Table 7. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.25	0.75	0.44
1	0.45	0.70	0.55
a	0.55	0.63	0.55
b	0.65	0.63	0.66
c	0.65	0.68	0.66

$$L_\ell(\mathcal{I}[I];\beta_I)^c = \begin{cases} \emptyset & \text{if } \beta_I \in [0.75, 1], \\ \{0\} & \text{if } \beta_I \in [0.70, 0.75), \\ \{0, 1\} & \text{if } \beta_I \in [0.68, 0.70), \\ \{0, 1, c\} & \text{if } \beta_I \in [0.63, 0.68), \\ X & \text{if } \beta_I \in [0, 0.63), \end{cases}$$

and

$$U_\ell(\mathcal{I}[F];\alpha_F)^c = \begin{cases} \emptyset & \text{if } \alpha_F \in [0, 0.44], \\ \{0\} & \text{if } \alpha_F \in (0.44, 0.55], \\ \{0, 1, a\} & \text{if } \alpha_F \in (0.55, 0.66], \\ X & \text{if } \alpha_F \in (0.66, 1] \end{cases}$$

are subalgebras of $(X, *, 0)$ whenever they are nonempty for all α_T, β_I, $\alpha_F \in [0, 1]$. But $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not a $(2, 3, 2)$-length neutrosophic subalgebra of $(X, *, 0)$ since

$$\mathcal{I}[T]_\ell(b * a) = \mathcal{I}[T]_\ell(c) = 0.65 > 0.55 = \min\{\mathcal{I}[T]_\ell(b), \mathcal{I}[T]_\ell(a)\},$$

$$\mathcal{I}[I]_\ell(b * c) = \mathcal{I}[I]_\ell(a) = 0.63 < 0.68 = \max\{\mathcal{I}[I]_\ell(b), \mathcal{I}[I]_\ell(c)\},$$

and/or

$$\mathcal{I}[F]_\ell(b * a) = \mathcal{I}[F]_\ell(c) = 0.66 > 0.55 = \min\{\mathcal{I}[F]_\ell(b), \mathcal{I}[F]_\ell(a)\}.$$

By the similar way to the proof of Theorem 3.14, we have the following theorem.

THEOREM 3.16. *Given an (i, j, k)-length neutrosophic subalgebra $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ of $(X, *, 0)$, the following assertions are valid.*

(1) *If* $(i,j,k) = (2,2,2)$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $U_\ell(\mathcal{I}[F];\alpha_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* α_T, α_I, $\alpha_F \in [0,1]$.

(2) *If* $(i,j,k) = (2,2,3)$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* α_T, α_I, $\beta_F \in [0,1]$.

(3) *If* $(i,j,k) = (2,3,3)$, *then* $U_\ell(\mathcal{I}[T];\alpha_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* α_T, β_I, $\beta_F \in [0,1]$.

(4) *If* $(i,j,k) = (3,2,2)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $U_\ell(\mathcal{I}[F];\alpha_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, α_I, $\alpha_F \in [0,1]$.

(5) *If* $(i,j,k) = (3,2,3)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $U_\ell(\mathcal{I}[I];\alpha_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, α_I, $\beta_F \in [0,1]$.

(6) *If* $(i,j,k) = (3,3,2)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ *and* $U_\ell(\mathcal{I}[F];\alpha_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, β_I, $\alpha_F \in [0,1]$.

(7) *If* $(i,j,k) = (3,3,3)$, *then* $L_\ell(\mathcal{I}[T];\beta_T)^c$, $L_\ell(\mathcal{I}[I];\beta_I)^c$ *and* $L_\ell(\mathcal{I}[F];\beta_F)^c$ *are subalgebras of* $(X,*,0)$ *whenever they are nonempty for all* β_T, β_I, $\beta_F \in [0,1]$.

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A General Model of Neutrosophic Ideals in BCK/BCI-Algebras Based on Neutrosophic Points

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Hashem Bordbar, Rajab Ali Borzooei, Florentin Smarandache, Young Bae Jun (2020). A General Model of Neutrosophic Ideals in BCK/BCI-Algebras Based on Neutrosophic Points. *Bulletin of the Section of Logic*, 17. DOI: 10.18778/0138-0680.2020.18

Abstract

More general form of $(\in, \in\vee q)$-neutrosophic ideal is introduced, and their properties are investigated. Relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal are discussed. Characterizations of $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal are discussed, and conditions for a neutrosophic set to be an $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal are displayed.

Keywords: Ideal, neutrosophic $\in$-subset, neutrosophic q_k-subset, neutrosophic $\in\vee q_k$-subset, $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal.

2010 Mathematical Subject Classification: 06F35, 03G25, 03B52.

1. Introduction

Smarandache [23, 24] introduced the concept of neutrosophic sets which is a more general platform to extend the notions of the classical set and (intuitionistic, interval valued) fuzzy set. Neutrosophic set theory is applied to several parts which are referred to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun [10] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras based on neutrosophic points. Borumand and Jun [22] studied several properties of $(\in, \in \vee q)$-neutrosophic subalgebras and $(q, \in \vee q)$-neutrosophic subalgebras in BCK/BCI-algebras. Jun et al. [11] discussed neutrosophic $\mathcal{N}$-structures with an application in BCK/BCI-algebras, and in [13, 14] introduced neutrosophic quadruple numbers based on a set and construct neutrosophic quadruple BCK/BCI-algebras.

Song et al. [25] introduced the notion of commutative $\mathcal{N}$-ideal in BCK-algebras and investigated several properties. Bordbar, Jun and et al. [21] and [17] introduced the notion of $(q, \in \vee q)$-neutrosophic ideal, and $(\in, \in \vee q)$-neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. Also in [7, 26], they discussed the notion of $BMBJ$-neutrosophic sets, subalgebra and ideals, as a generalisation of neutrosophic set, and investigated it's application and related properties to BCI/BCK-algebras.

For more information about the mentioned topics, please refer to [3, 4, 8, 12, 16, 18, 19, 20].

In this paper, we introduce a more general form of $(\in, \in \vee q)$-neutrosophic ideal, and investigate their properties. We discuss relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We consider characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We investigate conditions for a neutrosophic set to be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We find conditions for an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal.

2. Preliminaries

By a *BCI-algebra* we mean a set X with a binary operation $*$ and the special element 0 satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = y * x = 0 \Rightarrow x = y$,

for all x, y, $z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A subset I of a BCK/BCI-algebra X is called an *ideal* of X (see [9, 15]) if it satisfies:

$$0 \in I, \tag{2.1}$$

$$(\forall x, y \in X)\,(x * y \in I,\ y \in I \ \Rightarrow\ x \in I)\,. \tag{2.2}$$

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [9] and [15] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} = \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} = \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \{1, 2\}\}$ and $\bigwedge\{a_i \mid i \in \{1, 2\}\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [23]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets (see [10]):

$T_\in(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\}$,

$I_{\in}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta\}$,

$F_{\in}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\}$.

We say $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ and $F_{\in}(A;\gamma)$ are *neutrosophic $\in$-subsets.*

3. Generalizations of neutrosophic ideals based on neutrosophic points

In what follows, let k_T, k_I and k_F denote arbitrary elements of $[0,1)$ unless otherwise specified. If k_T, k_I and k_F are the same number in $[0,1)$, then it is denoted by k, i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$T_{q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) + \alpha + k_T > 1\}$,

$I_{q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) + \beta + k_I > 1\}$,

$F_{q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) + \gamma + k_F < 1\}$,

$T_{\in\vee\, q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha + k_T > 1\}$,

$I_{\in\vee\, q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1\}$,

$F_{\in\vee\, q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma + k_F < 1\}$.

We say $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are *neutrosophic q_k-subsets*; and $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are *neutrosophic $\in\vee\, q_k$-subsets.* For $\psi \in \{\in,\, q,\, q_k,\, q_{k_T},\, q_{k_I},\, q_{k_F},\, \in\vee\, q,\, \in\vee\, q_k,\, \in\vee\, q_{k_T},\, \in\vee\, q_{k_I},\, \in\vee\, q_{k_F}\}$, the element of $T_\psi(A;\alpha)$ (resp., $I_\psi(A;\beta)$ and $F_\psi(A;\gamma)$) is called a *neutrosophic T_ψ-point* (resp., *neutrosophic I_ψ-point* and *neutrosophic F_ψ-point*) with value α (resp., β and γ).

It is clear that

$$T_{\in\vee\, q_{k_T}}(A;\alpha) = T_{\in}(A;\alpha) \cup T_{q_{k_T}}(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q_{k_I}}(A;\beta) = I_{\in}(A;\beta) \cup I_{q_{k_I}}(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q_{k_F}}(A;\gamma) = F_{\in}(A;\gamma) \cup F_{q_{k_F}}(A;\gamma). \tag{3.3}$$

THEOREM 3.1. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following assertions are equivalent.*

(1) *The nonempty neutrosophic* $\in$*-subsets* $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ *and* $F_{\in}(A;\gamma)$ *are ideals of* X *for all* $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ *and* $\gamma \in [0, \frac{1-k_F}{2})$.

(2) $A = (A_T, A_I, A_F)$ *satisfies the following assertion.*

$$(\forall x \in X)\left(\begin{array}{l} A_T(x) \leq A_T(0) \vee \frac{1-k_T}{2} \\ A_I(x) \leq A_I(0) \vee \frac{1-k_I}{2} \\ A_F(x) \geq A_F(0) \wedge \frac{1-k_F}{2} \end{array}\right) \tag{3.4}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \vee \frac{1-k_T}{2} \geq A_T(x * y) \wedge A_T(y) \\ A_I(x) \vee \frac{1-k_I}{2} \geq A_I(x * y) \wedge A_I(y) \\ A_F(x) \wedge \frac{1-k_F}{2} \leq A_F(x * y) \vee A_F(y) \end{array}\right) \tag{3.5}$$

PROOF: Assume that the nonempty neutrosophic $\in$-subsets $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ and $F_{\in}(A;\gamma)$ are ideals of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. If there are $a, b \in X$ such that $A_T(a) > A_T(0) \vee \frac{1-k_T}{2}$, then $a \in T_{\in}(A;\alpha_a)$ and $0 \notin T_{\in}(A;\alpha_a)$ for $\alpha_a := A_T(a) \in (\frac{1-k_T}{2}, 1]$. This is a contradiction, and so $A_T(x) \leq A_T(0) \vee \frac{1-k_T}{2}$ for all $x \in X$. We also know that $A_I(x) \leq A_I(0) \vee \frac{1-k_I}{2}$ for all $x \in X$ by the similar way. Now, let $x \in X$ be such that $A_F(x) < A_F(0) \wedge \frac{1-k_F}{2}$. If we take $\gamma_x := A_F(x)$, then $\gamma_x \in [0, \frac{1-k_F}{2})$ and so $0 \in F_{\in}(A;\gamma_x)$ since $F_{\in}(A;\gamma_x)$ is an ideal of X. Hence $A_F(0) \leq \gamma_x = A_F(x)$, which is a contradiction. Hence $A_F(x) \geq A_F(0) \wedge \frac{1-k_F}{2}$ for all $x \in X$. Suppose that $A_I(x) \vee \frac{1-k_I}{2} < A_I(x*y) \wedge A_I(y)$ for some $x, y \in X$ and take $\beta := A_I(x * y) \wedge A_I(y)$. Then $\beta \in (\frac{1-k_I}{2}, 1]$ and $x * y, y \in I_{\in}(A;\beta)$. But $x \notin I_{\in}(A;\beta)$ which is a contradiction. Thus $A_I(x) \vee \frac{1-k_I}{2} \geq A_I(x * y) \wedge A_I(y)$ for all $x, y \in X$. Similarly, we have $A_T(x) \vee \frac{1-k_T}{2} \geq A_T(x * y) \wedge A_T(y)$ for all $x, y \in X$. Suppose that there exist $x, y \in X$ such that $A_F(x) \wedge \frac{1-k_F}{2} > A_F(x * y) \vee A_F(y)$. Taking $\gamma := A_F(x * y) \vee A_F(y)$ implies that $\gamma \in [0, \frac{1-k_F}{2})$, $x * y \in F_{\in}(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$, but $x \notin F_{\in}(A;\gamma)$. This is a contradiction, and so $A_F(x) \wedge \frac{1-k_F}{2} \leq A_F(x * y) \vee A_F(y)$ for all $x, y \in X$.

Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.4) and (3.5). Let $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ be such that $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ and $F_{\in}(A;\gamma)$ are nonempty. For any $x \in T_{\in}(A;\alpha)$, $y \in I_{\in}(A;\beta)$ and $z \in F_{\in}(A;\gamma)$, we get

$$A_T(0) \vee \tfrac{1-k_T}{2} \geq A_T(x) \geq \alpha > \tfrac{1-k_T}{2},$$
$$A_I(0) \vee \tfrac{1-k_I}{2} \geq A_I(y) \geq \beta > \tfrac{1-k_I}{2},$$
$$A_F(0) \wedge \tfrac{1-k_F}{2} \leq A_F(z) \leq \gamma < \tfrac{1-k_F}{2},$$

and so $A_T(0) \geq \alpha$, $A_I(0) \geq \beta$ and $A_F(0) \leq \gamma$. Hence $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. Let $a, b, x, y, u, v \in X$ be such that $a * b \in T_\in(A;\alpha)$, $b \in T_\in(A;\alpha)$, $x * y \in I_\in(A;\beta)$, $y \in I_\in(A;\beta)$, $u * v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$. It follows from (3.5) that

$$A_T(a) \vee \tfrac{1-k_T}{2} \geq A_T(a * b) \wedge A_T(b) \geq \alpha > \tfrac{1-k_T}{2},$$
$$A_I(x) \vee \tfrac{1-k_I}{2} \geq A_I(x * y) \wedge A_I(y) \geq \beta > \tfrac{1-k_I}{2},$$
$$A_F(u) \wedge \tfrac{1-k_F}{2} \leq A_F(u * v) \vee A_F(v) \leq \gamma < \tfrac{1-k_F}{2}.$$

Hence $A_T(a) \geq \alpha$, $A_I(x) \geq \beta$ and $A_F(u) \leq \gamma$, that is, $a \in T_\in(A;\alpha)$, $x \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Therefore $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. $\square$

Corollary 3.2 ([21]). Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following assertions are equivalent.

(1) The nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertion.

$$(\forall x \in X)\left(\begin{array}{l} A_T(x) \leq A_T(0) \vee 0.5 \\ A_I(x) \leq A_I(0) \vee 0.5 \\ A_F(x) \geq A_F(0) \wedge 0.5 \end{array}\right)$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \vee 0.5 \geq A_T(x * y) \wedge A_T(y) \\ A_I(x) \vee 0.5 \geq A_I(x * y) \wedge A_I(y) \\ A_F(x) \wedge 0.5 \leq A_F(x * y) \vee A_F(y) \end{array}\right)$$

Definition 3.3. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, \in \vee\, q_{(k_T, k_I, k_F)})$-*neutrosophic ideal* of X if the following assertions are valid.

$$(\forall x \in X)\begin{pmatrix} x \in T_{\in}(A;\alpha_x) \;\Rightarrow\; 0 \in T_{\in\vee\, q_{k_T}}(A;\alpha_x) \\ x \in I_{\in}(A;\beta_x) \;\Rightarrow\; 0 \in I_{\in\vee\, q_{k_I}}(A;\beta_x) \\ x \in F_{\in}(A;\gamma_x) \;\Rightarrow\; 0 \in F_{\in\vee\, q_{k_F}}(A;\gamma_x) \end{pmatrix}, \tag{3.6}$$

$$(\forall x, y \in X)\begin{pmatrix} x*y \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y) \Rightarrow x \in T_{\in\vee\, q_{k_T}}(A;\alpha_x \wedge \alpha_y) \\ x*y \in I_{\in}(A;\beta_x),\ y \in I_{\in}(A;\beta_y) \Rightarrow x \in I_{\in\vee\, q_{k_I}}(A;\beta_x \wedge \beta_y) \\ x* y \in F_{\in}(A;\gamma_x),\ y \in F_{\in}(A;\gamma_y) \Rightarrow x \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{pmatrix} \tag{3.7}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Example 3.4. Let $X = \{0,1,2,3,4\}$ be a set with the binary operation $*$ which is given in Table 1.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then $(X,*,0)$ is a BCK-algebra (see [15]). Consider a neutrosophic set $A = (A_T, A_I, A_F)$ in X which is given by Table 2.

Table 2: Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.5	0.45
1	0.5	0.3	0.93
2	0.3	0.7	0.67
3	0.4	0.3	0.93
4	0.1	0.2	0.74

Routine calculations show that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of X for $k_T = 0.24$, $k_I = 0.08$ and $k_F = 0.16$.

THEOREM 3.5. *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if $A = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X) \begin{pmatrix} A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2} \\ A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2} \\ A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2} \end{pmatrix}, \tag{3.8}$$

$$(\forall x, y \in X) \begin{pmatrix} A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\} \\ A_I(x) \geq \bigwedge\{A_I(x * y), A_I(y), \frac{1-k_I}{2}\} \\ A_F(x) \leq \bigvee\{A_F(x * y), A_F(y), \frac{1-k_F}{2}\} \end{pmatrix}. \tag{3.9}$$

PROOF: Assume that $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. If $A_T(0) < A_T(a) \wedge \frac{1-k_T}{2}$ for some $a \in X$, then there exists $\alpha_a \in (0, 1]$ such that $A_T(0) < \alpha_a \leq A_T(a) \wedge \frac{1-k_T}{2}$. It follows that $\alpha_a \in (0, \frac{1-k_T}{2}]$, $a \in T_\in(A; \alpha_a)$ and $0 \notin T_\in(A; \alpha_a)$. Also, $A_T(0) + \alpha_a + k_T < 2\alpha_a + k_T \leq 1$, i.e., $0 \notin T_{q_{k_T}}(A; \alpha_a)$. Hence $0 \notin T_{\in\vee\, q_{k_T}}(A; \alpha_a)$, a contradiction. Thus $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$ for all $x \in X$. Similarly, we have $A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2}$ for all $x \in X$. Suppose that $A_F(0) > A_F(z) \vee \frac{1-k_F}{2}$ for some $z \in X$ and take $\gamma_z := A_F(z) \vee \frac{1-k_F}{2}$. Then $\gamma_z \geq \frac{1-k_F}{2}$, $z \in F_\in(A; \gamma_z)$ and $0 \notin F_\in(A; \gamma_z)$. Also $A_F(0) + \gamma_z + k_F \geq 1$, that is, $0 \notin F_{q_{k_F}}(A; \gamma_z)$. This is a contradiction, and thus $A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2}$ for all $x \in X$. Suppose that $A_I(a) < \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$ for some $a, b \in X$ and take $\beta := \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\}$. Then $\beta \leq \frac{1-k_I}{2}$, $a * b \in I_\in(A; \beta)$, $b \in I_\in(A; \beta)$ and $a \notin I_\in(A; \beta)$. Also, we have $A_I(a) + \beta + k_I \leq 1$, i.e., $a \notin I_{q_{k_F}}(A; \beta)$. This is impossible, and therefore $A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\}$ for all $x, y \in X$. By the similar way, we can verify that $A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Now assume that $A_F(a) > \bigvee\{A_F(a * b), A_F(b), \frac{1-k_F}{2}\}$ for some $a, b \in X$. Then there exists $\gamma \in [0, 1)$ such that $A_F(a) > \gamma \geq \bigvee\{A_F(a * b), A_F(b), \frac{1-k_F}{2}\}$. Then $\gamma \geq \frac{1-k_F}{2}$, $a * b \in F_\in(A; \gamma)$, $b \in F_\in(A; \gamma)$ and $a \notin F_\in(A; \gamma)$. Also, $A_F(a) + \gamma + k_F \geq 1$, i.e., $a \notin F_{q_{k_F}}(A; \gamma)$. Thus $a \notin F_{\in\vee\, q_{k_F}}(A; \gamma)$, which is a contradiction. Hence $A_F(x) \leq \bigvee\{A_F(x * y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$.

Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.8) and (3.9). For any x, y, $z \in X$, let α_x, $\beta_y \in (0, 1]$ and $\gamma_z \in [0, 1)$ be such that $x \in T_\in(A; \alpha_x)$, $y \in I_\in(A; \beta_y)$ and $z \in F_\in(A; \gamma_z)$. Then $A_T(x) \geq \alpha_x$, $A_I(y) \geq \beta_y$ and $A_F(z) \leq \gamma_z$. Assume that $A_T(0) < \alpha_x$, $A_I(0) < \beta_y$ and $A_F(0) > \gamma_z$. If $A_T(x) < \frac{1-k_T}{2}$, then

$$A_T(0) \geq A_T(x) \wedge \tfrac{1-k_T}{2} = A_T(x) \geq \alpha_x,$$

a contradiction. Hence $A_T(x) \geq \frac{1-k_T}{2}$, and so

$$A_T(0) + \alpha_x + k_T > 2A_T(0) + k_T \geq 2\left(A_T(x) \wedge \tfrac{1-k_T}{2}\right) + k_T = 1.$$

Hence $0 \in T_{q_{k_T}}(A; \alpha_x) \subseteq T_{\in\vee\, q_{k_T}}(A; \alpha_x)$. Similarly, we get $0 \in I_{q_{k_I}}(A; \beta_y) \subseteq I_{\in\vee\, q_{k_I}}(A; \beta_y)$. If $A_F(z) > \frac{1-k_F}{2}$, then $A_F(0) \leq A_F(z) \vee \frac{1-k_F}{2} = A_F(z) \leq \gamma_z$ which is a contradiction. Hence $A_F(z) \leq \frac{1-k_F}{2}$, and thus

$$A_F(0) + \gamma_z + k_F < 2A_F(0) + k_F \leq 2\left(A_F(z) \vee \tfrac{1-k_F}{2}\right) + k_F = 1.$$

Hence $0 \in F_{q_{k_F}}(A; \gamma_z) \subseteq F_{\in\vee\, q_{k_F}}(A; \gamma_z)$. For any $a, b, p, q, x, y \in X$, let $\alpha_a, \alpha_b, \beta_p, \beta_q \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$ be such that $a * b \in T_\in(A; \alpha_a)$, $b \in T_\in(A; \alpha_b)$, $p * q \in I_\in(A; \beta_p)$, $q \in I_\in(A; \beta_q)$, $x * y \in F_\in(A; \gamma_x)$, and $y \in F_\in(A; \gamma_y)$. Then $A_T(a * b) \geq \alpha_a$, $A_T(b) \geq \alpha_b$, $A_I(p * q) \geq \beta_p$, $A_I(q) \geq \beta_q$, $A_F(x * y) \leq \gamma_x$, and $A_F(y) \leq \gamma_y$. Suppose that $a \notin T_\in(A; \alpha_a \wedge \alpha_b)$. Then $A_T(a) < \alpha_a \wedge \alpha_b$. If $A_T(a * b) \wedge A_T(b) < \frac{1-k_T}{2}$, then

$$A_T(a) \geq \bigwedge\{A_T(a * b), A_T(b), \tfrac{1-k_T}{2}\} = A_T(a * b) \wedge A_T(b) \geq \alpha_a \wedge \alpha_b.$$

This is a contradiction, and so $A_T(a * b) \wedge A_T(b) \geq \frac{1-k_T}{2}$. Thus

$$\begin{aligned} A_T(a) + (\alpha_a \wedge \alpha_b) + k_T &> 2A_T(a) + k_T \\ &\geq 2\left(\bigwedge\{A_T(a * b), A_T(b), \tfrac{1-k_T}{2}\}\right) + k_T = 1, \end{aligned}$$

which induces $a \in T_{q_{k_T}}(A; \alpha_a \wedge \alpha_b) \subseteq T_{\in\vee\, q_{k_T}}(A; \alpha_a \wedge \alpha_b)$. By the similarly way, we get $p \in I_{\in\vee\, q_{k_I}}(A; \beta_p \wedge \beta_q)$. Suppose that $x \notin F_\in(A; \gamma_x \vee \gamma_y)$, that is, $A_F(x) > \gamma_x \vee \gamma_y$. If $A_F(x * y) \vee A_F(y) > \frac{1-k_F}{2}$, then

$$A_F(x) \leq \bigvee\{A_F(x * y), A_F(y), \tfrac{1-k_F}{2}\} = A_F(x * y) \vee A_F(y) \leq \gamma_x \vee \gamma_y,$$

which is impossible. Thus $A_F(x * y) \vee A_F(y) \leq \frac{1-k_F}{2}$, and so

$$A_F(x) + (\gamma_x \vee \gamma_y) + k_F < 2A_F(x) \leq 2\left(\bigvee\{A_F(x*y), A_F(y), \tfrac{1-k_F}{2}\}\right) + k_F = 1.$$

This implies that $x \in F_{q_{k_F}}(A;\gamma_x \vee \gamma_y) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma_x \vee \gamma_y)$. Consequently, $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

COROLLARY 3.6 ([21]). For a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.

(1) $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertions.

$$(\forall x \in X)\begin{pmatrix} A_T(0) \geq A_T(x) \wedge 0.5 \\ A_I(0) \geq A_I(x) \wedge 0.5 \\ A_F(0) \leq A_F(x) \vee 0.5 \end{pmatrix},$$

$$(\forall x, y \in X)\begin{pmatrix} A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), 0.5\} \\ A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), 0.5\} \\ A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), 0.5\} \end{pmatrix}.$$

THEOREM 3.7. *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.*

PROOF: Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ and let $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ be such that $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are nonempty. Using (3.8), we get $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) \geq A_I(y) \wedge \frac{1-k_I}{2}$, and $A_F(0) \leq A_F(z) \vee \frac{1-k_F}{2}$ for all $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$. It follows that $A_T(0) \geq \alpha \wedge \frac{1-k_T}{2} = \alpha$, $A_I(0) \geq \beta \wedge \frac{1-k_I}{2} = \beta$, and $A_F(0) \leq \gamma \vee \frac{1-k_F}{2} = \gamma$, that is, $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. Let $x, y, a, b, u, v \in X$ be such that $x * y \in T_\in(A;\alpha)$, $y \in T_\in(A;\alpha)$, $a * b \in I_\in(A;\beta)$, $b \in I_\in(A;\beta)$, $u * v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$ for $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Then $A_T(x*y) \geq \alpha$, $A_T(y) \geq \alpha$, $A_I(a*b) \geq \beta$, $A_I(b) \geq \beta$, $A_F(u*v) \leq \gamma$, and $A_F(v) \leq \gamma$. It follows from (3.9) that

$A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\} \geq \alpha \wedge \frac{1-k_T}{2} = \alpha,$

$A_I(a) \geq \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\} \geq \beta \wedge \frac{1-k_I}{2} = \beta,$

$A_F(u) \leq \bigvee\{A_F(u*v), A_F(v), \frac{1-k_F}{2}\} \leq \gamma \vee \frac{1-k_F}{2} = \gamma$

and so that $x \in T_\in(A;\alpha)$, $a \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Therefore $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ such that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If there exist $x, y, z \in X$ such that $A_T(0) < A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) < A_I(y) \wedge \frac{1-k_I}{2}$, and $A_F(0) > A_F(z) \vee \frac{1-k_F}{2}$, then $0 \notin T_\in(A;\alpha_x)$, $0 \notin I_\in(A;\beta_y)$ and $0 \notin F_\in(A;\gamma_z)$ by taking $\alpha_x := A_T(x) \wedge \frac{1-k_T}{2}$, $\beta_y := A_I(y) \wedge \frac{1-k_I}{2}$, and $\gamma_z := A_F(z) \vee \frac{1-k_F}{2}$. This is a contradiction, and so $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2}$, and $A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2}$ for all $x \in X$. Now, suppose that there $x, y, a, b, u, v \in X$ be such that $A_T(x) < \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$, $A_I(a) < \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$, and $A_F(u) > \bigvee\{A_F(u*v), A_F(v), \frac{1-k_F}{2}\}$. If we take $\alpha := \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$, $\beta := \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$, and $\gamma := \bigvee\{A_F(u*v), A_F(v), \frac{1-k_F}{2}\}$, then $\alpha \leq \frac{1-k_T}{2}$, $\beta \leq \frac{1-k_I}{2}$, $\gamma \geq \frac{1-k_F}{2}$, $x*y \in T_\in(A;\alpha)$, $y \in T_\in(A;\alpha)$, $a*b \in I_\in(A;\beta)$, $b \in I_\in(A;\beta)$, $u*v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$. But $x \notin T_\in(A;\alpha)$, $a \notin I_\in(A;\beta)$ and $u \notin F_\in(A;\gamma)$, which induces a contradiction. Therefore $A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$, $A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\}$, and $A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Using Theorem 3.5, we conclude that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

Corollary 3.8 ([21]). A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.

It is clear that every $(\in, \in)$-neutrosophic ideal is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal. But the converse is not true in general. For example, the $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal $A = (A_T, A_I, A_F)$ with $k_T = 0.24$, $k_I = 0.08$ and $k_F = 0.16$ in Example 3.4 is not an $(\in, \in)$-neutrosophic ideal since $2 \in I_\in(A; 0.56)$ and $0 \notin I_\in(A; 0.56)$.

We now consider conditions for an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal.

THEOREM 3.9. *Let $A = (A_T, A_I, A_F)$ be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ such that*

$$(\forall x \in X)\left(A_T(x) < \tfrac{1-k_T}{2},\ A_I(x) < \tfrac{1-k_I}{2},\ A_F(x) > \tfrac{1-k_F}{2}\right).$$

Then $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

PROOF: Let $x, y, z \in X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ be such that $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$. Then $A_T(x) \geq \alpha$, $A_I(y) \geq \beta$ and $A_F(z) \leq \gamma$. It follows from (3.8) that

$$A_T(0) \geq A_T(x) \wedge \tfrac{1-k_T}{2} = A_T(x) \geq \alpha,$$

$$A_I(0) \geq A_I(y) \wedge \tfrac{1-k_I}{2} = A_I(y) \geq \beta,$$

$$A_F(0) \leq A_F(z) \vee \tfrac{1-k_F}{2} = A_F(z) \leq \gamma.$$

Hence $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. For any $x, y, a, b, u, v \in X$, let $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0,1]$ and $\gamma_u, \gamma_v \in [0,1)$ be such that $x * y \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a*b \in I_\in(A;\beta_a)$, $b \in I_\in(A;\beta_b)$, $u*v \in F_\in(A;\gamma_u)$, and $v \in F_\in(A;\gamma_v)$. Then $A_T(x*y) \geq \alpha_x$, $A_T(y) \geq \alpha_y$, $A_I(a*b) \geq \beta_a$, $A_I(b) \geq \beta_b$, $A_F(u*v) \leq \gamma_u$, and $A_F(v) \leq \gamma_v$. It follows from (3.9) that

$$A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \tfrac{1-k_T}{2}\} = A_T(x*y) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y,$$

$$A_I(a) \geq \bigwedge\{A_I(a*b), A_I(b), \tfrac{1-k_I}{2}\} = A_I(a*b) \wedge A_I(b) \geq \beta_a \wedge \beta_b,$$

$$A_F(u) \leq \bigvee\{A_F(u*v), A_F(v), \tfrac{1-k_F}{2}\} = A_F(u*v) \vee A_F(v) \leq \gamma_u \vee \gamma_v.$$

Thus $x \in T_\in(A;\alpha_x \wedge \alpha_y)$, $a \in I_\in(A;\beta_a \wedge \beta_b)$ and $u \in F_\in(A;\gamma_u \vee \gamma_v)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

COROLLARY 3.10 ([21]). Let $A = (A_T, A_I, A_F)$ be an $(\in, \in \vee q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$ such that

$$(\forall x \in X)\,(A_T(x) < 0.5,\ A_I(x) < 0.5,\ A_F(x) > 0.5).$$

Then $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

THEOREM 3.11. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee q_k$-subsets $T_{\in\vee q_{k_T}}(A;\alpha)$, $I_{\in\vee q_{k_I}}(A;\beta)$*

and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ *are ideals of* X *for all* $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ *and* $\gamma \in [\frac{1-k_F}{2}, 1)$, *then* $A = (A_T, A_I, A_F)$ *is an* $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$*-neutrosophic ideal of* X.

PROOF: Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ such that the nonempty neutrosophic $\in \vee\, q_k$-subsets $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If $A_T(0) < A_T(x) \wedge \frac{1-k_T}{2} := \alpha_x$, $A_I(0) < A_I(y) \wedge \frac{1-k_I}{2} := \beta_y$ and $A_F(0) > A_F(z) \vee \frac{1-k_F}{2} := \gamma_z$ for some $x, y, z \in X$, then $x \in T_\in(A;\alpha_x) \subseteq T_{\in\vee\, q_{k_T}}(A;\alpha_x)$, $y \in I_\in(A;\beta_y) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta_y)$, $z \in F_\in(A;\gamma_z) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma_z)$, $0 \notin T_\in(A;\alpha_x)$, $0 \notin I_\in(A;\beta_y)$, and $0 \notin F_\in(A;\gamma_z)$. Also, since $A_T(0) + \alpha_x + k_T < 2\alpha_x + k_T \leq 1$, i.e., $0 \notin T_{q_{k_T}}(A;\alpha_x)$, $A_I(0) + \beta_y + k_I < 2\beta_y + k_I \leq 1$, i.e., $0 \notin I_{q_{k_I}}(A;\beta_Y)$, $A_F(0) + \gamma_z + k_F > 2\gamma_z + k_F \geq 1$, i.e., $0 \notin F_{q_{k_F}}(A;\gamma_z)$, we get $0 \notin T_{\in\vee\, q_{k_T}}(A;\alpha_x)$, $0 \notin I_{\in\vee\, q_{k_I}}(A;\beta_y)$, and $0 \notin F_{\in\vee\, q_{k_F}}(A;\gamma_z)$. This is a contradiction, and thus (3.8) is valid. Suppose that there exist $a, b \in X$ such that $A_I(a) < \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$. Taking $\beta := \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$ implies that $a*b \in I_\in(A;\beta) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta)$, $b \in I_\in(A;\beta) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta)$. Since $I_{\in\vee\, q_{k_I}}(A;\beta)$ is an ideal of X, it follows that $a \in I_{\in\vee\, q_{k_I}}(A;\beta)$, i.e., $a \in I_\in(A;\beta)$ or $a \in I_{q_{k_I}}(A;\beta)$, and so that $a \in I_{q_{k_I}}(A;\beta)$, i.e., $A_I(a) + \beta + k_I > 1$, since $a \notin I_\in(A;\beta)$. But $A_I(a) + \beta + k_I < 2\beta + k_I \leq 1$, a contradiction. Hence $A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\}$ for all $x, y \in X$. Similarly, we can verify that $A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Assume that $A_F(a) > \bigvee\{A_F(a*b), A_F(b), \frac{1-k_F}{2}\} := \gamma$ for some $a, b \in X$. Then $a \notin F_\in(A;\gamma)$, $a*b \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma)$, $b \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma)$. Since $F_{\in\vee\, q_{k_F}}(A;\gamma)$ is an ideal of X, we have $a \in F_{\in\vee\, q_{k_F}}(A;\gamma)$. On the other hand, $A_F(a) + \gamma + k_F > 2\gamma + k_F \geq 1$, that is, $a \notin F_{q_{k_F}}(A;\gamma)$. Hence $a \notin F_{\in\vee\, q_{k_F}}(A;\gamma)$, a contradiction. Thus $A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Therefore (3.9) is valid, and consequently $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of X by Theorem 3.5. $\square$

COROLLARY 3.12 ([21]). Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of X.

4. Conclusions

More general form of $(\in, \in \vee q)$-neutrosophic ideal was introduced, and their properties were investigated. Relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were discussed. Characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were discussed, and conditions for a neutrosophic set to be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were displayed.

These results can be applied to characterize the neutrosophic ideals in a BCK/BCI-algebra. In our future research, we will focus on some properties of ideal such as intersections, unions, maximality, primeness and height, and try to find the relations between these properties of ideals and the results of this paper. For instance, how we can define the prime and maximal neutrosophic ideals? Whatis the meaning of height of these types of ideals? For information about the maximality, primeness and height of ideals, please refer to [1, 2, 6, 5].

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Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-)HyperAlgebra

Florentin Smarandache

Florentin Smarandache (2020). Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-)HyperAlgebra. *Neutrosophic Sets and Systems*, 33, 290-296

Abstract: We recall and improve our 2019 concepts of *n-Power Set of a Set, n-SuperHyperGraph, Plithogenic n-SuperHyperGraph,* and *n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyperAlgebra* respectively, and we present several properties and examples connected with the real world.

Keywords: n-Power Set of a Set, n-SuperHyperGraph (n-SHG), n-SHG-vertex, n-SHG-edge, Plithogenic n-SuperHyperGraph, n-ary HyperOperation, n-ary HyperAxiom, n-ary HyperAlgebra, n-ary NeutroHyperOperation, n-ary NeutroHyperAxiom, n-ary NeutroHyperAlgebra, n-ary AntiHyperOperation, n-ary AntiHyperAxiom, n-ary AntiHyperAlgebra

1. Introduction

In this paper, with respect to the classical HyperGraph (that contains HyperEdges), we add the SuperVertices (a group of vertices put all together form a SuperVertex), in order to form a SuperHyperGraph (SHG). Therefore, each SHG-vertex and each SHG-edge belong to $P(V)$, where V is the set of vertices, and $P(V)$ means the power set of V.

Further on, since in our world we encounter complex and sophisticated groups of individuals and complex and sophisticated connections between them, we extend the SuperHyperGraph to n-SuperHyperGraph, by extending $P(V)$ to $P^n(V)$ that is the n-power set of the set V (see below). Therefore, the n-SuperHyperGraph, through its n-SHG-vertices and n-SHG-edges that belong to $P^n(V)$, can the best (so far) to model our complex and sophisticated reality.

In the second part of the paper, we extend the classical HyperAlgebra to n-ary HyperAlgebra and its alternatives n-ary NeutroHyperAlgebra and n-ary AntiHyperAlgebra.

2. n-Power Set of a Set

Let U be a universe of discourse, and a subset $V \subseteq U$. Let $n \geq 1$ be an integer.

Let $P(V)$ be the *Power Set of the Set V* (i.e. all subsets of V, including the empty set ϕ and the whole set V). This is the classical definition of power set.

For example, if $V = \{a, b\}$, then $P(V) = \{\phi, a, b, \{a, b\}\}$.

But we have extended the power set to *n-Power Set of a Set* [1].

For $n = 1$, one has the notation (identity): $P^1(V) \equiv P(V)$.

For $n = 2$, the 2-Power Set of the Set V is defined as follows:

$P^2(V) = P(P(V)$.

In our previous example, we get:

$P^2(V) = P(P(V) = P(\{\phi, a, b, \{a, b\}\}) = \{\phi, a, b, \{a, b\}; \{\phi, a\}, \{\phi, b\}, \{\phi, \{a, b\}\}, \{a, \{a, b\}\}, \{b, \{a, b\}\}; \{\phi, a, b\}, \{\phi, a, \{a, b\}\}, \{\phi, b, \{a, b\}\}, \{a, b, \{a, b\}\}; \{\phi, a, b, \{a, b\}\}\}$.

Definition of n-Power Set of a Set

In general, the **n-Power Set of a Set V** is defined as follows:

$$P^{n+1}(V) = P(P^n(V)), \text{ for integer } n \geq 1.$$

3. Definition of SuperHyperGraph (*SHG)*

A **SuperHyperGraph** (*SHG)* [1] is an ordered pair $SHG = (G \subseteq P(V), E \subseteq P(V))$, where

(i) $V = \{V_1, V_2, \ldots, V_m\}$ is a finite set of $m \geq 0$ vertices, or an infinite set.

(ii) P(V) is the power set of V (all subset of V). Therefore, an ***SHG-vertex*** may be a *single* (classical) *vertex*, or a *super-vertex* (a subset of many vertices) that represents a group (organization), or even an *indeterminate-vertex* (unclear, unknown vertex); ϕ represents the *null-vertex* (vertex that has no element).

(iii) $E = \{E_1, E_2, \ldots, E_m\}$, for $m \geq 1$, is a family of subsets of V, and each E_j is an SHG-edge, $E_i \in P(V)$. An ***SHG-edge*** may be a (classical) *edge*, or a *super-edge* (edge between super-vertices) that represents connections between two groups (organizations), or *hyper-super-edge*) that represents connections between three or more groups (organizations), *multi-edge*, or even *indeterminate-edge* (unclear, unknown edge); ϕ represents the *null-edge* (edge that means there is no connection between the given vertices).

4. **Characterization of the SuperHyperGraph**

Therefore, a **SuperHyperGraph** (*SHG)* may have any of the below:

- *SingleVertices* (V_i), as in classical graphs, such as: V_1, V_2, etc.;
- *SuperVertices (or SubsetVertices)* (SV_i), belonging to $P(V)$, for example: $SV_{1,3} = V_1V_3$, $SV_{2,57} = V_2V_{57}$, etc. that we introduce now for the first time. A super-vertex may represent a group (organization, team, club, city, country, etc.) of many individuals;

  The comma between indexes distinguishes the single vertexes assembled together into a single SuperVertex. For example $SV_{12,3}$ means the single vertex S_{12} and single vertex S_3 are put together to form a super-vertex. But $SV_{1,23}$ means the single vertices S_1 and S_{23} are put together; while $SV_{1,2,3}$ means S_1, S_2, S_3 as single vertices are put together as a super-vertex.

  In no comma in between indexes, i.e. SV_{123} means just a single vertex V_{123}, whose index is 123, or $SV_{123} \equiv V_{123}$.
- *IndeterminateVertices (*i.e. unclear, unknown vertices*);* we denote them as: IV_1, IV_2, etc. that we introduce now for the first time;
- *NullVertex* (i.e. vertex that has no elements, let's for example assume an abandoned house, whose all occupants left), denoted by ϕV .

- *SingleEdges*, as in classical graphs, i.e. edges connecting only two single-vertices, for example: $E_{1,5} = \{V_1, V_5\}$, $E_{2,3} = \{V_2, V_3\}$, etc.;
- *HyperEdges*, i.e. edges connecting three or more single-vertices, for example $HE_{1,4,6} = \{V_1, V_4, V_6\}$, $HE_{2,4,5,7,8,9} = \{V_2, V_4, V_5, V_7, V_8, V_9\}$, etc. as in hypergraphs;
- *SuperEdges* (or *SubsetEdges*), i.e. edges connecting only two SHG-vertices (and at least one vertex is SuperVertex), for example $SE_{(13,6),(45,79)} = \{SV_{13,6}, SV_{45,79}\}$ connecting two SuperVertices, $SE_{9,(2,345)} = \{V_9, SV_{2,345}\}$ connecting one SingleVertex V_9 with one SuperVertex, $SV_{2,345}$, etc. that we introduce now for the first time;
- *HyperSuperEdges* (or *HyperSubsetEdges*), i.e. edges connecting three or more vertices (and at least one vertex is SuperVertex, for example $HSE_{3,45,236} = \{V_3, V_{45}, V_{236}\}$, $HSE_{1234,456789,567,5679} = \{SV_{1234}, SV_{456789}, SV_{567}, SV_{5679}\}$, etc. that we introduce now for the first time;
- *MultiEdges*, i.e. two or more edges connecting the same (single-/super-/indeterminate-) vertices; each vertex is characterized by many attribute values, thus with respect to each attribute value there is an edge, the more attribute values the more edges (= multiedge) between the same vertices;
- *IndeterminateEdges* (i.e. unclear, unknown edges; either we do not know their value, or we do not know what vertices they might connect): IE_1, IE_2, etc. that we introduce now for the first time;
- *NullEdge* (i.e. edge that represents no connection between some given vertices; for example two people that have no connections between them whatsoever): denoted by ϕE .

5. Definition of the n-SuperHyperGraph (*n-SHG*)

A **n-SuperHyperGraph** (*n-SHG*) [1] is an ordered pair $n\text{-}SHG = (G_n \subseteq P^n(V), E_n \subseteq P^n(V))$, where $P^n(V)$ is the n-power set of the set V, for integer $n \geq 1$.

6. Examples of 2-SuperHyperGraph, SuperVertex, IndeterminateVertex, SingleEdge, Indeterminate Edge, HyperEdge, SuperEdge, MultiEdge, 2-SuperHyperEdge

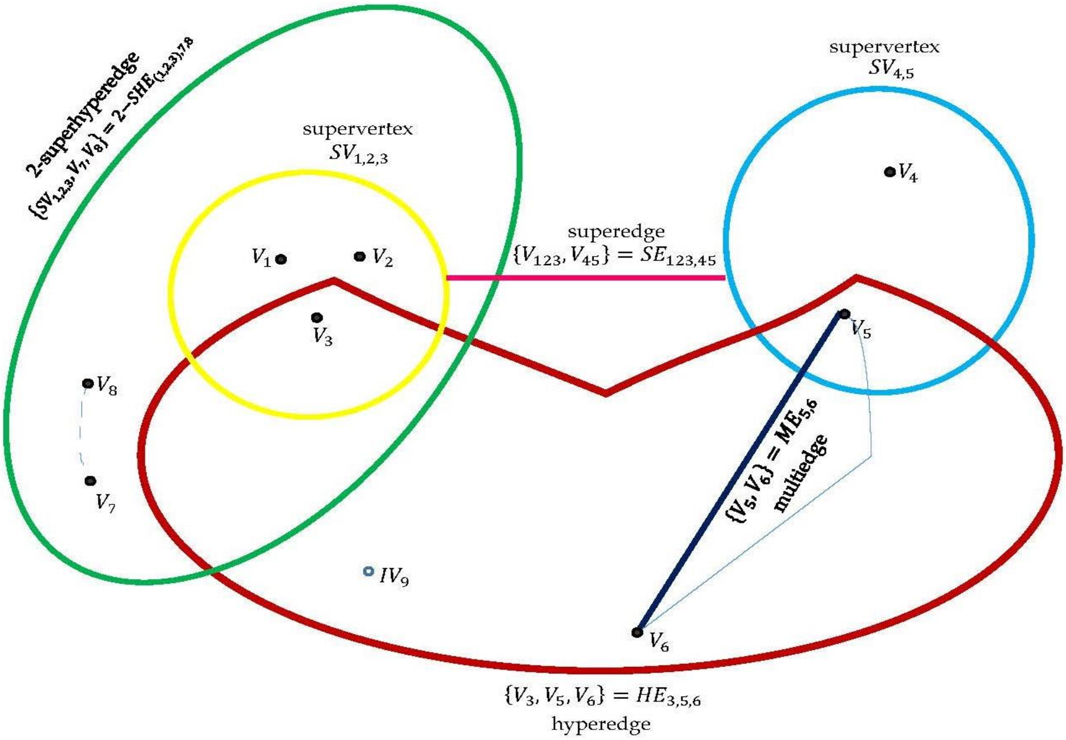

Figure 1. 2-SuperHyperGraph,
($IE_{7,8}$ = Indeterminate Edge between single vertices V_7 and V_8, since the connecting curve is dotted,
IV_9 is an Indeterminate Vertex (since the dot is not filled in),
while $ME_{5,6}$ is a MultiEdge (double edge in this case) between single vertices V_5 and V_6.

Let V_1 and V_2 be two single-vertices, characterized by the attributes a_1 = *size*, whose attribute values are *{short, medium, long}*, and a_2 = *color*, whose attribute values are *{red, yellow}.*
Thus we have the attributes values *(Size{short, medium, long}, Color{red, yellow})*, whence: $V_1(a_1\{s_1, m_1, l_1\}, a_2\{r_1, y_1\})$, where s_1 is the degree of short, m_1 degree of medium, l_1 degree of long, while r_1 is the degree of red and y_1 is the degree of yellow of the vertex V_1.
And similarly $V_2(\, a_1\{s_2, m_2, l_2\}, a_2\{r_2, y_2\}\,)$.
The degrees may be fuzzy, neutrosophic etc.

Example of fuzzy degree:
$V_1(\, a_1\{0.8, 0.2, 0.1\}, a_2\{0.3, 0.5\}\,)$.

Example of neutrosophic degree:
$V_1(\, a_1\{\, (0.7,0.3,0.0), (0.4,0.2,0.1),(0.3,0.1,0.1)\,\}, a_2\{\, (0.5,0.1,0.3), (0.0,0.2,0.7)\,\}\,)$.

Examples of the SVG-edges connecting single vertices V_1 and V_2 are below:

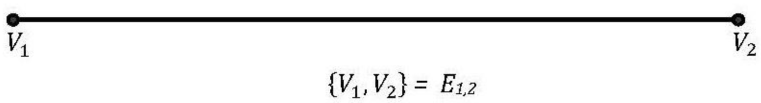

Figure 2. SingleEdge with respect to attributes a_1 and a_2 all together

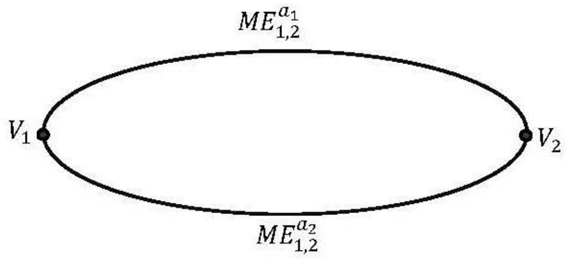

Figure 3. MultiEdge: top edge with respect to attribute a_1, and bottom edge with respect to attribute a_2

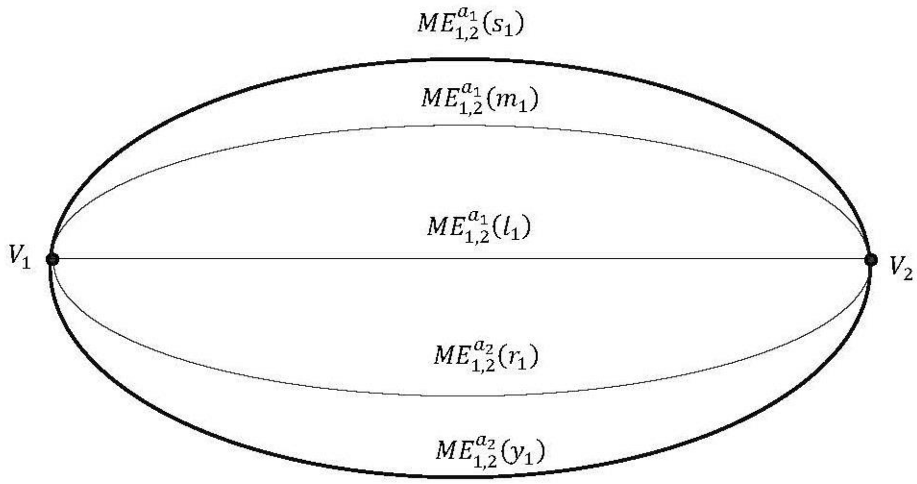

Figure 4. MultiEdge (= Refined MultiEdge from Figure 3):
the top edge from Figure 3, corresponding to the attribute a_1, is split into three sub-edges with respect to the attribute a_1 values s_1, m_1, and l_1;
while the bottom edge from Figure 3, corresponding to the attribute a_2, is split into two sub-edges with respect to the attribute a_2 values r_1, and y_1.

Depending on the application and on experts, one chooses amongst SingleEdge, MultiEdge, Refined-MultiEdge, Refined RefinedMultiEdge, etc.

7. Plithogenic n-SuperHyperGraph

As a consequence, we introduce for the first time the Plithogenic n-SuperHyperGraph.

A **Plithogenic n-SuperHyperGraph (n-PSHG)** is a n-SuperHyperGraph whose each *n-SHG-vertex* and each *n-SHG-edge* are characterized by many distinct attributes values $(a_1, a_2, \ldots, a_p, p \geq 1)$.

Therefore one gets *n-SHG-vertex*$(a_1, a_2, \ldots, a_p)$ and *n-SHG-edge*$(a_1, a_2, \ldots, a_p)$.

The attributes values degrees of appurtenance to the graph may be crisp / fuzzy / intuitionistic fuzzy / picture fuzzy / spherical fuzzy / etc. / neutrosophic / refined neutrosophic / degrees with respect to each *n-SHG-vertex* and each *n-SHG-edge* respectively.

For example, one has:

Fuzzy-*n-SHG-vertex*$(a_1(t_1), a_2(t_2), \ldots, a_p(t_p))$ and Fuzzy-*n-SHG-edge*$(a_1(t_1), a_2(t_2), \ldots, a_p(t_p))$;

Intuitionistic Fuzzy-*n-SHG-vertex*$(a_1(t_1, f_1), a_2(t_2, f_2), \ldots, a_p(t_p, f_p))$

and Intuitionistic Fuzzy-*n-SHG-edge*$(a_1(t_1, f_1), a_2(t_2, f_2), \ldots, a_p(t_p, f_p))$;
Neutrosophic-*n-SHG-vertex*$(a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \ldots, a_p(t_p, i_p, f_p))$
and Neutrosophic-*n-SHG-edge*$(a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \ldots, a_p(t_p, i_p, f_p))$;
etc.

Whence we get:

8. The Plithogenic (Crisp / Fuzzy / Intuitionistic Fuzzy / Picture Fuzzy / Spherical Fuzzy / etc. / Neutrosophic / Refined Neutrosophic) **n-SuperHyperGraph.**

9. Introduction to n-ary HyperAlgebra

Let U be a universe of discourse, a nonempty set $S \subset U$. Let $P(S)$ be the power set of S (i.e. all subsets of S, including the empty set ϕ and the whole set S), and an integer $n \geq 1$.

We formed [2] the following neutrosophic triplets, which are defined in below sections:
(n-ary HyperOperation, n-ary NeutroHyperOperation, n-ary AntiHyperOperation),
(n-ary HyperAxiom, n-ary NeutroHyperAxiom, n-ary AntiHyperAxiom), and
(n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyperAlgebra).

10. n-ary HyperOperation (n-ary HyperLaw)

A *n-ary HyperOperation (n-ary HyperLaw)* $*_n$ is defined as:

$*_n : S^n \to P(S)$, and

$\forall a_1, a_2, \ldots, a_n \in S$ one has $*_n(a_1, a_2, \ldots, a_n) \in P(S)$.

The n-ary HyperOperation (n-ary HyperLaw) is well-defined.

11. n-ary HyperAxiom

A *n-ary HyperAxiom* is an axiom defined of S, with respect the above *n-ary* operation $*_n$, that is true for all *n-plets* of S^n.

12. n-ary HyperAlgebra

A *n-ary HyperAlgebra* $(S, *_n)$, is the S endowed with the above n-ary well-defined HyperOperation $*_n$.

13. Types of n-ary HyperAlgebras

Adding one or more n-ary HyperAxioms to S we get different types of *n-ary* HyperAlgebras.

14. n-ary NeutroHyperOperation (n-ary NeutroHyperLaw)

A *n-ary NeutroHyperOperation* is a *n-ary* HyperOperation $*_n$ that is well-defined for some *n-plets* of S^n

[i.e. $\exists (a_1, a_2, \ldots, a_n) \in S^n, *_n(a_1, a_2, \ldots, a_n) \in P(S)$],

and indeterminate [i.e. $\exists (b_1, b_2, \ldots, b_n) \in S^n, *_n(b_1, b_2, \ldots, b_n) = indeterminate$]

or outer-defined [i.e. $\exists(c_1, c_2, ..., c_n) \in S^n, *_n(c_1, c_2, ..., c_n) \notin P(S)$] (or both), on other *n-plets* of S^n.

15. n-ary NeutroHyperAxiom

A *n-ary NeutroHyperAxiom* is an n-ary HyperAxiom defined of *S*, with respect the above *n-ary* operation $*_n$, that is true for some *n-plets* of S^n, and indeterminate or false (or both) for other *n-plets* of S^n.

16. n-ary NeutroHyperAlgebra is an n-ary HyperAlgebra that has some n-ary NeutroHyper-Operations or some n-ary NeutroHyperAxioms

17. n-ary AntiHyperOperation (n-ary AntiHyperLaw)

A *n-ary AntiHyperOperation* is a *n-ary* HyperOperation $*_n$ that is outer-defined for all *n-plets* of S^n [i.e. $\forall(s_1, s_2, ..., s_n) \in S^n, *_n(s_1, s_2, ..., s_n) \notin P(S)$].

18. n-ary AntiHyperAxiom

A *n-ary AntiHyperAxiom* is an n-ary HyperAxiom defined of *S*, with respect the above *n-ary* operation $*_n$ that is false for all *n-plets* of S^n.

19. n-ary AntiHyperAlgebra is an n-ary HyperAlgebra that has some n-ary AntiHyperOperations or some n-ary AntiHyperAxioms.

20. Conclusion

We have recalled our 2019 concepts of n-Power Set of a Set, n-SuperHyperGraph and Plithogenic n-SuperHyperGraph [1], afterwards the n-ary HyperAlgebra together with its alternatives n-ary NeutroHyperAlgebra and n-ary AntiHyperAlgebra [2], and we presented several properties, explanations, and examples inspired from the real world.

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On αω-closed sets and its connectedness in terms of neutrosophic topological spaces

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M. Parimala, M. Karthika, Florentin Smarandache, Said Broumi (2020). On αω-closed sets and its connectedness in terms of neutrosophic topological spaces. *International Journal of Neutrosophic Science*, 2(2), 82-88. DOI: 10.5281/zenodo.3728230

Abstract

The aim of this paper is to introduce the notion of neutrosophic $\alpha\omega$-closed sets and study some of the properties of neutrosophic $\alpha\omega$-closed sets. Further, we investigated neutrosophic $\alpha\omega$- continuity, neutrosophic $\alpha\omega$-irresoluteness, neutrosophic $\alpha\omega$ connectedness and neutrosophic contra $\alpha\omega$ continuity along with examples.

Keywords: neutrosophic topology, neutrosophic $\alpha\omega$-closed set, neutrosophic $\alpha\omega$-continuous function and neutrosophic contra $\alpha\omega$-continuous mappings.

1 Introduction

Zadeh [19] introduced truth (t) or the degree of membership of an object in fuzzy set theory. The falsehood (f) or the degree of non-membership of an object along with membership of an object introduced by Atanassov [4,5,6] in intuitionistic fuzzy set. Neutrosophic (i) or the degree of indeterminacy of an object along with membership and non-membership of an objects for incomplete, imprecise, indeterminate information was introduced by Smarandache [16,17] in 1998. The neutrosophic triplet set consist of three components (t, f, i) $= (truth, f\ alsehood, indeterminacy)$. The neutrosophic topological spaces introduced and developed by Salama et al., [15]. This leads to many investigation among researchers in the field of neutrosophic topology and their application in decision making algorithms [8,11,12,13,14]. Arokiarani et al.,[3] introduced and studied α-open sets in neutrosophic topological spaces. Devi et al., [7,9,10] introduced $\alpha\omega$-closed sets in general topology, fuzzy topology and intuitionistic fuzzy topology. In this article, we introduce neutrosophic $\alpha\omega$-closed sets in neutrosophic topological spaces. Also, we introduce and investigate neutrosophic $\alpha\omega$-continuous,neutrosophic $\alpha\omega$-irresoluteness, neutrosophic $\alpha\omega$ connectedness and neutrosophic contra $\alpha\omega$-continuous mappings.

2 Preliminaries

Let (X, τ) be the neutrosophic topological space(NTS). Each neutrosophic set(NS) in (X, τ) is called a neutrosophic open set(NOS) and its complement is called a neutrosophic closed set (NCS).

We provide some of the basic definitions in neutrosophic sets. These are very useful in the sequel.

Definition 2.1. [17] A neutrosophic set (NS) A is an object of the following form

$$U = \{\langle u, \mu_U(u), \nu_U(u), \omega_U(u)\rangle : u \in X\}$$

where the mappings $\mu_U : X \to I$, $\nu_U : X \to I$ and $\omega_U : X \to I$ denote the degree of membership (namely $\mu_U(u)$), the degree of indeterminacy (namely $\nu_U(u)$) and the degree of nonmembership (namely $\omega_U(u)$) for each element $u \in X$ to the set U, respectively and $0 \leq \mu_U(u) + \nu_U(u) + \omega_U(u) \leq 3$ for each $u \in X$.

Definition 2.2. [17] Let U and V be NSs of the form $U = \{\langle u, \mu_U(u), \nu_U(u), \omega_U(u)\rangle : u \in X\}$ and $V = \{\langle u, \mu_V(u), \nu_V(u), \omega_V(u)\rangle : u \in X\}$. Then

(i) $U \subseteq V$ if and only if $\mu_U(u) \leq \mu_V(u)$, $\nu_U(u) \geq \nu_V(u)$ and $\omega_U(u) \geq \omega_V(u)$;

(ii) $\overline{U} = \{\langle u, \nu_U(u), \mu_U(u), \omega_U(u)\rangle : u \in X\}$;

(iii) $U \cap V = \{\langle u, \mu_U(u) \wedge \mu_V(u), \nu_U(u) \vee \nu_V(u), \omega_U(u) \vee \omega_V(u)\rangle : u \in X\}$;

(iv) $U \cup V = \{\langle u, \mu_U(u) \vee \mu_V(u), \nu_U(u) \wedge \nu_V(u), \omega_U(u) \wedge \omega_V(u)\rangle : u \in X\}$.

We will use the notation $U = \langle u, \mu_U, \nu_U, \omega_U\rangle$ instead of $U = \{\langle u, \mu_U(u), \nu_U(u), \omega_U(u)\rangle : u \in X\}$. The NSs $0_\sim$ and $1_\sim$ are defined by $0_\sim = \{\langle u, \underline{0}, \underline{1}, \underline{1}\rangle : u \in X\}$ and $1_\sim = \{\langle u, \underline{1}, \underline{0}, \underline{0}\rangle : u \in X\}$.

Let $r, s, t \in [0, 1]$ such that $0 \leq r + s + t \leq 3$. A neutrosophic point (NP) $p_{(r,s,t)}$ is neutrosophic set defined by

$$p_{(r,s,t)}(u) = \begin{cases} (r, s, t)(x) & if\ u = p \\ (0, 1, 1) & otherwise \end{cases}$$

Let f be a mapping from an ordinary set X into an ordinary set Y, If $V = \{\langle y, \mu_V(y), \nu_V(y), \omega_V(y)\rangle : y \in Y\}$ is a NS in Y , then the inverse image of V under f is a NS defined by

$$f^{-1}(V) = \{\langle u, f^{-1}(\mu_V)(u), f^{-1}(\nu_V)(u), f^{-1}(\omega_V)(u)\rangle : u \in X\}$$

The image of NS $U = \{\langle v, \mu_U(v), \nu_U(v), \omega_U(v)\rangle : v \in Y\}$ under f is a NS defined by $f(U) = \{\langle v, f(\mu_U)(v), f(\nu_U)(v), f(\omega_U)$ $v \in Y\}$ where

$$f(\mu_U)(v) = \begin{cases} \sup\limits_{u \in f^{-1}(v)} \mu_U(u), & if\ f^{-1}(v) \neq 0 \\ 0 & otherwise, \end{cases}$$

$$f(\nu_U)(v) = \begin{cases} \inf\limits_{u \in f^{-1}(v)} \nu_U(u), & if\ f^{-1}(v) \neq 0 \\ 1 & otherwise, \end{cases}$$

$$f(\omega_U)(v) = \begin{cases} \inf\limits_{u \in f^{-1}(v)} \omega_U(u), & if\ f^{-1}(v) \neq 0 \\ 1 & otherwise, \end{cases}$$

for each $v \in Y$.

Definition 2.3. [15] A neutrosophic topology (NT) in a nonempty set X is a family τ of NSs in X satisfying the following axioms:

(NT1) $0_\sim, 1_\sim \in \tau$;

(NT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;

(NT3) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

Definition 2.4. [15] Let U be a NS in NTS X. Then
Nint(U) = $\cup\{O : O$ is an NOS in X and O $\subseteq U\}$ is called a neutrosophic interior of U;
Ncl(U) = $\cap\{O : O$ is an NCS in X and O $\supseteq U\}$ is called a neutrosophic closure of U.

Definition 2.5. [15] Let $p_{(r,s,t)}$ be a NP in NTS X. A NS U in X is called a neutrosophic neighborhood (NN) of $p_{(r,s,t)}$ if there exists a NOS V in X such that $p_{(r,s,t)} \in V \subseteq U$.

Definition 2.6. [3] A subset U of a neutrosophic space (X, τ) is called

1. a neutrosophic pre-open set if $U \subseteq Nint(Ncl(U))$ and a neutrosophic pre-closed set if $Ncl(Nint(U)) \subseteq U$,

2. a neutrosophic semi-open set if $U \subseteq Ncl(Nint(U))$ and a neutrosophic semi-closed set if $Nint(Ncl(U)) \subseteq U$,

3. a neutrosophic α-open set if $U \subseteq Nint(Ncl(Nint(U)))$ and a neutrosophic α-closed set if $Ncl(Nint(Ncl(U))) \subseteq U$,

The pre-closure (resp. semi-closure, α-closure) of a subset U of a neutrosophic space (X, τ) is the intersection of all pre-closed (resp. semi-closed, α-closed) sets that contain U and is denoted by $Npcl(U)$ (resp. $Nscl(U)$, $N\alpha cl(U)$).

3 On neutrosophic $\alpha\omega$-closed sets

Definition 3.1. A subset A of a neutrosophic topological space (X, τ) is called

1. a neutrosophic $N\omega$-closed set if $Ncl(U) \subseteq G$ whenever $U \subseteq G$ and G is neutrosophic semi-open in (X, τ).
2. a neutrosophic $\alpha\omega$-closed ($N\alpha\omega$-closed) set if $N\omega cl(U) \subseteq G$ whenever $U \subseteq G$ and G is an $N\alpha$-open set in (X, τ). Its complement is called a neutrosophic $\alpha\omega$-open ($N\alpha\omega$-open) set.

Definition 3.2. Let U be a NS in NTS X. Then
$N\alpha\omega int(U)$ = $\cup\{O : O$ is an $N\alpha\omega$OS in X and O $\subseteq U\}$ is said to be a neutrosophic $\alpha\omega$-interior of U;
$N\alpha\omega cl(U)$ = $\cap\{O : O$ is an $N\alpha\omega$CS in X and O $\supseteq U\}$ is said to be a neutrosophic $\alpha\omega$-closure of U.

Theorem 3.3. *Every $N\alpha$-closed set and N-closed set are $N\alpha\omega$-closed set.*
Proof. Let U be an $N\alpha$-closed set, then $U = N\alpha cl(U)$. Let $U \subseteq G$, G is $N\alpha$-open. Since U is $N\alpha$-closed, $N\omega cl(U) \subseteq N\alpha cl(U) \subseteq G$. Thus U is $N\alpha\omega$-closed.

Theorem 3.4. *Every neutrosophic semi-closed set in a neutrosophic set is an $N\alpha\omega$-closed.*
Proof. Let U be a Nsemi-closed set in (X, τ), then $U = Nscl(U)$. Let $U \subseteq G$, G is $N\alpha$-open in (X, τ). Since U is Nsemi-closed, $N\omega cl(U) \subseteq Nscl(U) \subseteq G$. This shows that U is $N\alpha\omega$-closed set.

The converses of the above theorems are not true as explained in Example 3.5.

Example 3.5. Let $X = \{u, v, w\}$ and neutrosophic sets A, B, C be defined by:

$$A = \langle(0.1, 0.4, 0.7), (0.9, 0.6, 0.3), (0.9, 0.6, 0.3)\rangle$$
$$B = \langle(0.6, 0.6, 0.4), (0.2, 0.7, 0.8), (1, 0.6, 0.5)\rangle$$
$$C = \langle(0.1, 0.4, 0.8), (0.2, 0.6, 0.4), (0.6, 0.5, 0.9)\rangle$$

Let $\tau = \{0_\sim, A, 1_\sim\}$. Then B is $N\alpha\omega$-closed in (X, τ) but not $N\alpha$-closed and thus it is not N-closed and C is $N\alpha\omega$-closed in (X, τ) but not $Nsemi$-closed.

Theorem 3.6. Let (X, τ) be a NTS and let $U \in NS(X)$. If U is $N\alpha\omega$-closed set and $U \subseteq V \subseteq N\omega cl(U)$, then V is $N\alpha\omega$-closed set.
Proof. Let G be a $N\alpha$-open set such that $V \subseteq G$. Since $U \subseteq V$, then $U \subseteq G$. But U is $N\alpha\omega$-closed, so $N\omega cl(U) \subseteq G$. Since $V \subseteq N\omega cl(U)$. Since $N\omega cl(V) \subseteq N\omega cl(U)$ and hence $N\omega cl(V) \subseteq G$. Therefore V is a $N\alpha\omega$-closed set.

Theorem 3.7. Let U be a $N\alpha\omega$-open set in X and $N\omega int(U) \subseteq V \subseteq U$, then V is $N\alpha\omega$-open.
Proof. Suppose U is $N\alpha\omega$-open in X and $N\omega int(U) \subseteq V \subseteq U$. Then $\overline{U}$ is $N\alpha\omega$-closed and $\overline{U} \subseteq \overline{V} \subseteq N\omega cl(\overline{U})$. Then $\overline{U}$ is a $N\alpha\omega$-closed set by theorem 3.5. Hence V is a $N\alpha\omega$-open set in X.

Theorem 3.8. A NS U in a NTS (X, τ) is a $N\alpha\omega$-open set if and only if $V \subseteq N\omega int(U)$ whenever V is a $N\alpha$-closed set and $V \subseteq U$.
Proof. Let U be a $N\alpha\omega$-open set and let V be a $N\alpha$-closed set such that $V \subseteq U$. Then $\overline{U} \subseteq \overline{V}$ and hence $N\omega cl(\overline{U}) \subseteq \overline{V}$, since $\overline{U}$ is $N\alpha\omega$-closed. But $N\omega cl(\overline{U}) = \overline{N\omega int(U)}$, thus $V \subseteq N\omega int(U)$.
Conversely, suppose that the condition is satisfied, then $\overline{N\omega int(U)} \subseteq \overline{V}$ whenever $\overline{V}$ is $N\alpha$-open set and $\overline{U} \subseteq \overline{V}$. This implies that $N\omega cl(\overline{U}) \subseteq \overline{V} = G$ where G is $N\alpha$-open set and $\overline{U} \subseteq G$. Therefore $\overline{U}$ is $N\alpha\omega$-closed set and hence U is $N\alpha\omega$-open.

Theorem 3.9. Let U be a $N\alpha\omega$-closed subset of (X, τ). Then $N\omega cl(U) - U$ does not contain any non-empty $N\alpha\omega$-closed set.
Proof. Assume that U is a $N\alpha\omega$-closed set. Let F be a non-empty $N\alpha\omega$-closed set, such that $F \subseteq$

$N\omega cl(U) - U = N\omega cl(U) \cap \overline{U}$. i.e., $F \subseteq N\omega cl(U)$ and $F \subseteq \overline{U}$. Therefore, $U \subseteq \overline{F}$. Since $\overline{F}$ is a $N\alpha\omega$-open set, $N\omega cl(U) \subseteq \overline{F} \Rightarrow F \subseteq (N\omega cl(U) - U) \cap (\overline{N\omega cl(U)}) \subseteq N\omega cl(U) \cap \overline{N\omega cl(U)}$. i.e., $F \subseteq \phi$. Therefore F is empty.

Corollary 3.10. Let U be a $N\alpha\omega$-closed set of (X, τ). Then $N\omega cl(U) - U$ does not contain no non-empty N-closed set.
Proof. The proof follows from the Theorem 3.9.

Theorem 3.11. If U is both $N\omega$-open and $N\alpha\omega$-closed set, then U is a $N\omega$-closed set.
Proof. Since U is both $N\omega$-open and $N\alpha\omega$-closed set in X, then $N\omega cl(U) \subseteq U$. Also we have $U \subseteq N\omega cl(U)$. This gives that $N\omega cl(U) = U$. Therefore U is a $N\omega$-closed set in X.

4 On neutrosophic $\alpha\omega$-continuity, connectedness and contra-continuity

Definition 4.1. Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces.

1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be a neutrosophic $\alpha\omega$-continuous (briefly, $N\alpha\omega$-continuous) function if the inverse image of every open set in Y is a $N\alpha\omega$-open set in X.
Equivalently, if the inverse image of every open set in (Y, σ) is $N\alpha\omega$-open in (X, τ);

2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be a neutrosophic $\alpha\omega$-irresolute (briefly, $N\alpha\omega$-irresolute) function if the inverse image of every $N\alpha\omega$-open set in Y is a $N\alpha\omega$-open set in X.
Equivalently, if the inverse image of every $N\alpha\omega$-open set in (Y, σ) is $N\alpha\omega$-open in (X, τ);

Definition 4.2. A NTS (X, τ) is said to be neutrosophic-$\alpha\omega T_{1/2}$($N\alpha\omega T_{1/2}$ in short) space if every $N\alpha\omega C$ in X is an NC in X.

Definition 4.3. Let (X, τ) be any neutrosophic topological space. (X, τ) is said to be neutrosophic $\alpha\omega$-disconnected (in shortly $N\alpha\omega$-disconnected) if there exists a $N\alpha\omega$-open and $N\alpha\omega$-closed set $\overline{F}$ such that $\overline{F} \neq 0_\sim$ and $\overline{F} \neq 1_\sim$. (X, τ) is said to be neutrosophic $\alpha\omega$-connected if it is not neutrosophic $\alpha\omega$-disconnected.

Theorem 4.4. Every $N\alpha\omega$-connected space is neutrosophic connected.
Proof. For a $N\alpha\omega$-connected (X, τ) space and let (X, τ) not be neutrosophic connected. Hence, there exists a proper neutrosophic set, $\overline{F} =< \mu_{\overline{F}(x)}, \sigma_{\overline{F}(x)}, \nu_{\overline{F}(x)} >$, $\overline{F} \neq 0_\sim$ and $\overline{F} \neq 1_\sim$, such that $\overline{F}$ is both neutrosophic open and neutrosophic closed in (X, τ). Since every neutrosophic open set is $N\alpha\omega$-open and neutrosophic closed set is $N\alpha\omega$-closed, X is not $N\alpha\omega$-connected. Therefore, (X, τ) is neutrosophic connected.
However, the converse is not true.

Example 4.5. Let $X = \{u, v, w\}$ and neutrosophic sets A, B and C be defined by:

$$A = \langle (0.4, 0.5, 0.5), (0.4, 0.5, 0.5), (0.5, 0.5, 0.5) \rangle$$
$$B = \langle (0.7, 0.6, 0.5), (0.7, 0.6, 0.5), (0.3, 0.4, 0.5) \rangle$$
$$C = \langle (0.5, 0.6, 0.5), (0.5, 0.6, 0.5), (0.5, 0.6, 0.5) \rangle$$

Let $\tau = \{0_\sim, A, B, 1_\sim\}$. It is obvious that (X, τ) is NTS. Now, (X, τ) is neutrosophic connected. However, it is not a $N\alpha\omega$-connected.

Theorem 4.6. Let (X, τ) be a neutrosophic $\alpha\omega T_{1/2}$ space. (X, τ) is neutrosophic connected iff (X, τ) is $N\alpha\omega$-connected.

Proof. Let (X, τ) is neutrosophic connected. Suppose that (X, τ) is not $N\alpha\omega$-connected, and there exists a neutrosophic set $\overline{F}$ which is both $N\alpha\omega$-open and $N\alpha\omega$-closed. Since (X, τ) is neutrosophic $\alpha\omega T_{1/2}$, $\overline{F}$ is both neutrosophic open and neutrosophic closed. Therefore, (X, τ) is not a neutrosophic connected which is contradiction to our hypothesis. Hence, (X, τ) is $N\alpha\omega$-connected.
Conversely, let (X, τ) is $N\alpha\omega$-connected. Suppose that (X, τ) is not neutrosophic connected, and there exists a neutrosophic set $\overline{F}$ such that $\overline{F}$ is both NCs and NOs $\in (X, \tau)$. Since the neutrosophic open set is $N\alpha\omega$-open and the neutrosophic closed set is $N\alpha\omega$-closed, (X, τ) is not $N\alpha\omega$-connected. Hence, (X, τ) is neutrosophic connected.

Theorem 4.7. Suppose (X,τ) and (Y,σ) are any two NTSs. If $g:(X,\tau)\to(Y,\sigma)$ is $N\alpha\omega$-continuous surjection and (X,τ) is $N\alpha\omega$-connected, then (Y,σ) is neutrosophic connected.
Proof. Suppose that (Y,σ) is not neutrosophic connected, such that the neutrosophic set $\overline{F}$ is both neutrosophic open and neutrosophic closed in (Y,σ). Since g is $N\alpha\omega$-continuous, $g^{-1}(\overline{F})$ is $N\alpha\omega$-open and $N\alpha\omega$-closed in (Y,σ). Thus, (Y,σ) is not $N\alpha\omega$-connected. Hence, (Y,σ) is neutrosophic connected.

Theorem 4.8. Let $g:(X,\tau)\to(Y,\sigma)$ be a function. Then the following conditions are equivalent.

(i) g is $N\alpha\omega$-continuous;

(ii) The inverse $f^{-1}(U)$ of each N-open set U in Y is $N\alpha\omega$-open set in X.

Proof. It is clear, since $g^{-1}(\overline{U})=\overline{g^{-1}(U)}$ for each N-open set U of Y.

Theorem 4.9. If $g:(X,\tau)\to(Y,\sigma)$ be a $N\alpha\omega$-continuous mapping, then the following statements holds:

(i) $g(N\alpha\omega Ncl(U))\subseteq Ncl(g(U))$, for all neutrosophic set U in X;

(ii) $N\alpha\omega Ncl(g^{-1}(V))\subseteq g^{-1}(Ncl(V))$, for all neutrosophic set V in Y.

Proof.

(i) Since $Ncl(g(U))$ is neutrosophic closed set in Y and g is $N\alpha\omega$-continuous, then $g^{-1}(Ncl(g(U)))$ is $N\alpha\omega$-closed in X. Now, since $U\subseteq g^{-1}(Ncl(g(U)))$. So, $N\alpha\omega cl(U)\subseteq g^{-1}(Ncl(g(U)))$. Therefore, $g(N\alpha\omega Ncl(U))\subseteq Ncl(g(U))$.

(ii) By replacing U with V in (i), we obtain $g(N\alpha\omega cl(g^{-1}(V)))\subseteq Ncl(g(g^{-1}(V)))\subseteq Ncl(V)$. Hence $N\alpha\omega cl(g^{-1}(V))\subseteq g^{-1}(Ncl(V))$.

Theorem 4.10. Let g be a function from a NTS (X,τ) to a NTS (Y,σ). Then the following statements are equivalent.

(i) g is a neutrosophic $\alpha\omega$-continuous function.

(ii) For every NP $p_{(r,s,t)}\in X$ and each NN U of $g(p_{(r,s,t)})$, there exists a $N\alpha\omega$-open set V such that $p_{(r,s,t)}\in V\subseteq g^{-1}(U)$.

(iii) For every NP $p_{(r,s,t)}\in X$ and each NN U of $g(p_{(r,s,t)})$, there exists a $N\alpha\omega$-open set V such that $p_{(r,s,t)}\in V$ and $g(V)\subseteq U$.

Proof. $(i)\Rightarrow(ii)$. If $p_{(r,s,t)}$ is a NP in X and also if U be a NN of $g(p_{(r,s,t)})$, then there exists a NOS W in Y such that $g(p_{(r,s,t)})\in W\subset U$. we have g is neutrosophic $\alpha\omega$-continuous, $V=g^{-1}(W)$ is an $N\alpha\omega OS$ and

$$p_{(r,s,t)}\in g^{-1}(g(p_{(r,s,t)}))\subseteq g^{-1}(W)=V\subseteq g^{-1}(U).$$

Thus (ii) is a valid statement.
$(ii)\Rightarrow(iii)$. Let $p_{(r,s,t)}$ be a NP in X and take U be a NN of $g(p_{(r,s,t)})$. Then there exists a $N\alpha\omega OS\ U$ such that $p_{(r,s,t)}\in V\subseteq g^{-1}(U)$ by (ii). Thus, we have $p_{(r,s,t)}\in V$ and $g(V)\subseteq g(g^{-1}(U))\subseteq U$. Hence (iii) is valid.
$(iii)\Rightarrow(i)$. Let V be a NOS in Y and let $p_{(r,s,t)}\in g^{-1}(V)$. Then $g(p_{(r,s,t)})\in g(g^{-1}(V))\subset V$. Since V is a NOS, it follows that V is a NN of $g(p_{(r,s,t)})$ so from (iii), there exists a $N\alpha\omega OS\ U$ such that $p_{(r,s,t)}\in U$ and $g(U)\subseteq V$. This implies that

$$p_{(r,s,t)}\in U\subseteq g^{-1}(g(U))\subseteq g^{-1}(V).$$

Then, we know that $g^{-1}(V)$ is a $N\alpha\omega OS$ in X. Thus g is neutrosophic $\alpha\omega$-continuous.

Definition 4.11. A function is said to be a neutrosophic contra $\alpha\omega$-continuous function if the inverse image of each NOS V in Y is a N$\alpha\omega$CS in X.

Theorem 4.12. Let $g:(X,\tau)\to(Y,\sigma)$ be a function. Then, the following assertions are equivalent:

(i) g is a neutrosophic contra $\alpha\omega$-continuous function;

(ii) $g^{-1}(V)$ is a N$\alpha\omega$ CS in X, for each NOS V in Y.

Proof. $(i) \Rightarrow (ii)$ Let g be any neutrosophic contra $\alpha\omega$-continuous function and let V be any NOS in Y. Then, $\overline{V}$ is a NCS in Y. By the assumption $g^{-1}(\overline{V})$ is a $N\alpha\omega OS$ in X. Hence, we get that $g^{-1}(V)$ is a $N\alpha\omega CS$ in X.

The converse of the theorem can be done in the same sense.

Theorem 4.13. Let $g : (X, \tau) \to (Y, \sigma)$ be a bijective mapping from an NTS X into an NTS Y. The mapping g is neutrosophic contra $\alpha\omega$-continuous if $Ncl(g(U)) \subseteq g(N\alpha\omega int(U))$, for each NS U in X.

Proof. Let V be any NCS in X. Then, $Ncl(V) = V$, and also g is onto, by assumption, it shows that $g(N\alpha\omega int(g^{-1}(V))) \supseteq Ncl(g(g^{-1}(V))) = Ncl(V) = V$. Hence $g^{-1}(g(N\alpha\omega int(g^{-1}(V)))) \supseteq g^{-1}(V)$. Since g is an into mapping, we have $N\alpha\omega int(g^{-1}(V)) = g^{-1}(g(N\alpha\omega int(g^{-1}(V)))) \supseteq g^{-1}(V)$. Therefore $N\alpha\omega int(g^{-1}(V))$
$= g^{-1}(V)$, so $g^{-1}(V)$ is a $N\alpha\omega$OS in X. Hence g is a neutrosophic contra $\alpha\omega$-continuous mapping.

Theorem 4.14. Let $g : (X, \tau) \to (Y, \sigma)$ be a mapping. Then the following statements are equivalent:

(i) g is a neutrosophic contra $\alpha\omega$-continuous mapping;

(ii) for each NP $p_{(r,s,t)}$ in X and NCS V containing $g(p_{(r,s,t)})$ there exists $N\alpha\omega OS\ U$ in X containing $p_{(r,s,t)}$ such that $A \subseteq f^{-1}(B)$;

(iii) for each NP $p_{(r,s,t)}$ in X and NCS V containing $p_{(r,s,t)}$ there exists $N\alpha\omega OS\ U\ in\ X$ containing $p_{(r,s,t)}$ such that $g(U) \subseteq V$.

Proof. $(i) \Rightarrow (ii)$ Let g be an neutrosophic contra $\alpha\omega$-continuous mapping, let V be any NCS in Y and let $p_{(r,s,t)}$ be a NP in X and such that $g(p_{(r,s,t)}) \in V$. Then $p_{(r,s,t)} \in g^{-1}(V) = N\alpha\omega int(g^{-1}(V))$. Let $U = N\alpha\omega int(g^{-1}(V))$. Then U is an $N\alpha\omega OS$ and $U = N\alpha\omega int(g^{-1}(V)) \subseteq g^{-1}(V)$.
$(ii) \Rightarrow (iii)$ The results follows from the evident relations $g(U) \subseteq g(g^{-1}(V)) \subseteq V$.
$(iii) \Rightarrow (i)$ Let V be any NCS in Y and let $p_{(r,s,t)}$ be a NP in X such that $p_{(r,s,t)} \in g^{-1}(V)$. Then $g(p_{(r,s,t)}) \in V$. According to the assumption, there exists an $N\alpha\omega OS\ U$ in X such that $p_{(r,s,t)} \in U$ and $g(U) \subseteq V$. Hence $p_{(r,s,t)} \in U \subseteq g^{-1}(g(U)) \subseteq g^{-1}(V)$. Therefore $p_{(r,s,t)} \in U = \alpha\omega int(U) \subseteq N\alpha\omega int(g^{-1}(V))$. Since, $p_{(r,s,t)}$ is an arbitrary NP and $g^{-1}(V)$ is the union of all NPs in $g^{-1}(V)$, we obtain that $g^{-1}(V) \subseteq N\alpha\omega int(g^{-1}(V))$. Thus g is a neutrosophic contra $N\alpha\omega$-continuous mapping.

Corollary 4.15. Let $X,\ X_1\ and\ X_2$ be NTSs, $p_1 : X \to X_1 \times X_2$ $(i = 1,\ 2)$ and $p_2 : X \to X_1 \times X_2$ are the projections of $X_1 \times X_2$ onto X_i, $(i = 1,\ 2)$. If $g : X \to X_1 \times X_2$ is a neutrosophic contra $\alpha\omega$-continuous, then $p_i g$ are also neutrosophic contra $\alpha\omega$-continuous mapping.

Proof. The proof follows from the fact that the projections are all neutrosophic continuous functions.

Theorem 4.16. Let $g : (X_1, \tau) \to (Y_1, \sigma)$ be a function. If the graph $h : X_1 \to X_1 \times Y_1$ of g is neutrosophic contra $\alpha\omega$-continuous, then g is neutrosophic contra $\alpha\omega$-continuous.

Proof. For every NOS V in Y_1 holds $g^{-1}(V) = 1 \wedge g^{-1}(V) = h^{-1}(1 \times V)$. Since h is a neutrosophic contra $\alpha\omega$-continuous mapping and $1 \times V$ is a NOS in $X_1 \times Y_1$, $g^{-1}(V)$ is a $N\alpha\omega CS$ in X_1, so g is a neutrosophic contra $\alpha\omega$-continuous mapping.

5 Conclusions

In this paper, we introduced and investigated the neutrosophic $\alpha\omega$ closed sets and its properties. Also, we investigated the continuity, irresolute, connectedness and contra-continuity in terms of neutrosophic $\alpha\omega$ closed sets.

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The Neutrosophic Triplet of BI-Algebras

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Akbar Rezaei, Florentin Smarandache (2020). The Neutrosophic Triplet of BI-Algebras. *Neutrosophic Sets and Systems*, 33, 313-321

Abstract: In this paper, the concepts of a Neutro-BI-algebra and Anti-BI-algebra are introduced, and some related properties are investigated. We show that the class of Neutro-BI-algebra is an alternative of the class of BI-algebras.

Keywords: BI-algebra; Neutro- BI -algebra; sub-Neutro- BI -algebra; Anti- BI -algebra; sub-Anti-BI-algebra; Neutrosophic Triplet of BI-algebra.

1. Introduction

1.1. BI-algebras

In 2017, A. Borumand Saeid et al. introduced BI-algebras as an extension of both a (dual) implication algebras and an implicative BCK-algebra, and they investigated some ideals and congruence relations [1]. They showed that every implicative BCK-algebra is a BI-algebra, but the converse is not valid in general. Recently, A. Rezaei et al. introduced the concept of a (branchwise) commutative BI-algebra and showed that commutative BI-algebras form a class of lower semilattices and showed that every commutative BI-algebra is a commutative BH-algebra [2].

1.2 Neutrosophy

Neutrosophy is a new branch of philosophy that generalized the dialectics and took into consideration not only the dynamics of opposites, but the dynamics of opposites and their neutrals introduced by Smarandache in 1998 [5]. Neutrosophic Logic / Set / Probability / Statistics etc. are all based on it.

One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, vague set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic vague set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been a very important tool in all various areas of data mining, decision making, e-learning, engineering, computer science, graph theory, medical diagnosis, probability theory, topology, social science, etc.

1.3 NeutroLaw, NeutroOperation, NeutroAxiom, and NeutroAlgebra

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

The Neutrosophy's Triplet is (<A>, <neutroA>, <antiA>), where <A> may be an item (concept, idea, proposition, theory, structure, algebra, etc.), <antiA> the opposite of <A>, while <neutroA> {also the notation <neutA> was employed before} the neutral between these opposites.

Based on the above triplet the following Neutrosophic Principle one has: a law of composition defined on a given set may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroLaw.

A law of composition defined on a given sets, such that the law is false (F) for set's elements is called AntiLaw.

Similarly, an operation defined on a given set may be well-defined for some set's elements, indeterminate for other set's elements, and outer-defined for the remainder of the set's elements; we call it NeutroOperation.

While, an operation defined on a given set that is outer-defined for all set's elements is called AntiOperation.

In classical algebraic structures, the laws of compositions or operations defined on a given set are automatically well-defined [i.e. true (T) for all set's elements], but this is idealistic.

Consequently, an axiom (let's say Commutativity, or Associativity, etc.) defined on a given set, may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroAxiom.

In classical algebraic structures, similarly an axiom defined on a given set is automatically true (T) for all set's elements, but this is idealistic too.

A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or some NeutroAxiom.

The NeutroLaw, NeutroOperation, NeutroAxiom, NeutroAlgebra and respectively AntiLaw, AntiOperation, AntiAxiom and AntiAlgebra were introduced by Smarandache in 2019 [4] and afterwards he recalled, improved and extended them in 2020 [5].

2. Neutro-BI-algebras, Anti-BI-Algebras

In this section, we apply Neutrosophic theory to generalize the concept of a BI-algebra. Some new concepts as, Neutro-sub- BI -algebra, Anti-sub- BI -algebra, Neutro- BI -algebra, sub-Neutro- BI -algebra, NutroLow-sub-Neutro- BI -algebra, AntiLow-sub-Neutro- BI -algebra, Anti- BI -algebra, sub-Anti- BI -algebra, NeutroLow-sub-Anti- BI -algebra and AntiLow-sub-Anti-BI-algebra are proposed.

Definition 2.1. (Definition of classical BI-algebras [1])

An algebra $(X,*,0)$ of type $(2,0)$ (i.e. X is a nonempty set, $*$ is a binary operation and 0 is a constant element of X) is said to be a *BI-algebra* if it satisfies the following axioms:

(B) $(\forall x \in X)(x * x = 0)$,

(BI) $(\forall x, y \in X)(x * (y * x) = x)$.

Example 2.2. ([1])

(i) Let X be a set with $0 \in X$. Define a binary operation $*$ on X By

$$x * y = \begin{cases} 0 & if\ x = y; \\ x & if\ x \neq y. \end{cases}$$

Then $(X,*,0)$ is a BI-algebra.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S), -, \emptyset)$ is a BI-algebra. Since $A - A = \emptyset$ and for every $A \in \mathcal{P}(S)$. Also, $A - (B - A) = A \cap (B \cap A^c)^c = A \cap (B^c \cup A^{cc}) = A$, for every $A, B \in \mathcal{P}(S)$. Thus, (B) and (BI) hold.

Definition 2.3. (Definition of classical sub-BI-algebras)

Let $(X,*,0)$ be a BI-algebra. A nonempty set S of X is said to be a *sub-BI-algebra* of X if $(\forall x,y \in S)(x * y \in S)$.

We note that X and $\{0\}$ are sub-BI-algebra.

Example 2.4. Let $X := \{0,a,b,c\}$ be a set with the following table.

Table 1

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $(X,*,0)$ is a BI-algebra. We can see that $S = \{0,a,b\}$ is a sub-algebra of X, $T = \{0,a,c\}$ is not a sub-algebra, since, $a,c \in T$, but $c * a = b \notin T$.

Definition 2.5. (Definition of Neutro-sub-BI-algebras)

Let $(X,*,0)$ be a BI-Algebra. A nonempty set NS of X is said to be a *Neutro-sub-BI-algebra* of X if $(\exists x,y \in NS)(x * y \in NS)$and $(\exists x,y \in NS)$ such that $x * y \notin NS$ or $x * y$ = indeterminate.

We note that X and $\{0\}$ are not Neutro-sub-BI-algebras. Since $*$ is a binary operation, and so $x * y \in X$, for all $x,y \in X$. Also, there are no $x,y \in \{0\}$ such that $x * y \notin \{0\}$.

Example 2.6. Consider the BI-algebra $(X,*,0)$ given in Example 2.4. $S = \{0,a,c\}$ is a Neutro-sub-BI-algebra, since $0 * a = 0 \in S$, $a * 0 = a \in S$ and $c * 0 = c \in S$, but $c * a = b \notin S$.

Definition 2.7. (Definition of Anti-sub-BI-algebras)

Let $(X,*,0)$ be a BI-algebra. A nonempty set AS of X is said to be an *Anti-sub-BI-algebra of X* if $(\forall x,y \in AS)(x * y \notin AS)$.

We note that X and $\{0\}$ are not Anti-sub-BI-algebra. Since $*$ is a binary operation, and so $x * y \in X$, for all $x,y \in X$. Also, $(\forall x,y \in \{0\})(x * y \in \{0\})$.

Example 2.8. Consider the BI-algebra $(X,*,0)$ given in Example 2.4. $S = \{c\}$ is an Anti-sub-BI-algebra, since $c * c = 0 \notin S$.

In classical algebraic structures, a Law (Operation) defined on a given set is automatically well-defined (i.e. true for all set's elements), but this is idealistic; in reality we have many more cases where the law (or operation) are not true for all set's elements. In NeutroAlgebra, a law (operation) may be well-defined (T) for some set's elements, indeterminate (I) for other set's elements, and outer-defined (F) for the other set's elements. We call it NeutroLaw (NeutroOperation).

In classical algebraic structures, an Axiom defined on a given set is automatically true for all set's elements, but this is idealistic too. In NeutroAlgebra, an axiom may be true for some of the set's elements, indeterminate (I) for other set's elements, and false (F) for other set's elements.

We call it NeutroAxiom.

A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or NeutroAxiom. NeutroAlgebra better reflects our imperfect, partial, indeterminate reality.

There are several NeutroAxioms that can be defined on a BI-algebra. We neutrosophically convert its first two classical axioms: (B) into (NB), and (BI) into (NBI). Afterwards, the classical axiom (BI) is completed negated in two different ways $(ABI1)$ and $(ABI2)$ respectively.

- (NB) $(\exists x \in NX)(x *_N x = 0)$ and $(\exists x \in NX)(x *_N x \neq 0)$,
- (NBI) $(\exists x, y \in NX)(x *_N (y *_N x) = x)$ and $(\exists x, y \in NX)(x *_N (y *_N x) \neq x)$,
- $(ABI1)$ $(\forall x \in NX, \exists y \in NX)(x *_N (y *_N x) \neq x)$,
- $(ABI2)$ $(\exists x \in NX, \forall y \in NX)(x *_N (y *_N x) \neq x)$.

In this paper we consider the following:

Definition 2.9. (Definition of Neutro-BI-algebras)

An algebra $(NX, *_N, 0_N)$ of type $(2, 0)$ (i.e. NX is a nonempty set, $*_N$ is a binary operation and 0_N is a constant element of X) is said to be a *Neutro-BI-algebra* if it satisfies the following NeutroAxioms:

(NB) $(\exists x \in NX)(x *_N x = 0_N)$ and $(\exists x \in NX)(x *_N x \neq 0_N$ or indeterminate),

(NBI) $(\exists x, y \in NX)(x *_N (y *_N x) = x)$ and $(\exists x, y \in NX)(x *_N (y *_N x) \neq x$ or indeterminate).

Example 2.10.

(i) Let $NX := \{0_N, a, b, c\}$ be a set with the following table.

Table 2

$*_N$	0_N	a	b	c
0_N	0_N	0_N	0_N	0_N
a	a	0_N	a	b
b	b	b	a	b
c	c	b	b	0_N

Then $(NX, *_N, 0_N)$ is a Neutro-BI-algebra. Since $a *_N a = 0_N$ and $b *_N b = a \neq 0_N$. Also, $a *_N (b *_N a) = a *_N b = a$ and $c *_N (b *_N c) = c *_N b = b \neq c$.

(ii) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_N$ on $\mathbb{R}$ by $x *_N y = x + y + 1$. Then $(\mathbb{R}, *_N, 0)$ is a Neutro-BI-algebra. Since if $x = 0$, then $0 *_N 0 = 0 + 0 + 1 = 1 \neq 0$, and if $x = -0.5$, then $x *_N x = x + x + 1 = 2x + 1 = -1 + 1 = 0$, so (NB) holds. For (NBI), let $x \in \mathbb{R}$. If $y = -x - 2$, then $x *_N (y *_N x) = x$, and if $y \neq -x - 2$, then $x *_N (y *_N x) \neq x$.

(iii) Consider the BI-algebra given in Example 2.2 (ii), it is not a Neutro-BI-algebra. Since (NB) and (NBI) are not valid.

(iv) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S), \cap, \emptyset)$ is a Neutro-BI-algebra. Since $\emptyset \cap \emptyset = \emptyset$, and for every $A \neq \emptyset$, $A \cap A = A \neq \emptyset$. Further, if $A \subseteq B$, then $A \cap (B \cap A) = A \cap A = A$. Also, since $A, A^c \in \mathcal{P}(S)$, we get $A \cap (A^c \cap A) = A \cap \emptyset = \emptyset \neq A$. Thus, (NB) and (NBI) hold. Moreover, by a similar argument $(\mathcal{P}(S), \cup, \emptyset)$, is not a BI-algebra, but is a Neutro-BI-algebra.

(v) Similarly, $(\mathcal{P}(S), \cap, S)$ and $(\mathcal{P}(S), \cup, S)$ are Neutro-BI-algebras.

(vi) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_N$ on $\mathbb{R}$ by $x *_N y = x^2 - y$. Then $(\mathbb{R}, *_N, 0)$ is not a BI-algebra. Since $3 *_N 3 = 3^2 - 3 = 6 \neq 0$, so (B) is not valid. If $x \in \{0,1\}$, then $x *_N x = 0$. If $x \notin \{0,1\}$, $x *_N x \neq 0$. Hence (NB) holds. If $x \in \{-y, y\}$, then $x *_N (y *_N x) = x$. If $x \notin \{-y, y\}$, then $x *_N (y *_N x) \neq x$. Thus, (NBI) is valid. Therefore, $(\mathbb{R}, *_N, 0)$ is a Neutro-BI-algebra.

(vii) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_N$ on $\mathbb{R}$ by $x *_N y = x^3 - y$. Then $(\mathbb{R}, *_N, 0)$ is not a BI-algebra. Since $3 *_N 3 = 3^3 - 3 = 24 \neq 0$, so (B) is not valid. If $x \in \{-1,0,1\}$, then $x *_N x = 0$. If $x \notin \{-1,0,1\}$, $x *_N x \neq 0$. Hence (NB) holds. If $x = y$, then $x *_N (y *_N x) = x$. If $x \neq y$, then $x *_N (y *_N x) \neq x$. Thus, (NBI) is valid. Therefore, $(\mathbb{R}, *_N, 0)$ is a Neutro-BI-algebra.

Definition 2.11. (Definition of sub-Neutro-BI-algebras)

Let $(NX, *_N, 0)$ be a Neutro-BI-algebra. A nonempty set NS of NX is said to be a *sub-Neutro-BI-algebra* of NX if $(\forall x, y \in NS)(x *_N y \in NS)$ and NS is itself a Neutro-BI-algebras.

Note that NX is a sub-Neutro-BI-algebra, because $*_N$ is a binary operation, and so it is close. $\{0_N\}$ is not a sub-Neutro-BI-algebra, since it is not a Neutro-BI-algebra because $0_N = 0_N *_N 0_N \in \{0_N\}$.

Example 2.12. Consider the Neutro-BI-algebra $(NX, *_N, 0_N)$ given in Example 2.10 (i). $NS = \{0_N, a, b\}$ is a sub-Neutro-BI-algebra of NX, but $NT = \{0_N, b, c\}$ is not a sub-Neutro-BI-algebra, since $b \in NT$, $b *_N b = a \notin NT$.

Definition 2.13. (Definition of NeutroLaw-sub-Neutro-BI-algebras)

Let $(NX, *_N, 0_N)$ be a Neutro-BI-algebra. A nonempty set NS of NX is said to be a *NeutroLaw-sub-Neutro-BI-algebra* of NX if $(\exists x, y \in NS)(x *_N y \in NS)$and $(\exists x, y \in NS)(x *_N y \notin NS)$.

{As a parenthesis, we recall that NS had to be itself a Neutro-BI-algebra, and this could occur by NS satisfying one or more of the following: the (NB) NeutroAxiom, the (NBI) NeutroAxiom, or the NeutroLaw. We chose, as a particular definition, the NeutroLaw.}

We note that neither NX nor $\{0\}$ are NeutroLaw-sub-Neutro-algebra.

Example 2.14. From Example 2.12, $NT = \{0_N, b, c\}$ is a NeutroLaw-sub-Neutro-BI-algebra. Since $b *_N c = b \in NT$ and $b *_N b = a \notin NT$.

Definition 2.15. (Definition of AntiLaw-sub-Neutro-BI-algebras)

Let $(NX, *_N, 0_N)$ be a Neutro-BI-algebra. A nonempty set AS of NX is said to be an *AntiLaw-sub-Neutro-BI-algebra* of X if $(\forall x, y \in AS)(x *_N y \notin AS)$.

{Similarly, as a parenthesis, we recall that AS had to be itself an Anti-BI-algebra, and this could occur by AS satisfying one or more of the following: the (AB) AntiAxiom, the (NBI) AntiAxiom, or the AntiLaw. We chose, as a particular definition, the AntiLaw.}

In this case NX is not an AntiLaw-sub-Neutro-BI-algebra, but $\{0_N\}$ may or may not be an AntiLaw-sub-Neutro-algebra. If $0_N *_N 0_N \in \{0_N\}$, then it is not an AntiLaw-sub-Neutro-algebra. If $0_N *_N 0_N \notin \{0_N\}$, then it is.

Example 2.16. Let $NX:=\{0_N, a, b, c\}$ be a set with the following table.

Table 3

$*_N$	0_N	a	b	c
0_N	0_N	0_N	0_N	0_N
a	a	0_N	a	b
b	b	b	a	a
c	c	b	a	a

Then $(NX, *_N, 0_N)$ is a Neutro-BI-algebra. $AS=\{b,c\}$ is an AntiLaw-sub-Neutro-BI-algebra, because $b *_N b = b *_N c = c *_N b = c *_N c = a \notin AS$.

Definition 2.17. (Definition of Anti-BI-algebras)

An algebra $(AX, *_A, 0_A)$ of type $(2,0)$ (i.e. AX is a nonempty set, $*_A$ is a binary operation and 0_A is a constant element of AX) is said to be an *Anti-BI-algebra* if it satisfies the following AntiAxioms,

(AB) $(\forall x \in AX)(x *_A x \neq 0_A)$,

(ABI) $(\forall x, y \in AX)(x *_A (y *_A x) \neq x)$.

Example 2.18.

(i) Let $\mathbb{N}$ be the natural number and $AX:=\mathbb{N}\cup\{0\}$. Define a binary operation $*$ on AX by $x *_A y = x+y+1$. Then $(AX, *_A, 0)$ is an Anti-BI-algebra. Since $x *_A x = x+y+1 \neq 0$, for all $x \in AX$, and $x *_A (y *_A x) = x *_A (y+x+1) = x+(x+y+1)+1 = 2x+y+2 \neq 0$, for all $x, y \in AX$.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Define the binary operation Δ (i.e. symmetric difference) by $A\Delta B = (A\cup B) - (A\cap B)$ for every $A, B \in \mathcal{P}(S)$. Then $(\mathcal{P}(S), \Delta, S)$ is not a BI-algebra neither Neutro-BI-algebra nor Anti-BI-algebra. Since $A\Delta A = \emptyset \neq S$ for every $A \in \mathcal{P}(S)$ we get (AB) hold, and so (B) and (NB) are not valid. Also, for every $A, B \in \mathcal{P}(S) - \{\emptyset\}$, we have $A\Delta(B\Delta A) = B \neq A$, and since $\emptyset \in \mathcal{P}(S)$, we get $\emptyset\Delta(\emptyset\Delta\emptyset) = \emptyset$. Thus, (ABI) is not valid.

(iii) Similarly, $(\mathcal{P}(S), \Delta, \emptyset)$ is not a BI-algebra neither Neutro-BI-algebra nor Anti-BI-algebra.

(iv) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Define the binary operation ∇ as $A\nabla B = (A\cup B)\cup C$, for every $A, B \in \mathcal{P}(S)$, where C is a given set of $P(S)$ and $C \notin \{\emptyset, A, B\}$. Then $(\mathcal{P}(S) - \{S\}, \nabla, \emptyset)$ is an Anti-BI-algebra. Since $A\nabla A = (A\cup A)\cup C = \mathrm{A}\cup C$, which can never be equal to $\emptyset$ since $\mathrm{C} \neq \emptyset$. Hence (AB) holds. Also, $A\nabla(B\nabla \mathrm{A}) \neq A$ and so (ABI) holds.

(v) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_A$ on $\mathbb{R}$ by $x *_A y = x^2+1$. Then $(\mathbb{R}, *_A, 0)$ is not a BI-algebra. Since $3 *_A 3 = 3^2+1 = 10 \neq 0$, so (B) is not valid. Let $x, y \in \mathbb{R}$, then $x *_A x = x^2+1 \neq 0$ and $x *_A (y *_A x) = x *_A (y^2+1) = x^2+1 \neq 0$. Thus, $(\mathbb{R}, *_A, 0)$ is an Anti-BI-algebra.

(vi) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_A$ on $\mathbb{R}$ by $x *_A y = x^2+1$. Then $(\mathbb{R}, *_A, 0)$ is not a BI-algebra. Since $3 *_A 3 = 3^2+1 = 10 \neq 0$, so (B) is not valid. Let $x, y \in \mathbb{R}$, then $x *_A x = x^2+1 \neq 0$, thus one has (AB), and $x *_A (y *_A x) = x *_A (y^2+1) = x^2+1 \neq 0$, or one has (ABI). Therefore, $(\mathbb{R}, *_A, 0)$ is an Anti-BI-algebra.

Definition 2.19. (Definition of sub-Anti-BI-algebras)

Let $(AX, *_A, 0_A)$ be an Anti-BI-algebra. A nonempty set AS of AX is said to be a *sub-Anti-BI-algebra* of X if $(\forall x, y \in AS)(x *_A y \in AS)$.

We note that AX is a sub-Anti-BI-algebra, but $\{0_A\}$ is not a sub-Anti-BI-algebra, since

$0_A *_A 0_A \notin \{0_A\}$.

Example 2.20. Consider the Anti-BI-algebra $(AX, *_A, 0)$ given in Example 2.18 (i). $\mathbb{N}$ is a sub-Anti-BI-algebra of AX. Since $x *_A y = x + y + 1 \in \mathbb{N}$, for all $x, y \in \mathbb{N}$.

Definition 2.21. (Definition of NeutroLaw-sub-Anti-BI-algebras)

Let $(AX, *_A, 0_A)$ be an Anti-BI-algebra. A nonempty set AS of AX is said to be a *NeutroLaw-sub-Anti-BI-algebra* of X if $(\exists x, y \in AS)(x *_A y \in AS)$ and $(\exists x, y \in AS)(x *_A y \notin AS)$.

In this case AX and $\{0_A\}$ are not NeutroLaw-sub-Anti-BI-algebras. Since $\nexists x, y \in AX$ such that $x *_A y \notin AX$, and similarly for $\{0_A\}$.

Example 2.22. Let $AX := \{0_A, a, b, c\}$ be a set with the following table.

Table 4

$*_A$	0_A	a	b	c
0_A	b	a	c	a
a	a	c	b	b
b	b	c	a	a
c	c	b	a	a

Then $(AX, *_A, 0_A)$ is an Anti-BI-algebra. $NS = \{a, b\}$ is a NeutroLaw-sub-Anti-BI-algebra, since $a *_A b = b \in NS$ and $b *_A a = c \notin NS$.

Definition 2.23. (Definition of AntiLaw-sub-Anti-BI-algebras)

Let $(AX, *_A, 0)$ be an Anti-BI-algebra. A nonempty set AS of AX is said to be an *AntiLaw-sub-Anti-BI-algebra* of X if $(\forall x, y \in AS)(x *_A y \notin AS)$.

In this case AX is not an AntiLaw-sub-Anti-BI-algebra, but $\{0_A\}$ may or may not be an AntiLaw-sub-Anti-BI-algebra. If $0_A *_A 0_A \in \{0_A\}$, then it is not an AntiLaw-sub-Anti-algebra. If $0_A *_A 0_A \notin \{0_A\}$, then it is.

Example 2.24. Consider the Anti-BI-algebra $(AX, *_A *, 0_A)$ given in Example 2.22. $AS = \{0_A\}$ is an AntiLaw-sub-Anti-BI-algebra of AX, since $0_A *_A 0_A = b \notin AS$.

Note. It is obvious that the concepts of BI-algebra and Anti-BI-algebra are different. In the following example we show that the concept of Neutro-BI-algebra is different from the concepts of BI-algebra and Anti-BI-algebra.

Example 2.25. Let $X = \mathbb{R} - \{0\}$, endowed with the real division $\div$ of numbers. $(X, \div)$ is well defined, since there is no division by zero. Put $x \coloneqq 3$ and $y \coloneqq 2$, we obtain $2 \div (3 \div 2) = \frac{4}{3} \neq 2$, and so (BI) is not valid. Then $(X, \div, -1)$ is not a BI-algebra, but it is a Neutro-BI-algebra, since if $x = y \coloneqq \pm 1$, then $x \div y = (\pm 1) \div (\pm 1) = 1 \neq -1$. If $x \coloneqq 3$ and $y \coloneqq -3$, then $x \div y = 3 \div (-3) = -1$, and so (NB) holds. For (NBI), again $x = y \coloneqq -1$, we get $(-1) \div \big((-1) \div (-1)\big) = -1$, and if $x \coloneqq 4$ and $y \coloneqq 7$, we have $4 \div (7 \div 4) = \frac{16}{7} \neq 4$, so (NBI) holds. Also, we can see that $(X, \div, -1)$ is not an Anti-BI-algebra, since (AB) and (ABI) are not valid.

3. The Neutrosophic Triplet of BI-algebra

In 2020, F. Smarandache defined a novel definition of Neutrosophic Triplet of (*Algebra*, *NeutroAlgebra*, *AntiAlgebra*) [4]. In this section we give a particular example, when the Algebra is replaced by a BI-algebra, and we get (*BI-algebra*, *Neutro-BI-algebra*, *Anti-BI-algebra*) as below.

Definition 3.1. Let $\mathcal{U}$ be a nonempty universe of discourse, and X, NX and AX be nonempty sets of $\mathcal{U}$, and an operation $*$ defined on the set X, and the same operation restrained to the set NX (denoted as $*_N$) and to the set AX (denoted as $*_A$) respectively. A triplet (X, NX, AX) endowed with a triplet of binary operations $(*, *_N, *_A)$ and a triplet of constants $(0, 0_N, 0_A)$ is said to be The ***Neutrosophic Triplet of BI-algebra*** for briefly ***NT-BI-algebra*** if it satisfies the following **Axioms** $\{(B), (BI)\}$, **NeutroAxioms** $\{(NB), (NBI)\}$, or **AntiAxioms** $\{(AB), (ABI)\}$ respectively:

(B) $(\forall x \in X)(x * x = 0)$,

(BI) $(\forall x, y \in X)(x * (y * x) = x)$,

(NB) $(\exists x \in NX)(x *_N x = 0_N)$ and $(\exists x \in NX)(x *_N x \neq 0_N$ or is $indeterminate)$,

(NBI) $(\exists x, y \in NX)(x *_N (y *_N x) = x)$ and

$(\exists x, y \in NX)(x *_N (y *_N x) \neq x$ or is $indeterminate)$,

(AB) $(\forall x \in AX)(x *_A x \neq 0_A)$,

(ABI) $(\forall x, y \in AX)(x *_A (y *_A x) \neq x)$.

Definition 3.2. A triplet $((S, *, 0), (NS, *_N, 0_N), (AS, *_A, *_A))$, where $S \subseteq X$, $NS \subseteq NX$ and $AS \subseteq AX$ is said to be a *sub-NT-BI-algebra* of NT-BI-algebra $((X, *, 0), (NX, *_N, 0_N), (AX, *_A, *_A))$ if:

(i) $(S, *, 0)$ is a sub-BI-algebra of $(X, *, 0)$,

(ii) $(NS, *_N, 0_N)$ is a sub-Neutro-BI-algebra of $(NX, *_N, 0_N)$,

(iii) $(NS, *_A, 0_A)$ is an sub-Anti-BI-algebra of $(AX, *_A, 0_A)$.

4. Conclusions

In this paper, we introduced the notions of new types of sub- BI -algebras. Also, Neutro- BI -algebras, sub-Neutro- BI -algebras, NeutroLow-sub-Neutro- BI -algebras, AntiLow-sub-Neutro- BI -algebras, Anti-BI-algebras, sub-Anti- BI -algebras, NeutroLow-sub-Anti- BI -algebras, AntiLow-sub-Anti- BI -algebras are studied and by several examples showed that the notions are different. Finally, the concept of a Neutrosophic Triplet of BI-algebra is defined. For future work we would define some types of NeutroFilters, NeutroIdeals, AntiFilters, AntiIdeals in the Neutrosophic Triplet of BI-algebras.

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NeutroAlgebra of Neutrosophic Triplets using {Zn , x}

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W. B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2020). NeutroAlgebra of Neutrosophic Triplets using {Zn, x}. *Neutrosophic Sets and Systems,* 38, 510-523

Abstract. Smarandache in 2019 has generalized the algebraic structures to NeutroAlgebraic structures and AntiAlgebraic structures. In this paper, authors, for the first time, define the NeutroAlgebra of neutrosophic triplets group under usual $+$ and $\times$, built using $\{Z_n, \times\}$, n a composite number, $5 < n < \infty$, which are not partial algebras. As idempotents in Z_n alone are neutrals that contribute to neutrosophic triplets groups, we analyze them and build NeutroAlgebra of idempotents under usual $+$ and $\times$, which are not partial algebras. We prove in this paper the existence theorem for NeutroAlgebra of neutrosophic triplet groups. This proves the neutrals assocaited with neutrosophic triplet groups in $\{Z_n, \times\}$ under product is a NeutroAlgebra of triplets. We also prove the non-existence theorem of NeutroAlgebra for neutrosophic triplets in case of Z_n when $n = 2p, 3p$ and $4p$ (for some primes p). Several open problems are proposed. Further, the NeutroAlgebras of extended neutrosophic triplet groups have been obtained.

Keywords: neutrosophic triplets; neutrosophic extended triplets; neutrosophic triplet group; neutrosophic extended triplet group; NeutroAlgebra; partial algebra; NeutroAlgebra of neutrosophic triplets; NeutroAlgebra of neutrosophic extended triplets; AntiAlgebra

1. Introduction

The neutrosophic theory proposed by Smarandache in [1] has become a powerful tool in the study/analysis of real-world data as they are dominated by uncertainty, inconsistency, and indeterminacy. Neutrosophy deals with the neutralities and indeterminacies of real-world problems. The innovative concept of neutrosophic triplet groups was introduced by [2], which gives for any element a in $(G, *)$, the $anti(a)$ and $neut(a)$ satisfying conditions

$$a * neut(a) = neut(a) * a = a$$

and

$$a * anti(a) = anti(a) * a = neut(a)$$

where $neut(a)$ is not the identity element or the classical identity of the group. They call $(a, neut(a), anti(a))$ as the neutrosophic triplet group. These neutrosophic triplets built using Z_n are always symmetric about the neutral elements. For if $(a,\ neut(a), anti(a))$ is neutrosophic triplet then $(anti(a), neut(a), a)$ there by giving a perfect symmetry of a and $anti(a)$ about the $neut(a)$. The study of neutralities have been carried out by several researchers in neutrosophic algebraic structures like neutrosophic triplet rings, groups, neutrosophic quadruple vector spaces, neutrosophic semi idempotents, duplets and triplets in neutrosophic rings, neutrosophic triplet in biaglebras, neutrosophic triplet classical group and their applications, triplet loops, subgroups, cancellable semigroups and Abel-Grassman groupoids [2–24].

[13] has defined a classical group structure on these neutrosophic triplet groups and has obtained several interesting properties and given open conjectures. Smarandache [2] defined the Neutrosophic Extended Triplet, when the neutral element is allowed to be the classical unit element. Zhang et al has defined neutrosophic extended triplet group and have obtained several results in [25]. Later [26] have obtained some results on neutrosophic extended triplet groups with partial order defined on it. More results about neutrosophic triplet groups and neutrosophic extended triplet groups can be found in [25–32] .

We in this paper study the very new notion of NeutroAlgebra introduced by [33]. Several interesting results are obtained in [12, 34–36], and they introduced Neutro BC Algebra and sub Neutro BI Algebra and so on. NeutroAlgebras and AntiAlgebras in the classical number systems were studied in [37].

Here we introduce NeutroAlgebra under the usual product and sum in case of idempotents in the semigroups $\{Z_n, \times\}$, n a composite number, $5 < n < \infty$. This study is very important for all the neutrosophic triplets in $\{Z_n, \times\}$, happen to be contributed only by the idempotents, which are the only neutrals in $\{Z_n, \times\}$. We obtain NeutroAlgebras under usual $+$ and $\times$ in the case of neutrosophic triplet groups and neutrosophic extended triplet groups. It is pertinent to keep on record we define classical product on neutrosophic triplets, and they are classical groups under product of these triplets. This paper has six sections. Section one is introductory in nature, and basic concepts are recalled in section two. Section three obtains the existence and non-existence theorem on NeutroAlgebras under usual $+$ or $\times$ using neutrosophic triplet groups. In section four, a similar study is carried out in the case of neutrosophic extended triplet groups. The fifth section provides a discussion on this topic, and the final section gives the conclusions based on our study and some open conjectures which will be taken for future research by the authors.

2. **Basic Concepts**

Here we recall some basic definitions which is important to make this paper a self contained one.

Definition 2.1. Let us assume that N is an empty set and with binary operation * defined on it. N is called a neutrosophic triplet set (NTS) if for any $a \in N$, there exists a neutral of "a" (denoted by neut(a)), and an opposite of "a" (denoted by anti(a)) satisfying the following conditions:

$$a * neut(a) = neut(a) * a = a$$

$$a * anti(a) = anti(a) * a = neut(a).$$

And, the neutrsophic triple is given by (a, neut(a), anti(a)).

In a neutrosophic triplet set (N, *), $a \in N$, neut(a) and anti(a) may not be unique.

In the definition given in [2], the neutral element cannot be an unit element in the usual sense, and then this restriction is removed, using the concept of a neutrosophic extended triplet in [26].

The classical unit element can be regarded as a special neutral element. The notion of neutrosophic triplet groups and that of neutrosophic extended triplet groups are distinctly dealt with in this paper.

Definition 2.2. Let us assume that (N, *) is a neutrosophic triplet set. Then, N is called a neutrosophic triplet group, if it satisfies:

(1) Closure Law, i.e., $a * b \in N, \forall a, b \in N$;

(2) Associativity, i.e., $(a * b) * c = a * (b * c), \forall a, b, c \in N$

A neutrosophic triplet group $(N, *)$ is said to be commutative, if $a * b = b * a, \forall a, b \in N$.

Let $\langle A \rangle$ be a concept (as in terms of attribute, idea, proposition, or theory). By the neutrosphication process, we split the non-empty space into three regions two opposite ones corresponding to $\langle A \rangle$ and $\langle anti\ A \rangle$, and one neutral (indeterminate) $\langle neut\ A \rangle$ (also denoted $\langle neutro\ A \rangle$) between the opposites, which may or may not be disjoint; depending on the application, but their union equals the whole space.

A NeutroAlgebra is an algebra that has at least one neutro operation or one neutro axiom (axiom that is true for some elements, indeterminate or false for the other elements) [33]. A partial algebra has at the minimum one partial operation, and all its axioms are classical. Through a theorem in [34], proved that NeutroAlgebra is a generalization of partial algebra, and also give illustrations of NeutroAlgebras that are not partial algebras. Boole has defined the Partial Algebra (based on Partial Function) as an algebra whole operation is partially well-defined, and partially undefined (this undefined goes under Indeterminacy with respect to NeutroAlgebra). Therefore, a Partial Algebra (Partial Function) has some elements for which the operation is undefined (not outer-defined). Similarly an AntiAlgebra is a nonempty set that is endowed with at least one anti-operation (or anti-function) or at least one anti-axiom.

3. NeutroAlgebras of neutrosophic triplets using $\{Z_n, \times\}$

Here for the first time authors build NeutroAlgebras using neutrosophic triplets group built using the modulo integers Z_n; n a composite number. Neutrosophic triplet groups and extended neutrosophic triplet groups were studied by [25, 26]. First we define NeutroAlgebra using the non-trivial idempotents of Z_n, n a composite number. This study is mandatory as all the neutral elements of neutrosophic triplets build using Z_n are only the non-trivial idempotents of Z_n. Next we give the existence and non existence theorems in case of NeutroAlgebras for these neutrosophic triplet sets. We give some interesting properties about them. Further it is important to note unless several open conjectures about idempotents in Z_n given in [13], are solved or some progress is made in that direction it will not be possible to completely characterize NeutroAlgebras of the neutrosophic triplet groups or extended neutrosophic triplet groups. We will be using [13] to get NeutroAlgebras of idempotents and NeutroAlgebra of neutrosophic triplet sets. First we provide examples of NeutroAlgebra using subsets of the semigroup $\{Z_n, \times\}$ and then NeutroAlgebra of idempotents in $\{Z_n, \times\}$.

Example 3.1. Let $S = \{Z_{15}, \times\}$ be a semigroup under product modulo 15. Now consider the subset $A = \{5, 10, 14\} \in S$. The Cayley table for A is given in Table 1, where outer-defined elements are denoted by *od*.

Table 1. Cayley Table for A

$\times$	5	10	14
5	10	5	10
10	5	10	5
14	10	5	od

We see the table has outer-defined elements denoted by *od*. So A is a NeutroAlgebra which is not a partial algebra, since the operation 14×14 is outer-defined. $14 \times 14 \equiv 1 \pmod{15}$, but $1 \notin \{5, 10, 14\}$. Therefore Table 1 is only a NeutroAlgebra. Every subset of S need not be a NeutroAlgebra. For take $B = \{3, 6, 9, 12\}$ a subset in S. Consider the Cayley table for B is given in Table 2.

B is not a NeutroAlgebra as every term in the cell is defined and associativity axiom is totally true..

Table 2. Cayley Table for B

$\times$	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

Clearly B is a subsemigroup of S, in fact a group under $\times$ modulo 15 with 6 as its multiplicative identity, so S is a Smarandache semigroup [10].

Consider $C = \{2, 7, 8\}$ a subset of S. The Cayley table for C is given in Table 3, this has every cell to be outer-defined.

Table 3. Cayley Table for C

$\times$	2	7	8
2	od	od	od
7	od	od	od
8	od	od	od

So C is not a NeutroAlgebra or a subsemigroup but an AntiAlgebra since the operation $\times$ is totally outer-defined under $\times$ modulo 15.

Thus we can categorically put forth the following facts.

Every classical algebraic structure A with binary operations defined on it is such that any proper subset B of A with inherited operation of A falls under the three categories;

(1) B can be a proper substructure of a stronger structure of A with the inherited operations of A.
(2) B can only be a NeutroAlgebra, which may be a Partial Algebra, when some operation is undefined, and all other operations are well-defined and all axioms are true.
(3) B can be an AntiAlgebra when at least one operation is totally outer-defined. or at least one axiom is totally false.

Under these circumstances if one wants to get a NeutroAlgebra which is not a partial algebra for a proper subset of a classical algebraic structure one should exploit the special axioms satisfied by them, to this end we study the property of idempotents in the semigroup $\{Z_n, \times\}$.

We also in case of neutrosophic triplet group obtain a NeutroAlgebra which is not a partial algebra.

First we give examples of NeutroAlgebra which are not partial algebras using idempotents of the semigroup S $= \{Z_n, \times\}$.

Example 3.2. Let S $= \{Z_6, \times\}$ be the semigroup under product modulo 6. The nontrivial idempotents of S are $V = \{3, 4\}$. The Cayley table for V is given in Table 4,

Table 4. Cayley Table for V

$\times$	3	4
3	3	od
4	od	4

So V is a NeutroAlgebra under $\times$ but not a partial algebra. For the same V define operation $+$ modulo 6, the Cayley table for V is given in Table 5 and V is AntiAlgebra and not a partial algebra either.

Table 5. Cayley Table for V

$+$	3	4
3	od	od
4	od	od

Suppose we take $W = \{0, 1, 3, 4\}$ the collection of trivial and non trivial idempotents of S, and if we take S as a whole set but study the idempotent axiom in W we see from Table 6.

Table 6. Cayley Table for W

$\times$	0	1	3	4
0	0	0	0	0
1	0	1	3	4
3	0	3	3	0
4	0	4	3	4

Suppose we find the Cayley table for W under $+$ we get the Cayley table given in the following Table 7.

W itself is a NeutroAlgebra under usual $+$ with several undefined terms. W under usual product is a subsemigroup of idempotents of S; where as S under sum of idempotents is a NeutroAlgebra which is not a partial algebra under the axiom of the property of idempotency.

Now if we take for any subset of S the axiom of idempotent property we get NeutroAlgebras which are not partial algebras.

To this effect we provide an example.

TABLE 7. Cayley Table for W

+	0	1	3	4
0	0	1	3	4
1	1	od	4	od
3	3	4	0	1
4	4	od	1	od

Example 3.3. Let S $= \{Z_{42}, \times\}$ be the semigroup under product modulo 42. The trivial and non trivial idempotents of S are $B = \{0, 1, 7, 15, 21, 22, 28, 36\}$. We define + modulo 42 on this set of idempotents keeping the resultant what we need is the axiom of idempotency. The Cayley table for B is given in Table 8.

TABLE 8. Cayley Table for B

+	0	1	7	15	21	22	28	36
0	0	1	7	15	21	22	28	36
1	1	od	od	od	22	od	od	od
7	7	od	od	22	28	od	od	1
15	15	od	22	od	36	od	1	od
21	21	22	28	36	0	1	7	15
22	22	od	od	od	1	od	od	od
28	28	od	od	1	7	od	od	22
36	36	od	1	od	15	od	22	od

Thus B is a NeutroAlgebra which is not a partial algebra under the axiom of idempotency. Thus we have a large class of NeutroAlgebras which are not partial algebras.

As the main theme of this paper is study of neutrosophic triplets using modulo integers $\{Z_n, \times\}$ and prove the existence theorem and non-existence theorem of NeutroAlgebra of neutrosophic triplet groups.

In view of all these we have the following existence theorem of NeutroAlgebra of neutrosophic triplets.

Theorem 3.4. *Let $S = \{Z_n, \times\}$, n not a prime, $5 < n < \infty$. Let V be the collection of all non trivial idempotents that is all neutrals of S, where 0 and 1 are not in S. Then V under product is a NeutroAlgebra of triplets.*

Proof. Let $W = \{w_1, w_2, \dots, w_t\}$ be the non trivial idempotents of S. It is proved in [13] that if $\underline{W_i}$ is the set of all neutrosophic triplets of a non trivial idempotent $\underline{w_i}$ in S which serves as the neutral for the collection W_i then $\{W_i, \times\}$ is a neutrosophic triplet classical group under usual product and i varies over all neutrals; $1 \leq i \leq t$. If V is the collection of all neutrosophic triplets (this V will include all W_i for different neutrals or non trivial idempotents in S), associated with $S = \{Z_n, \times\}$; then V is not closed under usual product [13] and there are many undefined elements under usual product so V is a NeutroAlgebra of neutrosophic triplets. Hence the claim. □

In view of this we have the following partial non existence theorem of NeutroAlgebra of neutrosophic triplets under + for Z_{np} where $n = 2, 3$ and 4 for some values of P provided in the Tables 9, 10 and 11. We have for Z_n, n a product of more than two primes can have NeutroAlgebra of neutrosophic triplets under +.

Theorem 3.5. *Let $S = \{Z_{np}, \times\}$; where n = 2, 3 and 4, (p a specific prime and np is not a square of a prime, prime values refer Tables 9, 10, and 11) be a semigroup under product modulo np. If V denotes the collection of all idempotents associated with the non trivial idempotents of Z_{np} then $\{V, +\}$ is never a NeutroAlgebra of triplets for $n = 2$, 3 and 4.*

Proof. Recall from [13] that there are two idempotents in all the three cases when $n = 2p$ or $3p$ or $4p$ given in Tables 9, 10 and 11. □

Table 9. Idempotent table for Z_{2p}

S.no	Z_{2p}	p	p+1
1	Z_6	3	4
2	Z_{10}	5	6
3	Z_{14}	7	8
4	Z_{22}	11	12
5	Z_{26}	13	14
6	Z_{34}	17	18
7	Z_{38}	19	20
8	Z_{46}	23	24
9	Z_{58}	29	30

We see any sum of the idempotents is 1 and product is 0.

Here in Z_{3p} and Z_{4p} also sum of idempotents is 1 and that product is 0. Tables are provided for them [13]. In case of $2p$ the nontrivial idempotents are p and $p+1$, clearly under sum this is a set. Thus we have proved the non-existence of NeutroAlgebra of idempotents under '+'.

To this effect first provide an example.

TABLE 10. Idempotent table for Z_{3p}

S. No.	Z_{3p}	p	p + 1	2p	2p + 1
1	Z_{15}	-	6	10	-
2	Z_{21}	7	-	-	15
3	Z_{33}	-	12	22	-
4	Z_{39}	13	-	-	27
5	Z_{51}	-	18	34	-
7	Z_{57}	19	-	-	39
8	Z_{69}	-	24	46	-
9	Z_{159}	-	54	106	-

TABLE 11. Idempotent table for Z_{4p}

S. No.	Z_{4p}	p	$p+1$	$3p$	$3p+1$
1	Z_{12}	-	4	9	-
2	Z_{20}	5	-	-	16
3	Z_{28}	-	8	21	-
4	Z_{44}	-	12	33	-
5	Z_{52}	13	-	-	40
6	Z_{76}	-	20	57	-
7	Z_{212}	53	-	-	160
8	Z_{388}	97	-	-	292
9	Z_{332}	-	84	249	-

Example 3.6. Consider the semigroup S = $\{Z_{10}, \times\}$. The nontrivial idempotents of S which contribute to the neutrosophic triplet set are; $\{6, 5\}$ in Z_{10}. Consider the neutrosophic triplet set $V = \{(5, 5, 5), (6, 6, 6), (8, 6, 2), (2, 6, 8), (4, 6, 4)\}$. It is proved $V \backslash \{(5, 5, 5)\}$ is a neutrosophic triplet classical group under $\times$ [13]. Now the Cayley table of V under usual product $\times$ is given in Table 12.

TABLE 12. Cayley Table for V

$\times$	(5,5,5)	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(5,5,5)	(5,5,5)	od	od	od	od
(6,6,6)	od	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(8,6,2)	od	(8,6,2)	(4,6,4)	(6,6,6)	(2,6,8)
(2,6,8)	od	(2,6,8)	(6,6,6)	(4,6,4)	(8,6,2)
(4,6,4)	od	(4,6,4)	(2,6,8)	(8,6,2)	(6,6,6)

Clearly V is a NeutroAlgebra under usual product and not a partial algebra. Since we have not included the neutrals that is non trivial idempotents like 0 and 1 we have this to be only a NeutroAlgebra of triplets.

TABLE 13. Cayley Table for V

+	(5,5,5)	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(5,5,5)	od	od	od	od	od
(6,6,6)	od	od	od	od	od
(8,6,2)	od	od	od	od	od
(2,6,8)	od	od	od	od	od
(4,6,4)	od	od	od	od	od

Thus the neutrosophic triplets collection yields only a set under addition where no pair of neutrosophic triplets gives under sum a neutrosophic triplet. Hence our claim no NeutroAlgebra neutrosophic triplets under addition. So V in Table 13 is an AntiAlgebra. Likewise the cases $3p$ and $4p$ from tables.

So if we include the non trivial idempotents 0 and 1 then we can get NeutroAlgebra of idempotents under + which is carried out in the following section.

Example 3.7. Consider the semigroup $S = \{Z_{105}, \times\}$ under $\times$ modulo 105. The non trivial idempotents are $V = \{15, 21, 36, 70, 85, 91\}$. Let M be the collection of all neutrosophic triplets using the idempotents in V. M contains elements say { (15, 15, 15), (21, 21, 21), (36, 36, 36), (30, 15, 60), (51, 36, 81)}, from the Cayley table of M under + we see there are some undefined terms also given in Table 14.

TABLE 14. Cayley Table for M

+	(15,15,15)	(21,21,21)	(36,36,36)	(30,15,60)	(51,36,81)
(15,15,15)	od	(36,36,36)	od	od	od
(21,21,21)	(36,36,36)	od	od	(51,36,81)	od
(36,36,36)	od	od	od	od	od
(30,15,60)	od	(51,36,81)	od	od	od
(51,36,81)	od	od	od	od	od

Hence we have a NeutroAlgebra of neutrosophic triplets under +.

We propose some open problems in this regard in the final section of this paper.
Now we find ways to get NeutroAlgebra of neutrosophic triplets under +. The possibility is by using extended neutrosophic triplets group we can have for all Z_n, n any composite number

NeutroAlgebra of neutrosophic triplets under +. Unless the conjectures proposed in [13] is solved complete characterization is not possible, only partial results and examples to that effect are possible.

In the following section we discuss NeutroAlgebra of extended neutrosophic triplet sets.

4. NeutroAlgebra of extended neutrosophic triplets using $\{Z_n, \times\}$

In this section we prove the existence of NeutroAlgebra of extended neutrosophic triplets using $\{Z_n, \times\}$, for more about extended neutrosophic triplets refer [2, 26] under both + and ×. Throughout this section we assume the collection of idempotents contains both the trivial idempotents 1 and 0. It is thus mandatory the neutrosophic triplet set collection contains (0, 0, 0) and (1, 1, 1) apart from the neutrosophic triplets of the form (a, 1, anti a = inverse of a), where a is in Z_n which has inverse in Z_n.

We first prove the collection of all trivial and non trivial idempotents in Z_n is a NeutroAlgebra under + and also under ×.

Theorem 4.1. *Let $S = \{Z_n, \times\}$ be the semigroup under product modulo $n, 5 < n < \infty$. Let $V = \{$Collection of all idempotents in Z_n including 0 and 1 $\}$.*

(1) *$V \setminus \{0, 1\}$ is a NeutroAlgebra of idempotents under × modulo n.*
(2) *V is a NeutroAlgebra of idempotents under + mod n.*

Proof. Consider $V \setminus \{0, 1\}$ for every x in $V \setminus \{1, 0\}$ is such that $x \times x = x$, so $V \setminus \{1, 0\}$ is a NeutroAlgebra under ×. Hence (1) is true.

Proof of (2): To show V is a NeutroAlgebra of idempotents under +. Since 0 is in V we have for every $x \in V$; $0 + x = x$ is in V, however we do not in general have the sum of two idempotents to be an idempotent. For instance 1 +1 = 2 is not an idempotent so $(V, +)$ has undefined elements, hence undefined. Thus (2) is proved. □

We provide an example to this effect.

Example 4.2. Let $S = \{Z_{10}$, n, $\times\}$ be the semigroup under × modulo 10. The trivial and non trivial idempotents are V= {0, 1, 5, 6}. It is easily verified V is a NeutroAlgebra under+, for 6 + 6 = 2 modulo 10. However V is not a NeutroAlgebra under ×, but $V \setminus$ {0, 1}is a NeutroAlgebra under × modulo 10. For 6 + 5 = 1 modulo 10, so $V \setminus${1, 0} is a NeutroAlgebra. Now the neutrosophic triplets of S associated with the idempotents V are N = { (0, 0, 0),(1, 1, 1), (5, 1, 5), (3, 1, 7), (7, 1, 3), (5, 5, 5), (6, 6, 6), (4 ,6, 4), (2, 6, 8) and (8, 6, 2) }. We see N under + is a NeutroAlgebra, for (1, 1, 1) + (7, 1, 3) = (8, 2, 4) is not in N. N is not a NeutroAlgebra under +. But $N \setminus$ { (0, 0, 0), (1, 1, 1), (5, 1, 5), (3, 1, 7), (7, 1, 3) } = W neutrosophic triplets formed by the non trivial idempotents 5 and 6 is a NeutroAlgebra as (5, 5, 5) × (2, 6, 8) = (0, 0, 0) which is not in W. Hence the claim. If $\{(0,0,0)\}$ is added, then the set V becomes a NeutroAlgebra under $+$.

TABLE 15. Cayley Table for V

+	(0, 0, 0)	(5,5,5)	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(0, 0, 0)	(0, 0, 0)	(5,5,5)	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(5,5,5)	(5,5,5)	od	od	od	od	od
(6,6,6)	(6,6,6)	od	od	od	od	od
(8,6,2)	(8,6,2)	od	od	od	od	od
(2,6,8)	(2,6,8)	od	od	od	od	od
(4,6,4)	(4,6,4)	od	od	od	od	od

Theorem 4.3. *Let $S = \{Z_n, \times\}$ be a semigroup under $\times$ modulo n, where n is not a prime and $5 < n < \infty$. Let $N = \{$collection of all extended neutrosophic triplet set including (0, 0, 0), and all neutrosophic triplets associated with the trivial idempotent 1$\}$.*

(1) *N is a NeutroAlgebra under $+$ of extended neutrosophic triplets set.*
(2) *$N \setminus \{(0,0,0)\}$ is a NeutroAlgebra of extended neutrosophic triplet set under product modulo n.*

Proof. Let N be the collection of all extended neutrosophic triplets including (0, 0, 0) and (1, 1, 1) and other triplets associated with the neutral 1.

Proof of (1): In the case extended triplet N we see sum of two idempotents need not be idempotent for (1, 1, 1) + (1, 1, 1) = (2, 2, 2) is not in N, hence N is the NeutroAlgebra of extended neutrosophic triplets which is not a partial algebra as the axiom of neutrosophic triplets is not satisfied.

Proof of (2): Consider $N \setminus \{(0,0,0)\}$. Clearly in general the product of any two idempotents is not an idempotent in Z_n, and several triplets are undefined and do not in general satisfy the triplet relation [13]. Hence the claim. □

5. **Discussions**

The study of NeutroAlgebra introduced by [33] is very new, here the authors built NeutroAlgebra using idempotents of $\{Z_n, \times\}$ a semigroup under $\times$ modulo n for appropriate n which are not partial algebras. Likewise NeutroAlgebra built using neutrosophic triplets set and extended neutrosophic triplets set. Some open problems based on our study is proposed in the section on conclusions.

6. **Conclusions**

For the first time authors have NeutroAlgebra using idempotents of a semigroup $S = \{Z_n, \times\}$; n a composite number $5 < n < \infty$, neutrosophic triplets and extended neutrosophic triplets. We have obtained NeutroAlgebras of idempotents which are not partial algebras under the classical operation of $+$ and $\times$ only using $S = \{Z_n, \times\}$, the semigroup under product for appropriate n. We have obtained both existence and non-existence theorem for NeutroAlgebras of idempotents in S. We suggest certain open problems for researchers as well as these problems will be taken by the authors for future study.

Problem 1: Does there exist a n (n a composite number) such that using $\{Z_n, \times\}$ there are no non trivial NeutroAlgebra of neutrosophic triplet set and NeutroAlgebra in extended neutrosophic triplet set?

Problem 2. Does there exist a n, n a composite number such that $\{Z_n, \times\}$ has its collection of trivial and non trivial idempotents denoted by N to be such that;

- $(N, +)$ is not NeutroAlgebra of idempotents ?
- $(N, \times)$ is not a NeutroAlgebra of idempotents?

Problem 3: Prove in case of $\{Z_{3p}, \times\}$ and $\{Z_{4p}, \times\}$, the idempotents are only of the form mentioned in Tables 10 and 11 respectively.

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Neutrosophic ℵ−bi-ideals in semigroups

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Porselvi, B. Elavarasan, Florentin Smarandache, Young Bae Jun (2020). Neutrosophic N-bi-ideals in semigroups. *Neutrosophic Sets and Systems*, 35, 422-434

Abstract: In this paper, we introduce the notion of neutrosophic ℵ-bi-ideal for a semigroup. We infer different semigroups using neutrosophic ℵ-bi-ideal structures. Moreover, for regular semigroups, neutrosophic ℵ-product and intersection of neutrosophic ℵ-ideals are identical.

Keywords: Semigroup, ideal, bi-ideal, neutrosophic ℵ− ideals, neutrosophic ℵ -bi-ideals, neutrosophic ℵ −product.

1. Introduction

In 1965, Zadeh [16] introduced the idea of fuzzy sets for modeling the ambiguous theories in the globe. In 1986, Atanassov [1] generalized fuzzy set and named as intuitionistic fuzzy set, and discussed it. Also from his view point, there are two degrees for any object in the world. They are degree of membership to a vague subset and degree of non-membership to that given subset.

Smarandache generalized fuzzy and intuitionistic fuzzy set, and referred as Neutrosophic set (see [2, 3, 6, 13-15]). It is identified by a truth, a falsity and an indeterminacy membership function. These sets are applied to many branches of mathematics to overcome the complexities arising from uncertain data. Neutrosophic set can distinguish between absolute membership and relative membership. Smarandache used this in non-standard analysis such as result of sport games (winning/defeating/tie), decision making and control theory, etc. This area has been studied by several authors (see [5, 10-12]).

In [8], M. Khan et al. presented and discussed the concepts of neutrosophic ℵ −subsemigroup of semigroup. In [5], Gulistan et al. have studied the idea of complex neutrosophic subsemigroups. They have introduced the notion of characteristic function of complex neutrosophic sets, direct product of complex neutrosophic sets.

In [4], B. Elavarasan et al. introduced the concepts of neutrosophic ℵ −ideal of semigroup and explored its properties. Also, the conditions are given for neutrosophic ℵ −structure becomes neutrosophic ℵ −ideal. Further, presented the notion of characteristic neutrosophic ℵ −structure over semigroup.

Throughout this article, X denotes a semigroup. Recall that for any subsets A and B of X, $AB = \{uw | u \in A \text{ and } w \in B\}$, the multiplication of A and B.

For a semigroup X,

(i) $\emptyset \neq U \subseteq X$ is a subsemigroup of X if $U^2 \subseteq U$.

(ii) A subsemigroup U of X is left (resp., right) ideal if $XU \subseteq U$ (resp., $UX \subseteq U$). U is an ideal of X if U is both left and right ideal of X.

(iii) X is left (resp., right) regular if for each $s \in X$, there exists $x \in X$ such that $s = xs^2$(resp., $s = s^2x$) [7].

(iv) X is regular if for each $s \in X$, there exists $x \in X$ such that $s = sxs$ [9].

(v) X is intra-regular if for every $s \in X$, there exist $x, y \in X$ such that $s = xs^2y$ [9].

(vi) A subsemigroup Y of X is bi-ideal if $YXY \subseteq Y$. For any $r' \in X$, $B(r') = \{r', r'^2, r'Xr'\}$ is the principal bi-ideal of X generated by r'.

2. Basics of neutrosophic ℵ – structures

In this section, we present the required basic definitions of neutrosophic ℵ −structures of X that we need in the sequel.

The collection of functions from a set X to $[-1, 0]$ is denoted by $\Im(X, [-1, 0])$. Note that $f \in \Im(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, ℵ −function on X). Here ℵ −structure means (X, f) of X.

Definition 2.1. [8] A neutrosophic ℵ − structure of X is defined to be the structure:

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{T_N(x), I_N(x), F_N(x)} \mid x \in X \right\}$$

where T_N is the negative truth membership function on X, I_N is the negative indeterminacy membership function on X and F_N is the negative falsity membership function on X.

Note that for any $x \in X$, X_N satisfies the condition $-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0$.

Definition 2.2. [8] A neutrosophic ℵ −structure X_N of X is called a neutrosophic ℵ −subsemigroup of X if the below condition is valid:

$$(\forall\, g_i, h_j \in X) \begin{pmatrix} T_N(g_ih_j) \leq T_N(g_i) \vee T_N(h_j) \\ I_N(g_ih_j) \geq I_N(g_i) \wedge I_N(h_j) \\ F_N(g_ih_j) \leq F_N(g_i) \vee F_N(h_j) \end{pmatrix}.$$

Let X_N be a neutrosophic ℵ − structure of X and let $\lambda, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq \lambda + \delta + \varepsilon \leq 0$. Then the set $X_N(\lambda, \delta, \varepsilon) := \{x \in X | T_N(x) \leq \lambda,\ I_N(x) \geq \delta,\ F_N(x) \leq \varepsilon\}$ is called a *(λ, δ, ε)* – level set of X_N.

Definition 2.3. [4] A neutrosophic ℵ −structure X_N of X is called a neutrosophic ℵ −left (resp., right) ideal of X if it satisfies:

$$(\forall\, g_i, h_j \in X) \begin{pmatrix} T_N(g_ih_j) \leq T_N(h_j)\ (resp., T_N(g_ih_j) \leq T_N(g_i)) \\ I_N(g_ih_j) \geq I_N(h_j)\ (resp., I_N(g_ih_j) \geq I_N(g_i)) \\ F_N(g_ih_j) \leq F_N(h_j)\ (resp., F_N(g_ih_j) \leq F_N(g_i)) \end{pmatrix}.$$

If X_N is both neutrosophic ℵ −left and neutrosophic ℵ −right ideal of X, then it is called a neutrosophic ℵ −ideal of X.

Definition 2.4. A neutrosophic ℵ −subsemigroup X_N of X is a neutrosophic ℵ −bi-ideal of X if the following condition is valid:

$$(\forall\, r,s,t\in X)\begin{pmatrix} T_N(rst)\leq T_N(r)\vee\ T_N(t)\\ I_N(rst)\geq I_N(r)\wedge I_N(t)\\ F_N(rst)\leq F_N(r)\vee F_N(t)\end{pmatrix}.$$

Clearly any neutrosophic $\aleph-$left (resp., right) ideal is neutrosophic $\aleph-$bi-ideal, but the neutrosophic $\aleph-$bi-ideal is not necessary to be a neutrosophic $\aleph-$left (resp., right) ideal.

Example 2.5. Consider the semigroup $X=\{0,a,b,c\}$ with binary operation as follows:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	b
c	b	b	b	c

Then $X_N=\left\{\frac{0}{(-0.9,-0.1,-0.7)},\frac{a}{(-0.8,-0.2,-0.5)},\frac{b}{(-0.7,-0.3,-0.3)},\frac{c}{(-0.5,-0.4,-0.1)}\right\}$ is a neutrosophic $\aleph-$bi-ideal of X, but X_N is not neutrosophic $\aleph-$left ideal as well as neutrosophic $\aleph-$right ideal of X. □

Definition 2.6. [8] For $\Phi\neq A\subseteq X$, the characteristic neutrosophic $\aleph-$structure of X is denoted by $\chi_A(X_N)$ and is defined to be neutrosophic $\aleph-$structure

$$\chi_A(X_N)=\frac{X}{(\chi_A(T)_N,\chi_A(I)_N,\ \chi_A(F)_N)}$$

where

$$\chi_A(T)_N: X\to[-1,0],\ x\to\begin{cases}-1 & if\ x\in A\\ 0 & otherwise,\end{cases}$$

$$\chi_A(I)_N: X\to[-1,0],\ x\to\begin{cases}0 & if\ x\in A\\ -1 & otherwise,\end{cases}$$

$$\chi_A(F)_N: X\to[-1,0],\ x\to\begin{cases}-1 & if\ x\in A\\ 0 & otherwise.\end{cases}$$

Definition 2.7. [8] Let $X_N:=\frac{X}{(T_N,\ I_N,\ F_N)}$ and $X_M:=\frac{X}{(T_M,\ I_M,\ F_M)}$.

(i) X_M is called a neutrosophic $\aleph-$ substructure of X_N over X, denoted by $X_N\subseteq X_M$, if
$$T_N(t)\geq T_M(t),\ I_N(t)\leq I_M(t),\ F_N(t)\geq F_M(t)\ \forall t\in X.$$
If $X_N\subseteq X_M$ and $X_M\subseteq X_N$, then we say that $X_N=X_M$.

(ii) The neutrosophic $\aleph-$ product of X_N and X_M is defined to be a neutrosophic $\aleph-$structure of X,
$$X_N\odot X_M:=\frac{X}{(T_{N\circ M},\ I_{N\circ M},\ F_{N\circ M})}=\left\{\frac{h}{T_{N\circ M}(h),\ I_{N\circ M}(h),\ F_{N\circ M}(h)}\mid h\in X\right\},$$
where
$$(T_N\circ T_M)(h)=T_{N\circ M}(h)=\begin{cases}\bigwedge_{h=rs}\{T_N(r)\vee T_M(s)\} & if\ \exists\, r,s\in X\ such\ that\ h=rs\\ 0 & otherwise,\end{cases}$$
$$(I_N\circ I_M)(h)=I_{N\circ M}(h)=\begin{cases}\bigvee_{h=rs}\{I_N(r)\wedge I_M(s)\} & if\ \exists\, r,s\in X\ such\ that\ h=rs\\ -1 & otherwise,\end{cases}$$
$$(F_N\circ F_M)(h)=F_{N\circ M}(h)=\begin{cases}\bigwedge_{h=rs}\{F_N(r)\vee F_M(s)\} & if\ \exists\, r,s\in X\ such\ that\ h=rs\\ 0 & otherwise.\end{cases}$$

(iii) For $t \in X$, the element $\frac{t}{(T_{N\circ M}(t),\ I_{N\circ M}(t),\ F_{N\circ M}(t))}$ is simply denoted by

$(X_N \odot X_M)(t) = (T_{N\circ M}(t),\ I_{N\circ M}(t),\ F_{N\circ M}(t))$ for the sake of convenience.

(iv) The union of X_N and X_M is a neutrosophic $\aleph$ −structure over X is defined as

$$X_N \cup X_M = X_{N\cup M} = (X; T_{N\cup M},\ I_{N\cup M},\ F_{N\cup M}),$$

where

$$(T_N \cup T_M)(h_i) = T_{N\cup M}(h_i) = T_N(h_i) \wedge T_M(h_i),$$
$$(I_N \cup I_M)(h_i) = I_{N\cup M}(h_i) = I_N(h_i) \vee I_M(h_i),$$
$$(F_N \cup F_M)(h_i) = F_{N\cup M}(h_i) = F_N(h_i) \wedge F_M(h_i) \ \forall h_i \in X.$$

(v) The intersection of X_N and X_M is a neutrosophic $\aleph$ −structure over X is defined as

$$X_N \cap X_M = X_{N\cap M} = (X; T_{N\cap M},\ I_{N\cap M},\ F_{N\cap M}),$$

where

$$(T_N \cap T_M)(h_i) = T_{N\cap M}(h_i) = T_N(h_i) \vee T_M(h_i),$$
$$(I_N \cap I_M)(h_i) = I_{N\cap M}(h_i) = I_N(h_i) \wedge I_M(h_i),$$
$$(F_N \cap F_M)(h_i) = F_{N\cap M}(h_i) = F_N(h_i) \vee F_M(h_i) \ \forall \ h_i \in X.$$

3. Neutrosophic $\aleph$ −bi-ideals of semigroups

In this section, we examine different properties of neutrosophic $\aleph$ −bi-ideals of X.

Theorem 3.1. For $\Phi \neq B \subseteq X$, the following assertions are equivalent:

(i) $\chi_B(X_N)$ is a neutrosophic $\aleph$ −bi-ideal of X,

(ii) B is a bi-ideal of X.

Proof: Suppose $\chi_B(X_N)$ is a neutrosophic $\aleph$ −bi-ideal of X. Let $r, t \in B$ and $s \in X$. Then

$$\chi_B(T)_N(rst) \le \chi_B(T)_N(r) \vee \chi_B(T)_N(t) = -1,$$
$$\chi_B(I)_N(rst) \ge \chi_B(I)_N(r) \wedge \chi_B(I)_N(t) = 0,$$
$$\chi_B(F)_N(rst) \le \chi_B(F)_N(r) \vee \chi_B(F)_N(t) = -1.$$

Thus $rst \in B$ and hence B is a bi-ideal of X,

Conversely, assume B is a bi-ideal of X. Let $r, s, t \in X$.

If $r \in B$ and $t \in B$, then $rst \in B$. Now

$$\chi_B(T)_N(rst) = -1 = \chi_B(T)_N(r) \vee \chi_B(T)_N(t),$$
$$\chi_B(I)_N(rst) = 0 = \chi_B(I)_N(r) \wedge \chi_B(I)_N(t),$$
$$\chi_B(F)_N(rst) = -1 = \chi_B(F)_N(r) \vee \chi_B(F)_N(t).$$

If $r \notin B$ or $t \notin B$, then

$$\chi_B(T)_N(rst) \le 0 = \chi_B(T)_N(r) \vee \chi_B(T)_N(t),$$
$$\chi_B(I)_N(rst) \ge -1 = \chi_B(I)_N(r) \wedge \chi_B(I)_N(t)$$
$$\chi_B(F)_N(rst) \le 0 = \chi_B(F)_N(r) \vee \chi_B(F)_N(t).$$

Therefore $\chi_B(X_N)$ is a neutrosophic $\aleph$ −bi-ideal of X. □

Theorem 3.2. Let $\lambda, \delta, \varepsilon \in [-1, 0]$ be such that $-3 \le \lambda + \delta + \varepsilon \le 0$. If X_N is a neutrosophic $\aleph$ −bi-ideal, then $(\lambda, \delta, \varepsilon)$ −level set of X_N is a neutrosophic bi- ideal of X whenever $X_N(\lambda, \delta, \varepsilon) \neq \emptyset$.

Proof: Suppose $X_N(\lambda, \delta, \varepsilon) \neq \emptyset$ for $\lambda, \delta, \varepsilon \in [-1, 0]$ with $-3 \le \lambda + \delta + \varepsilon \le 0$. Let X_N be a neutrosophic $\aleph$ −bi-ideal and let $x, y, z \in X_N(\lambda, \delta, \varepsilon)$. Then

$$T_N(xyz) \le T_N(x) \vee T_N(z) \le \lambda,$$
$$I_N(xyz) \ge I_N(x) \wedge I_N(z) \ge \delta,$$

$$F_N(xyz) \leq F_N(x) \vee F_N(z) \leq \varepsilon$$

which imply $xyz \in X_N(\lambda, \delta, \varepsilon)$. Therefore $X_N(\lambda, \delta, \varepsilon)$ is a neutrosophic $\aleph$ −bi-ideal of X. □

Theorem 3.3. Let X_M be a neutrosophic $\aleph -$ structure of X. Then the equivalent assertions are:

(i) $X_M \odot X_M \subseteq X_M$ and $X_M \odot \chi_X(X_N) \odot X_M \subseteq X_M$ for any neutrosophic $\aleph -$ structure X_N,

(ii) X_M is a neutrosophic $\aleph$ −bi-ideal of X.

Proof: Suppose (i) holds. Then X_M is neutrosophic $\aleph -$ subsemigroup of X by Theorem 4.6 of [8]. Let $r, s, t \in X$ and let $a = rst$. Then

$$(T_M)(rst) \leq (T_M \circ \chi_X(T)_N \circ T_M)(rst) = \bigwedge_{a=rst} \{(T_M \circ \chi_X(T)_N)(rs) \vee T_M(t)\}$$

$$= \bigwedge_{a=bt} \{\bigwedge_{b=rs} \{(T_M(r) \vee \chi_X(T)_N(s)\} \vee T_M(t)\}$$

$$\leq \bigwedge \{T_M(r) \vee T_M(t)\} \leq T_M(r) \vee T_M(t),$$

$$I_M(rst) \geq (I_M \circ \chi_X(I)_N \circ I_M)(rst) = \bigvee_{a=rst} \{(I_M \circ \chi_X(I)_N)(rs) \wedge I_M(t)\}$$

$$= \bigvee_{a=bt} \{\bigvee_{b=rs} \{ I_M(r) \wedge \chi_X(I)_N(s)\} \wedge I_M(t)\}$$

$$\geq \bigvee_{a=rst} \{ I_M(r) \wedge I_M(t)\} \geq I_M(r) \wedge I_M(t),$$

$$(F_M)(rst) \leq (F_M \circ \chi_X(F)_N \circ F_M)(rst) = \bigwedge_{a=rst} \{(F_M \circ \chi_X(F)_N)(rs) \vee F_M(t)\}$$

$$= \bigwedge_{a=bt} \{\bigwedge_{b=rs} \{(F_M(r) \vee \chi_X(F)_N(s)\} \vee F_M(t)\}$$

$$\leq \bigwedge_{a=rst} \{F_M(r) \vee F_M(t)\} \leq F_M(r) \vee F_M(t).$$

Therefore X_M is a neutrosophic $\aleph -$ bi-ideal of X.

For converse, suppose (ii) holds. Then $X_M \odot X_M \subseteq X_M$ by Theorem 4.6 of [8].

Let $x \in X$. If $x = rb$ and r$= st$ for some r$, b, s, t \in X$, then

$$(T_M \circ \chi_X(T)_N \circ T_M)(x) = \bigwedge_{x=rb} \{(T_M \circ \chi_X(T)_N)(r) \vee T_M(b)\}$$

$$= \bigwedge_{x=rb} \{\bigwedge_{r=st} \{T_M(s) \vee \chi_X(T)_N(t)\} \vee T_M(b)\}$$

$$= \bigwedge_{x=rb} \{\bigwedge_{r=st} \{(T_M(s)\} \vee T_M(b)\}$$

$= \wedge_{x=rb}\{T_M(s_i) \vee T_M(b)\}$ for some $s_i \in X$ and r$= s_i t_i$

$$\geq \bigwedge_{x=s_i t_i b} T_M(s_i t_i b) = T_M(x),$$

$$(I_M \circ \chi_X(I)_N \circ I_M)(x) = \bigvee_{x=rb} \{(I_M \circ \chi_X(I)_N)(r) \wedge I_M(b)\}$$

$$= \bigvee_{x=rb} \{ \bigvee_{r=pq} \{ I_M(s) \wedge \chi_X(I)_N(t)\} \wedge I_M(b)\}$$
$$= \bigvee_{x=rb} \{ \bigvee_{r=st} \{ I_M(s)\} \ \wedge I_M(b)\}$$

$= \vee_{x=ab}\{I_M(s_i) \wedge I_M(b)\}$, for some $s_i \in X$ and $r = s_i t_i$

$$\leq \bigvee_{x=s_i t_i b} I_M(s_i t_i b) = I_M(x),$$

$$(F_M \circ \chi_X(F)_N \circ F_M)(x) = \bigwedge_{x=rb} \{(F_M \circ \chi_X(F)_N)(r) \vee F_M(b)\}$$

$$= \bigwedge_{x=rb} \{ \bigwedge_{a=st} \{(F_M(s) \vee \chi_X(F)_N(t)\} \vee F_M(b)\}$$

$$= \bigwedge_{x=rb} \{ \bigwedge_{r=st} \{(F_M(s)\} \vee F_M(b)\}$$

$= \wedge_{x=rb}\{F_M(s_i) \vee F_M(b)\}$ for some $s_i \in X$ and $a = s_i t_i$

$$\geq \bigwedge_{x=s_i t_i b} F_M(s_i t_i b) = F_M(x).$$

Otherwise $x \neq rb$ or $a \neq st$ for all r, $b, s, t \in X$. Then

$$(T_M \circ \chi_X(T)_N \circ T_M)(x) = 0 \geq T_M(x),$$
$$(I_M \circ \chi_X(I)_N \circ I_M)(x) = -1 \leq I_M(x),$$
$$(F_M \circ \chi_X(F)_N \circ F_M)(x) = 0 \geq F_M(x).$$

Therefore $X_M \odot \chi_X(X_N) \odot X_M \subseteq X_M$ for any neutrosophic $\aleph -$ structure X_N over X. □

Definition 3.4. A semigroup X is called neutrosophic $\aleph -$left (resp., right) duo if every neutrosophic $\aleph -$left (resp., right) ideal is neutrosophic $\aleph -$ideal of X.

If X is both neutrosophic $\aleph -$left duo and neutrosophic $\aleph -$right duo, then X is called neutrosophic $\aleph -$duo

Theorem 3.5. If X is regular left duo (resp., duo, right duo), then the equivalent assertions are:

(i) X_M in X is neutrosophic $\aleph -$bi- ideal,

(ii) X_M in X is neutrosophic $\aleph -$right ideal (resp., ideal, left ideal).

Proof: $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Suppose X_M is a neutrosophic $\aleph -$bi- ideal and $g, h \in X$. As X is regular, we get $g = gtg \in gX \cap Xg$ for some $t \in X$ which gives $gh \in (gX \cap Xg)X \subseteq gX \cap Xg$ as X is left duo. So $gh = gs$ and $gh = s'g$ for some $s, s' \in X$. As X is regular, $\exists r \in X$: $gh = ghrgh = gsrs'g = g(srs')g$. Since X_M is neutrosophic $\aleph -$bi- ideal, we have

$$T_M(gh) = T_M(g(srs')g) \leq T_M(g) \vee T_M(g) = T_M(g),$$

$$I_M(gh) = I_M(g(srs')g) \geq I_M(g) \wedge I_M(g) = I_M(g),$$

$$F_M(gh) = F_M(g(srs')g) \leq F_M(g) \vee F_M(g) = F_M(g).$$

Therefore X_M is neutrosophic $\aleph -$right ideal.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Suppose X_M is neutrosophic $\aleph -$right ideal and let $x, y, z \in X$. Then

$$T_M(xyz) \leq T_M(x) \leq T_M(x) \vee T_M(z),$$

$$I_M(xyz) \geq I_M(x) \geq I_M(x) \wedge I_M(z),$$

$$F_M(xyz) \leq F_M(x) \leq F_M(x) \vee F_M(z).$$

Therefore X_M is a neutrosophic $\aleph$ −bi-ideal. □

Theorem 3.6. If X is regular, then the equivalent assertions are:
(i) X is left duo (resp., right duo, duo),
(ii) X is neutrosophic $\aleph$ −left duo (resp., right duo, duo).

Proof: $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Let r, s $\in X$, we have $rs \in (rXr)s \subseteq r(Xr)X \subseteq Xr$ as Xr is left ideal. Since X is regular, we have $rs = tr$ for some $t \in X$.

If X_M is neutrosophic $\aleph$ −left ideal, then $T_M(rs) = T_M(tr) \leq T_M(r), I_M(rs) = I_M(tr) \geq I_M(r)$ and $F_M(rs) = F_M(tr) \leq F_M(r)$. Thus X_M is neutrosophic $\aleph$ −right ideal and therefore X is neutrosophic $\aleph$ −left duo.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Let A be a left ideal of X. Then $\chi_A(X_M)$ is a neutrosophic $\aleph$ −left ideal by Theorem 3.5 of [4]. By assumption, $\chi_A(X_M)$ is neutrosophic $\aleph$ −ideal. Thus A is a right ideal of X. □

Theorem 3.7. If X is regular, then the equivalent assertions are:
(i) Every neutrosophic $\aleph$ −bi-ideal is a neutrosophic $\aleph$ −right (resp., left ideal, ideal) ideal,
(ii) Every bi-ideal of X is a right ideal (resp., left ideal, ideal).

Proof: $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Let A be a bi-ideal of X. Then *by* Theorem 3.1 $\chi_A(X_M)$ is neutrosophic $\aleph$ −bi-ideal for a neutrosophic $\aleph$ −structure X_M. Now by assumption, $\chi_A(X_M)$ is neutrosophic $\aleph$ −right ideal. So by Theorem 3.5 of [4], A is right ideal.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Let X_M be a neutrosophic $\aleph$ −bi-ideal and let $r, s \in X$. Then we get rXr is a bi-ideal of X. By hypothesis, we can have rXr is right ideal. Since X is regular, we can get r$\in rXr$. So $rs \in (rXr)X \subseteq rXr$ implies $rs = rxr$ for some $x \in X$. Now,

$$T_M(rs) = T_M(rxr) \leq T_M(r) \vee T_M(r) = T_M(r),$$
$$I_M(rs) = I_M(rxr) \geq I_M(r) \wedge I_M(r) = I_M(r)$$
$$F_M(rs) = F_M(rxr) \leq F_M(r) \vee F_M(r) = F_M(r).$$

Thus X_M is a neutrosophic $\aleph$ −right ideal of X. □

Theorem 3.8. For any X, the equivalent conditions are:

(i) X is regular,

(ii) $X_M \cap X_N = X_M \odot X_N \odot X_M$ for every neutrosophic $\aleph$ − bi-ideal X_M and neutrosophic $\aleph$ − ideal X_N of X.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose $X$is regular, X_M is a neutrosophic $\aleph$ − bi-ideal and X_N is a neutrosophic $\aleph$ − ideal of X. Then by Theorem 3.3, we have $X_M \odot X_N \odot X_M \subseteq X_M$ and $X_M \odot X_N \odot X_M \subseteq X_N$. So $X_M \odot X_N \odot X_M \subseteq X_M \cap X_N$.

Let $r' \in X$. As X is regular, there is $p \in X$ such that $r' = r'pr' = r'pr'pr'$. Now

$$T_{M\circ N\circ M}(r') = \bigwedge_{r'=de}\{T_M(d) \vee T_{N\circ M}(e)\}$$
$$= \bigwedge_{r'=r'e}\{T_M(r') \vee \{\bigwedge_{v=pr'pr'}\{T_N(pr'p) \vee T_M(r')\}\}$$
$$\leq \bigwedge_{r'=r'e}\{T_M(r') \vee T_N(r')\} \leq T_M(r') \vee T_N(r') = T_{M\cap N}(r'),$$

$$I_{M\circ N\circ M}(r') = \vee_{r'=de}\{I_M(d) \wedge I_{N\circ M}(e)\}$$

$$= \bigvee_{r'=r'e} \{I_M(r') \wedge \{ \bigvee_{v=pr'pr'} \{ I_N(pr'p) \wedge I_M(r')\}\}$$

$$\geq \bigvee_{r'=r'e} \{I_M(r') \wedge I_N(r')\} \geq I_M(r') \wedge I_N(r') = I_{M\cap N}(r'),$$

$$F_{M\circ N\circ M}(r') = \bigwedge_{r'=de} \{F_M(d) \vee F_{N\circ M}(e)\}$$
$$= \bigwedge_{r'=r'e} \{F_M(r') \vee \{ \bigwedge_{v=pr'pr'} \{F_N(pr'p) \vee F_M(r')\}\}$$
$$\leq \bigwedge_{r'=r'e} \{F_M(r') \vee F_N(r')\} \leq F_M(r') \vee F_N(r') = F_{M\cap N}(r').$$

Thus $X_{M\cap N} \subseteq X_M \odot X_N \odot X_M$ and hence $X_{M\cap N} = X_M \odot X_N \odot X_M$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds. Then $X_M \cap \chi_X(X_N) = X_M \odot \chi_X(X_N) \odot X_M$. But $X_M \cap \chi_X(X_N) = X_M$, so $X_M = X_M \odot \chi_X(X_N) \odot X_M$ for every neutrosophic $\aleph-$ bi-ideal X_M of X.

Let $u' \in X$. Then $\chi_{B(u')}(X_M)$ is neutrosophic $\aleph-$ bi-ideal by Theorem 3.1.

By assumption, we have

$$\chi_{B(u')}(T)_M = \chi_{B(u')}(T)_M \circ \chi_X(T)_N \circ \chi_{B(u')}(T)_M = \chi_{B(u')XB(u')}(T)_M,$$
$$\chi_{B(u')}(I)_M = \chi_{B(u')}(I)_M \circ \chi_X(I)_N \circ \chi_{B(u')}(I)_M = \chi_{B(u')XB(u')}(I)_M,$$
$$\chi_{B(u')}(F)_M = \chi_{B(u')}(F)_M \circ \chi_X(F)_N \circ \chi_{B(u')}(F)_M = \chi_{B(u')XB(u')}(F)_M.$$

Since $u' \in B(u')$, we have

$$\chi_{B(u')XB(u')}(T)_M(u') = \chi_{B(u')}(T)_M(u') = -1,$$
$$\chi_{B(u')XB(u')}(I)_M(u') = \chi_{B(u')}(I)_M(u') = 0,$$
$$\chi_{B(u')XB(u')}(F)_M(u') = \chi_{B(u')}(F)_M(u') = -1$$

Thus $u' \in B(u')XB(u')$ and hence X is regular. □

Theorem 3.9. For any X, the below statements are equivalent:

(i) X is regular,

(ii) $X_M \cap X_N = X_M \odot X_N$ for every neutrosophic $\aleph-$ bi-ideal X_M and neutrosophic $\aleph-$ left ideal X_N of X.

Proof:$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let X_M and X_N be neutrosophic $\aleph-$ bi-ideal and neutrosophic $\aleph-$left ideal of X respectively. Let $r \in X$. Then $\exists x \in X$: $r = rxr$. Now

$$T_{M\circ N}(r) = \bigwedge_{r=uv} \{T_M(u) \vee T_N(v)\} \leq T_M(r) \vee T_N(xr) \leq T_M(r) \vee T_N(r) = T_{M\cap N}(r),$$
$$I_{M\circ N}(r) = \bigvee_{r=uv} \{I_M(u) \wedge I_N(v)\} \geq I_M(r) \wedge I_N(xr) \geq I_M(r) \wedge I_N(r) = I_{M\cap N}(r),$$
$$F_{M\circ N}(r) = \bigwedge_{r=uv} \{F_M(u) \vee F_N(v)\} \leq F_M(r) \vee F_N(xr) \leq F_M(r) \vee F_N(r) = F_{M\cap N}(r).$$

Therefore $X_{M\cap N} \subseteq X_M \odot X_N$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds, and let X_M and X_N be neutrosophic $\aleph-$ right ideal and neutrosophic $\aleph-$ left ideal of X respectively. Since every neutrosophic $\aleph-$ right ideal is neutrosophic $\aleph-$ bi-ideal, X_M is neutrosophic $\aleph-$ bi-ideal. Then by assumption, $X_{M\cap N} \subseteq X_M \odot X_N$. By Theorem 3.8 and Theorem 3.9 of [4], we can get $X_M \odot X_N \subseteq X_N$ and $X_M \odot X_N \subseteq X_M$ and so $X_M \odot X_N \subseteq X_M \cap X_N = X_{M\cap N}$. Therefore $X_M \odot X_N = X_{M\cap N}$.

Let K and L be right and left ideals of X respectively, and $r \in K \cap L$. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$ which implies $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Since $r \in K \cap L$, we have

$\chi_{K\cap L}(T)_M(r) = -1 = \chi_{KL}(T)_M(r), \chi_{K\cap L}(I)_M(r) = 0 = \chi_{KL}(I)_M(r)$ and $\chi_{K\cap L}(F)_M(r) = -1 = \chi_{KL}(F)_M(r)$ which imply r $\in KL$.Thus $K \cap L \subseteq KL \subseteq K \cap L$. So $K \cap L = KL$. Thus X is regular. □

Theorem 3.10. For any X, the equivalent conditions are:

(i) X is regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\aleph -$ right ideal X_N and neutrosophic $\aleph -$ bi-ideal X_M of X.

Proof: It is same as Theorem 3.9. □

Theorem 3.11. For any X, the equivalent assertions are:

(i) X is regular,

(ii) $X_L \cap X_M \cap X_N \subseteq X_L \odot X_M \odot X_N$ for every neutrosophic$\aleph -$ right ideal $X_{L,}$ neutrosophic $\aleph -$ bi-ideal X_M and neutrosophic $\aleph -$ left ideal X_N of X.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose X is regular, and let X_L, X_M, X_N be neutrosophic $\aleph -$ right, bi-ideal, left ideals of X respectively. Let r $\in X$. Then there is $x \in X$ with r= $rxr = rxrxr$. Now

$$T_{L\circ M\circ N}(r) = \bigwedge_{r=uv}\{T_L(u) \vee T_{M\circ N}(v)\} \leq T_L(rx) \vee T_{M\circ N}(rxr) \leq T_L(r) \vee \{T_M(r) \vee T_N(xr)\}$$
$$\leq T_L(r) \vee T_M(r) \vee T_N(r) = T_{L\cap M\cap N}(r),$$
$$I_{L\circ M\circ N}(r) = \bigvee_{r=uv}\{I_L(u) \wedge I_{M\circ N}(v)\} \geq I_L(rx) \wedge I_{M\circ N}(rxr) \geq I_L(r) \wedge \{I_M(r) \wedge I_N(xr)\}$$
$$\geq I_L(r) \wedge I_M(r) \wedge I_N(r) = I_{L\cap M\cap N}(r),$$
$$F_{L\circ M\circ N}(r) = \bigwedge_{r=uv}\{F_L(u) \vee F_{M\circ N}(v)\} \leq F_L(rx) \vee F_{M\circ N}(rxr) \leq F_L(r) \vee F_M(r) \vee F_N(xr)$$
$$\leq F_L(r) \vee F_M(r) \vee F_N(r) = F_{L\cap M\cap N}(r).$$

Therefore $X_{L\cap M\cap N} \subseteq X_L \odot X_M \odot X_N$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds, and let X_L and X_N be neutrosophic $\aleph -$ right and neutrosophic $\aleph -$ left ideal of X respectively, and X_M a neutrosophic $\aleph -$bi-ideal of X. Then $\chi_X(X_M)$ is a neutrosophic $\aleph -$ bi-ideal by Theorem 3.1. Now $X_L \cap X_N = X_L \cap \chi_X(X_M) \cap X_N \subseteq X_L \odot \chi_X(X_M) \odot X_N \subseteq X_L \odot X_N$. Again by Theorem 3.8 and Theorem 3.9 of [4], we can get $X_L \odot X_N \subseteq X_L \cap X_N$and so $X_L \odot X_N = X_L \cap X_N$.

Let K and L be right and left ideals of X respectively. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$. By Theorem 3.6 of [4], we have $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Let r $\in K \cap L$. Then

$$\chi_{KL}(T)_M(r) = \chi_{K\cap L}(T)_M(r) = -1,$$
$$\chi_{KL}(I)_M(r) = \chi_{K\cap L}(I)_M(r) = 0,$$
$$\chi_{KL}(F)_M(r) = \chi_{K\cap L}(F)_M(r) = -1.$$

So r $\in KL$. Thus $K \cap L \subseteq KL \subseteq K \cap L$. Hence $K \cap L = KL$. Therefore X is regular. □

Theorem 3.12. For any X, the equivalent conditions are:

(i) X is regular and intra- regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\aleph -$ bi-ideals X_M, X_N of X.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let X_M and X_N be neutrosophic $\aleph -$ bi-ideals. Let $h \in X$. Then by regularity of X, h $= hxh = hxhxh$ for some x $\in X$. Since X is intra-regular, $\exists y, z \in X$: h $= yh^2z$. Then $h = hxyhhzxh$. Now

$$T_{M\circ N}(h) = \bigwedge_{h=rt}\{T_M(r) \vee T_N(t)\} \leq T_M(hxyh) \vee T_N(hzxh) \leq T_M(h) \vee T_N(h) = T_{M\cap N}(h),$$
$$I_{M\circ N}(h) = \bigvee_{h=rt}\{I_M(r) \wedge I_N(t)\} \geq I_M(hxyh) \wedge I_N(hzxh) \geq I_M(h) \wedge I_N(h) = I_{M\cap N}(h),$$
$$F_{M\circ N}(h) = \bigwedge_{h=rt}\{F_M(r) \vee F_N(t)\} \leq F_M(hxyh) \vee F_N(hzxh) \leq F_M(h) \vee F_N(h) = F_{M\cap N}(h).$$

Therefore $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds, and let X_M and X_N be neutrosophic $\aleph-$ right and left ideal of X respectively. Then X_M and X_N are neutrosophic $\aleph-$ bi-ideals. By assumption, $X_{M\cap N} \subseteq X_M \odot X_N$. By Theorem 3.8 and Theorem 3.9 of [4], we can get $X_M \odot X_N \subseteq X_N$ and $X_M \odot X_N \subseteq X_M$ and so $X_M \odot X_N \subseteq X_M \cap X_N = X_{M\cap N}$. Therefore $X_M \odot X_N = X_{M\cap N}$.

Let K, L be right, left ideals of X respectively. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$.

By Theorem 3.6 of [4], $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Let $r \in K \cap L$. Then $\chi_{K\cap L}(T)_M(r) = -1 = \chi_{KL}(T)_M(r), \chi_{K\cap L}(I)_M(r) = 0 = \chi_{KL}(I)_M(r)$ and $\chi_{K\cap L}(F)_M(r) = -1 = \chi_{KL}(F)_M(r)$ which imply $r \in KL$. Thus $K \cap L \subseteq KL \subseteq K \cap L$ and hence $K \cap L = KL$. Therefore X is regular.

Also, for $r \in X, \chi_{B(r)}(X_M) \cap \chi_{B(r)}(X_M) = \chi_{B(r)}(X_M) \odot \chi_{B(r)}(X_M)$. By Theorem 3.8 and Theorem 3.9 of [4], we get $\chi_{B(r)}(X_M) = \chi_{B(r)B(r)}(X_M)$.since$\chi_{B(r)}(T)_M(r) = -1 = \chi_{B(r)}(F)_M(r)$and $\chi_{B(r)}(I)_M(r) = 0$, we get $\chi_{B(r)B(r)}(T)_M(r) = -1 = \chi_{B(r)B(r)}(F)_M(r)$ and $\chi_{B(r)B(r)}(I)_M(r) = 0$ which imply $r \in B(r)B(r)$. Thus X is intra-regular. □

Theorem 3.13. For any X, the equivalent conditions are:

(i) X is intra-regular and regular,

(ii) $X_M \cap X_N \subseteq (X_M \odot X_N) \cap (X_N \odot X_M)$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

Proof:$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose X is regular and intra- regular, and let X_M and X_N be neutrosophic $\aleph-$ bi-ideals of X. Then by Theorem 3.12, $X_M \odot X_N \supseteq X_M \cap X_N$. Similarly we can prove that $X_N \odot X_M \supseteq X_N \cap X_M$.Therefore $(X_M \odot X_N) \cap (X_N \odot X_M) \supseteq X_M \cap X_N$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let X_M and X_N be neutrosophic $\aleph-$ bi-ideals of X. Then $X_M \cap X_N \subseteq X_M \odot X_N$ gives X is intra-regular and regular by Theorem 3.12. □

Theorem 3.14. For any X, the equivalent assertions are:

(i) X is intra-regular and regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N \odot X_M$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

Proof:$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let X_M and X_N be neutrosophic $\aleph-$ bi-ideals, and $a \in X$. As X is regular, $a = axa = axaxaxa$ for some $x \in X$. Since X is intra-regular, $a = ya^2z$ for some $y, z \in X$. Then $a = (axya)(azxya)(azxa)$. Now

$$\begin{aligned}
T_{M\circ N\circ M}(a) &= \bigwedge_{a=km}\{T_M(k) \vee T_{N\circ M}(m)\} \\
&= \bigwedge_{a=(axya)v}\{T_M(axya) \vee \{\bigwedge_{v=rt}\{T_N(r) \vee T_M(t)\}\} \\
&\leq T_M(axya) \vee T_N(azxya) \vee T_M(azxa) \\
&\leq T_M(a) \vee T_N(a) \vee T_M(a) = T_{M\cap N}(a),
\end{aligned}$$

$$\begin{aligned}
I_{M\circ N\circ M}(a) &= \bigvee_{a=km}\{I_M(k) \wedge I_{N\circ M}(m)\} \\
&= \bigvee_{a=(axya)v}\{I_M(axya) \wedge \{\bigvee_{v=rt}\{I_N(r) \wedge I_M(t)\}\}
\end{aligned}$$

$$\geq I_M(axya) \wedge I_N(azxya) \wedge I_M(azxa)$$
$$\geq I_M(a) \wedge I_N(a) \wedge I_M(a) = I_{M\cap N}(a),$$

and

$$F_{M\circ N\circ M}(a) = \bigwedge_{a=km}\{F_M(k) \vee F_{N\circ M}(m)\}$$
$$= \bigwedge_{a=(axya)v}\{F_M(axya) \vee \{\bigwedge_{v=rt}\{F_N(r) \vee F_M(t)\}\}$$
$$\leq F_M(axya) \vee F_N(azxya) \vee F_M(azxa)$$
$$\leq F_M(a) \vee F_N(a) \vee F_M(a) = F_{M\cap N}(a).$$

Therefore $X_M \cap X_N \subseteq X_M \odot X_N \odot X_M$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let $h_j \in X$. Then

$$\chi_{B(h_j)}(X_M) \subseteq \chi_{B(h_j)}(X_M) \cap \chi_{B(h_j)}(X_M) \subseteq \chi_{B(h_j)}(X_M) \odot \chi_{B(h_j)}(X_M) \odot \chi_{B(h_j)}(X_M).$$

So

$$\chi_{B(h_j)}(T)_M(h_j) \geq \chi_{B(h_j)B(h_j)B(h_j)}(T)_M(h_j),$$
$$\chi_{B(h_j)}(I)_M(h_j) \leq \chi_{B(h_j)B(h_j)B(h_j)}(I)_M(h_j),$$
$$\chi_{B(h_j)}(F)_M(h_j) \geq \chi_{B(h_j)B(h_j)B(h_j)}(F)_M(h_j).$$

Since $\chi_{B(h_j)}(T)_M(h_j) = -1 = \chi_{B(h_j)}(F)_M(h_j)$ and $\chi_{B(h_j)}(I)_M(h_j) = 0$, we get $\chi_{B(h_j)B(h_j)B(h_j)}(T)_M(h_j) = -1 = \chi_{B(h_j)B(h_j)B(h_j)}(F)_M(h_j)$ and $\chi_{B(h_j)B(h_j)B(h_j)}(I)_M(h_j) = 0$ which imply $h_j \in B(h_j)B(h_j)B(h_j)$. Therefore X is intra-regular and regular. □

Theorem 3.15. For any X, the equivalent assertions are:

(i) X is intra-regular,

(ii) For each neutrosophic $\aleph-$ideal X_M of X, $X_M(a) = X_M(a^2)$ $\forall a \in X$.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let $a \in X$. Then $a = ya^2z$ for some $y, z \in X$. For a neutrosophic $\aleph-$ideal X_M, we have

$$T_M(a) = T_M(ya^2z) \leq T_M(a^2z) \leq T_M(a^2) \leq T_M(a),$$
$$I_M(a) = I_M(ya^2z) \geq I_M(a^2z) \geq I_M(a^2) \geq I_M(a),$$
$$F_M(a) = F_M(ya^2z) \leq F_M(a^2z) \leq F_M(a^2) \leq F_M(a),$$

so $T_M(a) = T_M(a^2)$; $I_M(a) = I_M(a^2)$ and $F_M(a) = F_M(a^2)$ for all $a \in X$. Therefore $X_M(a) = X_M(a^2)$

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let $a \in X$. Then $I(a^2)$ is an ideal of X. Thus $\chi_{I(a^2)}(X_M)$ is neutrosophic $\aleph-$ideal by Theorem 3.5 of [4]. By assumption, $\chi_{I(a^2)}(X_M)(a) = \chi_{I(a^2)}(X_M)(a^2)$. Since $\chi_{I(a^2)}(T)_M(a^2) = -1 = \chi_{I(a^2)}(F)_M(a^2)$ and $\chi_{I(a^2)}(I)_M(a^2) = 0$, we get $\chi_{I(a^2)}(T)_M(a) = -1 = \chi_{I(a^2)}(F)_M(a)$ and $\chi_{I(a^2)}(I)_M(a) = 0$ imply $a \in I(a^2)$. Thus X is intra-regular. □

Theorem 3.16. For any X, the equivalent assertions are:

(i) X is left (resp., right) regular,

(ii) For each neutrosophic $\aleph-$left (resp., right) ideal X_M of X, $X_M(a) = X_M(a^2)$ $\forall a \in X$.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose X is left regular. Then $a = ya^2$ for some $y \in X$ Let X_M be neutrosophic $\aleph-$ left ideal. Then $T_M(a) = T_M(ya^2) \leq T_M(\mathrm{a}^2)$ and so $T_M(a) = T_M(a^2), I_M(a) =$

$I_M(ya^2) \geq I_M(a)$ and so $I_M(a) = I_M(a^2)$, and $F_M(a) = F_M(ya^2) \leq F_M(a)$ and so $F_M(a) = F_M(a^2)$. Therefore $X_M(a) = X_M(a^2)$ for all $a \in X$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let X_M be neutrosophic $\aleph$ −left ideal. Then for any $a \in X$, we have $\chi_{L(a^2)}(T)_M(a) = \chi_{L(a^2)}(T)_M(a^2) = -1, \chi_{L(a^2)}(I)_M(a) = \chi_{L(a^2)}(I)_M(a^2) = 0$ and $\chi_{L(a^2)}(F)_M(a) = \chi_{L(a^2)}(F)_M(a^2) = -1$ imply $a \in L(a^2)$. Thus X is left regular. □

Corollary 3.17. Let X be a regular right duo (resp., left duo). Then the equivalent conditions are:

(i) X is left regular,

(ii) For each neutrosophic $\aleph$ −bi- ideal X_M of X, we have $X_M(a) = X_M(a^2)$ for all $a \in X$.

Proof: It is evident from Theorem 3.5 and Theorem 3.16. □

Conclusions

In this paper, we have presented the concept of neutrosophic $\aleph$ − bi −ideals of semigroups and explored their properties, and characterized regular semigroups, intra-regular semigroups and semigroups using neutrosophic ℵ-bi-ideal structures. We have also shown that the neutrosophic ℵ-product of ideals and the intersection of neutrosophic ℵ-ideals are identical for a regular semigroup. In future, we will focus on the idea of neutrosophic $\aleph$ −prime ideals of semigroups and its properties.

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Neutrosophic Components Semigroups and Multiset Neutrosophic Components Semigroups

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W.B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2020). Neutrosophic Components Semigroups and Multiset Neutrosophic Components Semigroups. *Symmetry*, 12, 818; DOI: 10.3390/sym12050818

Abstract: Neutrosophic components (NC) under addition and product form different algebraic structures over different intervals. In this paper authors for the first time define the usual product and sum operations on NC. Here four different NC are defined using the four different intervals: (0, 1), [0, 1), (0, 1] and [0, 1]. In the neutrosophic components we assume the truth value or the false value or the indeterminate value to be from the intervals (0, 1) or [0, 1) or (0, 1] or [0, 1]. All the operations defined on these neutrosophic components on the four intervals are symmetric. In all the four cases the NC collection happens to be a semigroup under product. All of them are torsion free semigroups or weakly torsion free semigroups. The NC defined on the interval [0, 1) happens to be a group under addition modulo 1. Further it is proved the NC defined on the interval [0, 1) is an infinite commutative ring under addition modulo 1 and usual product with infinite number of zero divisors and the ring has no unit element. We define multiset NC semigroup using the four intervals. Finally, we define n-multiplicity multiset NC semigroup for finite n and these two structures are semigroups under + modulo 1 and $\{M(S), +, \times\}$ and $\{n\text{-}M(S), +, \times\}$ are NC multiset semirings. Several interesting properties are discussed about these structures.

Keywords: neutrosophic components (NC); NC semigroup; multiset NC; n-multiplicity; multiset NC semigroup; special zero divisors; torsion free semigroup; weakly torsion free semigroup; infinite commutative ring; group under addition modulo 1; infinite neutrosophic communicative ring; multiset NC semirings

1. Introduction

Semigroups play a vital role in algebraic structures [1–5] and they are applied in several fields and it is a generalization of groups, as all groups are semigroups and not vice versa. Neutrosophic sets proposed by Smarandache in [6] has become an interesting area of major research in recent days both in the area of algebraic structures [7–11] as well as in applications ranging from medical diagnosis to sentiment analysis [12,13]. The study of neutrosophic triplets happens to be a special form of neutrosophic sets. Extensive study in this direction have been carried out by several researchers in [8,14–17]. Here we are interested in the study of neutrosophic components (NC) over the intervals (0, 1), (0, 1], [0, 1) and [0, 1]. So far researchers have studied and applied NC only on the interval [0, 1] though they were basically defined by Smarandache [18] on all intervals. Further they have not studied them under the usual operation + and $\times$. Here we venture to study NC on all the four intervals and obtain several interesting algebraic properties about them.

Smarandache multiset semigroup studied in [19] is different from these semigroups. Further these multiset NC semigroups are also different from multi semigroups in [20] which deals with multi structures on semigroups.

Any algebraic structure becomes more efficient for application only when it enjoys some strong properties. In fact a set endowed with closed associative binary operation happens to be a semigroup. This semigroup structure does not yield many applications like algebraic codes or commutative rings or commutative semirings. Basically to have a vector space one needs at least the basic algebraic structure to be a group under addition. The same is true in case of algebraic codes. However none of the intervals [0, 1] or (0, 1) or (0, 1] can afford to have a group structure under +. One can not imagine of a group structure under product for no inverse element can be got for any element in these intervals. But when we consider the interval [0, 1) we see it is a group under addition modulo 1.

In fact for any collection of NC which are triplets to have a stronger structure than a semigroup we need to have a strong structure on the interval over which it is built. That is why this paper studies the NC on the interval [0, 1). These commutative rings in [0, 1) can be used to built both algebraic codes on the NC for which we basically need these NC to be at least a commutative ring. With this motivation, we have developed this paper.

This paper further proves that multiset NC built on the interval [0, 1) happens to be a commutative semiring paving way to build multiset NC algebraic codes and multiset neutrosophic algebraic codes which can be applied to cryptography with indeterminacy.

The paper is organized as follows. Section one is introductory in nature. Section 2 recalls the basic concepts of partial order, torsion free semigroup and neutrosophic set. Section 3 introduces NC on the four intervals [0, 1], (0,1), [0, 1) and (0, 1] and mainly prove they are infinite NC semigroups which are torsion free. The new notion of weakly torsion free elements in a semigroup is introduced in this paper and it is proved that NC semigroups built on intervals [0, 1] and [0, 1) are weakly torsion free under usual product $\times$. We further prove the NC built using the interval [0, 1) happens to be an infinite order commutative ring with infinite number of zero divisors and it has no unit. In Section 4 we prove multiset NC built using these four intervals are multiset neutrosophic semigroups under usual product $\times$. We prove only in case of [0, 1) the multiset NC is a ring with infinite number of zero divisors and in all the other interval, $M(S)$ is a torsion free or weakly torsion free semigroup under $\times$. Only in case of the interval [0, 1), $M(S)$ is semigroup under modulo addition 1. In Section 5 we define n-multiplicity multiset NC on all the intervals and obtain several interesting properties. Discussions about this study are given in Section 6 and the final section gives conclusions and future research based on their structures.

2. Basic Concepts

In this section we introduce the basic concepts needed to make this paper a self contained one. We first recall the definition of partially ordered set.

Definition 1. *There exist some distinct elements $a, b \in S$ such that $a < b$ or $a > b$, and other distinct elements $b, c \in S$ such that neither $b < c$ nor $b > c$, then we say $(S, <)$ is a partially ordered set. We say $(S, \leq)$ is a totally ordered set if for every pair $a, b, \in S$ we have $a \leq b$ or $b \geq a$.*

The set of integers is a totally ordered set and the power set of a set X; P(X) is only a partially ordered set.

Next we proceed on to define torsion free semigroup.

Definition 2. *A semigroup $\{S, \times\}$ is said to be a torsion free semigroup if for $a, b \in S, a \neq b, a^n \neq b^n$ for any $1 \leq n < \infty$.*

We recall the definition of semiring in the following from [21].

Definition 3. *For a non empty set S, $\{S, +, \times\}$ is defined as a semiring if the following conditions are true*

1. *$\{S, +\}$ is a commutative semigroup with 0 as its additive identity.*
2. *$\{S, \times\}$ is a semigroup.*

3. *$a \times (b+c) = a \times b + a \times c$ for all $a, b, c, \in S$ follows distribution law.*

   If $\{S, \times\}$ is a commutative semigroup we call $\{S, +, \times\}$ as a commutative semiring.

For more, see [21].

For example, set of integers under product is a torsion free semigroup. Finally we give the basic definition of neutrosophic set.

Definition 4. *The Neutrosophic components (NC) is a triplet (a, b, c) where a is the truth membership function from the unit interval [0, 1], b is the indeterminacy membership function and c is the falsity membership function all of them are from the unit interval [0, 1].*

For more about Neutrosophic components (NC), sets and their properties please refer [6].

Next we proceed onto define the notion of multiset.

Definition 5. *A neutrosophic multiset is a neutrosophic set where one or more elements are repeated with same neutrosophic components or with different neutrosophic components.*

Example 1. *$M = \{a(0.3, 0.4, 0.5), a(0.3, 0.4, 0.5), b(1, 0, 0.2), b(1, 0, 0.2), c(0.7, 1, 0)\}$ is a neutrosophic multiset. For more refer [18]. However we in this paper use the term multiset NC to denote elements of the form $\{5(0.3, 0.4, 1), 3(0.6, 0, 1), (0, 0.7, 0.5)\}$ so 5 is the multiplicity of the NC (0.3, 0.4, 1) and 3 is the multiplicity of the NC (0.6, 0, 1) and 1 is the multiplicity of the NC (0, 0.7, 0.5).*

For more about multisets and multiset graphs [18,22].

3. Neutrosophic Components (NC) Semigroups under Usual Product and Sum

Throughout this section $\{x, y, z\}$ will denote the truth value, indeterminate value, false value where x, y, z belongs to $[0, 1]$, the neutrosophic set. However we define special NC on the intervals (0, 1), (0, 1] and [0, 1). We first prove $S_1 = \{(x, y, z)/x, y, z \in (0, 1)\}$ is a semigroup under product and obtain several interesting properties about NC semigroups using the four intervals (0, 1), (0, 1], [0, 1) and [0, 1].

Example 2. *Let $a = (0.3, 0.8, 0.5)$ and $b = (0.9, 0.2, 0.7)$ be any two NC in S_1. We define product $a \times b$ = (0.3, 0.8, 0.5) $\times$ (0.9, 0.2, 0.7) = (0.3 $\times$ 0.9, 0.8 $\times$ 0.2, 0.5 $\times$ 0.7) = (0.27, 0.16, 0.35). It is again a neutrosophic set in S_1.*

Definition 6. *The four NC $S_1 = \{(x, y, z)/x, y, z \in (0, 1)\}$, $S_2 = \{(x, y, z)/x, y, z \in [0, 1)\}$, $S_3 = \{(x, y, z)/x, y, z \in (0, 1]\}$ and $S_4 = \{(x, y, z)/x, y, z \in [0, 1]\}$ are all only partially ordered sets for if a = (x, y, z) and b = (s, r, t) are in S_i then $a < b$ if and only if $x < s$, $y < r$, $z < t$; but not all elements are ordered in S_i, that is why we say S_i are only partially ordered sets, and denote it by $(S_i, \leq)$;where $\leq$ denotes the classical order relation over reals; $1 \leq i \leq 4$.*

For instance if $a = (0.3, 0.7, 0.5)$ and $b = (0.5, 0.2, 0.3)$ are in S_i then a and b cannot be compared. If $d = (0.8, 0.5, 0.7)$ and $c = (0.6, 0.2, 0.5)$, then $d > c$ or $c < d$.

In view of this we have the following theorem.

Theorem 1. *Let $S_1 = \{(x, y, z)/x, y, z \in (0, 1)\}$ be the collection of all NC which are such that the elements x, y and z do not take any extreme values.*

1. *$\{S_1, \times\}$ is an infinite order commutative semigroup which is not a monoid and has no zero divisors.*
2. *Every $a = (x, y, z)$ in S_1 will generate an infinite cyclic subsemigroup under product of S_1 denoted by $(P, \times)$.*

3. *The elements of P forms a totally ordered set, (for if $a = (x, y, z) \in P$ we see $a^2 = a \times a < a$).*
4. *$\{S_1, \times\}$ has no idempotents and $\{S_1, \times\}$ is a torsion free semigroup.*

Proof. Proof of 1: Clearly if $a = (x, y, z)$ and $b = (r, s, t)$ are in S_1, then $a \times b = (x \times r, y \times s, z \times t)$ is in S_1; as $x \times r, y \times s$ and $z \times t \in (0, 1)$. Hence, $\{S_1, \times\}$ is a semigroup under product. Further as number of elements in (0, 1) is infinite so is S_1. Finally as the product in (0, 1) is commutative so is the product in S_1. Hence the claim. (1, 1, 1) is not in S_1 as we have used only the open interval (0, 1), we see $\{S_1, \times\}$ is not a monoid. S_1 has no zero divisors as the elements are from the open interval which does not include 0, hence the claim.

Proof of 2: Let $a = (x, y, z)$ be in S, we see $a \times a = (x \times x, y \times y, z \times z) = a^2$, and so on $a \times a \times \ldots \times a = a^n = (x^n, y^n, z^n)$ and n can take values from $(0, \infty)$. Thus a in S generates a cyclic subsemigroup of infinite order, hence the claim.

Proof of 3: Let $P = \langle a \rangle$, a generates the semigroup under product, it is of infinite order and from the property of elements in (0, 1); $a > a^2 > a^3 >$ and so on $> a^n$. Hence the claim.

Proof of 4: If any $a = (x, y, z) \in S_1$ as $x, y, z \in (0, 1)$, and x, y and z are torsion free so is a. We see $a^2 \neq a$ for any $a \in S_1$. Further if $a \neq b$ for no $n \in (0, \infty)$; $a^n = b^n$. Hence the claim.
□

Definition 7. *The four NC S_1, S_2, S_3 and S_4 mentioned in definition 6 under the usual product $\times$ forms a commutative semigroup of infinite order defined as the NC semigroups.*

Theorem 2. *Let $S_2 = \{(x, y, z)/x, y, z \in [0, 1)\}$ be the collection of NC. $\{S_2, \times\}$ is only a semigroup and not a monoid and has infinite number of zero divisors. Further all other results mentioned in Theorem 1 are true with an additional property if $a \neq b$; $(a, b \in S_2)$ we have*

$$\lim_{n\to\infty} a^n = \lim_{n\to\infty} b^n = (0, 0, 0)$$

as (0, 0, 0) $\in S_2$.

Proof as in case of Theorem 1.

In view of this we define an infinite torsion free semigroup to be weakly torsion free if $a \neq b$; but

$$\lim_{n\to\infty} a^n = \lim_{n\to\infty} b^n$$

Thus S_2 is only a weakly torsion free semigroup.

It is interesting to note S_1 is contained in S_2 and in fact S_1 is a subsemigroup of S_2.The differences between S_1 and S_2 is that S_2 has infinite number of zero divisors and the $\lim_{n\to\infty} a^n = (0, 0, 0)$ exists in S_2 and S_1 is torsion free but S_2 is weakly torsion free.

Theorem 3. *Let $S_3 = \{(x, y, z)/x, y, z \in (0, 1]\}$ be the collection of NC. $\{S_3, \times\}$ is a monoid and has no zero divisors.*

Results 2 to 4 of Theorem 1 are true. Finally S_1 is a subset of S_3, in fact S_1 is a subsemigroup of S_3. The main difference between S_1 and S_3 is that S_3 is a monoid and S_1 is not a monoid. The difference between S_2 and S_3 is that S_3 has no zero divisors but S_2 has zero divisors and S_3 is a monoid.

Next we prove a theorem for S_4.

Theorem 4. *Let $S_4 = \{(x, y, z)/x, y, z \in [0, 1]\}$. $\{S_4, \times\}$ is a semigroup and is a monoid and has zero divisors. Other three conditions of Theorem 1 is true, but S_4 like S_2 is only a weakly torsion free semigroup.*

Proof as in case of Theorem 1. We have S_1 contained in S_2 and S_2 is contained in S_4 and S_1 contained in S_3 and S_3 is contained in S_4.

However, it is interesting to note S_2 and S_3 are not related in spite of the above relations.

Now we analyse all these four neutrosophic semigroups to find out, on which of them we can define addition modulo 1. S_1 does not include the element (0, 0, 0) as 0 is not in (0, 1), so S_1 is not even closed under addition modulo 1. So S_1 in not a semigroup or a group under plus modulo 1. Since S_3 and S_4 contains (1, 1, 1) we cannot define addition modulo 1; hence, they can not have any algebraic structure under addition modulo 1. Now consider $\{S_2, +\}$, clearly $\{S_2, +\}$ is a group under addition modulo 1.

In view of all these we have the following theorem.

Definition 8. *The NC $\{S_2, +\}$ under usual addition modulo 1 is a group defined as the NC group denoted by $\{S_2, +\}$.*

Theorem 5. *$\{S_2, +\}$ is a group under addition modulo 1.*

Proof. For any $y, x \in S_2$, x + y (mod 1) $\in S_2$. (0, 0, 0) $\in S_2$ acts as additive identity. Further for every x there is a unique $y \in S_2$ with $x + y =$ (0, 0, 0). Hence the theorem. □

Definition 9. *The NC S_2 under the operations of the usual addition + modulo 1 and usual product $\times$ forms a commutative ring of infinite order defined as the NC commutative ring denoted by $\{S_2, +, \times\}$.*

Theorem 6. *$\{S_2, +, \times\}$ is a commutative ring with infinite number of zero divisors and has no multiplicative identity (1, 1, 1).*

Proof. Follows from the Theorem 1 and the fact S_2 is closed under + modulo 1 by Theorem 5. The distributive property is inherited from the number theoretic properties of modulo integers. As 1 is not in $[0, 1)$; (1, 1, 1) is not in S_2, hence the result. □

Next we proceed on to define multiset NC semigroups in the following section.

4. Multiset NC Semigroups

In this section we proceed on to define multiset NC semigroups using S_1, S_2, S_3 and S_4. We see $M(S_1) = \{$Collection of all multiset NC using elements of $S_1\}$. On similar lines we define $M(S_2), M(S_3)$ and $M(S_4)$ using S_2, S_3 and S_4 respectively. We prove $\{M(S_2), +, \times\}$ is a multiset neutrosophic semiring of infinite order.

Recall [18], A is a multi neutrosophic set, then $A = \{5(0.3, 0.7, 0.9), 12(0.6.0.2, 0.7), 8(0.1, 0.5, 0.1), (0.6, 0.7, 0.5)\}$; that is in the multiset neutrosophic set A; (0.3, 0.7, 0.9) has occurred 5 times; (0.6, 0.2, 0.7) has occurred 12 times or its multiplicity is 12 in A and so on.

Let $M(S_1) = \{$Collection of all multisets using the elements from $S_1\}$, $M(S_1)$ is an infinite collection. We just show how the classical product is defined on $M(S_1)$.

Let $A = \{9(0.3, 0.2, 0.4), 2(0.6, 0.7, 0.1), (0.1, 0.3, 0.2)\}$ and $B = \{5(0.1, 0.2, 0.5), 10(0.8, 0.4, 0.5)\}$ in $M(S_1)$ be any two multisets. We define the classical product $\times$ of A and B as follows;

$$\begin{aligned} A \times B = \{&9(0.3, 0.2, 0.4) \times 5(0.1, 0.2, 0.5), 9(0.3, 0.2, 0.4) \times 10(0.8, 0.4, 0.5), \\ &2(0.6, 0.7, 0.1) \times 5(0.1, 0.2, 0.5), 2(0.6, 0.7, 0.1) \times 10(0.8, 0.4, 0.5), \\ &(0.1, 0.3, 0.2) \times 5(0.1, 0.2, 0.5), (0.1, 0.2, 0.5) \times 10(0.8, 0.4, 0.5)\} \\ = \{&45(0.03, 0.04, 0.2), 90(0.24, 0.08, 0.2), 10(0.06, 0.14, 0.05), \\ &20(0.48, 0.28, 0.05), 5(0.01, 0.06, 0.1), 10(0.08, 0.08, 0.25)\}; \end{aligned}$$

$A \times B$ is in $M(S_1)$, thus $\{M(S_1), \times\}$ is a commutative semigroup of infinite order defined as the multiset NC semigroup.

Definition 10. *Let $M(S_i)$ be the multi NC using elements of $S_i (i = 1,2,3,4)$, $\{M(S_i), \times\}$ on the usual product $\times$ is defined as the multiset neutrosophic semigroup for $i = 1,2,3$ and 4.*

Definition 11. *Let $\{S_2, \times\}$ be the multiset NC semigroup under $\times$, elements of the form $(a,0,0),(0,b,c)$ and so on which are infinite in number with $a,b,c \in S_2$ contribute to zero divisors. Hence multisets using these types of elements contribute to zeros of the form $n(0,0,0)$; $1 < n < \infty$. As the zeros are of varying multiplicity we call these zero divisors as special type of zero divisors.*

We will provide examples of them.

Example 3. *Let $R = \{(S_2), \times\}$ be the multiset NC semigroup under product. Let A = (0.6, 0, 0) and B = (0, 0.4, 0.5) be in R, A $\times$ B = (0, 0, 0). Take $D = \{9(0.6,0.9,0)\}$ and E = 9(0, 0, 0.4) in R; we get $D \times E = \{81(0,0,0)\}$. Take $W = \{7(0,0.5,0), 4(0,0.6,0)\}$ and $V = \{(0.7,0,0.4), 20(0.8,0,0)\}$ be two multisets in R; $W \times V = \{7 \times 44(0,0,0) + 7 \times 20(0,0,0) + 4 \times 44(0,0,0) + 4 \times 20(0,0,0)\} = \{704(0,0,0)\}$ is a special type of zero divisor of R.*

Thus $M(S_2)$ is closed under the binary operation $\times$.

Theorem 7. *The neutrosophic multiset semigroups $\{M(S_i), \times\}$ for i = 1, 2, 3, 4 are commutative and of infinite order satisfying, the following properties for each $M(S_i)$; i = 1, 2, 3, 4.*

1. *$\{M(S_1), \times\}$ has no trivial or non-trivial special type of zero divisors and no trivial or non-trivial idempotents.*
2. *$\{M(S_2), \times\}$ has infinite number of special type of zero divisors and no non-trivial idempotents.*
3. *$\{M(S_3), \times\}$ has no trivial or non-trivial special zero divisors but has (1, 1, 1) as identity and has no non trivial idempotents.*
4. *$\{M(S_4), \times\}$ has non-trivial special type of zero divisors and has (1, 1, 1) as its identity and has idempotents of the form $\{(0,1,0),(1,1,0),(0,0,1),(1,0,1)$ and so on $\}$.*

Proof. 1. Follows from the fact that S_1 has no zero divisors and idempotents as it is built on the interval (0, 1).
2. Evident from the fact S_2 is built on [0, 1) so has special type of zero divisors by definition but no idempotent.
3. True from the fact S_3 is built on (0, 1], so (1, 1, 1) $\in M(S_3)$.
4. S_4 which is built on [0, 1] has infinite special type of zero divisors as (0, 0, 0) $\in S_4$ by Definition 11 and (1, 1, 1) $\in M(S_4)$ and has idempotents of the form $\{(0,1,0),(1,1,0),(0,0,1),(1,0,1)$ and so on $\}$.

Hence the claims of the theorem. □

Now we proceed onto define usual addition on $M(S_1)$

$S_1 = \{(x,y,z)/x,y,z \in (0,1)\}$ in not even closed under addition. For there are $x,y \in (0,1)$ such that $x + y$ is 1 or greater than 1, so these elements are not in (0 , 1), hence our claim.

Recall $S_2 = \{(x,y,z)/x,y,z \in [0,1)\}$. We can define addition modulo 1 and product under that addition both S_2 and $[0,1)$ are closed.

Let $a = (0.7,0.6,0.9)$ and $b = (0.5,0.9,0.4)$ be in S_2, we find $a + b$ mod 1.

$a + b = (0.7,0.6,0.9) + (0.5,0.9,0.4) = (0.7 + 0.5(mod\ 1), 0.6 + 0.9(mod\ 1), 0.9 + 0.4(mod\ 1)) = (0.2,0.5,0.3)$ is in S_2. (0, 0, 0) in S_2 acts as the additive identity.

For every $a \in S_2$ there is a unique $b \in S_2$ such that $a + b = (0,0,0) mod\ 1$. Thus $(S_2, +)$ is a NC group of infinite under addition modulo 1. Further $(S_2, \times)$ is a semigroup under product of infinite order which is commutative and not a monoid as (1, 1, 1) is not in S_2.

Now we illustrate how addition is performed on any two neutrosophic multisets in $M(S_2)$.

Let $A = \{7(0.3,0.8,0.45), 9(0.02,0.41,0.9), (0.6,0.3,0.2)\}$ and $B = \{5(0.1,0,0.9), 2(0.6,0.5,0)\}$ be any two multisets of $M(S_2)$. To find the sum of A with B under addition modulo 1.

$A+B$ = { 35[(0.3, 0.8, 0.45) + (0.1, 0, 0.9)]mod 1, 45[(0.02, 0.41, 0.9) + (0.1, 0, 0.9)]mod 1, 5[(0.6, 0.3, 0.2) + (0.1,0, 0.9)]mod 1, 14[(0.3, 0.8, 0.45) + (0.6, 0.5,0)]mod 1, 18[(0.02, 0.41, 0.9) + (0.6, 0.5, 0)]mod 1, 2[(0.6, 0.3, 0.2) + (0.6, 0.5, 0)]mod 1$\}$ = $\{$35(0.4, 0.8, 0.35), 45(0.12, 0.41, 0.8), 5(0.7, 0.3, 0.1), 14(0.9, 0.3, 0.45), 18(0.62, 0.91, 0.9), 2(0.2, 0.8, 0.2)$\}$

is in $M(S_2)$. This is the way addition modulo 1 operation is performed. For $M(S_3)$ and $M(S_4)$ we can not define usual addition modulo 1 as (1, 1, 1) $\in M(S_3)$ and $M(S_4)$.

Next we proceed on to describe the product of any two elements in $M(S_2)$. We take the above A and B and find $A \times B$. $A \times B$ = {35[(0.3,0.8,0.45) $\times$ (0.1, 0, 0.9)], 45[(0.02, 0.41, 0.9) $\times$ (0.1, 0, 0.9)], 5[(0.6, 0.3, 02)$\times$ (0.1, 0, 0.9)], 14[(0.3, 0.8, 0.45)$\times$(0.6, 0.5 0)], 18[(0.02, 0.41. 0.9)$\times$ (0.0.6, 0.5, 0)], 2[(0.6, 0.3, 0.2) $\times$ (0.6, 0.5, 0)]$\}$ = {35(0.03, 0, 0.405), 45(0.002,0, 0.81), 5(0.06, 0, 0.18), 14(0.18, 0.4, 0), 18(0.012, 0.205, 0), 2(0.36, 0.15, 0)$\}$, is in $M(S_2)$.

Theorem 8. $\{M(S_2), +\}$ *is a multiset NC semigroup under addition modulo 1.*

Proof. $M(S_2)$ is closed under the binary operation addition modulo 1. Thus $M(S_2)$ is the neutrosophic multiset semigroup under + modulo 1. □

Now we proceed on to define a special type of zero divisors. In view of this we have the following theorem.

Theorem 9. $R = \{M(S_2), \times\}$ *is an infinite commutative multiset NC semigroup, which is not a monoid and has special type of zero divisors.*

Proof. We see $M(S_2)$ under the binary operation product is closed and is associative as the base set S_2 is associative and commutative and is closed under the binary operation product. Thus $\{(S_2), \times\}$ is commutative semigroup of infinite order. Further $M(S_2)$ does not contain (1, 1, 1) so $\{M(S_2), \times\}$ is not a monoid.

From the above definition and description of special zero divisors R has infinite number of them. □

We have the following theorem.

Theorem 10. $\{M(S_2), +, \times\}$ *is a NC multiset commutative semiring of infinite order which has infinite numbers of special type of zero divisors.*

Proof. Follows from Theorem 8 and Theorem 9. □

Next we proceed on to define n- multiplicity neutrosophic multisets and derive some properties related with them. $M(S_3)$ and $M(S_4)$ are just multiset NC semigroups under product and in fact they are monoids. Further $M(S_4)$ has infinite number of special zero divisors.

5. n-Multiplicity Neutrosophic Set Semigroups Using S_1, S_2, S_3 and S_4

In this section we define the new notion of n-multiplicity NC using S_1, S_2, S_3 and S_4. We prove these n-multiplicity NC are of infinite order but what is restricted is the multiplicity n, that is any element cannot exceed multiplicity n; it can maximum be n, where n is a positive finite integer. Finally we prove $\{M(S_2), +, \times\}$ where $S_2 = [0, 1)$ is a NC n-multiset commutative semiring of infinite order.

We will first illustrate this situation by some examples before we make an abstract definition of them.

Example 4. *Let 4-*$M(S_1)$ = $\{$*collection all multisets with entries from* $S_1 = \{(x, y, z)/x, y, z \in (0, 1)\}$*, such that any element in* S_1 *can maximum repeat itself only four times*$\}$. *Here* $n = 4$, $A = \{4(0.5, 0.7, 0.4), 3(0.1, 0.9, 0.7), 4(0.1, 0.2, 0.3), 4(0.7, 0.8, 0.4), 4(0.8, 0.8, 0.8), 2(0.9, 0.9, 0.9),$

$3(0.7, 0.9, 0.6), (0.6, 0.1, 0.1)\}$ *be a 4-multiplicity multiset from 4-*$M(S_1)$*. We see the NC* $(0.5, 0.7, 0.4), (0.1, 0.2, 0.3), (0.7, 0.8, 0.4)$ *and (0.8, 0.8,0.8) have multiplicity four which is the highest multiplicity an element of 4-*$M(S_1)$ *can have. The NC (0.1, 0.9, 0.7) and (0.7,0.9,0.6) have multiplicity 3. The multiplicity of (0.9, 0.9, 0.9) is two and that of (0.6, 0.1, 0.1) is one. Clearly* S_1 *does not contain the extreme values 0 and 1 as* S_1 *is built using the open interval (0, 1). However on* $M(S_1)$ *we can not define addition.*

Thus 4-$M(S_1)$ can not have the operation of addition defined on it. Now we show how the operation $\times$ is defined on 4-$M(S_1)$ for the some $A, B \in$ 4-$M(S_1)$. Now

$$\begin{aligned} A \times B &= \{3(0.3, 0.7, 0.8), 2(0.5, 0.9, 0.6), 4(0.2, 0.3, 0.4)\} \times \{(0.1, 0.3, 0.7), 2(0.5, 0.7, 0.1)\} \\ &= \{3(0.03, 0.21, 0.56), 2(0.05, 0.27, 0.42), 4(0.02, 0.09, 0.28), \\ &\quad 6(0.15, 0.49, 0.08), 4(0.25, 0.63, 0.06), 8(0.1, 0.21, 0.04)\} \end{aligned}$$

we now use the fact we can have maximum only 4 multiplicity of an element so we replace 6(0.15, 0.49, 0.08) by 4(0.15, 0.49, 0.08) and 8(0.1, 0.21, 0.04) by 4(0.1, 0.21, 0.04). Now the thresholded product is $\{$(3(0.03, 0.21, 0.56), 2(0.05, 0.27, 0.42), 4(0.02, 0.09, 0.28), 4(0.15, 0.49, 0.08), 4(0.25, 0.63, 0.06), 4(0.1, 0.21, 0.04))$\} \in$ 4-$M(S_1)$.

$\{$4-$M(S_1), \times\}$ is a commutative neutrosophic multiset semigroup of infinite order and the multiplicity of any element cannot exceed 4.

This semigroup is not a monoid and it has no special zero divisors or zero divisors or units.

Definition 12. *12 Let n-*$M(S_i)$ *=\{ collection of all multisets with entries from* S_i *of at-most multiplicity* $n; 2 \leq n < \infty\}(1 \leq i < 4)$*. n-*$M(S_i)$ *under usual product,* $\times$ *is defined as the n-multiplicity NC semigroup,* $1 \leq i \leq 4$.

In view of this we have the following theorem.

Theorem 11. *Let n-*$M(S_i) = \{t(x, y, z) | x, y, z \in S_i; 1 \leq t \leq n\}$ *be the n-multiplicity neutrosophic multisets* $(1 \leq i \leq 4)$*.*

1. *n-*$M(S_i)$ *is not closed under the binary operation '+' under usual addition, for i* $=$ *1, 3 and 4.*
2. *n-*$M(S_i)$ *is a (n-multiplicity neutrosophic multiset) semigroup under the usual product for i = 1, 2, 3 and 4.*
3. $\{$*n-*$M(S_i), \times\}$ *is a monoid for i = 3 and 4. .*
4. $\{$*n-*$M(S_i), \times\}$ *has no special zero divisors if* $S_i = S_1$ *and* S_3 *but they have no non trivial idempotents.* S_2 *and special zero divisors and no non trivial idempotents, but* S_4 *has both non trivial special zero divisors and non trivial idempotents.*

Proof. Proof of 1: If $A = \{(0.3, 0.8, 0.9)\}$ and $B = \{(0.4, 0.3, 0.1)\} \in$ n-$M(S_i)$. $A + B = \{$(0.7, 1.1, 1.0)$\} \notin$ n-$M(S_i)$ as S_i when built using S_3 and S_4 and by example 4 n-$M(S_1)$. Only $M(S_2)$ is closed under addition.

Proof of 2: Since $(S_i, \times)$ is closed under product so is n-$M(S_i)$ with replacing the numbers greater than n by n in the resultant product; $i = 1, 2, 3$ and 4 are semigroups, hence the claim.

Proof of 3: As $(1, 1, 1) \in S_3$ and S_4 so is in n-$M(S_3)$ and n-$M(S_4)$ respectively so they are monoids.

Proof of 4: n-$M(S_i)$ has no special zero divisors in case of S_1 and S_3. Finally $S_i = \{(x, y, z) | x, y, z \in S_i\}$, has zero divisors and special zero divisors in case of S_2 and S_4 for $i = 2$ and 4, and non trivial idempotents contributed by 0's and 1's only in case of S_4. Hence the theorem. □

Example 5. *Let 5-*$M(S_2)$ $=$ $\{$*Collection of all neutrosophic multisets which can occur at most 5-times that is the multiplicity is 5 with elements from* $S_2 = \{(x, y, z) | x, y, z \in [0, 1)\}\}$ *Let* $A =$

$4(0.2, 0.5, 0.7), 3(0.1, 0.2, 0.3), 5(0.3, 0.1, 0.2), (0.1, 0.2, 0.8) \in$ *5-*$M(S_2)$ *We see the multiplicity of (0.3, 0.1, 0.2) is 5 others are less than 5.*

Let $A = \{$*3(0.3, 0.2, 0), 4(0.5, 0.6, 0.9), 5(0.1, 0.2, 0.7)*$\}$ *and* $B = \{$*4(0.8, 0.1, 0.9), 2(0.6, 0.6, 0.6)*$\} \in$ *5-*$M(S_2)$*. Now we first find* $A \times B = \{$*5(0.24, 0.02, 0), 5(0.4, 0.06, 0.81), 5(0.08, 0.02, 0.63), 5(0.06, 0.12, 0.42)*$\} \in 5(M(S_2))$.

$A + B = \{$*5(0.1, 0.3, 0.9), 5(0.9, 0.8, 0.6), 5(0.3, 0.7, 0.8), 5(0.9, 0.3, 0.6), 5(0.1, 0.2, 0.5), 5(0.7, 0.8, 0.3)*$\} \in$ *5-*$M(S_2)$*. Addition is done modulo 1. However we have closure axiom to be true under + for elements in* S_2 *and in case of* S_1*;* $0 \notin S_1$ *= (0, 1)). This closure axiom is flouted.*

If addition modulo 1 is done we have to see that 1 is not included in the interval and 0 is included in that interval so we need to have only closed open interval [0, 1). Under these two constraints only we can make S_2 *as well as* $M(S_2)$ *and n-*$M(S_2)$ *as semigroups under addition modulo 1.*

We can built strong structure only using the [0, 1).

Theorem 12. *Let n-*$M(S_2)$ *= Collection of all multisets of S built using* $S_2 = \{(x, y, z) | x, y, z \in [0, 1)\}$ *with multiplicity less than or equal to n;* $2 \leq n \leq \infty$

$\{$ *n-*$M(S_2), \times\}$ *is a commutative neutrosophic multiset semigroup of infinite order and is not a monoid, n-*$M(S_2)$ *has infinite number of zero divisors.*

Proof. If A and $B \in$ n-$M(S_2)$ we find $A \times B$ and update the multiplicities in $A \times B$ to be less than or equal to n so that $A \times B \in$ n-$M(S_2)$. by Theorem 11(2).

Clearly $(1, 1, 1) \notin$ n-$M(S_2)$ so is not a monoid. □

Theorem 13. B = $\{$*n-*$M(S_2), +, \times\}$*, the n-multiplicity multiset NC is a commutative semiring of infinite order and has no unit, where* $S_2 = [0, 1)$.

Proof. Follows from the fact $\{$ n-$M(S_2)$, +$\}$ is a commutative semigroup under addition modulo 1, Theorem 11(1) and Theorem 12 and $\{$n-$M(S_2)$, $\times\}$ is a commutative semigroup under $\times$. Hence the claim. □

6. Discussions

The main motive of this paper is to construct strong algebraic structures with two binary operations on the NC. Here we are able to get a NC commutative ring structure using the base interval as [0, 1). This will lead to future research of constructing Smarandache neutrosophic vector spaces and Smarandache neutrosophic algebraic codes using the same interval [0, 1). Now using the same interval [0, 1), we construct multiset NC and n-multiset NC $2 \leq n < \infty$. On these we were able to built only neutrosophic multiset(n-multiplication set) commutative semiring structure. Now using these we can construct Smarandache multiset neutrosophic semi vector spaces which will be taken as future research. So this is significant first step to develop other strong structures and apply them to NC codes and NC cryptography.

7. Conclusions

In this paper, authors have made a study of NC on the 4-intervals (0, 1) (0, 1], [0, 1] and [0, 1). We define usual + and $\times$ on these intervals which is very different from the study taken so far. The main properties enjoyed by these NC semigroups are developed. Further of these intervals only the interval [0, 1) gives a nice algebraic structure viz an abelian group under usual addition modulo 1, which in turn helps in constructing NC commutative ring under usual addition modulo 1 and product, the ring has infinite number of zero divisors, whereas all the other intervals are semigroups/monoids which are torsion free or weakly torsion free of infinite order under $\times$. Further in this paper we introduce the notion of multiset NC semigroups using these four intervals under product. Furthermore, the multiset

NC forms a commutative semiring with zero divisors only when the interval [0, 1) is used. Finally we introduce n-multiplicity multiset using these NC. They are also semigroups which is torsion free or weakly torsion free under product.

For future research we will be using the product and addition modulo 1 in the place of min and max in Single Valued Neutrosophic Set (SVNS) and would compare the results with the existing ones when applied as SVNS models to real world problems.

Apart from all these we can use these NC, multiset NC and n-multiplicity multiset NC to built NC codes which is one of the applications to neutrosophic cryptography which will be taken up by the

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Further Theory of Neutrosophic Triplet Topology and Applications

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Mohammed A. Al Shumrani, Muhammad Gulistan, Florentin Smarandache (2020). Further Theory of Neutrosophic Triplet Topology and Applications. *Symmetry*, 12, 1207; DOI: 10.3390/sym12081207

Abstract: In this paper we study and develop the Neutrosophic Triplet Topology (NTT) that was recently introduced by Sahin et al. Like classical topology, the NTT tells how the elements of a set relate spatially to each other in a more comprehensive way using the idea of Neutrosophic Triplet Sets. This article is important because it opens new ways of research resulting in many applications in different disciplines, such as Biology, Computer Science, Physics, Robotics, Games and Puzzles and Fiber Art etc. Herein we study the application of NTT in Biology. The Neutrosophic Triplet Set (NTS) has a natural symmetric form, since this is a set of symmetric triplets of the form <A>, <anti(A)>, where <A> and <anti(A)> are opposites of each other, while <neuti(A)>, being in the middle, is their axis of symmetry. Further on, we obtain in this paper several properties of NTT, like bases, closure and subspace. As an application, we give a multicriteria decision making for the combining effects of certain enzymes on chosen DNA using the developed theory of NTT.

Keywords: neutrosophic triplet set; neutrosophic triplet topolgy; decision making; application

1. Introduction

The main aim of the paper is to introduce the Neutrosophic Triplet Topology (NTT) in various fields of research, due to its great potential of applicability. However, in order to do so, we first study its theoretical properties, such as open and closed sets, base and subspace, all extended from classical topology and neutrosophic topology to (NTT). In daily life we are witnessing many situations in which the role of neutralities is very important. To control neutralities Smarandache initiated the theme of neutrosophic logic in 1995, which later on proved to be a very handy tool to capture uncertainty. Thus Smarandache [1], generalizes almost all the existing logics like, fuzzy logic, intutionistic fuzzy logic etc. After this many reserchers used neutrosophic sets and logic in algebra, such as Kandasamy et al. [2–4], Agboola et al. [5–8], Ali et al. [9–12], Gulistan et al. [13–15]. More recently Smarandache et al. [16,17] introduced the idea of NT group which open a new research direction. Zhang et al. [18], Bal et al. [19], Jaiyeola el al. [20], Gulistan et al. [21] used NT set in different directions.

On the other hand Munkres [22], studied topology in detail. Chang [23] gave the concept of fuzzy topology in 1968. After this further study at fuzzy topology has been done by Thivagar [24], Lowen [25], Sarkar [26] and Palaniappan [27] , Onasanya et al. [28], Shumrani et al. [29]. Sahin et al. [30] presented the fresh idea of NTT.

Thus in this aricle, we further extended the theory of NT topology. We study some basic properties of NTT where we introduce NT base, NT closure and NT subspace and investigate these topological notions. Moreover, as an application, we give a multicriteria decision making for the combining effects of certain enzymes on chosen DNA.

2. Preliminaries

In this section we recall some helpful material from [1,16] and for basics of topology we refer the reader [22].

Definition 1. *[1] A neutrosophic set is of the form*

$$Ħ = \{(\flat, T(\flat), I(\flat), F(\flat)) :: \flat \in U\}$$

where $T, I, F : U \longmapsto]0^-, 1^+[$.

Definition 2. *[16] "Let Ħ be a set together with a binary operation* $\star$*. Then* $Ħ_T$ *is called a NT set if for any* $\flat \in Ħ$*, there exist a neutral of "*$\flat$*" called* $neut(\flat)$*, different from the classical algebraic unitary element, and an opposite of "*$\flat$*" called* $anti(\flat)$*, with* $neut(\flat)$ *and* $anti(\flat)$ *belonging to Ħ, such that:*

$\flat \star neut(\flat) = neut(\flat) \star \flat = \flat$

and

$\flat \star anti(\flat) = anti(\flat) \star \flat = neut(\flat)$."

3. Neutrosophic Triplet Topology (NTT)

In this section, we study NTT in detail.

Definition 3. *[30] Let* $Ħ_T$ *be a NT set and let* $Ħ_\tau$ *be a non-empty subset of* $\mathcal{P}(Ħ_T)$*. If* $Ħ_\tau$ *satisfy the following conditions:*

- $\emptyset$, $Ħ_T$ *in* $Ħ_\tau$,
- *The intersection of a finite number of sets in* $Ħ_\tau$ *is also in* $Ħ_\tau$,
- *The union of an arbitrary number of sets in* $Ħ_\tau$ *is also in* $Ħ_\tau$.

then $Ħ_\tau$ *is called a NTT.*

Remark 1. *The pair* ($Ħ_T$, $Ħ_\tau$) *is called a NT topological space. The elements of* $Ħ_\tau$ *which are subsets of* $Ħ_T$ *are called NT open sets of NT topological space* ($Ħ_T$,$Ħ_\tau$).

Example 1. *Let* $Ħ_T$ *be a NT set of Ħ and* $Ħ_\tau = \{\emptyset, Ħ_T\}$*. Then* $Ħ_\tau$ *is a topology for* $Ħ_T$ *and it is called the NT trivial (or indiscrete) topology.*

Example 2. *Let* $Ħ_T$ *be a NT set of Ħ and* $Ħ_\tau = \mathcal{P}(Ħ_T)$*. Then* τ *is a topology for* $Ħ_T$ *and it is called the NT discrete topology.*

Example 3. *Let* $Ħ_T$ *be a NT set and* $Ħ_\tau$ *be the collection of* $\emptyset$ *and those subsets of* $Ħ_T$ *whose complements are finite. Then* $Ħ_\tau$ *is called the neutrosophic triplet cofinite topology.*

Example 4. *Let* $Ħ = \{\flat_1, \flat_2, \flat_3\}$ *with the binary operation defined by the following table*

$*$	$\flat_1$	$\flat_2$	$\flat_3$
$\flat_1$	$\flat_3$	$\flat_2$	$\flat_1$
$\flat_2$	$\flat_2$	$\flat_2$	$\flat_3$
$\flat_3$	$\flat_1$	$\flat_3$	$\flat_2$

Then $(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2)$ *and* $(\flat_3, \flat_2, \flat_3)$ *are neutrosophic triplets of* $Ħ$*. Let* $Ħ_T = \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$ *be the set of triplets of* $Ħ$*. Then*

$$\mathcal{P}(Ħ_T) = \{\emptyset, \{(\flat_1, \flat_3, \flat_1)\}, \{(\flat_2, \flat_2, \flat_2)\}, \{(\flat_3, \flat_2, \flat_3)\}, \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2)\}, \\ \{(\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}, \{(\flat_1, \flat_3, \flat_1), (\flat_3, \flat_2, \flat_3)\}, Ħ_T\}.$$

Consider the following subsets

$$Ħ_{\tau 1} = \{\emptyset, \{(\flat_1, \flat_3, \flat_1)\}, Ħ_T\},$$
$$Ħ_{\tau 2} = \{\emptyset, \{(\flat_1, \flat_3, \flat_1)\}, \{(\flat_2, \flat_2, \flat_2)\}, Ħ_T\},$$
$$Ħ_{\tau 3} = \{\emptyset, \{(\flat_3, \flat_2, \flat_3)\}, \{(\flat_1, \flat_3, \flat_1)\}, \{(\flat_3, \flat_2, \flat_3), (\flat_1, \flat_3, \flat_1)\}, Ħ_T\}$$

then $Ħ_{\tau 1}$ *and* $Ħ_{\tau 3}$ *are NT topologies while* $Ħ_{\tau 2}$ *is not NTT.*

Definition 4. *Let* $(Ħ_T, Ħ_\tau)$ *be a topological space. A subset* $F \subseteq Ħ_T$ *is said to be NT closed if and only if its complement* $Ħ_T \setminus F$ *is NT open.*

Example 5. *Let* $Ħ_T = \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$ *be as in Example 4 with the NTT* $Ħ_\tau = \{\emptyset, Ħ_T, \{(\flat_3, \flat_2, \flat_3)\}, \{(\flat_1, \flat_3, \flat_1)\}, \{(\flat_3, \flat_2, \flat_3), (\flat_1, \flat_3, \flat_1)\}\}$*. Then the NT closed subsets of* $Ħ_T$ *are*

$$Ħ_T, \emptyset, \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2)\}, \{(\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}, \{(\flat_2, \flat_2, \flat_2)\}.$$

Remark 2. *The NT closed sets of a NT topological space* $(Ħ_T, Ħ_\tau)$ *has the following properties,*

1. $\emptyset$, $Ħ_T$ *are NT closed.*
2. *Finite union of NT closed sets is NT closed set.*
3. *The arbitrary intersection of NT closed sets is a NT closed set.*

Definition 5. *Two NT topologies* $Ħ_{\tau 1}$ *and* $Ħ_{\tau 2}$ *of the NT set* $Ħ_T$ *are said to be comparable if* $Ħ_{\tau 1} \subset Ħ_{\tau 2}$ *or* $Ħ_{\tau 2} \subset Ħ_{\tau 1}$*. Further* $Ħ_{\tau 1}$ *and* $Ħ_{\tau 2}$ *are said to be equal if* $Ħ_{\tau 1} \subset Ħ_{\tau 2}$ *and* $Ħ_{\tau 2} \subset Ħ_{\tau 1}$*. If* $Ħ_{\tau 1} \subset Ħ_{\tau 2}$ *holds, then we say that* $Ħ_{\tau 2}$ *is finer than* $Ħ_{\tau 1}$ *and* $Ħ_{\tau 1}$ *is coarser than* $Ħ_{\tau 2}$*.*

Example 6. *Let* $Ħ_T$ *be a NT set having more than one element as a triplet element then any topology on* $Ħ_T$ *is finer than the NT indiscrete topology on* $Ħ_T$ *and coarser than the NT discrete topology on* $Ħ_T$*.*

The intersection of two NT topologies is always a NTT while the union of two NT topologies is not in general a NTT as shown in the following example.

Example 7. *Let* $Ħ_T = \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$ *be as in Example 4. Consider the two NT topologies*

$$Ħ_{\tau 1} = \{\emptyset, \{(\flat_1, \flat_3, \flat_1)\}, Ħ_T\}$$

$$Ħ_{\tau 2} = \{\emptyset, \{(\flat_2, \flat_2, \flat_2)\}, Ħ_T\}.$$

Then

$$Ħ_{\tau 1} \cup Ħ_\tau = \{\emptyset, \{(\flat_1, \flat_3, \flat_1)\}, \{(\flat_2, \flat_2, \flat_2)\}, Ħ_T\}$$

is not a NTT.

Example 8. *Let* $(Ħ_T, Ħ_\tau)$ *be a NT topological space. If for some* $(\flat_1, \flat_2, \flat_3) \in Ħ_T$ *and* $M \in Ħ_\tau$*, we have* $(\flat_1, \flat_2, \flat_3) \in M$*, we say that* M *is a neighborhood of* $(\flat_1, \flat_2, \flat_3)$*. A set* $L \subseteq Ħ_T$ *is open if and only if for each* $(\flat_1, \flat_2, \flat_3) \in L$ *there exists a neighborhood* $M_{(\flat_1, \flat_2, \flat_3)}$ *of* $(\flat_1, \flat_2, \flat_3)$ *contained in* L*.*

Example 9. *Let $\hbar_T = \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$ be as in Example 4. Consider the following NTT*

$$\hbar_{\tau 1} = \{\emptyset, \{(\flat_1, \flat_3, \flat_1)\}, \hbar_T\}$$

Note that the NT $(\flat_1, \flat_3, \flat_1)$ has two neighborhoods, namely $\{(\flat_1, \flat_3, \flat_1)\}$ and $\hbar_T$ while $\hbar_T$ is the only neighborhood for both $(\flat_2, \flat_2, \flat_2)$ and $(\flat_3, \flat_2, \flat_3)$.

4. Neutrosophic Triplet Bases of Neutrosophic Triplet Topology (NTT)

In this section, we define and study bases of a NTT for generating NT topologies.

Definition 6. *Let $(\hbar_T, \hbar_\tau)$ be a NT topological space. A family $\hbar(\beta) \subset \hbar_\tau$ is called a NT basis (or NT base) for $\hbar_\tau$ if each NT open subset of $\hbar_T$ is the union of members of $\hbar(\beta)$. The members of $\hbar(\beta)$ are called basis open sets of the topology $\hbar_\tau$.*

Example 10. *Let $\hbar_T$ be any NT set. Then the collection of all NT subsets of $\hbar_T$ is a basis for the NT discrete topology on $\hbar_T$.*

Example 11. *Let $\hbar_T = \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$ be as in Example 4 with the NTT*

$$\hbar_\tau = \{\emptyset, \{(\flat_3, \flat_2, \flat_3)\}, \{(\flat_1, \flat_3, \flat_1)\}, \{(\flat_3, \flat_2, \flat_3), (\flat_1, \flat_3, \flat_1)\}, \hbar_T\}.$$

Then $\hbar(\beta) = \{\{(\flat_3, \flat_2, \flat_3)\}, \{(\flat_1, \flat_3, \flat_1)\}, \hbar_T\}$ is a NT basis for $(\hbar_T, \hbar_\tau)$.

Theorem 1. *Let $(\hbar_T, \hbar_\tau)$ be a NT topological space. A family*

$$\hbar(\beta) \subseteq \hbar_\tau$$

is a NT basis for $\hbar_\tau$ if and only if, for each

$$\hbar(O) \in \hbar_\tau$$

and

$$(\flat_o, \flat_o, \flat_o) \in \hbar(O),$$

there is a

$$\hbar(\Im) \in \hbar(\beta)$$

such that

$$(\flat_o, \flat_o, \flat_o) \in \hbar(\Im) \subseteq \hbar(O).$$

Proof. Suppose that $\hbar(\beta)$ is a NT base for NTT τ. By definition each $\hbar(O) \in \hbar_\tau$ is a union of members of $\hbar_\tau$. Let

$$\hbar(O) = \cup\{\hbar(\Im_\alpha) : \hbar(\Im_\alpha) \in \hbar(\beta)\}.$$

If $(\flat_o, \flat_o, \flat_o)$ is an arbitrary NT point of $\hbar(O)$, then $(\flat_o, \flat_o, \flat_o)$ belongs to at least one $\hbar(\Im_\alpha)$ in the union

$$\cup_\alpha \hbar(\Im_\alpha) = \hbar(O).$$

Hence

$$(\flat_o, \flat_o, \flat_o) \in \hbar(\Im_\alpha) \subseteq \cup_\alpha \hbar(\Im_{2\alpha}) = \hbar(O).$$

Thus

$$(\flat_o, \flat_o, \flat_o) \in \hbar(\Im_\alpha) \subseteq \hbar(O).$$

Conversly, suppose that for each

$$(\flat_o, \flat_o, \flat_o) \in Ħ(O),$$

there is a

$$Ħ\left(\Im_{(\flat_o,\flat_o,\flat_o)}\right) \in Ħ(\beta)$$

such that

$$(\flat_o, \flat_o, \flat_o) \in Ħ\left(\Im_{(\flat_o,\flat_o,\flat_o)}\right) \subseteq Ħ(O).$$

Thus

$$\begin{aligned} Ħ(O) &= \cup\{\{(\flat_o, \flat_o, \flat_o)\} : (\flat_o, \flat_o, \flat_o) \in Ħ(O)\} \\ &\subseteq \cup\left\{Ħ\left(\Im_{(\flat_o,\flat_o,\flat_o)}\right) : (\flat_o, \flat_o, \flat_o) \in Ħ(O)\right\} \subseteq Ħ(O). \end{aligned}$$

Therefore

$$Ħ(O) = \cup\left\{Ħ\left(\Im_{(\flat_o,\flat_o,\flat_o)}\right) : (\flat_o, \flat_o, \flat_o) \in Ħ(O)\right\}.$$

Thus $Ħ(O)$ is a union of members of $Ħ(\beta)$ and therefore $Ħ(\beta)$ is a NT bases for τ. □

Theorem 2. *A family $Ħ(\beta)$ of NT subsets of a neutrosophic triplet set(NTS) $Ħ_T$ is a NT bases for some NTT on $Ħ_T$ if and only if the following conditions are satisfied:*

(1) *Each $(\flat_o, \flat_o, \flat_o)$ in $Ħ_T$ is contained in some*

$$Ħ(\Im) \in Ħ(\beta)$$

i.e.,

$$Ħ_T = \cup\{Ħ(\Im) : Ħ(\Im) \in Ħ(\beta)\}.$$

(2) *For any $Ħ(\Im_1)$,$Ħ(\Im_2)$ belonging to $Ħ(\beta)$ the intersection*

$$Ħ(\Im_1) \cap Ħ(\Im_2)$$

is a union of members of $Ħ(\beta)$. Equivalently, for each

$$(\flat_o, \flat_o, \flat_o) \in Ħ(\Im_1) \cap Ħ(\Im_2)$$

there exist a

$$Ħ(\Im_3) \in Ħ(\beta)$$

such that

$$(\flat_o, \flat_o, \flat_o) \in Ħ(\Im_3) \subseteq Ħ(\Im_1) \cap Ħ(\Im_2).$$

Proof. Suppose that a family $Ħ(\beta)$ of a NT subsets of NT set $Ħ_T$ is a NT basis for some NTT on $Ħ_T$.Since $Ħ_T \in Ħ_\tau$ (is open), then by definition of NT basis, $Ħ_T$ can be written as union of members of $Ħ(\beta)$. Now let $Ħ(\Im_1)$, $Ħ(\Im_2)$ be members of $Ħ(\beta)$. Then $Ħ(\Im_1)$, $Ħ(\Im_2)$ are NT sets and so is $Ħ(\Im_1) \cap Ħ(\Im_2)$. By Theorem 1, for each

$$(\flat_o, \flat_o, \flat_o) \in Ħ(\Im_1) \cap Ħ(\Im_2)$$

there is a

$$Ħ(\Im_3) \in Ħ(\beta)$$

such that

$$(\flat_o, \flat_o, \flat_o) \in Ħ(\Im_3) \subseteq Ħ(\Im_1) \cap Ħ(\Im_2).$$

Conversly, Suppose that both conditions (1) and (2) hold. Let Ħ_τ be the family of NT subsets of Ħ_T. Which are obtained by taking union of members of $\text{Ħ}(\beta)$. We claim that Ħ_τ is a NTT on Ħ_T. We need to show that the conditions of NTT are satisfied by the member of Ħ_τ. Let

$$\{\text{Ħ}(O_\alpha) : \alpha \in \Omega\}$$

be a class of members of Ħ_τ. Each $\text{Ħ}(O_\alpha)$ is a union of members of $\text{Ħ}(\beta)$ and so

$$\cup\{\text{Ħ}(O_\alpha) : \alpha \in \Omega\}$$

is also a union of members of $\text{Ħ}(\beta)$. Hence

$$\cup_{\alpha\in\Omega}\text{Ħ}(O_\alpha) \in \text{Ħ}_\tau.$$

Next suppose that

$$\text{Ħ}(O_1), \text{Ħ}(O_2) \in \text{Ħ}_\tau.$$

We shall show that

$$N(O_1) \cap \text{Ħ}(O_2) \in \text{Ħ}_\tau.$$

Let

$$(\flat_o, \flat_o, \flat_o) \in \text{Ħ}(O_1) \cap \text{Ħ}(O_2).$$

There are sets $\text{Ħ}(\Im_1), \text{Ħ}(\Im_2)$ in $\text{Ħ}(\beta)$ such that

$$(\flat_o, \flat_o, \flat_o) \in \text{Ħ}(\Im_1) \subset \text{Ħ}(O_1)$$

and

$$(\flat_o, \flat_o, \flat_o) \in \text{Ħ}(\Im_2) \subset \text{Ħ}(O_2).$$

Let $\text{Ħ}(\Im_{23}) \in \text{Ħ}(\beta)$ be such that

$$(\flat_o, \flat_o, \flat_o) \in \text{Ħ}(\Im_3) \subset \text{Ħ}(\Im_1) \cap \text{Ħ}(\Im_{22}).$$

Then

$$(\flat_o, \flat_o, \flat_o) \in \text{Ħ}(\Im_3) \subset \text{Ħ}(\Im_1) \cap \text{Ħ}(\Im_2) \subset \text{Ħ}(O_1) \cap \text{Ħ}(O_2)$$

which means that

$$\text{Ħ}(O_1) \cap \text{Ħ}(O_2)$$

belong to τ. By (1)

$$\text{Ħ}_T = \cup\{\text{Ħ}(\Im) : \text{Ħ}(\Im) \in \text{Ħ}(\beta)\}$$

So $\text{Ħ}_T \in \text{Ħ}_\tau$. Also, if we take the union of empty class of members of $\text{Ħ}(\beta)$ we note that $\phi \in \text{Ħ}_\tau$. Hence Ħ_τ is a topology on Ħ_T. Since each member of Ħ_τ is a union of members of $\text{Ħ}(\beta)$ by definition, $\text{Ħ}(\beta)$ is a NT basis for Ħ_τ. □

5. Neutrosophic Triplet Closure

In this section, we define NT closure of neutrosophic triplet topological space.

Definition 7. *Let $(\text{Ħ}_T, \tau)$ be a NT topological space and let $\text{Ħ}(\Im)$ be any NT subset of Ħ_T. A NT $(\flat_o, \flat_o, \flat_o) \in \text{Ħ}_T$ is said to be NT adherent to $\text{Ħ}(\Im)$ if each NT neighbourhood of $(\flat_o, \flat_o, \flat_o)$ contain a NT point of $\text{Ħ}(\Im)$*

(which may be $(\flat_o, \flat_o, \flat_o)$ itself). The NT set of all NT points of $\hbar_T$ adherent to $\hbar(\Im)$ is called the NT closure of $\hbar(\Im)$ and is denoted by $\hbar\left(\overline{\Im}\right)$ in symbols,

$$\hbar\left(\overline{\Im}\right) = \left\{(\flat_o, \flat_o, \flat_o) \in \hbar_T : \text{ for all } \hbar_{(\flat_o,\flat_o,\flat_o)}, \ \hbar_{(\flat_o,\flat_o,\flat_o)} \cap \hbar(\Im)\right\} \neq \phi.$$

Equivalently, NT closure of $\hbar(\Im)$ is the smallest NT closed super set of $\hbar(\Im)$. Neutrosophic triplet closure of $\hbar(\Im)$ is denoted by $\overline{\hbar(\Im)}$ or $\hbar\left(\overline{\Im}\right)$.

Remark 3. *It is clear from the definition that $\hbar(\Im) \subset \hbar\left(\overline{\Im}\right)$.*

Example 12. *Let $\hbar_T = \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$ be as in Example 4 with the NTT $\tau = \{\phi, \{(\flat_1, \flat_3, \flat_1)\}, \hbar_T\}$. Let $\hbar(\Im_1) = \{(\flat_1, \flat_3, \flat_1)\}$ and $\hbar(\Im_2) = \{(\flat_2, \flat_2, \flat_2)\}$. We will find $\hbar\left(\overline{\Im_1}\right)$ and $\hbar\left(\overline{\Im_2}\right)$. Since $\hbar(\Im_1) \subset \hbar(\overline{\Im_1})$, we have $(\flat_1, \flat_3, \flat_1) \in \hbar\left(\overline{\Im_1}\right)$.*

Now

$$(\flat_2, \flat_2, \flat_2) \in \hbar_T.$$

Since the only neighborhood of $(\flat_2, \flat_2, \flat_2)$ is $\hbar_T$ and $\hbar_T \cap \hbar(\Im_1) \neq \phi$, we have that $(\flat_2, \flat_2, \flat_2) \in \hbar\left(\overline{\Im_1}\right)$. Similarly, we have that $(\flat_3, \flat_2, \flat_3) \in \hbar(\overline{\Im_1})$. Therefore, $\hbar\left(\overline{\Im_1}\right) = \hbar_T$.

Next we will find $\hbar(\overline{\Im_2})$. Since $\{(\flat_1, \flat_3, \flat_1)\}$ is a neighborhood of $(\flat_1, \flat_3, \flat_1)$ and $\{(\flat_1, \flat_3, \flat_1)\} \cap \hbar(\Im_2) = \phi$, we have that $(\flat_1, \flat_3, \flat_1) \notin \hbar\left(\overline{\Im_2}\right)$. Since the only neighborhood of $(\flat_2, \flat_2, \flat_2)$ is $\hbar_T$ and $\hbar_T \cap \hbar(\Im_2) \neq \phi$, we have $(\flat_2, \flat_2, \flat_2) \in \hbar\left(\overline{\Im_2}\right)$. Similarly, we have that $(\flat_3, \flat_2, \flat_3) \in \hbar\left(\overline{\Im_2}\right)$. Hence, $\hbar\left(\overline{\Im_2}\right) = \{(\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$.

Theorem 3. *$\hbar(\Im)$ is NT closed if and only if $\hbar(\Im) = \hbar\left(\overline{\Im}\right)$.*

Proof. Assume that $\hbar(\Im)$ is a NT closed. Then $\hbar(\Im)$ is a closed set containing $\hbar(\Im)$. Therefore, $\hbar\left(\overline{\Im}\right) \subset \hbar(\Im)$. However, by definition $\hbar(\Im) \subset \hbar\left(\overline{\Im}\right)$. Hence, $\hbar(\Im) = \hbar\left(\overline{\Im}\right)$. Conversely, assume that $\hbar(\Im) = \hbar\left(\overline{\Im}\right)$. Since $\hbar\left(\overline{\Im}\right)$ is the smallest NT superset of $\hbar(\Im)$, so $\hbar\left(\overline{\Im}\right)$ is NT closed, which implies that $\hbar(\Im)$ is NT closed. □

Theorem 4. *Let $(\hbar_T, \hbar_\tau)$ be a NT topological space and let $\hbar(\Im_1)$ and $\hbar(\Im_2)$ be arbitrary NT subsets of $\hbar_T$. Then*

- $\overline{\phi} = \phi$
- $\overline{\hbar_T} = \hbar_T$
- $\overline{\hbar(\Im_1) \cup \hbar(\Im_2)} = \overline{\hbar(\Im_1)} \cup \overline{\hbar(\Im_2)}$
- $\overline{\hbar(\Im_1) \cap \hbar(\Im_2)} \subset \overline{\hbar(\Im_1)} \cap \overline{\hbar(\Im_2)}$
- $\overline{\overline{\hbar(\Im_1)}} = \overline{\hbar(\Im_1)}$
- *If $\hbar(\Im_1) \subset \hbar(\Im_2)$, then $\overline{\hbar(\Im_1)} \subset \overline{\hbar(\Im_2)}$.*

Proof.

(1) It is trivial.

(2) $\hbar_T$ and $\overline{\hbar_T}$ are both closed sets and therefore $\hbar_T = \overline{\hbar_T}$ by Theorem 3.

(3) Let $(\flat_o, \flat_o, \flat_o) \in \overline{Ħ(\Im_1)}$. Then each NT neighbourhood $Ħ_{(\flat_o,\flat_o,\flat_o)}$ of $(\flat_o, \flat_o, \flat_o)$ contains some point of $Ħ(\Im_1)$ and hence $Ħ_{(\flat_o,\flat_o,\flat_o)}$ contains some point of $Ħ(\Im_1 \cup \Im_2)$. Thus $(\flat_o, \flat_o, \flat_o) \in \overline{Ħ(\Im_1 \cup \Im_2)}$.Therefore, $\overline{Ħ(\Im_1)} \subset \overline{Ħ(\Im_1 \cup \Im_2)}$. Similarly, $\overline{Ħ(\Im_2)} \subset \overline{Ħ(\Im_1 \cup \Im_2)}$. Thus

$$\overline{Ħ(\Im_1)} \cup \overline{Ħ(\Im_2)} \subset \overline{Ħ(\Im_1 \cup \Im_2)}.$$

For the converse inclusion, we have , by definition $Ħ(\Im_1) \subset \overline{Ħ(\Im_1)}$ and $Ħ(\Im_2) \subset \overline{Ħ(\Im_2)}$. Therefore

$$Ħ(\Im_1 \cup \Im_2) \subset \overline{Ħ(\Im_1)} \cup \overline{Ħ(\Im_2)}.$$

However, $\overline{Ħ(\Im_1)} \cup \overline{Ħ(\Im_2)}$ is a NT closed set containing $Ħ(\Im_1 \cup \Im_2)$. Hence by Theorem 3 we have

$$\overline{Ħ(\Im_1) \cup Ħ(\Im_2)} = \overline{Ħ(\Im_1)} \cup \overline{Ħ(\Im_2)}.$$

(4) Since $Ħ(\Im_1) \subset \overline{Ħ(\Im_1)}$, and $Ħ(\Im_2) \subset \overline{Ħ(\Im_2)}$ we have

$$Ħ(\Im_1) \cap Ħ(\Im_2) \subset \overline{Ħ(\Im_1)} \cap \overline{Ħ(\Im_2)}.$$

However, $\overline{Ħ(\Im_1)} \cap \overline{Ħ(\Im_2)}$ is a NT closed set and therefore by Theorem 3

$$\begin{aligned} Ħ(\Im_1) \cap Ħ(\Im_2) &\subset \overline{Ħ(\Im_1) \cap Ħ(\Im_2)} \\ &\subset \overline{Ħ(\Im_1)} \cap \overline{Ħ(\Im_2)}. \end{aligned}$$

Implies that

$$\overline{Ħ(\Im_1) \cap Ħ(\Im_2)} \subset \overline{Ħ(\Im_1)} \cap \overline{Ħ(\Im_2)}.$$

(5) We apply Theorem 3 to the NT closed set $\overline{N(\Im_1)}$ to obtain

$$\overline{\overline{Ħ(\Im_1)}} = \overline{Ħ(\Im_1)}.$$

(6) If $Ħ(\Im_1) \subset Ħ(\Im_2)$ then $Ħ(\Im_1) \cup Ħ(\Im_2) = Ħ(\Im_2)$. Taking closures on both sides and applying (3) we have

$$\overline{Ħ(\Im_1)} \cup \overline{Ħ(\Im_2)} = \overline{Ħ(\Im_2)}.$$

Hence, $\overline{Ħ(\Im_1)} \subset \overline{Ħ(\Im_2)}$.

□

Remark 4. *The equality*

$$\overline{Ħ(\Im_1) \cap Ħ(\Im_2)} = \overline{Ħ(\Im_1)} \cap \overline{Ħ(\Im_2)}$$

does not hold in general.

6. Neutrosophic Triplet Subspace

In this section, we define the NT subspace.

Definition 8. *Let $(Ħ_T, Ħ_\tau)$ be a NT topological space and $Ħ(\mathrm{Y}) \subset Ħ_T$, where $Ħ(\mathrm{Y}) \neq \phi$. Then*

$$\tau_{Ħ(\mathrm{Y})} = \{Ħ(V) \cap Ħ(\mathrm{Y}) : Ħ(V) \in Ħ_\tau\}$$

is a NTT on $Ħ(\mathrm{Y})$, called NT subspace topology. Open sets in $Ħ(\mathrm{Y})$ consist of all intersections of open sets of $Ħ_T$ with $Ħ(\mathrm{Y})$.

Let us check that the collection $Ħ_{\tau Ħ(Y)}$ is a NTT on $Ħ(Y)$.

We shall show that $Ħ_{\tau Ħ(Y)}$ satisfies the three properties of a NT topology on $Ħ(Y)$.

T_1: Suppose that

$$Ħ(O_1), Ħ(O_2), ..., Ħ(O_{Ħ})$$

belong to $Ħ_{\tau Ħ(Y)}$ then, there are subsets $Ħ(U_1), Ħ(U_2), ..., Ħ(U_{Ħ})$ of $Ħ_T$ belonging to $Ħ_\tau$ such that

$$Ħ(O_i) = Ħ(Y) \cap Ħ(U_i), \ \ i = 1, 2, ..., n.$$

Now $Ħ(O_1) \cap Ħ(O_2)$ ...

$$\begin{aligned} Ħ(O_n) &= (Ħ(Y) \cap Ħ(U_1)) \cap (Ħ(Y) \cap Ħ(U_2)) ... \cap (Ħ(Y) \cap Ħ(U_n)) \\ &= Ħ(Y) \cap (Ħ(U_1) \cap Ħ(U_2) .... \cap \ Ħ(U_n)) \end{aligned}$$

A NT open set in $Ħ(Y)$, since

$$Ħ(U_1) \cap Ħ(U_2) .... \cap Ħ(U_n) \in Ħ_\tau$$

Hence

$$Ħ(O_1) \cap Ħ(O_2) .... Ħ(O_n) \in \tau_{Ħ(Y)}.$$

This finite intersection of members of $Ħ_{\tau Ħ(Y)}$ is again in $\tau_{Ħ(Y)}$.

T_2: Let $\{Ħ(O_\alpha) : \alpha \in \Omega\}$ be an arbitrary family of members of $Ħ_{\tau Ħ(Y)}$. Then there exist a family $\{U_\alpha : \alpha \in \Omega\}$ of member of $Ħ_\tau$ such that $Ħ(O_\alpha) = Ħ(Y) \cap Ħ(U_\alpha)$ for all $\alpha \in \Omega$. Therefore,

$$\cup_{\alpha \in \Omega} Ħ(O_\alpha) = \cup_{\alpha \in \Omega} (Ħ(Y) \cap Ħ(U_\alpha)) = Ħ(Y) \cap (\cup_{\alpha \in \Omega} U_\alpha)$$

Since $Ħ_\tau$ is a NTT on $Ħ(Y)$.

T, $\cup \{Ħ(U_\alpha) : \alpha \in \Omega\}$ is in τ. Hence

$$Ħ(Y) \cap (\cup_{\alpha \in \Omega} U_\alpha) \in Ħ_{\tau Ħ(Y)}.$$

Thus, $\cup_{\alpha \in \Omega} Ħ(O_\alpha)$ belongs to $\tau_{Ħ(Y)}$. Hence arbitrary union of members of $Ħ_{\tau Ħ(Y)}$ is also in $Ħ_{\tau Ħ(Y)}$.

T_3: $Ħ(Y)$ and ϕ belong to $Ħ_{\tau Ħ(Y)}$ since

$$Ħ(Y) \cap Ħ_T = Ħ(Y)$$

and

$$Ħ(Y) \cap \phi = \phi$$

Hence, $Ħ_{\tau Ħ(Y)}$ is a NTT on $Ħ(Y)$.

Example 13. *Let* $Ħ_T = \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2), (\flat_3, \flat_2, \flat_3)\}$ *be as in Example 4 with the NTT*

$$Ħ_\tau = \{\phi, \{(\flat_1, \flat_3, \flat_1)\}, \{(\flat_2, \flat_2, \flat_2)\}, \{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2)\}, Ħ_T\}$$

and $Ħ(Y) = \{(\flat_1, \flat_3, \flat_1), (\flat_3, \flat_2, \flat_3)\}$

Taking intersection of each member of τ *with* $Ħ(Y)$ *. Then*

$$\phi \cap \text{Ħ}(Y) = \phi$$
$$\{(\flat_1, \flat_3, \flat_1)\} \cap \text{Ħ}(Y) = \{(\flat_1, \flat_3, \flat_1)\}$$
$$\{(\flat_2, \flat_2, \flat_2)\} \cap \text{Ħ}(Y) = \phi$$
$$\{(\flat_1, \flat_3, \flat_1), (\flat_2, \flat_2, \flat_2)\} \cap \text{Ħ}(Y) = \{(\flat_1, \flat_3, \flat_1)\}$$
$$\text{Ħ}_T \cap \text{Ħ}(Y) = \text{Ħ}(Y)$$
$$\tau_{\text{Ħ}(Y)} = \{\phi, \{(\flat_1, \flat_3, \flat_1)\}, \text{Ħ}(Y)\}.$$

7. Applications

In Mathematics, topology is concerned with the properties of space that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing. Like topology, the NTT tells how elements of a set relate spatially to each other in a more comprehensive way using the idea of Neutrosophic triplet sets. It has many application in different disciplines, Biology, Computer science, Physics, Robotics, Games and Puzzles and Fiber art etc. Here we study the application of NTT in Biology.

Suppose that we have a certain type of DNA and we are going to discuss the combine effects of certain enzymes like, $\Im_1, \Im_2, \Im_3$ on chosen DNA using the idea of NT sets. These enzymes cut, twist, and reconnect the DNA, causing knotting with observable effects. Assume the set Ħ$= \{\Im_1, \Im_2, \Im_3\}$ and assume that their mutual effect on each other is shown in the following table

$*$	$\Im_1$	$\Im_2$	$\Im_3$
$\Im_1$	$\Im_3$	$\Im_2$	$\Im_1$
$\Im_2$	$\Im_2$	$\Im_2$	$\Im_3$
$\Im_3$	$\Im_1$	$\Im_3$	$\Im_2$

Then $(\Im_1, \Im_3, \Im_1), (\Im_2, \Im_2, \Im_2)$ and $(\Im_3, \Im_2, \Im_3)$ are neutrosophic triplets of Ħ. Here $(\Im_1, \Im_3, \Im_1)$ means that the enzymes $\Im_1, \Im_3$ play the role of anti and neut of each other, $(\Im_2, \Im_2, \Im_2)$ means that the enzyme $\Im_2$ has no neut and anti and $\Im_1, \Im_3$ are anti and neut of each other in different situations. Let $\text{Ħ}_T = \{(\Im_1, \Im_3, \Im_1), (\Im_2, \Im_2, \Im_2), (\Im_3, \Im_2, \Im_3)\}$ be the set of triplets of Ħ. Then

$$\mathcal{P}(\text{Ħ}_T) = \{\emptyset, \{(\Im_{11}, \Im_3, \Im_1)\}, \{(\Im_2, \Im_2, \Im_2)\}, \{(\Im_3, \Im_2, \Im_3)\}, \{(\Im_1, \Im_3, \Im_1), (\Im_2, \Im_2, \Im_2)\},$$
$$\{(\Im_2, \Im_2, \Im_2), (\Im_3, \Im_2, \Im_3)\}, \{(\Im_1, \Im_3, \Im_1), (\Im_3, \Im_2, \Im_3)\}, \text{Ħ}_T\}.$$

Here $\mathcal{P}(\text{Ħ}_T)$ discuss the all possible outcomes of anti and neut. Consider the following two subsets of $\mathcal{P}(\text{Ħ}_T)$. $\tau_1 = \{\emptyset, \{(\Im_1, \Im_3, \Im_1)\}, \text{Ħ}_T\}$ and $\tau_2 = \{\emptyset, \{(\Im_3, \Im_2, \Im_3)\}, \{(\Im_1, \Im_3, \Im_1)\}, \{(\Im_3, \Im_2, \Im_3), (\Im_1, \Im_3, \Im_1)\}, \text{Ħ}_T\}$. Then τ_1 and τ_2 are NT topologies and stand for the combination of enzymes that effect the DNA. While $\tau_3 = \{\emptyset, \{(\Im_3, \Im_2, \Im_3)\}, \{(\Im_2, \Im_2, \Im_2)\}, \text{Ħ}_T\}$ is not NTT and stands for the combination of enzymes that does not effect the DNA as union of $\{(\Im_3, \Im_2, \Im_3)\}, \{(\Im_2, \Im_2, \Im_2)\}$ does not belongs to τ_3. As τ_1 and τ_2 neutrosophic triplet topologies so $\tau_1 \cap \tau_2 = \tau_1$ and $\tau_1 \cup \tau_2 = \tau_2$ is again a neutrosophic triplets topology which effects the DNA. The NTT $\emptyset$ stands for the combination of enzymes where we can not have any answer while neutrosophic triplet topology $\mathcal{P}(\text{Ħ}_T)$ stands for the strongest case of combination of enzymes which effects the DNA. Now if we want more insight of this problem we may use other concepts like, NT neighborhoods etc.

On the other hand Leonhard Euler demonstrated problem that it was impossible to find a route through the town that would cross each of its seven bridges exactly once. This problem leads us towards the NT graph theory using the concept of NTT as the route does not depend upon the any physical scenario, but it depends upon the spatially connectivity between the bridges.

Similarly to classify the letters correctly and the hairy ball theorem of algebraic topology can be discussed in a more practical way using the concept of NTT.

8. Conclusions

In this article, we used the idea of NTT and introduced some of their properties, such as NT base, NT closure and NT subspace. At the end we discuss an application of multicriteria decision making problem with the help of NTT.

Acknowledgments: This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. KEP-34-130-38. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

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Neutrosophic Fuzzy Matrices and Some Algebraic Operations

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Rakhal Das, Florentin Smarandache, Binod Chandra Tripathy (2020). Neutrosophic Fuzzy Matrices and Some Algebraic Operations. *Neutrosophic Sets and Systems*, 32, 401-409

Abstract: In this article, we study neutrosophic fuzzy set and define the subtraction and multiplication of two rectangular and square neutrosophic fuzzy matrices. Some properties of subtraction, addition and multiplication of these matrices and commutative property, distributive property have been examined.
Keywords: Neutrosophic fuzzy matrix, Neutrosophic set. Commutativity, Distributive, Subtraction of neutrosophic matrices.

1. Introduction

Neutrosophic set was introduced by Florentin Smarandache [1] in 1998, where each element had three associated defining functions, namely the membership function (T), the non-membership (F) function and the indeterminacy function (I) defined on the universe of discourse X, the three functions are completely independent. Relative to the natural problems sometimes one may not be able to decide. After the development of the Neutrosophic set theory, one can easily take decision and indeterminacy function of the set is the nondeterministic part of the situation. The applications of the theory has been found in various field for dealing with indeterminate and inconsistent information in real world one may refer to [2,3,4]. Neutrosophic set is a part of neutrosophy which studied the origin, nature and scope of neutralities, as well as their interactions with ideational spectra. The neutrosophic set generalizes the concept of classical fuzzy set [10, 11], interval valued fuzzy set, intuitionistic fuzzy set and so on. In the recent years, the concept of neutrosophic set has been applied successfully by Broumi et al. [12, 13, 14] and Abdel-Basset et al. [15, 16, 17, 18]

The single-valued neutrosophic number which is a generalization of fuzzy numbers and intuitionistic fuzzy numbers. A single-valued neutrosophic number is simply an ordinary number whose precise value is somewhat uncertain from a philosophical point of view. There are two special forms of single-valued neutrosophic numbers such as single-valued trapezoidal neutrosophic numbers and single-valued triangular neutrosophic numbers.

The neutrosophic interval matrices have been defined by Vasantha Kandasamy and Florentin Smarandache in their book "Fuzzy interval matrices, Neutrosophic interval matrices, and applications". A neutrosophic fuzzy matrix $[a_{ij}]_{n\times m}$, whose entries are of the form $a + Ib$ (neutrosophic number), where a, b are the elements of the interval [0,1] and I is an indeterminate such that $I^n = I$, n being a positive integer.

So the difference between the neutrosophic number of the form $a + Ib$ and the single-valued neutrosophic numbers is that the generalization of fuzzy number and the single-valued neutrosophic components <*T*, *I*, *F*> is the generalization of fuzzy numbers and intuitionistic fuzzy numbers. Since fuzzy number lies between 0 to 1 so the component neutrosophic fuzzy number a and b lies in [0,1]. In the case of single-valued neutrosophic matrix components will be the true value, indeterminacy and fails value with three components in each element of a matrix [3, 4, 8].

We know the important role of matrices in science and technology. However, the classical matrix theory sometimes fails to solve the problems involving uncertainties, occurring in an imprecise environment. Kandasamy and Smarandache [7] introduced fuzzy relational maps and neutrosophic relational maps. Thomason [8], introduced the fuzzy matrices to represent fuzzy relation in a system based on fuzzy set theory and discussed about the convergence of powers of fuzzy matrix. Dhar, Broumi and Smarandache [2] define Square Neutrosophic Fuzzy Matrices whose entries are of the form $a+Ib$, where a and b are fuzzy number from [0, 1] gives the definition of Neutrosophic Fuzzy Matrices multiplication.

In this paper our ambition is to define the subtraction of fuzzy neutrosophic matrices, rectangular fuzzy neutrosophic matrices and study some algebraic properties. We shall focus on all types of neutrosophic fuzzy matrices. The paper unfolds as follows. The next section briefly introduces some definitions related to neutrosophic set, neutrosophic matrices, Fuzzy integral neutrosophic matices and fuzzy matrix. Section 3 presents a new type of fuzzy neutrosophic matrices and investigated some properties such as subtraction, commutative property and distributive property.

2. Materials and Methods (proposed work with more details)

In this section we recall some concepts of neutrosophic set, neutrosophic matrices and fuzzy neutrosophic matrices proposed by Kandasamy and Smarandache in their monograph [3], and also the concept of fuzzy matrix (One may refer to [2])

Definition 2.1 (Smarandache [1]). Let U be an universe of discourse then the neutrosophic set A is an object having the form $A = \{< x{:}T_A(x), I_A(x), F_A(x)>, x \in U\}$, where the functions $T, I, F : U\rightarrow\ ]^-0, 1^+[$ define respectively the degree of membership (or Truthness), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $x \in U$ to the set A with the condition.

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]^-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.2 (Dhar et al. [3]). Let $M_{mxn}= \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$, is a neutrosophic field. We call M_{mxn} to be the neutrosophic matrix.

Example 2.1: Let $R(I) = \langle R \cup I \rangle$be the neutrosophic field

$$M_{4x3} = \begin{pmatrix} 5 & 0 & 2.1I \\ 3.5I & 3 & 5 \\ 7 & 4I & 0 \\ 8 & -5I & I \end{pmatrix}$$

M_{4x3} denotes the neutrosophic matrix, with entries from real and the indeterminacy.

Definition 2.3 (Kandasamy and Smarandache [5])

Let $N = [0, 1] \cup I$ where I is the indeterminacy. The $m \times n$ matrices $M_{mxn} = \{(a_{ij}) : a_{ij} \in [0, 1] \cup I\}$ is called the fuzzy integral neutrosophic matrices. Clearly the class of $m \times n$ matrices is contained in the class of fuzzy integral neutrosophic matrices.

The row vector $1 \times n$ and column vector $m \times 1$ are the fuzzy neutrosophic row matrices and fuzzy neutrosophic column matrices respectively.

Example 2.2: Let $M_{4x3} = \begin{pmatrix} 0.5 & 0 & 0.1I \\ I & 0.3 & 0.5 \\ 0.7 & 0.4I & 0 \\ 0.8 & 0.5I & I \end{pmatrix}$ be a 4 ×3 integral fuzzy neutrosophic matrix

Definition 2.5 (Kandasamy and Smarandache [5]).

Let $N_s = [0, 1] \cup \{bI : b \in [0, 1]\}$; we call the set N_s to be the fuzzy neutrosophic set. Let N_s be the fuzzy neutrosophic set. $M_{mxn} = \{(a_{ij}): a_{ij} \in N_s\ i= 1$ to m and $j = 1$ to $n\}$ we call the matrices with entries from N_s to be the fuzzy neutrosophic matrices.

Example 2.3: Let $N_s = [0,1] \cup \{bI: b \in [0,1]\}$ be the fuzzy neutrosophic set and

$$P = \begin{pmatrix} 0.5 & 0 & 0.1I \\ I & 0.3 & 0.5 \\ 0 & I & 0.01 \end{pmatrix}$$

be a 3 ×3 fuzzy neutrosophic matrix.

Definition 2.6 (Thomas [9]). A fuzzy matrix is a matrix which has its elements from the interval [0, 1], called the unit fuzzy interval. A_{mxn} fuzzy matrix for which $m = n$ (i.e. the number of rows is equal to the number of columns) and whose elements belong to the unit interval [0, 1] is called a fuzzy square matrix of order n. A fuzzy square matrix of order two is expressed in the following way

$A= \begin{pmatrix} x & y \\ t & z \end{pmatrix}$,

where the entries x, y, t, z all belongs to the interval [0,1].

Definition 2.7 (Kandasamy and Smarandache [5]). Let A be a neutrosophic fuzzy matrix, whose entries is of the form $a + Ib$ (neutrosophic number), where a, b are the elements of [0,1] and I is an indeterminate such that $I^n = I$, n being a positive integer.

$$A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}$$

Definition 2.8 Multiplication Operation of two Neutrosophic Fuzzy Matrices

Consider two neutrosophic fuzzy matrices, whose entries are of the form $a + Ib$ (neutrosophic number), where a, b are the elements of [0,1] and I is an indeterminate such that $I^n = I$, n being a positive integer, given by

$$A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}, \quad B = \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$$

The Multiplication Operation of two Neutrosophic Fuzzy Matrices is given by

$$AB = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}\begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$$

$$D_{11} = [\max\{ \min(x_1, m_1), \min(x_2, m_3)\} + I \max\{ \min\{(y_1, n_1), \min(y_2, n_3)\}]$$

$$D_{21} = [\max \{\min(x_1, m_2), \min (x_2, m_4)\} + I \max \{\min(y_1, n_2), \min(y_2, n_4)\}]$$

$$D_{21} = [\max \{\min\{(x_3, m_1), \min (x_4, m_3)\} + I \max \{\min\{(y_3, n_1), \min (y_4, n_3)\}]$$

$$D_{22} = [\max \{\min\{(x_3, m_2), \min (x_4, m_4)\} + I \max\{ \min\{(y_3, n_2), \min (y_4, n_4)\}]$$

Hence, $AB = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$.

3. Results (examples / case studies related to the proposed work)

In this section we define the subtraction and distributive property of neutrosophic fuzzy matrices along with some properties associated with such matrices.

3.1 Subtraction Operation of two Neutrosophic Fuzzy Matrices

Consider two neutrosophic fuzzy matrices given by

$$A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \\ x_5 + Iy_5 & x_6 + Iy_6 \end{pmatrix}$$

and $$B = \begin{pmatrix} t_1 + Iz_1 & t_2 + Iz_2 \\ t_3 + Iz_3 & t_4 + Iz_4 \\ t_5 + Iz_5 & t_6 + Iz_6 \end{pmatrix}.$$

Addition and multiplication between two neutrosophic fuzzy matrices have been defined in Smarandache [2]. We would like to define the subtraction of these two matrices as follows.

$A - B = C$,

where c_{ij} are as follows

$c_{11} = \min\{x_1, t_1\} + I \min\{y_1, z_1\}$

$c_{12} = \min\{x_2, t_2\} + I \min\{y_2, z_2\}$

$c_{21} = \min\{x_3, t_3\} + I \min\{y_3, z_3\}$

$c_{21} = \min\{x_4, t_4\} + I \min\{y_4, z_4\}$

$c_{31} = \min\{x_5, t_5\} + I \min\{y_5, z_5\}$

$c_{32} = \min\{x_6, t_6\} + I \min\{y_6, z_6\}$

Since $\min\{a, b\} = \min\{b, a\}$ so based on this we have the following properties.

Proposition 3.1. The following properties hold in the case of neutrosophic fuzzy matrix for subtraction

(*i*) $A-B = B-A$

(*ii*) $(A - B) - C = A - (B - C) = (B- C) – A = (C – B) – A$.

Proof. Consider three neutrosophic fuzzy matrices A, B and C as follows.

$$A = \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix},\ B = \begin{pmatrix} c_{11}+d_{11}I & c_{12}+d_{12}I \\ c_{21}+d_{21}I & c_{22}+d_{22}I \\ c_{31}+d_{31}I & c_{32}+d_{32}I \end{pmatrix}$$

and $$C = \begin{pmatrix} l_{11}+m_{11}I & l_{12}+m_{12}I \\ l_{21}+m_{21}I & l_{22}+m_{22}I \\ l_{31}+m_{31}I & l_{32}+m_{32}I \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix} - \begin{pmatrix} c_{11}+d_{11}I & c_{12}+d_{12}I \\ c_{21}+d_{21}I & c_{22}+d_{22}I \\ c_{31}+d_{31}I & c_{32}+d_{32}I \end{pmatrix} = D \text{ (say)},$$

where,

$D_{11} = \min\{a_{11}, c_{11}\}+I\min\{b_{11}, d_{11}\} = x_{11} + Iy_{11}$

$D_{12} = \min\{a_{12}, c_{12}\}+I\min\{b_{12}, d_{12}\} = x_{12} + Iy_{12}$

$D_{21} = \min\{a_{21}, c_{21}\}+I\min\{b_{21}, d_{21}\}= x_{21} + Iy_{21}$

$D_{22} = \min\{a_{22}, c_{22}\}+I\min\{b_{22}, d_{22}\} = x_{22} + Iy_{22}$

$D_{31} = \min\{a_{31}, c_{31}\}+I\min\{b_{31}, d_{31}\} = x_{31} + Iy_{31}$

$D_{32} = \min\{a_{32}, c_{32}\}+I\min\{b_{32}, d_{32}\} = x_{32} + Iy_{32}$

$$D = \begin{pmatrix} x_{11}+Iy_{11} & x_{12}+Iy_{12} \\ x_{21}+Iy_{21} & x_{22}+Iy_{22} \\ x_{31}+Iy_{31} & x_{32}+Iy_{32} \end{pmatrix} \text{ and } B - A = \begin{pmatrix} x_{11}+Iy_{11} & x_{12}+Iy_{12} \\ x_{21}+Iy_{21} & x_{22}+Iy_{22} \\ x_{31}+Iy_{31} & x_{32}+Iy_{32} \end{pmatrix} = D,$$

[$\because$ $\min(a, c) = \min(c, a)$]

Hence, $A – B = B – A$.

Now we have,

$D – C = (A – B) – C$

$$= \begin{pmatrix} x_{11}+Iy_{11} & x_{12}+Iy_{12} \\ x_{21}+Iy_{21} & x_{22}+Iy_{22} \\ x_{31}+Iy_{31} & x_{32}+Iy_{32} \end{pmatrix} - \begin{pmatrix} l_{11}+m_{11}I & l_{12}+m_{12}I \\ l_{21}+m_{21}I & l_{22}+m_{22}I \\ l_{31}+m_{31}I & l_{32}+m_{32}I \end{pmatrix}$$

$= F$ (say),

where,

F_{11} = min{x_{11}, l_{11}}+Imin{y_{11}, m_{11}} = min{a_{11}, c_{11}, l_{11}}+Imin{b_{11}, d_{11}, m_{11}} = $n_{11} + Ik_{11}$

F_{12} = min{x_{12}, l_{12}}+Imin{y_{12}, m_{12}} = min{a_{12}, c_{12}, l_{12}}+Imin{b_{11}, d_{12}, m_{12}} = $n_{12} + Ik_{12}$

F_{21} = min{x_{21}, l_{21}}+Imin{y_{21}, m_{21}} = min{a_{21}, c_{21}, l_{21}}+Imin{b_{21}, d_{21}, m_{21}}= $n_{21} + Ik_{21}$

F_{22} = min{x_{22}, l_{22}}+Imin{y_{22}, m_{22}}= min{a_{22}, c_{22}, l_{22}}+Imin{b_{22}, d_{22}, m_{22}}= $n_{22} + Ik_{22}$

F_{31} = min{x_{31}, l_{31}}+Imin{y_{31}, m_{31}} = min{a_{31}, c_{31}, l_{31}}+Imin{b_{31}, d_{31}, m_{31}}= $n_{31} + Ik_{31}$

F_{32} = min{x_{32}, l_{32}}+Imin{y_{32}, m_{32}} = min{a_{31}, c_{31}, l_{31}}+Imin{b_{31}, d_{31}, m_{31}}= $n_{32} + Ik_{32}$

$$(A-B)-C = F = \begin{pmatrix} n_{11}+Ik_{11} & n_{12}+Ik_{12} \\ n_{21}+Ik_{21} & n_{22}+Ik_{22} \\ n_{31}+Ik_{31} & n_{32}+Ik_{32} \end{pmatrix}.$$

Next we have,

$$B-C = \begin{pmatrix} c_{11}+d_{11}I & c_{12}+d_{12}I \\ c_{21}+d_{21}I & c_{22}+d_{22}I \\ c_{31}+d_{31}I & c_{32}+d_{32}I \end{pmatrix} - \begin{pmatrix} l_{11}+m_{11}I & l_{12}+m_{12}I \\ l_{21}+m_{21}I & l_{22}+m_{22}I \\ l_{31}+m_{31}I & l_{32}+m_{32}I \end{pmatrix} = E \text{ (say)},$$

where

E_{11} = min{c_{11}, l_{11}}+Imin{d_{11}, m_{11}} = $p_{11} + Iq_{11}$

E_{12} = min{c_{12}, l_{12}}+Imin{d_{12}, m_{12}} = $p_{12} + Iq_{12}$

E_{21} = min{c_{21}, l_{21}}+Imin{d_{21}, m_{21}} = $p_{21} + Iq_{21}$

E_{22} = min{c_{22}, l_{22}}+Imin{d_{22}, m_{22}}= $p_{22} + Iq_{22}$

E_{31} = min{c_{31}, l_{31}}+Imin{d_{31}, m_{31}} = $p_{31} + Iq_{31}$

E_{32} = min{c_{32}, l_{32}}+Imin{d_{32}, m_{32}} = $p_{32} + Iq_{32}$.

We have

$$B-C = E = \begin{pmatrix} p_{11}+Iq_{11} & p_{12}+Iq_{12} \\ p_{21}+Iq_{21} & p_{22}+Iq_{22} \\ p_{31}+Iq_{31} & p_{32}+Iq_{32} \end{pmatrix}$$

$$A-(B-C) = \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix} - \begin{pmatrix} p_{11}+Iq_{11} & p_{12}+Iq_{12} \\ p_{21}+Iq_{21} & p_{22}+Iq_{22} \\ p_{31}+Iq_{31} & p_{32}+Iq_{32} \end{pmatrix},$$

where

min{a_{11}, p_{11}}+Imin{b_{11}, q_{11}} = min{a_{11}, c_{11}, l_{11}}+Imin{b_{11}, d_{11}, m_{11}}

min{a_{12}, p_{12}}+*I*min{b_{12}, q_{12}} = min{a_{12}, c_{12}, l_{12}}+*I*min{b_{11}, d_{12}, m_{12}}

min{a_{21}, p_{21}}+*I*min{b_{21}, q_{21}} = min{a_{21}, c_{21}, l_{21}}+*I*min{b_{21}, d_{21}, m_{21}}

min{a_{22}, p_{22}}+*I*min{b_{22}, q_{22}}= min{a_{22}, c_{22}, l_{22}}+*I*min{b_{22}, d_{22}, m_{22}}

min{a_{31}, p_{31}}+*I*min{b_{31}, q_{31}} = min{a_{31}, c_{31}, l_{31}}+*I*min{b_{31}, d_{31}, m_{31}}

min{a_{32}, p_{32}}+*I*min{b_{32}, q_{32}} = min{a_{31}, c_{31}, l_{31}}+*I*min{b_{31}, d_{31}, m_{31}}

$$F = \begin{pmatrix} n_{11}+Ik_{11} & n_{12}+Ik_{12} \\ n_{21}+Ik_{21} & n_{22}+Ik_{22} \\ n_{31}+Ik_{31} & n_{32}+Ik_{32} \end{pmatrix}$$

Therefore, $A-(B-C) = F = (A-B)-C$.

3.2 Identity element for subtraction

In the group theory under the operation "*" the identity element I_N of a set is an element such that $I_N * A = A * I_N = A$.

Specially the identity element of neutrosophic set is $I_N = \{[a_{ij}+b_{ij}I]_{m\times n}: \ a_{ij} = 1 = b_{ij}$ for all $i, j\}$.

Result 3.1. For a neutrosophic fuzzy matrix, I_N is the identity matrix for subtraction.

Let $A=\begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix}$, and $I_N = \begin{pmatrix} 1+I & 1+I \\ 1+I & 1+I \\ 1+I & 1+I \end{pmatrix}$ be the neutrosophic identity matrix of order 3x2.

Then we have the following

$$A - I_N = \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix} - \begin{pmatrix} 1+I & 1+I \\ 1+I & 1+I \\ 1+I & 1+I \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix} = I_N - A = A,$$

where

min{a_{11}, 1}+Imin{b_{11},1} = $a_{11}+b_{11}I$

min{a_{12}, 1}+Imin{b_{12},1}= $a_{12}+b_{12}I$

min{a_{21}, 1}+Imin{b_{21},1} = $a_{21}+b_{21}I$

min{a_{22}, 1}+Imin{b_{22},1} = $a_{22}+b_{22}I$

min{a_{31}, 1}+Imin{b_{31},1} = $a_{31}+b_{31}I$

min{a_{32}, 1}+Imin{b_{32},1} = $a_{32}+b_{32}I$

3.3 Identity element for addition

In neutrosophic matrix addition we can define a identity element I_N such that $I_N = \{[a_{ij}+b_{ij}I]_{mxn}: \ a_{ij}=0 = b_{ij}$ for all $i, j\}$

Let $A = \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix}$, and $I_N = \begin{pmatrix} 0+0I & 0+0I \\ 0+0I & 0+0I \\ 0+0I & 0+0I \end{pmatrix}$ be the neutrosophic identity matrix of order 3x2.

Then we have the following

$$A - I_N = \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix} - \begin{pmatrix} 1+I & 1+I \\ 1+I & 1+I \\ 1+I & 1+I \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \\ a_{31}+b_{31}I & a_{32}+b_{32}I \end{pmatrix}$$

$$= I_N - A = A,$$

where

max{a_{11}, 0}+Imax{b_{11},0} = $a_{11}+b_{11}I$
max{a_{12}, 0}+Imax{b_{12},0}= $a_{12}+b_{12}I$
max{a_{21}, 0}+Imax{b_{21},0} = $a_{21}+b_{21}I$
max{a_{22}, 0}+Imax{b_{22},0} = $a_{22}+b_{22}I$
max{a_{31}, 0}+Imax{b_{31},0} = $a_{31}+b_{31}I$
max{a_{32}, 0}+Imax{b_{32},0} = $a_{32}+b_{32}I$.

Result 3.2. The neutrosophic set forms a groupoid, semigroup, monaid and is commutative under the neutrosophic matrix operation of subtraction. The distributive law also holds for subtraction, i.e. $A(B-C) = AB - AC$.

Result 3.3. The neutrosophic set forms a groupoid, semigroup, monaid and commutative under the operation of addition. The distributive law also holds for addition, i.e. $A(B+C) = AB + AC$.

Thus we have, $A(B \pm C) = AB \pm AC$.

4. Applications

The formation of neutrosophic group structure, neutrosophic matrix set and algebraic structure on this set, the results are applicable.

5. Conclusions

In this paper we have established some neutrosophic algebraic property, and subtraction operation addition and multiplication of these matrices and commutative property, distributive property had been examine. This result can be applied further application of neutrosophic fuzzy matric theory. For the development of neutrosophic group and its algebraic property the results of this paper would be helpful.

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Neutrosophic Quadruple Algebraic Codes over Z_2 and their Properties

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W.B. Vasantha Kandasamy, Ilanthenral Kandasamy, Florentin Smarandache (2020). Neutrosophic Quadruple Algebraic Codes over Z_2 and their Properties. *Neutrosophic Sets and Systems*, 33, 169-182

Abstract. In this paper we for the first time develop, define and describe a new class of algebraic codes using Neutrosophic Quadruples which uses the notion of known value, and three unknown triplets (T, I, F) where T is the truth value, I is the indeterminate and F is the false value. Using this Neutrosophic Quadruples several researchers have built groups, NQ-semigroups, NQ-vector spaces and NQ-linear algebras. However, so far NQ algebraic codes have not been developed or defined. These NQ-codes have some peculiar properties like the number of message symbols are always fixed as 4-tuples, that is why we call them as Neutrosophic Quadruple codes. Here only the check symbols can vary according to the wishes of the researchers. Further we find conditions for two NQ-Algebraic codewords to be orthogonal. In this paper we study these NQ codes only over the field Z_2. However, it can be carried out as a matter of routine in case of any field Z_p of characteristics p.

Keywords: Neutrosophic Quadruples; NQ-vector spaces; NQ-groups; Neutrosophic Quadruple Algebraic codes (NQ-algebraic codes); Dual NQ-algebraic codes; orthogonal NQ- algebraic codes; NQ generator matrix; parity check matrix; self dual NQ algebraic codes

1. Introduction

Neutrosophic Quadruples (NQ) was introduced by Smarandache [1] in 2015, it assigns a value to known part in addition to the truth, indeterminate and false values, it happens to be very interesting and innovative. NQ numbers was first introduced by [1] and algebraic operations like addition, subtraction and multiplication were defined. Neutrosophic Quadruple algebraic structures where studied in [2]. Smarandache and et al introduced Neutrosophic triplet groups, modal logic Hedge algebras in [3, 4]. Zhang and et al in [5–7] defined and described Neutrosophic duplet semigroup and triplet loops and strong AG(1, 1) loops. In [8–12], various structures like Neutrosophic triplet and neutrosophic rings application to mathematical modelling, classical group of neutrosophic triplets on $\{Z_{2p}, \times\}$ and neutrosophic duplets in neutrosophic rings were developed and analyzed.

Algebraic structures of neutrosophic duplets and triplets like quasi neutrosophic triplet loops, AG-groupoids, extended triplet groups and NT-subgroups were studied in [7, 13, 16, 17]. Various types of refined neutrosophic sets were introduced, developed and applied to real world problems by [18–24]. In 2015, [18] has obtained several algebraic structures on refined Neutrosophic sets. Neutrosophy has found immense applications in [25–28]. Neutrosophic algebraic structures in general were studied in [29–32]. The algebraic structure of Neutrosophic Quadruples, such as groups, monoids, ideals, BCI-algebras, BCI-positive implicative ideals, hyper structures and BCK/BCI algebras have been developed recently and studied in [34–39]. In 2016 [33] have developed some algebraic structures using Neutrosophic Quadruples $(NQ, +)$ groups and $(NQ, .)$ monoids and scalar multiplication on Neutrosophic Quadruples. [41] have recently developed the notion of NQ vector spaces over R(reals) (or Complex numbers C or Z_p the field of characteristic p, p a prime). They have also defined NQ dual vector subspaces and proved all these NQ-vectors though are distinctly different, yet they are of dimension 4.

The main aim of this paper is to introduce Neutrosophic Quadruple (NQ) algebraic codes over Z_2. (However it can be extended for any Z_p, p a prime). Any NQ codeword is an ordered quadruple with four message symbols which can be a real or complex value, truth value, indeterminate or complex value and the check symbols are combinations of these four elements. We have built a new class of NQ algebraic codes which can measure the four aspects of any code word.

The proposed work is important for Neutrosophic codes have been studied Neutrosophic codes have been studied by [42] but it has the limitations for it could involve only the indeterminacy present and not all the four factors which are present in Neutrosophic Quadruple codes. Hence when the codes are endowed with all the four features it would give in general a better result of detecting the problems while transmission takes place.

It is to be recalled any classical code gives us only the approximately received code word. However the degrees of truth or false or indeterminacy present in the correctness of the received code word is never studied. So our approach would not only be novel and innovative but give a better result when used in real channels.

The main objective of this study is to assess the quality of the received codeword for the received code word may be partially indeterminate or partially false or all the four, we can by this method assess the presence of these factors and accordingly go for re-transmission or rejection.

Hexi codes were defined in [43,44] which uses 16 symbols, 0 to 9 and A to F. Likewise these NQ codes uses the symbols 0, 1, T, I and F.

This paper is organized into six sections. Section one is introductory in nature. Basic concepts needed to make this paper a self-contained one is given in section two. Neutrosophic Quadruple algebraic codes (NQ-codes) are introduced and some interesting properties about them are given in section three. Section four defines the new notion of special orthogonal NQ codes using the inner product of two NQ codewords. The uses of NQ codes and comparison with classical linear algebraic codes are carried out in section five. The final section gives the conclusions based on our study.

2. **Basic Concepts**

In this section we first give the basic properties about the NQ algebraic structures needed for this study. Secondly we give some fundamental properties associated with algebraic codes in general. For NQ algebraic structures refer [29,33].

Definition 2.1. A Neutrosophic quadruple number is of the form (x, yT, zI, wF) where T, I, F are the usual truth value, indeterminate value and the false value respectively and $x, y, z, w \in Z_p$ (or R or C). The set NQ is defined by $NQ = \{(x, yT, zI, wF)|x, y, z, w \in R$ (or Z_p or C); p a prime$\}$ is defined as the Neutrosophic set of quadruple numbers.

A Neutrosophic quadruple number (x, yT, zI, wF) represents any entity or concept which may be a number an idea etc., x is called the known part and (yT, zI, wF) is called the unknown part. Addition, subtraction and scalar multiplication are defined in [33] in the following way. Let $x = (x_1, x_2T, x_3I, x_4F)$ and $y = (y_1, y_2T, y_3I, y_4F) \in NQ$.

$x + y = (x_1 + y_1, (x_2 + y_2)T, (x_3 + y_3)I, (x_4 + y_4)F)$

$x - y = (x_1 - y_1, (x_2 - y_2)T, (x_3 - y_3)I, (x_4 - y_4)F)$

For any $a \in R$ (or C or Z_p) and $x = (x_1, x_2T, x_3I, x_4F)$ where $a \in R$ (or C or Z_P) will be known as scalars and $x \in NQ$ the scalar product of a with x in defined by

$$a.x = a(x_1, x_2T, x_3I, x_4F)$$
$$= (ax_1, ax_2T, ax_3I, ax_4F).$$

If $a = 0$ then $a.x = (0, 0, 0, 0)$. $(0, 0, 0, 0)$ is the additive identity in $(NQ, +)$. For every $x \in NQ$ there exists a unique element $-x = (-x_1, -x_2T, -x_3I, -x_4F)$, in NQ such that $x + (-x) = (0, 0, 0, 0)$. x is called the additive inverse of $-x$ and vice versa.

Finally for $a, b \in C$ (or R or Z_p) and $x, y, \in NQ$ we have $(a + b).x = a.x + b.x$ and $(a \times b).x = a \times (b.x); a(x + y) = a.x + a.y$.

These properties are essential for us to build NQ-algebraic codes.

We use the following results; proofs of which can be had form [33].

Theorem 2.2. $(NQ,+)$ *is an abelian group.*

[33] defines product of any pair of elements $x,y \in NQ$ as follows. Let $x = (x_1, x_2T, x_3I, x_4F)$ and $y = (y_1, y_2T, y_3I, y_4F) \in NQ$.

$$x.y = (x_1, x_2T, x_3I, x_4F).(y_1, y_2T, y_3I, y_4F)$$
$$(x_1y_1, (x_1y_2 + x_2y_1 + x_2y_2)T,$$
$$(x_1y_3 + x_2y_3 + x_3y_1 + x_3y_2 + x_3y_3)I,$$
$$(x_1y_4 + x_2y_4 + x_3y_4 + x_4y_4 + x_4y_1 + x_4y_2 + x_4y_3)F).$$

Theorem 2.3. $(NQ,.)$ *is a commutative monoid.*

Now we just recall some of the properties associated with basic algebraic codes.

Through out this paper Z_2 will denote the finite field of characteristic two. V a finite dimensional vector space over $F = Z_2$ [40].

We call a n-tuple to be $C = C(n,k)$ codeword if C has k message symbols and $n-k$ check symbols. For $c = (c_1, c_2, \ldots, c_k, c_{k+1}, \ldots c_n)$ where $(c_1, c_2, \ldots, c_k) \in V$ (dimension of V over Z_2) and $c_{k+1}, \ldots, c_n$ are check symbols calculated using the $(c_1, c_2, \ldots, c_k) \in V$. To basically generate the code words we use the concept of generator matrix denoted by G and G is a $k \times n$ matrix with entries from Z_2 and to evaluate the correctness of the received codeword we use the parity check matrix H, which is a $n-k \times n$ matrix with entries from Z_2. We in this paper use only the standard form of the generator matrix and parity check matrix for any $C(n,k)$ code of length n with k message symbols. The standard form of the generator matrix G for an $C(n,k)$ code is as follows:

$$G = (I_k, -A^T)$$

where I_k is a $k \times k$ identity matrix and $-A^T$ is a $k \times n-k$ matrix with entries from Z_2. Here the standard form of the parity check matrix $H = (A, I_{n-k})$ where A is a $n-k \times k$ matrix with entries fromZ_2 and I_{n-k} is the $n-k \times n-k$ identity matrix. We have $GH^T = (0)$. In this paper, we use both the generator matrix and the parity check matrix of a NQ code to be only in the standard form.

3. Definition of NQ algebraic codes and their properties

In this section we proceed on to define the new class of algebraic codes called Neutrosophic Quadruple algebraic codes (NQ-algebraic codes) using the NQ vector spaces over the finite field Z_2. We have defined NQ vector spaces over Z_2 in [41].

$$NQ = \{(a, bT, cI, dF)|a, b, c, d \in Z_2\}$$

under + is an abliean group.

Now we proceed on to define $\times$ on NQ. Let

$$x = x_1 + x_2T + x_3I + x_4F$$

and

$$y = y_1 + y_2T + y_3I + y_4F$$

where $x_i, y_i \in R$ or C or Z_p (p a prime) and T, I and F satisfy the following table for product $\times$.

$\times$	T	I	F	0
T	T	0	0	0
I	0	I	0	0
F	0	0	F	0
0	0	0	0	0

So the set $\{T, I, F, 0\}$ under product is an idempotent semigroup. now we find

$$x \times y = (x_1 + x_2T + x_3I + x_4F) \times (y_1 + y_2T + y_3I + y_4F)$$

$$= x_1y_1 + (y_1x_2 + x_1y_2 + x_2y_2)T + (x_3y_1 + y_3x_1 + x_3y_3)I + (x_1y_4 + y_1x_4 + x_4y_4)F \in NQ$$

$\{NQ, \times\}$ is a semigroup which is commutative.

In this section we introduce the new notion of algebraic codes using the set NQ which is a group under '+'

$NQ = \{(0\ 0\ 0\ 0),\ (1\ 0\ 0\ 0),\ (0\ T\ 0\ 0),\ (0\ 0\ I\ 0),\ (0\ 0\ 0\ F),\ (1\ T\ 0\ 0),\ (1\ 0\ I\ 0),\ (1\ 0\ 0\ F),$
$(0\ T\ I\ 0),\ (0\ T\ 0\ F),\ (0\ 0\ I\ F),\ (1\ T\ I\ 0),\ (1\ T\ 0\ F),\ (1\ 0\ I\ F),\ (0\ T\ I\ F),\ (1\ T\ I\ F)\ \};$

$\{NQ, +\}$ is a NQ vector space over $Z_2 = \{0, 1\}$. NQ coding comprises of transforming a block of message symbols in NQ into a NQ code word $a_1a_2a_3a_4x_5x_6 \ldots x_n$, where $a_1a_2a_3a_4 \in NQ$ that is $a_1a_2a_3a_4 = (a_1a_2a_3a_4) \in NQ$ is a quadruple and $x_5, x_6, \ldots, x_n$ belongs to the set $T = \{a + bT + cI + dF/a, b, c, d$ takes its values from $Z_2 = \{0, 1\}\}$. The first four terms $a_1a_2a_3a_4$ symbols are always the message symbols taken from NQ and the remaining $n-4$ are the check symbols or the control symbols which are from T.

In this paper NQ codewords will be written as $a_1a_2a_3a_4x_5x_6x_7 \ldots x_n$, where $(a_1a_2a_3a_4) \in NQ$ and $x_i \in T, 4 < i \leq n$. The check symbols can be obtained from the NQ message symbols in such a way that the NQ code words $a = (a_1a_2a_3a_4)$ satisfy the system of linear equations $Ha^T = (0)$, where H is the $n-4 \times n$ parity check matrix in the standard form with elements from Z_2. Throughout this paper we assume $H = (A, I_{n-4})$, with A, a $n-4 \times 4$ matrix and I_{n-4} the $n-4 \times n-4$ identity matrix with entries from Z_2.

The matrix $G = (I_{4\times 4}, -A^T)$ is called the canonical generator matrix of the linear $(n, 4)$ NQ code with parity check matrix $H = (A, I_{n-4})$.

We use only standard form of the generator matrix and parity check matrix to generate the NQ-codewords for general matrix of appropriate order will not serve the purpose which is a limitation in this case.

We provide some examples of a HQ linear algebraic code.

Example 3.1. Let $C(7, 4)$ be a NQ code of length 7. G be the NQ generator matrix of the $(7, 4)NQ$ code.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

G takes the entries from Z_2, over which the NQ vector space is defined and the message symbols are from NQ. Consider the set of NQ message symbols, $P = \{$(0 0 0 0), (0 T 0 0), (0 0 I 0), (0 0 0 F), (0 0 I F), (0 T I F), (1 0 I F), (0 T I 0), (0 T 0 F)$\} \subseteq NQ$. We now give the NQ code words of

$C(7, 4) = \{$ (0 0 0 0 0 0 0), (0 T 0 0 0 T 0), (0 0 0 F 0 0 F), (0 0 I 0 I 0 0), (0 0 I F I 0 F), (0 T I F I T F), (1 0 I F $1+$ I I F) (0 T I 0 I T 0), (0 T 0 F 0 T F) $\}$ which are associated with $P \subseteq$ NQ. The NQ parity check matrix associated with this generator matrix G is as follows;

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

It is easily verified $Hx^t = (0)$; for all NQ code words $x \in C(7, 4)$. Suppose one receives a NQ code word $y = (0\ I\ 0\ T\ I\ 0\ 0)$; how to find out if the received NQ code word y is a correct one or not. For this we find out Hy^t, if $Hy^t = (0)$, then y is a correct code word; if $Hy^t \neq (0)$, then some error has occurred during transmission. Clearly $Hy^t \neq (0)$. Thus y is not a correct NQ code word.

How to correct it? These NQ code behave differently as these codewords, which is a $1 \times n$ row matrix does not take the values from Z_2, but from NQ and T; message symbols from NQ and check symbols from T. Hence, we cannot use the classical method of coset leader method for error correction, however we use the parity check matrix for error detection.

We have to adopt a special method to find the corrected version of the received NQ code word which has error.

Here we describe the procedure for error correction which is carried out in three steps; Suppose y is the received NQ code word;

(1) We first find Hy^t, if Hy^t is zero no error; on the other hand if Hy^t is not zero there is error so we go to step two for correction.

(2) Now consider the NQ received code word with error. We observe and correct only the first four component in the y that is we correct the message symbols; if the first component is 1 or 0 then it is accepted as the correct component in y; if on the other hand the first component is T (or I or F) and if 1 has occurred in the rest of any three components then replace T (or I or F) by one if 1 has not occurred in the 2nd or 3rd or 4th component replace the first component by 0.

Now observe the second component if it is T accept, if not T but 0 or 1 or I or F, then replace by zero if T has not occurred in the first or third or fourth place. If T has occurred in any of the 3 other components replace it by T. Next observe the third component if it is I accept else replace by I if I has occurred as first or second or fourth component. If in none of the first four places I has occurred, then fill the third place by zero. Now observe the fourth component if it is F accept it, if not replace by 0 if in none of the other places F has occurred or by F if F has occurred in first or second or third place, now the message word is in NQ by this procedure. If the corrected NQ code word z of y is such that $Hz^t = (0)$ then accept it if not we go for the next step. We check only for the correctness of the message symbols.

(3) For check symbols we use the table of codewords or check matrix H and find the check symbols.

Table of NQ codewords related to $P \subset NQ$ given in example 2.

TABLE 1. Table of NQ codewords related to P

Sno	Message symbols in P	NQ Codeword
1	(0 0 0 0)	(0 0 0 0 0 0 0)
2	(0 T 0 0)	(0 T 0 0 0 T 0)
3	(0 0 I 0)	(0 0 I 0 I 0 0)
4	(0 0 0 F)	(0 0 0 F 0 0 F)
5	(0 0 I F)	(0 0 I F I 0 F)
6	(0 T I F)	(0 T I F I T F)
7	(0 T I 0)	(0 T I 0 I T 0)
8	(0 T 0 F)	(0 T 0 F 0 T F)
9	(1 0 I F)	(1 0 I F 1+I 1 F)

We provide one example of the codeword given in Example 2.1. Let $y = (I\ 1\ F\ 0\ 1{+}I\ 1\ F)$, we see Hy^t is not zero, so we have found the error hence we proceed to next step. We see first component cannot be I so replace I by 1 for 1 has occurred as second component. As second component cannot be one we see in none of the four components T has occurred so we replace 1 by zero. In the second place. Third component is F which is incorrect so we replace it by I as I has occurred in the first place. We observe the fourth component it can be 0 or F; 0 only in case F has not occurred in the first three places but F has occurred as the third component so we replace the zero of the fourth component by F. So the corrected message symbol is (1 0 I F). In step three we check from the table of codes the check symbols and the check symbols matches with the check symbols of the corrected message symbols so we take this as the corrected version of corrected code word as (1 0 I F I+I 1 F).

We give the definition of the procedure.

Definition 3.2. Let C(n, 4) be a NQ code of length n defined over Z_2. The message symbols are always from the set NQ; whatever be n there are only 16 codewords only check symbols increase and not the message symbol length, for it is always four. If $y = (A_1\ A_2\ A_3\ A_4\ a_5\ a_6\ a_7 \ldots a_n)$ is a received NQ codeword and it has some error, then we define the rearrangement technique of error correction in the message symbols $A_1\ A_2\ A_3\ A_4$ only, where if $A_1\ A_2\ A_3\ A_4$ is to be in NQ then A_1 can only values 1 or 0, A_2 can take values 0 or T; A_3 can take values 0 or I and A_4 can take values 0 or F. If this is taken care of the message symbol will be correct and will be in NQ.

If not the following rearrangement process is carried out;

Observe if A_1 is different from 0 or 1 then see values in the 2nd, third and the fourth components if 1 has occurred in any one of them replace the first component by 1, if 1 has not occurred in any one of the four components fill the first component by zero. Now go for the second component A_2 if A_2 is T then it is correct ;if not and 1 or 0 or I or F has occurred and T has occurred in any one of the other three places replace the second component by T; if T has not occurred as any one of the four components replace the second component by 0. Inspect the third component if it is I then it is correct, if not I and if T or 0 or 1 or F has occurred and I has occurred in any of the four components replace the third component by I, if I has failed to occur in any of the four places replace the third component by zero. Now for the fourth component if it is F it is correct, if not and if F has occurred in any one of the other three components replace it by F, if not by zero. After this arrangement certainly the message symbols will be in NQ.

This method of getting the correct code word is defined as the rearrangement technique.

4. **Orthogonal NQ codes and special orthogonal NQ codes**

In this section we define the notion of orthogonality of two HQ code words and the special orthogonal HQ code words and suggest some open problems in this direction in the last section of this paper. Now we define first inner product on the NQ code words of the NQ algebraic code $C(n, 4)$ defined over Z_2.

Definition 4.1. Let $C(n, 4)$ be a NQ code of length n defined over Z_2. Let $x = (A_1\ A_2\ A_3\ A_4\ a_5\ a_6\ a_7\ \ldots\ a_n)$ and $y = (B_1\ B_2\ B_3\ B_4\ b_5\ b_6\ b_7 \ldots b_n)$ be any two NQ code words from $C(n, 4)$, where $A_i, B_i \in NQ$, $i = 1, 2, 3, 4$ and $a_j, b_j \in T$; $j = 5, 6, \ldots n$. We define the dot product of x and y as follows:

$$x.y = A_1 \times B_1 + A_2 \times B_2 + A_3 \times B_3 + A_4 \times B_4 + a_5 \times b_5 + \ldots + a_n \times b_n$$

.

If $x.y = 0$ then we say the two NQ codes words are orthogonal or dual with each other.

Example 4.2. Let $C(6, 4)$ be a NQ code of length 4 defined over Z_2; with associated generated matrix G in the standard form with entries from Z_2 given in the following:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The $C(6, 4)$ NQ code words generated by G is as follows; $C(6, 4) = \{$(0 0 0 0 0 0), (1 0 0 0 1 0), (0 T 0 0 0 T), (0 0 I 0 I I), (0 0 0 F F 0), (1 T 0 0 1 T), (1 0 I 0 1+I I), (1 0 0 F F + 1 0), (0 T I 0 I T + I), (0 T 0 F F T), (0 0 I F I+F I), (1 T I 0 1 +I T + I), (1 T 0 F 1+ F T), (1 0 I F 1 +I +F I),(0 T I F I + F I +T), (1 T I F 1 + I + F I + T)$\}$.

We see (0 0 0 0 0 0) is orthogonal with every other NQ code word in the NQ code (6, 4). Consider the NQ code word (1 0 0 0 1 0) in C (6, 4), NQ code words orthogonal to (1 0 0 0 1 0) are $\{$(1 0 0 0 1 0), (0 0 0 0 0 0), (0 T 0 0 0 T), (1 T 0 0 1 T) $\}$. The NQ codes orthogonal to (0 T 0 0 0 T) are given by

$\{$(0 0 0 0 0 0), (0 T 0 0 0 T), (1 0 0 0 1 0), (0 0 I 0 I I), (0 0 0 F F 0), (1 T 0 0 1 T), (1 0 0 F 1 + F 0), (0 T I 0 1 + I T + I), (1 0 I 0 1 +I I), (0 T 0 F F T), (0 0 I F I + F I), (1 I T 0 1 +I T + I), (1 T 0 F 1 + F T), (1 0 I F 1 + I + F I), (0 T I F I + F T + I), (1 T I F 1 + T +F T + I) $\} = C(6, 4)$.

Thus every element in $C(6, 4)$ is orthogonal with (0 T 0 0 0 T). However (1 0 0 0 0 1) is not orthogonal with every element in $C(6, 4)$. We call all those NQ codes words which are orthogonal to every code word in $C(6, 4)$ including it as the special orthogonal NQ code. A NQ code word which is orthogonal to itself is defined as the self orthogonal NQ code word.

We define them in the following;

Definition 4.3. Let C(n, 4) be a NQ code of length n. We say a NQ code word is self orthogonal if x . x = 0 for x in C(n, 4). A NQ code word x in C(n, 4) is defined as a special orthogonal NQ code word if x is self orthogonal and x is orthogonal with every NQ code word in C(n, 4). (0 0 0 ...0) is a trivial special NQ code word.

We give yet another example of a NQ code which has NQ special orthogonal code word.

Example 4.4. Let C(7, 4) be a NQ code word of length 7. Let G be the associated generator matrix of the NQ code C.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

It is easily verified that only the NQ code word (0 0 I 0 0 0 I) in C is the special orthogonal NQ code word. We have yet another extreme case where every NQ code word in that NQ code is a special orthogonal NQ code word.

We give examples of them.

Example 4.5. Let C(8, 4) be a NQ code generated by the following generator matrix G

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It is easily verified every NQ code word in $C(8,4)$ is a special orthogonal NQ code word. We call such NQ codes as special self orthogonal NQ code or self orthogonal NQ code.

Definition 4.6. Let $C = C(n,4)$ be a NQ code word defined over Z_2. We define C to be a NQ special self orthogonal code if every NQ code word in C is a special orthogonal NQ code word of C.

5. Uses of NQ codes and comparison of NQ codes with classical linear algebraic codes

NQ codes are best suited for data transmission where one does not require security. They are also very useful in data storage for one can easily retrieve the data even i f the data i s corrupted. The disadvantage of these NQ codes i s that they always have a fixed number of message symbols namely four. They are not compatible i n channels were one needs security. The only flexibility i s one can have any number of check symbols. NQ codes are entirely different from the classical linear algebraic code ; for these code words take the message symbols from NQ and the check symbols from T where as the later take their values from Z_2 (or Z_p).

Classical linear algebraic codes takes its code words from Z_p, p a prime or more commonly from Z_2; and are defined over Z_p or Z_2; but in case of NQ codes the code words take their values from NQ for message symbols and from T for their check symbols which is a big difference as we can only use the standard form of the generator matrix and the parity check matrix, in this case also both the matrices take their values from Z_2 (or Z_p) only. The similarity is both the codes take the entries of the matrices from the finite field over which they are defined. All NQ codes are only of a fixed form that is they can have only 4 message symbols from NQ, but the classical codes can have any value from 1 to m , m ¡ n, which is a major difference between the two class of codes. Both NQ codes and the classical linear code use parity matrix to detect the error in the received code word, that is error detection procedure for both of them is the same. For error correction we have to adopt a special technique of rearrangement of the message symbols once an error is detected in the received NQ code word, as the coset leader method of error correction cannot be carried out as the NQ code words do not belong to the field over which the NQ code words are defined.

6. Conclusions

In this paper for the first time we have defined the new class of codes called NQ codes which are distinctly different from the classical algebraic linear codes. All these NQ codes can have only fixed number of message symbols viz four. NQ codes are of the form $C(n, 4)$, n can vary from 5 to any finite integer. We have defined orthogonality of these NQ codes. This has lead us to define NQ special orthogonal code word and NQ special orthogonal codes. We suggest the following problems:

(1) Prove or disprove all NQ codes have a non trivial code word which is orthogonal to all codes in C (n, 4).
(2) Characterize all NQ codes C (n, 4) which are NQ special orthogonal codes.

For future research we would be defining super NQ structures and NQ codes over Z_p, p an odd prime. Also application of these codes can be done in case of Hexi codes [43] in McEliece Public Key crypto-systems [44] and in coding applications like T-Direct codes [45] and multi covering radius with rank metric [46].

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How we can extend the standard deviation notion with neutrosophic interval and quadruple neutrosophic numbers

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V. Christianto, F. Smarandache, Muhammad Aslam (2020). How we can extend the standard deviation notion with neutrosophic interval and quadruple neutrosophic numbers. *International Journal of Neutrosophic Science,* 2(2), 72-76; DOI: 10.5281/zenodo.3728218

Abstract

During scientific demonstrating of genuine specialized framework we can meet any sort and rate model vulnerability. Its reasons can be incognizance of modelers or information mistake. In this way, characterization of vulnerabilities, as for their sources, recognizes aleatory and epistemic ones. The aleatory vulnerability is an inalienable information variety related with the researched framework or its condition. Epistemic one is a vulnerability that is because of an absence of information on amounts or procedures of the framework or the earth [7]. Right now, we examine fourfold neutrosophic numbers and their potential application for practical displaying of physical frameworks, particularly in the unwavering quality evaluation of engineering structures. Contribution: we propose to extend the notion of standard deviation to by using symbolic quadruple operator.

Keywords: Standard deviation, Neutrosophic Interval, Quadruple Neutrosophic Numbers.

1.Introduction

We all know about uncertainty modelling of various systems, which usually is represented by:

$$X = x' + 1.64s \tag{1}$$

Or

$$X = x' + 1.96s \tag{2}$$

Here, the constants 1.64 or 1.96 can be replaced with k. What we mean is a constant corresponding to bell curve, the number is usually assumed to be 1.96 for 95% acceptance, or 1.64 for 90% acceptance, respectively.

But since s only takes account statistical uncertainty, there is lack of measure for indeterminacy. That is why we suggest to extend from

$$X = x' + k.\ s \tag{3}$$

To become neutrosophic quadruple numbers.

Before we move to next section, first we would mention other possibility, i.e. by expressing the relation as follow

$$(X_L + X_U\ I_N) = k.\ (\sigma_L + \sigma_U\ I_N), \text{ where } I_N \text{ is a measure of indeterminacy} \tag{4}$$

Actually, we we need to add some results for various I_N, for example I_N=0,0.1,0.2,0.3,0.4 etc. Nonetheless, because this paper is merely suggesting a conceptual framework, we don't explore it further here. Interested readers are suggested to consult ref. [1-2].

2. A short review on quaternions

We all know the quaternions, but quadruple neutrosophic numbers are different. In quaternions, a+bi + cj + dk you have i^2 = j^2 = k^2 = -1 = ijk, while on quadruple neutrosophic numbers we have:[3]

$$N = a + bT + cI + dF \text{ one has: } T^2 = T,\ I^2 = I,\ F^2 = F, \tag{5}$$

where a = known part of N, bT+cI+dF = unknown part of N, with T = degree of truth-membership, I = degree of indeterminate-membership, and F = degree of false-membership, and a, b, c, d are real (or complex) numbers, and an absorption law defined depending on expert and on application (so it varies); if we consider for example the neutrosophic order T > I > F, then the stronger absorbs the weaker, i.e.

$$TI = T,\ TF = T, \text{ and } IF = I,\ TIF = T. \tag{6}$$

Other orders can also be employed, for example T < I < F: (see book [1], at page 186.) Other interpretations can be given to T, I, F upon each application.

3. Application: statistical uncertainty and beyond

Designers must arrangement with dangers and vulnerabilities as a piece of their expert work and, specifically, vulnerabilities are intrinsic to building models. Models assume a focal job in designing. Models regularly speak to a dynamic and admired rendition of the scientific properties of an objective. Utilizing models, specialists can explore and gain comprehension of how an article or wonder will perform under specified conditions.[8]

Furthermore, according to Murphy & Gardoni & Harris Jr, which can be rephrased as follows: "For engineers, managing danger and vulnerability is a significant piece of their expert work. Vulnerabilities are associated with understanding the normal world, for example, knowing whether a specific occasion will happen, and in knowing the presentation of building works, for example, the conduct and reaction of a structure or foundation, the fluctuation in material properties (e.g., attributes of soil, steel, or solid), geometry, and outer limit conditions (e.g., loads or physical limitations). Such vulnerabilities produce dangers. In the standard record chance is the result of a lot of potential outcomes and their related probabilities of event (Kaplan and Gerrick 1981), where the probabilities measure the probability of event of the potential outcomes considering the hidden vulnerabilities. One significant utilization of models in designing danger investigation is to measure the probability or likelihood of the event of specific occasions or a lot of outcomes. Such models are regularly alluded to as probabilistic models to feature their specific capacity to represent and measure vulnerabilities."[8]

Uncertainties come in many forms, for example:

> "The uncertainties in developing a model are:
>
> • Model Inexactness. This kind of vulnerability emerges when approximations are presented in the plan of a model. There are two basic issues that may emerge: blunder as the model (e.g., a straight articulation is utilized when the real connection is nonlinear), and missing factors (i.e., the model contains just a subset of the factors that influence the amount of intrigue). …
>
> • Mistaken Assumptions. Models depend on a series of expectations. Vulnerabilities may be related with the legitimacy of such suspicions (e.g., issues emerge when a model accept typicality or homoskedasticity when these suppositions are disregarded).
>
> • Measurement Error. The parameters in a model are commonly aligned utilizing an example of the deliberate amounts of intrigue and the fundamental factors considered in the model. These watched qualities, in any case, could be inaccurate because of blunders in the estimation gadgets or systems, which at that point prompts mistakes in the alignment procedure. …
>
> • Statistical Uncertainty. Factual vulnerability emerges from the scantiness of information used to align a model. Specifically, the exactness of one's derivations relies upon the perception test size. The littler the example size, the bigger is the vulnerability in the evaluated estimations of the parameters. … However, the confidence in the model would probably increment on the off chance that it was adjusted utilizing one thousand examples. The factual vulnerability catches our level of confidence in a model considering the information used to adjust the model."[8]

With regards to statistical uncertainty, according to Ditlevsen and Madsen, which can rephrased as follows: "It is the reason for any estimating technique to produce data about an amount identified with the object of estimation. In the event that the amount is of a fluctuating nature with the goal that it requires a probabilistic model for its depiction, the estimating technique must make it conceivable to define quantitative data about the parameters of the picked probabilistic model. Clearly a deliberate estimation of a solitary result of a non-degenerate arbitrary variable X just is sufficient for giving a rough gauge of the mean estimation of X and is insufficient for giving any data about the standard deviation of X. In any case, if an example of X is given, that is, whenever estimated estimations of a specific number of freely produced results of X are given, these qualities can be utilized for figuring gauges for all parameters of the model. The reasons that such an estimation from an example of X is conceivable and bodes well are to be found in the numerical likelihood hypothesis. The most rudimentary ideas and rules of the hypothesis of insights are thought to be known to the peruser. To delineate the job of the measurable ideas in the unwavering quality examination it is beneficial to rehash the most fundamental highlights of the depiction of the data that an example of X of size n contains about the mean worth E[X]. It is sufficient for our motivation to make the streamlining supposition that X has a known standard deviation D[X] = σ."[5]

Now, it seems possible to extend it further to include not only statistical uncertainty but also modelling error etc. It can be a good application of Quadruple Neutrosophic Numbers.

4. Towards an improved model of standard deviation

Few days ago, we just got an idea regarding application of symbolic Neutrosophic quadruple numbers, where we can use it to extend the notion of *standard deviation.*

As we know usually people wrote:

$$X' = x + k.\sigma \tag{7}$$

Where X mean observation, σ standard deviation, and k is usually a constant to be determined by statistical bell curve, for example 1.64 for 95% accuracy.

We can extend it by using symbolic quadruple operator:

$$X' = x \pm (k.\sigma + m.i + n.f) \tag{8}$$

Where X' stands for actual prediction from a set of observed x data, σ is standard deviation, i is indeterminacy and f falsefood. That way modelling error (falsehood) and indeterminacy can be accounted for.

Alternatively, one can write a better expression:

$$X' = x \pm (T.\sigma + I.\sigma + F.\sigma) \tag{9}$$

where T = the truth degree of s (standard deviation), I = degree of indeterminacy about s, and F = degree of falsehood about s.

A slightly more general expression is the following:

$$X' = x \pm a\,(T.\sigma + I.\sigma + F.\sigma) \tag{10}$$

where T = the truth degree of s (standard deviation), I = degree of indeterminacy about s, and F = degree of falsehood about s.

Or

$$X' = x \pm (a.T.\sigma + b.I.\sigma + c.F.\sigma) \tag{11}$$

where T = the truth degree of s (standard deviation), I = degree of indeterminacy about s, and F = degree of falsehood about s, and a, b, c are constants to be determined.

That way we reintroduce quadruple Neutrosophic numbers into the whole of statistics estimate.

For further use in engineering fields especially in reliability methods, readers can consult [5-7].

5. Conclusion

In this paper, we reviewed existing use of standard deviation in various fields of science including engineering, and then we consider a plausible extension of standad deviation based on the notion of quadruple neutrosophic numbers. More investigation is recommended.

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Some Results on Single Valued Neutrosophic Hypergroup

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S. Rajareega, D. Preethi, J. Vimala, Ganeshsree Selvachandran, Florentin Smarandache (2020). Some Results on Single Valued Neutrosophic Hypergroup. *Neutrosophic Sets and Systems*, 31, 80-85

Abstract: We introduced the theory of Single valued neutrosophic hypergroup as the initial theory of single valued neutrosophic hyper algebra and also developed some results on single valued neutrosophic hypergroup.

Keywords: Hypergroup; Level sets; Single valued neutrosophic sets; Single valued neutrosophic hypergroup.

1. Introduction

Florentin Smarandache introduced Neutrosophic sets in 1998 [16], which is the generalization of the intuitionistic fuzzy sets. In some real time situations, decision makers faced some difficulties with uncertainty and inconsistency values. Neutrosophic sets helped the decision makers to deal with uncertainty values. Abdel-Basset et.al. used neutrosophic concept in real life decision-making problems [1-7]. The concept of single valued neutrosophic set was introduced by Wang. et. al [17].

As a generalization of classical algebraic structure, Algebraic hyper structure was introduced by F. Marty [11]. Corsini and Leoreanu-Fotea developed the applications of hyper structure [9]. Algebraic hyperstructures has many applications in fuzzy sets, lattices, artificial intelligence, automation, combinatorics. Corsini introduced hypergroup theory [8]. After while the hyperstructure theory has seen broader applications in many fields. Some of the recent works on hyperstructures related to vague soft groups, vague soft rings and vague soft ideals can be found in [12, 13].

In this paper we develop the theory of single valued neutrosophic hypergroup and also established some results on single valued neutrosophic hypergroup.

2. Preliminaries

Definition 2.1 [17] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminancy-membership function I_A and a falsity-membership function F_A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$.

$T_A: X \to]0^-, 1^+[$

$I_A: X \to]0^-, 1^+[$

$F_A: X \to]0^-, 1^+[$

There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.

Definition 2.2 [17] Let X be a space of points (objects),with a generic element of X denoted by x. A single valued neutrosophic set (SVNS) A in X is characterized by T_A, I_A and F_A. For each point x in X, $T_A, I_A, F_A \in [0,1]$.

Definition 2.3 [17] The complement of a SVNS A is denoted by c(A) and is defined by

$T_{c(A)}(x) = F_A(x)$

$I_{c(A)}(x) = 1 - I_A(x)$

$F_{c(A)}(x) = T_A(x)$, for all x in X.

Definition 2.4 [17] A SVNS A is contained in the other SVNS B, $A \subseteq B$, if and only if,

$T_A(x) \leq T_B(x)$

$I_A(x) \geq I_B(x)$

$F_A(x) \geq F_B(x)$, for all x in X.

Definition 2.5 [17] The union of two SVNS s A and B is a SVNS C, written as $C = A \cup B$, whose truth, indeterminancy and falsity-membership functions are defined by,

$T_C(x) = \max(T_A(x), T_B(x))$

$I_C(x) = \min(I_A(x), I_B(x))$

$F_C(x) = \min(F_A(x), F_B(x))$, for all x in X.

Definition 2.6 [17] The intersection of two SVNS s A and B is a SVNS C, written as $C = A \cap B$, whose truth, indeterminancy and falsity-membership functions are defined by,

$T_C(x) = \min(T_A(x), T_B(x))$

$I_C(x) = \max(I_A(x), I_B(x))$

$F_C(x) = \max(F_A(x), F_B(x))$, for all x in X.

Definition 2.7 [17] The falsity-favorite of a SVNS B, written as $B\nabla A$, whose truth and falsity-membership functions are defined by

$T_B(x) = T_A(x)$

$I_B(x) = 0$

$F_B(x) = \min\{F_A(x) + I_A(x), 1\}$, for all x in X.

Definition 2.8 [13] A hypergroup $\langle H, \circ \rangle$ is a set H equipped with an associative hyperoperation $(\circ): H \times H \to P(H)$ which satisfies $x \circ H = H \circ x = H$ for all $x \in H$ (Reproduction axiom)

Definition 2.9 [13] A hyperstructure $\langle H, \circ \rangle$ is called an H_v-group if the following axioms hold:

(i) $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$ for all $x, y, z \in H$,

(ii) $x \circ H = H \circ x = H$ for all $x \in H$.

If $\langle H, \circ \rangle$ only satisfies (i), then $\langle H, \circ \rangle$ is called a H_v- semigroup.

Definition 2.10 [13] A subset K of H is called a subhypergroup if $\langle K, \circ \rangle$ is a hypergroup of $\langle H, \circ \rangle$.

3. Single Valued Neutrosophic Hypergroup.

Throughout this section H denotes the hypergroup $< H, \circ >$

Definition 3.1 *Let $\mathcal{A}$ be a single valued neutrosophic set over H. Then $\mathcal{A}$ is called a single valued neutrosophic hypergroup over H, if the following conditions are satisfied* (i) $\forall\ p, q \in H$, $min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \leq inf\{T_{\mathcal{A}}(r):\ r \in p \circ q\}$,

$max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\} \geq sup\{I_{\mathcal{A}}(r):\ r \in p \circ q\}$ *and*

$$max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} \geq sup\{F_{\mathcal{A}}(r):\ r \in p \circ q\}$$

$(ii)\ \forall\ l, p \in H, there\ exists\ q \in H\ such\ that\ p \in l \circ q\ and$

$$min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(p)\} \leq T_{\mathcal{A}}(q),$$
$$max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(p)\} \geq I_{\mathcal{A}}(q)\ and$$
$$max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(p)\} \geq F_{\mathcal{A}}(q)$$

$(iii)\ \forall\ l, p \in H, there\ exists\ r \in H\ such\ that\ p \in r \circ l\ and$

$$min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(p)\} \leq T_{\mathcal{A}}(r),$$
$$max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(p)\} \geq I_{\mathcal{A}}(r)\ and$$
$$max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(p)\} \geq F_{\mathcal{A}}(r)$$

If $\mathcal{A}$ satisfies condition (i) then $\mathcal{A}$ is a single valued neutrosophic semihypergroup over H. Condition (ii) and (iii) represent the left and right reproduction axioms respectively. Then $\mathcal{A}$ is a single valued neutrosophic subhypergroup of H.

Example 3.2 If the family of t-level sets of SVNS $\mathcal{A}$ over H

$\mathcal{A}_{\mathrm{t}} = \{\mathrm{p} \in \mathrm{H}\ |\ \mathrm{T}_{\mathcal{A}}(\mathrm{p}) \geq \mathrm{t}, \mathrm{I}_{\mathcal{A}}(\mathrm{p}) \leq \mathrm{t}\ \text{and}\ \mathrm{F}_{\mathcal{A}}(\mathrm{p}) \leq \mathrm{t}\}$ is a subhypergroup of H then,

$\mathcal{A}$ is a single valued neutrosophic hypergroup over H.

Theorem 3.3 Let $\mathcal{A}$ be a SVNS over H. Then $\mathcal{A}$ is a single valued neutrosophic hypergroup over H iff $\mathcal{A}$ is a single valued neutrosophic semihypergroup over H and also $\mathcal{A}$ satisfies the left and right reproduction axioms.

Proof. The proof is obvious from Definition: 3.1

Theorem 3.4 Let $\mathcal{A}$ be a SVNS over H. If $\mathcal{A}$ is a single valued neutrosophic hypergroup over H ,then $\forall\ \mathrm{t} \in [0,1]\ \mathcal{A}_{\mathrm{t}} \neq \emptyset$ is a subhypergroup of H.

Proof. Let $\mathcal{A}$ be a single valued neutrosophic hypergroup over H and let $\mathrm{p}, \mathrm{q} \in \mathcal{A}_{\mathrm{t}}$, then

$\mathrm{T}_{\mathcal{A}}(\mathrm{p}), \mathrm{T}_{\mathcal{A}}(\mathrm{q}) \geq \mathrm{t}, \mathrm{I}_{\mathcal{A}}(\mathrm{p}), \mathrm{I}_{\mathcal{A}}(\mathrm{q}) \leq \mathrm{t}$ and $\mathrm{F}_{\mathcal{A}}(\mathrm{p}), \mathrm{F}_{\mathcal{A}}(\mathrm{q}) \leq \mathrm{t}$.

Then we have,

$\inf\{\mathrm{T}_{\mathcal{A}}(\mathrm{r}): \mathrm{r} \in \mathrm{p} \circ \mathrm{q}\} \geq \min\{\mathrm{T}_{\mathcal{A}}(\mathrm{p}), \mathrm{T}_{\mathcal{A}}(\mathrm{q})\} \geq \min\{\mathrm{t}, \mathrm{t}\} = \mathrm{t}$

$\sup\{\mathrm{I}_{\mathcal{A}}(\mathrm{r}): \mathrm{r} \in \mathrm{p} \circ \mathrm{q}\} \leq \mathrm{t}$ and

$\sup\{\mathrm{F}_{\mathcal{A}}(\mathrm{r}): \mathrm{r} \in \mathrm{p} \circ \mathrm{q}\} \leq \mathrm{t}$

This implies $\mathrm{r} \in \mathcal{A}_{\mathrm{t}}$. Then $\forall\ \mathrm{r} \in \mathrm{p} \circ \mathrm{q}\ , \mathrm{p} \circ \mathrm{q} \subseteq \mathcal{A}_{\mathrm{t}}$.

Thus $\forall \mathrm{r} \in \mathcal{A}_{\mathrm{t}}$, we obtain $\mathrm{r} \circ \mathcal{A}_{\mathrm{t}} \subseteq \mathcal{A}_{\mathrm{t}}$

Now, Let $\mathrm{l}, \mathrm{p} \in \mathcal{A}_{\mathrm{t}}$, then there exist $\mathrm{q} \in \mathrm{H}$ such that $\mathrm{p} \in \mathrm{l} \circ \mathrm{q}$ and

$\{\mathrm{T}_{\mathcal{A}}(\mathrm{q})\} \geq \min\{\mathrm{T}_{\mathcal{A}}(\mathrm{l}), \mathrm{T}_{\mathcal{A}}(\mathrm{p})\} \geq \min\{\mathrm{t}, \mathrm{t}\} = \mathrm{t}$

$\{\mathrm{I}_{\mathcal{A}}(\mathrm{q})\} \leq \mathrm{t}$ and

$\{\mathrm{F}_{\mathcal{A}}(\mathrm{q})\} \leq \mathrm{t}$. This implies $\mathrm{q} \in \mathcal{A}_{\mathrm{t}}$

This proves that $\mathcal{A}_{\mathrm{t}} \subseteq \mathrm{r} \circ \mathcal{A}_{\mathrm{t}}$. As such $\mathcal{A}_{\mathrm{t}} = \mathrm{r} \circ \mathcal{A}_{\mathrm{t}}$

Which proves that $\mathcal{A}_{\mathrm{t}}$ is a subhypergroup of H.

Theorem 3.5 Let $\mathcal{A}$ be a SVNS over H. Then the following are equivalent,

(i) $\mathcal{A}$ is a single valued neutrosophic hypergroup over H

(ii) $\forall\ \mathrm{t} \in [0,1]\ \mathcal{A}_{\mathrm{t}} \neq \emptyset$ is a subhypergroup of H.

Proof. (i) $\Rightarrow$ (ii) The proof is obvious from Theorem : 3.4.

(ii) ⇒ (i) Now assume that $\mathcal{A}_t$ is a subhypergroup of H.

Let $p, q \in \mathcal{A}_{t_0}$ and let $\min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} = \max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\} = \max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} = t_0$

Since $p \circ q \subseteq \mathcal{A}_{t_0}$, then for every $r \in p \circ q$, $T_{\mathcal{A}}(r) \geq t_0$, $I_{\mathcal{A}}(r) \leq t_0$, $F_{\mathcal{A}}(r) \leq t_0$

$$\min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \leq \inf\{T_{\mathcal{A}}(r):\ r \in p \circ q\},$$
$$\max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\} \geq \sup\{I_{\mathcal{A}}(r):\ r \in p \circ q\} \text{ and}$$
$$\max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} \geq \sup\{F_{\mathcal{A}}(r):\ r \in p \circ q\}$$

Condition (i) is verified.

Next, let $l, p \in \mathcal{A}_{t_1}$, for every $t_1 \in [0,1]$ and

let $\min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(q)\} = \max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(p)\} = \max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(q)\} = t_1$

Then there exist $q \in \mathcal{A}_{t_1}$ such that $p \in l \circ q \subseteq \mathcal{A}_{t_1}$. Since $q \in \mathcal{A}_{t_1}$,

$$T_{\mathcal{A}}(q) \geq t_1 = \min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(q)\}$$
$$I_{\mathcal{A}}(q) \leq t_1 = \max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(q)\}$$
$$F_{\mathcal{A}}(q) \leq t_1 = \max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(q)\}$$

Condition (ii) is verified. Similarly, (iii) .

Theorem 3.6 Let $\mathcal{A}$ be a SVNS over H. Then $\mathcal{A}$ be a single valued neutrosophic hypergroup over H iff $\forall\, \alpha, \beta, \gamma \in [0,1]$, $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a subhypergroup of H.

Proof. The proof is straight forward.

Theorem 3.7 Let $\mathcal{A}$ be a single valued neutrosophic hypergroup over H and $\forall\, t_1, t_2 \in [0,1]$ $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ be the t-level sets of $\mathcal{A}$ with $t_1 \geq t_2$, then $\mathcal{A}_{t_1}$ is a subhypergroup of $\mathcal{A}_{t_2}$.

Proof. $\forall t_1, t_2 \in [0,1]$, $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ be the t-level sets of $\mathcal{A}$ with $t_1 \geq t_2$

This implies that $\mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2}$

By Theorem 3.4. $\mathcal{A}_{t_1}$ is a subhypergroup of $\mathcal{A}_{t_2}$.

Theorem 3.8 Let $\mathcal{A}$ and $\mathcal{B}$ be single valued neutrosophic hypergroups over H. Then $\mathcal{A} \cap \mathcal{B}$ is a single valued neutrosophic hypergroup over H if it is non-null.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ be single valued neutrosophic hypergroups over H.

By Definition: 2.6. $\mathcal{A} \cap \mathcal{B} = \{< p, T_{\mathcal{A}\cap\mathcal{B}}(p), I_{\mathcal{A}\cap\mathcal{B}}(p), F_{\mathcal{A}\cap\mathcal{B}}(p) > :\ p \in H\}$

where $T_{\mathcal{A}\cap\mathcal{B}}(p) = T_{\mathcal{A}}(p) \wedge T_{\mathcal{B}}(p)$, $I_{\mathcal{A}\cap\mathcal{B}}(p) = I_{\mathcal{A}}(p) \vee I_{\mathcal{B}}(p)$ and $F_{\mathcal{A}\cap\mathcal{B}}(p) = F_{\mathcal{A}}(p) \vee F_{\mathcal{B}}(p)$

For all $p, q \in H$

(i) $\min\{T_{\mathcal{A}\cap\mathcal{B}}(p), T_{\mathcal{A}\cap\mathcal{B}}(q)\} = \min\{T_{\mathcal{A}}(p) \wedge T_{\mathcal{B}}(p), T_{\mathcal{A}}(q) \wedge T_{\mathcal{B}}(q)\}$

$$\leq \min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \wedge \min\{T_{\mathcal{B}}(p), T_{\mathcal{B}}(q)\}$$
$$\leq \inf\{T_{\mathcal{A}}(r):\ r \in p \circ q\} \wedge \inf\{T_{\mathcal{B}}(r):\ r \in p \circ q\}$$
$$\leq \inf\{T_{\mathcal{A}}(r) \wedge T_{\mathcal{B}}(r):\ r \in p \circ q\}$$
$$= \inf\{T_{\mathcal{A}\cap\mathcal{B}}(r):\ r \in p \circ q\}$$

Similarly, we can prove that $\max\{I_{\mathcal{A}\cap\mathcal{B}}(p), I_{\mathcal{A}\cap\mathcal{B}}(q)\} \geq \sup\{I_{\mathcal{A}\cap\mathcal{B}}(r):\ r \in p \circ q\}$

$$\max\{F_{\mathcal{A}\cap\mathcal{B}}(p), F_{A\cap B}(q)\} \geq \sup\{F_{\mathcal{A}\cap\mathcal{B}}(r):\ r \in p \circ q\}$$

(ii) $\forall\, l, p \in H$, there exists $q \in H$ such that $p \in l \circ q$,

$\min\{T_{\mathcal{A}\cap\mathcal{B}}(l), T_{\mathcal{A}\cap\mathcal{B}}(p)\} = \min\{T_{\mathcal{A}}(l) \wedge T_{\mathcal{B}}(l)\}, \{T_{\mathcal{A}}(p) \wedge T_{\mathcal{B}}(p)\}$

$$= \min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(p)\} \wedge \min\{T_{\mathcal{B}}(l), T_{B}(p)\}$$
$$\leq T_{\mathcal{A}}(q) \wedge T_{\mathcal{B}}(q) = T_{\mathcal{A}\cap\mathcal{B}}(q)$$

Therefore, $\mathcal{A} \cap \mathcal{B}$ is a single valued neutrosophic hypergroup over H.

Theorem 3.9 Let $\mathcal{A}$ and $\mathcal{B}$ be single valued neutrosophic hypergroups over H. Then $\mathcal{A} \cup \mathcal{B}$ is a single valued neutrosophic hypergroup over H.

Proof. By Definition: 2.5.

$$\mathcal{A} \cup \mathcal{B} = \{< p, T_{\mathcal{A}\cup\mathcal{B}}(p), I_{\mathcal{A}\cup\mathcal{B}}(p), F_{\mathcal{A}\cup\mathcal{B}}(p) > : \ p \in H\}$$

where $T_{\mathcal{A}\cup\mathcal{B}}(p) = T_{\mathcal{A}}(p) \vee T_{\mathcal{B}}(p)$, $I_{\mathcal{A}\cup\mathcal{B}}(p) = I_{\mathcal{A}}(p) \wedge I_{\mathcal{B}}(p)$ and $F_{\mathcal{A}\cup\mathcal{B}}(p) = F_{\mathcal{A}}(p) \wedge F_{\mathcal{B}}(p)$

For all $p, q \in H$,

$$\begin{aligned}
\min\{T_{\mathcal{A}\cup\mathcal{B}}(p), T_{\mathcal{A}\cup\mathcal{B}}(q)\} &= \min\{T_{\mathcal{A}}(p) \vee T_{\mathcal{B}}(p), T_{\mathcal{A}}(q) \vee T_{\mathcal{B}}(q)\} \\
&\leq \min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \vee \min\{T_{\mathcal{B}}(p), T_{\mathcal{B}}(q)\} \\
&\leq \inf\{T_{\mathcal{A}}(r): \ r \in p \circ q\} \vee \inf\{T_{\mathcal{B}}(r): \ r \in p \circ q\} \\
&\leq \inf\{T_{\mathcal{A}}(r) \vee T_{\mathcal{B}}(r): \ r \in p \circ q\} \\
&= \inf\{T_{\mathcal{A}\cup\mathcal{B}}(r): \ r \in p \circ q\}
\end{aligned}$$

Similarly, the other holds.

Theorem 3.10 Let $\mathcal{A}$ be a single valued neutrosophic hypergroup over H. Then the falsity- favorite of $\mathcal{A}$ (ie., $\nabla\mathcal{A}$) is also a single valued neutrosophic hypergroup over H.

Proof. By Definition: 2.7. $\mathcal{B} = \nabla\mathcal{A}$, where the membership values are $T_{\mathcal{B}}(x) = T_{\mathcal{A}}(x)$, $I_{\mathcal{B}}(x) = 0$ and $F_{\mathcal{B}}(x) = \min\{F_{\mathcal{A}}(x) + I_{\mathcal{A}}(x), 1\}$

Then we have to prove for $F_{\mathcal{B}}$, $\forall p, q \in H$

$$\begin{aligned}
\max\{F_{\mathcal{B}}(p), F_{\mathcal{B}}(q)\} &= \max\{F_{\mathcal{A}}(p) + I_{\mathcal{A}}(p) \wedge 1, F_{\mathcal{A}}(q) + I_{\mathcal{A}}(q) \wedge 1\} \\
&= \max\{F_{\mathcal{A}}(p) + I_{\mathcal{A}}(p), F_{\mathcal{A}}(q) + I_{\mathcal{A}}(q)\} \wedge 1 \\
&\geq (\max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} + \max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\}) \wedge 1 \\
&\geq (\sup\{F_{\mathcal{A}}(r) \ : \ r \in p \circ q\} + \sup\{I_{\mathcal{A}}(r) \ : \ r \in p \circ q\}) \wedge 1 \\
&= \sup\{F_{\mathcal{A}}(r) + I_{\mathcal{A}}(r) \wedge 1 \ : \ r \in p \circ q\} \\
&= \sup\{F_{\mathcal{B}}(r) \ : \ r \in p \circ q\})
\end{aligned}$$

In similar manner the other conditions holds.

4. Conclusions

In this paper, we have developed the theory of hypergroup for the single-valued neutrosophic set by introducing several hyperalgebraic structures and some results were verified. The future research related to this work involve the development of other hyperalgebraic theory for the single-valued neutrosophic sets and interval-valued neutrosophic sets.

Acknowledgments: The article has been written with the joint financial support of RUSA-Phase 2.0 grant sanctioned vide letter No.F 24-51/2014-U, Policy (TN Multi-Gen), Dept. of Edn. Govt. of India, Dt. 09.10.2018, UGC-SAP (DRS-I) vide letter No.F.510/8/DRS-I/2016(SAP-I) Dt. 23.08.2016, DST-PURSE 2nd Phase programme vide letter No. SR/PURSE Phase 2/38 (G) Dt. 21.02.2017 and DST (FST - level I) 657876570 vide letter No.SR/FIST/MS-I/2018/17 Dt. 20.12.2018.

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Polarity of generalized neutrosophic subalgebras in BCK/BCI-algebras

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Rajab Ali Borzooei, Florentin Smarandache, Young Bae Jun (2020). Polarity of generalized neutrosophic subalgebras in BCK/BCI-algebras. *Neutrosophic Sets and Systems*, 32, 123-145

Abstract: k-polar generalized neutrosophic set is introduced, and it is applied to BCK/BCI-algebras. The notions of k-polar generalized subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra are defined, and several properties are investigated. Characterizations of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra are discussed, and the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are are considered. We show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras by using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra. A k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra is established by using the generaliged neutrosophic $\in\vee q$-sets, conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra are provided.

Keywords: k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

1 Introduction

In the fuzzy set which is introduced by Zadeh [35], the membership degree is expressed by only one function so called the truth function. As a generalization of fuzzy set, intuitionistic fuzzy set is introduced by Atanassove by using membership function and nonmembership function. The membership (resp. nonmembership) function represents truth (resp. false) part. Smarandache introduced a new notion so called neutrosophic set by using three functions, i.e., membership function (t), nonmembership function (f) and neutalitic/indeterministic membership function (i) which are independent components. Neutrosophic set is applied to BCK/BCI-algebras which are discussed in the papers [13, 19, 20, 21, 22, 26, 27, 30]. Indeterministic membership function is leaning to one side, membership function or nonmembership function, in the application of neutrosophic set to algebraic structures. In order to divide the role of the indeterministic membership function, Song et al.

[31] introduced the generalized neutralrosophic set, and discussed its application in BCK/BCI-algebras. Borzooei et al. [8] introduced the notion of a commutative generalized neutrosophic ideal in a BCK-algebra, and investigated related properties. They considered characterizations of a commutative generalized neutrosophic ideal. Using a collection of commutative ideals in BCK-algebras, they established a commutative generalized neutrosophic ideal. They also introduced the notion of equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras, and investigated related properties. Zhang [36] introduced the notion of bipolar fuzzy sets as an extension of fuzzy sets, and it is applied in several (algebraic) structures such as (ordered) semigroups (see [12, 7, 10, 28]), (hyper) BCK/BCI-algebras (see [6, 14, 15, 23, 16, 17]) and finite state machines (see [18, 32, 33, 34]). The bipolar fuzzy set is an extension of fuzzy sets whose membership degree range is $[-1, 1]$. So, it is possible for a bipolar fuzzy set to deal with positive information and negative information at the same time. Chen et al. [9] raised a question: "How to generalize bipolar fuzzy sets to multipolar fuzzy sets and how to generalize results on bipolar fuzzy sets to the case of multipolar fuzzy sets?" To solve their question, they tried to fold the negative part into positive part, that is, they used positive part instead of negative part in bipolar fuzzy set. And then they introduced introduced an m-polar fuzzy set which is an extension of bipolar fuzzy sets. It is applied to BCK/BCI-algebra, graph theory and decision-making problems etc. (see [4, 2, 1, 3, 29, 5, 25]).

In this paper, we introduce k-polar generalized neutrosophic set and apply it to BCK/BCI-algebras to study. We define k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra and study various properties. We discuss characterization of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We show that the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are also a k-polar generalized neutrosophic subalgebra. Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras. Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras. Using the generaliged neutrosophic $\in\vee q$-sets, we establish a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We provide conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

2 Preliminaries

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I) $(\forall u, v, w \in X)\ (((u * v) * (u * w)) * (w * v) = 0)$,

(II) $(\forall u, v \in X)\ ((u * (u * v)) * v = 0)$,

(III) $(\forall u \in X)\ (u * u = 0)$,

(IV) $(\forall u, v \in X)\ (u * v = 0,\ v * u = 0 \ \Rightarrow\ u = v)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(V) $(\forall u \in X)\ (0 * u = 0)$,

then X is called a BCK*-algebra.*

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall u \in X)\,(u * 0 = u)\,, \tag{2.1}$$
$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; u * w \leq v * w,\; w * v \leq w * u)\,, \tag{2.2}$$
$$(\forall u, v, w \in X)\,((u * v) * w = (u * w) * v) \tag{2.3}$$

where $u \leq v$ if and only if $u * v = 0$. A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $u * v \in S$ for all $u, v \in S$.

See the books [11] and [24] for more information on BCK/BCI-algeebras.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if $\mu(u * v) \geq \min\{\mu(u), \mu(v)\}$ for all $u, v \in X$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

3 k-polar generalized neutrosophic subalgebras

A *k-polar generalized neutrosophic set* over a universe X is a structure of the form:

$$\widehat{\mathcal{L}} := \left\{ \frac{z}{\left(\widehat{\ell}_T(z), \widehat{\ell}_{IT}(z), \widehat{\ell}_{IF}(z), \widehat{\ell}_F(z)\right)} \mid z \in X,\; \widehat{\ell}_{IT}(z) + \widehat{\ell}_{IF}(z) \leq \hat{1} \right\} \tag{3.1}$$

where $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are mappings from X into $[0, 1]^k$. The membership values of every element $z \in X$ in $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are denoted by

$$\begin{aligned} \widehat{\ell}_T(z) &= \left((\pi_1 \circ \widehat{\ell}_T)(z), (\pi_2 \circ \widehat{\ell}_T)(z), \cdots, (\pi_k \circ \widehat{\ell}_T)(z)\right), \\ \widehat{\ell}_{IT}(z) &= \left((\pi_1 \circ \widehat{\ell}_{IT})(z), (\pi_2 \circ \widehat{\ell}_{IT})(z), \cdots, (\pi_k \circ \widehat{\ell}_{IT})(z)\right), \\ \widehat{\ell}_{IF}(z) &= \left((\pi_1 \circ \widehat{\ell}_{IF})(z), (\pi_2 \circ \widehat{\ell}_{IF})(z), \cdots, (\pi_k \circ \widehat{\ell}_{IF})(z)\right), \\ \widehat{\ell}_F(z) &= \left((\pi_1 \circ \widehat{\ell}_F)(z), (\pi_2 \circ \widehat{\ell}_F)(z), \cdots, (\pi_k \circ \widehat{\ell}_F)(z)\right), \end{aligned} \tag{3.2}$$

respectively, and satisfies the following condition

$$(\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}_{IF})(z) \leq 1$$

for all $i = 1, 2, \cdots, k$.

We shall use the ordered quadruple $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ for the k-polar generalized neutrosophic set in (3.1).

Note that for every k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X, we have

$$(\forall z \in X)\left(\hat{0} \leq \widehat{\ell}_T(z) + \widehat{\ell}_{IT}(z) + \widehat{\ell}_{IF}(z) + \widehat{\ell}_F(z) \leq \hat{3}\right),$$

that is, $0 \leq (\pi_i \circ \widehat{\ell}_T)(z) + (\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}_{IF})(z) + (\pi_i \circ \widehat{\ell}_F)(z) \leq 3$ for all $z \in X$ and $i = 1, 2, \cdots, k$.

Unless otherwise stated in this section, X will represent a BCK/BCI-algebra.

Definition 3.1. A k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X is called a *k-polar generalized neutrosophic subalgebra* of X if it satisfies:

$$(\forall z, y \in X)\left(\begin{array}{l} \widehat{\ell}_T(z * y) \geq \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \\ \widehat{\ell}_{IT}(z * y) \geq \widehat{\ell}_{IT}(z) \wedge \widehat{\ell}_{IT}(y) \\ \widehat{\ell}_{IF}(z * y) \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y) \\ \widehat{\ell}_F(z * y) \leq \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \end{array}\right), \tag{3.3}$$

that is,

$$\left\{\begin{array}{l} (\pi_i \circ \widehat{\ell}_T)(z * y) \geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y) \\ (\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) \\ (\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq (\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y) \\ (\pi_i \circ \widehat{\ell}_F)(z * y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \end{array}\right. \tag{3.4}$$

for $i = 1, 2, \cdots, k$.

Example 3.2. Consider a BCK-algebra $X = \{0, \alpha, \beta, \gamma\}$ with the binary operation "$*$" which is given below.

$*$	0	α	β	γ
0	0	0	0	0
α	α	0	α	α
β	β	β	0	β
γ	γ	γ	γ	0

Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a 4-polar neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^4,\ z \mapsto \left\{\begin{array}{ll} (0.6, 0.7, 0.8, 0.9) & \text{if } z = 0, \\ (0.4, 0.4, 0.8, 0.5) & \text{if } z = \alpha, \\ (0.5, 0.6, 0.7, 0.3) & \text{if } z = \beta, \\ (0.3, 0.5, 0.4, 0.7) & \text{if } z = \gamma, \end{array}\right.$$

$$\widehat{\ell}_{IT} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.9) & \text{if } z = 0, \\ (0.6, 0.4, 0.7, 0.5) & \text{if } z = \alpha, \\ (0.5, 0.5, 0.4, 0.8) & \text{if } z = \beta, \\ (0.2, 0.6, 0.5, 0.7) & \text{if } z = \gamma, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.2, 0.3, 0.4, 0.5) & \text{if } z = 0, \\ (0.4, 0.7, 0.5, 0.8) & \text{if } z = \alpha, \\ (0.5, 0.5, 0.8, 0.6) & \text{if } z = \beta, \\ (0.7, 0.3, 0.6, 0.7) & \text{if } z = \gamma, \end{cases}$$

$$\widehat{\ell}_{F} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.4, 0.4, 0.3, 0.2) & \text{if } z = 0, \\ (0.8, 0.7, 0.5, 0.3) & \text{if } z = \alpha, \\ (0.6, 0.5, 0.6, 0.6) & \text{if } z = \beta, \\ (0.4, 0.6, 0.8, 0.4) & \text{if } z = \gamma, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a 4-polar generalized neutrosophic subalgebra of X.

If we take $z = y$ in (3.3) and use (III), then we have the following lemma.

Lemma 3.3. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic subalgebra of a BCK/BCI-algebr X. Then*

$$(\forall z, y \in X) \quad \left(\begin{array}{l} \widehat{\ell}_T(0) \geq \widehat{\ell}_T(z),\ \widehat{\ell}_{IT}(0) \geq \widehat{\ell}_{IT}(z) \\ \widehat{\ell}_{IF}(0) \leq \widehat{\ell}_{IF}(z),\ \widehat{\ell}_F(0) \leq \widehat{\ell}_F(z) \end{array} \right). \tag{3.5}$$

Proposition 3.4. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. If there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} \widehat{\ell}_T(z_n) = \hat{1} = \lim_{n\to\infty} \widehat{\ell}_{IT}(z_n)$ and $\lim_{n\to\infty} \widehat{\ell}_{IF}(z_n) = \hat{0} = \lim_{n\to\infty} \widehat{\ell}_F(z_n)$, then $\widehat{\ell}_T(0) = \hat{1} = \widehat{\ell}_{IT}(0)$ and $\widehat{\ell}_{IF}(0) = \hat{0} = \widehat{\ell}_F(0)$.*

Proof. Using Lemma 3.3, we have

$$\begin{aligned} \hat{1} &= \lim_{n\to\infty} \widehat{\ell}_T(z_n) \leq \widehat{\ell}_T(0) \leq \hat{1} = \lim_{n\to\infty} \widehat{\ell}_{IT}(z_n) \leq \widehat{\ell}_{IT}(0) \leq \hat{1}, \\ \hat{0} &= \lim_{n\to\infty} \widehat{\ell}_{IF}(z_n) \geq \widehat{\ell}_{IF}(0) \geq \hat{0} = \lim_{n\to\infty} \widehat{\ell}_F(z_n) \geq \widehat{\ell}_F(0) \geq \hat{0}. \end{aligned}$$

This completes the proof. □

Proposition 3.5. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic subalgebra of X such that*

$$(\forall z, y \in X) \quad \left(\begin{array}{l} \widehat{\ell}_T(z * y) \geq \widehat{\ell}_T(y),\ \widehat{\ell}_{IT}(z * y) \geq \widehat{\ell}_{IT}(y) \\ \widehat{\ell}_{IF}(z * y) \leq \widehat{\ell}_{IF}(y),\ \widehat{\ell}_F(z * y) \leq \widehat{\ell}_F(y) \end{array} \right). \tag{3.6}$$

Then $\widehat{\mathcal{L}}$ is constant on X, that is, $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are constants on X.

Proof. Since $z * 0 = z$ for all $z \in X$, it follows from the condition (3.6) that

$$\widehat{\ell}_T(z) = \widehat{\ell}_T(z * 0) \geq \widehat{\ell}_T(0),\ \widehat{\ell}_{IT}(z) = \widehat{\ell}_{IT}(z * 0) \geq \widehat{\ell}_{IT}(0), \tag{3.7}$$

$$\widehat{\ell}_{IF}(z) = \widehat{\ell}_{IF}(z * 0) \leq \widehat{\ell}_{IF}(0),\ \widehat{\ell}_F(z) = \widehat{\ell}_F(z * 0) \leq \widehat{\ell}_F(0) \tag{3.8}$$

for all $z \in X$. Combining (3.5) and (3.7) induces $\widehat{\ell}_T(z) = \widehat{\ell}_T(0)$, $\widehat{\ell}_{IT}(z) = \widehat{\ell}_{IT}(0)$, $\widehat{\ell}_{IF}(z) = \widehat{\ell}_{IF}(0)$ and $\widehat{\ell}_F(z) = \widehat{\ell}_F(0)$ for all $z \in X$. Therefore $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are constants on X, that is, $\widehat{\mathcal{L}}$ is constant on X. □

Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, consider the following cut sets.

$$\begin{aligned}
U(\widehat{\ell}_T, \hat{n}_T) &:= \{z \in X \mid \widehat{\ell}_T(z) \geq \hat{n}_T\},\\
U(\widehat{\ell}_{IT}, \hat{n}_{IT}) &:= \{z \in X \mid \widehat{\ell}_{IT}(z) \geq \hat{n}_{IT}\},\\
L(\widehat{\ell}_{IF}, \hat{n}_{IF}) &:= \{z \in X \mid \widehat{\ell}_{IF}(z) \leq \hat{n}_{IF}\},\\
L(\widehat{\ell}_F, \hat{n}_F) &:= \{z \in X \mid \widehat{\ell}_F(z) \leq \hat{n}_F\}
\end{aligned}$$

for $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0, 1]^k$, that is,

$$\begin{aligned}
U(\widehat{\ell}_T, \hat{n}_T) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq \hat{n}_T^i \text{ for all } i = 1, 2, \cdots, k\},\\
U(\widehat{\ell}_{IT}, \hat{n}_{IT}) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq \hat{n}_{IT}^i \text{ for all } i = 1, 2, \cdots, k\},\\
L(\widehat{\ell}_{IF}, \hat{n}_{IF}) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq \hat{n}_{IF}^i \text{ for all } i = 1, 2, \cdots, k\},\\
L(\widehat{\ell}_F, \hat{n}_F) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq \hat{n}_F^i \text{ for all } i = 1, 2, \cdots, k\}
\end{aligned}$$

where $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k)$, $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k)$, $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k)$ and $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k)$. It is clear that $U(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k U(\widehat{\ell}_T, \hat{n}_T)^i$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^k U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$ and $L(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^k L(\widehat{\ell}_F, \hat{n}_F)^i$, where

$$\begin{aligned}
U(\widehat{\ell}_T, \hat{n}_T)^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq \hat{n}_T^i\},\\
U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq \hat{n}_{IT}^i\},\\
L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq \hat{n}_{IF}^i\},\\
L(\widehat{\ell}_F, \hat{n}_F)^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq \hat{n}_F^i\}
\end{aligned}$$

for $i = 1, 2, \cdots, k$.

We handle the characterization of k-polar generalized neutrosophic subalgebra.

Theorem 3.6. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X if and only if the cut sets $U(\widehat{\ell}_T, \hat{n}_T)$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $L(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0, 1]^k$.*

Proof. Assume that $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. Let $z, y \in X$. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$ for all $\hat{n}_T \in [0, 1]^k$, then $(\pi_i \circ \widehat{\ell}_T)(z) \geq n_T^i$ and $(\pi_i \circ \widehat{\ell}_T)(y) \geq n_T^i$ for $i = 1, 2, \cdots, k$. It follows that

$$(\pi_i \circ \widehat{\ell}_T)(z * y) \geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y) \geq n_T^i$$

$i = 1, 2, \cdots, k$. Hence $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$, and so $U(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. If $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$ for all $\hat{n}_F \in [0,1]^k$, then $(\pi_i \circ \widehat{\ell}_F)(z) \leq n_F^i$ and $(\pi_i \circ \widehat{\ell}_F)(y) \leq n_F^i$ for $i = 1, 2, \cdots, k$. Hence

$$(\pi_i \circ \widehat{\ell}_F)(z * y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \leq n_F^i$$

$i = 1, 2, \cdots, k$, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$. Therefore $L(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, we can verify that $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$ and $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ are subalgebras of X.

Conversely, suppose that the cut sets $U(\widehat{\ell}_T, \hat{n}_T)$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $L(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0,1]^k$. If there exists $\alpha, \beta \in X$ such that $\widehat{\ell}_{IT}(\alpha * \beta) < \widehat{\ell}_{IT}(\alpha) \wedge \widehat{\ell}_{IT}(\beta)$, that is,

$$(\pi_i \circ \widehat{\ell}_{IT})(\alpha * \beta) < (\pi_i \circ \widehat{\ell}_{IT})(\alpha) \wedge (\pi_i \circ \widehat{\ell}_{IT})(\beta)$$

for $i = 1, 2, \cdots, k$, then $\alpha, \beta \in U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $\alpha * \beta \notin U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ where $\hat{n}_{IT}^i = (\pi_i \circ \widehat{\ell}_{IT})(\alpha) \wedge (\pi_i \circ \widehat{\ell}_{IT})(\beta)$ for for $i = 1, 2, \cdots, k$. This is a contradiction, and so

$$\widehat{\ell}_{IT}(z * y) \geq \widehat{\ell}_{IT}(z) \wedge \widehat{\ell}_{IT}(y)$$

for all $z, y \in X$. By the similarly way, we know that $\widehat{\ell}_T(z * y) \geq \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y)$ for all $z, y \in X$. Now, suppose that $\widehat{\ell}_F(\alpha * \beta) > \widehat{\ell}_F(\alpha) \vee \widehat{\ell}_F(\beta)$ for some $\alpha, \beta \in X$. Then

$$(\pi_i \circ \widehat{\ell}_F)(\alpha * \beta) > (\pi_i \circ \widehat{\ell}_F)(\alpha) \vee (\pi_i \circ \widehat{\ell}_F)(\beta)$$

for $i = 1, 2, \cdots, k$. If we take $n_F^i = (\pi_i \circ \widehat{\ell}_F)(\alpha) \vee (\pi_i \circ \widehat{\ell}_F)(\beta)$ for $i = 1, 2, \cdots, k$, then $\alpha, \beta \in L(\widehat{\ell}_F, \hat{n}_F)^i$ but $\alpha * \beta \notin L(\widehat{\ell}_F, \hat{n}_F)^i$, a contradiction. Hence

$$\widehat{\ell}_F(z * y) \leq \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y)$$

for all $z, y \in X$. Similarly, we can check that $\widehat{\ell}_{IF}(z * y) \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y)$ for all $z, y \in X$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. $\square$

Theorem 3.7. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X if and only if the fuzzy sets $\pi_i \circ \widehat{\ell}_T$, $\pi_i \circ \widehat{\ell}_{IT}$, $\pi_i \circ \widehat{\ell}^c_F$ and $\pi_i \circ \widehat{\ell}^c_{IF}$ are fuzzy subalgebras of X where $(\pi_i \circ \widehat{\ell}^c_F)(z) = 1 - (\pi_i \circ \widehat{\ell}_F)(z)$ and $(\pi_i \circ \widehat{\ell}^c_{IF})(z) = 1 - (\pi_i \circ \widehat{\ell}_{IF})(z)$ for all $z \in X$ and $i = 1, 2, \cdots, k$.*

Proof. Suppose that $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. For any $i = 1, 2, \cdots, k$, it is clear that $\pi_i \circ \widehat{\ell}_T$ and $\pi_i \circ \widehat{\ell}_{IT}$ are fuzzy subalgebras of X. For any $z, y \in X$, we get

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_F)(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_F)(z * y) = 1 - (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \\ &= (1 - (\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_F)(y)) \\ &= (\pi_i \circ \widehat{\ell}^c_F)(z) \wedge (\pi_i \circ \widehat{\ell}^c_F)(y)\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_{IF})(z*y) &= 1-(\pi_i \circ \widehat{\ell}_{IF})(z*y) = 1-(\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)\\ &= (1-(\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_{IF})(y))\\ &= (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y).\end{aligned}$$

Hence $\pi_i \circ \widehat{\ell}^c_F$ and $\pi_i \circ \widehat{\ell}^c_{IF}$ are fuzzy subalgebras of X .

Conversely, suppose that the fuzzy sets $\pi_i \circ \widehat{\ell}_T$, $\pi_i \circ \widehat{\ell}_{IT}$, $\pi_i \circ \widehat{\ell}^c_F$ and $\pi_i \circ \widehat{\ell}^c_{IF}$ are fuzzy subalgebras of X for $i = 1, 2, \cdots, k$ and let $z, y \in X$. Then

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_T)(z*y) &\geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y),\\ (\pi_i \circ \widehat{\ell}_{IT})(z*y) &\geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)\end{aligned}$$

for all $i = 1, 2, \cdots, k$. Also we have

$$\begin{aligned}1-(\pi_i \circ \widehat{\ell}_F)(z*y) &= (\pi_i \circ \widehat{\ell}^c_F)(z*y) \geq (\pi_i \circ \widehat{\ell}^c_F)(z) \wedge (\pi_i \circ \widehat{\ell}^c_F)(y)\\ &= (1-(\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_F)(y))\\ &= 1-((\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y))\end{aligned}$$

and

$$\begin{aligned}1-(\pi_i \circ \widehat{\ell}_{IF})(z*y) &= (\pi_i \circ \widehat{\ell}^c_{IF})(z*y) \geq (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y)\\ &= (1-(\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_{IF})(y))\\ &= 1-((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y))\end{aligned}$$

which imply that $(\pi_i \circ \widehat{\ell}_F)(z*y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y)$ and

$$(\pi_i \circ \widehat{\ell}_{IF})(z*y) \leq (\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)$$

for all $i = 1, 2, \cdots, k$. Hence $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X . $\square$

Theorem 3.8. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X, then so are $\square\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$.*

Proof. Note that $(\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}^c_{IT})(z) = (\pi_i \circ \widehat{\ell}_{IT})(z) + 1 - (\pi_i \circ \widehat{\ell}_{IT})(z) = 1$ and $(\pi_i \circ \widehat{\ell}_F)(z) + (\pi_i \circ \widehat{\ell}^c_F)(z) = (\pi_i \circ \widehat{\ell}_F)(z) + 1 - (\pi_i \circ \widehat{\ell}_F)(z) = 1$, that is, $\widehat{\ell}_{IT}(z) + \widehat{\ell}^c_{IT}(z) = \hat{1}$ and $\widehat{\ell}_F(z) + \widehat{\ell}^c_F(z) = \hat{1}$ for all $z \in X$. Hence $\square\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$ are k-polar generalized neutrosophic sets over X. For any $z, y \in X$, we get

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_{IT})(z*y) &= 1-(\pi_i \circ \widehat{\ell}_{IT})(z*y) \leq 1-((\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y))\\ &= (1-(\pi_i \circ \widehat{\ell}_{IT})(z)) \vee (1-(\pi_i \circ \widehat{\ell}_{IT})(y))\\ &= (\pi_i \circ \widehat{\ell}^c_{IT})(z) \vee (\pi_i \circ \widehat{\ell}^c_{IT})(y),\end{aligned}$$

$$
\begin{aligned}
(\pi_i \circ \widehat{\ell}^c_T)(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_T)(z * y) \leq 1 - ((\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y)) \\
&= (1 - (\pi_i \circ \widehat{\ell}_T)(z)) \vee (1 - (\pi_i \circ \widehat{\ell}_T)(y)) \\
&= (\pi_i \circ \widehat{\ell}^c_T)(z) \vee (\pi_i \circ \widehat{\ell}^c_T)(y),
\end{aligned}
$$

$$
\begin{aligned}
(\pi_i \circ \widehat{\ell}^c_{IF})(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_{IF})(z * y) \geq 1 - ((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)) \\
&= (1 - (\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_{IF})(y)) \\
&= (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y),
\end{aligned}
$$

and

$$
\begin{aligned}
(\pi_i \circ \widehat{\ell}^c_F)(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_F)(z * y) \geq 1 - ((\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y)) \\
&= (1 - (\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_F)(y)) \\
&= (\pi_i \circ \widehat{\ell}^c_F)(z) \wedge (\pi_i \circ \widehat{\ell}^c_F)(y).
\end{aligned}
$$

Therefore $\Box\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\Diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$ are $k$polar generalized neutrosophic subalgebras of X. □

Theorem 3.9. *Let $\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_k \subseteq [0,1]^k$, that is, $\Lambda_i \subseteq [0,1]$ for $i = 1, 2, \cdots, k$. Let $\mathcal{S}_i := \{S_{t_i} \mid t_i \in \Lambda_i\}$ be a family of subalgebras of X for $i = 1, 2, \cdots, k$ such that*

$$
X = \bigcup_{t_i \in \Lambda_i} S_i, \tag{3.9}
$$

$$
(\forall s_i, t_i \in \Lambda_i)\,(s_i > t_i \;\Rightarrow\; S_{s_i} \subset S_{t_i}) \tag{3.10}
$$

for $i = 1, 2, \cdots, k$. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X defined by

$$
(\forall z \in X) \quad \left.\begin{array}{l} (\pi_i \circ \widehat{\ell}_T)(z) = \bigvee\{q_i \in \Lambda_i \mid z \in S_{q_i}\} = (\pi_i \circ \widehat{\ell}_{IT})(z), \\ (\pi_i \circ \widehat{\ell}_{IF})(z) = \bigwedge\{r_i \in \Lambda_i \mid z \in S_{r_i}\} = (\pi_i \circ \widehat{\ell}_F)(z) \end{array}\right) \tag{3.11}
$$

for $i = 1, 2, \cdots, k$. Then $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X.

Proof. For any $i = 1, 2, \cdots, k$, we consider the following two cases.

$$
t_i = \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\} \text{ and } t_i \neq \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\}.
$$

The first case implies that

$$
\begin{aligned}
z \in U(\widehat{\ell}_T, t_i) &\Leftrightarrow (\forall q_i < t_i)(z \in S_{q_i}) \Leftrightarrow z \in \bigcap_{q_i < t_i} S_{q_i}, \\
z \in U(\widehat{\ell}_{IT}, t_i) &\Leftrightarrow (\forall q_i < t_i)(z \in S_{q_i}) \Leftrightarrow z \in \bigcap_{q_i < t_i} S_{q_i}.
\end{aligned}
$$

Hence $U(\widehat{\ell}_T, t_i) = \bigcap_{q_i<t_i} S_{q_i} = U(\widehat{\ell}_{IT}, t_i)$, and so $U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X for all $i = 1, 2, \ldots, k$. Hence $U(\widehat{\ell}_T, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X. For the second case, we will show that $U(\widehat{\ell}_T, t_i) = \bigcup_{q_i\geq t_i} S_{q_i} = U(\widehat{\ell}_{IT}, t_i)$ for all $i = 1, 2, \ldots, k$. If $z \in \bigcup_{q_i\geq t_i} S_{q_i}$, then $z \in S_{q_i}$ for some $q_i \geq t_i$. Hence $(\pi_i \circ \widehat{\ell}_{IT})(z) = (\pi_i \circ \widehat{\ell}_T)(z) \geq q_i \geq t_i$, and so $z \in U(\widehat{\ell}_T, t_i)$ and $z \in U(\widehat{\ell}_{IT}, t_i)$. If $z \notin \bigcup_{q_i\geq t_i} S_{q_i}$, then $z \notin S_{q_i}$ for all $q_i \geq t_i$. The condition $t_i \neq \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\}$ induces $(t_i - \varepsilon_i, t_i) \cap \Lambda_i = \emptyset$ for some $\varepsilon_i > 0$. Hence $z \notin S_{q_i}$ for all $q_i > t_i - \varepsilon_i$, which means that if $z \in S_{q_i}$ then $q_i \leq t_i - \varepsilon_i$. Hence $(\pi_i \circ \widehat{\ell}_{IT})(z) = (\pi_i \circ \widehat{\ell}_T)(z) \leq t_i - \varepsilon_i < t_i$ and so $z \notin U(\widehat{\ell}_{IT}, t_i) = U(\widehat{\ell}_T, t_i)$. Therefore $U(\widehat{\ell}_T, t_i) = U(\widehat{\ell}_{IT}, t_i) \subseteq \bigcup_{q_i\geq t_i} S_{q_i}$. Consequently, $U(\widehat{\ell}_T, t_i) = U(\widehat{\ell}_{IT}, t_i) = \bigcup_{q_i\geq t_i} S_{q_i}$ which is a subalgebra of X, and therefore $U(\widehat{\ell}_T, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X. Now, we consider the following two cases.

$$s_i = \bigwedge\{r_i \in \Lambda_i \mid r_i > s_i\} \text{ and } s_i \neq \bigwedge\{r_i \in \Lambda_i \mid r_i > s_i\}.$$

For the first case, we get

$$z \in L(\widehat{\ell}_{IF}, s_i) \Leftrightarrow (\forall s_i < r_i)(z \in S_{r_i}) \Leftrightarrow z \in \bigcap_{r_i>s_i} S_{r_i},$$
$$z \in L(\widehat{\ell}_F, s_i) \Leftrightarrow (\forall s_i < r_i)(z \in S_{r_i}) \Leftrightarrow z \in \bigcap_{r_i>s_i} S_{r_i}.$$

It follows that $L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i) = \bigcap_{r_i>s_i} S_{r_i}$, which is a subalgebra of X. The second case induces $(s_i, s_i + \varepsilon_i) \cap \Lambda_i = \emptyset$ for some $\varepsilon_i > 0$. If $z \in \bigcup_{r_i\leq s_i} S_{r_i}$, then $z \in S_{r_i}$ for some $r_i \leq s_i$, and thus $(\pi_i \circ \widehat{\ell}_{IF})(z) = (\pi_i \circ \widehat{\ell}_F)(z) \leq r_i \leq s_i$, i.e., $z \in L(\widehat{\ell}_{IF}, s_i)$ and $z \in L(\widehat{\ell}_F, s_i)$. Hene $\bigcup_{r_i\leq s_i} S_{r_i} \subseteq L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i)$. If $z \notin \bigcup_{r_i\leq s_i} S_{r_i}$, then $z \notin S_{r_i}$ for all $r_i \leq s_i$ which implies that $z \notin S_{r_i}$ for all $r_i \leq s_i + \varepsilon_i$, that is, if $z \in S_{r_i}$ then $r_i \geq s_i + \varepsilon_i$. Thus $(\pi_i \circ \widehat{\ell}_{IF})(z) = (\pi_i \circ \widehat{\ell}_F)(z) \geq s_i + \varepsilon_i \geq s_i$ and so $z \notin L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i)$. This shows that $L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i) = \bigcup_{r_i\leq s_i} S_{r_i}$, which is a subalgebra of X. Therefore $L(\widehat{\ell}_F, \hat{s}) = \bigcap_{i=1,2,\ldots,k} L(\widehat{\ell}_F, s_i)$ and $U(\widehat{\ell}_{IF}, \hat{s}) = \bigcap_{i=1,2,\ldots,k} L(\widehat{\ell}_{IF}, s_i)$ are subalgebras of X. Using Theorem 3.6, we know that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X. □

4 k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebras

Let $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k)$, $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k)$, $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k)$ and $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k)$ in $[0, 1]^k$. Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, we consider the following sets.

$$\begin{aligned}
T_q(\widehat{\ell}_T, \hat{n}_T) &:= \{z \in X \mid \widehat{\ell}_T(z) + \hat{n}_T > \hat{1}\}, \\
IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) &:= \{z \in X \mid \widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}\}, \\
IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) &:= \{z \in X \mid \widehat{\ell}_{IF}(z) + \hat{n}_{IF} < \hat{1}\}, \\
F_q(\widehat{\ell}_F, \hat{n}_F) &:= \{z \in X \mid \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}\},
\end{aligned}$$

which are called *generaliged neutrosophic* q*-sets*, and

$$\begin{aligned}
T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) &:= \{z \in X \mid \widehat{\ell}_T(z) \geq \hat{n}_T \text{ or } \widehat{\ell}_T(z) + \hat{n}_T > \hat{1}\}, \\
IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) &:= \{z \in X \mid \widehat{\ell}_{IT}(z) \geq \hat{n}_{IT} \text{ or } \widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}\}, \\
IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) &:= \{z \in X \mid \widehat{\ell}_{IF}(z) \leq \hat{n}_{IF} \text{ or } \widehat{\ell}_{IF}(z) + \hat{n}_{IF} < \hat{1}\}, \\
F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) &:= \{z \in X \mid \widehat{\ell}_F(z) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}\}
\end{aligned}$$

which are called *generaliged neutrosophic* $\in\vee q$*-sets*. Then

$$T_q(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^{k} T_q(\widehat{\ell}_T, \hat{n}_T)^i, \;\; IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^{k} IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})^i,$$
$$IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^{k} IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i, \;\; F_q(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^{k} F_q(\widehat{\ell}_F, \hat{n}_F)^i$$

and

$$T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^{k} T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i, \;\; IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^{k} IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})^i,$$
$$IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^{k} IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})^i, \;\; F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^{k} F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)^i$$

where

$$\begin{aligned}
T_q(\widehat{\ell}_T, \hat{n}_T)^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1\}, \\
IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) + n_{IT}^i > 1\}, \\
IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1\}, \\
F_q(\widehat{\ell}_F, \hat{n}_F)^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1\}
\end{aligned}$$

and

$$\begin{aligned}
T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq n_T^i \text{ or } (\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1\},\\
IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq n_{IT}^i \text{ or } (\pi_i \circ \widehat{\ell}_{IT})(z) + n_{IT}^i > 1\},\\
IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq n_{IF}^i \text{ or } (\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1\},\\
F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq n_F^i \text{ or } (\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1\}.
\end{aligned}$$

It is clear that $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = U(\widehat{\ell}_T, \hat{n}_T) \cup T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = U(\widehat{\ell}_{IT}, \hat{n}_{IT}) \cup IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \cup IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$, and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = L(\widehat{\ell}_F, \hat{n}_F) \cup F_q(\widehat{\ell}_F, \hat{n}_F)$.

By routine calculations, we have the following properties.

Proposition 4.1. *Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, we have*

1. *If $\hat{n}_T, \hat{n}_{IT} \in [0, 0.5]^k$, then $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = U(\widehat{\ell}_T, \hat{n}_T)$ and $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = U(\widehat{\ell}_{IT}, \hat{n}_{IT})$.*

2. *If $\hat{n}_F, \hat{n}_{IF} \in [0.5, 1]^k$, then $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = L(\widehat{\ell}_F, \hat{n}_F)$.*

3. *If $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$, then $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = T_q(\widehat{\ell}_T, \hat{n}_T)$ and $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$.*

4. *If $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$, then $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = F_q(\widehat{\ell}_F, \hat{n}_F)$.*

Unless otherwise stated in this section, X will represent a BCK/BCI-algebra.

Definition 4.2. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is called a *k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra* of X if it satisfies:

$$\begin{aligned}
z \in U(\widehat{\ell}_T, \hat{n}_T),\ y \in U(\widehat{\ell}_T, \hat{n}_T) &\Rightarrow z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T),\\
z \in U(\widehat{\ell}_{IT}, \hat{n}_{IT}),\ y \in U(\widehat{\ell}_{IT}, \hat{n}_{IT}) &\Rightarrow z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}),\\
z \in L(\widehat{\ell}_{IF}, \hat{n}_{IF}),\ y \in L(\widehat{\ell}_{IF}, \hat{n}_{IF}) &\Rightarrow z * y \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}),\\
z \in L(\widehat{\ell}_F, \hat{n}_F),\ y \in L(\widehat{\ell}_F, \hat{n}_F) &\Rightarrow z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)
\end{aligned} \tag{4.1}$$

for all $z, y \in X$, $\hat{n}_T, \hat{n}_{IT} \in (0, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 1)^k$.

Example 4.3. Consider a BCI-algebra $X = \{0, 1, 2, \alpha, \beta\}$ with the binary operation "$*$" which is given below.

$*$	0	1	2	α	β
0	0	0	0	α	α
1	1	0	1	β	α
2	2	2	0	α	α
α	α	α	α	0	0
β	β	α	β	1	0

Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a 3-polar neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.5, 0.5) & \text{if } z = 0, \\ (0.7, 0.7, 0.2) & \text{if } z = 1, \\ (0.7, 0.8, 0.5) & \text{if } z = 2, \\ (0.3, 0.4, 0.5) & \text{if } z = \alpha, \\ (0.3, 0.4, 0.2) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IT} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.5, 0.6) & \text{if } z = 0, \\ (0.4, 0.3, 0.7) & \text{if } z = 1, \\ (0.6, 0.8, 0.4) & \text{if } z = 2, \\ (0.7, 0.4, 0.1) & \text{if } z = \alpha, \\ (0.4, 0.3, 0.1) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.3, 0.1, 0.5) & \text{if } z = 0, \\ (0.8, 0.3, 0.7) & \text{if } z = 1, \\ (0.3, 0.8, 0.5) & \text{if } z = 2, \\ (0.7, 0.9, 0.6) & \text{if } z = \alpha, \\ (0.8, 0.9, 0.7) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_F : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.2, 0.5) & \text{if } z = 0, \\ (0.3, 0.9, 0.8) & \text{if } z = 1, \\ (0.5, 0.2, 0.4) & \text{if } z = 2, \\ (0.6, 0.4, 0.6) & \text{if } z = \alpha, \\ (0.6, 0.9, 0.8) & \text{if } z = \beta, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is 3-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.4. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0,1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0,1)^k$.*

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Then $\widehat{\ell}_T(z) + \hat{n}_T > \hat{1}$ and $\widehat{\ell}_T(y) + \hat{n}_T > \hat{1}$, that is, $(\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1$ and $(\pi_i \circ \widehat{\ell}_T)(y) + n_T^i > 1$ for $i = 1, 2, \cdots, k$. It follows that

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_T)(z * y) + n_T^i &\geq ((\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y)) + n_T^i \\ &= ((\pi_i \circ \widehat{\ell}_T)(z) + n_T)^i \wedge ((\pi_i \circ \widehat{\ell}_T)(y) + n_T)^i > 1\end{aligned}$$

for $i = 1, 2, \cdots, k$. Hence $\widehat{\ell}_T(z * y) + \hat{n}_T > \hat{1}$, that is, $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Therefore $T_q(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. Let $z, y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Then $(\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1$ and $(\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF}^i < 1$ for $i = 1, 2, \cdots, k$.

Hence

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_{IF})(z * y) + n_{IF}^i &\leq ((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)) + n_{IF}^i \\ &= ((\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF})^i \vee ((\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF})^i < 1\end{aligned}$$

for $i = 1, 2, \cdots, k$ and so $\widehat{\ell}_{IF}(z * y) + \hat{n}_{IF} < \hat{1}$. Thus $z * y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X. By the similar way, we can verify that $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X. □

We handle characterizations of a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.5. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall z, y \in X) \begin{pmatrix} \widehat{\ell}_T(z * y) \geq \bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} \\ \widehat{\ell}_{IT}(z * y) \geq \bigwedge\{\widehat{\ell}_{IT}(z), \widehat{\ell}_{IT}(y), \widehat{0.5}\} \\ \widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} \\ \widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \end{pmatrix}, \tag{4.2}$$

that is,

$$\begin{cases} (\pi_i \circ \widehat{\ell}_T)(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(z), (\pi_i \circ \widehat{\ell}_T)(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(z), (\pi_i \circ \widehat{\ell}_{IF})(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_F)(z * y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\} \end{cases} \tag{4.3}$$

for all $z, y \in X$ and $i = 1, 2, \cdots, k$.

Proof. Suppose that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X and let $z, y \in X$. For any $i = 1, 2, \ldots, k$, assume that $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) < 0.5$. Then

$$(\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)$$

because if $(\pi_i \circ \widehat{\ell}_{IT})(z * y) < (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)$, then there exists $n_{IT}^i \in (0, 0.5)$ such that

$$(\pi_i \circ \widehat{\ell}_{IT})(z * y) < n_{IT}^i \leq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y).$$

It follows that $z \in U(\widehat{\ell}_{IT}, n_{IT})^i$ and $y \in U(\widehat{\ell}_{IT}, n_{IT})^i$ but $z * y \notin U(\widehat{\ell}_{IT}, n_{IT})^i$. Also $(\pi_i \circ \widehat{\ell}_{IT})(z * y) + n_{IT}^i < 1$, i.e., $z * y \notin IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Hence $z * y \notin IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ which is a contradiction. Therefore

$$(\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$$

for all $z, y \in X$ with $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) < 0.5$. Now suppose that $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) \geq 0.5$. Then $z \in U(\widehat{\ell}_{IT}, 0.5)^i$ and $y \in U(\widehat{\ell}_{IT}, 0.5)^i$, and so $z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, 0.5)^i = U(\widehat{\ell}_{IT}, 0.5)^i \cup IT_q(\widehat{\ell}_{IT}, 0.5)^i$.

Hence $z * y \in U(\widehat{\ell}_{IT}, 0.5)^i$. Otherwise, $(\pi_i \circ \widehat{\ell}_{IT})(z * y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently,

$$(\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$$

for all $z, y \in X$. Similarly, we know that

$$(\pi_i \circ \widehat{\ell}_{T})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{T})(z), (\pi_i \circ \widehat{\ell}_{T})(y), 0.5\}$$

for all $z, y \in X$. Suppose that $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) > \widehat{0.5}$. If $\widehat{\ell}_F(z * y) > \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) := \hat{n}_F$, then $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, $z * y \notin L(\widehat{\ell}_F, \hat{n}_F)$ and $\widehat{\ell}_F(z * y) + \hat{n}_F > 2\hat{n}_F > 1$, i.e., $z * y \notin F_q(\widehat{\ell}_F, \hat{n}_F)$. This is a contradiction, and so $\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ whenever $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) > \widehat{0.5}$. Now assume that $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \leq \widehat{0.5}$. Then $z, y \in L(\widehat{\ell}_F, \widehat{0.5})$ and thus $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \widehat{0.5}) = L(\widehat{\ell}_F, \widehat{0.5}) \cup F_q(\widehat{\ell}_F, \widehat{0.5})$. If $z * y \notin L(\widehat{\ell}_F, \widehat{0.5})$, that is, $\widehat{\ell}_F(z * y) > \widehat{0.5}$, then $\widehat{\ell}_F(z * y) + \widehat{0.5} > \widehat{0.5} + \widehat{0.5} = \hat{1}$, i.e., $z * y \notin F_q(\widehat{\ell}_F, \widehat{0.5})$. This is a contradiction. Hence $\widehat{\ell}_F(z * y) \leq \widehat{0.5}$ and so $\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ whenever $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \leq \widehat{0.5}$. Therefore $\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ for all $z, y \in X$. By the similar way, we have $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\}$ for all $z, y \in X$.

Conversely, let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X which satisfies the condition (4.2). Let $z, y \in X$ and $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k) \in [0,1]^k$. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) \geq \hat{n}_T$ and $\widehat{\ell}_T(y) \geq \hat{n}_T$. If $\widehat{\ell}_T(z * y) < \hat{n}_T$, then $\widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \geq \widehat{0.5}$. Otherwise, we get

$$\widehat{\ell}_T(z * y) \geq \bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} = \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \geq \hat{n}_T,$$

which is a contradiction. Hence

$$\widehat{\ell}_T(z * y) + \hat{n}_T > 2\widehat{\ell}_T(z * y) \geq 2\bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} = \hat{1}$$

and so $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T) \subseteq T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Similarly, if $z, y \in U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, then $z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ for $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k) \in [0,1]^k$. Now, let $z, y \in L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ for $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k) \in [0,1]^k$. Then $\widehat{\ell}_{IF}(z) \leq \hat{n}_{IF}$ and $\widehat{\ell}_{IF}(y) \leq \hat{n}_{IF}$. If $\widehat{\ell}_{IF}(z * y) > \hat{n}_{IF}$, then $\widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(z) \leq \widehat{0.5}$ because if not, then $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y) \leq \hat{n}_{IF}$, which is a contradiction. Thus

$$\widehat{\ell}_{IF}(z * y) + \hat{n}_{IF} < 2\widehat{\ell}_{IF}(z * y) \leq 2\bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} = \hat{1}$$

and so $z * y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) \subseteq IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Similarly, we know that if $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k) \in [0,1]^k$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets subalgebras.

Theorem 4.6. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Suppose that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. Let $z, y \in X$. If $z, y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ for $\hat{n}_{IT} \in (0.5, 1]^k$, then $\widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}$ and $\widehat{\ell}_{IT}(y) + \hat{n}_{IT} > \hat{1}$. It follows from Theorem 4.5 that

$$\begin{aligned}\widehat{\ell}_{IT}(z * y) + \hat{n}_{IT} &\geq \bigwedge\{\widehat{\ell}_{IT}(z), \widehat{\ell}_{IT}(y), \widehat{0.5}\} + \hat{n}_{IT} \\ &= \bigwedge\{\widehat{\ell}_{IT}(z) + \hat{n}_{IT}, \widehat{\ell}_{IT}(y) + \hat{n}_{IT}, \widehat{0.5} + \hat{n}_{IT}\} \\ &> \hat{1},\end{aligned}$$

i.e., $z * y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Thus $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X. Suppose that $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F \in [0, 0.5)^k$. Then $(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1$ and $(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1$. Using Theorem 4.5, we have

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_F)(z * y) + n_F^i &\leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\} + n_F^i \\ &= \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i, (\pi_i \circ \widehat{\ell}_F)(y) + n_F^i, 0.5 + n_F^i\} \\ &< 1\end{aligned}$$

and thus $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)^i$ for all $i = 1, 2, \cdots, k$. Hence $z * y \in \bigcap_{i=1}^k F_q(\widehat{\ell}_F, \hat{n}_F)^i = F_q(\widehat{\ell}_F, \hat{n}_F)$, and therefore $F_q(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, we can induce that $T_q(\widehat{\ell}_T, \hat{n}_T)$ and $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ are subalgebras of X for $\hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F \in [0, 0.5)^k$. □

Using the generaliged neutrosophic $\in\vee q$-sets, we establish a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.7. *Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X, if the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 1)^k$, then $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X.*

Proof. Assume that there exist $\alpha, \beta \in X$ such that

$$(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) < \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(\alpha), (\pi_i \circ \widehat{\ell}_T)(\beta), 0.5\}$$

for $i = 1, 2, \cdots, k$. Then there exists $n_T^i \in (0, 0.5]$ such that

$$(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) < n_T^i \leq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(\alpha), (\pi_i \circ \widehat{\ell}_T)(\beta), 0.5\}.$$

Hence $\alpha, \beta \in U(\widehat{\ell}_T, \hat{n}_T)^i$, and so $\alpha, \beta \in \bigcap_{i=1}^k U(\widehat{\ell}_T, \hat{n}_T)^i = U(\widehat{\ell}_T, \hat{n}_T) \subseteq T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Since $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X, it follows that $\alpha * \beta \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i$. Thus $(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) \geq n_T^i$ or $(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) + n_T^i > 1$ for $i = 1, 2, \cdots, k$. This is a contradiction, and thus $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(z), (\pi_i \circ \widehat{\ell}_T)(y), 0.5\}$ for all $z, y \in X$ and $i = 1, 2, \cdots, k$. Now, if there exist $\alpha, \beta \in X$ such that

$$(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) > \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(\alpha), (\pi_i \circ \widehat{\ell}_{IF})(\beta), 0.5\}$$

for $i = 1, 2, \cdots, k$, then

$$(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) > n^i_{IF} \geq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(\alpha), (\pi_i \circ \widehat{\ell}_{IF})(\beta), 0.5\} \tag{4.4}$$

for some $n^i_{IF} \in [0.5, 1)$. Hence $\alpha, \beta \in L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$, and so $\alpha, \beta \in \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \subseteq IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$. This implies that $\alpha * \beta \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$, and (4.4) induces $\alpha * \beta \notin L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$ and $(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) + n^i_{IF} > 2n^i_{IF} > 1$ for $i = 1, 2, \cdots, k$. Thus $\alpha * \beta \notin \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $\alpha * \beta \notin \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Hence $\alpha * \beta \notin IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ which is a contradiction. Therefore

$$(\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(z), (\pi_i \circ \widehat{\ell}_{IF})(y), 0.5\}$$

for for all $z, y \in X$ and $i = 1, 2, \cdots, k$, i.e., $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\}$ for all $z, y \in X$. Similarly, we show that $(\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$ and $(\pi_i \circ \widehat{\ell}_F)(z * y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\}$ for all $z, y \in X$ and $i = 1, 2, \cdots, k$. Using Theorem 4.5, we conclude that $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic $\in\vee q$-sets subalgebras.

Theorem 4.8. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0, 0.5]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0.5, 1)^k$.*

Proof. Let $z, y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Then

$$z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i \text{ or } z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$$

and

$$y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i \text{ or } y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$$

for $i = 1, 2, \cdots, k$. Thus we get the following four cases:

(i) $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(ii) $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(iii) $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(iv) $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$.

For the first case, we have $z * y \in IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$ and so

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

In the the case (ii) (resp., (iii)), $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ (resp., $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i)$ induce $\widehat{\ell}_{IT}(y) > 1 - n^i_{IT} \geq n^i_{IT}$ (resp., $\widehat{\ell}_{IT}(z) > 1 - n^i_{IT} \geq n^i_{IT}$), that is, $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ (resp., $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i)$. Thus $z * y \in IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$ which implies that

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

The last case induces $\widehat{\ell}_{IT}(z) > 1 - n^i_{IT} \geq n^i_{IT}$ and $\widehat{\ell}_{IT}(y) > 1 - n^i_{IT} \geq n^i_{IT}$, i.e., $z, y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$. It follows that

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

Therefore $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X for all $\hat{n}_{IT} \in (0, 0.5]^k$. Similarly, we can show that the set $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X for all $\hat{n}_T \in (0, 0.5]^k$. Let $z, y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Then

$$\widehat{\ell}_F(z) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$$

and

$$\widehat{\ell}_F(y) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(y) + \hat{n}_F < \hat{1}.$$

If $\widehat{\ell}_F(z) \leq \hat{n}_F$ and $\widehat{\ell}_F(y) \leq \hat{n}_F$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq \hat{n}_F \vee \widehat{0.5} = \hat{n}_F$$

by Theorem 4.5, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $\widehat{\ell}_F(z) \leq \hat{n}_F$ or $\widehat{\ell}_F(y) + \hat{n}_F < \hat{1}$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq \bigvee\{\hat{n}_F, \hat{1} - \hat{n}_F, \widehat{0.5}\} = \hat{n}_F$$

by Theorem 4.5. Hence $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Similarly, if $\widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) \leq \hat{n}_F$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $\widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) + \hat{n}_F < \hat{1}$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq (\hat{1} - \hat{n}_F) \vee \widehat{0.5} = \widehat{0.5} < \hat{n}_F$$

by Theorem 4.5. Thus $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Consequencly, $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X for all $\hat{n}_F \in [0.5, 1)^k$. By the similar way, we can verify that $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X for all $\hat{n}_{IF} \in [0.5, 1)^k$. □

5 k-polar generalized $(q,\ \in\vee q)$-neutrosophic subalgebras

Definition 5.1. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is called a *k-polar generalized $(q,\ \in\vee q)$-neutrosophic subalgebra* of X if it satisfies:

$$\begin{aligned}
&z \in T_q(\widehat{\ell}_T, \hat{n}_T),\ y \in T_q(\widehat{\ell}_T, \hat{n}_T) \ \Rightarrow\ z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T),\\
&z \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}),\ y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) \ \Rightarrow\ z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}),\\
&z \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}),\ y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) \ \Rightarrow\ z * y \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}),\\
&z \in F_q(\widehat{\ell}_F, \hat{n}_F),\ y \in F_q(\widehat{\ell}_F, \hat{n}_F) \ \Rightarrow\ z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)
\end{aligned} \tag{5.1}$$

for all $z, y \in X$, $\hat{n}_T, \hat{n}_{IT} \in (0,1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0,1)^k$.

Example 5.2. Let $X = \{0, 1, 2, \alpha, \beta\}$ be the BCI-algebra which is given in Example 4.3. Let $\widehat{\mathcal{L}} := (\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F)$ be a 3-polar generalized neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.7, 0.8) & \text{if } z = 0,\\ (0.7, 0.0, 0.0) & \text{if } z = 1,\\ (0.0, 0.0, 0.9) & \text{if } z = 2,\\ (0.0, 0.0, 0.0) & \text{if } z = \alpha,\\ (0.0, 0.0, 0.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IT} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.7, 0.8) & \text{if } z = 0,\\ (0.7, 0.0, 0.0) & \text{if } z = 1,\\ (0.5, 0.8, 0.9) & \text{if } z = 2,\\ (0.0, 0.0, 0.7) & \text{if } z = \alpha,\\ (0.0, 0.0, 0.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.3, 0.1) & \text{if } z = 0,\\ (1.0, 1.0, 0.2) & \text{if } z = 1,\\ (0.3, 0.4, 1.0) & \text{if } z = 2,\\ (0.4, 1.0, 1.0) & \text{if } z = \alpha,\\ (1.0, 1.0, 1.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_F : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.4, 0.4) & \text{if } z = 0,\\ (0.4, 1.0, 1.0) & \text{if } z = 1,\\ (1.0, 0.2, 0.1) & \text{if } z = 2,\\ (1.0, 0.3, 1.0) & \text{if } z = \alpha,\\ (1.0, 1.0, 1.0) & \text{if } z = \beta, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a 3-polar generalized $(q,\ \in\vee q)$-neutrosophic subalgebra of X.

Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras.

Theorem 5.3. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Then $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, and so $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$ or $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. If $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq n_T^i > 1 - n_T^i$ since $n_T^i > 0.5$ for all $i = 1, 2, \cdots, k$. Hence $z*y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, and so $T_q(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. By the similar way, we can verify that $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X. Let $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$ of $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. If $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $(\pi_i \circ \widehat{\ell}_F)(z * y) \leq n_F^i < 1 - n_F^i$ since $n_F^i < 0.5$ for all $i = 1, 2, \cdots, k$. Thus $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, and hence $F_q(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, the set $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X. □

Theorem 5.4. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Let $z, y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ for $\hat{n}_T \in (0.5, 1]^k$. If $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, then obviously $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. If $z \in U(\widehat{\ell}_T, \hat{n}_T)$ and $y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$, i.e., $z \in T_q(\widehat{\ell}_T, \hat{n}_T)$. It follows that $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. We can prove $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ whenever $y \in U(\widehat{\ell}_T, \hat{n}_T)$ and $z \in T_q(\widehat{\ell}_T, \hat{n}_T)$ in the same way. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$ and $\widehat{\ell}_T(y) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$ and so $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Thus $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Therefore $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X for $\hat{n}_T \in (0.5, 1]^k$. Now, let $z, y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F \in [0, 0.5)^k$. If $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then obviously $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $z \in L(\widehat{\ell}_F, \hat{n}_F)$ and $y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $\widehat{\ell}_F(z) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$, i.e., $z \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Hence $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Similarly, we can prove that if $y \in L(\widehat{\ell}_F, \hat{n}_F)$ and $z \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $\widehat{\ell}_F(z) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$, that is, $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Hence $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Therefore $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X for all $\hat{n}_F \in [0, 0.5)^k$. In the same way, we can show that $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X for $\hat{n}_{IT} \in (0.5, 1]^k$ and $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X for all $\hat{n}_{IF} \in [0, 0.5)^k$. □

We provide conditions for a k-polar generalized neutrosophic set to be a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

Theorem 5.5. *For a subalgebra S of X, let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X such that*

$$(\forall z \in S)(\widehat{\ell}_T(z) \geq \widehat{0.5},\ \widehat{\ell}_{IT}(z) \geq \widehat{0.5},\ \widehat{\ell}_{IF}(z) \leq \widehat{0.5},\ \widehat{\ell}_F(z) \leq \widehat{0.5}), \tag{5.2}$$

$$(\forall z \in X \setminus S)(\widehat{\ell}_T(z) = \widehat{0} = \widehat{\ell}_{IT}(z),\ \widehat{\ell}_{IF}(z) = \widehat{1} = \widehat{\ell}_F(z)). \tag{5.3}$$

Then $\widehat{\mathcal{L}}$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X.

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k T_q(\widehat{\ell}_T, \hat{n}_T)^i$. Then $(\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1$ and $(\pi_i \circ \widehat{\ell}_T)(y) + n_T^i > 1$ for all $i = 1, 2, \cdots, k$. If $z * y \notin S$, then $z \in X \setminus S$ or $y \in X \setminus S$ since S is a subalgebra of X. Hence $(\pi_i \circ \widehat{\ell}_T)(z) = 0$ or $(\pi_i \circ \widehat{\ell}_T)(y) = 0$, which imply that $n_T^i > 1$, a contradiction. Thus $z * y \in S$ and so $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq 0.5$ by (5.2). If $n_T^i > 0.5$, then $(\pi_i \circ \widehat{\ell}_T)(z * y) + n_T^i > 1$, ie., $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)^i$ for all $i = 1, 2, \cdots, k$. Hence $z * y \in \bigcap_{i=1}^k T_q(\widehat{\ell}_T, \hat{n}_T)^i = T_q(\widehat{\ell}_T, \hat{n}_T)$. Similarly, if $z, y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, then $z * y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Let $z, y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$. Then $(\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1$ and $(\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF}^i < 1$ for all $i = 1, 2, \cdots, k$, which implies that $z * y \in S$. If $n_{IF}^i \geq 0.5$, then $(\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq 0.5 \leq n_{IF}^i$ for all $i = 1, 2, \cdots, k$ which shows that $z * y \in \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$. If $n_{IF}^i < 0.5$, then $(\pi_i \circ \widehat{\ell}_{IF})(z * y) + n_{IF}^i < 1$ for all $i = 1, 2, \cdots, k$ and so $z * y \in \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Similarly way is to show that if $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X. □

Combining Theorems 5.3 and 5.5, we have the following corollary.

Corollary 5.6. *If a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ satisfies two conditions* (5.2) *and* (5.3) *for a subalgebra S of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

6 Conclusions

We have introduced k-polar generalized neutrosophic set and have applied it to BCK/BCI-algebras. We have defined k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra and have studid various properties. We have discussed characterization of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We have shown that the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are also a k-polar generalized neutrosophic subalgebra. Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we have shown that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras. Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we have shown that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras. Using the generaliged neutrosophic $\in\vee q$-sets, we have established a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We have provided conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

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Neutrosophic $\mathcal{N}$-Structures Applied to Sheffer Stroke BL-Algebras

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Tugce Katican, Tahsin Oner, Akbar Rezaei, Florentin Smarandache (2021). Neutrosophic N-Structures Applied to Sheffer Stroke BL-Algebras. *Computer Modeling in Engineering & Sciences,* 129(1), 355-372. DOI: 10.32604/cmes.2021.016996

ABSTRACT

In this paper, we introduce a neutrosophic $\mathcal{N}$-subalgebra, a (ultra) neutrosophic $\mathcal{N}$-filter, level sets of these neutrosophic $\mathcal{N}$-structures and their properties on a Sheffer stroke BL-algebra. By defining a quasi-subalgebra of a Sheffer stroke BL-algebra, it is proved that the level set of neutrosophic $\mathcal{N}$-subalgebras on the algebraic structure is its quasi-subalgebra and vice versa. Then we show that the family of all neutrosophic $\mathcal{N}$-subalgebras of a Sheffer stroke BL-algebra forms a complete distributive lattice. After that a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is described, we demonstrate that every neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its neutrosophic $\mathcal{N}$-subalgebra but the inverse is generally not true. Finally, it is presented that a level set of a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is also its (ultra) filter and the inverse is always true. Moreover, some features of neutrosophic $\mathcal{N}$-structures on a Sheffer stroke BL-algebra are investigated.

KEYWORDS

Sheffer stroke BL-algebra; (ultra) filter; neutrosophic $\mathcal{N}$-subalgebra; (ultra) neutrosophic $\mathcal{N}$-filter

1 Introduction

Fuzzy set theory, which has the truth (t) (membership) function and state positive meaning of information, is introduced by Zadeh [1] as a generalization the classical set theory. This led scientists to find negative meaning of information. Hence, intuitionistic fuzzy sets [2] which are fuzzy sets with the falsehood (f) (nonmembership) function were introduced by Atanassov. However, there exist uncertainty and vagueness in the language, as well as positive ana negative meaning of information. Thus, Smarandache defined neutrosophic sets which are intuitionistic fuzzy sets with the indeterminacy/neutrality (i) function [3,4]. Thereby, neutrosophic sets are determined on three components: $(t, i, f) : (truth,\ indeterminacy,\ falsehood)$ [5]. Since neutrosophy enables that information in language can be comprehensively examined at all points, many researchers applied neutrosophy to different theoretical areas such as BCK/BCI-algebras, BE-algebras, semigroups, metric spaces, Sheffer stroke Hilbert algebras and strong Sheffer stroke non-associative MV-algebras [6–15] so as to improve devices imitating human behaviours and thoughts, artificial intelligence and technological tools.

Sheffer stroke (or Sheffer operation) was originally introduced by Sheffer [16]. Since Sheffer stroke can be used by itself without any other logical operators to build a logical system which is easy to control, Sheffer stroke can be applied to many logical algebras such as Boolean algebras [17], ortholattices [18], Sheffer stroke Hilbert algebras [19]. On the other side, BL-algebras were introduced by Hájek as an axiom system of his Basic Logic (BL) for fuzzy propositional logic, and he widely studied many types of filters [20]. Moreover, Oner et al. [21] introduced BL-algebras with Sheffer operation and investigated some types of (fuzzy) filters.

We give fundamental definitions and notions about Sheffer stroke BL-algebras, $\mathcal{N}$-functions and neutrosophic $\mathcal{N}$-structures defined by these functions on a crispy set X. Then a neutrosophic $\mathcal{N}$-subalgebra and a (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-structure are presented on Sheffer stroke BL-algebras. By defining a quasi-subalgebra of a Sheffer stroke BL-algebra, it is proved that every (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-subalgebra of the algebra is the quasi-subalgebra and the inverse is true. Also, we show that the family of all neutrosophic $\mathcal{N}$-subalgebras of this algebraic structure forms a complete distributive lattice. Some properties of neutrosophic $\mathcal{N}$-subalgebras of Sheffer stroke BL-algebras are examined. Indeed, we investigate the case which $\mathcal{N}$-functions defining a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra are constant. Moreover, we define a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra by $\mathcal{N}$-functions and analyze many features. It is demonstrated that (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its filter but the inverse does not hold in general. In fact, we propound that (τ, γ, ρ)-level set of a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its (ultra) filter and the inverse is true. Finally, new subsets of a Sheffer stroke BL-algebra are defined by the $\mathcal{N}$-functions and special elements of the algebra. It is illustrated that these subsets are (ultra) filters of a Sheffer stroke BL-algebra for the (ultra) neutrosophic $\mathcal{N}$-filter but the special conditions are necessary to prove the inverse.

2 Preliminaries

In this section, basic definitions and notions on Sheffer stroke BL-algebras and neutrosophic $\mathcal{N}$-structures.

Definition 2.1. [18] Let $\mathcal{H} = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke (or Sheffer operation)* if it satisfies the following conditions:

$(S1)$ $x \mid y = y \mid x$,

$(S2)$ $(x \mid x) \mid (x \mid y) = x$,

$(S3)$ $x \mid ((y \mid z) \mid (y \mid z)) = ((x \mid y) \mid (x \mid y)) \mid z$,

$(S4)$ $(x \mid ((x \mid x) \mid (y \mid y))) \mid (x \mid ((x \mid x) \mid (y \mid y))) = x$.

Definition 2.2. [21] A Sheffer stroke BL-algebra is an algebra $(C, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ satisfying the following conditions:

$(sBL-1)$ $(C, \vee, \wedge, 0, 1)$ is a bounded lattice,

$(sBL-2)$ $(C, |)$ is a groupoid with the Sheffer stroke,

$(sBL-3)$ $c_1 \wedge c_2 = (c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))$,

$(sBL-4)$ $(c_1 \mid (c_2 \mid c_2)) \vee (c_2 \mid (c_1 \mid c_1)) = 1$,

for all $c_1, c_2 \in C$.

$1 = 0 \mid 0$ is the greatest element and $0 = 1 \mid 1$ is the least element of C.

Proposition 2.1. [21] In any Sheffer stroke BL-algebra C, the following features hold, for all $c_1, c_2, c_3 \in C$:

(1) $c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3))) = c_2 \mid ((c_1 \mid (c_3 \mid c_3)) \mid (c_1 \mid (c_3 \mid c_3)))$,
(2) $c_1 \mid (c_1 \mid c_1) = 1$,
(3) $1 \mid (c_1 \mid c_1) = c_1$,
(4) $c_1 \mid (1 \mid 1) = 1$,
(5) $(c_1 \mid 1) \mid (c_1 \mid 1) = c_1$,
(6) $(c_1 \mid c_2) \mid (c_1 \mid c_2) \leq c_3 \Leftrightarrow c_1 \leq c_2 \mid (c_3 \mid c_3)$
(7) $c_1 \leq c_2$ iff $c_1 \mid (c_2 \mid c_2) = 1$,
(8) $c_1 \leq c_2 \mid (c_1 \mid c_1)$,
(9) $c_1 \leq (c_1 \mid c_2) \mid c_2$,
(10) (a) $(c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2))) \leq c_1$,
(b) $(c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2))) \leq c_2$.
(11) If $c_1 \leq c_2$, then
(i) $c_3 \mid (c_1 \mid c_1) \leq c_3 \mid (c_2 \mid c_2)$,
(ii) $(c_1 \mid c_3) \mid (c_1 \mid c_3) \leq (c_2 \mid c_3) \mid (c_2 \mid c_3)$,
(iii) $c_2 \mid (c_3 \mid c_3) \leq c_1 \mid (c_3 \mid c_3)$.
(12) $c_1 \mid (c_2 \mid c_2) \leq (c_3 \mid (c_1 \mid c_1)) \mid ((c_3 \mid (c_2 \mid c_2)) \mid (c_3 \mid (c_2 \mid c_2)))$,
(13) $c_1 \mid (c_2 \mid c_2) \leq (c_2 \mid (c_3 \mid c_3)) \mid ((c_1 \mid (c_3 \mid c_3)) \mid (c_1 \mid (c_3 \mid c_3)))$,
(14) $((c_1 \vee c_2) \mid c_3) \mid ((c_1 \vee c_2) \mid c_3) = ((c_1 \mid c_3) \mid (c_1 \mid c_3)) \vee ((c_2 \mid c_3) \mid (c_2 \mid c_3))$,
(15) $c_1 \vee c_2 = ((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2)) \wedge ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))$.

Lemma 2.1. [21] Let C be a Sheffer stroke BL-algebra. Then

$$(c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2) = (c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1),$$

for all $c_1, c_2 \in C$.

Corollary 2.1. [21] Let C be a Sheffer stroke BL-algebra. Then

$$c_1 \vee c_2 = (c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2),$$

for all $c_1, c_2 \in C$.

Lemma 2.2. [21] Let C be a Sheffer stroke BL-algebra. Then

$$c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3))) = (c_1 \mid (c_2 \mid c_2)) \mid ((c_1 \mid (c_3 \mid c_3)) \mid (c_1 \mid (c_3 \mid c_3))),$$

for all $c_1, c_2, c_3 \in C$.

Definition 2.3. [21] A filter of C is a nonempty subset $P \subseteq C$ satisfying

$(SF-1)$ if $c_1, c_2 \in P$, then $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in P$,

$(SF-2)$ if $c_1 \in P$ and $c_1 \leq c_2$, then $c_2 \in P$.

Proposition 2.2. [21] Let P be a nonempty subset of C. Then P is a filter of C if and only if the following hold:

$(SF-3)$ $1 \in P$,

$(SF-4)$ $c_1 \in P$ and $c_1 \mid (c_2 \mid c_2) \in P$ imply $c_2 \in P$.

Definition 2.4. [21] Let P be a filter of C. Then P is called an ultra filter of C if it satisfies $c \in P$ or $c \mid c \in P$, for all $c \in C$.

Lemma 2.3. [21] A filter P of C is an ultra filter of C if and only if $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$, for all $c_1, c_2 \in C$.

Definition 2.5. [8] $\mathcal{F}(X, [-1, 0])$ denotes the collection of functions from a set X to $[-1, 0]$ and an element of $\mathcal{F}(X, [-1, 0])$ is called a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). An $\mathcal{N}$-structure refers to an ordered pair (X, f) of X and $\mathcal{N}$-function f on X.

Definition 2.6. [12] A neutrosophic $\mathcal{N}$-structure over a nonempty universe X is defined by

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X \right\}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}$-structure X_N over X satisfies the condition $(\forall x \in X)(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$.

Definition 2.7. [13] Let X_N be a neutrosophic $\mathcal{N}$-structure on a set X and τ, γ, ρ be any elements of $[-1, 0]$ such that $-3 \leq \tau + \gamma + \rho \leq 0$. Consider the following sets:

$T_N^\tau := \{x \in X : T_N(x) \leq \tau\}$,

$I_N^\gamma := \{x \in X : I_N(x) \geq \gamma\}$

and

$F_N^\rho := \{x \in X : F_N(x) \leq \rho\}$.

The set

$X_N(\tau, \gamma, \rho) := \{x \in X : T_N(x) \leq \tau, I_N(x) \geq \gamma \text{ and } T_N(x) \leq \rho\}$

is called the (τ, γ, ρ)-level set of X_N. Moreover, $X_N(\tau, \gamma, \rho) = T_N^\tau \cap I_N^\gamma \cap F_N^\rho$.

Consider sets

$X_N^{c_t} := \{x \in X : T_N(x) \leq T_N(c_t)\}$,

$X_N^{c_i} := \{x \in X : I_N(x) \geq I_N(c_i)\}$

and

$X_N^{c_f} := \{x \in X : F_N(x) \leq F_N(c_f)\}$,

for any $c_t, c_i, c_f \in X$. Obviously, $c_t \in X_N^{c_t}, c_i \in X_N^{c_i}$ and $c_f \in X_N^{c_f}$ [13].

3 Neutrosophic $\mathcal{N}$-Structures

In this section, neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-filters on Sheffer stroke BL-algebras. Unless otherwise specified, C denotes a Sheffer stroke BL-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}$-structure C_N on a Sheffer stroke BL-algebra C is called a neutrosophic $\mathcal{N}$-subalgebra of C if the following condition is valid:

$$\begin{aligned}&\min\{T_N(c_1), T_N(c_2)\} \leq T_N(c_1 \mid (c_2 \mid c_2)),\\ &\max\{I_N(c_1), I_N(c_2)\} \geq I_N(c_1 \mid (c_2 \mid c_2)) \text{ and}\\ &\max\{F_N(c_1), F_N(c_2)\} \geq F_N(c_1 \mid (c_2 \mid c_2)),\end{aligned} \tag{1}$$

for all $c_1, c_2 \in C$.

Example 3.1. Consider a Sheffer stroke BL-algebra C where the set $C = \{0, a, b, c, d, e, f, 1\}$ and the Sheffer operation $|$, the join operation $\vee$ and the meet operation $\wedge$ on C has the Cayley tables in Tab. 1 [21]. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{x}{(-0.08, -0.999, -0.26)} : x = d, 1\right\} \cup \left\{\frac{x}{(-0.92, -0.52, -0.0012)} : x \in C - \{d, 1\}\right\}$$

on C is a neutrosophic $\mathcal{N}$-subalgebra of C.

Table 1: Tables of the Sheffer operation $|$, the join operation $\vee$ and the meet operation $\wedge$ on C

$\mid$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	1
$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	d	e	d	e	1	1
b	b	d	b	f	d	1	f	1
c	c	e	f	c	1	e	f	1
d	d	d	d	1	d	1	1	1
e	e	e	1	e	1	e	1	1
f	f	1	f	f	1	1	f	1
1	1	1	1	1	1	1	1	1
$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

Definition 3.2. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and τ, γ, ρ be any elements of $[-1,0]$ such that $-3 \leq \tau+\gamma+\rho \leq 0$. For the sets

$T_N^{\tau} := \{c \in C : T_N(c) \geq \tau\}$,

$I_N^{\gamma} := \{c \in C : I_N(c) \leq \gamma\}$

and

$F_N^{\rho} := \{c \in C : F_N(c) \leq \rho\}$,

the set

$C_N(\tau,\gamma,\rho) := \{c \in C : T_N(c) \geq \tau, I_N(c) \leq \gamma \text{ and } F_N(c) \leq \rho\}$

is called the (τ,γ,ρ)-level set of C_N. Moreover, $C_N(\tau,\gamma,\rho) = T_N^{\tau} \cap I_N^{\gamma} \cap F_N^{\rho}$.

Definition 3.3. A subset D of a Sheffer stroke BL-algebra C is called a quasi-subalgebra of C if $c_1 \mid (c_2 \mid c_2) \in D$, for all $c_1, c_2 \in D$. Obviously, C itself and $\{1\}$ are quasi-subalgebras of C.

Example 3.2. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then $\{0, a, f, 1\}$ is a quasi-subalgebra of C.

Theorem 3.1. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and τ, γ, ρ be any elements of $[-1,0]$ such that $-3 \leq \tau+\gamma+\rho \leq 0$. If C_N is a neutrosophic $\mathcal{N}$-subalgebra of C, then the nonempty level set $C_N(\tau,\gamma,\rho)$ of C_N is a quasi-subalgebra of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-subalgebra of C and c_1, c_2 be any elements of $C_N(\tau,\gamma,\rho)$, for $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Then $T_N(c_1), T_N(c_2) \geq \tau, I_N(c_1), I_N(c_2) \leq \gamma$ and $F_N(c_1), F_N(c_2) \leq \rho$. Since

$\tau \leq \min\{T_N(c_1), T_N(c_2)\} \leq T_N(c_1 \mid (c_2 \mid c_2))$,

$I_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\} \leq \gamma$

and

$F_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\} \leq \rho$,

for all $c_1, c_2 \in C$, we obtain that $c_1 \mid (c_2 \mid c_2) \in T_N^{\tau}$, $c_1 \mid (c_2 \mid c_2) \in I_N^{\gamma}$ and $c_1 \mid (c_2 \mid c_2) \in F_N^{\rho}$, and so, $c_1 \mid (c_2 \mid c_2) \in T_N^{\tau} \cap I_N^{\gamma} \cap F_N^{\rho} = C_N(\tau,\gamma,\rho)$. Hence, $C_N(\tau,\gamma,\rho)$ is a quasi-subalgebra of C.

Theorem 3.2. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and T_N^{τ}, I_N^{γ} and F_N^{ρ} be quasi-subalgebras of C, for all $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Then C_N is a neutrosophic $\mathcal{N}$-subalgebra of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C, and T_N^{τ}, I_N^{γ} and F_N^{ρ} be quasi-subalgebras of C, for all $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Suppose that c_1 and c_2 be any elements of C such that $w_1 = T_N(c_1 \mid (c_2 \mid c_2)) < \min\{T_N(c_1), T_N(c_2)\} = w_2$, $t_1 = \max\{I_N(c_1), I_N(c_2)\} < I_N(c_1 \mid (c_2 \mid c_2)) = t_2$ and $r_1 = \max\{F_N(c_1), F_N(c_2)\} < F_N(c_1 \mid (c_2 \mid c_2)) = r_2$. If $\tau_1 = \dfrac{1}{2}(w_1+w_2) \in [-1,0)$, $\gamma_1 = \dfrac{1}{2}(t_1+t_2) \in [-1,0)$ and $\rho_1 = \dfrac{1}{2}(r_1+r_2) \in [-1,0)$, then $w_1 < \tau_1 < w_2$, $t_1 < \gamma_1 < t_2$ and $r_1 < \rho_1 < r_2$. Thus, $c_1, c_2 \in T_N^{\tau_1}$, $c_1, c_2 \in I_N^{\gamma_1}$ and $c_1, c_2 \in F_N^{\rho_1}$ but $c_1 \mid (c_2 \mid c_2) \notin T_N^{\tau_1}$, $c_1 \mid (c_2 \mid c_2) \notin I_N^{\gamma_1}$ and $c_1 \mid (c_2 \mid c_2) \notin F_N^{\rho_1}$, which are contradictions. Hence, $\min\{T_N(c_1), T_N(c_2)\} \leq T_N(c_1 \mid (c_2 \mid c_2))$, $I_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$ and $F_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\}$, for all $c_1, c_2 \in C$. Thereby, C_N is a neutrosophic $\mathcal{N}$-subalgebra of C.

Theorem 3.3. Let $\{C_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}$-subalgebras of a Sheffer stroke BL-algebra C. Then $\{C_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive lattice.

Proof. Let D be a nonempty subset of $\{C_{N_i} : i \in \mathbb{N}\}$. Since C_{N_i} is a neutrosophic $\mathcal{N}$-subalgebra of C, for all $i \in \mathbb{N}$, it satisfies the condition (1). Then $\bigcap D$ satisfies the condition (1). Thus, $\bigcap D$ is a neutrosophic $\mathcal{N}$-subalgebra of C. Let E be a family of all neutrosophic $\mathcal{N}$-subalgebras of C containing $\bigcup\{C_{N_i} : i \in \mathbb{N}\}$. Thus, $\bigcap E$ is also a neutrosophic $\mathcal{N}$-subalgebra of C. If $\bigwedge_{i\in\mathbb{N}} C_{N_i} = \bigcap_{i\in\mathbb{N}} C_{N_i}$ and $\bigvee_{i\in\mathbb{N}} C_{N_i} = \bigcap E$, then $(\{C_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ forms a complete lattice. Also, it is distibutive by the definitions of $\bigvee$ and $\bigwedge$.

Lemma 3.1. Let C_N be a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra C. Then $T_N(c) \leq T_N(1)$, $I_N(c) \geq I_N(1)$ and $F_N(c) \geq F_N(1)$, for all $c \in C$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-subalgebra of C. Then it follows from Poposition 2.1 (2) that

$$T_N(c) = \min\{T_N(c), T_N(c)\} \leq T_N(c \mid (c \mid c)) = T_N(1),$$

$$I_N(1) = I_N(c \mid (c \mid c)) \leq \max\{I_N(c), I_N(c)\} = I_N(c)$$

and

$$F_N(1) = F_N(c \mid (c \mid c)) \leq \max\{F_N(c), F_N(c)\} = F_N(c),$$

for all $c \in C$.

The inverse of Lemma 3.1 is not true in general.

Example 3.3. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{x}{(-0.01, -0.1, -0.11)} : x = a, b, 1\right\} \cup \left\{\frac{x}{(-0.1, -0.01, -0.01)} : x \in C - \{a, b, 1\}\right\}$$

on C is not a neutrosophic $\mathcal{N}$-subalgebra of C since $\max\{F_N(a), F_N(b)\} = -0.11 < -0.01 = F_N(f) = F_N(a \mid (b \mid b))$.

Lemma 3.2. A neutrosophic $\mathcal{N}$-subalgebra C_N of a Sheffer stroke BL-algebra C satisfies $T_N(c_1) \leq T_N(c_1 \mid (c_2 \mid c_2))$, $I_N(c_1) \geq I_N(c_1 \mid (c_2 \mid c_2))$ and $F_N(c_1) \geq F_N(c_1 \mid (c_2 \mid c_2))$, for all $c_1, c_2 \in C$ if and only if T_N, I_N and F_N are constant.

Proof. Let C_N be a a neutrosophic $\mathcal{N}$-subalgebra of C such that $T_N(c_1) \leq T_N(c_1 \mid (c_2 \mid c_2))$, $I_N(c_1) \geq I_N(c_1 \mid (c_2 \mid c_2))$ and $F_N(c_1) \geq F_N(c_1 \mid (c_2 \mid c_2))$, for all $c_1, c_2 \in C$. Since $T_N(1) \leq T_N(1 \mid (c \mid c)) = T_N(c)$, $I_N(1) \geq I_N(1 \mid (c \mid c)) = I_N(c)$ and $F_N(1) \geq F_N(1 \mid (c \mid c)) = F_N(c)$ from Proposition 2.1 (3), it is obtained from Lemma 3.1 that $T_N(c) = T_N(1)$, $I_N(c) = I_N(1)$ and $F_N(c) = F_N(1)$, for all $c \in C$. Hence, T_N, I_N and F_N are constant.

Conversely, it is obvious since T_N, I_N and F_N are constant.

Definition 3.4. A neutrosophic $\mathcal{N}$-structure C_N on a Sheffer stroke BL-algebra C is called a neutrosophic $\mathcal{N}$-filter of C if

1. $c_1 \leq c_2$ implies $T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1)$ and $F_N(c_2) \leq F_N(c_1)$,
2. $\min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2))$, $I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$ and $F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\}$,

for all $c_1, c_2 \in C$.

Example 3.4. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{x}{(-0.3,-1,-0.15)} : x = c,e,f,1\right\} \cup \left\{\frac{x}{(-1,-0.7,0)} : x = 0,a,b,d\right\}$$

on C is a neutrosophic $\mathcal{N}$-filter of C.

Theorem 3.4. *Let C_N be a a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C. Then C_N is a neutrosophic $\mathcal{N}$-filter of C if and only if*

$$\begin{aligned}
&\min\{T_N(c_1), T_N(c_1 \mid (c_2 \mid c_2))\} \leq T_N(c_2) \leq T_N(1),\\
&I_N(1) \leq I_N(c_2) \leq \max\{I_N(c_1), I_N(c_1 \mid (c_2 \mid c_2))\} \textit{ and}\\
&F_N(1) \leq F_N(c_2) \leq \max\{F_N(c_1), F_N(c_1 \mid (c_2 \mid c_2))\},
\end{aligned} \tag{2}$$

for all $c_1, c_2 \in C$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C. Then it follows from (sBL-3) and Definition 3.4 that

$$\min\{T_N(c_1),T_N(c_1 \mid (c_2 \mid c_2))\} \leq T_N((c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))) = T_N(c_1 \wedge c_2) \leq T_N(c_2) \leq T_N(1),$$

$$I_N(1) \leq I_N(c_2) \leq I_N(c_1 \wedge c_2) = I_N((c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))) \leq \max\{I_N(c_1),I_N(c_1 \mid (c_2 \mid c_2))\}$$

and

$$F_N(1) \leq F_N(c_2) \leq F_N(c_1 \wedge c_2) = F_N((c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))) \leq \max\{F_N(c_1),F_N(c_1 \mid (c_2 \mid c_2))\},$$

for all $c_1, c_2 \in C$.

Conversely, let C_N be a a neutrosophic $\mathcal{N}$-structure on C satisfying the condition (2). Assume that $c_1 \leq c_2$. Then $c_1 \mid (c_2 \mid c_2) = 1$ from Proposition 2.1 (7). Thus,

$$T_N(c_1) = \min\{T_N(c_1), T_N(1)\} = \min\{T_N(c_1), T_N(c_1 \mid (c_2 \mid c_2))\} \leq T_N(c_2),$$

$$I_N(c_2) \leq \max\{I_N(c_1), I_N(c_1 \mid (c_2 \mid c_2))\} = \max\{I_N(c_1), I_N(1)\} = I_N(c_1)$$

and

$$F_N(c_2) \leq \max\{F_N(c_1), F_N(c_1 \mid (c_2 \mid c_2))\} = \max\{F_N(c_1), F_N(1)\} = F_N(c_1),$$

for all $c_1, c_2 \in C$. Also, it follows from Proposition 2.1 (9), (S1) and (S2) that

$$\begin{aligned}
\min\{T_N(c_1), T_N(c_2)\} &\leq \min\{T_N(c_1), T_N(c_1 \mid (c_1 \mid c_2))\}\\
&= \min\{T_N(c_1), T_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\}\\
&\leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),
\end{aligned}$$

$$\begin{aligned}
I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) &\leq \max\{I_N(c_1), I_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\}\\
&= \max\{I_N(c_1), I_N(c_1 \mid (c_1 \mid c_2))\}\\
&\leq \max\{I_N(c_1), I_N(c_2)\}
\end{aligned}$$

and

$$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le \max\{F_N(c_1), F_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\}$$
$$= \max\{F_N(c_1), F_N(c_1 \mid (c_1 \mid c_2))\}$$
$$\le \max\{F_N(c_1), F_N(c_2)\},$$

for all $c_1, c_2 \in C$. Thus, C_N is a neutrosophic $\mathcal{N}$-filter of C.

Corollary 3.1. Let C_N be a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C. Then

1. $\min\{T_N(c_3), T_N(c_3 \mid (((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1)) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))))\} \le T_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))$,
   $I_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2)) \le \max\{I_N(c_3), I_N(c_3 \mid (((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1)) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))))\}$
   and $F_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2)) \le \max\{F_N(c_3), F_N(c_3 \mid (((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1)) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))))\}$,
2. $\min\{T_N(c_3), T_N(c_3 \mid ((c_1 \mid (c_2 \mid c_2)) \mid\mid (c_1 \mid (c_2 \mid c_2))))\} \le T_N(c_1 \mid (c_2 \mid c_2))$,
   $I_N(c_1 \mid (c_2 \mid c_2)) \le \max\{I_N(c_3), I_N(c_3 \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\}$ *and*
   $F_N(c_1 \mid (c_2 \mid c_2)) \le \max\{F_N(c_3), F_N(c_3 \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\}$,
3. $\min\{T_N(c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid\mid (c_2 \mid (c_3 \mid c_3)))), T_N(c_1 \mid (c_2 \mid c_2))\} \le T_N(c_1 \mid (c_3 \mid c_3))$,
   $I_N(c_1 \mid (c_3 \mid c_3)) \le \max\{I_N(c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3)))), I_N(c_1 \mid (c_2 \mid c_2))\}$ *and*
   $F_N(c_1 \mid (c_3 \mid c_3)) \le \max\{F_N(c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3)))), F_N(c_1 \mid (c_2 \mid c_2))\}$,
4. $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1)$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1)$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1)$ imply $T_N(c_1) \le T_N(c_2)$, $I_N(c_2) \le I_N(c_1)$ and $F_N(c_2) \le F_N(c_1)$,

for all $c_1, c_2, c_3 \in C$.

Proof. It is proved from Theorem 3.4, Lemma 2.1 and Lemma 2.2.

Lemma 3.3. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C. Then C_N is a neutrosophic $\mathcal{N}$-filter of C if and only if

$$c_1 \le c_2 \mid (c_3 \mid c_3) \text{ implies } \begin{pmatrix} \min\{T_N(c_1), T_N(c_2)\} \le T_N(c_3), \\ I_N(c_3) \le \max\{I_N(c_1), I_N(c_2)\} \quad and \\ F_N(c_3) \le \max\{F_N(c_1), F_N(c_2)\}, \end{pmatrix} \tag{3}$$

for all $c_1, c_2, c_3 \in C$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C and $c_1 \le c_2 \mid (c_3 \mid c_3)$. Then it is obtained from Definition 3.4 (1) and Theorem 3.4 that

$$\min\{T_N(c_1), T_N(c_2)\} \le \min\{T_N(c_2), T_N(c_2 \mid (c_3 \mid c_3))\} \le T_N(c_3),$$

$$I_N(c_3) \le \max\{I_N(c_2), I_N(c_2 \mid (c_3 \mid c_3))\} \le \max\{I_N(c_1), I_N(c_2)\}$$

and

$$F_N(c_3) \le \max\{F_N(c_2), F_N(c_2 \mid (c_3 \mid c_3))\} \le \max\{F_N(c_1), F_N(c_2)\},$$

for all $c_1, c_2, c_3 \in C$.

Conversely, let C_N be a neutrosophic $\mathcal{N}$-structure on C satisfying the condition (3). Since it is known from Proposition 2.1 (4) that $c \le 1 = c \mid (1 \mid 1)$, for all $c \in C$, we get that $T_N(c) = \min\{T_N(c), T_N(c)\} \le T_N(1)$, $I_N(1) \le \max\{I_N(c), I_N(c)\} = I_N(c)\}$ and $F_N(1) \le \max\{F_N(c), F_N(c)\} = F_N(c)\}$, for all $c \in C$. Suppose that $c_1 \le c_2$. Since we have $c_1 \le c_2 = 1 \mid (c_2 \mid c_2)$ from Proposition 2.1

(3), it is obtained that $T_N(c_1) = \min\{T_N(c_1), T_N(1)\} \leq T_N(c_2)$, $I_N(c_2) \leq \max\{I_N(c_1), I_N(1)\} = I_N(c_1)$ and $F_N(c_2) \leq \max\{F_N(c_1), F_N(1)\} = F_N(c_1)$. Since $c_1 \leq (c_1 \mid c_2) \mid c_2 = c_2 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2)))$ from Proposition 2.1 (9), (S1) and (S2), it follows that

$$\min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$$

$$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$$

and

$$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\},$$

for all $c_1, c_2 \in C$. Thus, C_N is a neutrosophic $\mathcal{N}$-filter of C.

Lemma 3.4. Every neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C is a neutrosophic $\mathcal{N}$-subalgebra of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C. Since

$$\begin{aligned}
&((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))) \\
&\quad = c_1 \mid ((((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid (c_2 \mid c_2)) \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid (c_2 \mid c_2))) \\
&\quad = c_1 \mid ((c_1 \mid ((c_2 \mid (c_2 \mid c_2)) \mid (c_2 \mid (c_2 \mid c_2)))) \mid (c_1 \mid ((c_2 \mid (c_2 \mid c_2)) \mid (c_2 \mid (c_2 \mid c_2))))) \\
&\quad = c_1 \mid ((c_1 \mid (1 \mid 1)) \mid (c_1 \mid (1 \mid 1))) \\
&\quad = c_1 \mid (1 \mid 1) \\
&\quad = 1
\end{aligned}$$

from Proposition 2.1 (1), (2), (4) and (S3), it follows from Proposition 2.1 (7) that $(c_1 \mid c_2) \mid (c_1 \mid c_2) \leq c_1 \mid (c_2 \mid c_2)$, for all $c_1, c_2 \in C$. Then

$$\min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq T_N(c_1 \mid (c_2 \mid c_2)),$$

$$I_N(c_1 \mid (c_2 \mid c_2)) \leq I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$$

and

$$F_N(c_1 \mid (c_2 \mid c_2)) \leq F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\},$$

for all $c_1, c_2 \in C$. Thereby, C_N is a neutrosophic $\mathcal{N}$-subalgebra of C.

The inverse of Lemma 3.4 is usually not true.

Example 3.5. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{0}{(-1,0,0)}, \frac{1}{(0,-1,-1)}\right\} \cup \left\{\frac{x}{(-0.5,-0.5,-0.5)} : x \in C - \{0,1\}\right\}$$

on C is a neutrosophic $\mathcal{N}$-subalgebra of C whereas it is not a neutrosophic $\mathcal{N}$-filter of C since $\min\{T_N(a), T_N(b)\} = -0.5 > -1 = T_N((a \mid b) \mid (a \mid b))$.

Definition 3.5. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C. Then an ultra neutrosophic $\mathcal{N}$-filter C_N of C is a neutrosophic $\mathcal{N}$-filter of C satisfying $T_N(c) = T_N(1)$, $I_N(c) = I_N(1)$, $F_N(c) = F_N(1)$ or $T_N(c \mid c) = T_N(1)$, $I_N(c \mid c) = I_N(1)$, $F_N(c \mid c) = F_N(1)$, for all $c \in C$.

Example 3.6. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{ \frac{x}{(-0.02, -0.77, -0.6)} : x = b, d, f, 1 \right\} \cup \left\{ \frac{x}{(-0.79, -0.05, -0.41)} : x = 0, a, c, e \right\}$$

on C is an ultra neutrosophic $\mathcal{N}$-filter of C.

Remark 3.1. By Definition 3.5, every ultra neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C is a neutrosophic $\mathcal{N}$-filter of C but the inverse does not generally hold.

Example 3.7. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-filter

$$C_N = \left\{ \frac{x}{(-0.18, -0.82, -0.57)} : \ x = e, 1 \right\} \cup \left\{ \frac{x}{(-1, -0.64, -0.43)} : x \in C - \{e, 1\} \right\}$$

of C is not ultra since $T_N(a) \neq T_N(1) \neq T_N(a \mid a) = T_N(f)$, $I_N(a) \neq I_N(1) \neq I_N(a \mid a) = I_N(f)$ and $F_N(a) \neq F_N(1) \neq TF_N(a \mid a) = F_N(f)$.

Lemma 3.5. Let C_N be a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C. Then C_N is an ultra neutrosophic $\mathcal{N}$-filter of C if and only if $T_N(c_1) \neq T_N(1), T_N(c_2) \neq T_N(1)$, $I_N(c_1) \neq I_N(1), I_N(c_2) \neq I_N(1)$ and $F_N(c_1) \neq F_N(1), F_N(c_2) \neq F_N(1)$ imply $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$.

Proof. Let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C, and $T_N(c_1) \neq T_N(1), T_N(c_2) \neq T_N(1)$, $I_N(c_1) \neq I_N(1), I_N(c_2) \neq I_N(1)$ and $F_N(c_1) \neq F_N(1), F_N(c_2) \neq F_N(1)$, for any $c_1, c_2 \in C$. Then $T_N(c_1 \mid c_1) = T_N(1) = T_N(c_2 \mid c_2)$, $I_N(c_1 \mid c_1) = I_N(1) = I_N(c_2 \mid c_2)$ and $F_N(c_1 \mid c_1) = F_N(1) = F_N(c_2 \mid c_2)$. Since

$$(c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))) = (c_2 \mid c_2) \mid ((c_1 \mid (c_1 \mid c_1)) \mid (c_1 \mid (c_1 \mid c_1))) = (c_2 \mid c_2) \mid (1 \mid 1) = 1$$

and

$$(c_2 \mid c_2) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_2 \mid (c_1 \mid c_1))) = (c_1 \mid c_1) \mid ((c_2 \mid (c_2 \mid c_2)) \mid (c_2 \mid (c_2 \mid c_2))) = (c_1 \mid c_1) \mid (1 \mid 1) = 1$$

from (S1), (S3), Proposition 2.1 (2) and (4), it follows from Theorem 3.4 that

$$T_N(1) = \min\{T_N(1), T_N(1)\} = \min\{T_N(c_1 \mid c_1), T_N((c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\} \leq T_N(c_1 \mid (c_2 \mid c_2)),$$

$$I_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{I_N(c_1 \mid c_1), I_N((c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\} = \max\{I_N(1), I_N(1)\} = I_N(1),$$

$$F_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{F_N(c_1 \mid c_1), F_N((c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\} = \max\{F_N(1), F_N(1)\} = F_N(1),$$

and similarly, $T_N(1) \leq T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_2 \mid (c_1 \mid c_1)) \leq I_N(1)$, $F_N(c_2 \mid (c_1 \mid c_1)) \leq F_N(1)$. Hence, we obtain from Theorem 3.4 that $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$.

Conversely, let C_N be a neutrosophic $\mathcal{N}$-filter of C such that $T_N(c_1) \neq T_N(1), T_N(c_2) \neq T_N(1)$, $I_N(c_1) \neq I_N(1), I_N(c_2) \neq I_N(1)$ and $F_N(c_1) \neq F_N(1), F_N(c_2) \neq F_N(1)$ imply $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$. Assume that $T_N(c) \neq T_N(1) \neq T_N(0) = T_N(1 \mid 1)$, $I_N(c) \neq I_N(1) \neq I_N(0) = I_N(1 \mid 1)$ and $F_N(c) \neq F_N(1) \neq F_N(0) = F_N(1 \mid 1)$. Hence, $T_N(c \mid c) = T_N(1 \mid ((c \mid c) \mid (c \mid c))) = T_N(c \mid 1) = T_N(c \mid ((1 \mid 1) \mid (1 \mid 1))) = T_N(1)$, $T_N((1 \mid 1) \mid (c \mid c)) = T_N(1)$, $I_N(c \mid c) = I_N(1 \mid ((c \mid c) \mid (c \mid c))) = I_N(c \mid 1) = I_N(c \mid ((1 \mid 1) \mid (1 \mid 1))) = I_N(1)$, $I_N((1 \mid 1) \mid (c \mid c)) = I_N(1)$ and $F_N(c \mid c) = F_N(1 \mid ((c \mid c) \mid (c \mid c))) = F_N(c \mid 1) = F_N(c \mid ((1 \mid 1) \mid (1 \mid 1))) = F_N(1)$, $F_N((1 \mid 1) \mid$

$(c \mid c)) = F_N(1)$ from Proposition 2.1 (3), (4), (S1) and (S2). Suppose that $T_N(c \mid c) \neq T_N(1) \neq T_N(0) = T_N(1 \mid 1)$, $I_N(c) \neq I_N(1) \neq I_N(0) = I_N(1 \mid 1)$ and $F_N(c) \neq F_N(1) \neq F_N(0) = F_N(1 \mid 1)$. Thus, $T_N(c) = T_N(1 \mid (c \mid c)) = T_N((c \mid c) \mid ((1 \mid 1) \mid (1 \mid 1))) = T_N(1)$, $T_N((1 \mid 1) \mid ((c \mid c) \mid (c \mid c))) = T_N(1)$, $I_N(c) = I_N(1 \mid (c \mid c)) = I_N((c \mid c) \mid ((1 \mid 1) \mid (1 \mid 1))) = I_N(1)$, $I_N((1 \mid 1) \mid ((c \mid c) \mid (c \mid c))) = I_N(1)$ and $F_N(c) = F_N(1 \mid (c \mid c)) = F_N((c \mid c) \mid ((1 \mid 1) \mid (1 \mid 1))) = F_N(1)$, $F_N((1 \mid 1) \mid ((c \mid c) \mid (c \mid c))) = F_N(1)$ from Proposition 2.1 (3), (4), (S1) and (S2). Therefore, C_N is an ultra neutrosophic $\mathcal{N}$-filter of C.

Lemma 3.6. Let C_N be a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C. Then C_N is an ultra neutrosophic $\mathcal{N}$-filter of C if and only if $T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2)$, for all $c_1, c_2 \in C$.

Proof. Let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C. If $T_N(c_1) = T_N(1), I_N(c_1) = I_N(1)$, $F_N(c_1) = F_N(1)$ or $T_N(c_2) = T_N(1), I_N(c_2) = I_N(1)$, $F_N(c_2) = F_N(1)$, then the proof is completed from Theorem 3.4. Assume that $T_N(c_1) \neq T_N(1) \neq T_N(c_2), I_N(c_1) \neq I_N(1) \neq I_N(c_2)$ and $F_N(c_1) \neq F_N(1) \neq F_N(c_2)$. Thus, we have from Lemma 3.5 that $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$. Since

$$T_N(c_1 \vee c_2) = \min\{T_N(1), T_N(c_1 \vee c_2)\} = \min\{T_N(c_1 \mid (c_2 \mid c_2)), T_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))\} \leq T_N(c_2),$$

$$I_N(c_2) \leq \max\{I_N(c_1 \mid (c_2 \mid c_2)), I_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))\} = \max\{I_N(1), I_N(c_1 \vee c_2)\} = I_N(c_1 \vee c_2),$$

$$F_N(c_2) \leq \max\{F_N(c_1 \mid (c_2 \mid c_2)), F_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))\} = \max\{F_N(1), I_N(c_1 \vee c_2)\} = F_N(c_1 \vee c_2),$$

and similarly, $T_N(c_1 \vee c_2) = T_N(c_2 \vee c_1) \leq T_N(c_1)$, $I_N(c_1) \leq I_N(c_2 \vee c_1) = I_N(c_1 \vee c_2)$, $F_N(c_1) \leq F_N(c_2 \vee c_1) = F_N(c_1 \vee c_2)$ from Corollary 2.1 and Theorem 3.4, it follows that $T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2)$, for all $c_1, c_2 \in C$.

Conversely, let C_N be a neutrosophic $\mathcal{N}$-filter of C satisfying that $T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2)$, for any $c_1, c_2 \in C$. Since

$$T_N(1) = T_N(c \mid (c \mid c)) = T_N((c \mid ((c \mid c) \mid (c \mid c))) \mid ((c \mid c) \mid (c \mid c))) = T_N(c \vee (c \mid c)) \leq T_N(c) \vee T_N(c \mid c),$$

$$I_N(c) \vee I_N(c \mid c) \leq I_N(c \vee (c \mid c)) = I_N((c \mid ((c \mid c) \mid (c \mid c))) \mid ((c \mid c) \mid (c \mid c))) = I_N(c \mid (c \mid c)) = I_N(1)$$

and

$$F_N(c) \vee F_N(c \mid c) \leq F_N(c \vee (c \mid c)) = F_N((c \mid ((c \mid c) \mid (c \mid c))) \mid ((c \mid c) \mid (c \mid c))) = F_N(c \mid (c \mid c)) = F_N(1)$$

from Proposition 2.1 (2), (S1), (S2) and Corollary 2.1, it is obtained from Theorem 3.4 that $T_N(c) \vee T_N(c \mid c) = T_N(1)$, $I_N(c) \vee I_N(c \mid c) = I_N(1)$ and $F_N(c) \vee F_N(c \mid c) = F_N(1)$, and so, $T_N(c) = T_N(1)$, $I_N(c) = I_N(1)$, $F_N(c) = F_N(1)$ or $T_N(c \mid c) = T_N(1)$, $I_N(c \mid c) = I_N(1)$, $F_N(c \mid c) = F_N(1)$, for all $c \in C$. Thus, C_N is an ultra neutrosophic $\mathcal{N}$-filter of C.

Theorem 3.5. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and τ, γ, ρ be any elements of $[-1, 0]$ with $-3 \leq \tau + \gamma + \rho \leq 0$. If C_N is a (ultra) neutrosophic $\mathcal{N}$-filter of C, then the nonempty subset $C_N(\tau, \gamma, \rho)$ is a (ultra) filter of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C and $C_N(\tau, \gamma, \rho) \neq \emptyset$, for $\tau, \gamma, \rho \in [-1, 0]$ with $-3 \leq \tau + \gamma + \rho \leq 0$. Asumme that $c_1, c_2 \in C_N(\tau, \gamma, \rho)$. Since $\tau \leq T_N(c_1), \tau \leq T_N(c_2)$, $I_N(c_1) \leq \gamma, I_N(c_2) \leq \gamma$, $F_N(c_1) \leq \rho$ and $F_N(c_2) \leq \rho$, it follows that

$$\tau \leq \min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$$

$$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\} \leq \gamma$$

and

$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), f_N(c_2)\} \leq \rho.$

Then $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in T_N^\tau, I_N^\gamma, F_N^\rho$, and so, $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in C_N(\tau,\gamma,\rho)$. Suppose that $c_1 \in C_N(\tau,\gamma,\rho)$ and $c_1 \leq c_2$. Since $\tau \leq T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1) \leq \gamma$ and $F_N(c_2) \leq F_N(c_1) \leq \rho$, we have that $c_2 \in T_N^\tau, I_N^\gamma, F_N^\rho$, and so, $c_2 \in C_N(\tau,\gamma,\rho)$. Hence, $C_N(\tau,\gamma,\rho)$ is a filter of C. Moreover, let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C. Assume that $c_1 \vee c_2 \in C_N(\tau,\gamma,\rho)$. Since $\tau \leq T_N(c_1 \vee c_2)$, $I_N(c_1 \vee c_2) \leq \gamma$ and $F_N(c_1 \vee c_2) \leq \rho$, it is obtained from Lemma 3.6 that $\tau \leq T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2) \leq \gamma$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2) \leq \rho$, for all $c_1, c_2 \in C$. Thus, $\tau \leq T_N(c_1)$, $I_N(c_1) \leq \gamma$, $F_N(c_2) \leq \rho$ or $\tau \leq T_N(c_2)$, $I_N(c_2) \leq \gamma$, $F_N(c_2) \leq \rho$, and so, $c_1 \in C_N(\tau,\gamma,\rho)$ or $c_2 \in C_N(\tau,\gamma,\rho)$. By Lemma 2.3, $C_N(\tau,\gamma,\rho)$ is an ultra filter of C.

Theorem 3.6. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C, and T_N^τ, I_N^γ and F_N^ρ be (ultra) filters of C, for all $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Then C_N is a (ultra) neutrosophic $\mathcal{N}$-filter of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-structure on C, and T_N^τ, I_N^γ and F_N^ρ be filters of C, for all $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Assume that

$\tau_1 = T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) < \min\{T_N(c_1), T_N(c_2)\} = \tau_2,$

$\gamma_1 = \max\{I_N(c_1), I_N(c_2)\} < I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) = \gamma_2$

and

$\rho_1 = \max\{F_N(c_1), f_N(c_2)\} < F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) = \rho_2,$

for some $c_1, c_2 \in C$. If $\tau_0 = \frac{1}{2}(\tau_1+\tau_2)$, $\gamma_0 = \dfrac{1}{2}(\gamma_1+\gamma_2)$, $\rho_0 = \dfrac{1}{2}(\rho_1+\rho_2) \in [-1,0)$, then $\tau_1 < \tau_0 < \tau_2$, $\gamma_1 < \gamma_0 < \gamma_2$ and $\rho_1 < \rho_0 < \rho_2$. So, $(c_1 \mid c_2) \mid (c_1 \mid c_2) \notin T_N^{\tau_0}, I_N^{\gamma_0}, F_N^{\rho_0}$ when $c_1, c_2 \in T_N^{\tau_0}, I_N^{\gamma_0}, F_N^{\rho_0}$, which contradict with (SF-1). Thus

$\min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$

$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$

and

$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), f_N(c_2)\},$

for all $c_1, c_2 \in C$. Let $c_1 \leq c_2$. Suppose that $T_N(c_2) < T_N(c_1)$, $I_N(c_1) < I_N(c_2)$ and $F_N(c_1) < F_N(c_2)$, for some $c_1, c_2 \in C$. If $\tau^* = \dfrac{1}{2}(T_N(c_1)+T_N(c_2))$, $\gamma^* = \dfrac{1}{2}(I_N(c_1)+I_N(c_2))$, $\rho^* = \dfrac{1}{2}(F_N(c_1)+F_N(c_2)) \in [-1,0)$, then $T_N(c_2) < \tau^* < T_N(c_1)$, $I_N(c_1) < \gamma^* < I_N(c_2)$ and $F_N(c_1) < \rho^* < F_N(c_2)$. Hence, $c_1 \in T_N^{\tau^*}, I_N^{\gamma^*}, F_N^{\rho^*}$ but $c_2 \notin T_N^{\tau^*}, I_N^{\gamma^*}, F_N^{\rho^*}$ which is a contradiction with (SF-2). Therefore, $T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1)$ and $F_N(c_2) \leq F_N(c_1)$, for all $c_1, c_2 \in C$. Thereby, C_N is a neutrosophic $\mathcal{N}$-filter of C.

Also, let T_N^τ, I_N^γ and F_N^ρ be ultra filters of C, for all $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$, and $T_N(c_1 \vee c_2) = \tau$, $I_N(c_1 \vee c_2) = \gamma$ and $F_N(c_1 \vee c_2) = \rho$. Since $c_1 \vee c_2 \in T_N^\tau, I_N^\gamma, F_N^\rho$, it follows from Lemma 2.3 that $c_1 \in T_N^\tau, I_N^\gamma, F_N^\rho$ or $c_2 \in T_N^\tau, I_N^\gamma, F_N^\rho$. Thus, $T_N(c_1 \vee c_2) = \tau \leq T_N(c_1), T_N(c_2)$,

$I_N(c_1), I_N(c_2) \le \gamma = I_N(c_1 \vee c_2)$ and $F_N(c_1), F_N(c_2) \le \rho = F_N(c_1 \vee c_2)$, and so, $T_N(c_1 \vee c_2) \le T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \le I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \le F_N(c_1 \vee c_2)$, for all $c_1, c_2 \in C$. By Lemma 3.6, C_N is an ultra neutrosophic $\mathcal{N}$-filter of C.

Definition 3.6. Let C be a Sheffer stroke BL-algebra. Define

$$C_N^{c_t} := \{c \in C : T_N(c_t) \le T_N(c)\},$$

$$C_N^{c_i} := \{c \in C : I_N(c) \le I_N(c_i)\}$$

and

$$C_N^{c_f} := \{c \in C : F_N(c) \le F_N(c_f)\},$$

for all $c_t, c_i, c_f \in C$. It is obvious that $c_t \in C_N^{c_t}, c_i \in C_N^{c_i}$ and $c_f \in C_N^{c_f}$.

Example 3.8. Consider the Sheffer stroke BL-algebra C in Example 3.1. Let $c_t = a, c_i = b$, $c_f = c \in C$,

$$T_N(x) = \begin{cases} -0.18 & \text{if } x = 0, a, f, 1 \\ -0.29 & \text{otherwise,} \end{cases} \quad I_N(x) = \begin{cases} 0 & \text{if } x = d, e, f \\ -1 & \text{otherwise} \end{cases} \quad \textit{and } F_N(x) = \begin{cases} -0.55 & \text{if } x = 0, 1 \\ -0.56 & \text{if } x = a, b, c \\ -0.57 & \text{if } x = d, e, f. \end{cases}$$

Then

$$C_N^{a} = \{x \in C : T_N(a) \le T_N(x)\} = \{x \in C : -0.18 \le T_N(x)\} = \{0, a, f, 1\},$$

$$C_N^{xb} = \{x \in C : I_N(x) \le I_N(b)\} = \{x \in C : I_N(x) \le -1\} = \{0, a, b, c, 1\}$$

and

$$C_N^{c} = \{x \in C : F_N(x) \le F_N(c)\} = \{x \in C : F_N(x) \le -0.56\} = \{a, b, c, d, e, f\}.$$

Theorem 3.7. Let c_t, c_i and c_f be any elements of a Sheffer stroke BL-algebra C. If C_N is a (ultra) neutrosophic $\mathcal{N}$-filter of C, then $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are (ultra) filters of C.

Proof. Let c_t, c_i and c_f be any elements of C and C_N be a neutrosophic $\mathcal{N}$-filter of C. Assume that $c_1, c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since $T_N(c_t) \le T_N(c_1), T_N(c_t) \le T_N(c_2)$, $I_N(c_1) \le I_N(c_i), I_N(c_2) \le I_N(c_i)$ and $F_N(c_1) \le F_N(c_f), F_N(c_2) \le F_N(c_f)$, we get that

$$T_N(c_t) \le \min\{T_N(c_1), T_N(c_2)\} \le T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$$

$$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le \max\{I_N(c_1), I_N(c_2)\} \le I_N(c_i)$$

and

$$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le \max\{F_N(c_1), F_N(c_2)\} \le F_N(c_f).$$

Then $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Suppose that $c_1 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ and $c_1 \le c_2$. Since $T_N(c_t) \le T_N(c_1) \le T_N(c_2)$, $I_N(c_2) \le I_N(c_1) \le I_N(c_i)$ and $F_N(c_2) \le F_N(c_1) \le F_N(c_f)$, it is obtained that $c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Thus, $C_N^{c_t}, C_N^{c_i}$, $C_N^{c_f}$ are filters of C.

Let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C and $c_1 \vee c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since
$T_N(c_t) \leq T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$,
$I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2) \leq I_N(c_i)$
and
$F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2) \leq F_N(c_f)$
from Lemma 3.6, it follows that $T_N(c_t) \leq T_N(c_1)$, $I_N(c_1) \leq I_N(c_i)$, $F_N(c_1) \leq F_N(c_f)$ or $T_N(c_t) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_i)$, $F_N(c_2) \leq F_N(c_f)$. Hence, $c_1 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ or $c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Therefore, $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are ultra filters of C from Lemma 2.3.

Example 3.9. Consider the Sheffer stroke BL-algebra C in Example 3.1. For a neutrosophic $\mathcal{N}$-filter

$$C_N = \left\{ \frac{x}{(-0.21, -0.41, -0.61)} : x = 0, a, b, d \right\} \cup \left\{ \frac{x}{(-0.13, -0.53, -0.93)} : x = c, e, f, 1 \right\}$$

of C, $c_t = b$, $c_i = c$ and $c_f = f \in C$, the subsets

$C_N^b = \{x \in C : T_N(b) \leq T_N(x)\} = \{x \in C : -0.21 \leq T_N(x)\} = C$,

$C_N^c = \{x \in C : I_N(x) \leq I_N(c)\} = \{x \in C : I_N(x) \leq -0.53\} = \{c, e, f, 1\}$

and

$C_N^f = \{x \in C : F_N(x) \leq F_N(f)\} = \{x \in C : F_N(x) \leq -0.93\} = \{c, e, f, 1\}$

of C are filters of C. Also, C_N^b, C_N^c and C_N^f are ultra since C_N is ultra.

The inverse of Theorem 3.7 does not hold in general.

Example 3.10. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then

$C_N^c = \{x \in C : T_N(c) \leq T_N(x)\} = \{x \in C : -0.11 \leq T_N(x)\} = C$,

$C_N^d = \{x \in C : I_N(x) \leq I_N(d)\} = \{x \in C : I_N(x) \leq 0\} = C$

and

$C_N^e = \{x \in C : F_N(x) \leq F_N(e)\} = \{x \in C : F_N(x) \leq -0.12\} = C$

of C are filters of C but a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{ \frac{x}{(-0.11, 0, -0.12)} : x = 0, c, d, e \right\} \cup \left\{ \frac{x}{(0, -1, -0.87)} : x = a, b, f, 1 \right\}$$

is not a neutrosophic $\mathcal{N}$-filter of C since $T_N(d) = -0.11 < 0 = T_N(a)$ when $a \leq d$.

Theorem 3.8. Let c_t, c_i and c_f be any elements of a Sheffer stroke BL-algebra C and C_N be a neutrosophic $\mathcal{N}$-structure on C.

1. If $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C, then

$$T_N(c_1) \leq \min\{T_N(c_2 \mid (c_3 \mid c_3)), T_N(c_2)\} \Rightarrow T_N(c_1) \leq T_N(c_3),$$
$$\max\{I_N(c_2 \mid (c_3 \mid c_3)), I_N(c_2)\} \leq I_N(c_1) \Rightarrow I_N(c_3) \leq I_N(c_1) \text{ and} \tag{4}$$

$\max\{F_N(c_2 \mid (c_3 \mid c_3)), F_N(c_2)\} \leq F_N(c_1) \Rightarrow F_N(c_3) \leq F_N(c_1)$,

for all $c_1, c_2, c_3 \in C$.

2. If C_N satisfies the condition (4) and

$$c_1 \leq c_2 \textit{ implies } T_N(c_1) \leq T_N(c_2), I_N(c_2) \leq I_N(c_1) \textit{ and } F_N(c_2) \leq F_N(c_1), \tag{5}$$

for all $c_1, c_2, c_3 \in C$, then $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C, for all $c_t \in T_N^{-1}$, $c_i \in I_N^{-1}$ and $c_f \in F_N^{-1}$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-structure on C.

1. Assume that $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C, for all $c_t, c_i, c_f \in C$, and c_1, c_2 and c_3 are any elements of C such that $T_N(c_1) \leq \min\{T_N(c_2 \mid (c_3 \mid c_3)), T_N(c_2)\}$, $\max\{I_N(c_2 \mid (c_3 \mid c_3)), I_N(c_2)\} \leq I_N(c_1)$ and $\max\{F_N(c_2 \mid (c_3 \mid c_3)), F_N(c_2)\} \leq F_N(c_1)$. Since $c_2 \mid (c_3 \mid c_3), c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ where $c_t = c_i = c_f = c_1$, we have from (SF-4) that $c_3 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ where $c_t = c_i = c_f = c_1$. So, $T_N(c_1) \leq T_N(c_3)$, $I_N(c_3) \leq I_N(c_1)$ and $F_N(c_3) \leq F_N(c_1)$, for all $c_1, c_2, c_3 \in C$.
2. Suppose that C_N be a neutrosophic $\mathcal{N}$-structure on C satisfying the conditions (4) and (5), for any $c_t \in T_N^{-1}, c_i \in I_N^{-1}$ and $c_f \in F_N^{-1}$. Let $c_1, c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since $c_2 \leq (c_2 \mid c_1) \mid c_1 = c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2)))$ from Proposition 2.1 (9), (S1)–(S2), and $T_N(c_t) \leq T_N(c_1), T_N(c_t) \leq T_N(c_2)$, $I_N(c_1) \leq I_N(c_i), I_N(c_2) \leq I_N(c_i)$, $F_N(c_1) \leq F_N(c_f)$ and $F_N(c_2) \leq F_N(c_f)$, it follows from the condition (5) that

$$T_N(c_t) \leq \min\{T_N(c_1), T_N(c_2)\} \leq \min\{T_N(c_1), T_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\},$$

$$\max\{I_N(c_1), I_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\} \leq \max\{I_N(c_1), I_N(c_2)\} \leq I_N(c_i)$$

and $\max\{F_N(c_1), F_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\} \leq \max\{F_N(c_1), F_N(c_2)\} \leq F_N(c_f)$.

Thus, $T_N(c_t) \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2))$, $I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq I_N(c_i)$ and $F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq F_N(c_f)$ from the condition (4), and so, $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Let $c_1 \leq c_2$ and $c_1 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since $T_N(c_t) \leq T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1) \leq I_N(c_i)$ and $F_N(c_2) \leq F_N(c_1) \leq F_N(c_f)$ from condition (5), it is obtained that $c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Thereby, $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C.

Example 3.11. Consider the Sheffer stroke BL-algebra C in Example 3.1. Let

$$T_N(x) = \begin{cases} -0.07 & \text{if } x = 1 \\ -0.77 & \text{otherwise,} \end{cases} \quad I_N(x) = \begin{cases} -0.63 & \text{if } x = e, 1 \\ 0 & \text{otherwise,} \end{cases} \quad \textit{and } F_N(x) = \begin{cases} -0.84 & \text{if } x = a, d, e, 1 \\ -0.42 & \text{otherwise.} \end{cases}$$

Then the filters $C_N^{c_t} = C, C_N^{c_i} = \{e.1\}$ and $C_N^{c_f} = \{a, d, e, 1\}$ of C satisfy the condition (4), for the elements $c_t = a, c_i = e$ and $c_f = d$ of C.

Also, let

$$C_N = \left\{\frac{x}{(-0.91, -0.23, -0.001)} : x \in C - \{1\}\right\} \cup \left\{\frac{1}{(-0.17, -0.86, -0.79)}\right\}$$

be a neutrosophic $\mathcal{N}$-structure on C satisfying the conditions (4) and (5). Then the subsets

$C_N^{c_t} = \{x \in C : T_N(f) \leq T_N(x)\} = \{x \in C : -0.91 \leq T_N(x)\} = C,$

$C_N^{c_i} = \{x \in C : I_N(x) \leq I_N(b)\} = \{x \in A : I_N(x) \leq -0.23\} = C$

and

$C_N^{c_f} = \{x \in C : F_N(x) \leq F_N(1)\} = \{x \in C : F_N(x) \leq -0.79\} = \{1\}$

of C are filters of C, where $c_t = f, c_i = b$ and $c_f = 1$ of C.

4 Conclusion

In the study, neutrosophic $\mathcal{N}$-structures defined by $\mathcal{N}$-functions on Sheffer stroke BL-algebras have been examined. By giving basic definitions and notions of Sheffer stroke BL-algebras and neutrosophic $\mathcal{N}$-structures on a crispy set X, a neutrosophic $\mathcal{N}$-subalgebra and a (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-structure are defined on Sheffer stroke BL-algebras. We determine a quasi-subalgebra of a Sheffer stroke BL-algebra and prove that the (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra is its quasi-subalgebra and vice versa. Besides, it is stated that the family of all neutrosophic $\mathcal{N}$-subalgebras of the algebra forms a complete distributive lattice. It is illustrated that every neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra satisfies $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$, for all elements x of the algebra but the inverse does not generally hold. We interpret the case which $\mathcal{N}$-functions defining a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra are constant. Also, a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is described and some properties are analysed. Indeed, it is proved that every neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is the neutrosophic $\mathcal{N}$-subalgebra but the inverse is not true in general, and that the (τ, γ, ρ)-level set of a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its (ultra) filter and the inverse is always true. After that the subsets $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ of a Sheffer stroke BL-algebra are described by means of $\mathcal{N}$-functions and any elements c_t, c_i and c_f of this algebraic structure, it is demonstrated that these subsets are (ultra) filters of a Sheffer stroke BL-algebra if C_N is the (ultra) neutrosophic $\mathcal{N}$-filter.

In future works, we wish to study on plithogenic structures and relationships between neutrosophic $\mathcal{N}$-structures on some algebraic structures.

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Applications of (Neutro/Anti)sophications to Semihypergroups

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Akbar Rezaei, Florentin Smarandache, S. Mirvakili (2021). Applications of (Neutro/Anti)sophications to Semihypergroups. *Journal of Mathematics*, 6649349, 7. DOI: 10.1155/2021/6649349

In this paper, we extend the notion of semi-hypergroups (resp. hypergroups) to neutro-semihypergroups (resp. neutro-hypergroups). We investigate the property of anti-semihypergroups (resp. anti-hypergroups). We also give a new alternative of neutro-hyperoperations (resp. anti-hyperoperations), neutro-hyperoperation-sophications (resp. anti-hypersophications). Moreover, we show that these new concepts are different from classical concepts by several examples.

1. Introduction

A hypergroup, as a generalization of the notion of a group, was introduced by F. Marty [1] in 1934. The first book in hypergroup theory was published by Corsini [2]. Nowadays, hypergroups have found applications to many subjects of pure and applied mathematics, for example, in geometry, topology, cryptography and coding theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets and automata theory, physics, and also in biological inheritance [3–7]. The first book in semi-hypergroup theory was published by Davvaz in 2016 (see [8]). In recent years, several other valuable books in hyperstructures have been written by Davvaz et al. [6, 9, 10].

M. Al-Tahan et al. introduced the Corsini hypergroup and studied its properties as a special hypergroup that was defined by Corsini. They investigated a necessary and sufficient condition for the productional hypergroup to be a Corsini hypergroup, and they characterized all Corsini hypergroups of orders 2 and 3 up to isomorphism [3]. Semi-hypergroup, hypergroup, and fuzzy hypergroup of order 2 are enumerated in [7, 11, 12]. S. Hoskova-Mayerova et al. used the fuzzy multisets to introduce the concept of fuzzy multi-hypergroups as a generalization of fuzzy hypergroups, defined the different operations on fuzzy multi-hypergroups, and extended the fuzzy hypergroups to fuzzy multi-hypergroups [13].

In 2019 and 2020, within the field of neutrosophy, Smarandache [14–16] generalized the classical algebraic structures to neutroalgebraic structures (or neutroalgebras) (whose operations and axioms are partially true, partially indeterminate, and partially false) as extensions of partial algebra and to antialgebraic structures (or antialgebras) (whose operations and axioms are totally false). Furthermore, he extended any classical structure, no matter what field of knowledge, to a neutrostructure and an antistructure. These are new fields of research within neutrosophy. Smarandache in [16] revisited the notions of neutroalgebras and anti-algebras, where he studied partial algebras, universal algebras, effect algebras, and Boole's partial algebras and showed that neutroalgebras are the generalization of partial algebras. Also, with respect to the classical hypergraph (that contains hyperedges), Smarandache added the supervertices (a group of vertices put together to form a supervertex), in order to form a super-hypergraph. Then, he extended the super-hypergraph to n-super-hypergraph, by extending the power set $P(V)$ to $P^n(V)$ that is the n-power set of the set V (the n-super-hypergraph, through its n -super-hypergraph-vertices and n -superhypergraph-edges that belong to $P^n(V)$, can be the best (so far) to model our complex and sophisticated reality). Furthermore, he extended the classical hyperalgebra to n-ary hyperalgebra and its alternatives n -ary neutro-hyperalgebra and n -ary anti-hyperalgebra [17]. The notion of neutrogroup was defined and studied by Agboola in [18].

Recently, M. Al-Tahan et al. studied neutro-ordered algebra and some related terms such as neutro-ordered subalgebra and neutro-ordered homomorphism in [19].

In this paper, the concept of neutro-semihypergroup and anti-semihypergroup is formally presented. And, new alternatives are introduced, such as neutro-hyperoperations (resp. anti-hyperoperations), neutro-hyperaxioms, and anti-hyperaxioms. We show that these definitions are different from classical definitions by presenting several examples. Also, we enumerate neutro-hypergroup and anti-hypergroup of order 2 (see Table 1) and obtain some known results (see Table 2).

2. Preliminaries

In this section, we recall some basic notions and results regarding hyperstructures.

Definition 1 (see [2, 8]). A hypergroupoid $(H, \circ)$ is a nonempty set H together with a map $\circ\colon H \times H \longrightarrow P^*(H)$ called (binary) hyperoperation, where $P^*(H)$ denotes the set of all nonempty subsets of H. The hyperstructure $(H, \circ)$ is called a hypergroupoid, and the image of the pair (x, y) is denoted by $x \circ y$.

If A and B are nonempty subsets of H and $x \in H$, then by $A \circ B$, $A \circ x$, and $x \circ B$ we mean $A \circ B = \cup_{a\in A, b\in B} a \circ b$, $A \circ x = A \circ \{x\}$, and $x \circ B = \{x\} \circ B$.

Definition 2 (see [2, 8]). A hypergroupoid $(H, \circ)$ is called a semi-hypergroup if it satisfies the following:

(A) $(\forall a, b, c \in H)\ (a \circ (b \circ c) = (a \circ b) \circ c)$ (associativity).

Definition 3 (see [2, 8]). A hypergroupoid $(H, \circ)$ is called a quasi-hypergroup if reproduction axiom is valid. This means that, for all a of H, we have

(R) $(\forall a \in H)\ (H \circ a = a \circ H = H)$ (i.e. $(\forall a, b \in H)$ $(\exists c, d \in H)$ s.t. $b \in c \circ a, b \in a \circ d$).

Definition 4 (see [2, 8]). A hypergroupoid $(H, \circ)$ which is both a semi-hypergroup and a quasi-hypergroup is called a hypergroup.

Example 1 (see [2, 8])

(i) Let H be a nonempty set, and for all $x, y \in H$, we define $x \circ y = H$. Then, $(H, \circ)$ is a hypergroup, called the total hypergroup.

(ii) Let G be a group and H a normal subgroup of G, and for all $x, y \in G$, we define $x \circ y = xyH$. Then, $(G, \circ)$ is a hypergroup.

Definition 5 (see [2, 12]). Let $(H, \circ)$ be a hypergroupoid. The commutative law on $(H, \circ)$ is defined as follows:

(C) $(\forall a, b \in H)\ (a \circ b = b \circ a)$.

$(H, \circ)$ is called a commutative hypergroupoid.

Table 1: Classification of the hypergroupoids of order 2.

		A	NA	AA
C	R	6	4	—
	NR	—	—	—
	AR	—	—	—
	Etc.	3	2	—
NC	R	—	—	—
	NR	—	—	—
	AR	—	—	—
	Etc.	—	—	—
AC	R	2	8	—
	NR	—	—	—
	AR	—	—	—
	Etc.	6	10	4

Table 2: Classification of the semi-hypergroups of order 2.

	Com	Noncom	N
Semigroup	3	2	5
Group	1	—	1
Semi-hypergroup	9	8	17
Hypergroup	6	2	8

Example 2 (see [13]). Let $\mathbb{Z}$ be the set of integers, and define $\circ_1$ on $\mathbb{Z}$ as follows. For all $x, y \in \mathbb{Z}$,

$$x \circ_1 y = \begin{cases} 2\mathbb{Z}, & \text{if } x, y \text{ have same partiy}, \\ 2\mathbb{Z}+1, & \text{otherwise}. \end{cases} \tag{1}$$

Then, $(\mathbb{Z}, \circ_1)$ is a commutative hypergroup.

3. On Neutro-hypergroups and Anti-hypergroups

F. Smarandache generalized the classical algebraic structures to the neutroalgebraic structures and antialgebraic structures. Neutro-sophication of an item C (that may be a concept, a space, an idea, a hyperoperation, an axiom, a theorem, a theory, an algebra, etc.) means to split C into three parts (two parts opposite to each other, and another part which is the neutral/indeterminacy between the opposites), as pertinent to neutrosophy $((\langle A\rangle, \langle \text{neut}A\rangle, \langle \text{anti}A\rangle)$, or with other notation $(T, I, F))$, meaning cases where C is partially true (T), partially indeterminate (I), and partially false (F), while antisophication of C means to totally deny C (meaning that C is made false on its whole domain) (see [14, 15, 17, 20]).

Neutrosophication of an axiom on a given set X means to split the set X into three regions such that, on one region, the axiom is true (we say the degree of truth T of the axiom), on another region, the axiom is indeterminate (we say the degree of indeterminacy I of the axiom), and on the third region, the axiom is false (we say the degree of falsehood F of the axiom), such that the union of the regions covers the whole set, while the regions may or may not be disjoint, where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$.

Antisophication of an axiom on a given set X means to have the axiom false on the whole set X (we say total degree of falsehood F of the axiom) or $(0, 0, 1)$.

Neutrosophication of a hyperoperation defined on a given set X means to split the set X into three regions such that, on one region, the hyperoperation is well-defined (or inner-defined) (we say the degree of truth T of the hyperoperation), on another region, the hyperoperation is indeterminate (we say the degree of indeterminacy I of the hyperoperation), and on the third region, the hyperoperation is outer-defined (we say the degree of falsehood F of the hyperoperation), such that the union of the regions covers the whole set, while the regions may or may not be disjoint, where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$.

Antisophication of a hyperoperation on a given set X means to have the hyperoperation outer-defined on the whole set X (we say total degree of falsehood F of the axiom) or $(0, 0, 1)$.

In this section, we will define the neutro-hypergroups and anti-hypergroups.

Definition 6. A neutro-hyperoperation is a map $\circ: H \times H \longrightarrow P(U)$, where U is a universe of discourse that contains H that satisfies the below neutrosophication process.

The neutrosophication (degree of well-defined, degree of indeterminacy, and degree of outer-defined) of the hyperoperation is the following neutrohyperoperation (NH):

(NR) $(\exists x, y \in H)\ (x \circ y \in P^*(H))$ and $(\exists x, y \in H)(x \circ y$ is an indeterminate subset, or $x \circ y \notin P^*(H))$.

The neutrosophication (degree of truth, degree of indeterminacy, and degree of falsehood) of the hypergroup axiom of associativity is the following neutroassociativity (NA):

(NA) $(\exists a, b, c \in H)\ (a \circ (b \circ c) = (a \circ b) \circ c)$ and $(\exists d, e, f \in H)(d \circ (e \circ f) \neq (d \circ e) \circ f$ or $d \circ (e \circ f) =$ indeterminate, or $(d \circ e) \circ f =$ indeterminate).

Neutroreproduction axiom (NR):

(NR) $(\exists a \in H)(H \circ a = a \circ H = H)$ and $(\exists b \in H)$ $(H \circ b, b \circ H$, and H are not all three equal, or some of them are indeterminate).

Also, we define the neutrocommutativity (NC) on $(H, \circ)$ as follows:

(NC) $(\exists a, b \in H)\,(a \circ b = b \circ a)$ and $(\exists c, d \in H)$ $(c \circ d \neq d \circ c$, or $c \circ d =$ indeterminate, or $d \circ c =$ indeterminate).

Now, we define a neutro-hyperalgebraic system $S = \langle H, F, A \rangle$, where H is a set or neutrosophic set, F is a set of the hyperoperations, and A is the set of hyperaxioms, such that there exists at least one neutro-hyperoperation or at least one neutro-hyperaxiom and no anti-hyperoperation and no anti-hyperaxiom.

Definition 7. The anti-hypersophication (totally outer-defined) of the hyperoperation defines anti-hyperoperation (AH): (AH) $(\forall x, y \in H)\ (x \circ y \notin P^*(H))$.

The anti-hypersophication (totally false) of the hypergroup is as follows:

(AA) $(\forall x, y, z \in H)\ (x \circ (y \circ z) \neq (x \circ y) \circ z)$ (antiassociativity)

(AR) $(\forall a \in H)(H \circ a,\ a \circ H$, and H are not equal) (antireproduction axiom)

Also, we define the anticommutativity (AC) on $(H, \circ)$ as follows:

(AC) $(\forall a, b \in H$ with $a \neq b)\ (a \circ b \neq b \circ a)$.

Definition 8. A neutro-semihypergroup is an alternative of semi-hypergroup that has at least (NH) or (NA), which does not have (AA).

Example 3

(i) Let $H = \{a, b, c\}$ and $U = \{a, b, c, d\}$ be a universe of discourse that contains H. Define the neutro-hyperoperation $\circ_2$ on H with Cayley's table.

$\circ_2$	a	b	c
a	a	a	a
b	b	$\{a, b\}$	$\{a, b, d\}$
c	c	?	H

Then, $(H, \circ_2)$ is a neutro-semihypergroup. Since $a \circ_2 b \in P^*(H)$, $b \circ_2 c = \{a, b, d\} \notin P^*(H)$, and $c \circ_2 b =$ indeterminate, so (NH) holds.

(ii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_3$ on H with Cayley's table.

$\circ_3$	a	b	c
a	a	a	a
b	b	$\{a, b\}$	$\{a, b\}$
c	c	$\{b, c\}$	H

Then, $(H, \circ_3)$ is a neutro-semihypergroup. (NA) is valid, since $(b \circ_3 c) \circ_3 a = \{a, b\} \circ_3 a = (a \circ_3 a) \cup (b \circ_3 a) = \{a\} \cup \{b\} = \{a, b\}$ and $b \circ_3 (c \circ_3 a) = b \circ_3 \{c\} = b \circ_3 c = \{a, b\}$.

Hence, $(b \circ_3 c) \circ_3 a = b \circ_3 (c \circ_3 a)$. Also, $\{b \circ_3 a\} \circ_3 c = \{b\} \circ_3 c = b \circ_3 c = \{a, b\}$ and $b \circ_3 (a \circ_3 c) = b \circ_3 \{a\} = b \circ_3 a = \{b\}$, so $(b \circ_3 a) \circ_3 c \neq b \circ_3 (a \circ_3 c)$.

Definition 9. A neutrocommutative semi-hypergroup is a semi-hypergroup that satisfies (NC).

Example 4. Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_4$ on H with Cayley's table.

$\circ_4$	a	b	c
a	$\{a, c\}$	a	a
b	a	b	c
c	a	$\{b, c\}$	$\{b, c\}$

Then, $(H, \circ_4)$ is a semi-hypergroup, but not a hypergroup, since $a \circ_4 H = H \circ_4 a = \{a, c\} \neq H$. (NC) is valid, since $a \circ_4 b = \{a\} = b \circ_4 a$ and $c \circ_4 b = \{b, c\} \neq b \circ_4 c = \{c\}$.

Definition 10. A neutrocommutative hypergroup is a hypergroup that satisfies (NC).

Example 5. Let $H = \{a, b, c, d, e, f\}$. Define the operation $\circ_5$ on H with Cayley's table.

$\circ_5$	a	b	c	d	e	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d	e	b	a
d	d	c	f	a	e	b
f	f	d	c	b	a	e

Then, $(H, \circ_5, e)$ is a group and so is a natural hypergroup. Also, it is a neutrocommutative hypergroup, since $a \circ_5 b = e = b \circ_5 a$ and $a \circ_5 c = d \neq c \circ_5 a = f$.

Definition 11. A neutrohypergroup is an alternative of hypergroup that has at least (NH) or (NA) or (NR), which does not have (AA) and (AR).

Example 6. Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_6$ on H with Cayley's table.

$\circ_6$	a	b	c
a	a	b	c
b	b	b	b
c	c	c	a

Then, $(H, \circ_6)$ is a neutrohypergroup. The hyperoperation $\circ_6$ is associative. (NR) is valid, since $a \circ_6 H = (a \circ_6 a) \cup (a \circ_6 b) \cup (a \circ_6 c) = H = (a \circ_6 a) \cup (b \circ_6 a) \cup (c \circ_6 a) = H \circ_6 a$, $b \circ_6 H = (b \circ_6 a) \cup (b \circ_6 b) \cup (b \circ_6 c) = \{b\} \neq H \neq \{c, b\} = (a \circ_6 b) \cup (b \circ_6 b) \cup (c \circ_6 b) = H \circ_6 b$, and $c \circ_6 H = (c \circ_6 a) \cup (c \circ_6 b) \cup (c \circ_6 c) = \{a, c\} \neq H$, but $H \circ_6 c = (a \circ_6 c) \cup (b \circ_6 c) \cup (c \circ_6 c) = \{a, b, c\} = H$.

Note that every neutro-semihypergroup, neutrohypergroup, neutrocommutative semi-hypergroup, and neutrocommutative hypergroup are neutro-hyperalgebraic systems.

Definition 12. An anti-semihypergroup is an alternative of semi-hypergroup that has at least (AH) or (AA).

Example 7

(i) Let $\mathbb{N}$ be the set of natural numbers except 0. Define hyperoperation $\circ_7$ on $\mathbb{N}$ by $x \circ_7 y = \{(x^2/x^2 + 1), y\}$. Then, $(\mathbb{N}, \circ_7)$ is an anti-semihypergroup. (AH) is valid, since, for all $x, y \in \mathbb{N}$, $x \circ_7 y \notin P^*(\mathbb{N})$. Thus, (AH) holds.

(ii) Let $H = \{a, b\}$. Define the hyperoperation $\circ_8$ on H with Cayley's table.

$\circ_8$	a	b
a	b	a
b	b	a

Then, $(H, \circ_8)$ is an anti-semihypergroup. (AA) is valid, since, for all $x, y, z \in H$, $x \circ_8 (y \circ_8 z) \neq (x \circ_8 y) \circ_8 z$.

(iii) Let $H = \{a, b\}$. Define the hyperoperation $\circ_9$ on H with Cayley's table.

$\circ_9$	a	b
a	b	H
b	a	a

Then, $(H, \circ_9)$ is an anticommutative semi-hypergroup. (AC) is valid, since $a \circ_9 b = H \neq b \circ_9 a = \{a\}$.

Definition 13. An anti-hypergroup is an anti-semihypergroup, or it satisfies (AR).

Example 8

(i) Let $\mathbb{R}$ be the set of real numbers. Define hyperoperation $\circ_{10}$ on $\mathbb{R}$ by $x \circ_{10} y = \{x^2 + 1, x^2 - 1\}$. Then, $(\mathbb{R}, \circ_{10})$ is an anti-semihypergroup, since, for all $x, y, z \in \mathbb{R}$, $x \circ_{10} (y \circ_{10} z) \neq (x \circ_{10} y) \circ_{10} z$. Because $x \circ_{10} (y \circ_{10} z) = x \circ_{10} \{y^2 + 1, y^2 - 1\} = \{x \circ_{10} (y^2 + 1), x \circ_{10} (y^2 - 1)\} = \{x^2 + 1, x^2 - 1\}$, but $(x \circ_{10} y) \circ_{10} z = \{x^2 + 1, x^2 - 1\} \circ_{10} z = ((x^2 + 1) \circ_{10} z) \cup ((x^2 - 1) \circ_{10} z) = \{(x^2 + 1)^2 + 1, (x^2 - 1)^2 + 1\}$. Hence, (AA) is valid.

(ii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_{11}$ on H with Cayley's table.

$\circ_{11}$	a	b	c
a	a	a	b
b	a	a	a
c	c	c	c

Then, $(H, \circ_{11})$ is an anti-semihypergroup. The hyperoperation $\circ_{11}$ is associative. Also, (AR) holds, since $a \circ_{11} H = (a \circ_{11} a) \cup (a \circ_{11} b) \cup (a \circ_{11} c) = \{c\} \neq H \neq \{b, c\} = (a \circ_{11} a) \cup (b \circ_{11} a) \cup (c \circ_{11} a) = H \circ_{11} a$, $b \circ_{11} H = (b \circ_{11} a) \cup (b \circ_{11} b) \cup (b \circ_{11} c) = \{b\} \neq H \neq \{b, c\} = (a \circ_{11} b) \cup (b \circ_{11} b) \cup (c \circ_{11} b) = H \circ_{11} b$, and $c \circ_{11} H = (c \circ_{11} a) \cup (c \circ_{11} b) \cup (c \circ_{11} c) = \{c\} \neq H \neq \{b, c\} = (a \circ_{11} c) \cup (b \circ_{11} c) \cup (c \circ_{11} c) = H \circ_{11} c$.

(iii) Let $\mathbb{R}$ be the set of real numbers. Define hyperoperation $\circ_{12}$ on $\mathbb{R}$ by $x \circ_{12} y = \{x, 1\}$. Then, $(\mathbb{R}, \circ_{12})$ is an anti-semihypergroup. The hyperoperation $\circ_{12}$ is associative, since, for all $x, y, z \in \mathbb{R}$, we have $x \circ_{12} (y \circ_{12} z) = x \circ_{12} \{y, 1\} = (x \circ_{12} y) \cup (x \circ_{12} 1) = \{x, 1\} \cup \{x, 1\} = \{x, 1\}$ and $(x \circ_{12} y) \circ_{12} z = \{x, 1\} \circ_{12} z = (x \circ_{12} z) \cup (1 \circ_{12} z) = \{x, 1\} \cup \{1, 1\} = \{x, 1\}$, so $x \circ_{12} (y \circ_{12} z) = (x \circ_{12} y) \circ_{12} z$. However, for $a \in \mathbb{R}$, we have $a \circ_{12} \mathbb{R} = \cup_{x \in \mathbb{R}} a \circ_{12} x = \cup_{x \in \mathbb{R}}$

$\{a, 1\} = \{a, 1\} \neq \mathbb{R}$ and $R \circ_{12} a = \cup_{x\in\mathbb{R}} x \circ_{12} a = \cup_{x\in R}\{x, 1\} = \mathbb{R}$. Thus, $a \circ_{12} \mathbb{R} \neq \mathbb{R} \circ_{12} a$.

Definition 14. An anticommutative semi-hypergroup is a semi-hypergroup that satisfies (AC).

Example 9

(i) Let $H = \{a, b\}$. Define the hyperoperation $\circ_{13}$ on H with Cayley's table.

$\circ_{13}$	a	b
a	a	a
b	H	b

Then, $(H, \circ_{13})$ is a semi-hypergroup and (AC) is valid, since $a \circ_{13} b = \{a\} \neq b \circ_{13} a = H$. Thus, $(H, \circ_{13})$ is an anticommutative semi-hypergroup.

(ii) Let $H = \{a, b\}$. Define the hyperoperation $\circ_{14}$ on H with Cayley's table.

$\circ_{14}$	a	b
a	b	a
b	b	a

Then, $(H, \circ_{14})$ is an anticommutative semi-hypergroup, and the hyperoperation $\circ_{14}$ is not associative, since $(a \circ_{14} a) \circ_{14} a = \{b\} \circ_{14} a = \{b\} \neq a \circ_{14} (a \circ_{14} a) = a \circ_{14} \{b\} = \{a\}$. (AC) is valid, since $a \circ_{14} b = \{a\} \neq b \circ_{14} a = \{b\}$.

Definition 15. An anticommutative hypergroup is a hypergroup that satisfies (AR).

Example 10

(i) Let $H = \{a, b\}$. Define the hyperoperation $\circ_{15}$ on H with Cayley's table.

$\circ_{15}$	a	b
a	H	a
b	H	H

Then, $(H, \circ_{15})$ is an anticommutative hypergroup. (AC) is valid, since $a \circ_{15} b = \{a\} \neq b \circ_{15} a = H$.

(ii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_{16}$ on H with Cayley's table.

$\circ_{16}$	a	b	c
a	a	a	H
b	b	b	H
c	c	c	H

Then, $(H, \circ_{16})$ is an anticommutative hypergroup. The hyperoperation $\circ_{16}$ is associative. Also, (AC) holds, since $a \circ_{16} b = \{a\} \neq b \circ_{16} a = \{b\}$, $a \circ_{16} c = H \neq c \circ_{16} a = \{c\}$, and $b \circ_{16} c = H \neq c \circ_{16} b = \{c\}$.

(iii) Let $H = \{a, b, c\}$. Define the hyperoperation $\circ_{17}$ on H with Cayley's table.

$\circ_{17}$	a	b	c
a	a	b	c
b	a	b	c
c	H	H	H

Then, $(H, \circ_{17})$ is an anticommutative hypergroup, (AC) holds, since $a \circ_{17} b = \{b\} \neq b \circ_{17} a = \{a\}$, $a \circ_{17} c = \{c\} \neq c \circ_{17} a = H$, and $b \circ_{17} c = \{c\} \neq c \circ_{17} b = H$.

Note that every anti-semihypergroup, antihypergroup, anticommutative semi-hypergroup, and anticommutative hypergroup are anti-hyperalgebraic systems.

In the following results, we use hyperoperation instead of neutro-hyperoperation.

Note that if $(H, \circ)$ is a neutro-semihypergroup and $(G, \circ)$ is an anti-semihypergroup, then $(H \cap G, \circ)$ is not a neutro-semihypergroup, but it is an anti-semihypergroup. Also, let $(H, \circ_H)$ be a neutro-semihypergroup, $(G, \circ_G)$ be an anti-semihypergroup, and $H \cap G = \varnothing$. Define hyperoperation $\circ$ on $H \uplus G$ by

$$x \circ y = \left\{ \begin{array}{ll} x \circ_H y, & \text{if } x, y \in H, \\ x \circ_G y, & \text{if } x, y \in G, \\ \{x, y\}, & \text{otherwise.} \end{array} \right\}. \tag{2}$$

Then, $(H \uplus G, \circ)$ is a neutro-semihypergroup, but it is not an anti-semihypergroup.

Proposition 1. *Let $(H, \circ)$ be an antisemihypergroup and $e \in H$. Then, $(H \cup \{e\}, *)$ is a neutrosemihypergroup, where $*$ is defined on $H \cup \{e\}$ by*

$$x * y = \left\{ \begin{array}{ll} x \circ_H y, & \text{if } x, y \in H, \\ \{e, x, y\}, & \text{otherwise.} \end{array} \right\}. \tag{3}$$

Proof. It is straightforward.

Proposition 2. *Let $(H, \circ)$ be a commutative hypergroupoid. Then, $(H, \circ)$ cannot be an anti-semihypergroup.*

Proof. Let $a \in H$. Then, $a \circ (a \circ a) = (a \circ a) \circ a$, so $(H, \circ)$ cannot be an anti-semihypergroup.

Corollary 1. *Let $(H, \circ)$ be a hypergroupoid, and there exists $a \in H$ such that $a^\circ a$ commuted with a. Then, $(H, \circ)$ cannot be an anti-semihypergroup.*

Corollary 2. *Let $(H, \circ)$ be a hypergroupoid with a scalar idempotent, i.e., there exists $a \in H$ such that $a^\circ a = a$. Then, $(H, \circ)$ cannot be an anti-semihypergroup.*

Proposition 3. *Let $(H, \circ_H)$ and $(G, \circ_G)$ be two neutro-semihypergroups (resp. anti-semihypergroups). Then, $(H \times$*

$G, *)$ *is a neutro-semihypergroup (resp. anti-semi-hypergroups), where* $*$ *is defined on* $H \times G$. *For any* $(x_1, y_1), (x_2, y_2) \in H \times G$,

$$(x_1, y_1) * (x_2, y_2) = (x_1 \circ_H x_2, y_1 \circ_G y_2). \quad (4)$$

Note that if $(H, \circ)$ is a neutro-semihypergroup, then if there is a nonempty set $H_1 \subseteq H$, such that $(H_1, \circ)$ is a semi-hypergroup, we call it Smarandache semi-hypergroup.

Suppose $(H, \circ_H)$ and $(G, \circ_G)$ are two hypergroupoids. A function $f: H \longrightarrow G$ is called a homomorphism if, for all $a, b \in H$, $f(a \circ_H b) = f(a) \circ_G f(b)$ (see [21, 22], for details).

Proposition 4. *Let* $(H, \circ_H)$ *be a semi-hypergroup,* $(G, \circ_G)$ *be a neutro-hypergroup, and* $f: H \longrightarrow G$ *be a homomorphism. Then,* $(f(H), \circ_G)$ *is a semi-hypergroup, where* $f(H) = \{f(h): h \in H\}$.

Proof. Assume that $(H, \circ_H)$ is a semi-hypergroup and $x, y, z \in f(H)$. Then, there exist $h_1, h_2, h_3 \in f(H)$ such that $f(h_1) = x$, $f(h_2) = y$, and $f(h_3) = z$, so we have

$$\begin{aligned} x \circ_G (y \circ_G z) &= f(h_1) \circ_G (f(h_2) \circ_G (h_3)) \\ &= f(h_1) \circ_G f(h_2 \circ_H h_3) = f(h_1 \circ_H (h_2 \circ_H h_3)) \\ &= f((h_1 \circ_H h_2) \circ_H h_3) = f(h_1 \circ_H h_2) \circ_G f(h_3) \\ &= (f(h_1) \circ_G f(h_2)) \circ_G f(h_3) = (x \circ_G y) \circ_G z. \end{aligned} \quad (5)$$

Then, $(f(H), \circ_G)$ is a semi-hypergroup. □

Definition 16. Let $(H, \circ_H)$ and $(G, \circ_G)$ be two hypergroupoids. A bijection $f: H \longrightarrow G$ is an isomorphism if it conserves the multiplication (i.e., $f(a \circ_H b) = f(a) \circ_G f(b)$) and write $H \cong G$. A bijection $f: H \longrightarrow G$ is an antiisomorphism if for all $a, b \in H$, $f(a \circ_H b) \neq f(b) \circ_G f(a)$. A bijection $f: H \longrightarrow G$ is a neutroisomorphism if there exist $a, b \in H$, $f(a \circ_H b) = f(b) \circ_G f(a)$, i.e., degree of truth (T), there exist $c, d \in H$ and $f(c \circ_H d)$ or $f(c) \circ_G f(d)$ are indeterminate, i.e., degree of indeterminacy (I), and there exist $e, h \in H$, $f(e \circ_H h) \neq f(e) \circ_G f(h)$, i.e., degree of falsehood (F), where (T, I, F) are different from $(1, 0, 0)$ and $(0, 0, 1)$, and $T, I, F \in [0, 1]$.

Let $\circ$ be a hyperoperation on $H = \{a, b\}$ and $(A_{11}, A_{12}, A_{21}, A_{22})$ inside of Cayley's table.

$\circ$	a	b
a	A_{11}	A_{12}
b	A_{21}	A_{22}

Lemma 1 (see [5]). *Let* $(H = \{a, b\}, \circ_H)$ *and* $(G = \{a', b'\}, \circ_G)$ *be hypergroupoids with Cayley's tables* (A, B, C, D) *and* (A', B', C', D')*, respectively. Then,* $H \cong G$ *if and only if, for all* $i, j \in \{1, 2\}$, $A_{ij} = A'_{ij}$ *or*

$$A'_{ij} = \left\{ \begin{array}{ll} A^d_{ij}, & \text{if } A_{ij} = H, \\ G \setminus A'_{ij}, & \text{if } A_{ij} \neq H, \end{array} \right\}, \quad (6)$$

where $A^d_{11} = A_{22}$, $A^d_{12} = A_{12}$, $A^d_{21} = A_{21}$, *and* $A^d_{22} = A_{11}$.

Lemma 2 (see [6]). *If* $(H, \circ)$ *is a hypergroupoid, then* $(H, *)$ *is a hypergroupoid when* $x * y = y \circ x$ *for all* $x, y \in H$.

$(H, *)$ in Lemma 2 is called dual hypergroupoid of $(H, \circ)$.

Theorem 1. *Let* $(H = \{a, b\}, \circ)$. *Then,* $(H, \circ) \cong (H, *)$ *if and only if* $(H, \circ)$ *is anticommutative.*

Lemma 3. *There exist 4 anticommutative anti-semi-hypergroup of order 2 (up to isomorphism).*

Proof. Let $(H, \circ)$ be an anticommutative anti-semihypergroup. By Corollary 2, we have $a \circ a \neq a$ and $b \circ b \neq b$. Also, $a \circ b \neq b \circ a$. Consider the following.

If $a \circ a = H$, then $a \circ (a \circ a) = a \circ H = H = H \circ a = (a \circ a) \circ a$, a contradiction. Then, we get $a \circ a = b$ and $b \circ b = a$.

Now, we have

Case 1. If $a \circ b = a$, then $b \circ a = H$ or $b \circ a = b$, so we get (b, a, b, a) and (b, a, H, a) are two anti-semihypergroups

Case 2. If $a \circ b = b$, then $b \circ a = H$ or $b \circ a = a$, so we get (b, b, a, a) and (b, b, H, a) are two anti-semihypergroups

Case 3. If $a \circ b = H$, then $b \circ a = a$ or $b \circ a = b$, so we get (b, H, a, a) and (b, H, b, a) are two anti-semihypergroups

It can be see that $(b, a, H, a) \cong (b, H, b, a)$ and $(b, H, a, a) \cong (b, b, H, a)$. Therefore, (b, b, a, a), (b, a, b, a), (b, a, H, a), and (b, H, a, a) are 4 nonisomorphic anti-semihypergroups of order 2. □

Corollary 3. *There exists two nonisomorphic anti-semigroups of order 2:* (b, b, a, a) *and* (b, a, b, a). *Anti-semigroup* (b, b, a, a) *is the dual form of the anti-semigroup* (b, a, b, a).

Corollary 4. *There exists two nonisomorphic anti-semihypergroups of order 2:* (b, a, H, a) *and* (b, H, a, a). *Anti-semihypergroup* (b, a, H, a) *is the dual form of the anti-semihypergroup* (b, H, a, a).

Theorem 2. *Let* $(H, \circ)$ *be a hypergroupoid of order 2. Then,* $(H, \circ)$ *does not have (NR) or (AR).*

Proof. Let $H = \{a, b\}$. Suppose $Ha \neq H$, $aH \neq H$, and $Ha \neq aH$. Hence, $Ha = \{a\}$ or $Ha = \{b\}$. First, give $Ha = \{a\}$, then $aH \neq H$ and $Ha \neq aH$ implies that $aH = \{b\}$. Then, $a \circ a \subseteq Ha = \{b\}$ and $a \circ a \subseteq Ha = \{a\}$. Therefore, $\{b\} = a \circ a \circ a = \{a\}$, and this is a contradiction. In the similar way, we obtain $Hb \neq H$, $bH \neq H$, and $Hb \neq bH$, a contradiction.

Using Lemmas 1 and 2 and Theorem 1, we can find 45 nonisomorphic classes hypergroupoids of the order 2. We characterize these 45 classes in Table 1.

Note that semi-hypergroups, hypergroups, and fuzzy hypergroups of order 2 are enumerated in [7, 11, 12].

We obtain anti-semihypergroups and neutro-semihypergroups of order 2 and the classification of the hypergroupoids of order 2 (classes up to isomorphism).

R, NR, AR, A, NA, AA, C, NC, and AC in Table 1 are denoted in Sections 2 and 3.

A result from Table 1 confirms the enumeration of the hyperstructure of order 2 [11, 23, 24], which is summarized as follows. □

4. Conclusion and Future Work

In this paper, we have studied several special types of hypergroups, neutro-semihypergroups, anti-semihypergroups, neutro-hypergroups, and anti-hypergroups. New results and examples on these new algebraic structures have been investigated. Also, we characterize all neutro-hypergroups and anti-hypergroups of order two up to isomorphism. These concepts can further be generalized.

Future research to be done related to this topic are

(a) Define neutro-quasihypergroup, anti-quasihypergroup, neutrocommutative quasi-hypergroup, and anticommutative quasi-hypergroup

(b) Define neutro-hypergroups, anti-hypergroups, neutrocommutative hypergroups, and anti-commutative hypergroups

(c) Define and investigate neutroHv-groups, antiHv-groups, neutroHv-rings, and antiHv-rings

(d) It will be interesting to characterize infinite neutro-hypergroups and anti-hypergroups up to isomorphism

(e) These results can be applied to other hyper-algebraic structures, such as hyper-rings, hyper-spaces, hyper-BCK-algebra, hyper-BE-algebras, and hyper-K-algebras.

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NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries

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Florentin Smarandache (2021). NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries. *Neutrosophic Sets and Systems* 46, 457-476

Abstract
In this paper we extend the NeutroAlgebra & AntiAlgebra to the geometric space, by founding the NeutroGeometry & AntiGeometry.
While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.), and the NeutroAxiom results from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system.
Therefore, the NeutroGeometry and AntiGeometry are respectively alternatives and generalizations of the Non-Euclidean Geometries.
In the second part, we recall the evolution from Paradoxism to Neutrosophy, then to NeutroAlgebra & AntiAlgebra, afterwards to NeutroGeometry & AntiGeometry, and in general to NeutroStructure & AntiStructure that naturally arise in any field of knowledge.
At the end, we present applications of many NeutroStructures in our real world.

Keywords: Non-Euclidean Geometries, Euclidean Geometry, Lobachevski-Bolyai-Gauss Geometry, Riemannian Geometry, NeutroManifold, AntiManifold, NeutroAlgebra, AntiAlgebra, NeutroGeometry, AntiGeometry, NeutroAxiom, AntiAxiom, Partial Function, NeutroFunction, AntiFunction, NeutroOperation, AntiOperation, NeutroAttribute, AntiAttribute, NeutroRelation, AntiRelation, NeutroStructure, AntiStructure

1. Introduction

In our real world, the spaces are not homogeneous, but mixed, complex, even ambiguous. And the elements that populate them and the rules that act upon them are not perfect, uniform, or complete - but fragmentary and disparate, with unclear and conflicting information, and they do not apply in the same degree to each element.

The perfect, idealistic ones exist just in the theoretical sciences. We live in a multi-space endowed with a multi-structure [35]. Neither the space's elements nor the regulations that govern them are egalitarian, all of them are characterized by degrees of diversity and variance. The indeterminate (vague, unclear, incomplete, unknown, contradictory etc.) data and procedures are surrounding us.

That's why, for example, the classical algebraic and geometric spaces and structures were extended to more realistic spaces and structures [1], called respectively NeutroAlgebra & AntiAlgebra [2019] and respectively NeutroGeometry & AntiGeometry [1969, 2021], whose elements do not necessarily behave the same, while the operations and rules onto these spaces may only be partially (not totally) true.

While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's five postulates, Hilbert's 20 axioms, etc.), and the NeutroAxiom results from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system.
Therefore, the NeutroGeometry and AntiGeometry are respectively alternatives and generalizations of the Non-Euclidean Geometries.
In the second part, we recall the evolution from Paradoxism to Neutrosophy, then to NeutroAlgebra & AntiAlgebra, afterwards to NeutroGeometry & AntiGeometry, and in general to NeutroStructure & AntiStructure that naturally arise in any field of knowledge. At the end, we present applications of many NeutroStructures in our real world.

On a given space, a classical Axiom is totally (100%) true. While a NeutroAxiom is partially true, partially indeterminate, and partially false. Also, an AntiAxiom is totally (100%) false.

A classical Geometry has only totally true Axioms. While a NeutroGeometry is a geometry that has at least one NeutroAxiom and no AntiAxiom. Also, an AntiGeometry is a geometry that has at least one AntiAxiom.

Below we introduce, in the first part of this article, the construction of NeutroGeometry & AntiGeometry, together with the Non-Euclidean geometries, while in the second part we recall the evolution from paradoxism to neutrosophy, and then to NeutroAlgebra & AntiAlgebra, culminating with the most general form of NeutroStructure & AntiStructure in any field of knowledge.

A classical (100%) true statement on a given classical structure, may or may not be 100% true on its corresponding NeutroStructure or AntiStructure, it depends on the neutrosophication or antisophication procedures [1 – 24].

Further on, the neutrosophic triplet (Algebra, NeutroAlgebra, AntiAlgebra) was restrained or extended to all fuzzy and fuzzy extension theories (FET) triplets of the form (Algebra, Neutro$_{FET}$Algebra, Anti$_{FET}$Algebra), where FET may be: Fuzzy, Intuitionistic Fuzzy, Inconsistent Intuitionistic Fuzzy (Picture Fuzzy, Ternary Fuzzy), Pythagorean Fuzzy (Atanassov's Intuitionistic Fuzzy of second type), q-Rung Orthopair Fuzzy, Spherical Fuzzy, n-HyperSpherical Fuzzy, Refined Neutrosophic, etc.

1.1. Concept, NeutroConcept, AntiConcept

Let us consider on a given geometric space a *classical geometric concept* (such as: axiom, postulate, operator, transformation, function, theorem, property, theory, etc.).

We form the following geometric neutrosophic triplet:

Concept(1, 0, 0), *NeutroConcept*(T, I, F), *AntiConcept* (0, 0, 1),

where (T, I, F) $\notin$ {(1, 0, 0), (0, 0, 1)}.

{ Of course, we consider only the neutrosophic triplets (Concept, NeutroConcept, AntiConcept) that make sense in our everyday life and in the real world. }

Concept(1, 0, 0) means that the degree of truth of the concept is T = 1, I = 0, F = 0, or the Concept is 100% true, 0% indeterminate, and 0% false in the given geometric space.

NeutroConcept (T, I, F) means that the concept is T% true, I% indeterminate, and 0% false in the given geometric space, with (T, I, F) $\in$ [0, 1], and (T, I, F) $\notin$ {(1, 0, 0), (0, 0, 1)}.

AntiConcept (0, 0, 1) means that T = 0, I = 0, and F = 1, or the Concept is 0% true, 0% indeterminate, and 100% false in the given geometric space.

1.2. Geometry, NeutroGeometry, AntiGeometry

We go from the neutrosophic triplet (*Algebra*, *NeutroAlgebra*, *AntiAlgebra*) to a similar neutrosophic triplet (*Geometry, NeutroGeometry, AntiGeometry*), in the same way.

Correspondingly from the algebraic structuires, with respect to the geometries, one has:

In the *classical (Euclidean) Geometry*, on a given space, all classical geometric *Concepts* are 100% true (i.e. true for all elements of the space).

While in a *NeutroGeometry*, on a given space, there is at least one *NeutroConcept* (and no *AntiConcept*).

In the *AntiGeometry*, on a given space, there is at least one *AntiConcept*.

1.3. Geometric NeutroSophication and Geometric AntiSophication

Similarly, as to the algebraic structures, using the process of *NeutroSophication* of a classical geometric structure, a *NeutroGeometry* is produced; while through the process of *AntiSophication* of a classical geometric structure produces an *AntiGeometry*.

Let *S* be a classical *geometric space*, and <A> be a *geometric concept* (such as: postulate, axiom, theorem, property, function, transformation, operator, theory, etc.). The <antiA> is the opposite of <A>, while <neutA> (also called <neutroA>) is the neutral (or indeterminate) part between <A> and <antiA>.

The neutrosophication tri-sections *S* into three subspaces:

- the first subspace, denoted just by <A>, where the *geometric concept* is totally true [degree of truth *T* = 1]; we denote it by *Concept(1,0,0).*

- the second subspace, denoted by <neutA>, where the *geometric concept* is partially true [degree of truth T], partially indeterminate [degree of indeterminacy I], and partially false [degree of falsehood F], denoted as *NeutroConcept(T,I,F)*, where $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$;

- the third subspace, denoted by <antiA>, where the *geometric concept* is totally false [degree of falsehood $F = 1$], denoted by *AntiConcept(0,0,1).*

The three subspaces may or may not be disjoint, depending on the application, but they are exhaustive (their union equals the whole space S).

1.4. Non-Euclidean Geometries

1.4.1. The *Lobachevsky* (also known as *Lobachevsky-Bolyai-Gauss*) *Geometry*, and called *Hyperbolic Geometry*, is an AntiGeometry, because the Fifth Euclidean Postulate (in a plane, through a point outside a line, only one parallel can be drawn to that line) is 100% invalidated in the following AntiPostulate (first version) way: in a plane through a point outside of a line, there can be drawn infinitely many parallels to that line. Or (T, I, F) = (0, 0, 1).

1.4.2. The *Riemannian Geometry*, which is called *Elliptic Geometry*, is an AntiGeometry too, since the Fifth Euclidean Postulate is 100% invalidated in the following AntiPostulate (second version) way: in a place, through a point outside of a line, no parallel can be drawn to that line. Or (T, I, F) = (0, 0, 1).

1.4.3. The *Smarandache Geometries (SG)* are more complex [30 – 57]. Why this type of mixed non-Euclidean geometries, and sometimes partially Non-Euclidean and partially Euclidean? Because the real geometric spaces are not pure but hybrid, and the real rules do not uniformly apply to all space's elements, but they have degrees of diversity – applying to some geometrical concepts (point, line, plane, surface, etc.) in a smaller or bigger degree.

From Prof. Dr. Linfan Mao's arXiv.org paper *Pseudo-Manifold Geometries with Applications* [57], Cornell University, New York City, USA, 2006, https://arxiv.org/abs/math/0610307 :

> "A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways and a Smarandache n-manifold is a n-manifold that support a Smarandache geometry.
>
> Iseri provided a construction for Smarandache 2-manifolds by equilateral triangular disks on a plane and a more general way for Smarandache 2-manifolds on surfaces, called map geometries was presented by the author (…).
>
> However, few observations for cases of $n \geq 3$ are found on the journals. As a kind of Smarandache geometries, a general way for constructing dimensional n pseudo-manifolds are presented for any integer n

≥ 2 in this paper. Connection and principal fiber bundles are also defined on these manifolds. Following these constructions, nearly all existent geometries, such as those of Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kahler geometry and Finsler geometry, etc. are their sub-geometries."

Iseri ([34], [39 - 40]) has constructed some Smarandache Manifolds (S-manifolds) that topologically are piecewise linear, and whose geodesics have elliptic, Euclidean, and hyperbolic behavior. An SG geometry may exhibit one or more types of negative, zero, or positive curvatures into the same given space.

1.4.3.1) If at least one axiom is validated (partially true, T > 0) and invalidated (partially false, F > 0), and no other axiom is only invalidated (*AntiAxiom*), then this first class of SG geometry is a *NeutroGeometry*.

1.4.3.2) If at least one axiom is only invalidated (or F = 1), no matter if the other axioms are classical or *NeutroAxioms* or *AntiAxioms* too, then this second class of SG geometry is an *AntiGeometry*.

1.4.3.3) *The model of an SG geometry that is a NeutroGeometry:*

Bhattacharya [38] has constructed the following SG model:

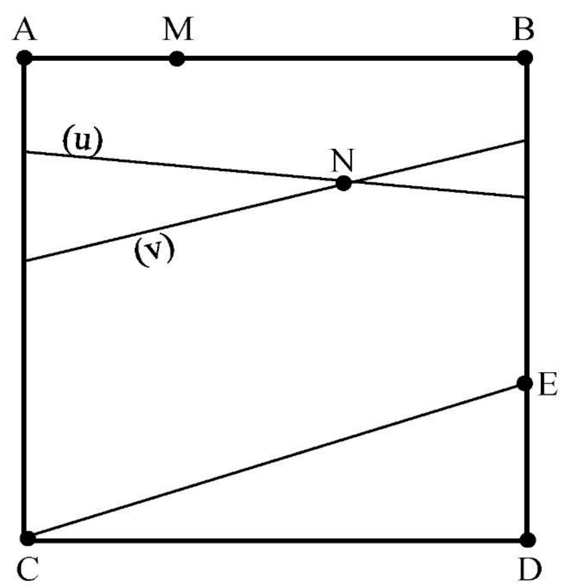

Fig. 1. Bhattacharya's Model for the SG geometry as a NeutroGeometry

The geometric space is a square ABCD, comprising all points inside and on its edges.

"Point" means the classical point, for example: A, B, C, D, E, N, and M.

"Line" means any segment of line connecting two points on the opposite square sides AC and BD, for example: AB, CD, CE, (u), and (v).

"Parallel lines" are lines that do not intersect.

Let us take a line CE and an exterior point N to it. We observe that there is an infinity of lines passing through N and parallel to CE [all lines passing through N and in between the lines (u) and (v) for example] – the *hyperbolic case*.

Also, taking another exterior point, D, there is no parallel line passing through D and parallel to CE because all lines passing through D intersects CE – the *elliptic case*.

Taking another exterior point $M \in AB$, then we only have one line AB parallel to CE, because only one line passes through the point M – the *Euclidean case*.

Consequently, the Fifth Euclidean Postulate is twice invalidated, but also once validated.

Being partially hyperbolic Non-Euclidean, partially elliptic Non-Euclidean, and partially Euclidean, therefore we have here a SG.

This is not a Non-Euclidean Geometry (since the Euclid's Fifth Postulate is not totally false, but only partially), but it is a NeutroGeometry.

Theorem 1.4.3.3.1

If a statement (proposition, theorem, lemma, property, algorithm, etc.) is (totally) true (degree of truth T = 1, degree of indeterminacy I = 0, and degree of falsehood F = 0) in the classical geometry, the statement may get any logical values (i.e. T, I, F may be any values in [0, 1]) in a NeutroGeometry or in an AntiGeometry

Proof.

The logical value the statement gets in a NeutroGeometry or in an AntiGeometry depends on what classical axioms the statement is based upon in the classical geometry, and how these axioms behave in the NeutroGeometry or AntiGeometry models.

Let's consider the below classical geometric proposition P(L1, L2, L3) that is 100% true:

In a 2D-Euclidean geometric space, if two lines L1 and L2 are parallel with the third line L3, then they are also parallel (i.e. L1 // L2).

In Bhattacharya's Model of an SG geometry, this statement is partially true and partially false.

For example, in Fig. 1:

- degree of truth: the lines AB and (u) are parallel to the line CE, then AB is parallel to (u);
- degree of falsehood: the lines (u) and (v) are parallel to the line CE, but (u) and (v) are not parallel since they intersect in the point N.

1.4.3.4) The Model of a SG geometry that is an AntiGeometry

Let us consider the following rectangular piece of land PQRS,

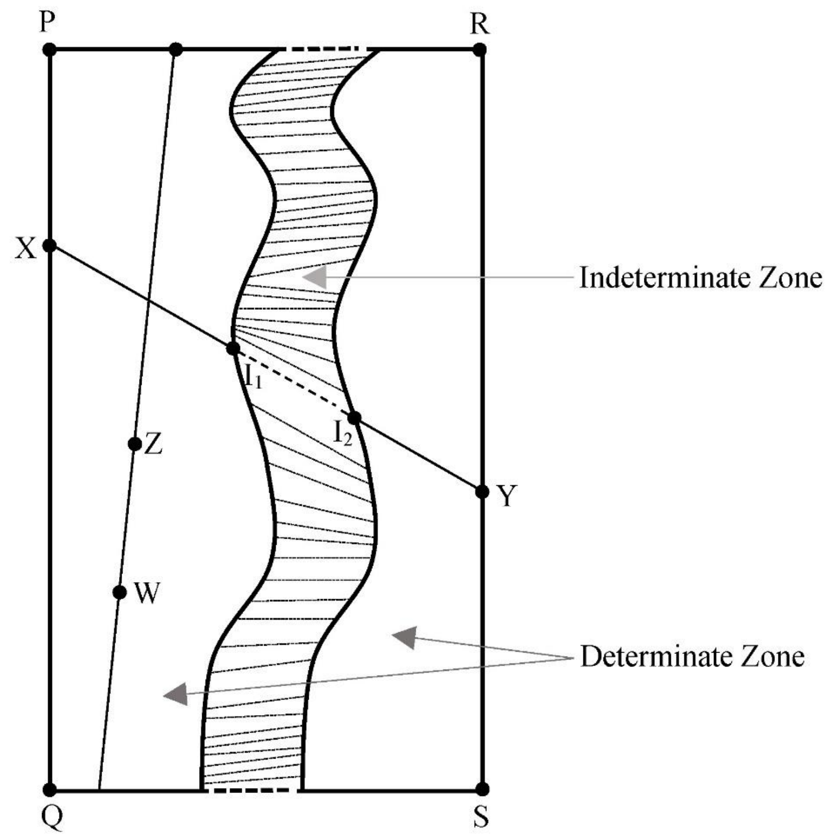

Fig. 2. Model for an SG geometry that is an AntiGeometry

whose middle (shaded) area is an indeterminate zone (a river, with swamp, canyons, and no bridge) that is impossible to cross over on the ground. Therefore, this piece of land is composed from a determinate zone and an indeterminate zone (as above).

"Point" means any classical (usual) point, for example: P, Q, R, S, X, Y, Z, and W that are determinate well-known (classical) points, and I_1, I_2 that are indeterminate (not well-known) points [in the indeterminate zone].

"Line" is any segment of line that connects a point on the side PQ with a point on the side RS. For example, PR, QS, XY. However, these lines have an indeterminate (not well known, not clear) part that is the indeterminate zone. On the other hand, ZW is not a line since it does not connect the sides PQ and RS.

The following geometric classical axiom: *through two distinct points there always passes one single line*, is totally (100%) denied in this model in the following two ways:

through any two distinct points, in this given model, either no line passes (see the case of ZW), or only one partially determinate line does (see the case of XY) - therefore no fully determinate line passes. Thus, this SG geometry is an AntiGeometry.

1.5. Manifold, NeutroManifold, AntiManifold

1.5.1. Manifold

The classical **Manifold** [29] is a topological space that, on the small scales, near each point, resembles the classical (Euclidean) Geometry Space [i.e. in this space there are only *classical Axioms (totally true)*]. Or each point has a neighborhood that is homeomorphic to an open unit ball of the Euclidean Space R^n (where R is the set of real numbers). Homeomorphism is a continuous and bijective function whose inverse is also continuous.

"In general, any object that is near 'flat' on the small scale is a manifold" [29].

1.5.2. NeutroManifold

The **NeutroManifold** is a topological space that, on the small scales, near each point, resembles the NeutroGeometry Space [i.e. in this space there is *at least a NeutroAxiom (partially true, partially indeterminate, and partially false) and no AntiAxiom*].

For example, Bhattacharya's Model for a SG geometry (Fig. 1) is a NeutroManifold, since the geometric space ABCD has a NeutroAxiom (i.e. the Fifth Euclidean Postulate, which is partially true and partially false), and no AntiAxiom.

1.5.3. AntiManifold

The Anti**Manifold** is a topological space that, on the small scales, near each point, resembles the AntiGeometry Space [i.e. in this space there is *at least one AntiAxiom (totally false)*].

For example, the Model for a SG geometry (Fig. 2) is an AntiManifold, since the geometric space PQRS has an AntiAxiom (i.e., through two distinct points there always passes a single line - which is totally false).

***

2. Evolution from Paradoxism to Neutrosophy then to NeutroAlgebra/AntiAlgebra and now to NeutroGeometry/AntiGeometry

Below we recall and revise the previous foundations and developments that culminated with the introduction of NeutroAlgebra & AntiAlgebra as new field of research, extended then to NeutroStructure

& AntiStructure, and now particularized to NeutroGeometry & AntiGeometry that are extensions of the Non-Euclidean Geometries.

2.1. From Paradoxism to Neutrosophy

Paradoxism [58] is an international movement in science and culture, founded by Smarandache in 1980s, based on excessive use of antitheses, oxymoron, contradictions, and paradoxes. During three decades (1980-2020) hundreds of authors from tens of countries around the globe contributed papers to 15 international paradoxist anthologies.

In 1995, he extended the *paradoxism* (based on opposites) to a new branch of philosophy called *neutrosophy* (based on opposites and their neutral) [59], that gave birth to many scientific branches, such as: neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics, neutrosophic algebraic structures, and so on with multiple applications in engineering, computer science, administrative work, medical research, social sciences, etc.

Neutrosophy is an extension of Dialectics that have derived from the Yin-Yan Ancient Chinese Philosophy.

2.2. From Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures

In 2019 Smarandache [1] generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) {whose operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) {whose operations and axioms are totally false} and on 2020 he continued to develop them [2,3,4].

The NeutroAlgebras & AntiAlgebras are a *new field of research*, which is inspired from our real world.

In classical algebraic structures, all operations are 100% well-defined, and all axioms are 100% true, but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some operations or some laws.

By substituting Concept with Operation, Axiom, Theorem, Relation, Attribute, Algebra, Structure etc. respectively, into the above (Concept, NeutroConcept, AntiConcept), we get the below neutrosophic triplets:

2.3. Operation, NeutroOperation, AntiOperation

When we define an operation on a given set, it does not automatically mean that the operation is well-defined. There are three possibilities:

1) The operation is well-defined (also called inner-defined) for all set's elements [degree of truth T = 1] (as in classical algebraic structures; this is a classical Operation). Neutrosophically we write: Operation(1,0,0).

2) The operation if well-defined for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and outer-defined for the other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is a NeutroOperation). Neutrosophically we write:

NeutroOperation(T,I,F).
3) The operation is outer-defined for all set's elements [degree of falsehood F = 1] (this is an AntiOperation). Neutrosophically we write: AntiOperation(0,0,1).

An operation * on a given non-empty set *S* is actually a *n-ary* function, for integer $n \geq 1$, $f : S^n \to S$.

2.4. Function, NeutroFunction, AntiFunction

Let *U* be a universe of discourse, *A* and *B* be two non-empty sets included in *U*, and *f* be a function: $f : A \to B$

Again, we have three possibilities:

1) The function is well-defined (also called inner-defined) for all elements of its domain *A* [degree of truth T = 1] (this is a classical Function), i.e. $\forall x \in A, f(x) \in B$. Neutrosophically we write: Function(1,0,0).
2) The function if well-defined for some elements of its domain, i.e. $\exists x \in A, f(x) \in B$ [degree of truth T], indeterminate for other elements, i.e. $\exists x \in A, f(x) = indeterminate$ [degree of indeterminacy I], and outer-defined for the other elements, i.e. $\exists x \in A, f(x) \notin B$ [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1). This is a NeutroFunction. Neutrosophically we write: NeutroFunction(T,I,F).
3) The function is outer-defined for all elements of its domain *A* [degree of falsehood F = 1] (this is an AntiFunction), i.e. $\forall x \in A, f(x) \notin B$ (all function's values are outside of its codomain *B*; they may be outside of the universe of discourse too). Neutrosophically we write: AntiFunction(0,0,1).

2.5. NeutroFunction & AntiFunction vs. Partial Function

We prove that the NeutroFunction & AntiFunction are extensions and alternatives of the Partial Function.

Definition of Partial Function [60]

A function f: A→B is sometimes called a *total function*, to signify that f(a) is defined for every a ∈ A. If C is any set such that C ⊇ A then f is also a *partial function* from C to B.

Clearly if f is a function from A to B then it is a partial function from A to B, but a partial function need not be defined for every element of its domain. The set of elements of A for which f is defined is sometimes called the *domain of definition*.

From other sites, the Partial Function means: for any a ∈ A one has: f(a) ∈ B or f(a) = undefined.

Comparison

i) "Partial" is mutually understood as there exist at least one element $a_1 \in A$ such that $f(a_1) \in B$, or the function is defined for at least one element (therefore T > 0).
Such restriction is released in the NeutroFunction and AntiFunction (where T is allowed to be 0).
Example 1.
Let's consider the set of positive integers $Z = \{1, 2, 3, \ldots\}$, included into the universe of discourse *R*, which is the set of real numbers. Let's define the function
$f_1 : Z \to Z$, $f_1(x) = \frac{x}{0}$, for all $x \in Z$.
Clearly, the function f_1 is *100%* undefined, therefore the indeterminacy I = 1, while T = 0 and F = 0.
Hence f_1 is a NeutroFunction, but not a Partial Function.
Example 2.
Let's take the set of odd positive integers $D = \{1, 3, 5, \ldots\}$, included in the universe of discourse *R*. Let's define the function $f_2 : D \to D, f_2(x) = \frac{x}{2}$, for all $x \in D$.
The function f_2 is *100%* outer-defined, since $\frac{x}{2} \notin D$ for all $x \in D$. Whence F = 1, T = 0, and I = 0. Hence this is an AntiFunction, but not a partial Function.

ii) The Partial Function does not catch ***all types of indeterminacies*** that are allowed in a NeutroFunction. Indeterminacies may occur with respect to: the function's domain, codomain, or relation that connects the elements in the domain with the elements in the codomain.
Example 3.
Let's consider the function $g: \{1, 2, 3, \ldots, 9, 10, 11\} \to \{12, 13, \ldots, 19\}$, about whom we only have vague, unclear information as below:
g(1 or 2) = 12, i.e. we are not sure if g(1) = 12 or g(2) = 12;
g(3) = 18 or 19, i.e. we are not sure if g(3) = 18 or g(3) = 19;
g(4 or 5 or 6) = 13 or 17;
g(7) = unknown;
g(unknown) = 14.
All the above values represent the function's degree of indeterminacy (I > 0).
g(10) = 20 that does not belong to the codomain; (outer-defined, or degree of falsehood F > 0);
g(11) = 15 that belongs to the codomain; (inner-defined, or degree of truth, hence T > 0).
Function *g* is a NeutroFunction (with I > 0, T > 0, F > 0), but not a Partial Function since such types of indeterminacies are not characteristic to it.

2.6. Axiom, NeutroAxiom, AntiAxiom

Similarly for an axiom, defined on a given set, endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set's elements. We have three possibilities again:
1) The axiom is true for all set's elements (totally true) [degree of truth T = 1] (as in classical algebraic

structures; this is a classical Axiom). Neutrosophically we write: Axiom(1,0,0).

2) The axiom if true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is NeutroAxiom). Neutrosophically we write NeutroAxiom(T,I,F).

3) The axiom is false for all set's elements [degree of falsehood F = 1](this is AntiAxiom). Neutrosophically we write AntiAxiom(0,0,1).

2.7. Theorem, NeutroTheorem, AntiTheorem

In any science, a classical Theorem, defined on a given space, is a statement that is 100% true (i.e. true for all elements of the space). To prove that a classical theorem is false, it is sufficient to get a single counter-example where the statement is false. Therefore, the classical sciences do not leave room for *partial truth* of a theorem (or a statement). But, in our world and in our everyday life, we have many more examples of statements that are only partially true, than statements that are totally true. The NeutroTheorem and AntiTheorem are generalizations and alternatives of the classical Theorem in any science.

Let's consider a theorem, stated on a given set, endowed with some operation(s). When we construct the theorem on a given set, it does not automatically mean that the theorem is true for all set's elements. We have three possibilities again:

1) The theorem is true for all set's elements [totally true] (as in classical algebraic structures; this is a classical Theorem). Neutrosophically we write: Theorem(1,0,0).

2) The theorem if true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for the other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is a NeutroTheorem). Neutrosophically we write: NeutroTheorem(T,I,F).

3) The theorem is false for all set's elements (this is an AntiTheorem). Neutrosophically we write: AntiTheorem(0,0,1).

And similarly for (Lemma, NeutroLemma, AntiLemma), (Consequence, NeutroConsequence, AntiConsequence), (Algorithm, NeutroAlgorithm, AntiAlgorithm), (Property, NeutroProperty, AntiProperty), etc.

2.8. Relation, NeutroRelation, AntiRelation

1) A classical Relation is a relation that is true for all elements of the set (degree of truth T = 1). Neutrosophically we write Relation(1,0,0).

2) A NeutroRelation is a relation that is true for some of the elements (degree of truth T), indeterminate for other elements (degree of indeterminacy I), and false for the other elements (degree of falsehood F). Neutrosophically we write Relation(T,I,F), where (T,I,F) is different from (1,0,0) and (0,0,1).

3) An AntiRelation is a relation that is false for all elements (degree of falsehood F = 1). Neutrosophically we write Relation(0,0,1).

2.9. Attribute, NeutroAttribute, AntiAttribute

1) A classical Attribute is an attribute that is true for all elements of the set (degree of truth T = 1). Neutrosophically we write Attribute(1,0,0).

2) A NeutroAttribute is an attribute that is true for some of the elements (degree of truth T), indeterminate for other elements (degree of indeterminacy I), and false for the other elements (degree of falsehood F). Neutrosophically we write Attribute(T,I,F), where (T,I,F) is different from (1,0,0) and (0,0,1).

3) An AntiAttribute is an attribute that is false for all elements (degree of falsehood F = 1). Neutrosophically we write Attribute(0,0,1).

2.10. Algebra, NeutroAlgebra, AntiAlgebra

1) An algebraic structure who's all operations are well-defined and all axioms are totally true is called a classical Algebraic Structure (or Algebra).
2) An algebraic structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called a NeutroAlgebraic Structure (or NeutroAlgebra).
3) An algebraic structure that has at least one AntiOperation or one Anti Axiom is called an AntiAlgebraic Structure (or AntiAlgebra).
Therefore, a neutrosophic triplet is formed: <Algebra, NeutroAlgebra, AntiAlgebra>, where "Algebra" can be any classical algebraic structure, such as: a groupoid, semigroup, monoid, group, commutative group, ring, field, vector space, BCK-Algebra, BCI-Algebra, etc.

2.11. Algebra, Neutro$_{FET}$Algebra, Anti$_{FET}$Algebra

The neutrosophic triplet (Algebra, NeutroAlgebra, AntiAlgebra) was further on restrained or extended to all fuzzy and fuzzy extension theories (FET), making triplets of the form: (Algebra, Neutro$_{FET}$Algebra, Anti$_{FET}$Algebra), where FET may be: Fuzzy, Intuitionistic Fuzzy, Inconsistent Intuitionistic Fuzzy (Picture Fuzzy, Ternary Fuzzy), Pythagorean Fuzzy (Atanassov's Intuitionistic Fuzzy of second type), q-Rung Orthopair Fuzzy, Spherical Fuzzy, n-HyperSpherical Fuzzy, Refined Neutrosophic, etc. See several examples below.

2.11.1. The Intuitionistic Fuzzy Triplet (Algebra, Neutro$_{IF}$Algebra, Anti$_{IF}$Algebra)

Herein "IF" stands for intuitionistic fuzzy.

When Indeterminacy (I) is missing, only two components remain, T and F.

1) The Algebra is the same as in the neutrosophic environment, i.e. a classical Algebra where all operations are totally well-defined and all axioms are totally true (T = 1, F = 0).
2) The Neutro$_{IF}$Algebra means that at least one operation or one axiom is partially true (degree of truth T) and partially false (degree of partially falsehood F), with $T, F \in [0,1], 0 \leq T + F \leq 1,$ with $(T, F) \neq (1,0)$ that represents the classical Axiom, and $(T, F) \neq (0,1)$ that represents the Anti$_{IF}$Axiom,

and no Anti_{IF}Operation (operation that is totally outer-defined) and no Anti_{IF}Axiom.

3) The Anti_{IF}Algebra means that at least one operation or one axiom is totally false (T = 0, F = 1), no matter how the other operations or axioms are.

Therefore, one similarly has the triplets: (Operation, $\text{Neutro}_{\text{IF}}$Operation, Anti_{IF}Operation) and (Axiom, $\text{Neutro}_{\text{IF}}$Axiom, Anti_{IF}Axiom).

2.11.2. The Fuzzy Triplet (Algebra, $\text{Neutro}_{\text{Fuzzy}}$Algebra, $\text{Anti}_{\text{Fuzzy}}$Algebra)

When the Indeterminacy (I) and the Falsehood (F) are missing, only one component remains, T.

1) The Algebra is the same as in the neutrosophic environment, i.e. a classical Algebra where all operations are totally well-defined and all axioms are totally true (T = 1).
2) The $\text{Neutro}_{\text{Fuzzy}}$Algebra means that at least one operation or one axiom is partially true (degree of truth T), with $T \in (0,1)$,
   and no $\text{Anti}_{\text{Fuzzy}}$Operation (operation that is totally outer-defined) and no $\text{Anti}_{\text{Fuzzy}}$Axiom.
3) The Anti_{IF}Algebra means that at least one operation or one axiom is totally false (F = 1), no matter how the other operations or axioms are.

Therefore, one similarly has the triplets: (Operation, $\text{Neutro}_{\text{Fuzzy}}$Operation, $\text{Anti}_{\text{Fuzzy}}$Operation) and (Axiom, $\text{Neutro}_{\text{Fuzzy}}$Axiom, $\text{Anti}_{\text{Fuzzy}}$Axiom).

2.12. Structure, NeutroStructure, AntiStructure in any field of knowledge

In general, by NeutroSophication, Smarandache extended any classical *Structure*, in no matter what field of knowledge, to a *NeutroStructur*e, and by AntiSophication to an *AntiStructure*.

i) A classical Structure, in any field of knowledge, is composed of: a non-empty space, populated by some elements, and both (the space and all elements) are characterized by some relations among themselves (such as: operations, laws, axioms, properties, functions, theorems, lemmas, consequences, algorithms, charts, hierarchies, equations, inequalities, etc.), and by their attributes (size, weight, color, shape, location, etc.).

Of course, when analysing a structure, it counts with respect to what relations and what attributes we do it.

ii) A NeutroStructure is a structure that has at least one NeutroRelation or one NeutroAttribute, and no AntiRelation and no AntiAttribute.

iii) An AntiStructure is a structure that has at least one AntiRelation or one AntiAttribute.

2.13. Almost all real Structures are NeutroStructures

The Classical Structures in science mostly exist in theoretical, abstract, perfect, homogeneous, idealistic spaces - because in our everyday life almost all structures are NeutroStructures, since they are neither perfect nor applying to the whole population, and not all elements of the space have the same relations and same attributes in the same degree (not all elements behave in the same way).

The indeterminacy and partiality, with respect to the space, to their elements, to their relations or to their attributes are not taken into consideration in the Classical Structures. But our Real World is full of structures with indeterminate (vague, unclear, conflicting, unknown, etc.) data and partialities.

There are exceptions to almost all laws, and the laws are perceived in different degrees by different people.

2.14. Applications of NeutroStructures in our Real World

(i) In the Christian society the marriage law is defined as the union between a male and a female (*degree of truth*).

But, in the last decades, this law has become less than 100% true, since persons of the same sex were allowed to marry as well (*degree of falsehood*).

On the other hand, there are transgender people (whose sex is indeterminate), and people who have changed the sex by surgical procedures, and these people (and their marriage) cannot be included in the first two categories (*degree of indeterminacy*).

Therefore, since we have a NeutroLaw (with respect to the Law of Marriage) we have a Christian NeutroStructure.

(ii) In India, the law of marriage is not the same for all citizen: Hindi religious men may marry only one wife, while the Muslims may marry up to four wives.

(iii) Not always the difference between good and bad may be clear, from a point of view a thing may be good, while from another point of view bad. There are things that are partially good, partially neutral, and partially bad.

(iv) The laws do not equally apply to all citizens, so they are NeutroLaws. Some laws apply to some degree to a category of citizens, and to a different degree to another category. As such, there is an American folkloric joke: All people are born equal, but some people are more equal than others!

- There are powerful people that are above the laws, and other people that benefit of immunity with respect to the laws.

- For example, in the court of law, privileged people benefit from better defense lawyers than the lower classes, so they may get a lighter sentence.

- Not all criminals go to jail, but only those caught and proven guilty in the court of law. Nor the criminals that for reason of insanity cannot stand trail and do not go to jail since they cannot make a difference between right and wrong.

- Unfortunately, even innocent people went and may go to jail because of sometimes jurisdiction mistakes...

- The Hypocrisy and Double Standard are widely spread: some regulation applies to some people, but not to others!

(v) Anti-Abortion Law does not apply to all pregnant women: the incest, rapes, and women whose life is threatened may get abortions.

(vi) Gun-Control Law does not apply to all citizen: the police, army, security, professional hunters are allowed to bear arms.

Etc.

Conclusion

In this paper we have extended the Non-Euclidean Geometries to NeutroGeometry (a geometric space that has at least one NeutroAxiom and no AntiAxiom) and to AntiGeometry (a geometric space that has at least one AntiAxiom) similarly to the NeutroAlgebras and AntiAlgebras.

A NeutroAxiom is an axiom that is partially true, partially indeterminate, and partially false in the same space. While the AntiAxiom is an axiom that is totally false in the given space.

While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry (1969) results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.), and the NeutroGeometry results from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system.
Therefore, the NeutroGeometry and AntiGeometry are respectively alternatives and generalizations of the Non-Euclidean Geometries.
In the second part, we recall the evolution from Paradoxism to Neutrosophy, then to NeutroAlgebra & AntiAlgebra, afterwards to NeutroGeometry & AntiGeometry, and in general to NeutroStructure & AntiStructure that naturally arise in any field of knowledge.
At the end, we present applications of many NeutroStructures in our real world.

Further on, we have recalled and reviewed the evolution from Paradoxism to Neutrosophy, and from the classical algebraic structures to NeutroAlgebra and AntiAlgebra structures, and in general to the NeutroStructure and AntiStructure in any field of knowledge. Then many applications of NeutroStructures from everyday life were presented.

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On Neutrosophic Quadruple Groups

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Abstract
As generalizations and alternatives of classical algebraic structures there have been introduced in 2019 the NeutroAlgebraic structures (or NeutroAlgebras) and AntiAlgebraic structures (or AntiAlgebras). Unlike the classical algebraic structures, where all operations are well defined and all axioms are totally true, in NeutroAlgebras and AntiAlgebras, the operations may be partially well defined and the axioms partially true or, respectively, totally outer-defined and the axioms totally false. These NeutroAlgebras and AntiAlgebras form a new field of research, which is inspired from our real world. In this paper, we study neutrosophic quadruple algebraic structures and NeutroQuadrupleAlgebraicStructures. NeutroQuadrupleGroup is studied in particular and several examples are provided. It is shown that $(NQ(\mathbb{Z}), \div)$ is a NeutroQuadrupleGroup. Substruc-tures of NeutroQuadrupleGroups are also presented with examples.

Keywords Neutrosophic quadruple number · NeutroAlgebra · NeutroQuadrupleGroup · NeutroQuadrupleSubgroup

1 Introduction

It was started from Paradoxism, then to Neutrosophy, and afterwards to Neutrosophic Set and Neutrosophic Algebraic Structures. Paradoxism [21] is an international movement in science and culture, founded by Smarandache in 1980 s, based on excessive use of antitheses, oxymoron, contradictions, and paradoxes. During the 3 decades (1980–2020), hundreds of authors from tens of countries around the globe contributed papers to 15 international paradoxist anthologies. In 1995, Smarandache extended the paradoxism (based on opposites) to a new branch of philosophy called neutrosophy (based on opposites and their neutrals) that gave birth to many scientific branches, such as neutrosophic logic, neutrosophic set, neutrosophic probability and statistics, neutrosophic algebraic structures, and so on with multiple applications in engineering, computer science, administrative work, medical research etc. Neutrosophy is an extension of Yin-Yang Ancient Chinese Philosophy and of course of Dialectics. From Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraicStructures. In 2019 and 2020, Smarandache [16–18] generalized the classical Algebraic Structures to NeutroAlgebraicStructures (or NeutroAlgebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebra) whose operations and axioms are totally false. By considering a space and an operation defined on, in general, it does not mean that the operation is well defined for all elements of the space. We have three cases, as in neutrosophy: either the operation is well defined (as in classical algebraic structures), or partially defined and partially undefined, or partially outer-defined. Similarly, in general by defining an axiom on a given space under some given operations it does not mean that the axion is true for all elements of the space. Again we gave three cases as in neutrosophy: the axiom is true for all elements (as in classical algebraic structures), or the axiom is partially true and partially false, or the axiom is false for all elements. Motivation is the fact that in mathematics, in general, by defining an operation on a given set it does not mean that the operation is automatically well defined, but many times it is only partially well defined. Similarly, by defining an axiom on a given set, in general it does not mean that the axiom is true for all elements, but only partially true (i.e. true for some elements and maybe false for other elements). In the present paper, we study neutrosophic quadruple algebraic structures and NeutroQuadrupleAlgebraicStructures. NeutroQuadrupleGroup is studied in particular and several examples are provided. It is shown that $(NQ(\mathbb{Z}), \div)$ is a NeutroQuadrupleGroup. Substructures of NeutroQuadrupleGroups are also presented with examples.

1.1 Operation, NeutroOperation, AntiOperation

When we define an operation on a given set, it does not automatically mean that the operation is well defined. There are three possibilities:

- The operation is well-defined (or inner-defined) for all set's elements (as in classical algebraic structures this is classical Operation).
- The operation if well-defined for some elements, indeterminate for other elements, and outer-defined for others elements (this is NeutroOperation).
- The operation is outer-defined for all set's elements (this is AntiOperation).

1.2 Axiom, NeutroAxiom, AntiAxiom

Similarly for an axiom, defined on a given set, endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set's elements. We have three possibilities again:

- The axiom is true for all set's elements (totally true) (as in classical algebraic structures; this is a classical Axiom).
- The axiom if true for some elements, indeterminate for other elements, and false for other elements (this is NeutroAxiom).
- The axiom is false for all set's elements (this is AntiAxiom).

1.3 Algebra, NeutroAlgebra, AntiAlgebra

- An algebraic structure whose all operations are well-defined and all axioms are totally true is called Classical Algebraic Structure (or Algebra).
- An algebraic structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called NeutroAlgebraic Structure (or NeutroAlgebra).
- An algebraic structure that has at least one AntiOperation or Anti Axiom is called AntiAlgebraic Structure (or AntiAlgebra).

Therefore, a neutrosophic triplet structure is formed (see [1–8]):

$< Algebra, NeutroAlgebra, AntiAlgebra >$.

"Algebra" can be: groupoid, semigroup, monoid, group, commutative group, ring, field, vector space, BCK-Algebra, BCI-Algebra, K-algebra, BE-algebra, H_v-rings, etc. (see [9–15] and [20]).

The sets of natural/integer/rational/real/complex numbers are, respectively, denoted by

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

The Neutrosophic Quadruple Numbers and the Absorbance Law were introduced by Smarandache [19]; they have the general form:

$N = a + bT + cI + dF$, where a, b, c, d may be numbers of any type (natural, integer, rational, irrational, real, complex, etc.), where "a" is the known part of the neutrosophic quadruple number N, while "$bT + cI + dF$" is the unknown part of the neutrosophic quadruple number N; then the unknown part is split into three subparts: degree of confidence (T), degree of indeterminacy of confidence–nonconfidence (I), and degree of nonconfidence (F). N is a four-dimensional vector that can also be written as: $N = (a, b, c, d)$.

There are transcendental, irrational, etc. numbers that are not well known, they are only partially known and partially unknown, they may have infinitely many decimals. Not even the most modern supercomputers can compute more than a few thousands decimals, but the infinitely many left decimals still remain unknown. Therefore, such numbers are very little known (because only a finite number of decimals are known), and infinitely unknown (because an infinite number of decimals are unknown). Take for example: $\sqrt{2} = 1.4142\ldots$.

2 Arithmetic Operations on the Neutrosophic Set of Quadruple Numbers

Definition 1 A neutrosophic set of quadruple numbers denoted by $NQ(X)$ is a set defined by

$$NQ(X) = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\},$$

where T, I, F have their usual neutrosophic logic meanings.

Definition 2 A neutrosophic quadruple number is a number of the form $(a, bT, cI, dF) \in NQ(X)$. For any neutrosophic quadruple number (a, bT, cI, dF) representing any entity which may be a number, an idea, an object, etc., a is called the known part and (bT, cI, dF) is called the unknown part. Two neutrosophic quadruple numbers $x = (a, bT, cI, dF)$ and $y = (e, fT, gI, hF)$ are said to be equal written $x = y$ if and only if $a = e, b = f, c = g, d = h$.

Example 1 $NQ(\mathbb{N})$, $NQ(\mathbb{Z})$, $NQ(\mathbb{Q})$, $NQ(\mathbb{Z})$ and $NQ(\mathbb{C})$ are neutrosophic sets of quadruple natural, integers, rationals, real and complex numbers respectively.

Example 2 The following

$$x = 2 - 3T + 4I - 5F \in NQ(\mathbb{Z}),$$
$$y = \sqrt{2} - \frac{3}{4}T - 11I - \frac{5}{6}F \in NQ(\mathbb{R}),$$
$$z = (3 + 2i) - (-4 + 3i)T + (4i)I - \left(\frac{1}{5} - \frac{1}{6}i\right)F \in NQ(\mathbb{C})$$

are examples of neutrosophic quadruple of integers, real and complex numbers, respectively.

Definition 3 Let $a = (a_1, a_2T, a_3I, a_4F)$, $b = (b_1, b_2T, b_3I, b_4F) \in NQ(X)$. We define the following:

$$a + b = (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F)$$
$$a - b = (a_1 - b_1, (a_2 - b_2)T, (a_3 - b_3)I, (a_4 - b_4)F).$$

Definition 4 Let $a = (a_1, a_2T, a_3I, a_4F) \in NQ(X)$ and let α be any scalar which may be real or complex, the scalar product $\alpha.a$ is defined by

$$\begin{aligned}\alpha.a =& \alpha.(a_1, a_2T, a_3I, a_4F)\\ =& (\alpha a_1, \alpha a_2T, \alpha a_3I, \alpha a_4F).\end{aligned}$$

If $\alpha = 0$, then we have $0.a = (0, 0, 0, 0)$ and for any non-zero scalars m and n and $b = (b_1, b_2T, b_3I, b_4F)$, we have

$$\begin{aligned}(m + n)a &= ma + na,\\ m(a + b) &= ma + mb,\\ mn(a) &= m(na),\\ -a &= (-a_1, -a_2T, -a_3I, -a_4F).\end{aligned}$$

Example 3 From Example 2, we obtain the following:

$$\begin{aligned}x + y &= (2 + \sqrt{2}) - \frac{15}{4}T - 7I - \frac{35}{6}F.\\ x - y &= (2 - \sqrt{2}) - \frac{9}{4}T + 15I - \frac{25}{6}.\\ 2iz &= (-4 + 6i) + (6 + 8i)T - 8I - \left(\frac{1}{3} + \frac{2}{5}i\right)F.\end{aligned}$$

Multiplication of two neutrosophic quadruple numbers cannot be carried out like multiplication of two real or complex numbers. To multiply two neutrosophic quadruple numbers $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ(X)$, the prevalence order of $\{T, I, F\}$ is required. Consider the following prevalence orders:

(i) Suppose in an optimistic way we consider the prevalence order $T \succ I \succ F$. Then we have

$$\begin{aligned}TI &= IT = \max\{T, I\} = T,\\ TF &= FT = \max\{T, F\} = T,\\ IF &= FI = \max\{I, F\} = I,\\ TT &= T^2 = T,\\ II &= I^2 = I,\\ FF &= F^2 = F.\end{aligned}$$

Then

$$\begin{aligned}a \times b &= (a_1, a_2T, a_3I, a_4F).(b_1, b_2T, b_3I, b_4F)\\ &= (a_1b_1, (a_1b_2 + a_2b_1, a_2b_2 + a_2b_3\\ &\quad + a_2b_4 + a_3b_2 + a_4b_2)T, (a_1b_3 + a_3b_1\\ &\quad + a_3b_3 + a_3b_4 + a_4b_3)I, (a_1b_4 + a_4b_1 + a_4b_4)F).\end{aligned}$$

(ii) Suppose in a pessimistic way we consider the prevalence order $T \prec I \prec F$. Then we have

$$TI = IT = \max\{T, I\} = I,$$
$$TF = FT = \max\{T, F\} = F,$$
$$IF = FI = \max\{I, F\} = F,$$
$$TT = T^2 = T,$$
$$II = I^2 = I,$$
$$FF = F^2 = F.$$

Then

$$\begin{aligned} a \times b &= (a_1, a_2T, a_3I, a_4F).(b_1, b_2T, b_3I, b_4F) \\ &= (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, \\ &\quad (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, \\ &\quad (a_1b_4 + a_2b_4, a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F). \end{aligned}$$

Example 4 From Example 2, we obtain the following:

(i) For the prevalence order $T \succ I \succ F$, we have

$$x \times y = \left(2\sqrt{2}, \left(37 - 3\sqrt{2}\right)T, \left(-\tfrac{43}{3} + 4\sqrt{2}\right)I, \left(\tfrac{15}{6} - 5\sqrt{2}\right)F\right).$$

(ii) For the prevalence order $T \prec I \prec F$, we have

$$x \times y = \left(2\sqrt{2}, \left(\tfrac{3}{4} - 3\sqrt{2}\right)T, \left(-36 + 4\sqrt{2}\right)I, \left(\tfrac{695}{12} - 5\sqrt{2}\right)F\right).$$

Two neutrosophic quadruple numbers $m = (a_1, b_1T, c_1I, d_1F)$ and $n = (a_2, b_2T, c_2I, d_2F)$ cannot be divided as we do for real and complex numbers. Since the literal neutrosophic components T, I and F are not invertible, the inversion of a neutrosophic quadruple number or the division of a neutrosophic quadruple number by another neutrosophic quadruple number must be carried out a systematic way. Suppose we are to evaluate m/n. Then we must look for a neutrosophic quadruple number $p = (x, yT, zI, wF)$ equivalent to m/n. In this way, we write $m/n = p$. Then

$$\frac{(a_1, b_1T, c_1I, d_1F)}{(a_2, b_2T, c_2I, d_2F)} = (x, yT, zI, wF)$$

if and only if

$$\begin{aligned} &(a_2, b_2T, c_2I, d_2F)(x, yT, zI, wF) \\ &\quad \equiv (a_1, b_1T, c_1I, d_1F). \end{aligned}$$

Assuming the prevalence order $T \succ I \succ F$ and from the equality of two neutrosophic quadruple numbers, we obtain from Eq. (1)

$$a_2x = a_1$$
$$b_2x + (a_2 + b_2 + c_2 + d_2)y + b_2z + b_2w = b_1$$
$$c_2x + (a_2 + c_2 + d_2)z + c_2w = c_1$$
$$d_2x + (a_2 + d_2)w = d_1$$

a system of linear equations in unknowns x, y, z and w.

By similarly assuming the prevalence order $T \prec I \prec F$, we obtain from Eq. (1)

$$a_2x = a_1$$
$$b_2x + (a_2 + b_2)y = b_1$$
$$c_2x + c_2y + (a_2 + b_2 + c_2)z = c_1$$
$$d_2x + d_2y + d_2z + (a_2 + b_2 + c_2 + d_2)w = d_1$$

a system of linear equations in unknowns x, y, z and w.

Example 5 Let $a = (2, -T, I, 2F)$ and $b = (1, 2T, -I, F)$ be two neutrosophic quadruple numbers in $NQ(\mathbb{R})$.

(i) For the prevalence order $T \succ I \succ F$, we obtain

$$\frac{(2, -T, I, 2F)}{(1, 2T, -I, F)} = \left(2, -\frac{11}{3}T, 3I, 0F\right).$$

(ii) For the prevalence order $T \prec I \prec F$, we obtain

$$\frac{(2, -T, I, 2F)}{(1, 2T, -I, F)} = \left(2, -\frac{5}{3}T, \frac{2}{3}I, \frac{1}{3}F\right).$$

Theorem 1 *Let $a, b, c, d, n \neq 0$. Then:*

(i) $\frac{(na, nbT, ncI, ndF)}{(a, bT, cI, dF)} = n.$

(ii) $\frac{(na, nbT, ncI, ndF)}{(n, 0T, 0I, 0F)} = (a, bT, cI, dF).$

Proof Straightforward. □

3 Neutrosophic Quadruple Algebraic Structures, Neutrosophic Quadruple Algebraic Hyper-structures and NeutroQuadrupleAlgebraicStructures

3.1 Neutrosophic Quadruple Algebraic Structures and Neutrosophic Quadruple Algebraic Hyper-structures

Let $NQ(X)$ be a neutrosophic quadruple set and let $*: NQ(X) \times NQ(X) \to NQ(X)$ be a classical binary operation on $NQ(X)$. The couple $(NQ(X), *)$ is called a neutrosophic quadruple algebraic structure. The structure $(NQ(X), *)$ is named according to the classical laws and axioms satisfied or obeyed by $*$.

If $*: NQ(X) \times NQ(X) \to \mathbb{P}(NQ(X))$ is the classical hyper operation on $NQ(X)$. Then the couple $(NQ(X), *)$ is called a neutrosophic quadruple hyper-algebraic structure; and the hyper-structure $(NQ(X), *)$ is named according to the classical laws and axioms satisfied by $*$.

If $(NQ(X), *)$ and $(NQ(Y), \circ)$ are two neutrosophic quadruple algebraic structures. The mapping $\phi : (NQ(X), *) \to (NQ(Y), \circ)$ is called a neutrosophic quadruple homomorphism if ϕ preserves $*$, $\circ$ and literal neutrosophic components T, I and F that is if

(i) $\phi(x * y) = \phi(x)\circ\phi(y) \quad \forall\, x, y \in NQ(X)$.
(ii) $\phi(T) = T$.
(iii) $\phi(I) = I$.
(iv) $\phi(F) = F$.

Theorem 2

(i) $(NQ(\mathbb{Z}), +), (NQ(\mathbb{Q}), +), (NQ(\mathbb{R}), +)$ *and* $(NQ(\mathbb{C}), +)$ *are abelian groups.*
(ii) $(NQ(\mathbb{Z}), +, \times)$, $(NQ(\mathbb{Q}), +, \times)$, $(NQ(\mathbb{R}), +, \times)$ *and* $(NQ(\mathbb{C}), +, \times)$ *are commutative rings.*
(iii) $(NQ(\mathbb{Z}), \times)$ *is a commutative monoid.*
(iv) $(NQ(\mathbb{Z}), \times)$ *is not a group.*
(v) $(NQ(\mathbb{Z}), \div)$ *is not a group.*

Proof See [7]. □

3.2 NeutroQuadrupleAlgebraicStructures

In this section, unless otherwise stated, the optimistic prevalence order $T \succ I \succ F$ will be assumed.

Definition 5 Let $NQ(G)$ be a nonempty set and let $*: NQ(G) \times NQ(G) \to NQ(G)$ be a binary operation on $NQ(G)$. The couple $(NQ(G), *)$ is called a neutrosophic quadruple group if the following conditions hold:

(QG1) $x * y \in G\ \forall x, y \in NQ(G)$ [closure law].
(QG2) $x * (y * z) = (x * y) * z\ \forall x, y, z \in G$ [axiom of associativity].
(QG3) There exists $e \in NQ(G)$ such that $x * e = e * x = x$ $\forall x \in NQ(G)$ [axiom of existence of neutral element].
(QG4) There exists $y \in NQ(G)$ such that $x * y = y * x = e$ $\forall x \in NQ(G)$ [axiom of existence of inverse element], where e is the neutral element of $NQ(G)$. If in addition $\forall x, y \in NQ(G)$, we have
(QG5) $x * y = y * x$, then $(NQ(G), *)$ is called a commutative neutrosophic quadruple group.

Definition 6 [NeutroSophication of the law and axioms of the neutrosophic quadruple].

(NQ(G)1) There exist some duplets (x, y), (u, v), $(p, q), \in NQ(G)$ such that $x * y \in G$ (inner-defined with degree of truth T) and $[u * v =$ indeterminate (with degree of indeterminacy I) or $p * q \notin NQ(G)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosureLaw].
(NQ(G)2) There exist some triplets (x, y, z), (p, q, r), $(u, v, w) \in NQ(G)$ such that $x * (y * z) = (x * y) * z$ (inner-defined with degree of truth T) and $[[p * (q * r)]$ or $[(p * q) * r] =$ indeterminate (with degree of indeterminacy I) or $u * (v * w) \neq (u * v) * w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].
(NQ(G)3) There exists an element $e \in NQ(G)$ such that $x * e = e * x = x$ (inner-defined with degree of truth T) and $[[x * e]$ or $[e * x] =$ indeterminate (with degree of indeterminacy I) or $x * e \neq x \neq e * x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in NQ(G)$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].
(NQ(G)4) There exists an element $u \in NQ(G)$ such that $x * u = u * x = e$ (inner-defined with degree of truth T) and $[[x * u]$ or $[u * x)] =$ indeterminate (with degree of indeterminacy I) or $x * u \neq e \neq u * x$ (outer-defined/falsehood with degre of falsehood F)] for at least one $x \in G$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)] where e is a NeutroNeutralElement in $NQ(G)$.
(NQ(G)5) There exist some duplets $(x, y), (u, v), (p, q) \in NQ(G)$ such that $x * y = y * x$ (inner-defined with degree of truth T) and $[[u * v]$ or $[v * u] =$ indeterminate (with degree of indeterminacy I) or $p * q \neq q * p$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 7 A NeutroQuadrupleGroup $NQ(G)$ is an alternative to the neutrosophic quadruple group $Q(G)$ that has at least one NeutroLaw or at least one of $\{NQ(G)1, NQ(G)2, NQ(G)3, NQ(G)4\}$ with no AntiLaw or AntiAxiom.

Definition 8 A NeutroCommutativeQuadrupleGroup $NQ(G)$ is an alternative to the commutative neutrosophic quadruple group $Q(G)$ that has at least one NeutroLaw or at least one of $\{NQ(G)1, NQ(G)2, NQ(G)3, NQ(G)4\}$ and $NQ(G)5$ with no AntiLaw or AntiAxiom.

Theorem 3 [15] *Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

Theorem 4 *Let $(NQ(G), *)$ be a neutrosophic quadruple group. Then*

(i) *there are 15 types of NeutroQuadrupleGroups,*
(ii) *there are 31 types of NeutroCommutativeQuadrupleGroups.*

Proof Follows from Theorem 3. □

Theorem 5 *For positive integers $n = 2, 3, 4, \cdots$,*

(i) *$(NQ(\mathbb{Z}_n), -)$ is a NeutroQuadrupleGroup.*
(ii) *$(NQ(\mathbb{Z}_n), \times)$ is a NeutroCommutativeQuadrupleGroup.*

Proof Follows from the definition of NeutroQuadrupleGroup and subtraction and multiplication of neutrosophic quadruple of integers modulo n. □

Theorem 6

(i) *$(NQ(\mathbb{Z}), -)$ is a NeutroQuadrupleGroup.*
(ii) *$(NQ(\mathbb{Z}), \times)$ is a NeutroCommutativeQuadrupleGroup.*
(iii) *$(NQ(\mathbb{Z}), \div)$ is a NeutroCommutativeQuadrupleGroup.*

Proof (i) and (ii) are easy. For (iii), let us consider the following:

NeutroClosure of $\div$ over $NQ(\mathbb{Z})$

For the degree of truth, let $a = (0, 0T, I, 0F) \in NQ(\mathbb{Z})$. Then $a \div a = \left(1 - k_1 - k_2, 0T, k_1 I, k_2 F\right) \in NQ(\mathbb{Z}), k_1, k_2 \in \mathbb{Z}$.

For the degree of indeterminacy, let $a = (4, 5T, -2I, -7F), b = (0, -6T, I, 3F) \in NQ(\mathbb{Z})$. Then $a \div b = \left(\frac{4}{0}, ?T, ?I, ?F\right) \notin NQ(\mathbb{Z})$.

For the degree of falsehood, let $a = (0, 0T, 0I, F)$, $b = (0, 0T, 0I, 2F) \in NQ(\mathbb{Z})$. Then $a \div b = \left(\frac{1}{2} - k, 0T, 0I, kF\right) \notin NQ(\mathbb{Z}), k \in \mathbb{Z}$.

NeutroAssociativity of $\div$ over $NQ(\mathbb{Z})$

For the degree of truth, let $a = (6, 6T, 6I, 6F)$, $b = (2, 2T, 2I, 2F), c = (-1, 0T, 0I, 0F) \in NQ(\mathbb{Z})$. Then $a \div (b \div c) = (-3, 0T, 0I, 0F)$, but $(a \div b) \div c = (-3, 0T, 0I, 0F)$.

For the degree of indeterminacy, let $a = (4, -T, 2I, -7F)$, $b = (0, T, 0I, -8F)$, $c = (0, 0T, 9I, -F) \in NQ(\mathbb{Z})$. Then
$a \div (b \div c) = (?, ?T, ?I, ?F)$.
$(a \div b) \div c = (?, ?T, ?I, ?F)$.

For the degree of falsehood, let $a = (0, 5T, 0I, 0F)$, $b = (0, T, 0I, 0F), c = (5, 0T, 0I, 0F) \in NQ(\mathbb{Z})$. Then
$a \div (b \div c) = \left(25 - k_1 - k_2 - k_3, k_1 T, k_2 I, k_3 F\right) \in NQ(\mathbb{Z}), k_1, k_2, k_3 \in \mathbb{Z}$.
$(a \div b) \div c = \left(\frac{1}{5}(5 - k_1 - k_2 - k_3), \frac{1}{5}k_1 T, \frac{1}{5}k_2 I, \frac{1}{5}k_3 F\right) \notin NQ(\mathbb{Z})$.

Existence of NeutroUnitaryElement and NeutroInverseElement in $NQ(\mathbb{Z})$ w.r.t. $\div$

Let $a = (0, T, 0I, 0F)$, $b = (0, 0T, I, 0F)$, $c = (0, 0T, 0I, F) \in NQ(\mathbb{Z})$. Then

$$a \div a = \left(1 - k_1 - k_2 - k_3, k_1 T, k_2 I, k_3 F\right) \tag{1}$$

$$b \div b = \left(1 - k_1 - k_2, 0T, k_1 I, k_2 F\right) \tag{2}$$

$$c \div c = (1 - k, 0T, 0I, kF) \tag{3}$$

$$a \div b = \left(-(k_1 + k_2), T, k_1 I, k_2 F\right) \tag{4}$$

$$b \div a = \left(-(k_1 + k_2 + k_3), k_1 T, k_2 I, k_3 F\right) \tag{5}$$

where $k, k_1, k_2, k_3 \in \mathbb{Z}$.

For the degree of truth, putting $k_1 = 1, k_2 = k_3 = 0$ in Eq. (1), $k_1 = 1, k_2 = 0$ in Eq. (2) and $k = 1$ in Eq. (3) we will obtain $a \div a = a, b \div b = b$ and $c \div c = c$. These show that a, b, c are, respectively, NeutroUnitaryElements and NeutroInverseElements in $NQ(\mathbb{Z})$.

For the degree of falsehood, putting $k_1 \neq 1, k_2 \neq k_3 \neq 0$ in Eq. (1), $k_1 \neq 1, k_2 \neq 0$ in Eq. (2) and $k \neq 1$ in Eq. (3) we will obtain $a \div a \neq a, b \div b \neq b$ and $c \div c \neq c$. These show that a, b, c are, respectively, not NeutroUnitaryElements and NeutroInverseElements in $NQ(\mathbb{Z})$.

NeutroCommtativity of $\div$ over $NQ(\mathbb{Z})$

For the degree of truth, putting $k_1 = 1, k_2 = k_3 = 0$ in Eq. (1), $k_1 = 1, k_2 = 0$ in Eq. (2) and $k = 1$ in Eq. (3) we will obtain $a \div a = a, b \div b = b$ and $c \div c = c$. These show the commutativity of $\div$ wrt a, b and c $NQ(\mathbb{Z})$.

For the degree of falsehood, putting $k_1 = k_2 = k_3 = 1$ in Eqs. (4) and (5), we will obtain $a \div b = (-2, T, I, F)$ and $b \div a = (-3, T, I, F) \neq a \div b$. Hence, $\div$ is NeutroCommutative in $NQ(\mathbb{Z})$.

The proof is complete. □

Definition 9 Let $(NQ(G), *)$ be a neutrosophic quadruple group. A nonempty subset $NQ(H)$ of $NQ(G)$ is called a NeutroQuadrupleSubgroup of $NQ(G)$ if $(NQ(H), *)$ is a neutrosophic quadruple group of the same type as $(NQ(G), *)$.

Example 6

(i) For $n = 2, 3, 4, \cdots (NQ(n\mathbb{Z}), -)$ is a NeutroQuadrupleSubgroup of $(NQ(\mathbb{Z}), -)$.

(ii) For $n = 2, 3, 4, \cdots (NQ(n\mathbb{Z}), \times)$ is a NeutroQuadrupleSubgroup of $(NQ(\mathbb{Z}), \times)$.

Example 7

(i) Let $NQ(H) = \{(a, bT, cI, dF) : a, b, c, d \in \{1, 2, 3\}\}$ be a subset of the NeutroQuadrupleGroup $(NQ(\mathbb{Z}_4), -)$. Then $(NQ(H), -)$ is a NeutroQuadrupleSubgroup of $(NQ(\mathbb{Z}_4), -)$.

(ii) Let $NQ(K) = \{(w, xT, yI, zF) : a, b, c, d \in \{1, 3, 5\}\}$ be a subset of the NeutroQuadrupleGroup $(NQ(\mathbb{Z}_6), \times)$. Then $(NQ(H), \times)$ is a NeutroQuadrupleSubgroup of $(NQ(\mathbb{Z}_6), \times)$.

4 Conclusion

We have in this paper studied neutrosophic quadruple algebraicstructures and NeutroQuadrupleAlgebraicStructures. NeutroQuadrupleGroup was studied in particular and several examples were provided. It was shown that $(NQ(\mathbb{Z}), \div)$ is a NeutroQuadrupleGroup. Substructures of NeutroQuadrupleGroups were also presented with examples.

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Universal NeutroAlgebra and Universal AntiAlgebra

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Florentin Smarandache (2021). Universal NeutroAlgebra and Universal AntiAlgebra.
NeutroAlgebra Theory, I, 11-15

ABSTRACT

This paper introduces the Universal NeutroAlgebra that studies the common properties of the NeutroAlgebra structures, and the Universal AntiAlgebra that studies the common properties of the AntiAlgebraic structures.

Keywords: NeutroAlgebra, AntiAlgebra, Universal NeutroAlgebra, Universal AntiAlgebra

INTRODUCTION

In 2019 and 2020 Smarandache [1, 2, 3, 4] generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or *NeutroAlgebra*) {whose operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or *AntiAlgebra*) {whose operations and axioms are totally false}.

The NeutroAlgebras & AntiAlgebras are a *new field of research*, which is inspired from our real world.

In classical algebraic structures, all axioms are 100%, and all operations are 100% well-defined,

but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some laws or some operations.

Using the process of *NeutroSophication* of a classical algebraic structure we produce a NeutroAlgebra, while the process of *AntiSophication* of a classical algebraic structure produces an AntiAlgebra.

BACKGROUND

1. (Operation, NeutroOperation, AntiOperation)

01. A classical Operation ($*_m$) is an operation that is well-defined (inner-defined) for all elements of the set S, i.e. $*_m(x_1, x_2, \ldots, x_m) \in S$ for all $x_1, x_2, \ldots, x_m \in S$.

02. An AntiOperation ($*_m$) is an operation that is not well-defined (i.e. it is outer-defined) for all elements for the set S; or $*_m(x_1, x_2, \ldots, x_m) \in U \setminus S$ for all $x_1, x_2, \ldots, x_m \in S$.

03. A NeutroOperation ($*_m$) is an operation that is partially well-defined (the degree of well-defined is T), partially indeterminate (the degree of indeterminacy is I), and partially outer-defined (the degree of outer-defined is F); where $(T,I,F) \neq (1,0,0)$ that represents the classical Operation, and $(T,I,F) \neq (0,0,1)$ that represents the AntiOperation.

An operation ($*_m$) is indeterminate if there exist some elements $a_1, a_2, \ldots, a_n \in S$ such that $*_m(a_1, a_2, \ldots, a_m)$ = undefined, or unknown, or unclear, etc.

2. (Axiom, NeutroAxiom, AntiAxiom)

A1. A classical Axiom is an axiom that is true for all elements of the set S.

A2. An AntiAxiom is an axiom that is false for all elements of the set S.

A3. A NeutroAxiom is an axiom that is partially true (the degree of truth is T), partially indeterminate (the degree of indeterminacy is I), and partially false (the degree of falsehood is F), where $(T,I,F) \neq (1,0,0)$ that represents the classical Axiom, and $(T,I,F) \neq (0,0,1)$ that represents the AntiAxiom.

3. (Algebra, NeutroAlgebra, AntiAalgebra)

S1. A classical Algebra (or Algebraic Structure) is a set S endowed only with classical Operations and classical Axioms.

S2. An AntiAlgebra (or AntiAlgebraic Structure) is a set S endowed with at least one AntiOperation or one AntiAxiom

S3. A NeutroAlgebra (or NeutroAlgebraic Structure) is a set S endowed with at least one NeutroOperation or one NeutroAxiom, and no AntiOperation and no AntiAxiom.

UNIVERSAL NEUTROALGEBRA AND UNIVERSAL ANTIALGEBRA

1. A Universe of Discourse, a Set, some Operations, and some Axioms

Let's consider a non-empty set S included in a universe of discourse U, or $S \subset U$.

The set S is endowed with *n* operations, $1 \leq n \leq \infty$, $*_1$, $*_2$, …, $*_n$.

Each operation $*_i$, for $i \in \{1, 2, ..., \infty\}$, is an m_i-ary operation, where $0 \leq m_i \leq \infty$. {A o-ary operation, where "0" stands for zero (or null-ary operation), simply denotes a constant.}

Then a number of α axioms, $0 \leq \alpha \leq \infty$, is defined on S.

The axioms may take the form of identities (or equational laws), quantifications {universal quantification ($\forall$) except before an identity, existential quantification ($\exists$)}, inequalities, inequations, and other relations.

With the condition that there exist at least one m-ary operation, with $m \geq 1$, or at least one axiom.

We have taken into consideration the possibility of infinitary operations, as well as infinite number of axioms.

2. The Structures, almost all, are NeutroStructures

A classical Structure, in any field of knowledge, is composed of: a non-empty space, populated by some elements, and both (the space and all elements) are characterized by some relations among themselves, and by some attributes.

Classical Structures are mostly in theoretical, abstract, imaginary spaces.

Of course, when analysing a structure, it counts with respect to what relations and attributes we analyse it.

In our everyday life almost all structures are NeutroStructures, governed by Universal NeutroAlgebras and Universal AntiAlgebras, since they are neither perfect nor uniform, and not all elements of the structure's space have the same relations and same attributes in the same degree (not all elements behave in the same way).

Conclusions

Since our world is full of indeterminacies, uncertainties, vagueness, contradictory information almost all existing structures are NeutroStructures, since either their spaces, or their elements or their relationships between elements or between are characterized by such indeterminacies.

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On Some NeutroHyperstructures

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Madeleine Al-Tahan, Bijan Davvaz, Florentin Smarandache, Osman Anis (2021). On Some NeutroHyperstructures. *Symmetry*, 13, 535. DOI: 10.3390/sym13040535

Abstract: Neutrosophy, the study of neutralities, is a new branch of Philosophy that has applications in many different fields of science. Inspired by the idea of Neutrosophy, Smarandache introduced NeutroAlgebraicStructures (or NeutroAlgebras) by allowing the partiality and indeterminacy to be included in the structures' operations and/or axioms. The aim of this paper is to combine the concept of Neutrosophy with hyperstructures theory. In this regard, we introduce NeutroSemihypergroups as well as NeutroH$_v$-Semigroups and study their properties by providing several illustrative examples.

Keywords: NeutroHypergroupoid; NeutroSemihypergroup; NeutroH$_v$-semigroup; NeutroHyperideal; NeutroStrongIsomorphism

1. Introduction

In 1995 and inspired by the existence of neutralities, Smarandache introduced Neutrosophy as a new branch of Philosophy that deals with indeterminacy. During the past, ideas were viewed as "True" or "False"; however, if we view an idea from a neutrosophic point of view, it will be "True", "False", or "Indeterminate". The indeterminacy is the key that distinguishes Neutrosophy from other approaches. In the past twenty years, this field demonstrated important progress in which it grabbed the attention of many researchers and different works were done from both a theoretical point of view and from an applicative view. Unlike our real world that is full of imperfections and partialities, abstract systems are constructed on a given perfect space (set), where the operations are totally well-defined and the axioms are totally true for all spacial elements. Starting from the latter idea, Smarandache [1–3] introduced NeutroAlgebra, whose operations are partially well-defined, partially indeterminate, and partially outer-defined, and the axioms are partially true, partially indeterminate, and partially false. Many researchers worked on special types of NeutroAlgebras by applying them to different types of algebraic structures such as groups, rings, BE-Algebras, BCK-Algebras, etc. For more details, we refer to [4–10].

On the other hand, hyperstructure theory is a generalization of classical algebraic structures and was introduced in 1934 at the eighth Congress of Scandinavian Mathematicians by Marty [11]. Marty generalized the notion of groups by defining hypergroups. The class of algebraic hyperstructures is larger than that of algebraic structures where the operation on two elements in the latter is again an element, whereas the hyperoperation of two elements in the first class is a non-void set. For details about hyperstructure theory and its applications, we refer to the articles [12–15] and the books [16–18]. A generalization of algebraic hyperstructures, known as weak hyperstructures (H_v-structures), was introduced in 1994 by Vougiouklis [19]. The axioms in the latter are weaker than that of algebraic hyperstructures. For details about H_v-structures, we refer to [19–22].

As a natural extension of NeutroAlgebraicStructure, NeutroHyperstructure was defined recently [23,24] where Ibrahim and Agboola [23] defined NeutroHypergroups and studied a special type. Our paper is concerned about some NeutroHyperstructures and is organized as follows: Section 2 presents some basic preliminaries related to hyperstructure theory. Section 3 defines NeutroSemihypergroups, NeutroH$_v$-Semigroups, and some related new concepts and illustrates these new concepts via examples. Moreover, we study some properties of their subsets under NeutroStrongHomomorphism.

2. Algebraic Hyperstructures

In this section, we present some definitions and examples about (weak) algebraic hyperstructures that are used throughout the paper. For more details about hyperstructure theory, we refer to [16–20].

Definition 1 ([16])**.** *Let H be a non-empty set and $\mathcal{P}^*(H)$ be the family of all non-empty subsets of H. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a binary hyperoperation on H. The couple $(H, \circ)$ is called a hypergroupoid.*

If A and B are two non-empty subsets of H and $h \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ h \circ A = \{h\} \circ A \text{ and } A \circ h = A \circ \{h\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if the associative axiom is satisfied. i.e., for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$. In other words,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

An element h in a hypergroupoid $(H, \circ)$ is called *idempotent* if $h \circ h = h$.

Example 1. *Let H be any non-empty set and define "$\star$" on H as follows. For all $x, y \in H$, $x \star y = \{x, y\}$. Then $(H, \star)$ is a semihypergroup.*

Example 2. *Let $H_0 = \{e, b, c\}$ and $(H_0, +)$ be defined by the following table.*

$+$	e	b	c
e	e	$\{e, b\}$	$\{e, c\}$
b	e	$\{e, b\}$	$\{e, c\}$
c	e	$\{e, b\}$	$\{e, c\}$

Then $(H_0, +)$ is a semihypergroup and e is an idempotent element in H_0.

As a generalization of algebraic hyperstructures, Vougiouklis [19,20] introduced H_v-structures. Weak axioms in H_v-structures replace some axioms of classical algebraic hyperstructures.

Definition 2 ([19,20])**.** *A hypergroupoid $(H, \circ)$ is called an H_v-semigroup if the weak associative axiom is satisfied. i.e., $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$.*

Example 3. *Let $H_1 = \{0, 1, 2, 3\}$ and "$+$" be the hyperoperation on H_1 defined by the following table.*

$+$	0	1	2	3
0	0	1	$\{0, 2\}$	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Then $(H_1, +)$ *is an* H_v*-semigroup.*

Remark 1. *Every semigroup is a semihypergroup and every semihypergroup is an* H_v*-semigroup.*

Definition 3 ([17])**.** *Let* $(H, \circ)$ *be a semihypergroup (*H_v*-semigroup) and* $M \neq \emptyset \subseteq H$*. Then* M *is a*

1. *subsemihypergroup (*H_v*-subsemigroup) of* H *if* $(M, \circ)$ *is a semihypergroup (*H_v*-semigroup).*
2. *left hyperideal of* H *if* M *is a subsemihypergroup (*H_v*-subsemigroup) of* H *and* $h \circ a \subseteq M$ *for all* $h \in H$*.*
3. *right hyperideal of* H *if* M *is a subsemihypergroup (*H_v*-subsemigroup) of* H *and* $a \circ h \subseteq M$ *for all* $h \in H$*.*
4. *hyperideal of* H *if* M *is both: a left hyperideal of* H *and a right hyperideal of* H*.*

Remark 2. *Let* $(H, \circ)$ *be a semihypergroup (*H_v*-semigroup) and* $M \neq \emptyset \subseteq H$*. To prove that* M *is subsemihypergroup (*H_v*-subsemigroup) of* H*, it suffices to show that* $a \circ b \subseteq M$ *for all* $a, b \in M$*.*

3. NeutroHyperstructures

In this section, we define NeutroSemihypergroups and NeutroH$_v$-Semigroups, present some illustrative examples, and study several properties of some important subsets of NeutroSemihypergroups and NeutroH$_v$-Semigroups.

Definition 4. *Let* A *be any non-empty set and "$\cdot$" be a hyperoperation on* A*. Then "$\cdot$" is called a NeutroHyperoperation on* A *if some (or all) of the following conditions hold in a way that* $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$*.*

1. *There exist* $x, y \in A$ *with* $x \cdot y \subseteq A$*. (This condition is called degree of truth, "T").*
2. *There exist* $x, y \in A$ *with* $x \cdot y \nsubseteq A$*. (This condition is called degree of falsity, "F").*
3. *There exist* $x, y \in A$ *with* $x \cdot y$ *is indeterminate in* A*. (This condition is called degree of indeterminacy, "I").*

Definition 5. *Let* A *be any non-empty set and "$\cdot$" be a hyperoperation on* A*. Then "$\cdot$" is called an AntiHyperoperation on* A *if* $x \cdot y \nsubseteq A$ *for all* $x, y \in A$*.*

Definition 6. *Let* A *be any non-empty set and "$\cdot$" be a hyperoperation on* A*. Then "$\cdot$" is called NeutroAssociative on* A *if there exist* $x, y, z, a, b, c, e, f, g \in A$ *satisfying some (or all) of the following conditions in a way that* $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$*.*

1. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$*; (This condition is called degree of truth, "T").*
2. $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c$*; (This condition is called degree of falsity, "F").*
3. $e \cdot (f \cdot g)$ *is indeterminate or* $(e \cdot f) \cdot g$ *is indeterminate or we cannot find if* $e \cdot (f \cdot g)$ *and* $(e \cdot f) \cdot g$ *are equal. (This condition is called degree of indeterminacy, "I").*

Definition 7. *Let* A *be any non-empty set and "$\cdot$" be a hyperoperation on* A*. Then "$\cdot$" is called AntiAssociative on* A *if* $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c$ *for all* $a, b, c \in A$*.*

Definition 8. *Let* A *be any non-empty set and "$\cdot$" be a hyperoperation on* A*. Then "$\cdot$" is called a NeutroWeakAssociative on* A *if there exist* $x, y, z, a, b, c, e, f, g \in A$ *satisfying some (or all) of the following conditions in a way that* $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$*.*

1. $[x \cdot (y \cdot z)] \cap [(x \cdot y) \cdot z] \neq \emptyset$*; (This condition is called degree of truth, "T").*
2. $[a \cdot (b \cdot c)] \cap [(a \cdot b) \cdot c] = \emptyset$*; (This condition is called degree of falsity, "F").*
3. $e \cdot (f \cdot g)$ *is indeterminate or* $(e \cdot f) \cdot g$ *is indeterminate or we cannot find if* $e \cdot (f \cdot g)$ *and* $(e \cdot f) \cdot g$ *have common elements. (This condition is called degree of indeterminacy, "I").*

Definition 9. *Let* A *be a non-empty set and "$\cdot$" be a hyperoperation on* A*. Then* $(A, \cdot)$ *is called a*

1. *NeutroHypergroupoid if "$\cdot$" is a NeutroHyperoperation.*
2. *NeutroSemihypergroup if "$\cdot$" is NeutroAssociative but not an AntiHyperoperation.*

3. *NeutroH$_v$-Semigroup if "$\cdot$" is NeutroWeakAssociative but not an AntiHyperoperation.*

Example 4. *Let $A = \{0,1\}$ and $(A,+)$ be defined by the following table.*

$+$	0	1
0	$\{0,1\}$	0
1	1	0

Then $(A,+)$ is a NeutroSemihypergroup and NeutroH$_v$-Semigroup. This is clear as

$$0+(0+0) = \{0,1\} = (0+0)+0 \text{ and } (1+1)+1 = 0 \neq 1 = 1+(1+1).$$

Example 5. *Let $\mathbb{R}$ be the set of real numbers and define "$\star$" on $\mathbb{R}$ as follows.*

$$x \star y = \begin{cases} [x,y] & \text{if } x < y; \\ [y,x] & \text{if } y < x; \\ 0 & \text{if } x = y = 0; \\ \frac{1}{x} & \text{if } x = y \neq 0. \end{cases}$$

Then $(\mathbb{R},\star)$ is a NeutroSemihypergroup. This is clear as $(1 \star 1) \star 1 = 1 = 1 \star (1 \star 1)$ and $(1 \star 2) \star 2 = \{\frac{1}{2}\} \cup [1,2] \neq [\frac{1}{2},1] = 1 \star (2 \star 2)$.

Example 6. *Let $M = \{m,a,d\}$ and $(M,\cdot)$ be defined by the following table.*

$\cdot$	m	a	d
m	m	m	m
a	m	$\{m,a\}$	d
d	m	d	d

Then $(M,\cdot)$ is a NeutroSemihypergroup. This is clear as $m \cdot (m \cdot m) = m = (m \cdot m) \cdot m$ and $a \cdot (a \cdot d) = d \neq \{m,d\} = (a \cdot a) \cdot d$.

Remark 3. *It is well known in classical algebraic hyperstructures that every semihypergroup is a hypergroupoid. This may fail to occur in NeutroHyperstructures. In Example 6, $(M,\cdot)$ is a NeutroSemihypergroup that is not a NeutroHypergroupoid.*

Proposition 1. *Every H$_v$-semigroup that is not a semihypergroup and has an idempotent element is a NeutroSemihypergroup.*

Proof. Let $(H,\circ)$ be an H_v-semigroup with $h^2 = h$ for some $h \in H$. Then $h \circ (h \circ h) = h = (h \circ h) \circ h$. Since $(H,\circ)$ is not a semihypergroup, it follows that there exist $x,y,z \in H$ with $x \circ (y \circ z) \neq (x \circ y) \circ z$. Therefore, $(H,\circ)$ is a NeutroSemihypergroup. □

Example 7. *Let $M = \{m,a,d\}$ and $(M,\diamond)$ be defined by the following table.*

$\diamond$	m	a	d
m	m	$\{a,d\}$	d
a	$\{a,d\}$	d	m
d	d	m	a

Then $(M,\diamond)$ is an H$_v$-semigroup having m as an idempotent element and hence, it is a NeutroSemihypergroup.

Remark 4. *It is well known in algebraic hyperstructures that every semihypergroup is an H_v-semigroup. This may not hold in NeutroHyperstructures. i.e., A NeutroSemihypergroup may not be a NeutroH$_v$-Semigroup.*

The H_v-semigroup $(M, \diamond)$ in Example 7 is a NeutroSemihypergroup that is not NeutroH$_v$-Semigroup.

Example 8. *Let $\mathbb{Z}$ be the set of integers and define "$\oplus$" on $\mathbb{Z}^2$ as follows. For all $m, n, p, q \in \mathbb{Z}$,*

$$(m, 0) \oplus (0, 0) = (0, 0) \oplus (m, 0) = \{(0, 0), (m, 0)\},$$

$$(0, n) \oplus (0, 0) = (0, 0) \oplus (0, n) = \{(0, 0), (0, n)\},$$

and if $(n, p, q) \neq (0, 0, 0), (m, p, q) \neq (0, 0, 0)$

$$(m, n) \oplus (p, q) = (p, q) \oplus (m, n) = (m + p, n + q).$$

Then $(\mathbb{Z}^2, \oplus)$ is a NeutroSemihypergroup. This is clear as

$$[(1, 2) \oplus (1, 3)] \oplus (1, 4) = (3, 9) = (1, 2) \oplus [(1, 3) \oplus (1, 4)]$$

and

$$[(1, 0) \oplus (1, 0)] \oplus (0, 0) = \{(2, 0), (0, 0)\} \neq \{(2, 0), (1, 0), (0, 0)\} = (1, 0) \oplus [(1, 0) \oplus (0, 0)].$$

Example 9. *Let $\mathbb{Z}$ be the set of integers and define "$\odot$" on $\mathbb{Z}^2$ as follows. For all $m, n, p, q \in \mathbb{Z}$,*

$$(m, n) \odot (p, q) = \begin{cases} (mp, nq) & \text{if } (m, n) \neq (1, 1) \text{ and } (p, q) \neq (1, 1); \\ \{(p, q), (1, 1)\} & \text{if } (m, n) = (1, 1); \\ \{(m, n), (1, 1)\} & \text{if } (p, q) = (1, 1). \end{cases}$$

Then $(\mathbb{Z}^2, \odot)$ is a NeutroSemihypergroup. This is clear as

$$[(1, 2) \odot (1, 3)] \odot (1, 4) = (1, 24) = (1, 2) \odot [(1, 3) \odot (1, 4)]$$

and

$$(1, 1) \odot [(2, 2) \odot (3, 3)] = \{(1, 1), (6, 6)\} \neq \{(1, 1), (3, 3), (6, 6)\} = [(1, 1) \odot (2, 2)] \odot (3, 3).$$

Example 10. *Let $\mathbb{Z}_6$ be the set of integers under addition modulo 6 and define "$\boxplus$" on $\mathbb{Z}_6$ as follows.*

$$x \boxplus y = (x + y) \mod 6 \text{ for all } (x, y) \notin \{(\bar{0}, \bar{3}), (\bar{0}, \bar{5})\},$$

$$\bar{0} \boxplus \bar{3} = \{\bar{0}, \bar{3}\}, \text{ and } \bar{0} \boxplus \bar{5} = \{\bar{0}, \bar{5}\}.$$

Then $(\mathbb{Z}_6, \boxplus)$ is a NeutroSemihypergroup. This is clear as $\bar{0} \boxplus (\bar{0} \boxplus \bar{0}) = \bar{0} = (\bar{0} \boxplus \bar{0}) \boxplus \bar{0}$ and $\bar{0} \boxplus (\bar{1} \boxplus \bar{2}) = \{\bar{0}, 3\} \neq \bar{3} = (\bar{0} \boxplus \bar{1}) \boxplus \bar{2}$.

Example 11. *Let $M = \{m, a, d\}$ and $(M, \bullet)$ be defined by the following table.*

$\bullet$	m	a	d
m	a	a	d
a	$\{m, a\}$	m	d
d	d	d	m

Then $(M, \bullet)$ is a NeutroH$_v$-Semigroup. This is clear as

$$[m \bullet (m \bullet m)] \cap [(m \bullet m) \bullet m] = \{a\} \cap \{m, a\} \neq \emptyset$$

and

$$[m \bullet (d \bullet d)] \cap [(m \bullet d) \bullet d] = \{a\} \cap \{m\} = \emptyset.$$

Moreover, $(M, \bullet)$ is a NeutroSemihypergroup as $d \bullet (d \bullet d) = (d \bullet d) \bullet d$.

Remark 5. *Every NeutroSemigroup is both: a NeutroSemihypergroup and a NeutroH_v-Semigroup. So, the results related to NeutroSemihypergroups (NeutroH_v-Semigroups) are more general than that related to NeutroSemigroups and as a result, we can deal with NeutroSemigroups as a special case of NeutroSemihypergroups (NeutroH_v-Semigroups).*

Example 12. *Let $S_1 = \{s, a, m\}$ and $(S_1, \cdot_1)$ be defined by the following table.*

$\cdot_1$	s	a	m
s	s	m	s
a	m	a	m
m	m	m	m

In [6], Al-Tahan et al. proved that $(S_1, \cdot_1)$ is a NeutroSemigroup. Thus, $(S_1, \cdot_1)$ is a NeutroSemihypergroup.

Theorem 1. *Let $(H, \circ)$ be a NeutroSemihypergroup (NeutroH_v-Semigroup) and "$\star$" be defined on H as $x \star y = y \circ x$ for all $x, y \in H$. Then $(H, \star)$ is a NeutroSemihypergroup (NeutroH_v-Semigroup).*

Proof. The proof is straightforward. □

Example 13. *Let $M = \{m, a, d\}$ and $(M, \bullet)$ be the NeutroSemihypergroup defined in Example 11. By applying Theorem 1, we get that $(M, \circledast)$ defined in the following table is a NeutroSemihypergroup and a NeutroH_v-Semigroup.*

$\circledast$	m	a	d
m	a	$\{m, a\}$	d
a	a	m	d
d	d	d	m

Definition 10. *Let $(H, \circ)$ be a NeutroSemihypergroup (NeutroH_v-Semigroup) and $S \neq \emptyset \subseteq H$. Then S is a NeutroSubsemihypergroup (NeutroH_v-Subsemigroup) of H if $(S, \circ)$ is a NeutroSemihypergroup (NeutroH_v-Semigroup).*

Remark 6. *Let $(H, \circ)$ be a NeutroSemihypergroup (NeutroH_v-Semigroup) and $S \neq \emptyset \subseteq H$. Unlike the case in algebraic hyperstructures (Remark 2), proving that $a \circ b \subseteq S$ for all $a, b \in S$ does not imply that S is a NeutroSubsemihypergroup (NeutroH_v-Subsemigroup) of H.*

As an illustration of Remark 6, $0 \star 0 = \{0\} \subseteq \{0\}$ in Example 5 but $\{0\}$ is not a NeutroSubsemihypergroup of $\mathbb{R}$.

Definition 11. *Let $(H, \circ)$ be a NeutroSemihypergroup (NeutroH_v-Semigroup) and $S \neq \emptyset \subseteq H$ be a NeutroSubsemihypergroup (NeutroH_v-Subsemigroup). Then*

(1) S is a NeutroLeftHyperideal of H if there exists $x \in S$ such that $r \circ x \subseteq S$ for all $r \in H$.
(2) S is a NeutroRightHyperideal of S if there exists $x \in S$ such that $x \circ r \subseteq S$ for all $r \in H$.
(3) S is a NeutroHyperideal of H if there exists $x \in S$ such that $r \circ x \subseteq S$ and $x \circ r \subseteq S$ for all $r \in H$.

A NeutroSemihypergroup (NeutroH_v-Semigroup) is called simple if it has no proper NeutroSubsemihypergroups (NeutroH_v-Subsemigroups).

Example 14. *Let $(A,+)$ be the NeutroSemihypergroup defined in Example 4. Then A is simple. This is clear as $\{0\}$ and $\{1\}$ are the only options for any possible proper NeutroSubsemihypergroup and $(\{0\},+)$ and $(\{1\},+)$ are AntiHypergroupoids.*

Example 15. *Let $(M,\bullet)$ be the NeutroSemihypergroup defined in Example 11. Then $\{m,a\}$ is a NeutroSubsemihypergroup of M.*

Example 16. *Let $(\mathbb{Z}^2,\oplus)$ be the NeutroSemihypergroup defined in Example 8, $M_1=\{(x,0): x\in\mathbb{Z}\}$, and $M_2=\{(0,x): x\in\mathbb{Z}\}$. Then M_1,M_2 are NeutroSubsemihypergroups of $\mathbb{Z}^2$.*

Remark 7. *The intersection of NeutroSubsemihypergroups may fail to be a NeutroSubsemihypergroup. This is clear from Example 16 as $\{(0,0)\}=M_1\cap M_2$ is not a NeutroSubsemihypergroup of $\mathbb{Z}^2$.*

Lemma 1. *Let $(H,\circ)$ be a NeutroSemihypergroup (NeutroH$_v$-Semigroup) and A,B be hypergroupoids. If A,B are NeutroSubsemihypergroups (NeutroH$_v$-Subsemigroups) of H then $A\cup B$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H.*

Proof. Let A,B be NeutroSubsemihypergroups. Since A and B are hypergroupoids, it follows that "$\circ$" is NeutroAssociative on both of A and B. The latter implies that there exist $x,y,z,a,b,c,e,f,g\in A\subseteq A\cup B$ satisfying some (or all) of the following conditions in a way that $(T,I,F)\notin\{(1,0,0),(0,0,1)\}$.

1. T: $x\circ(y\circ z)=(x\circ y)\circ z$;
2. F: $a\circ(b\circ c)\neq(a\circ b)\circ c$;
3. I: $e\circ(f\circ g)$ is indeterminate or $(e\circ f)\circ g$ is indeterminate or we cannot find if $e\circ(f\circ g)$ and $(e\circ f)\circ g$ are equal.

Therefore, $A\cup B$ is a NeutroSubsemihypergroup of H. The proof of (NeutroH$_v$-Subsemigroup is done similarly. □

Example 17. *Let $(\mathbb{Z}^2,\odot)$ be the NeutroSemihypergroup defined in Example 9, $N_1=\{(x,y)\in\mathbb{Z}^2: x,y\geq 1\}\cup\{(0,0)\}$, and $N_2=\{(x,y)\in\mathbb{Z}^2: x,y\leq 1\}\cup\{(0,0)\}$. Then N_1,N_2 are NeutroHyperideals of $\mathbb{Z}^2$. We show that N_1 is a NeutroHyperideal of $\mathbb{Z}^2$ and N_2 may be done similarly. Since*

$$[(1,2)\odot(1,3)]\odot(1,4)=(1,24)=(1,2)\odot[(1,3)\odot(1,4)]$$

and

$$(1,1)\odot[(2,2)\odot(3,3)]=\{(1,1),(6,6)\}\neq\{(1,1),(3,3),(6,6)\}=[(1,1)\odot(2,2)]\odot(3,3),$$

it follows that N_1 is a NeutroSubsemihypergroup of $\mathbb{Z}^2$. Having $(0,0)\in N_1$ and for all $(r,s)\in\mathbb{Z}^2$,

$$(r,s)\odot(0,0)=(0,0)\odot(r,s)=\begin{cases}(0,0) & \text{if } (r,s)\neq(1,1);\\ \{(0,0),(1,1)\} & \text{otherwise.}\end{cases}\subseteq N_1$$

implies that N_1 is a NeutroHyperideal of $\mathbb{Z}^2$.

Remark 8. *The intersection of NeutroHyperideals may fail to be a NeutroHyperideal. This is clear from Example 17 as $\{(0,0),(1,1)\}=N_1\cap N_2$ is not a NeutroHyperideal of $\mathbb{Z}^2$.*

Lemma 2. *Let $(H,\circ)$ be a NeutroSemihypergroup (NeutroH$_v$-Semigroup) and A,B be hypergroupoids. If A,B are NeutroLeftHyperideals (NeutroRightHyperideals or NeutroHyperideals) of H. Then $A\cup B$ is a NeutroLeftHyperideal (NeutroRightHyperideal or NeutroHyperideal) of H.*

Proof. Let A,B be NeutroLeftHyperideals of H. Lemma 1 asserts that $A\cup B$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H. Since A is a NeutroLeftHyperideal of H, it follows that there exists $a \in A$ such that $r \circ a \subseteq A$ for all $r \in H$. The latter implies that there exists $a \in A \cup B$ such that $r \circ a \subseteq A \cup B$ for all $r \in H$. Thus, $A \cup B$ is a NeutroLeftHyperideal of H. □

Definition 12. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be a function. Then*

(1) ϕ is called NeutroHomomorphism if $\phi(x \circ y) = \phi(x) \star \phi(y)$ for some $x, y \in A$.
(2) ϕ is called NeutroIsomomorphism if ϕ is a bijective NeutroHomomorphism.
(3) ϕ is called NeutroStrongHomomorphism if for all $x, y \in A$, $\phi(x \circ y) = \phi(x) \star \phi(y)$ when $x \circ y \subseteq H$, $\phi(x) \star \phi(y) \nsubseteq H'$ when $x \circ y \nsubseteq H$, and $\phi(x) \star \phi(y)$ is indeterminate when $x \circ y$ is indeterminate.
(4) ϕ is called NeutroStrongIsomomorphism if ϕ is a bijective NeutroOrderedStrongHomomorphism. In this case we say that $(H, \circ) \cong_{SI} (H', \star)$.

Example 18. *Let $(M, \bullet)$ and $(M, \circledast)$ be the NeutroSemihypergroups defined in Examples 11 and 13, respectively. Then $(M, \bullet) \cong_{SI} (M, \circledast)$ as $\phi : (M, \bullet) \to (M, \circledast)$ is a NeutroStongIsomorphism. Here,*

$$\phi(m) = a, \phi(a) = m, \text{ and } \phi(d) = d.$$

Theorem 2. *The relation "$\cong_{SI}$" is an equivalence relation on the set of NeutroSemihypergroups (NeutroH$_v$-Semigroups).*

Proof. By taking the identity map, we can easily prove that "$\cong_{SI}$" is a reflexive relation. Let $A \cong_{SI} B$. Then there exists a NeutroStrongIsomorphism $\phi : (A, \star) \to (B, \circledast)$. We prove that the inverse function $\phi^{-1} : B \to A$ of ϕ is a NeutroStrongIsomorphism. For all $b_1, b_2 \in B$, there exist $a_1, a_2 \in A$ with $\phi(a_1) = b_1$ and $\phi(a_2) = b_2$. We have

$$\phi^{-1}(b_1 \circledast b_2) = \phi^{-1}(\phi(a_1) \circledast \phi(a_2))$$

We consider the following cases for $\phi(a_1) \circledast \phi(a_2)$.

Case $\phi(a_1) \circledast \phi(a_2) \subseteq B$. Having ϕ a NeutroStrongIsomorphism and $\phi(a_1) \circledast \phi(a_2) \subseteq B$ imply that $a_1 \star a_2 \subseteq A$ and hence,

$$\phi^{-1}(b_1 \circledast b_2) = \phi^{-1}(\phi(a_1) \circledast \phi(a_2)) = \phi^{-1}(\phi(a_1 \star a_2)) = a_1 \star a_2 = \phi^{-1}(b_1) \star \phi^{-1}(b_2).$$

Case $\phi(a_1) \circledast \phi(a_2) \nsubseteq B$. Suppose, to get contradiction, that $\phi^{-1}(\phi(a_1)) \star \phi^{-1}(\phi(a_2)) = a_1 \star a_2 \subseteq A$ or indeterminate. Then by using our hypothesis that ϕ is NeutroStrongIsomorphism, we get that $\phi(a_1) \circledast \phi(a_2) \subseteq B$ or indeterminate.

Case $\phi(a_1) \circledast \phi(a_2)$ is indeterminate. Suppose, to get contradiction, that $\phi^{-1}(\phi(a_1)) \star \phi^{-1}(\phi(a_2)) = a_1 \star a_2 \subseteq A$ or $a_1 \star a_2 \nsubseteq A$. Then by using our hypothesis that ϕ is NeutroStrongIsomorphism, we get that $\phi(a_1) \circledast \phi(a_2) \subseteq B$ or $\phi(a_1) \circledast \phi(a_2) \nsubseteq B$.

Thus, $B \cong_{SI} A$ and hence, "$\cong_{SI}$" is a symmetric relation. Let $A \cong_{SI} B$ and $B \cong_{SI} C$. Then there exist NeutroStrongIsomorphisms $\phi : A \to B$ and $\psi : B \to C$. One can easily see that the composition function $\psi \circ \phi : A \to C$ of ψ and ϕ is a NeutroStrongIsomorphism. Thus, $A \cong_{SI} C$ and hence, "$\cong_{SI}$" is a transitive relation. □

Lemma 3. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be an injective NeutroStrongHomomorphism. If $M \subset H$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H then $\phi(M)$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H'.*

Proof. Let M be a NeutroSubsemihypergroup of H. If "$\circ$" is NeutroHyperoperation on M then it is clear that "$\star$" is NeutroHyperoperation on $\phi(M)$. If "$\circ$" is NeutroAssociative then there exist $x, y, z, a, b, c, d, e, f \in M$satisfying some (or all) of the following conditions in a way that $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

1. T: $x \circ (y \circ z) = (x \circ y) \circ z$;
2. F: $a \circ (b \circ c) \neq (a \circ b) \circ c$;
3. I: $e \circ (f \circ g)$ is indeterminate or $(e \circ f) \circ g$ is indeterminate or we cannot find if $e \circ (f \circ g)$ and $(e \circ f) \circ g$ are equal.

The latter and having ϕ an injective NeutroStrongHomomorphism imply that some (or all) of the following conditions are satisfied in a way that $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

1. T: $\phi(x) \star (\phi(y) \star \phi(z)) = (\phi(x) \star \phi(y)) \star \phi(z)$;
2. F: $\phi(a) \star (\phi(b) \star \phi(c)) \neq (\phi(a) \star \phi(b)) \star \phi(c)$;
3. I: $\phi(e) \star (\phi(f) \star \phi(g))$ is indeterminate or $(\phi(e) \star \phi(f)) \star \phi(g)$ is indeterminate or we cannot find if $\phi(e) \star (\phi(f) \star \phi(g))$ and $(\phi(e) \star \phi(f)) \star \phi(g)$ are equal.

Thus, $\phi(M)$ is a NeutroSubsemihypergroup. The proof that $\phi(M)$ is a NeutroH$_v$-Subsemigroup of H' is done similarly. □

Example 19. *Let $(M, \bullet)$ and $(M, \circledast)$ be the NeutroSemihypergroups defined in Examples 11 and 13, respectively. Example 15 asserts that $\{m, a\}$ is a NeutroSubsemihypergroup of $(M, \bullet)$. Using Example 18 and Lemma 3, we get that $\{a, m\} = \{\phi(m), \phi(a)\}$ is a NeutroSubsemihypergroup of $(M, \circledast)$.*

Lemma 4. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be a NeutroStrongIsomomorphism. If $N \subseteq H'$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H' then $\phi^{-1}(N)$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H.*

Proof. Let $N \subset H'$ be a NeutroSubsemihypergroup of H'. If "$\star$" is NeutroHyperoperation on N then it is clear that "$\circ$" is NeutroHyperoperation on $\phi^{-1}(N)$. Let "$\star$" be NeutroAssociative. Having ϕ is an onto NeutroStrongHomomorphism implies that there exist $\phi(x), \phi(y), \phi(z), \phi(a), \phi(b), \phi(c), \phi(d), \phi(e), \phi(f) \in N$ satisfying some (or all) of the following conditions in a way that $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

1. T: $\phi(x) \star (\phi(y) \star \phi(z)) = (\phi(x) \star \phi(y)) \star \phi(z)$;
2. F: $\phi(a) \star (\phi(b) \star \phi(c)) \neq (\phi(a) \star \phi(b)) \star \phi(c)$;
3. I: $\phi(e) \star (\phi(f) \star \phi(g))$ is indeterminate or $(\phi(e) \star \phi(f)) \star \phi(g)$ is indeterminate or we cannot find if $\phi(e) \star (\phi(f) \star \phi(g))$ and $(\phi(e) \star \phi(f)) \star \phi(g)$ are equal.

Having ϕ be an injective NeutroStrongHomomorphism implies that there exist $x, y, z, a,$ $b, c, d, e,f \in \phi^{-1}(N)$ satisfying some (or all) of the following conditions in a way that $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

1. T: $x \circ (y \circ z) = (x \circ y) \circ z$;
2. F: $a \circ (b \circ c) \neq (a \circ b) \circ c$;
3. I: $e \circ (f \circ g)$ is indeterminate or $(e \circ f) \circ g$ is indeterminate or we cannot find if $e \circ (f \circ g)$ and $(e \circ f) \circ g$ are equal.

Thus, $\phi^{-1}(N)$ is a NeutroSubsemihypergroup of H. The proof that $\phi^{-1}(N)$ is a NeutroH$_v$-Subsemigroup of H may be done similarly. □

Theorem 3. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be a NeutroStrongIsomorphism. Then $M \subseteq H$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H if and only if $\phi(M)$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H'.*

Proof. The proof follows from Theorem 2 and Lemmas 3 and 4. □

Corollary 1. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be a NeutroStrongIsomorphism. Then H is simple if and only if H' is simple.*

Proof. The proof follows from Theorem 3. □

Lemma 5. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be a NeutroStrongIsomorphism. If $M \subseteq H$ is a NeutroLeftHyperideal (NeutroRightHyperideal or NeutroHyperideal) of H then $\phi(M)$ is a NeutroLeftHyperideal (NeutroRightHyperideal or NeutroHyperideal) of H'.*

Proof. Let $M \subseteq H$ be a NeutroLeftHyperideal of H. Lemma 3 asserts that $\phi(M)$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H'. Having M a NeutroLeftHyperideal of H implies that there exists $x \in M$ such that $r \circ x \subseteq M$ for all $r \in H$. Having ϕ an onto NeutroStrongHomomorphism implies that $\phi(r) \star \phi(x) \subseteq \phi(M)$ for all $s = \phi(r) \in H'$. Thus, $\phi(M)$ is a NeutroLeftHyperideal of H'. The proofs of NeutroRightHyperideal and NeutroHyperideal are done similarly. □

Lemma 6. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be a NeutroStrongIsomorphism. If $N \subseteq H'$ is a NeutroLeftHyperideal (NeutroRightHyperideal or NeutroHyperideal) of H' then $\phi^{-1}(N)$ is a NeutroLeftHyperideal (NeutroRightHyperideal or NeutroHyperideal) of H.*

Proof. Let $N \subseteq H'$ be a NeutroLeftHyperideal of H. Lemma 3 asserts that $\phi^{-1}(N)$ is a NeutroSubsemihypergroup (NeutroH$_v$-Subsemigroup) of H. Having N a NeutroLeftHyperideal of H' implies that there exists $y \in N$ such that $s \star y \subseteq N$ for all $s \in H'$. Since ϕ is an NeutroStrongHomomorphism, it follows that $\phi(r \circ x) \subseteq N$ for all $r \in H$ where $y = \phi(x)$. The latter implies that there exists $x \in \phi^{-1}(N)$ with $r \circ x \subseteq \phi^{-1}(N)$ for all $r \in H$. Thus, $\phi^{-1}(N)$ is a NeutroLeftHyperideal of H. The proofs of NeutroRightHyperideal and NeutroHyperideal are done similarly. □

Theorem 4. *Let $(H, \circ)$, $(H', \star)$ be NeutroSemihypergroups (NeutroH$_v$-Semigroups) and $\phi : H \to H'$ be a NeutroStrongIsomorphism. Then $M \subseteq H$ is a NeutroLeftHyperideal (NeutroRightHyperideal or NeutroHyperideal) of H if and only if $\phi(M)$ is a NeutroLeftHyperideal (NeutroRightHyperideal or NeutroHyperideal) of H'.*

Proof. The proof follows from Theorem 2, Lemmas 5 and 6. □

Let H_α be any non-empty set for all $\alpha \in \Gamma$ and "$\cdot_\alpha$" be a hyperoperation on H_α. We define "$\circ$" on $\prod_{\alpha \in \Gamma} H_\alpha$ as follows: For all $(x_\alpha), (y_\alpha) \in \prod_{\alpha \in \Gamma} H_\alpha$, $(x_\alpha) \circ (y_\alpha) = \{(t_\alpha) : t_\alpha \in x_\alpha \cdot_\alpha y_\alpha\}$.

Theorem 5. *Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be hypergroupoids. Then $(H_1 \times H_2, \circ)$ is a NeutroSemihypergroup (NeutroH$_v$-Semigroup) if and only if either $(H_1, \circ_1)$ is a NeutroSemihypergroup (NeutroH$_v$-Semigroup) or $(H_2, \circ_2)$ is a NeutroSemihypergroup (NeutroH$_v$-Semigroup) or both are NeutroSemihypergroups (NeutroH$_v$-Semigroups).*

Proof. The proof is straightforward. □

Example 20. *Let $(\mathbb{R}, \circledast)$ be the semihypergroup defined as: $x \circledast y = \{x, y\}$ for all $x, y \in \mathbb{R}$ and $(M, \cdot)$ be the NeutroSemihypergroup defined in Example 6. Then the following are true.*

1. *$(\mathbb{R} \times M, \circ)$ is a NeutroSemihypergroup,*
2. *$(M \times \mathbb{R}, \circ)$ is a NeutroSemihypergroup, and*
3. *$(M \times M, \circ)$ is a NeutroSemihypergroup.*

In what follows, we present a way to construct a new NeutroSemihypergroup (NeutroH$_v$-Semigroup) from an existing one. This tool is of great importance to prove that for any positive integer $n \geq 2$, there exists at least one NeutroSemihypergroup (NeutroH$_v$-Semigroup) of order n.

Let $(H,\circ)$ be a NeutroSemihypergroup (NeutroH$_v$-Semigroup) and J be any non-empty set such that $H \cap J = \emptyset$ and $(H \circ H) \cap J = \emptyset$. The extension $H[J]$ of H by J is given as $H[J] = H \cup J$. We define the hyperoperation "$\circledcirc$" on $H[J]$ as follows.

$$x \circledcirc y = \begin{cases} x \circ y & \text{if } x,y \in H; \\ H \cup J & \text{otherwise.} \end{cases}$$

Theorem 6. *Let $(H,\circ)$ be a NeutroSemihypergroup (NeutroH$_v$-Semigroup) and J be any non-empty set such that $H \cap J = \emptyset$ and $(H \circ H) \cap J = \emptyset$. Then $(H[J],\circledcirc)$ is a NeutroSemihypergroup (NeutroH$_v$-Semigroup).*

Proof. Let $(H,\circ)$ be a NeutroSemihypergroup. If "$\circ$" is a NeutroHyperoperation then there exist $u,v,w,x,y,z \in H$ with $u \circ v \subseteq H$ representing "T", $w \circ x \nsubseteq H$ representing "F", $y \circ z$ is indeterminate representing "I". Where $(T,I,F) \notin \{(1,0,0),(0,0,1)\}$. Since $(H \circ H) \cap J = \emptyset$, it follows that there exist $u,v,w,x,y,z \in H$ with $u \circ v \subseteq H[J]$ representing "T", $w \circ x \nsubseteq H[J]$ representing "F" (as $w \circ x \nsubseteq H$ and $w \circ x \nsubseteq J$), $y \circ z$ is indeterminate representing "I". Where $(T,I,F) \notin \{(1,0,0),(0,0,1)\}$. Thus, "$\circledcirc$" is NeutroHyperoperation on $H[J]$. If "$\circ$" is NeutroAssociative on H then it is clear that "$\circledcirc$" is NeutroAssociative on $H[J]$. Therefore, $(H[J],\circledcirc)$ is a NeutroSemihypergroup. The case $(H[J],\circledcirc)$ is a NeutroH$_v$-Semigroup is done similarly. □

Example 21. *Let $(M,\cdot)$ be the NeutroSemihypergroup defined in Example 6 and $N = \{n\}$. Then $M[N] = \{m,a,d,n\}$ and $(M[N],\circledcirc)$ is the NeutroSemihypergroup defined by the following table.*

$\circledcirc$	m	a	d	n
m	m	m	m	$\{m,a,d,n\}$
a	m	$\{m,a\}$	d	$\{m,a,d,n\}$
d	m	d	d	$\{m,a,d,n\}$
n	$\{m,a,d,n\}$	$\{m,a,d,n\}$	$\{m,a,d,n\}$	$\{m,a,d,n\}$

Theorem 7. *Let $n \geq 2$ be an integer. Then there is at least one NeutroSemihypergroup of order n.*

Proof. The proof follows from Example 4 and Theorem 6. □

Corollary 2. *There are infinitely many NeutroSemihypergroups up to NeutroStrongIsomorphism.*

Proof. The proof follows from Theorem 7. □

Theorem 8. *Let $n \geq 2$ be any integer. Then there is at least one NeutroH$_v$-Semigroup of order n.*

Proof. The proof follows from Example 4 and Theorem 6. □

Corollary 3. *There are infinitely many NeutroH$_v$-Semigroups up to NeutroStrongIsomorphism.*

Proof. The proof follows from Theorem 8. □

4. Conclusions

In this paper, we discussed the properties of some NeutroHyperstructures. More precisely, we introduced NeutroSemihypergroups (NeutroH$_v$-Semigroups), constructed several examples, and studied some of their important subsets under NeutroStrongIsomorphism. It was shown through examples that some of the well known results for algebraic hyperstructures do not hold for NeutroHyperstructures. Moreover, it was proved that there is at least one NeutroSemihypergroup (NeutroH$_v$-Semigroups) of order n where n is any integer greater than one. The results in this paper may be considered as a base for any possible study in the field of NeutroHyperstructures.

For future research, we raise the following ideas.

1. Find all NeutroSemihypergroups (NeutroH$_v$-Semigroups) of small order (up to NeutroStrongIsomorphism).
2. Find bounds for the number of finite NeutroSemihypergroups (NeutroH$_v$-Semigroups) of arbitrary order n (up to NeutroStrongIsomorphism).
3. Classify simple NeutroSemihypergroups (NeutroH$_v$-Semigroups) up to NeutroStrongIsomorphism.
4. Define other NeutroHyperstructures such as NeutroPolygroup, NeutroHyperring, etc.
5. Find applications of NeutroHyperstructures in some fields like Biology, Physics, Chemistry, etc.

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Single-Valued Neutro Hyper BCK-Subalgebras

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M. Hamidi, Florentin Smarandache (2021). Single-Valued Neutro Hyper BCK-Subalgebras. *Journal of Mathematics,* Article ID 6656739, 11; DOI: 10.1155/2021/6656739

ABSTRACT
The purpose of this paper is to introduce the notation of single-valued neutrosophic hyper BCK-subalgebras and a novel concept of neutro hyper BCK-algebras as a generalization and alternative of hyper BCK-algebras, that have a larger applicable field. In order to realize the article's goals, we construct single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras on a given nonempty set. The result of the research is the generalization of single-valued neutrosophic BCK-subalgebras and neutro BCK-algebras to single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras, respectively. Also, some results are obtained between extended (extendable) single-valued neutrosophic BCK-subalgebras and single-valued neutrosophic hyper BCK-subalgebras via fundamental relation. The paper includes implications for the development of single-valued neutrosophic BCK-subalgebras and neutro BCK-algebras and for modelling the uncertainty problems by single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras. The new conception of single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras was given for the first time in this paper. We find a method that can apply these concepts in some complex networks.

1. Introduction

The theory of logical (hyper) algebra is related to the study of certain propositional calculi and tries to solve logical problems using (hyper) algebraic methods. Jun et al. [1] has introduced a logical (hyper) algebra named hyper BCK-algebras as development of BCK-algebras, which were initiated by Imai and Iseki [2] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. The theory of neutrosophic set as an extension of classical set and (intuitionistic) fuzzy set [3], and interval-valued (intuitionistic) fuzzy set, is introduced by Smarandache for the first time in 1998 [4] and mentioned second time in 2005 [5]. This concept handles problems involving imprecise, indeterminacy, and inconsistent data and describes an important role in the modelling of unsure hypernetworks in all sciences. Recently, due to the importance of these subjects, by combining the neutrosophic sets and (hyper) BCK-algebras, some researchers worked in more branches of neutrosophic (hyper) BCK-algebras such as MBJ-neutrosophic hyper BCK-ideals in hyper BCK-algebras, an approach to BMBJ-neutrosophic hyper BCK-ideals of hyper BCK-algebras, structures on doubt neutrosophic ideals of (BCK/BCI)-algebras under (S, T)-norms, BMBJ-neutrosophic subalgebras in (BCI/BCK)-algebras, MBJ-neutrosophic ideals of (BCK/BCI)-algebras, implicative neutrosophic quadruple BCK-algebras and ideals, neutrosophic hyper BCK-ideals, implicative neutrosophic quadruple BCK-algebras and ideals, bipolar-valued fuzzy soft hyper BCK ideals in hyper BCK-algebras, single-valued neutrosophic ideals in Sostak's sense, and multipolar intuitionistic fuzzy hyper BCK-ideals in hyper BCK-algebras [6–16]. Recently, a novel concept of neutrosophy theory titled neutro (hyper) algebra as development of classical (hyper) algebra and partial (hyper) algebra is introduced by Smarandache [17].

A neutro (hyper) algebra is a system that has at least one neutro (hyper) operation or one neutro axiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements), while a partial (hyper) algebra is a (hyper) algebra that has at least one partial (hyper) operation, and all its axioms are classical (i.e., axioms true for all elements). Smarandache proved that a neutron (hyper) algebra is a generalization of a partial (hyper) algebra and showed that neutro (hyper) algebras are not partial (hyper) algebras, necessarily. Hamidi and Smarandache [18] introduced the concept of neutro BCK-subalgebras as a generalization of BCK-algebras and presented main results in neutro BCK-subalgebras as an extension of BCK-algebras structures and their applications. In addition, the concept of neutro (hyper) algebra is studied in different branches such as neutro algebra structures and neutro (hyper) graph [19, 20].

Regarding these points, one of the aims of this paper is to introduce the concept of single-valued neutrosophic hyper BCK-subalgebras and extendable single-valued neutrosophic BCK-subalgebras and generalize the notion of single-valued neutrosophic hyper BCK-subalgebras by considering the notion of single-valued neutrosophic BCK-subalgebras. Also, we want to establish the relationship between single-valued neutrosophic BCK-algebras and single-valued neutrosophic hyper BCK – algebras. So a strongly regular relation is applied on any hyper BCK-algebras using the concept of single-valued neutrosophic hyper BCK-subalgebras, and a quotient hyper BCK-algebras (BCK – algebras) can be obtained. The main aim of this study is to introduce the notation of neutro hyper BCK-algebras as a generalization of neutro BCK-algebras in regard to single-valued neutrosophic hyper BCK-subalgebras. In the study of neutro hyper BCK-algebra, despite having key mathematical tools, there are some limitations. The union of two neutro hyper BCK-algebra is not necessarily a neutro hyper BCK-algebra so the class of neutro hyper BCK-algebra is not closed under any given algebraic operation. In addition, neutro hyper BCK-algebras are different with (intuitionistic fuzzy) hyper BCK-algebras and single-valued neutrosophic hyper BCK-algebras so could not generalize the capabilities of (intuitionistic fuzzy) single-valued neutrosophic hyper BCK-algebras to neutro hyper BCK-algebras.

2. Preliminaries

Definition 1 (see [2]) Let $X \neq \varnothing$. Then a universal algebra $(X, \vartheta, 0)$ of type $(2,0)$ is called a BCK-algebra if, for all, $x, y, z \in X$:

(BCI – 1) $((x\varrho y)\varrho(x\varrho z))\varrho(z\varrho y) = 0$,

(BCI – 2) $(x\varrho(x\varrho y))\varrho y = 0$,

(BCI – 3) $x\varrho x = 0$,

(BCI – 4) $x\varrho y = 0$ and $y\varrho x = 0$ imply $x = y$,

(BCK – 5) $0\varrho x = 0$, where $\varrho(x, y)$ is denoted by $x\varrho y$.

Definition 2 (see [1]). Let $X \neq \varnothing$ and $P^*(X) = \{Y \mid \varnothing \neq Y \subseteq X\}$. Then for a map $\vartheta: X^2 \longrightarrow P^*(X)$, a hyperalgebraic system $(X, \vartheta, 0)$ is called a hyper BCK-algebra if, for all, $x, y, z \in X$:

$(H1)\,(x\,\vartheta\, z)\,\vartheta\,(y\,\vartheta\, z) \ll x\,\vartheta\, y$,

$(H2)\,(x\,\vartheta\, y)\,\vartheta\, z = (x\,\vartheta\, z)\,\vartheta\, y$,

$(H3)\,x\,\vartheta\, X \ll x$,

$(H4)\,x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined b y $0 \in x\,\vartheta\, y$, $\forall A, B \subseteq H$, $A \ll B \Longleftrightarrow \forall a \in A\, \exists b \in B$ s.t $a \ll b$, $(A\,\vartheta\, B) = \cup_{a\in A, b\in B}(a\,\vartheta\, b)$, and $\vartheta(x, y)$ is denoted by $x\,\vartheta\, y$.

We will call X is a weak commutative hyper BCK-algebra if $\forall x, y \in X$, $(x\,\vartheta\,(x\,\vartheta\, y)) \cap (y\,\vartheta\,(y\,\vartheta\, x)) \neq \varnothing$ [21].

Theorem 1 (see [1]). *Let $(X, \vartheta, 0)$ be a hyper BCK-algebra. Then $\forall x, y, z \in X$ and $A, B \subseteq X$:*

(i) $(0\,\vartheta\, 0) = 0$, $0 \ll x$, $(0\,\vartheta\, x) = 0$, $x \in (x\,\vartheta\, 0)$ *and* $A \ll 0 \Rightarrow A = 0$

(ii) $x \ll x$, $x\,\vartheta\, y \ll x$ *and* $y \ll z$ *implies that* $x\,\vartheta\, z \ll x\,\vartheta\, y$

(iii) $A\,\vartheta\, B \ll A$, $A \ll A$ *and* $A \subseteq B$ *implies* $A \ll B$

Definition 3 (see [22]). Let $(X, \vartheta, 0)$ be a hyper BCK-algebra. A fuzzy set $\mu: X \longrightarrow [0,1]$ is called a fuzzy hyper BCK-subalgebra if $\forall x, y \in X$, $\wedge(\mu(x\,\vartheta\, y)) \geq T_{\min}(\mu(x), \mu(y))$.

Definition 4 (see [5]). Let V be a universal set. A neutrosophic subset (NS) X in V is an object having the following form: $X = \{(x, T_X(x), I_X(x), F_X(x)) \mid x \in V\}$, or $X: V \longrightarrow [0,1]\times[0,1]\times[0,1]$, which is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X, and a falsity-membership function F_X. There is no restriction on the sum of $T_X(x), I_X(x)$, and $F_X(x)$.

3. Single-Valued Neutrosophic Hyper BCK-Subalgebras

In this section, the concept of single-valued neutrosophic hyper BCK-subalgebras will be considered as a generalization of single-valued neutrosophic BCK-subalgebras, and some of its properties will be investigated. We will also prove that single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic BCK-subalgebras are related, and single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic BCK-subalgebras can be constructed from single-valued neutrosophic hyper BCK-subalgebras via a fundamental relation. We will define the concept of extendable single-valued neutrosophic BCK-subalgebras and will show that any infinite set is an extended single-valued neutrosophic BCK-subalgebra.

Throughout this section, we denote hyper BCK-algebra $(X, \vartheta, 0)$ by X. From now on, for all, $x, y \in [0,1]$, $T_{\min}(x, y) = \min\{x, y\}$ and $S_{\max}(x, y) = \max\{x, y\}$ are considered as triangular norm and triangular conorm, respectively. In the following definition, the notation of single-valued neutrosophic hyper BCK-subalgebra of any given nonempty is defined.

Definition 5 . A single-valued neutrosophic set $A = (T_A, I_A, F_A)$ in an X is called a single-valued neutrosophic hyper BCK-subalgebra of X, if

(i) $\wedge (T_A(x \vartheta y)) \geq T_{\min}(T_A(x), T_A(y))$

(ii) $\vee (I_A(x \vartheta y)) \leq S_{\max}(I_A(x), I_A(y))$

(iii) $\vee (F_A(x \vartheta y)) \leq S_{\max}(F_A(x), F_A(y))$

The importance of the following theorems is to determine the role and the effect of truth-membership function T_A, indeterminacy-membership function I_A, and falsity-membership function F_A on the element $0 \in A$.

Theorem 2. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $T_A(0) \geq T_A(x)$

(ii) $\wedge (T_A(x \vartheta 0)) = T_A(x)$

(iii) $\wedge (T_A(0 \vartheta x)) = T_A(0)$

Proof

(i) Let $x \in X$. Since $0 \in x \vartheta x$, we get that $T_A(0) \geq \wedge (T_A(x \vartheta x)) \geq T_{\min}(T_A(x), T_A(x)) = T_A(x)$.

(ii) Let $x \in X$. Since $x \in x \vartheta 0$, we get that $T_A(x) \geq \wedge (T_A(x \vartheta 0)) \geq T_{\min}(T_A(x), T_A(0)) = T_A(x)$. So $\wedge (T_A(x \vartheta 0)) = T_A(x)$.

(iii) Immediate by Theorem 1. □

Theorem 3. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $I_A(0) \leq I_A(x)$

(ii) $\vee (I_A(x \vartheta 0)) = I_A(x)$

(iii) $\vee (I_A(0 \vartheta x)) = I_A(0)$

Proof

(i) Let $x \in X$. Since $0 \in x \vartheta x$, we get that $I_A(0) \leq \vee (I_A(x \vartheta x)) \leq S_{\max}(I_A(x), I_A(x)) = I_A(x)$.

(ii) Let $x \in X$. Since $x \in x \vartheta 0$, we get that $I_A(x) \leq \vee (I_A(x \vartheta 0)) \leq S_{\max}(I_A(x), I_A(0)) = I_A(x)$. So $\vee (I_A(x \vartheta 0)) = I_A(x)$.

(iii) Immediate by Theorem 1. □

Corollary 1. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $F_A(0) \leq F_A(x)$

(ii) $\vee (F_A(x \vartheta 0)) = F_A(x)$

(iii) $\vee (F_A(0 \vartheta x)) = F_A(0)$

(iv) $T_{\min}(T_A(x), I_A(0), F_A(0)) \leq T_{\min}(T_A(0), I_A(x), F_A(x))$

In the following theorem, we construct single-valued neutrosophic subset on any nonempty set.

Theorem 4. *Let $0 \notin X \neq \emptyset$. Then there exist a hyperoperation "ϑ," a single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of $X' = X \cup \{0\}$ such that $(X', \vartheta, 0)$ is a hyper BCK-algebra and A is a single-valued neutrosophic hyper BCK-subalgebra of X'.*

Proof. Let $x, y \in X'$. Define "ϑ" on X' by $x \vartheta y = \begin{cases} 0, & \text{if } x = 0, \\ \{0, x\}, & \text{if } x = y, x \neq 0, \\ x, & \text{otherwise} \end{cases}$. Clearly, $(X', \vartheta, 0)$ is a hyper BCK-algebra. Now, it is easy to see that every single-valued neutrosophic set $A = (T_A, I_A, F_A)$ that $T_A(0) = 1$, $I_A(0) = F_A(0) = 0$ is a single-valued neutrosophic hyper BCK-subalgebra of X'.

Let $\mathrm{SVN}h = \{A = (T_A, I_A, F_A) \mid A \text{ is a single–valued neutrosophic hyper BCK – subalgebra of } X\}$, whence X is a hyper BCK-algebra and $|X| \geq 1$. □

Corollary 2. *Let $X \neq \emptyset$. Then X can be extended to a hyper BCK-algebra that $|\mathrm{SVN}h| = |\mathbb{R}|$.*

Proof. Let $X = \{x\}$. Then (X, ϑ, x) is a hyper BCK-algebra such that $x \vartheta x = \{x\}$. Then for a single-valued neutrosophic set, $A = (T_A, I_A, F_A)$ by $T_A(x) = I_A(x) = F_A(x) = \alpha$ is a single-valued neutrosophic hyper BCK-subalgebra of X, where $\alpha \in [0, 1]$. If $|X| \geq 2$; then by Theorem 4, we can construct at least a hyper BCK-subalgebra on X. Now, $\forall \alpha \in [0, 1]$ define $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ by

$$T_{A_\alpha}(x) = \begin{cases} 1, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0, \end{cases}$$
$$I_{A_\alpha}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0, \end{cases} \quad (1)$$
$$F_{A_\alpha}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0. \end{cases}$$

Obviously, $A = (T_{A_\alpha}, I_{A_\alpha}, F_{A_\alpha})$ a single-valued neutrosophic hyper BCK-subalgebra of X and so $|\mathrm{SVN}h| = |[0, 1]|$.

Let X be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ a single-valued neutrosophic hyper BCK-subalgebra of X and $\alpha, \beta, \gamma \in [0, 1]$. Define $T_A^\alpha = \{x \in X \mid T_A(x) \geq \alpha\}$, $I_A^\beta = \{x \in X \mid I_A(x) \leq \beta\}$, $F_A^\gamma = \{x \in X \mid F_A(x) \leq \gamma\}$, and $A^{(\alpha,\beta,\gamma)} = \{x \in X \mid T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}$.

Considering the relation between single-valued neutrosophic hyper BCK-subalgebras and (fuzzy) hyper BCK-subalgebra is the main aim of the following results via the level subsets. □

Theorem 5. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then*

(i) $0 \in A^{(\alpha,\beta,\gamma)} = T_A^\alpha \cap I_A^\beta \cap F_A^\gamma$

(ii) $A^{(\alpha,\beta,\gamma)}$ *is a hyper BCK-subalgebra of X*

(iii) If $0 \leq \alpha \leq \alpha' \leq 1$, *then* $T_A^{\alpha'} \subseteq T_A^\alpha, I_A^{\alpha'} \supseteq I_A^\alpha$ *and* $F_A^{\alpha'} \supseteq F_A^\alpha$

Proof

(i) Clearly, $A^{(\alpha,\beta,\gamma)} = A^\alpha \cap A^\beta \cap A^\gamma$ and by Theorems 2 and 3, and Corollary 1, we get that $0 \in A^{(\alpha,\beta,\gamma)}$.

(ii) Let $x, y \in T_A^{\alpha}$. Then $T_{\min}(T_A(x), T_A(y)) \geq \alpha$. Now, for any, $z \in x \vartheta y, T_A(z) \geq \inf(T_A(x \vartheta y)) \geq T_{\min}(T_A(x), T_A(y)) \geq \alpha$. Hence, $z \in T_A^{\alpha}$, and so $x \vartheta y \subseteq T_A^{\alpha}$. In similar a way, $x, y \in I_A^{\beta} \cap F_A^{\gamma}$ implies that $x \vartheta y \subseteq (I_A^{\beta} \cap F_A^{\gamma})$. Then $A^{(\alpha,\beta,\gamma)}$ is a hyper BCK-subalgebra of X.

(iii) Immediate. □

Corollary 3. *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. If $0 \leq \alpha \leq \alpha' \leq 1$, then $A^{(\alpha',\alpha,\alpha)}$ is a hyper BCK-subalgebra of $A^{(\alpha,\alpha',\alpha')}$.*

Let X be a hyper BCK-algebra, S be a hyper BCK-subalgebra of X and $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in [0,1]$. Define

$$T_A^{[\alpha,\alpha']}(x) = \begin{cases} \alpha', & \text{if } x \in S, \\ \alpha, & \text{if } x \notin S, \end{cases}$$
$$I_A^{[\beta,\beta']}(x) = \begin{cases} \beta', & \text{if } x \in S, \\ \beta, & \text{if } x \notin S, \end{cases} \tag{2}$$
$$F_A^{[\gamma,\gamma']}(x) = \begin{cases} \gamma', & \text{if } x \in S, \\ \gamma, & \text{if } x \notin S. \end{cases}$$

Thus, we have the following theorem.

Theorem 6. *Let X be a hyper BCK-algebra and S be a hyper BCK-subalgebra of X. Then*

(i) $T_A^{[\alpha,\alpha']}$ is a fuzzy hyper BCK-subalgebra of X
(ii) $I_A^{[\beta,\beta']}$ is a fuzzy hyper BCK-subalgebra of X
(iii) $F_A^{[\gamma,\gamma']}$ is a fuzzy hyper BCK-subalgebra of X
(iv) $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A^{[\gamma,\gamma']})$ is a single-valued neutrosophic hyper BCK-subalgebra of X

Proof

(i) Let $x, y \in X$. If $x, y \in S$, since S is a hyper subalgebra of X, we get that $x \vartheta y \subseteq S$ and so

$$\wedge T_A^{[\alpha,\alpha']}(x \vartheta y) \geq \wedge T_A^{[\alpha,\alpha']}(S) = \alpha' \geq T_{\min}\left(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y)\right). \tag{3}$$

If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$), then $\wedge T_A^{[\alpha,\alpha']}(x \vartheta y) \in \{\alpha, \alpha'\}$. Thus, $\wedge T_A^{[\alpha,\alpha']}(x \vartheta y) \geq T_{\min}(T_A^{[\alpha,\alpha']}(x), T_A^{[\alpha,\alpha']}(y))$, and so $T_A^{[\alpha,\alpha']}$ is a fuzzy hyper BCK-subalgebra of X.

(ii) and (iii) They are similar to (i).

(iv) Let $x, y \in X$. If $x, y \in S$, since S is a hyper BCK-subalgebra of X, we get that $x \vartheta y \subseteq S$, and so $\vee I_A^{[\beta,\beta']}(x \vartheta y) \leq \vee I_A^{[\beta,\beta']}(S) = \alpha' \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. If ($x \in S$ and $y \notin S$) or ($x \notin S$ and $y \in S$) or ($x \notin S$ and $y \notin S$), then $\vee I_A^{[\beta,\beta']}(x \vartheta y) \in \{\beta, \beta'\}$. Thus, $\vee T_A^{[\beta,\beta']}(x \vartheta y) \leq S_{\max}(I_A^{[\beta,\beta']}(x), I_A^{[\beta,\beta']}(y))$. In a similar way, we can see that $\vee F_A^{[\gamma,\gamma']}(x \vartheta y) \leq S_{\max}(F_A^{[\gamma,\gamma']}(x), F_A^{[\gamma,\gamma']}(y))$ an by item (i), $A = (T_A^{[\alpha,\alpha']}, I_A^{[\beta,\beta']}, F_A[\gamma,\gamma'])$ is a single-valued neutrosophic hyper BCK-subalgebra of X.

Let X be a hyper BCK-algebra and $x, y \in X$. Then $x\beta y \Longleftrightarrow \exists n \in \mathbb{N}, (a_1, \ldots, a_n) \in X^n$ and $\exists u \in \vartheta(a_1, \ldots, a_n)$ such that $\{x, y\} \subseteq u$. The relation β is a reflexive and symmetric relation but not transitive relation. Let $C(\beta)$ be the transitive closure of β (the smallest transitive relation such that contains β). Borzooei et al. in [21], proved that for any given weak commutative hyper BCK-algebra X, $C(\underline{\beta})$ is a strongly regular relation on X, and $((X/C(\beta)), \varrho, 0)$ is a BCK-algebra, where $C(\beta)(x)\varrho C(\beta)(y) = C(\beta)(x \vartheta y)$ and $\bar{0} = C(\beta)(0)$.

Considering the relation between single-valued neutrosophic hyper BCK-subalgebras and single-valued neutrosophic BCK-subalgebras has very important, especially in extension of single-valued neutrosophic BCK-subalgebras. So we prove the following theorems and corollaries. □

Theorem 7. *Let X be a weak commutative hyper BCK-subalgebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of $\underline{X}$. Then t here e x istsa single-valued neutrosophic set $\overline{A} = (T_A, \overline{I_A}, \overline{F_A})$ of BCK-algebra $((X/C(\beta)), \vartheta, \overline{0})$ that $\forall x, y \in X$,*

(i) $\overline{T_A}(C(\beta)(0)) \geq \overline{T_A}(C(\beta)(x))$
(ii) if $yC(\beta)x$, then $\overline{T_A}(C(\beta)(x)) = \overline{T_A}(C(\beta)(y))$
(iii) $\overline{I_A}(C(\beta)(0)) \leq \overline{I_A}(C(\beta)(x))$
(iv) if $yC(\beta)x$, then $\overline{I_A}(C(\beta)(x)) = \overline{I_A}(C(\beta)(y))$
(v) $\overline{F_A}(C(\beta)(0)) \leq \overline{F_A}(C(\beta)(x))$
(vi) if $yC(\beta)x$, then $\overline{F_A}(C(\beta)(x)) = \overline{F_A}(C(\beta)(y))$

Proof. Let $x, y, t \in X$. Then on $(X/C(\beta))$, define $\overline{T_A}(C(\beta)(t)) = \begin{cases} T_A(0), & \text{if } 0 \in C(\beta)(x), \\ \wedge_{tC(\beta)x} T_A(x), & \text{otherwise}, \end{cases}$, $\overline{I_A}(C(\beta)(t)) = \begin{cases} I_A(0), & \text{if } 0 \in C(\beta)(x), \\ \vee_{tC(\beta)x} I_A(x), & \text{otherwise}, \end{cases}$, and $\overline{F_A}(C(\beta)(t)) = \begin{cases} F_A(0), & \text{if } 0 \in C(\beta)(x), \\ \vee_{tC(\beta)x} F_A(x), & \text{otherwise}, \end{cases}$. Using Theorems 2 and 3, we get that:

(i) $\overline{T_A}(C(\beta)(0)) = T_A(0) \geq \wedge_{t'C(\beta)x} T_A(t') = \overline{T_A}(C(\beta)(x))$

(ii) Since $xC(\beta)y$ and $C(\beta)$ is transitive, we get that $\overline{T_A}(C(\beta)(x)) = \wedge_{tC(\beta)x} T_A(t) \geq \wedge_{tC(\beta)y} T_A(t) = \overline{T_A}(C(\beta)(y))$

(iii) $\overline{I_A}(C(\beta)(0)) = I_A(0) \leq \vee_{t'C(\beta)x} I_A(t') = \overline{I_A}(C(\beta)(x))$

(iv) Since $xC(\beta)y$ and $C(\beta)$ is transitive, we get that $\overline{I_A}(C(\beta)(x)) = \vee_{tC(\beta)x} I_A(t) = \vee_{tC(\beta)y} I_A(t) = \overline{I_A}(C(\beta)(y))$

(v) and (vi) They are similar to (iii) and (iv), respectively. □

Theorem 8. *Let X be a weak commutative hyper BCK-subalgebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then there exists a single-valued neutrosophic subset $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ of BCK-algebra $((X/C(\beta)), \vartheta, \overline{0})$ that $\forall x, y \in X$:*

(i) There exists $t \in x \vartheta y$ such that $\overline{T_A}(C(\beta)\ (x \vartheta y)) = T_A(t)$

(ii) There exists $t' \in x \vartheta y$ such that $\overline{I_A}(C(\beta)(x \vartheta y)) = I_A(t')$

(iii) There exists $t'' \in x \vartheta y$ such that $\overline{F_A}(C(\beta)(x \vartheta y)) = F_A(t'')$

Proof

(i) Let $x, y \in X$. Applying Theorem 7,

$$\begin{aligned}&\overline{T_A}(C(\beta)(x)\varrho C(\beta)(y)) = \overline{T_A}(C(\beta)(x \vartheta y))\\ &= \overline{T_A}\{C(\beta)(m) \mid m \in x \vartheta y\} = \bigwedge_{\substack{sC(\beta)m\\ m\in x\vartheta y}} T_A(s).\end{aligned} \tag{4}$$

Now, since $sC(\beta)m$ and $m \in x \vartheta y$, then $s \in x \vartheta y$, and so there exists $t \in x \vartheta y$ such that $T_A(t) = \wedge_{\substack{sC(\beta)m\\ m\in x\vartheta y}} T_A(s)$.

(ii) Let $x, y \in X$. Then

$$\begin{aligned}&\overline{I_A}(C(\beta)(x)\varrho C(\beta)(y)) = \overline{I_A}(C(\beta)(x \vartheta y))\\ &= \overline{I_A}\{C(\beta)(n) \mid n \in x \vartheta y\} = \bigvee_{\substack{tC(\beta)n\\ n\in x\vartheta y}} I_A(t).\end{aligned} \tag{5}$$

Now, since $tC(\beta)n$ and $n \in x \vartheta y$, then $t \in x \vartheta y$, and so there exists $t' \in x \vartheta y$ such that $I_A(t') = \wedge_{\substack{tC(\beta)n\\ n\in x\vartheta y}} I_A(t)$.

(iii) It is similar to item (ii).

Some categorical properties of single-valued neutrosophic BCK-subalgebras is investigated in the following theorem based on the categorical properties of single-valued neutrosophic hyper BCK-subalgebras. □

Theorem 9. *Let X be a weak commutative hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X. Then there exists a single-valued neutrosophic BCK-subalgebra $B = (T_B, I_B, F_B)$ of $((X/C(\beta)), F_B, C(\beta)(0))$ that $((T_B \vartheta \pi) \le T_A$, $(I_B \vartheta \pi) \ge I_A$ and $(I_B \vartheta F_B) \ge F_A)$ or the following diagrams are quasi commutative:*

$$X \quad \longrightarrow^{T_A} [0 \ \ 1]_\pi \downarrow \nearrow_{T_B} \frac{X}{C(\beta)}, \ X \quad \longrightarrow^{I_A} [0 \ \ 1]_\pi \downarrow \nearrow_{I_B} \frac{X}{C(\beta)}, \ X \quad \longrightarrow^{F_A} [0 \ \ 1]_\pi \downarrow \nearrow_{F_B} \frac{X}{C(\beta)}. \tag{6}$$

Proof. Choice $T_B = \overline{T_A}$, $I_B = \overline{I_A}$ and $F_B = \overline{F_A}$. Then by Theorem 7, (i) $\forall x \in X$,

$$\begin{aligned}T_B(C(\beta)(0)) &\ge T_B(C(\beta)(x)),\\ I_B(C(\beta)(0)) &\le I_B(C(\beta)(x)),\\ F_B(C(\beta)(0)) &\le F_B(C(\beta)(x)).\end{aligned} \tag{7}$$

(ii) By Theorem 8, $\forall x, y \in X$; there exists $\{t, t', t''\} \subseteq x \vartheta y$ that

$$\begin{aligned}T_B(C(\beta)(x \vartheta y)) &= T_A(t),\\ I_B(C(\beta)(x \vartheta y)) &= I_A(t'),\\ F_B(C(\beta)(x \vartheta y)) &= F_A(t'').\end{aligned} \tag{8}$$

So

$$\begin{aligned}T_B(C(\beta)(x)\varrho C(\beta)(y)) &= T_B(C(\beta)(x \vartheta y)) = T_A(t) \ge \wedge\left(T_A(x \vartheta y)\right)\\ &\ge T_{\min}(T_A(x), T_A(y)) \ge T_{\min}(T_B(C(\beta)(x)), T_B(C(\beta)(y))),\\ I_B(C(\beta)(x)\varrho C(\beta)(y)) &= I_B(C(\beta)(x \vartheta y)) = I_A(t') \le \vee\left(I_A(x \vartheta y)\right)\\ &\le S_{\max}(I_A(x), I_A(y)) \le S_{\max}(I_B(C(\beta)(x)), I_B(C(\beta)(y))),\\ F_B(C(\beta)(x)\varrho C(\beta)(y)) &= F_B(C(\beta)(x \vartheta y)) = F_A(t'') \le \vee\left(F_A(x \vartheta y)\right)\\ &\le S_{\max}(F_A(x), F_A(y)) \le S_{\max}(F_B(C(\beta)(x)), F_B(C(\beta)(y))).\end{aligned} \tag{9}$$

Therefore, $B = (T_B, I_B, F_B)$ is a single-valued neutrosophic BCK-subalgebra of $(X/C(\beta))$, $(T_B \vartheta \pi) \le T_A$, $(I_B \vartheta \pi) \ge I_A$, and $(I_B \vartheta \pi) \ge F_A$.

Based on the fundamental relation, we can obtain the single-valued neutrosophic BCK-subalgebras, and single-valued neutrosophic BCK-subalgebras are derived from some single-valued neutrosophic hyper BCK-subalgebras. In this regard, it is important that single-valued neutrosophic BCK-subalgebras are derived from single-valued neutrosophic hyper BCK-subalgebra with minimal order. So the concepts of (extended) extendable single-valued neutrosophic BCK-subalgebra are introduced as follows. □

Definition 6

(i) Let $(X, \varrho, 0)$ be a BCK-algebra and $(Y, \vartheta, 0)$ be a hyper BCK-algebra. We say that the BCK-algebra X is derived from the hyper BCK-algebra Y if X is isomorphic to a nontrivial quotient of Y $(X \cong (Y/C(\beta)))$.

(ii) A single-valued neutrosophic BCK-subalgebra $A = (T_A, I_A, F_A)$ of X is called an extendable single-valued neutrosophic BCK-subalgebra, if there exist a hyper BCK-algebra $(Y, \vartheta, 0)$, a single-valued neutrosophic hyper BCK-subalgebra $B = (T_B, I_B, F_B)$ of Y, and $n \in \mathbb{N}$ such that $|(X, \vartheta, A)| = |(Y, \vartheta, B)| - n$, and BCK-algebra X is derived of hyper BCK-algebra Y. If $X = Y$ and almost everywhere $(T_A, I_A, F_A) = (T_B, I_B, F_B)$ $((T_A, I_A, F_A) = (T_B, I_B, F_B)$a.e that means $|\{x; T_A(x) \neq T_B(x), I_A(x) \neq I_B(x), F_A(x) \neq F_B(x)\}| = 1)$, we will say that it is an extended single-valued neutrosophic BCK-subalgebra.

The following example introduces an extendable single-valued neutrosophic BCK-subalgebra.

Example 1. Let $X = \{-1, -2, -3, -4\}$. Then $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic BCK-subalgebra of BCK-algebra $(X, \vartheta, -1)$ (see Table 1).

Now, set $Y = \{0, -1, -2, -3, -4\} = X \cup \{0\}$. Then $B = (T_B, I_B, F_B)$ is a single-valued neutrosophic hyper BCK-subalgebra of $(Y, \vartheta, 0)$ (see Table 2).

Clearly, $(Y/C(\beta)) \cong X$, $|Y| = |X| + 1$, and so $A = (T_A, I_A, F_A)$ is an extendable single-valued neutrosophic BCK-subalgebra of $(X, \vartheta, -1)$.

In the following theorem, we try to generate BCK-algebras based on single-valued neutrosophic hyper BCK-subalgebras.

Theorem 10. *Let $(X, \vartheta, 0)$ be a hyper BCK-algebra, $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of X, and $\overline{X} = \{(T_A(x), I_A(x), F_A(x)) \mid x \in X\}$. If A is one to one map, then:*

(i) There exists a hyperoperation "ϑ'" on $\overline{X}$ such that $(\overline{X}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ is a hyper BCK-algebra

(ii) There exists a single-valued neutrosophic hyper BCK-subalgebra $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ of $\overline{X}$ related to $A = (T_A, I_A, F_A)$

(iii) There exists an operation "ϱ" (related to ϑ) on $\overline{X}$ that $(\overline{X}, \varrho, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra

Proof

(i) Let $x, y \in X$. Define a hyperoperation ϑ' on $\overline{X}$, by

$$(T_A(x), I_A(x), F_A(x))\,\vartheta'\,(T_A(y), I_A(y), F_A(y)) = (T_A(x\vartheta y), I_A(x\vartheta y), F_A(x\vartheta y)). \tag{10}$$

It can be easily seen that $(T_A(x), I_A(x), F_A(x)) \ll\prime (T_A(y), I_A(y), F_A(y)) \Longleftrightarrow x \ll y$. It is easy to see that $(\overline{X}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ is a hyper BCK-algebra.

(ii) Let $x \in X$. Define $\overline{A}(A(x)) = A(x)$. Clearly, $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ is a single-valued neutrosophic hyper BCK-subalgebra of $(\overline{X}, \vartheta')$.

(iii) Assume $x, y \in X$. Define an operation ϱ on $\overline{X}$ by

$$(T_A(x), I_A(x), F_A(x))\varrho(T_A(y), I_A(y), F_A(y)) = \begin{cases} (T_A(x), I_A(x), F_A(x)), & \text{if } y = 0,\\ (\vee T_A(x\vartheta y), \wedge I_A(x\vartheta y), \wedge F_A(x\vartheta y)) & \text{otherwise.}\end{cases} \tag{11}$$

We just prove BCI-4. Let $x, y \in X$ and

$$\begin{aligned}&(T_A(x), I_A(x), F_A(x))\varrho(T_A(y), I_A(y), F_A(y))\\ &\quad = (T_A(x), I_A(x), F_A(x))\varrho(T_A(y), I_A(y), F_A(y))\\ &\quad = (T_A(0), I_A(0), F_A(0)).\end{aligned} \tag{12}$$

Since A is a one to one map, $0 \in x\vartheta y$ and $0 \in y\vartheta x$. It follows that $(T_A(x), I_A(x), F_A(x)) = (T_A(y), I_A(y), F_A(y))$. It is easy to see that BCI-1, BCI-2, BCI-3, and BCK-5 are valid, and so $(\overline{X}, \varrho, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra. □

Corollary 4. *Let $(\overline{X}, \vartheta, (T_A(0), I_A(0), F_A(0)))$ be a hyper BCK-algebra and $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic hyper BCK-subalgebra of $\overline{X}$. Then there exists a binary operation "ϱ" on $\overline{X}$, such that $(\overline{X}, \varrho, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra.*

In the following theorem, we try to generate hyper BCK-algebras based on single-valued neutrosophic hyper BCK-subalgebras.

Theorem 11. *Let X be a nonempty set, $0 \notin X$ and $X' = X \cup \{0\}$. Then there exist a hyperoperation "ϑ" on X', a hyperoperation "ϑ'" on X', a binary operation "ϱ" on X', a single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of X', and a single-valued neutrosophic subset $B = (T_B, I_B, F_B)$ of X' that:*

(i) $(X', \vartheta, 0)$ is a hyper BCK-algebra, and $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of X'

Table 1

ϱ	-1	-2	-3	-4
-1	-1	-1	-1	-1
-2	-2	-1	-2	-2
-3	-3	-3	-1	-3
-4	-4	-4	-4	-1
	-1	-2	-3	-4
T_A	1	0.2	0.4	0.6
I_A	0.1	0.3	0.7	0.9
F_A	0.05	0.25	0.45	0.65

Table 2

ϑ	0	-1	-2	-3	-4
e	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
-1	$\{-1\}$	$\{0,-1\}$	$\{0,-1\}$	$\{e,-1\}$	$\{0,-1\}$
-2	$\{-2\}$	$\{-2\}$	$\{0,-1\}$	$\{-2\}$	$\{-2\}$
-3	$\{-3\}$	$\{-3\}$	$\{-3\}$	$\{0,-1\}$	$\{-3\}$
-4	$\{-4\}$	$\{-4\}$	$\{-4\}$	$\{-4\}$	$\{0,-1\}$
	0	-1	-2	-3	-4
T_B	1	1	0.2	0.4	0.6
I_B	0.1	0.1	0.3	0.7	0.9
F_B	0.05	0.05	0.25	0.45	0.65

(ii) $(\overline{X'}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ *is a hyper BCK-algebra, and* $A = (T_A, I_A, F_A)$ *is a single-valued neutrosophic hyper BCK-subalgebra of* $\overline{X'}$

(iii) $(\overline{X'}, \varrho, (T_A(0), I_A(0), F_A(0)))$ *is a BCK-algebra, and* $B = (T_B, I_B, F_B)$ *is a single-valued neutrosophic BCK-subalgebra of* $\overline{X'}$

(iv) $|X'| = |\overline{X'}| + 1$

Proof. Let $|X| \geq 2$ and $b \in X$ be fixed. For any $x, y \in X'$, define a binary hyperoperation ϑ on X' as follows:

$$x \vartheta y = \begin{cases} 0, & \text{if } x = 0, \\ \{0, b\}, & \text{if } x = y \text{ and } x \neq 0, \\ \{b\}, & \text{if } x = b \text{ and } y = 0, \\ \{0, b\}, & \text{if } x = b \text{ and } y \neq 0, \\ x, & \text{otherwise.} \end{cases} \tag{13}$$

Now, we show that $(X', \vartheta, 0)$ is a hyper BCK-algebra. We just check that conditions (H1) and (H2) are valid.

(H1): Let $x, y, z \in X'$. If $x = 0$, then $(x \vartheta z) \vartheta (y \vartheta z) = \{0\} \vartheta (y \vartheta z) = \{0\} \ll x \vartheta y$. If $x = b$, then $(x \vartheta z) \vartheta (y \vartheta z) \subseteq \{0, b\} \vartheta (y \vartheta z) \subseteq \{0, b\} \ll x \vartheta y$. If $x \notin \{0, b\}$, we consider the following cases:

Case 1: $x = y \neq z$. Then $(x \vartheta z) \vartheta (y \vartheta z) = x \vartheta y = x \vartheta x = \{0, b\} \ll \{0, b\} = x \vartheta y$.

Case 2: $x = z \neq y$. Then $(x \vartheta z) \vartheta (y \vartheta z) = \{0, b\} \vartheta (y \vartheta z) = \{0, b\} \ll x = x \vartheta y$.

Case 3: $y = z \neq x$. Then $(x \vartheta z) \vartheta (y \vartheta z) \subseteq x \vartheta \{0, b\} = \{0, b\} \ll x = x \vartheta y$.

Case 4: $x \neq y \neq z$. Then $(x \vartheta z) \vartheta (y \vartheta z) = x \vartheta y = x \ll x = x \vartheta y$.

Case 5: $x = y = z$. Then $(x \vartheta z) \vartheta (y \vartheta z) = \{0, b\} \ll \{0, b\} = x \vartheta y$.

(H2): Let $x, y, z \in X$. The proof of $(x \vartheta y) \vartheta z = (x \vartheta z) \vartheta y$ is similar to that of (H1), and then it is easy to see that $(X', \vartheta, 0)$ is a hyper BCK-algebra. Consider a single-valued neutrosophic subset $A = (T_A, I_A, F_A)$ of X' such that $T_A(0) = T_A(b) = 1$, $I_A(0) = I_A(b) = F_A(0) = F_A(b) = 0$; by equation (2) and some modifications, we get that

$$\begin{aligned} &\wedge (T_A(x \vartheta y)) \geq T_{\min}(T_A(x), T_A(y)), \\ &\vee (I_A(x \vartheta y)) \leq S_{\max}(I_A(x), I_A(y)), \\ &\vee (F_A(x \vartheta y)) \leq S_{\max}(F_A(x), F_A(y)). \end{aligned} \tag{14}$$

Hence, $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic hyper BCK-subalgebra of $(\overline{X'}, \vartheta, 0)$. Now, $\forall x, y \in X$; define a hyperoperation ϑ' on $\overline{X'}$ by

$$\begin{aligned} A(x) \vartheta' A(y) &= (T_A(x), I_A(x), F_A(x)) \vartheta' (T_A(y), I_A(y), F_A(y)) \\ &= (T_A(x \vartheta y), I_A(x \vartheta y), F_A(x \vartheta y)). \end{aligned} \tag{15}$$

Define a single-valued neutrosophic subset $B = (T_B, I_B, F_B)$ of $\overline{X'}$ by

$$B(A(x)) = A(x),$$
$$\text{or } (T_B(T_A(x)), I_B(I_A(x)), F_B(F_A(x))) = (T_A(x), I_A(x), F_A(x)), \tag{16}$$

and an operation ϱ on $\overline{X'}$ by

$$\begin{aligned} &(T_A(x), I_A(x), F_A(x)) \varrho (T_A(y), I_A(y), F_A(y)) \\ &= (\vee(T_A(x) \vartheta' T_A(y)), \wedge(I_A(x) \vartheta' I_A(y)), \wedge(F_A(x) \vartheta' F_A(y))). \end{aligned} \tag{17}$$

It can be easily seen that $(T_A(x), I_A(x), F_A(x)) \ll' (T_A(y), I_A(y), F_A(y)) \Longleftrightarrow x \ll y$, $(\overline{X'}, \vartheta', (T_A(0), I_A(0), F_A(0)))$ is a hyper BCK-algebra, $A = (T_A(x), I_A(x), F_A(x))$ is a single-valued neutrosophic hyper BCK-subalgebra of $\overline{X'}$, $(\overline{X'}, \vartheta, (T_A(0), I_A(0), F_A(0)))$ is a BCK-algebra, and $B = (T_B(x), I_B(x), F_B(x))$ is a single-valued neutrosophic BCK-subalgebra of $\overline{X'}$, and since $T_A(0) = T_A(b) = 1$, $I_A(0) = I_A(b) = F_A(0) = F_A(b) = 0$, we get that $|X'| = |\overline{X'}| + 1$. □

Corollary 5. *Each nonempty set can be constructed to an extendable single-valued neutrosophic BCK-subalgebra.*

4. Neutro Hyper BCK-Algebras

Smarandache in [17] introduced the concept of neutro hyper operation. An n-ary (for integer $n \geq 1$) hyperoperation $\vartheta: X^n \longrightarrow P(Y)$ is called a neutro hyper operation if it has n-plets in X^n for which the hyperoperation is well-defined $\vartheta(a_1, a_2, \ldots, a_n) \in P(Y)$ (degree of truth (T)), n-plets in X^n for which the hyperoperation is indeterminate (degree of indeterminacy (I)), and n-plets in X^n for which the hyperoperation is outer-defined $\vartheta(a_1, a_2, \ldots, a_n) \notin P(Y)$ (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with

$(T, I, F) \neq (1,0,0)$ that represents the n-ary (total) hyper operation and $(T, I, F) \neq (0,0,1)$ that represents the n-ary anti hyper operation.

In this section, we introduce a novel concept of neutro hyper BCK-algebras as a generalization of neutro BCK-algebras and analyze their properties. The main motivation of the concept of neutro hyper BCK-algebra is a generalization of neutro BCK-algebra, which is defined as follows.

Definition 7. Let $X \neq \varnothing$ and $P^*(X) = \{Y \mid \varnothing \neq Y \subseteq X\}$. Then for a map $\vartheta: X^2 \longrightarrow P^*(X)$, a hyperalgebraic system $(X, \vartheta, 0)$ is called a neutro hyper BCK-algebra if it satisfies in the following neutro axioms:

(H1) ($\exists x, y, z \in X$ that $(x \vartheta z) \vartheta (y \vartheta z) \ll x \vartheta y$) and ($\exists x', y', z' \in X$ that $(x' \vartheta z') \vartheta (y' \vartheta z') \not\ll x' \vartheta y'$ or indeterminate)

(H2) ($\exists x, y, z \in X$ that $(x \vartheta y) \vartheta z = (x \vartheta z) \vartheta y$) and ($\exists x', y', z' \in X$ that $(x' \vartheta y') \vartheta z\prime \neq (x' \vartheta z') \vartheta y'$ or indeterminate)

(H3) ($\exists x \in X$ that $x \vartheta X \ll x$) and ($\exists x' \in X$ that $x' \vartheta X \not\ll x'$ or indeterminate)

(H4) ($\exists x, y \in X$ that if $x \ll y$ and $y \ll x$ imply $x = y$) and ($\exists x', y' \in X$ that if $x' \ll y'$ and $y' \ll x'$ imply $x' \neq y'$ or indeterminate),

where $a \ll b$ is defined by $0 \in a \vartheta b$, and $\forall A, B \subseteq H$, $A \ll B \Longleftrightarrow \forall a \in A \, \exists b \in B \, \text{s.t} \, a \ll b$

If $(X, \vartheta, 0)$ is a neutro hyperalgebra and satisfies in condition (H1) to (H4), then we will call it is a neutro hyper BCK-algebra of type 4 (i.e., it satisfies 4 neutro axioms).

Investigation of partial order relation on neutro hyper BCK-algebra plays a main role in Hass diagram, so we have the following results.

Theorem 12. *Let $(X, \vartheta, 0)$ be a neutro hyper BCK-algebra, $x, y, z \in X$ and $A, B, C \subseteq X$. Then*

(i) $\exists x, y \in X$ *such that* $(x \vartheta y) \ll x$

(ii) $\exists x, y \in X$ *such that* $(x \vartheta y) \not\ll x$

(iii) $\exists x \in X$ *such that* $x \ll x$

(iv) $\exists x \in X$ *such that* $x \not\ll x$

(v) $\exists A, B \subseteq X$ *such that* $A \ll A$

(vi) $\exists A, B \subseteq X$ *such that* $A \not\ll A$

Proof. We prove only the item (ii), and other items are similar to it. Since $(X, \vartheta, 0)$ is a neutro hyper BCK-algebra, there exists $x \in X$such that $(x \vartheta X) \not\ll X$. It follows that there exist $a, y \in X$ such that $a \in x \vartheta y$ and $a \not\ll x$. Hence, $(x \vartheta y) \not\ll x$. □

Theorem 13. *Let $(X, \vartheta, 0)$ be a neutro hyper BCK-algebra, $x, y, z \in X$ and $A, B, C \subseteq X$. Then*

(i) if $A \ll B$, *then* $(A \cup C) \ll (B \cup C)$

(ii) if $A \not\ll B$, *then* $(A \cup C) \not\ll (B \cup C)$

Proof

(i) Let $a \in A$ be arbitrary. Since $A \ll B$, there exists $b \in B$ such that $a \ll b$. Hence, for $a \in (A \cup C)$, there exists $b \in (B \cup C)$ such that $a \ll b$ and so $(A \cup C) \ll (B \cup C)$.

(ii) Since $A \not\ll B$, there exists $a \in A$ such that for all, $b \in B$, we have $a \not\ll b$. Hence, there exists $a \in (A \cup C)$ such that for all, $b \in (B \cup C)$, we get that $a \not\ll b$ and so $(A \cup C) \not\ll (B \cup C)$. □

Example 2. (i) Every neutro BCK-algebra $(X, \vartheta, 0)$ is a neutro hyper BCK-algebra. Since, for all, $x, y \in X$, can define a hyperoperation ϑ on X by $x \vartheta y = \{x \varrho y\}$.

(ii) Consider $\mathbb{N}^* = \{0, 1, 2, 3, \ldots\}$. Define $x \vartheta y = \begin{cases} \{0, x\} & \text{if } x \leq y \\ 0 & (x, y) = (2,3) \text{ or } (x, y) = (3,2) \\ 2 & x = y = 1 \text{ or } (x, y) = (0,1) \\ x & \text{otherwise} \end{cases}$. Clearly, $(\mathbb{N}^*, \vartheta, 0)$ is a neutro hyper BCK-algebra.

The following theorem shows that neutro hyper BCK-algebras are the generalization of hyper BCK-algebras.

Theorem 14. *Every hyper BCK-algebra can be extended to a neutro hyper BCK-algebra.*

Proof. Let $(X, \vartheta, 0)$ be a hyper BCK-algebra and $\alpha \notin X$. For all, $x, y \in X \cup \{\alpha\}$, define ϑ_α on $X \cup \{\alpha\}$ by $x \vartheta_\alpha y = x \vartheta y$, where, $x, y \in X$ and whence $\alpha \in \{x, y\}$, define $x \vartheta_\alpha y$ is indeterminate or $x \vartheta_\alpha y \in X \cup \{\alpha\}$.

We show that how to construct neutro hyper BCK-algebras from BCK-algebras. □

Example 3. Let $X = \{0, 1, 2, 3, 4\}$ and consider Table 3.

Then

(i) If $a = 0$, then $(X, \vartheta_1, 0)$ is a neutro hyper BCK-algebra and if $a = 1$, then $(X \backslash \{3, 4, 5\}, \vartheta_1, 0)$ is a hyper BCK-algebra

(ii) $(X, \vartheta_2, 0)$ is a neutro hyper BCK-algebra and $(X \backslash \{4, 5\}, \vartheta_2, 0)$ is a hyper BCK-algebra

(iii) If $s = z = 0, w = 3$, then $(X, \vartheta_3, 0)$ is a neutro hyper BCK-algebra, and for $s = 1$, $z = 3$, $(X \backslash \{5\}, \vartheta_3, 0)$ is a hyper BCK-algebra. If $s = z = 0$, $w = \sqrt{2}$, then $(X, \vartheta_3, 0)$ is a neutro hyper BCK-algebra of type 4

The importance of the following theorem is to construct of neutro hyper BCK-algebra from any given nonempty set.

Theorem 15. *Let $0 \notin X \neq \varnothing$. Then there exists a hyperoperation "ϑ" on $X' = X \cup \{0\}$ such that $(X', \vartheta, 0)$ is a neutro hyper BCK-algebra.*

Proof. Let $0 \notin X \neq \varnothing$. Using Theorem 4, there exist a hyperoperation "ϑ" on $X' = X \cup \{0\}$ such that $(X', \vartheta, 0)$ is a hyper BCK-algebra. Now, apply Theorem 14; there exist a hyperoperation "ϑ'" on $X' = X \cup \{0\}$ such that $(X', \vartheta', 0)$ is a neutro hyper BCK-algebra.

Table 3: Neutro hyper BCK-algebras.

ϑ_1	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	a	2	4	3
2	2	2	0, 2	0	2	0
3	3	0	1	2	4	5
4	1	4	2	1	4	3
5	0	4	0	1	4	0
ϑ_2	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0, 1	0	0, 1	4	5
2	2	2	0	2	5	0
3	3	3	3	0	0	0
4	2	1	2	4	1	2
5	5	0	4	0	0	x
ϑ_3	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0, 2	1	1	s	0
2	2	0, 2	0, 2	0, 2	0, 2	3
3	3	3	3	0, 2	z	0
4	4	4	4	4	0, 2	1
5	2	0	2	2	2	w

Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras. Define ϑ on $X_1 \times X_2$ by $(x, y)\vartheta(x', y') = (x\vartheta_1 x', y\vartheta_2 y')$, where $(x, y), (x', y') \in X_1 \times X_2$ and say that $(x, y) \ll (x', y') \iff (0_1, 0_2) \in (x, y)\vartheta(x', y')$. The following theorem investigates the properties of partial order relation on product of Neutro hyper BCK algebras. □

Theorem 16. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras. Then*

(i) $\forall (x, y), (x', y') \in X_1 \times X_2, (x, y) \ll (x', y') \iff (x \not\ll_1 x')$ and $(y \not\ll_2 y')$

(ii) $\forall (x, y), (x', y') \in X_1 \times X_2, (x, y) \not\ll (x', y') \iff (x \ll_1 x')$ or $(y \ll_2 y')$

(iii) $\exists (x, y), (x', y') \in X_1 \times X_2, (0_1, 0_2) \in ((x, y)\vartheta(x', y'))\vartheta(x, y)$

(iv) $\exists (x, y), (x', y') \in X_1 \times X_2, (0_1, 0_2) \notin ((x, y)\vartheta(x', y'))\vartheta(x, y)$

Proof

(i) Immediate

(ii) Let $(x, y), (x', y') \in X_1 \times X_2$. Then $(0_1, 0_2) \in (x, y)\vartheta(x', y')$, if and only if $(0_1, 0_2) \in (x\vartheta_1 x', y\vartheta_2 y')$, if and if only $0_1 \notin x\vartheta x'$ or $0_2 \notin y\vartheta y'$, and if and only if $(x \ll_1 x')$ or $(y \ll_2 y')$

(iii) Since $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras, there exist $x, y \in X_1, x', y' \in X_2$ such that $0_1 \in (x\vartheta y)\vartheta x$ and $0_2 \in (x'\vartheta y')\vartheta x'$. It follows that $\exists (x, y), (x', y') \in X_1 \times X_2$, $(0_1, 0_2) \in ((x, y)\vartheta(x', y'))\vartheta(x, y)$

(iv) Since $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras, there exist $x, y \in X_1, x', y' \in X_2$ such that $0_1 \notin (x\vartheta y)\vartheta x$ and $0_2 \notin (x'\vartheta y')\vartheta x'$. It follows that $\exists (x, y), (x', y') \in X_1 \times X_2$, $(0_1, 0_2) \notin ((x, y)\vartheta(x', y'))\vartheta(x, y)$

We need to extend neutro hyper BCK-algebras to a larger class of neutro hyper BCK-algebras, so we apply the notation of product on neutro hyper BCK-algebras as follows. □

Theorem 17. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be two neutro hyper BCK-algebras. Then $(X_1 \times X_2, \vartheta, (0_1, 0_2))$ is a neutro hyper BCK-algebra.*

Proof. We prove only the item (H4), and other items by Theorem 16 are valid. Since $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ are neutro hyper BCK-algebras, there exist $(x_1, x_2), (y_1, y_2), (x'_1, x'_2), (y'_1, y'_2) \in X_1 \times X_2$ that if $(x_1 \ll_1 y_1, y_1 \ll_1 x_1)$, then $x_1 = y_1$, and if $(x_2 \ll_2 y_2, y_2 \ll_2 x_2)$, then $x_2 = y_2$. Also, if $(x'_1 \ll_1 y'_1, y'_1 \ll_1 x'_1)$, then $x_1 \neq y_1$, and if $(x_2' \ll_2 y'_2, y'_2 \ll_2 x'_2)$, then $x_2 \neq y_2$. By (i), it follows that there exist $(x_1, x_2), (y_1, y_2), (x'_1, x'_2), (y'_1, y'_2) \in X_1 \times X_2$ that if $(x_1, x_2) \ll (y_1, y_2)$, $(y_1, y_2) \ll (x_1, x_2)$, we have $(x_1, x_2) = (y_1, y_2)$, and if $(x'_1, x'_2) \ll (y'_1, y'_2)$, $(y'_1, y'_2) \ll (x'_1, x'_2)$, we have $(x'_1, x'_2) \neq (y'_1, y'_2)$.

Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be hyper BCK-algebras, where $X_1 \cap X_2 = \varnothing$. For some $x, y \in X$, define a hyperoperations ϑ_t, ϑ_s as follows:

$$x\vartheta_t y = \begin{cases} (x\vartheta_1 y)\backslash\{0_1\}, & \text{if } x, y \in X_1\backslash X_2, \\ x\vartheta_2 y, & \text{if } x, y \in X_2\backslash X_1, \\ t, & \text{if } x \in X_1,\ y \in X_2, \\ 0_2, & \text{if } x \in X_2,\ y \in X_1, \end{cases}$$
$$x\vartheta_s y = \begin{cases} x\vartheta_1 y, & \text{if } x, y \in X_1\backslash X_2, \\ (x\vartheta_2 y)\backslash\{0_2\}, & \text{if } x, y \in X_2\backslash X_1, \\ s, & \text{if } x \in X_1,\ y \in X_2, \\ 0_1, & \text{if } x \in X_2,\ y \in X_1, \end{cases} \tag{18}$$

and $0_1\vartheta_t 0_1 = 0_1$ $\vartheta_t 0_2 = 0_2\vartheta_t 0_1 = 0_1$, $0_1\vartheta_s 0_2 = 0_2\vartheta_s 0_1 = 0_2\vartheta_s 0_2 = 0_2$, where $0_2 \neq t \in X_2$, $0_1 \neq s \in X_1$. Thus, we have the following theorem.

We want to extend neutro hyper BCK-algebras to a larger class of neutro hyper BCK-algebras, so we apply the notation of union on neutro hyper BCK-algebras as follows. □

Theorem 18. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be hyper BCK-algebras, where $X_1 \cap X_2 = \varnothing$ and $X = X_1 \cup X_2$. Then*

(i) For all, $A \subseteq X_1, A \not\ll \{0_1, t\}$

(ii) For all, $A \subseteq X_1, A \not\ll 0_2$

(iii) For all, $A \subseteq X_1, A \not\ll A$, and for all, $B \subseteq X_2, B \not\ll B$

(iv) For all, $A \subseteq X_2, A \not\ll \{0_2, s\}$

(v) For all, $A \subseteq X_2, A \not\ll 0_1$

Proof

(i) Let $A \subseteq X_1$. Then $A\vartheta_t 0_1 = \cup_{a\in A}(a\vartheta_t 0_1) = \cup_{a\in A}((a\vartheta 0_1)\backslash\{0_1\})$. It follows that $0_1 \notin A\vartheta_t 0_1$, so $A \not\ll \{0_1\}$. In addition, $A \vartheta_t t = \cup_{a \in A} (a \vartheta_t t) = \{t\}$ and $0_1 \notin t \vartheta_t 0_1$. It follows that $0_1 \notin A \vartheta_t 0_1$, so $A \not\ll \{t\}$.

(ii) Let $A \subseteq X_1$. Then $A \vartheta_t 0_2 = \cup_{a \in A} (a \vartheta_t 0_t) = \{t\}$ and $0_1 \notin t \vartheta_t 0_2$. It follows that $0_1 \notin A \vartheta_t 0_1$, so $A \not\ll \{0_2\}$. In addition, $A \vartheta_t t = \cup_{a \in A} (a \vartheta_t t) = \{t\}$ and $0_1 \notin t \vartheta_t 0_1$. It follows that $0_1 \notin A \vartheta_t 0_1$, so $A \not\ll \{t\}$.

(iii) Let $A \subseteq X_1$ and $B \subseteq X_2$. Since $A \vartheta_t A = \cup_{a,a' \in A} (a \vartheta_t a') = \cup_{a,a_I \in A} ((a \vartheta_t a') \backslash \{0_1\})$ and $B \vartheta_s S = \cup_{b,b' \in B} (b \vartheta_t b') = \cup_{b,b' \in B} ((b \vartheta_s b') \backslash \{0_2\})$, we get that $0_1 \not\in A \vartheta_t A$ and $0_2 \not\in B \vartheta_s B$. Thus $A \not\ll A$ and $B \not\ll B$.

(iv) and (v) are similar to (i) and (ii), respectively. □

Theorem 19. *Let $(X_1, \vartheta_1, 0_1)$ and $(X_2, \vartheta_2, 0_2)$ be hyper BCK-algebras, where $X_1 \cap X_2 = \varnothing$ and $X = X_1 \cup X_2$. Then*

(i) $(X, \vartheta_t, 0_1)$ is a neutro hyper BCK-algebra

(ii) $(X, \vartheta_s, 0_2)$ is a neutro hyper BCK-algebra

Proof

(i) $\underline{(H_1:)}$ For some, $x, y, z \in X_2 \backslash X_1$, $(x \vartheta_t z) \vartheta_t (y \vartheta_t z) \ll (x \vartheta_t y)$. Since, for $x \in X_1$, $(((x \vartheta 0_1) \backslash \{0_1\}) \backslash \{0_1\}) \vartheta_t 0_2 = t \neq 0_2$, we get that

$$\begin{aligned}(x \vartheta_t 0_1) \vartheta_t (0_2 \vartheta_t 0_1) &= ((x \vartheta 0_1) \backslash \{0_1\}) \vartheta_t 0_1 \\ &= ((x \vartheta 0_1) \backslash \{0_1\}) \backslash \{0_1\} \ll 0_2 = 0_1 \vartheta_t 0_2.\end{aligned} \tag{19}$$

$\underline{(H_2:)}$ For some, $x, y, z \in X_2 \backslash X_1$, $(x \vartheta_t y) \vartheta_t z = (x \vartheta_t z) \vartheta_t y$. In addition, for $x \in X_1$,

$$\begin{aligned}(x \vartheta_t 0_2) \vartheta_t 0_1 &= t \vartheta_t 0_1 = 0_2 \neq t = ((x \vartheta 0_1) \backslash \{0_1\}) \vartheta_t 0_2 \\ &= (x \vartheta_t 0_1) \vartheta_t 0_2.\end{aligned} \tag{20}$$

$\underline{(H_3:)}$ For some, $x \in X_2 X_1, x \vartheta_t X = x \vartheta X_2 \ll X_2 = X$. Since $t \vartheta_t 0_1 = 0_2$ and $(\cup_{x \in X_1} ((0_1 \vartheta x)\ \{0_1\})) \vartheta_t 0_1 = (\cup_{x \in X_1} ((0_1 \vartheta x) \backslash \{0_1\})) \backslash \{0_1\}$, we get that

$$\begin{aligned}0_1 \vartheta_t X = (0_1 \vartheta_t X_1) \cup (0_1 \vartheta_t X_2) &= \left(\underset{x \in X_1}{\cup} (0_1 \vartheta_t x) \right) \cup \left(\underset{y \in X_2}{\cup} (0_1 \vartheta_t y) \right) \\ &= \left(\underset{x \in X_1}{\cup} (0_1 \vartheta x) \backslash \{0_1\} \right) \cup \{t\} \ll 0_1.\end{aligned} \tag{21}$$

$\underline{(H_3:)}$ Because $0_1 \ll 0_1$ and $0_1 \in 0_1 \vartheta_t 0_2$ and $0_1 \in 0_2 \vartheta_t 0_1$, while $0_1 \neq 0_2$, we get the item $\underline{(H_3:)}$ is valid. Therefore, $(X, \vartheta_t, 0_1)$ is a neutro hyper BCK-algebra.

(ii) It is similar to item (i). □

4.1. Application of Neutro Hyper BCK-Algebras and Single-Valued Neutrosophic Hyper BCK-Subalgebras. In this subsection, we describe some applications of neutro hyper BCK-algebra and single-valued neutrosophic hyper BCK-subalgebra in some complex (hyper) networks.

Table 4: Neutro hyper BCK-algebra of an economic network.

ϑ	a	b	c	d	e	f
a	a	a	a	a	a	f
b	b	a, c	b	b	a	a
c	c	a, c	a, c	a, c	a, c	d
d	d	d	d	a, c	a	a
e	e	e	e	e	a, c	b
f	c	a	c	c	c	???

Table 5: Single-valued neutrosophic hyper BCK-subalgebra of a data network.

ϑ	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{a, b\}$	$\{a, b\}$	$\{e, b\}$	$\{a, b\}$
c	$\{c\}$	$\{c\}$	$\{a, b\}$	$\{c\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{a, b\}$	$\{d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{a, b\}$
	a	b	c	d	e
T_B	1	1	0.2	0.4	0.6
I_B	0.1	0.1	0.3	0.7	0.9
F_B	0.05	0.05	0.25	0.45	0.65

Example 4 (economic network). Let $X = \{a = \text{China}, b = \text{Italy}, c = \text{Iran}, d = \text{Spain}, e = \text{Germany}, f = \text{USA}\}$ be a set of top countries, which are in an economic network. Suppose ϑ is the relations on X, which is described in Table 4, and for $x \neq y$, $x * y = D$ means that D is the set of countries that benefit from this economic partnership, whence the country x starts to country y, and for $x = y$, it means that the country x maintains its capital.

Clearly, $(X, *, \text{China})$ is a neutro hyper BCK-algebra in this model. We obtain that the USA is main source of this network; since if the USA starts to any other country, it does not benefit. In addition, if the USA starts to itself, this participation becomes indeterminate. Also, if any country starts to China, we conclude that China loss, else with USA, and if China starts to any other country, then China benefit else USA.

Example 5 (data network). Let $Y = \{a, b, c, d, e\}$ be a set of mobile sets, which are in a data network. Suppose ϑ is the relations on Y, which is described in Table 3, and for all, $x \neq, x * y = D$ means that D is a set of mobile sets that receive contents of messages that mobile set x starts to mobile set y, and for $x = y$, it means that the mobile set x retains its information. In addition, for any $y \in Y, T_B(y), I_B(y), F_B(y)$ are the cryptographic power, battery life, and RAM of mobile set y, respectively. Then $B = (T_B, I_B, F_B)$ is a single-valued neutrosophic hyper BCK-subalgebra of (Y, ϑ, a) in Table 5.

It is clear that if mobile set named "a" starts, then none of the devices receive the message, and if other devices start to name a mobile set "a", then this device (mobile set a) cannot receive their messages; hence, it is not suitable node in this network, since furthermore to its complex cryptography, its battery life, and RAM is weak. Also, one can see that the mobile set b is the best in this regard.

5. Conclusion

To conclude, the current paper has presented and analyzed the notion of single-valued neutrosophic hyper BCK-subalgebras and neutro hyper BCK-algebras and investigated some of their new useful properties. We defined the concept of the extended single-valued neutrosophic BCK-subalgebras and showed that for any $\alpha \in [0,1]$ and a single-valued neutrosophic subset hyper BCK-subalgebra, $A = (T_A, I_A, F_A)$, $A = (T_{A\alpha}, I_{A\alpha}, F_{A\alpha})$ is a hyper BCK-subalgebra. Through the concept of fundamental relation $C(\beta)$, we have generated the single-valued neutrosophic BCK-subalgebras from single-valued neutrosophic hyper BCK-subalgebras, so some categorical properties of single-valued neutrosophic BCK-subalgebras are investigated based on the categorical properties of single-valued neutrosophic hyper BCK-subalgebras. In addition, on any nonempty set, we have constructed at least one single-valued neutrosophic BCK-subalgebra and one extendable single-valued neutrosophic BCK-subalgebra. The concept of neutro hyper BCK-algebra as a generalization of neutro BCK-algebra is introduced in this study, and it is constructed the class of product of neutro hyper BCK-algebras and union of neutro hyper BCK-algebras via hyper BCK-algebras. In study of neutro hyper BCK-algebras, despite having key mathematical tools, there are some limitations. The union of two neutro hyper BCK-algebras is not necessarily; a neutro hyper BCK-algebras so the class of neutro hyper BCK-algebras is not closed under any given algebraic operation. In addition, neutro hyper BCK-algebras are different from single-valued neutrosophic hyper BCK-subalgebras so could not generalize the capabilities of single-valued neutrosophic hyper BCK-subalgebras to neutro hyper BCK-algebras and conversely. In final, we can apply these concepts in real world, especially in some complex (hyper) networks.

We hope that these results are helpful for further studies in single-valued neutrosophic logical algebras. In our future studies, we hope to obtain more results regarding single-valued neutrosophic (hyper) logical-subalgebras, neutro (hyper) logical-subalgebras, and their applications.

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On Complex Neutrosophic Lie Algebras

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M. Parimala, Florentin Smarandache, Madeleine Al-Tahan, Cenap Özel (2022). On Complex Neutrosophic Lie Algebras. *Palestine Journal of Mathematics,* 11(1), 235-242

Abstract Complex neutrosophic Lie subalgebras and complex neutrosophic ideals of Lie algebras are defined in this paper. Each component in complex neutrosophic Lie algebra has magnitude and phase terms. Some characteristics of complex neutrosophic Lie subalgebras (ideals) and some of their operations like intersection and Cartesian product are also discussed. Moreover, the relationship between complex neutrosophic Lie subalgebras (ideals) and neutrosophic Lie subalgebras (ideals) is investigated. Finally, the image and the inverse image of complex neutrosophic Lie subalgebra under Lie algebra homomorphisms are defined and the properties of complex neutrosophic Lie subalgebras and complex neutrosophic ideals under homomorphisms of Lie algebras are studied.

1 Introduction

L. Zadeh's [18] fuzzy sets and fuzzy logic have been implemented in vague, unclear situations of real world problems. Atanassov's Intuitionistic fuzzy set [3] have been developed from fuzzy set by including one more component called non-membership function into fuzzy set. His theory gained an extensive recognition as a very valuable tool in area of science, Technology, Engineering, Medicine, etc. Smarandache [14] further extended Atanassov's theory and he named it as neutrosophic theory, in which he included a third component called indeterminacy into Atanassov's theory. Smarandache's neutrosophic theory deals with imprecision, indeterminacy, and inconsistent data. Later, Ali and Smarandache [1] developed novel complex neutrosophic sets and this theory extends the range of components from unit interval to the unit disc in com-plex plane. Each of its components has amplitude values and phase values. Simultaneously, complex neutrosophic set has been appLied in science and engineering field. Lie algebras are a special case of general linear algebra and was named after being developed by Sophus Lie (1842-1899). Lie groups classifies the smooth subgroups. After the development of this theory, it was appLied in mathematics and physics. Lie subalgebras and their properties were developed and investigated further in [2, 6, 12, 13, 15].

This paper is concerned about complex neutrosophic sets in Lie algebras and it is constructed as follows: After an Introduction, in Section 2, we present some definitions that are used throughout the paper. In Section 3, we extend neutrosophic Lie algebra by including some components into complex neutrosophic Lie algebra and further we extend each component range from unit interval to unit disc in complex plane. Additionally, we introduce complex neutrosophic Lie subalgebras (ideals) and investigate their properties such as their intersection and their Cartesian product. Finally, in Section 4, we study complex neutrosophic Lie subalgebras (ideals) under homomorphism of Lie algebras.

2 Preliminaries

We include some descriptions, comments and findings in this section, that are important and are used all over the paper regularly.
A description of complex neutrosophic structure was introduced by M. Ali and F. Smarandache [1] and is as follows.

Definition 2.1. [1] An object $\mathfrak{S}$ defined on a universe of discourse $\mathfrak{U}$ is called complex neutrosophic set (CNS), if it can be expressed as $\mathfrak{S} = \{(\zeta, \langle \mathfrak{M}(\zeta), \mathfrak{I}(\zeta), \mathfrak{F}(\zeta)\rangle) : \zeta \in \mathfrak{U}\}$. The values $\mathfrak{M}(\zeta), \mathfrak{I}(\zeta), \mathfrak{F}(\zeta)$ and their number can be in the complex plane all inside the unit circle, and so is in the following form, $\mathfrak{M}(\zeta) = p(\zeta)e^{j\mu(\zeta)}, \mathfrak{I}(\zeta) = q(\zeta)e^{j\nu(\zeta)}, \mathfrak{F}(\zeta) = r(\zeta)e^{j\omega(\zeta)}$ where $p(\zeta), q(\zeta), r(\zeta)$ and $\mu(\zeta), \nu(\zeta), \omega(\zeta)$ are respectively the amplitude terms and the phase terms, $\mu(\zeta), \nu(\zeta), \omega(\zeta) \in [0,1]$, with $^{-}0 \leq p(\zeta)+q(\zeta)+r(\zeta) \leq 3^{+}$ and $\mu(\zeta), \nu(\zeta), \omega(\zeta)$ are real valued with $j = \sqrt{-1}$. The scaling factors μ, ν and $\omega \in [0, 2\pi]$.

Definition 2.2. A vector space $\mathfrak{L}$ over a field $\mathfrak{G}$ (equal to $\mathfrak{R}$ or $\mathfrak{D}$) on which $\mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ denoted by $(\alpha, \beta) \to [\alpha, \beta]$ is defined as a Lie algebra, if the following axioms are satisfied:

(i) $[\alpha, \beta]$ is bilinear,

(ii) $[\alpha, \alpha] = 0$ for all $\alpha \in \mathfrak{L}$,

(iii) $[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0$ for all $\alpha, \beta, \gamma \in \mathfrak{L}$, (Jacobi identity).

$\mathfrak{L}$ is used to denote a Lie algebra(LA). It is noted that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[\alpha, \beta], \gamma] = [\alpha, [\beta, \gamma]]$. But it is anti commutative, i.e. $[\alpha, \beta] = -[\beta, \alpha]$.
A subspace $\mathfrak{H}$ of $\mathfrak{L}$ that is closed under $[','\,]$ is a Lie subalgebra. We define a subspace $\mathfrak{G}$ of $\mathfrak{L}$ as a Lie ideal of $\mathfrak{L}$, if $\mathfrak{G}$ is with the property $[\mathfrak{G}, \mathfrak{L}] \subseteq \mathfrak{G}$. Clearly, any Lie ideal is a Lie subalgebra.

3 Complex Neutrosophic Lie Algebra

In this section, we introduce new concepts related to complex neutrosophic sets. In particular, we define and study complex neutrosophic Lie subalgebras as well as complex neutrosophic Lie ideals of Lie algebra.

Definition 3.1. A complex neutrosophic triplet set $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$ on $\mathfrak{L}$ is said to be a complex neutrosophic Lie subalgebra if it satisfies the following conditions:

(i) $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\beta)), \mathfrak{I}_{\mathfrak{C}}(\alpha+\beta) \leq \vee(\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{I}_{\mathfrak{C}}(\beta)), \mathfrak{F}_{\mathfrak{C}}(\alpha+\beta) \leq \vee(\mathfrak{F}_{\mathfrak{C}}(\alpha), \mathfrak{F}_{\mathfrak{C}}(\beta))$,

(ii) $\mathfrak{M}_{\mathfrak{C}}(\zeta\alpha) \geq \mathfrak{M}_{\mathfrak{C}}(\alpha)$, $\mathfrak{I}_{\mathfrak{C}}(\zeta\alpha) \leq \mathfrak{I}_{\mathfrak{C}}(\alpha)$, $\mathfrak{F}_{\mathfrak{C}}(\zeta\alpha) \leq \mathfrak{F}_{\mathfrak{C}}(\alpha)$,

(iii) $\mathfrak{M}_{\mathfrak{C}}([\alpha, \beta]) \geq \wedge\{\mathfrak{M}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\beta)\}, \mathfrak{I}_{\mathfrak{C}}([\alpha, \beta]) \leq \vee\{\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{I}_{\mathfrak{C}}(\beta)\}, \mathfrak{F}_{\mathfrak{C}}([\alpha, \beta]) \leq \vee\{\mathfrak{F}_{\mathfrak{C}}(\alpha), \mathfrak{F}_{\mathfrak{C}}(\beta)\}$,

where,

$$\begin{aligned}
\wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\beta)) &= [p_{\mathfrak{C}}(\alpha) \wedge p_{\mathfrak{C}}(\beta)]e^{j[\mu_{\mathfrak{C}}(\alpha) \wedge \mu_{\mathfrak{C}}(\beta)]} \\
\vee(\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{I}_{\mathfrak{C}}(\beta)) &= [q_{\mathfrak{C}}(\alpha) \vee q_{\mathfrak{C}}(\beta)]e^{j[\nu_{\mathfrak{C}}(\alpha) \vee \nu_{\mathfrak{C}}(\beta)]} \\
\vee(\mathfrak{F}_{\mathfrak{C}}(\alpha), \mathfrak{F}_{\mathfrak{C}}(\beta)) &= [r_{\mathfrak{C}}(\alpha) \vee r_{\mathfrak{C}}(\beta)]e^{j[\omega_{\mathfrak{C}}(\alpha) \vee \omega_{\mathfrak{C}}(\beta)]}
\end{aligned}$$

for all $\alpha, \beta \in \mathfrak{L}$ and $\zeta \in \mathcal{F}$

Definition 3.2. A complex neutrosophic triplet set $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$ on $\mathfrak{L}$ is said to be a complex neutrosophic Lie subalgebra if it satisfies the following conditions:

(i) $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\beta)), \mathfrak{I}_{\mathfrak{C}}(\alpha+\beta) \leq \vee(\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{I}_{\mathfrak{C}}(\beta)), \mathfrak{F}_{\mathfrak{C}}(\alpha+\beta) \leq \vee(\mathfrak{F}_{\mathfrak{C}}(\alpha), \mathfrak{F}_{\mathfrak{C}}(\beta))$,

(ii) $\mathfrak{M}_{\mathfrak{C}}(\zeta\alpha) \geq \mathfrak{M}_{\mathfrak{C}}(\alpha)$, $\mathfrak{I}_{\mathfrak{C}}(\zeta\alpha) \leq \mathfrak{I}_{\mathfrak{C}}(\alpha)$, $\mathfrak{F}_{\mathfrak{C}}(\zeta\alpha) \leq \mathfrak{F}_{\mathfrak{C}}(\alpha)$,

(iii) $\mathfrak{M}_{\mathfrak{C}}([\alpha, \beta]) \geq \wedge\{\mathfrak{M}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\beta)\}, \mathfrak{I}_{\mathfrak{C}}([\alpha, \beta]) \leq \vee\{\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{I}_{\mathfrak{C}}(\beta)\}, \mathfrak{F}_{\mathfrak{C}}([\alpha, \beta]) \leq \vee\{\mathfrak{F}_{\mathfrak{C}}(\alpha), \mathfrak{F}_{\mathfrak{C}}(\beta)\}$,

where,

$$\begin{aligned}
\wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\beta)) &= [p_{\mathfrak{C}}(\alpha) \wedge p_{\mathfrak{C}}(\beta)]e^{j[\mu_{\mathfrak{C}}(\alpha) \wedge \mu_{\mathfrak{C}}(\beta)]} \\
\vee(\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{I}_{\mathfrak{C}}(\beta)) &= [q_{\mathfrak{C}}(\alpha) \vee q_{\mathfrak{C}}(\beta)]e^{j[\nu_{\mathfrak{C}}(\alpha) \vee \nu_{\mathfrak{C}}(\beta)]} \\
\vee(\mathfrak{F}_{\mathfrak{C}}(\alpha), \mathfrak{F}_{\mathfrak{C}}(\beta)) &= [r_{\mathfrak{C}}(\alpha) \vee r_{\mathfrak{C}}(\beta)]e^{j[\omega_{\mathfrak{C}}(\alpha) \vee \omega_{\mathfrak{C}}(\beta)]}
\end{aligned}$$

for all $\alpha, \beta \in \mathfrak{L}$ and $\zeta \in \mathcal{F}$.

Remark 3.3. If $\mathfrak{C}$ is a complex neutrosophic subalgebra of $\mathfrak{L}$ then it may not be a complex neutrosophic ideal of $\mathfrak{L}$. (See Example 3.4.)

Example 3.4. The set of all 3-dimensional real vectors $\mathbb{R}^3 = \{(\alpha,\beta,\gamma)|\alpha,\beta,\gamma \in \mathfrak{R}\}$ forms a Lie algebra over $\mathfrak{F} = \mathbb{R}$ and with the usual cross product $\times$. We define the set $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$, where $\mathfrak{M},\mathfrak{I},\mathfrak{F} : \mathbb{R}^3 \to \mathcal{E}^2$ ($\mathcal{E}^2$ is the unit disc), by

$$\mathfrak{M}_{\mathfrak{C}}(\alpha) = \begin{cases} 0.8e^{j\frac{3\pi}{4}}, if\alpha = \beta = \gamma = 0 \\ 0.5e^{j\frac{\pi}{3}}, if\alpha \neq 0, \beta = \gamma = 0 \\ 0, otherwise \end{cases}$$

$$\mathfrak{I}_{\mathfrak{C}}(\alpha) = \begin{cases} 0, if\alpha = \beta = \gamma = 0 \\ 0.6e^{j\frac{\pi}{2}}, if\alpha \neq 0, \beta = \gamma = 0 \\ 07e^{j\frac{2\pi}{3}}, otherwise \end{cases}$$

$$\mathfrak{F}_{\mathfrak{C}}(\alpha) = \begin{cases} 0, if\alpha = \beta = \gamma = 0 \\ 0.6e^{j\frac{\pi}{2}}, if\alpha \neq 0, \beta = \gamma = 0 \\ 07e^{j\frac{2\pi}{3}}, otherwise \end{cases}$$

Then it is clear that $\mathfrak{C}$ is a complex neutrosophic subalgebra of $\mathfrak{L} = \mathbb{R}^3$. But it is not a complex neutrosophic Lie ideal since $\mathfrak{M}_{\mathfrak{C}} = ([(1,0,0),(1,1,1)]) = \mathfrak{M}_{\mathfrak{C}}(0,-1,1) = 0 \not\geq \mathfrak{I}_{\mathfrak{C}}(1,0,0)$,$\mathfrak{I}_{\mathfrak{C}} = ([(1,0,0),(1,1,1)]) = \mathfrak{I}_{\mathfrak{C}}(0,-1,1) = 1 \not\leq \mathfrak{I}_{\mathfrak{C}}(1,0,0)$, and $\mathfrak{F}_{\mathfrak{C}} = ([(1,0,0),(1,1,1)]) = \mathfrak{F}_{\mathfrak{C}}(0,-1,1) = 1 \not\leq \mathfrak{F}_{\mathfrak{C}}(1,0,0)$.

Remark 3.5. Every complex neutrosophic Lie ideal is a complex neutrosophic Lie subalgebra.

Theorem 3.6. *Let $\mathfrak{L}$ be a neutrosophic Lie algebra and $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ be a complex neutrosophic set on it. Then $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ is a complex neutrosophic Lie subalgebra $\mathfrak{L}$ if and only if the non-empty complex neutrosophic upper s-level cut(NCU s-lc)*

$$\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s}) = \{\alpha \in \mathfrak{L}|\mathfrak{M}(\alpha) \geq \mathfrak{s}\}$$

and the non-empty complex neutrosophic lower t-level cut(NCL t-lc)

$$\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t}) = \{\alpha \in \mathfrak{L}|\mathfrak{I}(\alpha) \leq \mathfrak{t}\}, \mathfrak{V}_{\mathfrak{F}}(\mathfrak{t}) = \{\alpha \in \mathfrak{L}|\mathfrak{F}(\alpha) \leq \mathfrak{t}\}$$

are Lie subalgebras of $\mathfrak{L}$, for all $\mathfrak{s},\mathfrak{t}$ lies in the complex unit disk in the plane.

Proof: *Let $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ be a complex neutrosophic Lie subalgebra on $\mathfrak{L}$ and $\mathfrak{s},\mathfrak{t}$ lies in the complex unit disk in the plane, be such that $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s}) \neq \emptyset$. Let $\alpha,\beta \in \mathfrak{L}$ be such that $\alpha \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ and $\beta \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$. It follows that*

$$\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta)) \geq \mathfrak{s},$$
$$\mathfrak{M}_{\mathfrak{C}}(\zeta\alpha) \geq \mathfrak{M}_{\mathfrak{C}}(\alpha) \geq \mathfrak{s},$$
$$\mathfrak{M}_{\mathfrak{C}}([\alpha,\beta]) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta)) \geq \mathfrak{s},$$

and hence , $\alpha+\beta \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$, $\zeta\alpha \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ and $[\alpha,\beta] \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$, Thus, $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ forms a Lie subalgebra of $\mathfrak{L}$. For the case of $\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t})$, and $\mathfrak{V}_{\mathfrak{F}}(\mathfrak{t})$ the proof is similar.
Conversely, suppose that $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s}) \neq \emptyset$ is a Lie subalgebra of $\mathfrak{L}$ for every $\mathfrak{s} \in [0,1]e^{j\pi[0,1]}$. Assume that $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) < \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))$, for some $\alpha,\beta \in \mathfrak{L}$. Now taking $\mathfrak{s}_o := \frac{1}{2}\{\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) + \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))\}$.

Then we have that $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) < \mathfrak{s}_o < \mathfrak{M}_{\mathfrak{C}}(\wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha)\beta))\}$. and hence $\alpha+\beta \notin \mathfrak{M}_{\mathfrak{C}}(\mathfrak{s})$, $\alpha \in \mathfrak{M}_{\mathfrak{C}}(\mathfrak{s})$ and $\beta \in \mathfrak{M}_{\mathfrak{C}}(\mathfrak{s})$. However, this is clearly a contradiction. Therefore $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))$
for all $\alpha,\beta \in \mathfrak{L}$. Similarly we can show that $\mathfrak{M}_{\mathfrak{C}}(\zeta\alpha) \geq \mathfrak{M}_{\mathfrak{C}}(\alpha)$,
$\mathfrak{M}_{\mathfrak{C}}([\alpha,\beta]) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))$, hence $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}$
For the case of $\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t})$, and $\mathfrak{V}_{\mathfrak{F}}(\mathfrak{t})$ the proof is similar. □

Theorem 3.7. *Let $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ be a complex neutrosophic subset of $\mathfrak{L}$. Then the following statements are equivalent:*

(i) $\mathfrak{C}$ is a complex neutrosophic ideal of $\mathfrak{L}$,

(ii) The complex neutrosophic upper s-level cut $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ is an ideal of $\mathfrak{L}$ for every $\mathfrak{s} \in Im(\mathfrak{M}_{\mathfrak{C}})$.

(iii) The complex neutrosophic lower t-level cuts $\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t})$ and $\mathfrak{V}_{\mathfrak{F}}(\mathfrak{t})$ are ideals of $\mathfrak{L}$ for every $\mathfrak{t} \in Im(\mathfrak{I}_{\mathfrak{C}})$ and $\mathfrak{t} \in Im(\mathfrak{F}_{\mathfrak{C}})$ respectively.

Theorem 3.8. *Let $\mathfrak{C}_1 = (\mathfrak{M}_1, \mathfrak{I}_1, \mathfrak{F}_1)$ and $\mathfrak{C}_2 = (\mathfrak{M}_2, \mathfrak{I}_2, \mathfrak{F}_2)$ be two neutrosophic complex Lie subalgebras over $\mathfrak{L}$, then the intersection $\mathfrak{C}_3 = \mathfrak{C}_1 \cap \mathfrak{C}_2 = (\mathfrak{M}_3, \mathfrak{I}_3, \mathfrak{F}_3)$ is a complex neutrosophic Lie subalgebra over $\mathfrak{L}$.*
Proof. *For each $\alpha, \beta \in \mathfrak{L}$ and $\zeta \in \mathcal{F}$.*

$$
\begin{gathered}
\mathfrak{M}_{\mathfrak{C}_3}(\alpha+\beta) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha+\beta), \mathfrak{M}_{\mathfrak{C}_2}(\alpha+\beta)\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_1}(\beta)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\alpha)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_3}(\alpha), \mathfrak{M}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}(\alpha+\beta) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha+\beta), \mathfrak{I}_{\mathfrak{C}_2}(\alpha+\beta)\} \\
\leq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{I}_{\mathfrak{C}_3}(\alpha), \mathfrak{I}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{F}_{\mathfrak{C}_3}(\alpha+\beta) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha+\beta), \mathfrak{F}_{\mathfrak{C}_2}(\alpha+\beta)\} \\
\leq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{F}_{\mathfrak{C}_3}(\alpha), \mathfrak{F}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{M}_{\mathfrak{C}_3}(\zeta\alpha) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\zeta\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\zeta\alpha)\} \geq \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\alpha)\} = \mathfrak{M}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{I}_{\mathfrak{C}_3}(\zeta\alpha) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\zeta\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\zeta\alpha)\} \leq \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\alpha)\} = \mathfrak{I}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{F}_{\mathfrak{C}_3}(\zeta\alpha) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\zeta\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\zeta\alpha)\} \leq \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\alpha)\} = \mathfrak{F}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{M}_{\mathfrak{C}_3}([\alpha,\beta]) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}([\alpha,\beta]), \mathfrak{M}_{\mathfrak{C}_2}([\alpha,\beta])\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_1}(\beta)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\alpha)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_3}(\alpha), \mathfrak{M}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}([\alpha,\beta]) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}([\alpha,\beta]), \mathfrak{I}_{\mathfrak{C}_2}([\alpha,\beta])\} \\
\geq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{I}_{\mathfrak{C}_3}(\alpha), \mathfrak{I}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{F}_{\mathfrak{C}_3}([\alpha,\beta]) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}([\alpha,\beta]), \mathfrak{F}_{\mathfrak{C}_2}([\alpha,\beta])\} \\
\geq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{F}_{\mathfrak{C}_3}(\alpha), \mathfrak{F}_{\mathfrak{C}_3}(\beta)\} \ \square
\end{gathered}
$$

Theorem 3.9. *Let $\{\mathfrak{C}_i | i \in \Delta\}$ be a collection of complex neutrosophic subalgebras of $\mathfrak{L}$ such that $\mathfrak{C}_i$ is homogenous with $\mathfrak{C}_\mathfrak{k}$ for all $j, \mathfrak{k} \in \Delta$. Then $\bigcap_{i\in\Delta} \mathfrak{C}_i = (\mathfrak{M}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{I}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{F}_{\cap_{i\in\Delta}\mathfrak{C}_i})$ is a complex neutrosophic subalgebra of $\mathfrak{L}$, where*
$\bigcap_{i\in\Delta} \mathfrak{C}_i = (\mathfrak{M}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{I}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{F}_{\cap_{i\in\Delta}\mathfrak{C}_i}) = ((\wedge_{i\in\Delta} p_{\mathfrak{C}_i})e^{j\wedge_{i\in\Delta}\mu_{\mathfrak{C}_i}}, (\vee_{i\in\Delta} q_{\mathfrak{C}_i})e^{j\vee_{i\in\Delta}\nu_{\mathfrak{C}_i}},$
$(\vee_{i\in\Delta} r_{\mathfrak{C}_i})e^{j\vee_{i\in\Delta}\omega_{\mathfrak{C}_i}})$

We omit the proof as it is similar to the proof of Theorem 3.8. $\square$

Theorem 3.10. *Let $\mathfrak{C}_1 = (\mathfrak{M}_1, \mathfrak{I}_1, \mathfrak{F}_1)$ and $\mathfrak{C}_2 = (\mathfrak{M}_2, \mathfrak{I}_2, \mathfrak{F}_2)$ be two neutrosophic complex Lie subalgebras over $\mathfrak{L}$, then the cartesian product $\mathfrak{C}_3 = \mathfrak{C}_1 \times \mathfrak{C}_2 = (\mathfrak{M}_3, \mathfrak{I}_3, \mathfrak{F}_3) = (\mathfrak{M}_1 \times \mathfrak{M}_2, \mathfrak{I}_1 \times \mathfrak{I}_2, \mathfrak{F}_1 \times \mathfrak{F}_2)$ is a complex neutrosophic Lie subalgebra over $\mathfrak{L} \times \mathfrak{L}$.*
Proof. *For each $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathfrak{L} \times \mathfrak{L}$ and $\zeta \in \mathcal{F}$. Then*

$$
\begin{gathered}
\mathfrak{M}_{\mathfrak{C}_3}(\alpha+\beta) = (\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2})(\alpha+\beta) = (\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2})((\alpha_1,\alpha_2)+(\beta_1,\beta_2)) = \\
\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1+\beta_1), \mathfrak{M}_{\mathfrak{C}_2}(\alpha_2+\beta_2)\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{M}_{\mathfrak{C}_1}(\beta_1)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2), \mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{M}_{\mathfrak{C}_2}(\alpha_2)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta_1), \mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\alpha_1,\alpha_2), \mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\beta_1,\beta_2)\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\alpha), \mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}(\alpha+\beta) = (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\alpha+\beta) = (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})((\alpha_1,\alpha_2)+(\beta_1,\beta_2)) = \\
\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1+\beta_1), \mathfrak{I}_{\mathfrak{C}_2}(\alpha_2+\beta_2)\} \\
\leq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{I}_{\mathfrak{C}_1}(\beta_1)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2), \mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{I}_{\mathfrak{C}_2}(\alpha_2)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta_1), \mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\alpha_1,\alpha_2), (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\beta_1,\beta_2)\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\alpha), (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I})_{\mathfrak{C}_2}(\beta)\}
\end{gathered}
$$

$$\begin{aligned}
\mathfrak{F}_{\mathfrak{C}_3}(\alpha+\beta) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha+\beta) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})((\alpha_1,\alpha_2)+(\beta_1,\beta_2)) = \\
\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1+\beta_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2+\beta_2)\} \\
\leq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_1}(\beta_1)\},\vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2)\},\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta_1),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha_1,\alpha_2),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\beta_1,\beta_2)\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F})_{\mathfrak{C}_2}(\beta)\} \\
\mathfrak{M}_{\mathfrak{C}_3}(\zeta\alpha) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\zeta\alpha) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\zeta(\alpha_1,\alpha_2)) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\zeta\alpha_1),\mathfrak{M}_{\mathfrak{C}_2}(\zeta\alpha_2))\} \\
\geq \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2))\} = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\alpha_1,\alpha_2) = \mathfrak{M}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{I}_{\mathfrak{C}_3}(\zeta\alpha) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\zeta\alpha) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\zeta(\alpha_1,\alpha_2)) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\zeta\alpha_1),\mathfrak{I}_{\mathfrak{C}_2}(\zeta\alpha_2))\} \\
\leq \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2))\} = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\alpha_1,\alpha_2) = \mathfrak{I}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{F}_{\mathfrak{C}_3}(\zeta\alpha) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\zeta\alpha) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\zeta(\alpha_1,\alpha_2)) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\zeta\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\zeta\alpha_2))\} \\
\leq \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2))\} = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha_1,\alpha_2) = \mathfrak{F}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{M}_{\mathfrak{C}_3}([\alpha,\beta]) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([\alpha,\beta]) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([(\alpha_1,\alpha_2),(\beta_1,\beta_2)]) = \\
\wedge\{\mathfrak{M}_{\mathfrak{C}_1}([\alpha_1,\beta_1]),\mathfrak{M}_{\mathfrak{C}_2}([\alpha_2,\beta_2])\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{M}_{\mathfrak{C}_1}(\beta_1)\},\wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2)\},\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta_1),\mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([\alpha_1,\alpha_2]),(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([\beta_1,\beta_2])\} \\
= \wedge\{(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\alpha),(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}([\alpha,\beta]) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([\alpha,\beta]) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([(\alpha_1,\alpha_2),(\beta_1,\beta_2)]) = \\
\vee\{\mathfrak{I}_{\mathfrak{C}_1}([\alpha_1,\beta_1]),\mathfrak{I}_{\mathfrak{C}_2}([\alpha_2,\beta_2])\} \\
\leq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{I}_{\mathfrak{C}_1}(\beta_1)\},\vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2)\},\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta_1),\mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([\alpha_1,\alpha_2]),(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([\beta_1,\beta_2])\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\alpha),(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\beta)\} \\
\mathfrak{F}_{\mathfrak{C}_3}([\alpha,\beta]) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([\alpha,\beta]) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([(\alpha_1,\alpha_2),(\beta_1,\beta_2)]) = \\
\vee\{\mathfrak{F}_{\mathfrak{C}_1}([\alpha_1,\beta_1]),\mathfrak{F}_{\mathfrak{C}_2}([\alpha_2,\beta_2])\} \\
\leq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_1}(\beta_1)\},\vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2)\},\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta_1),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([\alpha_1,\alpha_2]),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([\beta_1,\beta_2])\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\beta)\}
\end{aligned}$$

This shows that $\mathfrak{C}_1\times\mathfrak{C}_2$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}\times\mathfrak{L}$. □

4 On complex neutrosophic Lie algebra homomorphisms

In this section, we investigate the properties of complex neutrosophic Lie subalgebras and complex neutrosophic ideals under homomorphisms of Lie algebras.

Definition 4.1. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two Lie algebras over a field $\mathfrak{F}$. Then a linear transformation $\mathfrak{f}:\mathfrak{L}_1\to\mathfrak{L}_2$ is called a Lie homomorphism if $\mathfrak{f}([\alpha,\beta]) = [\mathfrak{f}(\alpha),\mathfrak{f}(\beta)]$ holds for all $\alpha,\beta\in\mathfrak{L}_1$.

For the Lie algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$, it can be easily observed that if $\mathfrak{f}:\mathfrak{L}_1\to\mathfrak{L}_2$ is a Lie homomorphism and $\mathfrak{C}=(\mathfrak{M},\mathfrak{I},\mathfrak{F})$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}_2$, then the complex neutrosophic set $\mathfrak{f}^{-1}(\mathfrak{C})$ of $\mathfrak{L}_1$ is also a neutrosophic Lie subalgebra, where

$$\mathfrak{f}^{-1}(\mathfrak{M}_{\mathfrak{C}})(\alpha) = \mathfrak{M}_{\mathfrak{C}}(\mathfrak{f}(\alpha)) = \mathfrak{p}_{\mathfrak{C}}(\mathfrak{f}(\alpha))e^{j\mu(\mathfrak{f}(\alpha))},\ \mathfrak{f}^{-1}(\mathfrak{I}_{\mathfrak{C}})(\alpha) = \mathfrak{I}_{\mathfrak{C}}(\mathfrak{f}(\alpha)) = \mathfrak{q}_{\mathfrak{C}}(\mathfrak{f}(\alpha))e^{j\nu(\mathfrak{f}(\alpha))}$$
$$\mathfrak{f}^{-1}(\mathfrak{F}_{\mathfrak{C}})(\alpha) = \mathfrak{F}_{\mathfrak{C}}(\mathfrak{f}(\alpha)) = \mathfrak{r}_{\mathfrak{C}}(\mathfrak{f}(\alpha))e^{j\omega(\mathfrak{f}(\alpha))}$$

Theorem 4.2. *Let $\xi:\mathfrak{L}\to\mathfrak{L}'$ be a Lie algebra homomorphism. If $\mathfrak{C}=(\mathfrak{M},\mathfrak{I},\mathfrak{F})$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}'$ with a membership, indeterminacy and non-membership functions are $\mathfrak{M}_{\mathfrak{C}}(\beta)=\mathfrak{p}_{\mathfrak{C}}(\beta)e^{j\mu_{\mathfrak{C}}(\beta)}$, $\mathfrak{I}_{\mathfrak{C}}(\beta)=\mathfrak{q}_{\mathfrak{C}}(\beta)e^{j\nu_{\mathfrak{C}}(\beta)}$, and $\mathfrak{F}_{\mathfrak{C}}(\beta)=\mathfrak{r}_{\mathfrak{C}}(\beta)e^{j\omega_{\mathfrak{C}}(\beta)}$, respectively , then the complex neutrosophic set $\xi^{-1}(\mathfrak{C})$ is also a complex neutrosophic Lie subalgebra of $\mathfrak{L}$.*

Proof. *First, we need to show that $\xi^{-1}(\mathfrak{C})$ is homogeneous. Note that if $\alpha\in\mathfrak{L}$, then $\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha) = \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{p}_{\mathfrak{C}}(\xi(\alpha))e^{j\mu_{\mathfrak{C}}(\xi(\alpha))} = (\mathfrak{p}_{\mathfrak{C}}\xi(\alpha))e^{j\mu_{\mathfrak{C}}(\xi(\alpha))}$, $\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha) = \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{q}_{\mathfrak{C}}(\xi(\alpha))e^{j\nu_{\mathfrak{C}}(\xi(\alpha))} = (\mathfrak{q}_{\mathfrak{C}}\xi(\alpha))e^{j\nu_{\mathfrak{C}}(\xi(\alpha))}$, and $\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha) = \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{r}_{\mathfrak{C}}(\xi(\alpha))e^{j\omega_{\mathfrak{C}}(\xi(\alpha))} = (\mathfrak{r}_{\mathfrak{C}}\xi(\alpha))e^{j\omega_{\mathfrak{C}}(\xi(\alpha))}$. Now, if $\alpha_1,\alpha_2\in\mathfrak{L}$ with $(\mathfrak{p}_{\mathfrak{C}}\xi)(\alpha_1)\leq(\mathfrak{p}_{\mathfrak{C}}\xi)(\alpha_2)$, that is $\mathfrak{p}_{\mathfrak{C}}(\xi(\alpha_1))\leq\mathfrak{p}_{\mathfrak{C}}(\xi(\alpha_2))$, $(\mathfrak{q}_{\mathfrak{C}}\xi)(\alpha_1)\geq(\mathfrak{q}_{\mathfrak{C}}\xi)(\alpha_2)$, that is $\mathfrak{q}_{\mathfrak{C}}(\xi(\alpha_1))\geq\mathfrak{q}_{\mathfrak{C}}(\xi(\alpha_2))$, $(\mathfrak{r}_{\mathfrak{C}}\xi)(\alpha_1)\geq(\mathfrak{r}_{\mathfrak{C}}\xi)(\alpha_2)$, that is $\mathfrak{r}_{\mathfrak{C}}(\xi(\alpha_1))\geq\mathfrak{r}_{\mathfrak{C}}(\xi(\alpha_2))$,* *then from the homogeneity of* $\mathfrak{C}$*, we have* $(\mu_{\mathfrak{C}}\xi)(\alpha_1) \leq (\mu_{\mathfrak{C}}\xi)(\alpha_2)$*, that is* $\mu_{\mathfrak{C}}(\xi(\alpha_1)) \leq \mu_{\mathfrak{C}}(\xi(\alpha_2))$*,* $(\nu_{\mathfrak{C}}\xi)(\alpha_1) \geq (\nu_{\mathfrak{C}}\xi)(\alpha_2)$*, that is* $\nu_{\mathfrak{C}}(\xi(\alpha_1)) \geq \nu_{\mathfrak{C}}(\xi(\alpha_2))$*,* $(\omega_{\mathfrak{C}}\xi)(\alpha_1) \geq (\omega_{\mathfrak{C}}\xi)(\alpha_2)$*, that is* $\omega_{\mathfrak{C}}(\xi(\alpha_1)) \geq \omega_{\mathfrak{C}}(\xi(\alpha_2))$*. Thus shows* $\xi^{-1}(\mathfrak{C})$ *is homogenous. Let* $\alpha_1, \alpha_2 \in \mathfrak{L}$ *and* $\zeta \in \mathcal{F}$*. Then*

$$\begin{gathered}
\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_1+\alpha_2) = \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1+\alpha_2)) \\
= \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1)+\xi(\alpha_2)) \\
\geq \wedge\{\mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_2))\} \\
= \wedge\{\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\} \\
\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_1+\alpha_2) = \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1+\alpha_2)) \\
= \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1)+\xi(\alpha_2)) \\
\leq \wedge\{\mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_2))\} \\
= \vee\{\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\} \\
\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_1+\alpha_2) = \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1+\alpha_2)) \\
= \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1)+\xi(\alpha_2)) \\
\leq \wedge\{\mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_2))\} \\
= \vee\{\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}, \textit{ (ξ is linear).} \\
\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\zeta\alpha) = \mathfrak{M}_{\mathfrak{C}}(\xi(\zeta\alpha)) = \mathfrak{M}_{\mathfrak{C}}(\zeta\xi(\alpha)) \\
\geq \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha) \\
\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\zeta\alpha) = \mathfrak{I}_{\mathfrak{C}}(\xi(\zeta\alpha)) = \mathfrak{I}_{\mathfrak{C}}(\zeta\xi(\alpha)) \\
\leq \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha) \\
\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\zeta\alpha) = \mathfrak{F}_{\mathfrak{C}}(\xi(\zeta\alpha)) = \mathfrak{F}_{\mathfrak{C}}(\zeta\xi(\alpha)) \\
\leq \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha), \textit{ (ξ is linear).} \\
\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}([\alpha_1,\alpha_2]) = \mathfrak{M}_{\mathfrak{C}}(\xi([\alpha_1,\alpha_2])) \\
= \mathfrak{M}_{\mathfrak{C}}([\xi(\alpha_1),\xi(\alpha_2)]) \\
\geq \wedge\{\mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_2))\} \\
= \wedge\{\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}, \\
\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}([\alpha_1,\alpha_2]) = \mathfrak{I}_{\mathfrak{C}}(\xi([\alpha_1,\alpha_2])) \\
= \mathfrak{I}_{\mathfrak{C}}([\xi(\alpha_1),\xi(\alpha_2)]) \\
\leq \vee\{\mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_2))\} \\
= \vee\{\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}, \\
\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}([\alpha_1,\alpha_2]) = \mathfrak{F}_{\mathfrak{C}}(\xi([\alpha_1,\alpha_2])) \\
= \mathfrak{F}_{\mathfrak{C}}([\xi(\alpha_1),\xi(\alpha_2)]) \\
\leq \vee\{\mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_2))\} \\
= \vee\{\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}, \textit{ (ξ is homomorphism).} \ \square
\end{gathered}$$

Theorem 4.3. *Let* $\xi : \mathfrak{L} \to \mathfrak{L}'$ *be a Lie algebra homomorphism. If* $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$ *is a complex neutrosophic ideal of* $\mathfrak{L}'$ *with a membership, indeterminacy and non-membership functions are* $\mathfrak{M}_{\mathfrak{C}}(\beta) = \mathfrak{p}_{\mathfrak{C}}(\beta)e^{j\mu_{\mathfrak{C}}(\beta)}$*,* $\mathfrak{I}_{\mathfrak{C}}(\beta) = \mathfrak{q}_{\mathfrak{C}}(\beta)e^{j\nu_{\mathfrak{C}}(\beta)}$*, and* $\mathfrak{F}_{\mathfrak{C}}(\beta) = \mathfrak{r}_{\mathfrak{C}}(\beta)e^{j\omega_{\mathfrak{C}}(\beta)}$*, respectively , then the complex neutrosophic set* $\xi^{-1}(\mathfrak{C})$ *is also a complex fuzzy ideal of* $\mathfrak{L}$*.*
Proof. *The proof is similar to that of Theorem 4.2.*

Theorem 4.4. *Let* $\xi : \mathfrak{L} \to \mathfrak{L}'$ *be a surjective Lie algebra homomorphism. If* $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$*, where* $\mathfrak{M}_{\mathfrak{C}}(\alpha) = \mathfrak{p}_{\mathfrak{C}}(\alpha)e^{j\mu_{\mathfrak{C}}(\alpha)}$*,* $\mathfrak{I}_{\mathfrak{C}}(\alpha) = \mathfrak{q}_{\mathfrak{C}}(\alpha)e^{j\nu_{\mathfrak{C}}(\alpha)}$*, and* $\mathfrak{F}_{\mathfrak{C}}(\alpha) = \mathfrak{r}_{\mathfrak{C}}(\alpha)e^{j\omega_{\mathfrak{C}}(\alpha)}$*, for any* $\alpha \in \mathfrak{L}$*, is a complex neutrosophic Lie subalgebra of* $\mathfrak{L}$*, then* $\xi(\mathfrak{C})$ *is also a complex neutrosophic Lie subalgebra of* $\mathfrak{L}'$*.*
Proof. *We prove that* $\xi(\mathfrak{C})$ *is homogenous. Suppose* $\beta \in \mathfrak{L}'$*. Then*

$$\begin{gathered}
\mathfrak{M}_{\xi(\mathfrak{C})}(\beta) = sup_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{M}_{\mathfrak{C}}(\alpha)\} = sup_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{p}_{\mathfrak{C}}(\alpha)e^{j\mu_{\mathfrak{C}}(\alpha)}\} \\
= sup_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{p}_{\mathfrak{C}}(\beta)\}e^{j(sup_{\alpha\in\xi^{-1}(\beta)}\{\mu_{\mathfrak{C}}(\beta)\}} = \mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)e^{j\mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)}. \\
\mathfrak{I}_{\xi(\mathfrak{C})}(\beta) = inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{I}_{\mathfrak{C}}(\alpha)\} = inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{q}_{\mathfrak{C}}(\alpha)e^{j\nu_{\mathfrak{C}}(\alpha)}\} \\
= inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{q}_{\mathfrak{C}}(\beta)\}e^{j(sup_{\alpha\in\xi^{-1}(\beta)}\{\nu_{\mathfrak{C}}(\beta)\}} = \mathfrak{q}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)e^{j\nu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)}. \\
\mathfrak{F}_{\xi(\mathfrak{C})}(\beta) = inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{F}_{\mathfrak{C}}(\alpha)\} = inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{r}_{\mathfrak{C}}(\alpha)e^{j\omega_{\mathfrak{C}}(\alpha)}\} \\
= inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{r}_{\mathfrak{C}}(\beta)\}e^{j(sup_{\alpha\in\xi^{-1}(\beta)}\{\omega_{\mathfrak{C}}(\beta)\}} = \mathfrak{r}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)e^{j\omega_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)}.
\end{gathered}$$

Now let $\beta_1, \beta_2 \in \mathfrak{L}'$ *with* $\mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_1) \leq \mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2)$ *and* $\mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2) < \mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_1)$*. Then there exist a* $\alpha_1 \in \xi^{-1}(\{\beta_1\})$*, such that* $\mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2) < \mu_{\mathfrak{C}}(\alpha_1)$*. Therefore, If* $\alpha \in \xi^{-1}(\{\beta_2\})$*, then* $\mu_{\mathfrak{C}}(\alpha) < \mu_{\mathfrak{C}}(\alpha_1)$*, and so, from the homogeneity of* $\mathfrak{C}$*, we obtain* $\mathfrak{p}_{\mathfrak{C}}(\alpha) < \mathfrak{p}_{\mathfrak{C}}(\alpha_1)$*. Thus,*

$sup_{\alpha\in\xi^{-1}(\beta_2)}\{\mathfrak{p}_{\mathfrak{C}}(\alpha)\} < \mathfrak{p}_{\mathfrak{C}}(\alpha_1)$ and so, $\mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2) \leq \mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_1)$, which is a contradiction. Similarly we can prove for indeterminacy and non-membership functions. This shows $\xi(\mathfrak{C})$ is homogenous.
Since $\mathfrak{C}$ is a complex neutrosophic subalgebra, $\overline{\mathfrak{C}} = \{(\alpha, \langle\mathfrak{F}_{\mathfrak{C}}(\alpha), 1-\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\alpha)\rangle)|\alpha\in\mathfrak{L}\}$ is a neutrosophic subalgebra of $\mathfrak{L}$, and so the images of the components are neutrosophic subalgebra of $\mathfrak{L}'$. Hence, for $\beta_1,\beta_2\in\mathfrak{L}'$ and $\zeta\in\mathcal{F}$, we have

(i) $\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1+\beta_2) \geq \wedge(\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{M}_{\xi(\mathfrak{C})}(\beta_2))$,
$\mathfrak{M}_{\xi(\mathfrak{C})}(\zeta\beta_1) \geq \wedge\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1)$,
$\mathfrak{M}_{\xi(\mathfrak{C})}([\beta_1,\beta_2]) \geq \wedge(\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{M}_{\xi(\mathfrak{C})}(\beta_2))$

(ii) $\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1+\beta_2) \leq \vee(\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{I}_{\xi(\mathfrak{C})}(\beta_2))$,
$\mathfrak{I}_{\xi(\mathfrak{C})}(\zeta\beta_1) \leq \vee\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1)$,
$\mathfrak{I}_{\xi(\mathfrak{C})}([\beta_1,\beta_2]) \leq \vee(\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{I}_{\xi(\mathfrak{C})}(\beta_2))$

(iii) $\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1+\beta_2) \leq \vee(\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{F}_{\xi(\mathfrak{C})}(\beta_2))$,
$\mathfrak{F}_{\xi(\mathfrak{C})}(\zeta\beta_1) \leq \vee\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1)$,
$\mathfrak{F}_{\xi(\mathfrak{C})}([\beta_1,\beta_2]) \leq \vee(\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{F}_{\xi(\mathfrak{C})}(\beta_2))$

Now our result follows from the homogeneity of $\xi(\mathfrak{C})$. □

Theorem 4.5. *Let $\xi : \mathfrak{L} \to \mathfrak{L}'$ be a surjective Lie algebra homomorphism. If $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$, where $\mathfrak{M}_{\mathfrak{C}}(\alpha) = \mathfrak{p}_{\mathfrak{C}}(\alpha)e^{j\mu_{\mathfrak{C}}(\alpha)}$, $\mathfrak{I}_{\mathfrak{C}}(\alpha) = \mathfrak{q}_{\mathfrak{C}}(\alpha)e^{j\nu_{\mathfrak{C}}(\alpha)}$, and $\mathfrak{F}_{\mathfrak{C}}(\alpha) = \mathfrak{r}_{\mathfrak{C}}(\alpha)e^{j\omega_{\mathfrak{C}}(\alpha)}$, for any $\alpha\in\mathfrak{L}$, is a complex neutrosophic ideal of $\mathfrak{L}$, then $\xi(\mathfrak{C})$ is also a complex neutrosophic ideal of $\mathfrak{L}'$. .*

Theorem 4.6. *Let $\xi : \mathfrak{L} \to \mathfrak{L}'$ be a surjective Lie homomorphism. If $\mathfrak{C}_1 = (\mathfrak{M}_1, \mathfrak{I}_1, \mathfrak{F}_1)$ and $\mathfrak{C}_2 = (\mathfrak{M}_2, \mathfrak{I}_2, \mathfrak{F}_2)$ are complex neutrosophic ideals of $\mathfrak{L}$ such that $\mathfrak{C}_1$ is homogeneous of $\mathfrak{C}_2$, then $\xi(\mathfrak{C}_1+\mathfrak{C}_2) = \xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)$.*
Proof. *For $\beta\in\mathfrak{L}'$, we have*

(i) $\mathfrak{M}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta) = sup_{\beta=\xi(\alpha)}\{\mathfrak{M}_{\mathfrak{C}_1+\mathfrak{C}_2}(\alpha)\}$
$= sup_{\beta=\xi(\alpha)}\{sup_{\alpha=\mathfrak{a}+\mathfrak{b}}\{\mathfrak{M}_{\mathfrak{C}_1}(\mathfrak{a}) \wedge \mathfrak{M}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= sup_{\beta=\xi(\mathfrak{a})+\xi(\mathfrak{b})}\{\mathfrak{M}_{\mathfrak{C}_1}(\mathfrak{a}) \wedge \mathfrak{M}_{\mathfrak{C}_2}(\mathfrak{b})\}$
$= sup_{\beta=\mathfrak{m}+\mathfrak{n}}\{sup_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{M}_{\mathfrak{C}_1}(\mathfrak{a})\} \wedge sup_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{M}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= sup_{\beta=\mathfrak{m}+\mathfrak{n}}\{\mathfrak{M}_{\xi(\mathfrak{C}_1)}(\mathfrak{m}) \wedge \mathfrak{M}_{\xi(\mathfrak{C}_1)}(\mathfrak{n})\}$
$=\mathfrak{M}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta)$.

(ii) $\mathfrak{I}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta) = inf_{\beta=\xi(\alpha)}\{\mathfrak{I}_{\mathfrak{C}_1+\mathfrak{C}_2}(\alpha)\}$
$= inf_{\beta=\xi(\alpha)}\{inf_{\alpha=\mathfrak{a}+\mathfrak{b}}\{\mathfrak{I}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{I}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\xi(\mathfrak{a})+\xi(\mathfrak{b})}\{\mathfrak{I}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{I}_{\mathfrak{C}_2}(\mathfrak{b})\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{I}_{\mathfrak{C}_1}(\mathfrak{a})\} \vee inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{I}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{\mathfrak{I}_{\xi(\mathfrak{C}_1)}(\mathfrak{m}) \vee \mathfrak{I}_{\xi(\mathfrak{C}_1)}(\mathfrak{n})\}$
$=\mathfrak{I}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta)$.

(iii) $\mathfrak{F}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta) = inf_{\beta=\xi(\alpha)}\{\mathfrak{F}_{\mathfrak{C}_1+\mathfrak{C}_2}(\alpha)\}$
$= inf_{\beta=\xi(\alpha)}\{inf_{\alpha=\mathfrak{a}+\mathfrak{b}}\{\mathfrak{F}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{F}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\xi(\mathfrak{a})+\xi(\mathfrak{b})}\{\mathfrak{F}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{F}_{\mathfrak{C}_2}(\mathfrak{b})\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{F}_{\mathfrak{C}_1}(\mathfrak{a})\} \vee inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{F}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{\mathfrak{F}_{\xi(\mathfrak{C}_1)}(\mathfrak{m}) \vee \mathfrak{F}_{\xi(\mathfrak{C}_1)}(\mathfrak{n})\}$
$=\mathfrak{F}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta)$. □

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Introduction to SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra

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Florentin Smarandache (2022). Introduction to SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 3(2), 17-24. DOI: 10.52547/HATEF.JAHLA.3.2.2

Abstract

In this paper we recall our concepts of n^{th}-Power Set of a Set, SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation, Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra. In general, in any field of knowledge, one actually encounters SuperHyperStructures (or more accurately (m, n)-SuperHyperStructures).

1 Introduction

One recalls the SuperHyperAgebra and Neutrosophic SuperHyperAlgebra introduced and developed by Smarandache [16, 18, 19] between 2016–2022.

1. **Definition of classical HyperOperations:**

Let U be a universe of discourse and H be a non-empty set, $H \subset U$.
A **classical Binary HyperOperation** $\circ_2^*$ is defined as follows:

$$\circ_2^* : H^2 \to P_*(H),$$

where H is a discrete or continuous set, and $P_*(H)$ is the powerset of H without the empty-set $\emptyset$, or $P_*(H) = P(H) \setminus \{\emptyset\}$.

A **classical m-ary HyperOperation** $\circ_m^*$ is defined as:

$$\circ_m^* : H^m \to P_*(H),$$

for integer $m \geq 1$. For $m = 1$ one gets a **Unary HyperOperation.**
The **classical HyperStructures** are structures endowed with classical HyperOperations.
The classical HyperOperations and classical HyperStructures were introduced by F. Marty [12] in 1934.

2. **Definition of the n^{th}-Power Set of a Set:**
The n^{th}-Powerset of a Set was introduced in [16, 18, 19] in the following way:
$P^n(H)$, as the n^{th}-Powerset of the Set H, for integer $n \geq 1$, is recursively defined as:
$P^2(H) = P(P(H))$, $P^3(H) = P(P^2(H)) = P(P(P(H)))$, $\cdots$,
$P^n(H) = P(P^{(n-1)}(H))$, where $P^\circ(H) \stackrel{\text{def}}{=} H$, and $P^1(H) \stackrel{\text{def}}{=} P(H)$.

The n^{th}-Powerset of a Set better reflects our complex reality, since a set H (that may represent a group, a society, a country, a continent, etc.) of elements (such as: people, objects, and in general any items) is organized onto subsets $P(H)$, and these subsets are again organized onto subsets of subsets $P(P(H))$, and so on. That's our world.

3. **Neutrosophic HyperOperation and Neutrosophic HyperStructures [12]:**

In the classical HyperOperation and classical HyperStructures, the empty-set $\emptyset$ does not belong to the power set, or $P_*(H) = P(H) \setminus \{\emptyset\}$.
However, in the real world we encounter many situations when a HyperOperation $\circ$ is *indeterminate*, for example $a \circ b = \emptyset$ (unknown, or undefined),
or *partially indeterminate*, for example: $c \circ d = \{[0.2, 0.3], \emptyset\}$.
In our everyday life, there are many more operations and laws that have some degrees of indeterminacy (vagueness, unclearness, unknowingness, contradiction, etc.), than those that are totally determinate.
That's why in 2016 we have extended the classical HyperOperation to the Neutrosophic HyperOperation, by taking the whole power $P(H)$ (that includes the empty-set $\emptyset$ as well), instead of $P_*(H)$ (that does not include the empty-set $\emptyset$), as follows.

3.1 **Definition of Neutrosophic HyperOperation:**

Let U be a universe of discourse and H be a non-empty set, $H \subset U$.
A **Neutrosophic Binary HyperOperation** $\circ_2$ is defined as follows:

$$\circ_2 : H^2 \to P(H),$$

where H is a discrete or continuous set, and $P(H)$ is the powerset of H that includes the empty-set $\emptyset$.

A **Neutrosophic m-ary HyperOperation** $\circ_m$ is defined as:

$$\circ_m : H^m \to P(H),$$

for integer $m \geq 1$. Similarly, for $m = 1$ one gets a **Neutrosophic Unary HyperOperation.**

3.2 **Neutrosophic HyperStructures:**

A Neutrosophic HyperStructure is a structured endowed with Neutrosophic HyperOperations.

4. **Definition of SuperHyperOperations:**

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [16].

Let $P_*^n(H)$ be the n^{th}-powerset of the set H such that none of $P(H)$, $P^2(H)$, $\cdots$, $P^n(H)$ contain the empty set $\emptyset$.

Also, let $P^n(H)$ be the n^{th}-powerset of the set H such that at least one of the $P(H)$, $P^2(H)$,$\cdots$, $P^n(H)$ contain the empty set $\emptyset$.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has **classical-type SuperHyperOperations**, or $P^n(H)$ and in this case one has **Neutrosophic SuperHyperOperations**, for integer $n \geq 2$.

4.1 A **classical-type Binary SuperHyperOperation** $\circ^*_{(2,n)}$ is defined as follows:

$$\circ^*_{(2,n)} : H^2 \to P_*^n(H),$$

where $P_*^n(H)$ is the n^{th}-power set of the set H, with no empty-set.

4.2 **Examples of classical-type Binary SuperHyperOperation:**

1) Let $H = \{a, b\}$ be a finite discrete set; then its power set, without the empty-set $\emptyset$, is: $P(H) = \{a, b, \{a, b\}\}$, and:

$$P^2(H) = P(P(H)) = P(\{a, b, \{a, b\}\}) = \{a, b, \{a, b\}, \{a, \{a, b\}\}, \{b, \{a, b\}\}, \{a, b, \{a, b\}\}\},$$

$$\circ^*_{(2,2)} : H^2 \to P_*^2(H).$$

$\circ^*_{(2,2)}$	a	b
a	$\{a, \{a, b\}\}$	$\{b, \{a, b\}\}$
b	a	$\{a, b, \{a, b\}\}$

Table 1: Example 1 of classical-type Binary SuperHyperOperation

2) Let $H = [0, 2]$ be a continuous set.
$P(H) = P([0, 2]) = \{A \mid \ A \subseteq [0, 2],\ A = \text{subset}\}$,
$P^2(H) = P(P([0, 2]))$.
Let $c, d \in H$.

$$\circ^*_{(2,2)} : H^2 \to P_*^2(H).$$

$\circ^*_{(2,2)}$	c	d
c	$\{[0,0.5],[1,2]\}$	$\{0.7,0.9,1.8\}$
d	$\{2.5\}$	$\{(0.3,0.6),\{0.4,1.9\},2\}$

Table 2: Example 2 of classical-type Binary SuperHyperOperation

4.2 **Classical-type m-ary SuperHyperOperation** (or a more accurate denomination (m,n)**-SuperHyperOperation**)

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:

$$\circ^*_{(m,n)} : H^m \to P^n_*(H),$$

where the integers $m, n \geq 1$,

$$H^m = \underbrace{H \times H \times \cdots \times H}_{m \text{ times}},$$

and $P^n_*(H)$ is the n^{th}-powerset of the set H that includes the empty-set.

This SuperHyperOperation is an m-ary operation defined from the set H to the n^{th}-powerset of the set H.

4.3 **Neutrosophic m-ary SuperHyperOperation** (or more accurate denomination **Neutrosophic** (m,n)**-SuperHyperOperation**):

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:

$$\circ_{(m,n)} : H^m \to P^n(H),$$

where the integers $m, n \geq 1$,
and $P^n(H)$ is the n-th powerset of the set H that includes the empty-set.

5. **SuperHyperAxiom:**

A **classical-type SuperHyperAxiom** or more accurately a (m,n)**-SuperHyperAxiom** is an axiom based on classical-type SuperHyperOperations.

Similarly, a **Neutrosophic SuperHyperAxiom** (or Neutrosphic (m,n)-SuperHyperAxiom) is an axiom based on Neutrosophic SuperHyperOperations.

There are:

• **Strong SuperHyperAxioms**, when the left-hand side is equal to the right-hand side as in non-hyper axioms,

• and **Week SuperHyperAxioms**, when the intersection between the left-hand side and the right-hand side is non-empty.

For examples, one has:

• Strong SuperHyperAssociativity, when $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H^m$, where the law $\circ^*_{(m,n)} : H^m \to P^n_*(H)$;

• and Week SuperHyperAssociativity, when $[(x \circ y) \circ z] \cap [x \circ (y \circ z)] \neq \emptyset$, for all $x, y, z \in H^m$.

6. **SuperHyperAlgebra and SuperHyperStructure:**

A **SuperHyperAlgebra** or more accurately $(m-n)$**-SuperHyperAlgebra** is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a **Neutrosophic SuperHyperAlgebra** (or Neutrosphic (m, n)-SuperHyperAlgebra) is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperAxioms.

In general, we have **SuperHyperStructures** (or $(m-n)$-SuperHyperStructures), and corresponding **Neutrosophic SuperHyperStructures.**

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

7. **Distinction between SuperHyperAlgebra vs. Neutrosophic SuperHyperAlgebra:**

i. If none of the power sets $P^k(H)$, $1 \leq k \leq n$, do not include the empty set $\emptyset$, then one has a classical-type SuperHyperAlgebra;

ii. If at least one power set, $P^k(H)$, $1 \leq k \leq n$, includes the empty set $\emptyset$, then one has a Neutrosophic SuperHyperAlgebra.

8. **SuperHyperGraph** (or n**-SuperHyperGraph):**

The SuperHyperAlgebra resembles the n-SuperHyperGraph [17, 18, 19], introduced by Smarandache in 2019, defined as follows:

8.1 **Definition of the n-SuperHyperGraph:**

Let $V = \{v_1, v_2, \cdots, v_m\}$, for $1 \leq m \leq \infty$, be a set of vertices, that contains Single Vertices (the classical ones), Indeterminate Vertices (unclear, vague, partially known), and Null Vertices (totally unknown, empty).

Let $P(V)$ be the power of set V, that includes the empty set $\emptyset$, too.
Then $P^n(V)$ be the n-powerset of the set V, defined in a recurent way, i.e.:
$P(V)$, $P^2(V) = P(P(V))$, $P^3(V) = P(P^2(V)) = P(P(P(V)))$, $\cdots$,
$P^n(V) = P(P^{(n-1)}(V))$, for $1 \leq n \leq \infty$, where by definition $P^0(V) \stackrel{\text{def}}{=} V$.

Then, the **n-SuperHyperGraph (n-SHG)** is an ordered pair:

$$\text{n-SHG} = (G_n, E_n),$$

where $G_n \subseteq P^n(V)$, and $E_n \subseteq P^n(V)$, for $1 \leq n \leq \infty$.
G_n is the set of vertices, and E_n is the set of edges.

The **set of vertices** G_n contains the following types of vertices:

■ *Singles Vertices* (the classical ones);

■ *Indeterminate Vertices* (unclear, vagues, partially unkwnown);

■ *Null Vertices* (totally unknown, empty);
and:

■ *SuperVertex* (or SubsetVertex), i.e. two ore more (single, indeterminate, or null) vertices put together as a group (organization).

■ *n-SuperVertex* that is a collection of many vertices such that at least one is a $(n-1)$-SuperVertex and all other r-SuperVertices into the collection, if any, have the order $r \leq n-1$.

The **set of edges** E_n contains the following types of edges:

■ Singles Edges (the classical ones);

■ *Indeterminate Edges* (unclear, vague, partially unknown);

■ *Null Edges* (totally unknown, empty);
and:

■ *HyperEdge* (connecting three or more single vertices);

■ *SuperEdge* (connecting two vertices, at least one of them being a SuperVertex);

■ *n-SuperEdge* (connecting two vertices, at least one being an n-SuperVertex, and the other of order r-SuperVertex, with $r \leq n$);

■ *SuperHyperEdge* (connecting three or more vertices, at least one being a SuperVertex);

■ *n-SuperHyperEdge* (connecting three or more vertices, at least one being an n-SuperVertex, and the other r-SuperVertices with $r \leq n$;

■ *MultiEdges* (two or more edges connecting the same two vertices);

■ *Loop* (and edge that connects an element with itself).
and:

■ *Directed Graph* (classical one);

■ *Undirected Graph* (classical one);

■ *Neutrosophic Directed Graph* (partially directed, partially undirected, partially indeterminate direction).

2 Conclusion

We recalled the most general form of algebras, called SuperHyperAlgebra (or more accurate denomination (m, n)-SuperHyperAlgebra) and the Neutrososophic SuperHyperAlgebra, and their extensions to SuperHyperStructures and respectively Neutrosophic SuperHyperAlgebra in any field of knowledge.

They are based on the n^{th}-Powerset of a Set, which better reflects our complex reality, since a set H (that may represent a group, a society, a country, a continent, etc.) of elements (such as: people, objects, and in general any items) is organized onto subsets $P(H)$, and these subsets are again organized onto subsets of subsets $P(P(H))$, and so on. That's our world.

Hoping that this new field of SuperHyperAlgebra will inspire researchers to studying several interesting particular cases, such as the SuperHyperGroupoid, SuperHyperMonoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

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On Single Valued Neutrosophic Regularity Spaces

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Yaser Saber, Fahad Alsharari, Florentin Smarandache, Mohammed Abdel-Sattar (2022). On Single Valued Neutrosophic Regularity Spaces. *Computer Modeling in Engineering & Sciences*, 130(3), 1625-1648; DOI: 10.32604/cmes.2022.017782

ABSTRACT

This article aims to present new terms of single-valued neutrosophic notions in the Šostak sense, known as single-valued neutrosophic regularity spaces. Concepts such as r-single-valued neutrosophic semi £-open, r-single-valued neutrosophic pre-£-open, r-single valued neutrosophic regular-£-open and r-single valued neutrosophic α£-open are defined and their properties are studied as well as the relationship between them. Moreover, we introduce the concept of r-single valued neutrosophic θ£-cluster point and r-single-valued neutrosophic γ£-cluster point, r-θ£-closed, and θ£-closure operators and study some of their properties. Also, we present and investigate the notions of r-single-valued neutrosophic θ£-connectedness and r-single valued neutrosophic δ£-connectedness and investigate relationship with single-valued neutrosophic almost £-regular. We compare all these forms of connectedness and investigate their properties in single-valued neutrosophic semiregular and single-valued neutrosophic almost regular in neutrosophic ideal topological spaces in Šostak sense. The usefulness of these concepts are incorporated to multiple attribute groups of comparison within the connectedness and separateness of θ£ and δ£.

KEYWORDS

Single valued neutrosophic θ£-closed; single valued neutrosophic θ£-separated; single valued neutrosophic δ£-separated; single-valued neutrosophic δ£-connected; single valued neutrosophic δ£-connected; single valued neutrosophic almost £-egular

1 Introduction

A neutrosophic set can be practical in addressing problems with indeterminate, imperfect, and inconsistent materials. The concept of neutrosophic set theory was introduced by Smarandache [1] as a new mathematical method that corresponds to the indeterminacy degree (uncertainty, etc.). Bakbak et al. [2] and Mishra et al. [3] applied the soft set theory successfully applied in several areas, such as the smoothness of functions, as well as architecture-based, neuro-linguistic programming. Wang et al. [4] proposed single-valued neutrosophic sets (SVNSs). Meanwhile, Kim et al. [5,6] inspected the single valued neutrosophic relations (SVNRs) and symmetric closure of SVNR, respectively. Recently, Saber et al. [7–9] introduced the concepts of single-valued neutrosophic ideal open local function and single-valued neutrosophic topological space. Many of their applications appear in the studies of Das et al. [10]. Alsharari et al. [11–13]. Riaz et al. [14]. Salama et al. [15–17]. Hur et al. [18,19]. Yang et al. [20]. El-Gayyar [21], AL-Nafee et al. [22]. Muhiuddin et al. [23,24] and Mukherjee et al. [25].

First, we define single-valued neutrosophic θ£-closed and single-valued neutrosophic δ£-closed sets as well as some of their core properties. We also present and explore the properties and characterizations of single valued neutrosophic operators namely θ£-closure ($\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$) and δ£-closure ($\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$) in the single valued neutrosophic ideal topological space $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$. We then define the concept of single valued neutrosophic regularity spaces. Next, we study single-valued neutrosophic θ£-separated and single-valued neutrosophic δ£-separated with giving some definitions and theorems. Furthermore, we also introduce single-valued neutrosophic θ£-connected and single valued neutrosophic δ£-connected relying on the single valued neutrosophic θ£-closure and δ£-closure operators.

We define a fixed universe $\tilde{\mathcal{F}}$ to be a finite set of objects and ζ a closed unit interval [0, 1]. Additionally, we denote $\zeta^{\tilde{\mathcal{F}}}$ as the set of all single-valued neutrosophic subsets of $\tilde{\mathcal{F}}$.

2 Preliminaries

This section provides a complete survey, some previous studies, and concepts associated with this study.

Definition 1. [1] Let $\tilde{\mathcal{F}}$ be a non-empty set. A neutrosophic set (briefly, $\mathcal{NS}$) in $\tilde{\mathcal{F}}$ is an object having the form $\alpha_n = \{\langle \upsilon, \tilde{\varrho}_{\alpha_n}(\upsilon), \tilde{\sigma}_{\sigma_n}(\omega), \tilde{\varsigma}_{\alpha_n}(\upsilon)\rangle : \upsilon \in \tilde{\mathcal{F}}\}$ where

$$\tilde{\varrho} : \tilde{\mathcal{F}} \to \rfloor^{-}\mathbf{0}, \mathbf{1}^{+}\lfloor, \tilde{\sigma} : \tilde{\mathcal{F}} \to \rfloor^{-}\mathbf{0}, \mathbf{1}^{+}\lfloor, \tilde{\varsigma} : \tilde{\mathcal{F}} \to \rfloor^{-}\mathbf{0}, \mathbf{1}^{+}\lfloor \text{ and } {}^{-}\mathbf{0} \leq \tilde{\varrho}_{\alpha_n}(\upsilon) + \tilde{\sigma}_{\alpha_n}(\upsilon) + \tilde{\varsigma}_{\alpha_n}(\upsilon) \leq \mathbf{3}^{+} \quad (1)$$

Represent the degree of membership ($\tilde{\varrho}_{\alpha_n}$), the degree of indeterminacy ($\tilde{\sigma}_{\alpha_n}$), and the degree of non-membership ($\tilde{\varsigma}_{\alpha_n}$) respectively of any $\upsilon \in \tilde{\mathcal{F}}$ to the set α_n.

Definition 2. [4] Suppose that $\tilde{\mathcal{F}}$ is a universal set a space of points (objects), with a generic element in $\tilde{\mathcal{F}}$ denoted by υ. Then α_n is called a single valued neutrosophic set (briefly, $\mathcal{SVNS}$) in $\tilde{\mathcal{F}}$, if α_n has the form $\alpha_n = \{\langle \upsilon, \tilde{\varrho}_{\alpha_n}(\upsilon), \tilde{\sigma}_{\alpha_n}(\upsilon), \tilde{\varsigma}_{\alpha_n}(\upsilon)\rangle : \upsilon \in \tilde{\mathcal{F}}\}$. Now, $\tilde{\varrho}_{\alpha_n}, \tilde{\sigma}_{\sigma_n}, \tilde{\varsigma}_{\alpha_n}$ indicate the degree of non-membership, the degree of indeterminacy, and the degree of membership, respectively of any $\upsilon \in \tilde{\mathcal{F}}$ to the set α_n.

Definition 3. [4] Let $\alpha_n = \{\langle \upsilon, \tilde{\varrho}_{\alpha_n}(\upsilon), \tilde{\sigma}_{\sigma_n}(\upsilon), \tilde{\varsigma}_{\alpha_n}(\upsilon)\rangle : \upsilon \in \tilde{\mathcal{F}}\}$ be an SVNS on $\tilde{\mathcal{F}}$. The complement of the set α_n (briefly, α_n^c) defined as follows: $\tilde{\varrho}_{\alpha_n^c}(\upsilon) = \tilde{\varsigma}_{\alpha_n}(\upsilon), \tilde{\sigma}_{\alpha_n}(\upsilon) = [\tilde{\sigma}_{\alpha_n}]^c(\upsilon), \tilde{\varsigma}_{\alpha_n^c}(\upsilon) = \tilde{\varrho}_{\alpha_n}(\upsilon)$.

Definition 4. [26] Let $\tilde{\mathcal{F}}$ be a non-empty set and $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ be in the form: $\alpha_n = \{\langle \upsilon, \tilde{\varrho}_{\alpha_n}(\upsilon), \tilde{\sigma}_{\alpha_n}(\upsilon), \tilde{\varsigma}_{\alpha_n}(\upsilon)\rangle : \upsilon \in \tilde{\mathcal{F}}\}$ and $\varepsilon_n = \{\langle \upsilon, \tilde{\varrho}_{\varepsilon_n}(\upsilon), \tilde{\sigma}_{\varepsilon_n}(\upsilon), \tilde{\varsigma}_{\varepsilon_n}(\upsilon)\rangle : \upsilon \in \tilde{\mathcal{F}}\}$ on $\tilde{\mathcal{F}}$ then,

(a) $\alpha_n \subseteq \varepsilon_n$ for every $\upsilon \in \tilde{\mathcal{F}}$; $\tilde{\varrho}_{\alpha_n}(\upsilon) \leq \tilde{\varrho}_{\varepsilon_n}(\upsilon), \tilde{\sigma}_{\alpha_n}(\upsilon) \geq \tilde{\sigma}_{\varepsilon_n}(\upsilon), \tilde{\varsigma}_{\alpha_n}(\upsilon) \geq \tilde{\varsigma}_{\varepsilon_n}(\upsilon)$.
(b) $\alpha_n = \varepsilon_n$ iff $\sigma_n \subseteq \varepsilon_n$ and $\sigma_n \supseteq \varepsilon_n$.
(c) $\tilde{0} = \langle 0, 1, 1\rangle$ and $\tilde{1} = \langle 1, 0, 0\rangle$.

Definition 5. [20] Let $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Then,

(a) $\alpha_n \cap \varepsilon_n$ is an SVNS, if for every $\upsilon \in \tilde{\mathcal{F}}$,

$$\boldsymbol{\alpha_n \cap \varepsilon_n = \langle \left(\tilde{\varrho}_{\alpha_n} \cap \tilde{\varrho}_{\varepsilon_n}\right)(\upsilon), \left(\tilde{\sigma}_{\alpha_n} \cup \tilde{\sigma}_{\varepsilon_n}\right)(\upsilon), \left(\tilde{\varsigma}_{\alpha_n} \cup \tilde{\varsigma}_{\varepsilon_n}\right)(\upsilon)\rangle}, \tag{2}$$

where, $(\tilde{\varrho}_{\alpha_n} \cap \tilde{\varrho}_{\varepsilon_n})(\upsilon) = \tilde{\varrho}_{\alpha_n}(\upsilon) \cap \tilde{\varrho}_{\varepsilon_n}(\upsilon)$ and $(\tilde{\varsigma}_{\alpha_n} \cup \tilde{\varsigma}_{\varepsilon_n})(\upsilon) = \tilde{\varsigma}_{\alpha_n}(\upsilon) \cup \tilde{\varsigma}_{\varepsilon_n}(\upsilon)$, for all $\upsilon \in \tilde{\mathcal{F}}$,

(b) $\alpha_n \cup \varepsilon_n$ is an SVNS, if for every $\upsilon \in \tilde{\mathcal{F}}$,

$$\boldsymbol{\alpha_n \cup \varepsilon_n = \langle \left(\tilde{\varrho}_{\alpha_n} \cup \tilde{\varrho}_{\varepsilon_n}\right)(\upsilon), \left(\tilde{\sigma}_{\alpha_n} \cap \tilde{\sigma}_{\varepsilon_n}\right)(\upsilon), \left(\tilde{\varsigma}_{\alpha_n} \cap \tilde{\varsigma}_{\varepsilon_n}\right)(\upsilon)\rangle}. \tag{3}$$

Definition 6. [15] For an any arbitrary family $\{\alpha_n\}_{i \in j} \in \zeta^{\tilde{\mathcal{F}}}$ of SVNS the union and intersection are given by

(a) $\bigcap_{i \in j} [\alpha_n]_i = \langle \cap_{i \in j} \tilde{\varrho}_{[\alpha_n]_i}(\upsilon), \cup_{i \in j} \tilde{\sigma}_{[\alpha_n]_i}(\upsilon), \cup_{i \in j} \tilde{\varsigma}_{[\alpha_n]_i}(\upsilon)\rangle$,
(b) $\bigcup_{i \in j} [\alpha_n]_i = \langle \cup_{i \in j} \tilde{\varrho}_{[\alpha_n]_i}(\upsilon), \cap_{i \in j} \tilde{\sigma}_{[\alpha_n]_i}(\upsilon), \cap_{i \in j} \tilde{\varsigma}_{[\alpha_n]_i}(\upsilon)\rangle$.

Definition 7. [21] A single-valued neutrosophic topological spaces is an ordered $(\tilde{\mathcal{F}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\varsigma}})$ where $\tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\varsigma}} : \zeta^{\tilde{\mathcal{F}}} \to \zeta$ is a mapping satisfying the following axioms:

(SVNT1) $\tilde{\tau}^{\tilde{\varrho}}(\tilde{0}) = \tilde{\tau}^{\tilde{\varrho}}(\tilde{1}) = \tilde{\tau}^{\tilde{\sigma}}(\tilde{0}) = \tilde{\tau}^{\tilde{\sigma}}(\tilde{1}) = 0$ and $\tilde{\tau}^{\tilde{\varsigma}}(\tilde{0}) = \tilde{\tau}^{\tilde{\varsigma}}(\tilde{1}) = 1$.

(SVNT2) $\tilde{\tau}^{\tilde{\varrho}}(\alpha_n \cap \varepsilon_n) \geq \tilde{\tau}^{\tilde{\varrho}}(\alpha_n) \cap \tilde{\tau}^{\tilde{\varrho}}(\varepsilon_n), \tilde{\tau}^{\tilde{\sigma}}(\alpha_n \cap \varepsilon_n) \leq \tau^{\tilde{\sigma}}(\alpha_n) \cup \tilde{\tau}^{\tilde{\sigma}}(\varepsilon_n), \tilde{\tau}^{\tilde{\varsigma}}(\alpha_n \cap \varepsilon_n) \leq \tilde{\tau}^{\tilde{\varsigma}}(\alpha_n) \cup \tilde{\tau}^{\tilde{\varsigma}}(\varepsilon_n)$ for every, $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$

(SVNT3) $\tilde{\tau}^{\tilde{\varrho}}(\cup_{j \in \Gamma}[\alpha_n]_j) \geq \cap_{j \in \Gamma} \tilde{\tau}^{\tilde{\varrho}}([\alpha_n]_j), \tilde{\tau}^{\tilde{\sigma}}(\cup_{i \in \Gamma}[\alpha_n]_j) \leq \cup_{j \in \Gamma} \tilde{\tau}^{\tilde{\sigma}}([\alpha_n]_j), \tilde{\tau}^{\tilde{\varsigma}}(\cup_{j \in \Gamma}[\alpha_n]_j) \leq \cup_{j \in \Gamma} \tilde{\tau}^{\tilde{\varsigma}}([\alpha_n]_j)$, for every $[\alpha_n]_j \in \zeta^{\tilde{\mathcal{F}}}$.

The quadruple $(\tilde{\mathcal{F}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\varsigma}})$ is called a single-valued neutrosophic topological spaces (briefly, SVNT, for short). Occasionally write $\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ for $(\tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\varsigma}})$ and it will cause no ambiguity.

Definition 8. [7] Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an **SVNTS.** Then, for every $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$. Then the single valued neutrosophic closure and single valued neutrosophic interior of α_n are define by:

$$\mathbf{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\boldsymbol{\alpha_n, r}) = \bigcap \{\boldsymbol{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} : \alpha_n \leq \varepsilon_n, \tau^{\tilde{\varrho}}([\varepsilon_n]^c) \geq r, \tau^{\tilde{\sigma}}([\varepsilon_n]^c) \leq 1 - r, \tau^{\tilde{\varsigma}}([\varepsilon_n]^c) \leq 1 - r}\} \tag{4}$$

$$\mathbf{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\boldsymbol{\alpha_n, r}) = \bigcup \{\boldsymbol{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} : \alpha_n \geq \varepsilon_n, \tau^{\tilde{\varrho}}(\varepsilon_n) \geq r, \tau^{\tilde{\sigma}}(\varepsilon_n) \leq 1 - r, \tau^{\tilde{\varsigma}}(\varepsilon_n) \leq 1 - r}\} \tag{5}$$

Definition 9. [7] Let $(\tilde{\mathcal{F}})$ be a nonempty set and $\upsilon \in \tilde{\mathcal{F}}$, let $s \in (0, 1]$, $t \in [0, 1)$ and $k \in [0, 1)$, then the single-valued neutrosophic point $x_{s,t,k}$ in $\tilde{\mathcal{F}}$ given by

$$x_{s,t,k}(\upsilon) = \begin{cases} (s, t, k), & \text{if } x = \upsilon \\ (0, 1, 1), & \text{otherwise.} \end{cases}$$

We define that, $x_{s,t,p} \in \alpha_n$ iff $s < \tilde{\varrho}_{\alpha_n}(\upsilon)$, $t \geq \tilde{\sigma}_{\alpha_n}(\upsilon)$ and $k \geq \tilde{\tilde{\varsigma}}_{\alpha_n}(\upsilon)$. We indicate the set of all single-valued neutrosophic points in $\tilde{\mathcal{F}}$ as $\mathrm{P}_{x_{s,t,k}}(\tilde{\mathcal{F}})$. A single-valued neutrosophic set α_n is said to be quasi-coincident with another single-valued neutrosophic set ε_n, denoted by $\alpha_n q \varepsilon_n$, if there exists an element $\upsilon \in \tilde{\mathcal{F}}$ such that $\tilde{\varrho}_{\alpha_n}(\upsilon) + \tilde{\varrho}_{\varepsilon_n}(\upsilon) > 1, \tilde{\sigma}_{\alpha_n}(\upsilon) + \tilde{\sigma}_{\varepsilon_n}(\upsilon) \leq 1, \tilde{\varsigma}_{\alpha_n}(\upsilon) + \tilde{\varsigma}_{\varepsilon_n}(\upsilon) \leq 1$.

Definition 10. [7] A mapping $£^{\tilde{\varrho}}, £^{\tilde{\sigma}}, £^{\tilde{\varsigma}}: \zeta^{\tilde{\mathcal{F}}} \to \zeta$ is called single-valued neutrosophic ideal (SVNI) on $\tilde{\mathcal{F}}$ if, it satisfies the following conditions:

($£_1$) $£^{\tilde{\varrho}}(\tilde{0}) = 1$ and $£^{\tilde{\sigma}}(\tilde{0}) = £^{\tilde{\varsigma}}(\tilde{0}) = 0$.

($£_2$) If $\sigma_n \leq \gamma_n$, then $£^{\tilde{\varrho}}(\varepsilon_n) \leq £^{\tilde{\varrho}}(\alpha_n)$, $£^{\tilde{\sigma}}(\varepsilon_n) \geq £^{\tilde{\sigma}}(\alpha_n)$ and $£^{\tilde{\varsigma}}(\varepsilon_n) \geq £^{\tilde{\varsigma}}(\alpha_n)$, for $\varepsilon_n, \alpha_n \in \zeta^{\tilde{\mathcal{F}}}$.

($£_3$) $£^{\tilde{\varrho}}(\alpha_n \cup \varepsilon_n) \geq £^{\tilde{\varrho}}(\alpha_n) \cap £^{\tilde{\varrho}}(\varepsilon_n)$, $£^{\tilde{\sigma}}(\alpha_n \cup \varepsilon_n) \leq £^{\tilde{\sigma}}(\alpha_n) \cup £^{\tilde{\sigma}}(\varepsilon_n)$ and $£^{\tilde{\varsigma}}(\alpha_n \cup \varepsilon_n) \leq £^{\tilde{\varsigma}}(\alpha_n) \cup £^{\tilde{\varsigma}}(\varepsilon_n)$, for $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$.

The tribal $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is called a single valued neutrosophic ideal topological space in Šostak sense (briefly, SVNITS).

Definition 11. [7] Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS for each $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. Then, the single valued neutrosophic ideal open local function $[\alpha_n]_r^{\odot}(\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ of α_n is the union of all single-valued neutrosophic points $x_{s,t,k}$ such that if $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ and $£^{\tilde{\varrho}}(\omega_n) \geq r$, $£^{\tilde{\sigma}}(\omega_n) \leq 1-r$, $£^{\tilde{\varsigma}}(\omega_n) \leq 1-r$, then there is at least one $\upsilon \in \tilde{\mathcal{F}}$ for which

$$\tilde{\varrho}_{\alpha_n}(\upsilon) + \tilde{\varrho}_{\varepsilon_n}(\upsilon) - 1 > \tilde{\varrho}_{\omega_n}(\upsilon), \quad \tilde{\sigma}_{\alpha_n}(\upsilon) + \tilde{\sigma}_{\varepsilon_n}(\upsilon) - 1 \leq \tilde{\sigma}_{\omega_n}(\upsilon), \quad \tilde{\varsigma}_{\alpha_n}(\upsilon) + \tilde{\varsigma}_{\varepsilon_n}(\upsilon) - 1 \leq \tilde{\varsigma}_{\omega_n}(\upsilon) \quad (6)$$

Occasionally, we will write $[\alpha_n]_r^{\odot}$ for $[\alpha_n]_r^{\odot}(\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ herein to avoid ambiguity.

Remark 1. [7] Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. Hence, we can write

$$\mathbf{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \alpha_n \cup [\alpha_n]_r^{\odot}, \quad \mathbf{int}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \alpha_n \cap [(\alpha_n^c)_r^{\odot}]^c \quad (7)$$

Clearly, $\mathrm{CI}^{\odot}_{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ is a single-valued neutrosophic closure operator and $(\tau^{\tilde{\varrho}\odot}(£), \tau^{\tilde{\sigma}\odot}(£), \tau^{\tilde{\varsigma}\odot}(£))$ is the single-valued neutrosophic topology generated by $\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$, i.e., $\tau^{\odot}(\mathcal{J})(\alpha_n) = \bigcup\{r | \mathrm{CI}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n^c, r) = \alpha_n^c\}$.

Theorem 1. [7] Let $\{[\alpha_n]_i\}_{i \in J} \subset \zeta^{\tilde{\mathcal{F}}}$ be a family of single-valued neutrosophic sets on $\tilde{\mathcal{F}}$ and $(\tilde{\mathcal{F}}, \tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be a SVNITS. Then,

(a) $(\cup([\alpha_n]_i)_r^{\odot} : i \in J) \leq (\cup[\alpha_n]_i : i \in j)_r^{\odot}$,
(b) $(\cap([\alpha_n]_i) : i \in j)_r^{\odot} \geq (\cap([\alpha_n]_i)_r^{\odot} : i \in J)$.

Theorem 2. [7] Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS and $r \in \zeta, \alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Then,

(a) $\mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n \vee \varepsilon_n, r) \leq \mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \vee \mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$,

(b) $\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq \mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq \alpha_n \leq \mathrm{CI}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$,

(c) $\mathrm{CI}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]^c, r) = [\mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)]^c$,

(d) $[\mathrm{CI}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)]^c = \mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]^c, r)$,

(e) $\mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n \wedge \varepsilon_n, r) = \mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \wedge \mathrm{int}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$.

Definition 12. [8] Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. For every $\alpha_n, \varepsilon_n, \omega_n \in \zeta^{\tilde{\mathcal{F}}}$, α_n and ε_n are called r-single-valued neutrosophic separated if for $r \in \zeta_0$,

$$\mathbf{CI}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \cap \varepsilon_n = \mathbf{CI}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \cap \alpha_n = \tilde{0} \quad (8)$$

An $\mathcal{SVNS}$, ω_n is called r-single-valued neutrosophic connected if r-$\mathcal{SVNSEP}$ $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} - \{\tilde{0}\}$ such that $\omega_n = \alpha_n \cup \varepsilon_n$ does not exist. A $\mathcal{SVNS}$ α_n is said to be r-single-valued neutrosophic connected if it is r-single-valued neutrosophic connected for any $r \in \zeta_0$. A $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is said to be r-single-valued neutrosophic connected if $\tilde{1}$ is r-single-valued neutrosophic connected.

3 Single Valued Neutrosophic δ£-Cluster Point and Single Valued Neutrosophic θ£-Cluster Point

In this section, we introduce the r-single-valued neutrosophic δ£-cluster point (abbreviated *SVN*δ£-cluster point) and r-single-valued neutrosophic £-closed set (abbreviated **SVN£C**). Furthermore, we analyze the single-valued neutrosophic δ£-closure operator (δ£-closure operator for brevity) and single-valued neutrosophic θ£-closure operator (θ£-closure operator for brevity).

Definition 13. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $r \in \zeta_0$. Then,

(a) α_n is said to be r-single valued neutrosophic £-open (briefly, *r-SVN£O*), if and only if $\alpha_n \leq \mathrm{int}_{\tilde{\tau}\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^{\odot}_r, r)$,
(b) α_n is said to be r-single valued neutrosophic semi-£-open (briefly, *r-SVNS£O*) if and only if $\alpha_n \leq \mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^{\odot}_r, r), r)$,
(c) α_n is called r-single valued neutrosophic pre-£-open (briefly, *r-SVNP£O*) if and only if $\alpha_n \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^{\odot}_r, r), r)$,
(d) α_n is called r-single valued neutrosophic regular-£-open (briefly, *r-SVNR£O*) if and only if $\alpha_n = \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^{\odot}_r, r), r)$,
(e) α_n is said to be r-single valued neutrosophic α£-open (briefly, r-*SVNα£O*) if and only if $\alpha_n \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}_{\tilde{\tau}\tilde{\varrho}}\left([\alpha_n]^{\odot}_r, r\right), r\right)$,
(f) α_n is said to be r-single valued neutrosophic $\star$-open set (briefly, *r-SVN* $\star$ O) if and only if $\alpha_n = \mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

The complement of an r−SVN£O (resp, r-SVNS£O, r-SVNP£O, r-SVNR£O, r-SVNα£O, r-SVN$\star$O) is said to be an r−SVN£C (resp, r-SVNS£C, r-SVNP£C, r-SVNR£C, r-SVNα£C, r-SVN$\star$C) respectively.

Remark 2. r-single valued neutrosophic open set (r − $\mathcal{SVN}$O) and r-SVN£O are independent notions as shown by the following example.

Example 1. Let $\tilde{\mathcal{F}} = \{a, b, c\}$ be a set. Define $\varepsilon_n, \pi_n, \omega_n \in \zeta^{\tilde{\mathcal{F}}}$ as follows:

$\varepsilon_n = \langle (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3) \rangle$; $\pi_n = \langle (0.4, 0.4, 0.4), (0.4, 0.4, 0.4), (0.4, 0.4, 0.4) \rangle$,

$\omega_n = \langle (0.5, 0.5, 0.5), (0.2, 0.2, 0.2), (0.1, 0.1, 0.1) \rangle$.

We define an SVNITS$(\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ on $\tilde{\mathcal{F}}$ as follows: for each $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$,

$$\tilde{\tau}^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & \mathit{if}\ \alpha_n = \left\{\tilde{0}, \tilde{1}\right\}, \\ \frac{2}{3}, & \mathit{if}\ \alpha_n = \{\varepsilon_n, \pi_n\}, \\ 0, & \mathit{otherwise}, \end{cases} \quad £^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & \mathit{if}\ \alpha_n = \tilde{0}, \\ \frac{2}{3}, & \mathit{if}\ 0 < \alpha_n \leq \omega_n \\ 0, & \mathit{otherwise}, \end{cases}$$

$$\tilde{\tau}^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & if\ \alpha_n = \left\{\tilde{0}, \tilde{1}\right\}, \\ \frac{1}{3}, & if\ \alpha_n = \{\varepsilon_n, \pi_n\}, \\ 1, & otherwise, \end{cases} \quad \pounds^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & if\ \alpha_n = \tilde{0}, \\ \frac{1}{3}, & if\ 0 < \alpha_n \leq \omega_n, \\ 1, & otherwise, \end{cases}$$

$$\tilde{\tau}^{\tilde{\varsigma}}(\alpha_n) = \begin{cases} 0, & if\ \alpha_n = \left\{\tilde{0}, \tilde{1}\right\}, \\ \frac{1}{3}, & if\ \alpha_n = \{\varepsilon_n, \pi_n\}, \\ 1, & otherwise, \end{cases} \quad \pounds^{\tilde{\varsigma}}(\alpha_n) = \begin{cases} 0, & if\ \alpha_n = \tilde{0}, \\ \frac{1}{3}, & if\ 0 < \alpha_n \leq \omega_n, \\ 1, & otherwise. \end{cases}$$

Based on $\varepsilon_n = \langle(0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3)\rangle$, it's clear that, $\frac{2}{3} - \mathcal{SVNO}$ is set because $\tau^{\tilde{\varrho}}(\langle(0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3)\rangle) \geq \frac{2}{3}, \tau^{\tilde{\sigma}}(\langle(0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3)\rangle) \leq \frac{1}{3}, \tau^{\tilde{\varsigma}}(\langle(0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3)\rangle) \leq \frac{1}{3}$.

However ε_n is not an r-SVN£O set, and for that, we must prove that $\varepsilon_n \nleq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^{\odot}_{\frac{2}{3}}, \frac{2}{3})$. So, we must first obtain $[\varepsilon_n]^{\odot}_{\frac{2}{3}}$. Based on Eq. (11), $\tilde{1}, \varepsilon_n, \pi_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, \frac{2}{3})$ and $\pounds^{\tilde{\varrho}}(\langle(0.5, 0.5, 0.5), (0.2, 0.2, 0.2), (0.1, 0.1, 0.1)\rangle) \geq \frac{2}{3}$, $\pounds^{\tilde{\sigma}}(\langle(0.5, 0.5, 0.5), (0.2, 0.2, 0.2), (0.1, 0.1, 0.1)\rangle) \leq \frac{1}{3}$, $\pounds^{\tilde{\varsigma}}(\langle(0.5, 0.5, 0.5), (0.2, 0.2, 0.2), (0.1, 0.1, 0.1)\rangle) \leq \frac{1}{3}$,
such that by using Eqs. (2), (3) and (6) we obtain,

$$\tilde{\varrho}_{\varepsilon_n}(\upsilon) + \tilde{\varrho}_{\tilde{1}}(\nu) - 1 > \tilde{\varrho}_{\omega_n}(\upsilon), \tilde{\sigma}_{\varepsilon_n}(\upsilon) + \tilde{\sigma}_{\tilde{1}}(\upsilon) - 1 \leq \tilde{\sigma}_{\omega_n}(\upsilon), \tilde{\varsigma}_{\varepsilon_n}(\upsilon) + \tilde{\varsigma}_{\tilde{1}}(\upsilon) - 1 \leq \tilde{\varsigma}_{\omega_n}(\upsilon).$$

$$(0.3, 0.3, 0.3)(\upsilon) + (1, 1, 1)(\nu) - 1 \ngtr (0.5, 0.5, 0.5)(\upsilon),$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0, 0, 0)(\upsilon) - 1 \leq (0.2, 0.2, 0.2)(\upsilon),$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0, 0, 0)(\upsilon) - 1 \leq (0.1, 0.1, 0.1)(\upsilon),$$

$$\tilde{\varrho}_{\varepsilon_n}(\upsilon) + \tilde{\varrho}_{\pi_n}(\nu) - 1 > \tilde{\varrho}_{\omega_n}(\upsilon), \tilde{\sigma}_{\varepsilon_n}(\upsilon) + \tilde{\sigma}_{\pi_n}(\upsilon) - 1 \leq \tilde{\sigma}_{\omega_n}(\upsilon), \tilde{\varsigma}_{\varepsilon_n}(\upsilon) + \tilde{\varsigma}_{\pi_n}(\upsilon) - 1 \leq \tilde{\varsigma}_{\omega_n}(\upsilon).$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0.4, 0.4, 0.4)(\nu) - 1 \ngtr (0.5, 0.5, 0.5)(\upsilon),$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0.4, 0.4, 0.4)(\upsilon) - 1 \leq (0.2, 0.2, 0.2)(\upsilon),$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0.4, 0.4, 0.4)(\upsilon) - 1 \leq (0.1, 0.1, 0.1)(\upsilon)$$

$$\tilde{\varrho}_{\varepsilon_n}(\upsilon) + \tilde{\varrho}_{\varepsilon_n}(\nu) - 1 > \tilde{\varrho}_{\omega_n}(\upsilon), \tilde{\sigma}_{\varepsilon_n}(\upsilon) + \tilde{\sigma}_{\varepsilon_n}(\upsilon) - 1 \leq \tilde{\sigma}_{\omega_n}(\upsilon), \tilde{\varsigma}_{\varepsilon_n}(\upsilon) + \tilde{\varsigma}_{\varepsilon_n}(\upsilon) - 1 \leq \tilde{\varsigma}_{\omega_n}(\upsilon).$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0.3, 0.3, 0.3)(\nu) - 1 \ngtr (0.5, 0.5, 0.5)(\upsilon),$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0.3, 0.3, 0.3)(\upsilon) - 1 \leq (0.2, 0.2, 0.2)(\upsilon),$$

$$(0.3, 0.3, 0.3)(\upsilon) + (0.3, 0.3, 0.3)(\upsilon) - 1 \leq (0.1, 0.1, 0.1)(\upsilon)$$

Therefore, $[\varepsilon_n]^{\odot}_{\frac{2}{3}} = \tilde{0}$. Subsequently, using Eq. (7) we obtain $\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^{\odot}_{\frac{2}{3}}, \frac{2}{3}) = int_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\tilde{0}, \frac{2}{3}) = \tilde{0}$, which implies that

$$\langle(0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3)\rangle = \varepsilon_n \nleq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left([\varepsilon_n]^{\odot}_{\frac{2}{3}}, \frac{2}{3}\right) = \tilde{0}.$$

Hence, ε_n is not an r-SVN£O set.

Definition 14. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $x_{s,t,k} \in \mathrm{P}_{s,t,k}(\tilde{\mathcal{F}})$ and $r \in \zeta_0$. Then,

(a) α_n is an r-single valued neutrosophic $Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$-neighborhood of $x_{s,t,k}$ if $x_{s,t,k} q \alpha_n$ with $\tau^{\tilde{\varrho}}(\alpha_n) \geq r, \tau^{\tilde{\sigma}}(\alpha_n) \leq 1-r, \tau^{\tilde{\varsigma}}(\alpha_n) \leq 1-r$;

(b) $x_{s,t,k}$ is an r-single valued neutrosophic θ£-cluster point (r-δ£-*cluster point*) of α_n if for every $\varepsilon_n \in Q_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$, we have $\alpha_n q \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{CI}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r), r)$;

(c) δ£-closure operator is the mapping of $\mathrm{CI}^{\delta£}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} : \zeta^{\tilde{\mathcal{F}}} \times \zeta_0 \to \zeta^{\tilde{\mathcal{F}}}$ defined as

$$\mathrm{CI}^{\delta£}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \cup \left\{ x_{s,t,k} \in \mathrm{P}_{s,t,k}\left(\tilde{\mathcal{F}}\right) : x_{s,t,k} \textit{ is } r{-}\delta£{-}\textit{cluster point of } \alpha_n \right\}.$$

Definition 15. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $x_{s,t,k} \in \mathrm{P}_{s,t,k}(\tilde{\mathcal{F}})$ and $r \in \zeta_0$. Then,

(a) α_n is called r-Single valued neutrosophic $\mathfrak{R}^{£}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$-neighborhood of $x_{s,t,k}$ if $x_{s,t,k} q \alpha_n$ and αn is *r-SVNRIO*. We denote $\mathfrak{R}^{£}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} = \left\{ \alpha_n \in \zeta^{\tilde{\mathcal{F}}} | x_{s,t,k} q \alpha_n, \alpha_n \textit{ is } \mathrm{r-SVNRIO} \right\}$,

(b) $x_{s,t,k}$ is called r-single valued neutrosophic θ£-cluster point (r-θ£-*cluster point*) of α_n if for any $\varepsilon_n \in Q_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}\left(x_{s,t,k}, r\right)$, we have $\alpha_n q \mathrm{CI}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$,

(c) θ£-closure operator is mapping $\mathrm{CI}^{\theta£}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} : \zeta^{\tilde{\mathcal{F}}} \times \zeta_0 \to \zeta^{\tilde{\mathcal{F}}}$ defined as

$$\boldsymbol{\mathrm{CI}^{\theta£}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \cup \left\{ x_{s,t,k} \in P_{s,t,k}\left(\tilde{\mathcal{F}}\right) : x_{s,t,k} \textit{ is } r{-}\theta£{-}\textit{cluster point of } \alpha_n \right\}} \tag{9}$$

Example 2. Let $\tilde{\mathcal{F}} = \{a, b, c\}$ be a set. Define $\varepsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{F}}}$ as follows:

$$\varepsilon_n = \langle (0.4, 0.4, 0.4), (0.4, 0.4, 0.4), (0.4, 0.4, 0.4) \rangle ; \pi_n = \langle (0.2, 0.2, 0.2), (0.2, 0.2, 0.2), (0.2, 0.2, 0.2) \rangle .$$

We define an SVNITS$(\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ on $\tilde{\mathcal{F}}$ as follows: for each $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$,

$$\tilde{\tau}^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & \textit{if } \alpha_n = \tilde{0}, \\ 1, & \textit{if } \alpha_n = \tilde{1}, \\ \dfrac{2}{3}, & \textit{if } \alpha_n = \varepsilon_n, \\ 0, & \textit{otherwise}, \end{cases} \qquad £^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & \textit{if } \alpha_n = \tilde{0}, \\ \dfrac{1}{3}, & \textit{if } \pi_n = \varepsilon_n \\ \dfrac{2}{3}, & \textit{if } 0 < \alpha_n < \pi_n \\ 0, & \textit{otherwise}, \end{cases}$$

$$\tilde{\tau}^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & \textit{if } \alpha_n = \tilde{0}, \\ 0, & \textit{if } \alpha_n = \tilde{1}, \\ \dfrac{1}{3}, & \textit{if } \alpha_n = \varepsilon_n, \\ 1, & \textit{otherwise}, \end{cases} \qquad £^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & \textit{if } \alpha_n = \tilde{0}, \\ \dfrac{2}{3}, & \textit{if } \pi_n = \varepsilon_n \\ \dfrac{1}{3}, & \textit{if } 0 < \alpha_n < \pi_n \\ 1, & \textit{otherwise}, \end{cases}$$

$$\tilde{\tau}^{\tilde{\varsigma}}(\alpha_n) = \begin{cases} 0, & if \;\; \alpha_n = \tilde{0}, \\ 0, & if \;\; \alpha_n = \tilde{1}, \\ \dfrac{1}{3}, & if \;\; \alpha_n = \varepsilon_n, \\ 1, & otherwise, \end{cases} \qquad £^{\tilde{\varsigma}}(\alpha_n) = \begin{cases} 0, & if \;\; \alpha_n = \tilde{0}, \\ \dfrac{2}{3}, & if \;\; \pi_n = \varepsilon_n \\ \dfrac{1}{3}, & if \;\; 0 < \alpha_n < \pi_n \\ 1, & otherwise, \end{cases}$$

From using (9) we get, we obtain

$$\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \begin{cases} \tilde{0}, & if \;\; \alpha_n = \tilde{0}, \\ \varepsilon_n^c, & if \;\; \tilde{0} \neq \alpha_n \leq \varepsilon_n^c, r \leq \dfrac{1}{3}, 1 - r \geq \dfrac{2}{3}, \\ 1, & otherwise. \end{cases}$$

Theorem 3. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, $r \in \zeta_0$ and $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Then the following properties are holds:

(a) $\alpha_n \leq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(b) If $\alpha_n \leq \varepsilon_n$, then $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$,

(c) $\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$ is r-*SVNRIO*,

(d) $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \cap\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} | \alpha_n \leq \varepsilon_n, \varepsilon_n$ is r-*SVNRIC*$\}$,

(e) $\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Proof. (a) and (b) are easily proved from (9).

(c) Let $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and $\varepsilon_n = \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$. Then, we have

$$\begin{aligned} \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r\right) &= \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r\right), r\right), r\right) \\ &\leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r), r) \\ &= \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r\right) = \varepsilon_n. \end{aligned}$$

Since $\varepsilon_n = \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)$, we have $\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r) = \varepsilon_n$.

(d) Based on $\mathcal{P} = \cap\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} | \alpha_n \leq \varepsilon_n, \varepsilon_n$ is *r-SVNRIC*$\}$, let $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \ngeq \mathcal{P}$; therefore, $\upsilon \in \tilde{\mathcal{F}}$ and $s \in (0, 1]$, $t \in [0, 1), k \in [0, 1)]$ exist such that

$$\left.\begin{aligned} &\tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n, r)}(\upsilon) < s < \tilde{\varrho}_{\mathcal{P}}(\upsilon) \\ &\tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n, r)}(\upsilon) \geq t \geq \tilde{\sigma}_{\mathcal{P}}(\upsilon) \\ &\tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n, r)}(\upsilon) \geq k \geq \tilde{\varsigma}_{\mathcal{P}}(\upsilon) \end{aligned}\right\} \tag{10}$$

Therefore, $x_{s,t,k}$ is not an *r-δ£-cluster point* of α_n. As such, $\varepsilon_n \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(x_{s,t,k}, r\right)$ and $\alpha_n \leq [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)]^c$. Consequently, $\alpha_n \leq [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)]^c = \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r), r)$.

Since $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r), r)$ is *r-SVNRIC*, we have $\tilde{\varrho}_{\mathcal{P}}(\upsilon) \leq \tilde{\varrho}_{\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}}([\varepsilon_n]^c,r),r)}(\upsilon) < s, \tilde{\sigma}_{\mathcal{P}}(\upsilon) \geq \tilde{\sigma}_{Cl_{\tilde{\tau}\tilde{\sigma}}(int^{\odot}_{\tilde{\tau}\tilde{\sigma}}([\varepsilon_n]^c,r),r)}(\upsilon) > t$ and $\tilde{\varsigma}_{\mathcal{P}}(\upsilon) \geq \tilde{\varsigma}_{Cl_{\tilde{\tau}\tilde{\varsigma}}(int^{\odot}_{\tilde{\tau}\tilde{\varsigma}}([\varepsilon_n]^c,r),r)}(\upsilon) > k$. This is a contradiction to Eq. (10). Therefore, $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \geq \mathcal{P}$.

Meanwhile, by setting $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \not\leq \mathcal{P}$, then an *r-δ£-cluster point* of $y_{s_1,t_1,k_1} \in P_{s,t,k}(\tilde{\mathcal{F}})$ of α_n exists such that

$$\left.\begin{array}{l} \tilde{\varrho}_{\mathbf{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(y) > s_1 > \tilde{\varrho}_{\mathcal{P}}(y) \\ \tilde{\sigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(y) \leq t_1 \leq \tilde{\sigma}_{\mathcal{P}}(y) \\ \tilde{\varsigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(y) \leq k_1 \leq \tilde{\varsigma}_{\mathcal{P}}(y) \end{array}\right\} \tag{11}$$

Owing to $\mathcal{P}$, there exists *r-SVNRIC* $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ *with* $\alpha_n \leq \varepsilon_n$ such that $\tilde{\varrho}_{\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(y) > s_1 > \tilde{\varrho}_{\varepsilon_n} \geq \tilde{\varrho}_{\mathcal{P}}(y)$, $\tilde{\sigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(y) \leq t_1 \leq \tilde{\varrho}_{\varepsilon_n} \leq \tilde{\sigma}_{\mathcal{P}}(y)$ and $\tilde{\varsigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(y) \leq k_1 \leq \tilde{\varrho}_{\varepsilon_n} \leq \tilde{\varsigma}_{\mathcal{P}}(y)$. Therefore, $[\varepsilon_n]^c \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(y_{s_1,t_1}, k_1)$. So, $\alpha_n \leq \varepsilon_n = [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r), r)]^c$. Hence, $\alpha_n \overline{q} \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r), r)$.

Additionally, y_{s_1,t_1,k_1} is not an *r-δ£-cluster point* of α_n, that is, $\tilde{\varrho}_{\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(y) < s_1, \tilde{\sigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(y) \geq t_1, \tilde{\varsigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(y) \geq k_1$. This is a contradiction to Eq. (11). Therefore, $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathcal{P}$,

(e) Suppose that $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \not\leq \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$; therefore, $\upsilon \in \tilde{\mathcal{F}}$ and $[s \in (0,1], t \in [0,1), k \in [0,1)]$ exist such that

$$\left.\begin{array}{l} \tilde{\varrho}_{\mathbf{CI}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) > s > \tilde{\varrho}_{\mathbf{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) \\ \tilde{\sigma}_{CI_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \leq t \leq \tilde{\sigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \\ \tilde{\varsigma}_{CI_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \leq k \leq \tilde{\varsigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon), \end{array}\right\} \tag{12}$$

Since, $\tilde{\varrho}_{\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s, \tilde{\sigma}_{\mathrm{CI}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \geq t, \tilde{\varsigma}_{\mathrm{CI}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \geq k$, we have $x_{s,t,k}$ not *r-δ£-cluster point* of α_n. Therefore, there exists $\varepsilon_n \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ and $\alpha_n \leq [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)]^c$. Hence, $\tilde{\varrho}_{\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) \leq \tilde{\varrho}_{[\mathrm{int}_{\tilde{\tau}\tilde{\varrho}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}}(\varepsilon_n,r),r)]^c}(\upsilon) < s$, $\tilde{\sigma}_{Cl_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \leq \tilde{\varrho}_{[int_{\tilde{\tau}\tilde{\sigma}}(Cl^{\odot}_{\tilde{\tau}\tilde{\sigma}}(\varepsilon_n,r),r)]^c}(\upsilon) \geq t$ and $\tilde{\varsigma}_{Cl_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \leq \tilde{\varrho}_{[int_{\tilde{\tau}\tilde{\varsigma}}(Cl^{\odot}_{\tilde{\tau}\tilde{\varsigma}}(\varepsilon_n,r),r)]^c}(\upsilon) \geq k$. It is a contradiction for Eq. (12). Thus $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Theorem 4. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, for each $r \in \zeta_0$ and $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Then the following properties hold:

(a) $\alpha_n \leq \mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(b) If $\alpha_n \leq \varepsilon_n$, then $\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$,

(c) $\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \cup\{x_{s,t,k} \in P_{s,t,k}(\tilde{\mathcal{F}}) | x_{s,t,k}$ is r-δ£-cluster point of $\alpha_n\}$,

(d) $\mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \cap\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} | \alpha_n \leq \mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r),\ \tau^{\tilde{\varrho}}([\varepsilon_n]^c) \geq r,\ \tau^{\tilde{\sigma}}([\varepsilon_n]^c) \leq 1-r,\ \tau^{\tilde{\varsigma}}([\varepsilon_n]^c) \leq 1-r\}$,

(e) $\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \cap\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} | \alpha_n \leq \varepsilon_n, \varepsilon_n$ is r-δ£-cluster point of $\alpha_n\}$

(f) $x_{s,t,k}$ is r-θ£-cluster point of α_n iff $x_{s,t,k} \in \mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(g) $x_{s,t,k}$ is r-δ£-cluster point of α_n iff $x_{s,t,k} \in \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(h) If $\alpha_n = \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{intl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$, then $\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \alpha_n$,

(i) $\alpha_n \leq \mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(j) $\mathrm{W}(\alpha_n \vee \varepsilon_n, r) = \mathrm{W}(\alpha_n, r) \vee \mathrm{W}(\varepsilon_n, r)$ for each $\mathrm{W} = \left\{CI^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, CI^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\right\}$,

(k) $\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r) = \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Proof. (a) and (b) are easily proved from Definition 14.

(c) Set $\mathcal{P} = \cup\{x_{s,t,k} \in P_{s,t,k}(\tilde{\mathcal{F}}) | x_{s,t,k}$ as an r-δ£-cluster point of $\alpha_n\}$. Suppose that $\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \nleq \mathcal{P}$. Then there exists $\upsilon \in \tilde{\mathcal{F}}$, and $[s \in (0,1],\ t \in [0,1),\ k \in [0,1)]$ such that

$$\left.\begin{array}{l} \tilde{\varrho}_{\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) > s > \tilde{\varrho}_{\mathcal{P}}(\upsilon) \\ \tilde{\sigma}_{CI_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \leq t \leq \tilde{\sigma}_{\mathcal{P}}(\upsilon) \\ \tilde{\varsigma}_{CI_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \leq k \leq \tilde{\varsigma}_{\mathcal{P}}(\upsilon) \end{array}\right\} \tag{13}$$

Consequently, $x_{s,t,k}$ is not r-δ£-cluster point of α_n. So, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ and $\alpha_n \leq \left[\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{CI}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r\right)\right]^c \leq [\varepsilon_n]^c$

Based on Eq. (4), $\tilde{\varrho}_{\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) \leq \tilde{\varrho}_{[\varepsilon_n]^c}(\upsilon) < s, \tilde{\sigma}_{CI_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \geq \tilde{\sigma}_{[\varepsilon_n]^c}(\upsilon) \geq t$ and $\tilde{\varsigma}_{CI_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \geq \tilde{\varsigma}_{[\varepsilon_n]^c}(\upsilon) \geq k$.

It is a contradiction for Eq. (13). Thus $\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathcal{P}$.

(d) $\gamma = \cap\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} | \alpha_n \leq \mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r),\ \tau^{\tilde{\varrho}}([\varepsilon_n]^c) \geq r,\ \tau^{\tilde{\sigma}}([\varepsilon_n]^c) \leq 1-r,\ \tau^{\tilde{\varsigma}}([\varepsilon_n]^c) \leq 1-r\}$.

Suppose that $\mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \ngeq \gamma$, then there exists $\upsilon \in \tilde{\mathcal{F}}$ and $[s \in (0,1],\ t \in [0,1),\ k \in [0,1)]$ such that

$$\left.\begin{array}{l} \tilde{\varrho}_{\mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) < s \leq \tilde{\varrho}_{\gamma}(\upsilon) \\ \tilde{\sigma}_{CI^{\theta\pounds}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t \geq \tilde{\sigma}_{\gamma}(\upsilon) \\ \tilde{\varsigma}_{CI^{\theta\pounds}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) > k \geq \tilde{\varsigma}_{\gamma}(\upsilon) \end{array}\right\} \tag{14}$$

Consequently, $x_{s,t,k}$ is not r-θ£-cluster point of α_n. So, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$, $\alpha_n \leq [(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)]^c$. Thus, $\alpha_n \leq [(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)]^c = (\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r), r)$, $\tau^{\tilde{\varrho}}(\varepsilon_n) \geq r$, $\tau^{\tilde{\sigma}}(\varepsilon_n) \leq 1-r$, $\tau^{\tilde{\varsigma}}(\varepsilon_n) \leq 1-r\}$. Hence, $\tilde{\varrho}_{\gamma}(\upsilon) \leq \tilde{\varrho}_{[\varepsilon_n]^c}(\upsilon) < s, \tilde{\sigma}_{\gamma}(\upsilon) \leq \tilde{\sigma}_{[\varepsilon_n]^c}(\upsilon) < t, \tilde{\varsigma}_{\gamma}(\upsilon) \leq \tilde{\varsigma}_{[\varepsilon_n]^c}(\upsilon) < k$.

It is a contradiction to Eq. (14). Thus $\mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \geq \gamma$.

Suppose that $\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \nleq \gamma$, then there exists r-$\theta£$-cluster point of α_n. $y_{s_1,t_1,k_1} \in P_{s,t,k}(\tilde{\mathcal{F}})$ of α_n, such that

$$\left.\begin{aligned} &\tilde{\varrho}_{\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(y) > s_1 > \tilde{\varrho}_{\gamma}(y) \\ &\tilde{\sigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(y) < t_1 \leq \tilde{\sigma}_{\gamma}(y) \\ &\tilde{\varsigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(y) < k_1 \leq \tilde{\varsigma}_{\gamma}(y) \end{aligned}\right\} \tag{15}$$

By the definition of γ, there exists $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\tau^{\tilde{\varrho}}(\varepsilon_n) \geq r$, $\tau^{\tilde{\sigma}}(\varepsilon_n) \leq 1-r, \tau^{\tilde{\varsigma}}(\varepsilon_n) \leq 1-r$ and $\alpha_n \leq \mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$, s.t $\tilde{\varrho}_{\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(y) > s_1 > \tilde{\varrho}_{\varepsilon_n}(y) \geq \tilde{\varrho}_{\gamma}(y), \tilde{\sigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(y) < t_1 \leq \tilde{\sigma}_{\varepsilon_n}(y) \leq \tilde{\sigma}_{\gamma}(y)$ and $\tilde{\varsigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(y) < k_1 \leq \tilde{\varsigma}_{\varepsilon_n}(y) \leq \tilde{\varsigma}_{\gamma}(y)$. Additionally, $[\varepsilon_n]^c \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(y_{s_1,t_1,k_1}, r)$. $\alpha_n \leq \mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) = [\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r)]^c$, implies $\alpha_n \overline{q} \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r)$. Hence y_{s_1,t_1,k_1} is not an r-$\theta£$-cluster point of α_n. It is a contradiction for Eq. (15). Thus $\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \gamma$.

(e) Similar results are shown in (c) and (d).

(f) ($\Rightarrow$), clear.

($\Leftarrow$) Suppose that $x_{s,t,k}$ is not an r-$\theta£$-cluster point of α_n. There exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \alpha_n$. Thus $\alpha_n \leq [\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)]^c = \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r)$. By (d), $\tilde{\varrho}_{\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) \leq \tilde{\varrho}_{[\varepsilon_n]^c}(\upsilon) < s, \tilde{\sigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \geq \tilde{\sigma}_{[\varepsilon_n]^c}(\upsilon) > t$ and $\tilde{\varsigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \geq \tilde{\varsigma}_{[\varepsilon_n]^c}(\upsilon) > t$. Hence $x_{s,t,k} \notin \mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

(g) is similarly proved as in (f).

(h) The validity of this axiom is obvious from Theorem 3 (4).

(i) Based on Theorem 3(e), we show that $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$. Suppose that $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \nleq \mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$, then there exists $\upsilon \in \zeta$ and $[s \in (0,1], t \in [0,1), k \in [0,1)]$ such that

$$\left.\begin{aligned} &\tilde{\varrho}_{\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) > s > \tilde{\varrho}_{\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) \\ &\tilde{\sigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t \geq \tilde{\sigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \\ &\tilde{\varsigma}_{CI^{\delta£}_{\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) > k \geq \tilde{\varsigma}_{CI^{\theta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \end{aligned}\right\} \tag{16}$$

Since $\tilde{\varrho}_{\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s, \tilde{\sigma}_{\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \leq t$ and $\tilde{\varsigma}_{\mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \leq k$, then we have $x_{s,t.k}$ is not r-$\theta£$-cluster point of α_n So, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(y_{s_1,t_1,k_1}, r)$, $\alpha_n \leq [\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)]^c$, implies $A\overline{q}\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)$. Hence, $x_{s,t,k}$ is not r-$\delta£$-cluster point of α_n, by (7), we can get than, $\tilde{\varrho}_{\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s, \sigma_{\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \geq t, \tilde{\varsigma}_{\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \geq k$. It is a contradiction for Eq. (16). Thus, $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\theta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

(j) Let $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \vee \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \ngeq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n \vee \varepsilon_n, r)$. Then there exists $\upsilon \in \tilde{\mathcal{F}}$ such that

$$\left.\begin{array}{l} \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\varepsilon_n,r)}(\upsilon) \vee \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s < \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n \vee \varepsilon_n,r)}(\upsilon) \\ \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\varepsilon_n,r)}(\upsilon) \vee \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t > \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n \vee \varepsilon_n,r)}(\upsilon) \\ \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\varepsilon_n,r)}(\upsilon) \vee \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t > \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n \vee \varepsilon_n,r)}(\upsilon) \end{array}\right\} \tag{17}$$

Since $\tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s, \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t,$ $\tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) > k$ and $\tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\varepsilon_n,r)}(\upsilon) < s, \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\varepsilon_n,r)}(\upsilon) >$ $t,$ $\tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\varepsilon_n,r)}(\upsilon) > k$. We obtain, $x_{s,t,k}$ is not r-$\delta £$-cluster point of α_n and ε_n So, there exists $[\alpha_\mathrm{n}]_1, [\varepsilon_n]_1 \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$, and $\alpha_n \le [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]_1, r), r)]^c, \varepsilon_n \le [int_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(Cl^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]_1, r), r)]^c$. Thus, $[\alpha_n]_1 \wedge [\varepsilon_n]_1 \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(x_{s,t,k}, r\right)$.

Using Eqs. (4) and (5) we obtain,

$$\begin{aligned} \alpha_n \vee \varepsilon_n &\le \left[\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left([\alpha_n]_1, r\right), r\right) \wedge int_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(Cl^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left([\varepsilon_n]_1, r\right), r\right)\right]^c \\ &\le [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]_1, r) \wedge \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]_1, r), r)]^c \\ &\le \left[\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left([\alpha_n]_1 \wedge [\varepsilon_n]_1, r\right), r\right)\right]^c. \end{aligned}$$

Therefore, $\alpha_n \vee \varepsilon_n \overline{q}$ $\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]_1 \wedge [\varepsilon_n]_1, r), r)$. Hence, $x_{s,t,k}$ is not r-$\delta £$-cluster point of $\alpha_n \vee \varepsilon_n$, by (g), $\tilde{\varrho}_{\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n \vee \varepsilon_n, r)}(\upsilon) < s, \tilde{\sigma}_{Cl^{\odot}_{\tilde{\tau}\tilde{\sigma}}\alpha_n \vee \varepsilon_n(,r)}(\upsilon) > t$ and $\tilde{\varsigma}_{Cl^{\odot}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n \vee \varepsilon_n, r)}(\upsilon) > k$. It is a contradiction for Eq. (17), and hence, $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n \vee \varepsilon_n, r) \le \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \vee \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Meanwhile, $\alpha_n \vee \varepsilon_n \ge \alpha_n$ and $\alpha_n \vee \varepsilon_n \ge \varepsilon_n$. Hence $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n \vee \varepsilon_n, r) \ge \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \vee \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$. Therefore, $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \vee \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n \vee \varepsilon_n, r)$.

(k) Since $\alpha_n \le \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$, we have $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \le \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$. On the other hand, suppose that $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \ngeq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$. Then there exists $\upsilon \in \tilde{\mathcal{F}}$ and $[s \in (0,1], t \in [0,1), k \in [0,1)]$ such that

$$\left.\begin{array}{l} \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s < \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}\left(\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r), r\right)}(\upsilon) \\ \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t \ge \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varrho}}\left(CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r), r\right)}(\upsilon) \\ \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) > k \ge \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r), r)}(\upsilon) \end{array}\right\} \tag{18}$$

Since $\tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s, \tilde{\sigma}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t, \tilde{\varsigma}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) > k$, we have $x_{s,t,k}$ is not an r-$\delta £$-cluster point of α_n. So, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $\alpha_n \le [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_{n,r}),r)]^c =$ $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r, r)$, since, $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r, r)$ is r−SVNRIC. Then by Theorem 3(d), $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \le \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r, r)$.

Similarly, $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r) \le \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r), r) = \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)$. Hence,

$\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r) \leq \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r) < x_{s,t,k}$. It is a contradiction for Eq. (18).

Theorem 5. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, for $r \in \zeta_0$ and $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Then the following properties hold:

(a) α_n is r-*SVNPIC* iff $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(b) α_n is r-*SVNSIC* iff $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(c) α_n is r-*SVNαIO* iff $\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Proof. (a) Let α_n be an *r-SVNPIC*. Then $\alpha_n \leq \mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$, and by Theorem 3 (3) and (4), we have

$$\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r\right), r\right) = \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r\right) \leq \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$$
$$\leq \mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r).$$

Conversely, suppose that there exist $\upsilon \in \tilde{\mathcal{F}}$ and $[s \in (0,1],\ t \in [0,1),\ k \in [0,1)]$ such that $\tilde{\varrho}_{\mathrm{CI}^{\delta\pounds}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) > s > \tilde{\varrho}_{\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon), \tilde{\sigma}_{CI^{\delta\pounds}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) < t \leq \tilde{\sigma}_{Cl_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon)$ and $\tilde{\varsigma}_{CI^{\delta\pounds}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) < k \leq \tilde{\varsigma}_{Cl_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon)$. Then $x_{s,t,k}$ is not r-δ-cluster point of α_n. So, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$, with $\alpha_n \leq [\varepsilon_n]^c$ Since $x_{s,t,k}$ is r-$\delta\pounds$-cluster point of α_n, for $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$, we have $\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r) q \alpha_n$. Since,

$$\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r\right) \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r), r\right),$$

we obtain, $\alpha_n \geq [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)]^c \geq [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r), r)]^c = \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n], r), r)$.

Hence, α_n is not *r-SVNIC* set.

(b) Let α_n is an *r-SVNSIC* set. Then, $\alpha_n \leq \mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r), r)$ and$\tau^{\tilde{\varrho}}([\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}}([\alpha_n, r)]^c \geq r, \tau^{\tilde{\sigma}}([\mathrm{Cl}_{\tilde{\tau}\tilde{\sigma}}([\alpha_n, r)]^c \leq r, \tau^{\tilde{\varsigma}}([\mathrm{Cl}_{\tilde{\tau}\tilde{\varsigma}}([\alpha_n, r)]^c \leq r$. By Theorem 4(d), we have $\mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

Conversely, suppose that there exist $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}, r \in \zeta_0, \upsilon \in \tilde{\mathcal{F}}$ and $[s \in (0,1], t \in [0,1),\ k \in [0,1)]$ such that $\tilde{\varrho}_{\mathrm{CI}^{\theta\pounds}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) > t > \tilde{\varrho}_{\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon), \tilde{\sigma}_{CI^{\theta\pounds}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) < t \leq \tilde{\sigma}_{Cl_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon)$ and $\tilde{\varsigma}_{CI^{\theta\pounds}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) < t \leq \tilde{\varsigma}_{Cl_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon)$. Then $[\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r]^c) = \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r) \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ Since $x_{s,t,k}$ is *r*-$\theta\pounds$-cluster point of α_n, we have $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r), r) q \alpha_n$. It implies $\alpha_n \nleq [\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r), r)]^c = \mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$. Thus, α_n is not an *r-SVNSIC*.

(c) Similar results are shown in (a) and (b).

4 *r*-δ£-Closed and *r*-θ£-Closed

In this section, we firstly introduce and analyze the r-δ£-closed and r-θ£-closed of an SVNITS$(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$. Subsequently, we define and analyze the single-valued neutrosophic £-regular and the single-valued neutrosophic almost £-regular of $\tilde{\mathcal{F}}$. The findings have resulted in many theorems.

Definition 16. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. For $r \in \zeta_0$ and $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Therefore,

(a) α_n is said to be r-δ£-closed ($[\alpha_n]_{\delta £}$) [resp. r-θ£-closed $[\alpha_n]_{\theta £}$] iff $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \alpha_n$ (resp. $\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \alpha_n$). We define

$$\boldsymbol{\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \cap\{\varepsilon_n | \alpha_n \leq \varepsilon_n, \varepsilon_n = \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)\}} \tag{19}$$

$$\boldsymbol{\Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \cap\{\varepsilon_n | \alpha_n \leq \varepsilon_n, \varepsilon_n = \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)\}} \tag{20}$$

(b) The complement of r-δ£-closed (resp. r-θ£-closed) set is called r-δ£-open (resp. r-θ£-open).

Theorem 6. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. For $r \in \zeta_0$ and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. Then the following properties are holds:

(c). $\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$,

(d). $\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$ is r-δ£-closed,

(e). $\Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\Theta^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$,

(f). $\Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$ is r-θ£-closed,

(g). $\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Proof. (1) Based on Theorem 4(i,j), $\alpha_n \leq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$, which implies $\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$. Suppose that $\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \ngeq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$. Then there exist $\upsilon \in \tilde{\mathcal{F}}$ and $[s \in (0,1], t \in [0,1),\ k \in [0,1)]$ such that $\tilde{\varrho}_{\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n, r)}(\upsilon) < s < \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n, r)}(\upsilon), \tilde{\sigma}_{\Delta^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n, r)}(\upsilon) > t > \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n, r)}(\upsilon)$ and $\tilde{\varsigma}_{\Delta^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n, r)}(\upsilon) > k > \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n, r)}(\upsilon)$. Based on Eq. (19), there exist $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and $\alpha_n \leq \varepsilon_n = \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$ such that $\tilde{\varrho}_{\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n, r)}(\upsilon) \leq \tilde{\varrho}_{\varepsilon_n}(\upsilon) < s < \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n, r)}(\upsilon), \tilde{\sigma}_{\Delta^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n, r)}(\upsilon) \geq \tilde{\varrho}_{\varepsilon_n}(\upsilon) > t > \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n, r)}(\upsilon)$ and $\tilde{\varsigma}_{\Delta^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n, r)}(\upsilon) \geq \tilde{\varrho}_{\varepsilon_n}(\upsilon) > k > \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n, r)}(\upsilon)$.

Meanwhile, $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) = \varepsilon_n$, which is a contradiction. Hence, $\Delta^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \geq \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

(b) is similar to Theorem 4 (k).

(c) Let $\alpha_n \leq [\varepsilon_n]_i = \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]_i, r)$. Therefore, $\bigwedge_{i \in \Gamma}[\varepsilon_n]_i \leq \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\bigwedge_{i \in \Gamma}[\varepsilon_n]_i, r) \leq \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]_i, r) = [\varepsilon_n]_i$. Consequently, $\wedge_{i \in \Gamma}[\varepsilon_n]_i \leq C_{\theta \mathcal{J}\tau}(\wedge_{i \in \Gamma}[\varepsilon_n]_i, r)$. Hence, $\Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$.

(d) It is directly obtained from (c).

(e) Since $\alpha_n \leq \Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$, by (c) and Eq. (19), $\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r) = \Theta^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Definition 17. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$, and $r \in \zeta_0$. Then $\tilde{\tilde{\mathcal{F}}}$ is called,

(a) single valued neutrosophic £-regular ($SVN£$-regular) if for any $\alpha_n \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$, there exists $\varepsilon_n \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \alpha_n$,

(b) single valued neutrosophic almost £-regular ($SVNA£$-regular), if for any $\alpha_n \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$, then there exists $\varepsilon_n \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \alpha_n$.

Theorem 7. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$. Then the following statements are equivalent:

(a) $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is called $SVN£$-regular,

(b) For each $x_{s,t,k} \in P_{s,t,k}\left(\tilde{\mathcal{F}}\right)$ and $\alpha_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(x_{s,t,k}, r\right)$, there exists $\varepsilon_n \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(x_{s,t,k}, r\right)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r\right)$,

(c) For each $x_{s,t,k} \in P_{s,t,k}\left(\tilde{\mathcal{F}}\right)$ and each $\alpha_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(x_{s,t,k}, r\right)$, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(x_{s,t,k}, r\right)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r\right)$,

(d) For each $x_{s,t,k} \in P_{s,t,k}(\tilde{\mathcal{F}})$ and r-$SVNRIC$ set $\omega_n \in \zeta^{\tilde{\mathcal{F}}}$ with $x_{s,t,k} \notin \omega_n$, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ and α_n is r-$SVN\star$-open set such that $\omega_n \leq \alpha_n$ and $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)\overline{q}\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$,

(e) For each $x_{s,t,k} \in P_{s,t,k}\left(\tilde{\mathcal{F}}\right)$ and r-$SVNRIC$ set $\omega_n \in \zeta^{\tilde{\mathcal{F}}}$ with $x_{s,t,k} \notin \omega_n$, there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(x_{s,t,k}, r\right)$ and α_n is r-$SVN\star$-open set such that $\omega_n \leq \alpha_n$ and $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)\,\overline{q}\alpha_n$,

(f) For each r-$SVNRIO$ set $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\omega_n q \alpha_n$, there exists r-$SVNRIO$ set $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ such that $\omega_n q \varepsilon_n \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \alpha_n$.

(g) For each r-$SVNRIC$ set $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\omega_n \nleq \alpha_n$, there exists r-$SVNRIO$ set $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and is r-$SVN\star$-open set $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$ such that $\omega_n q \varepsilon_n$, $\alpha_n \leq \pi_n$ and $\varepsilon_n \overline{q} \pi_n$.

Proof. The proof of (a)⇒(b) and (b)⇒(c) are clear.

(c)⇒(a): $x_{s,t,k} \in P_{s,t,k}(\tilde{\mathcal{F}})$ and $\alpha_n \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$. Then, by (c), there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}((\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r) = \alpha_n$. since, $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ we have $\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r) \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$.

Moreover, since, $\omega_n = \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r) \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$, we have $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\omega_n, r) \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$, and hence $x_{s,t,k} q \omega_n \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\omega_n, r) \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \alpha_n$ where $\omega_n \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$.

(c)⇒(d): Let ω_n be an r-$SVNRIC$ set in $\tilde{\mathcal{F}}$ and $x_t \in \mathrm{P}_{s,t,k}(\tilde{\mathcal{F}})$ with $x_{s,t,k} \notin \omega_n$. Then $x_{s,t,k} q [\omega_n]^c$ and $[\omega_n]^c \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r) \subset Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$. By (c), there exists $\pi_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that

$$\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\pi_n, r) \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left([\omega_n]^c, r\right), r\right) = [\omega_n]^c.$$

Next, $x_{s,t,k} q \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} \left(\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} (\pi_n, r), r \right)$, then $\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} \left(\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} (\pi_n, r), r \right) \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} \left(x_{s,t,k}, r \right)$, and hence by hypothesis, there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\pi_n, r), r)$. Then, $\omega_n \leq [\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\pi_n, r), r)]^c$. Put $\alpha_n = [\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\pi_n, r), r)]^c$ then α_n is r-$SVN\star O$ set. Hence

$$\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} (\alpha_n, r) \leq \left[\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} \left(\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} (\pi_n, r), r \right) \right]^c \leq \left[\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} (\varepsilon_n, r) \right].$$

Therefore, $\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \overline{q} \mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$.

(d)⇒(e): It is trivial.

(e)⇒(f): Suppose that α_n is an r-$SVNRIO$ set with $\omega_n q \alpha_n$, then $\omega_n \nleq [\alpha_n]^c$. Hence there exists $x_{s,t,k} \in \mathrm{P}_{s,t,k}(\tilde{\mathcal{F}})$ such that $x_{s,t,k} \in \omega_n$ and $\omega_n \nleq [\alpha_n]^c$ where $[\alpha_n]^c$ is r-$SVNRIC$ set. By (e), there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ and $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$ is r-$SVN\star O$ set such that $[\alpha_n]^c \leq \pi_n$ and $\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \overline{q} \pi_n$. From $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ we have $x_{s,t,k} q \varepsilon_n \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r), r)$.

By setting $[\varepsilon_n]_1 = \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r), r)$, we have $\omega_n q [\varepsilon_n]_1$ and $[\varepsilon_n]_1$ is r-$SVNRIO$ set such that

$$\omega_n q \, [\varepsilon_n]_1 \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} \left([\varepsilon_n]_1, r \right) \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} (\varepsilon_n, r) \leq \underline{1} - \pi_n \leq \alpha_n$$

(f)⇒(g): Let α_n be an r-$SVNRIC$ set $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\omega_n \nleq \alpha_n$. Therefore, $\omega_n q [\alpha_n]^c$ and hence by, then there exists an r-$SVNRIO$ set $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ such that $\omega_n q \varepsilon_n \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \leq [\alpha_n]^c$. Then, ε_n is an r-$SVNRIO$ set and $[\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)]^c$ is an r-$SVN\star O$ set such that $\omega_n q \varepsilon_n$, $\alpha_n \leq [\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)]^c$ and $\varepsilon_n \overline{q} [\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)]^c$.

(g)⇒(a): Let $\alpha_n \in \mathfrak{R}^{\pounds}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ Then $x_{s,t,k} \nleq [\alpha_n]^c$ and $[\alpha_n]^c$ is an r-$SVNRIC$ set. By (g), there exist r-$SVNRIO$ set $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and it is r-$SVN\star O$ set $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$ such that $x_{s,t,k} q \varepsilon_n$, $[\alpha_n]^c \leq \pi_n$ and $\varepsilon_n \overline{q} \pi_n$. Then, $\varepsilon_n \in \mathfrak{R}^{\pounds}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$. Since, π_n is r-$SVN\star O$ set, $\mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \overline{q} \pi_n$. Therefore, $x_{s,t,k} q \varepsilon_n \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \leq [\pi_n]^c \leq \alpha_n$. Hence $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is $SVN\pounds$-regular.

Theorem 8. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$. Then the following statements are equivalent:

(a) $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is called $SVN\pounds$-regular,

(b) For each $x_{s,t,k} \in P_{s,t,k}(\tilde{\mathcal{F}})$, $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\tau^{\tilde{\varrho}}([\alpha_n]^c) \geq r, \tau^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r, \tau^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$, and $x_{s,t,k} \notin \alpha_n$, there exists $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ with ε_n is r-$SVN\star O$ such that $x_{s,t,k} \notin \mathrm{Cl}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$ and $\alpha_n \leq \varepsilon_n$,

(c) For each $x_{s,t,k} \in P_{s,t,k}(\tilde{\mathcal{F}})$, $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\tau^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\tau^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\tau^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$, and $x_{s,t,k} \notin \alpha_n$, there exists, $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ and $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$ with π_n is r-$SVN\star O$ such that $\alpha_n \leq \varepsilon_n$ and $\varepsilon_n \overline{q} \pi_n$,

(d) For each $\omega_n, \alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\tau^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\tau^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\tau^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$, and $\omega_n \nleq \alpha_n$, then there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ and $\varepsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\tau^{\tilde{\varrho}}(\varepsilon_n) \geq r, \tau^{\tilde{\sigma}}(\varepsilon_n) \leq 1 - r, \tau^{\tilde{\varsigma}}(\varepsilon_n) \leq 1 - r$ and π_n is r-$SVN\star O$ sets such that $\omega_n q \varepsilon_n$, $\alpha_n \leq \pi_n$ and $\varepsilon_n \overline{q} \pi_n$.

Proof. Similar to the proof of Theorem 7.

Theorem 9. An SVNITS $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is $SVNA£$-regular iff for each $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0, \mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Proof. From Theorem 4(i), we only show that $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \geq \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Suppose that $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \ngeq \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$. Then there exist $\upsilon \in \tilde{\mathcal{F}}$ and $[s \in (0,1], t \in [0,1), k \in [0,1)]$ such that

$$\left.\begin{array}{l} \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s < \tilde{\varrho}_{\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) \\ \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t > \tilde{\sigma}_{CI^{\theta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \\ \tilde{\varsigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) > k > \tilde{\varsigma}_{CI^{\theta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \end{array}\right\} \tag{21}$$

Because $\tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n,r)}(\upsilon) < s, \tilde{\sigma}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) > t, \tilde{\varsigma}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) > k$, and $x_{s,t,k}$ is not an r-$\delta£$-cluster point of α_n. So, there exists $\varepsilon_n \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ with $\alpha_n \leq [\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)]^c$ Since $\varepsilon_n \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ we have $\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r) \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$. By $SVNA£$-regularity of $\tilde{\mathcal{F}}$, there exists $\omega_n \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\omega_n, r), r) \leq \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r)$. Thus,

$$\alpha_n \leq \left[\mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r), r\right)\right]^c \leq \left[\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\omega_n, r)\right]^c = \mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\left([\omega_n]^c, r\right),$$

and $\tau^{\tilde{\varrho}}(\omega_n) \geq r, \tau^{\tilde{\sigma}}(\omega_n) \leq 1 - r, \tau^{\tilde{\varsigma}}(\omega_n) \leq 1 - r$. By Theorem 4(d), $\tilde{\varrho}_{\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}}(\alpha_n)}(\upsilon) \leq \tilde{\varrho}_{[\omega_n]^c}(\upsilon) < s, \tilde{\sigma}_{CI^{\theta £}_{\tilde{\tau}\tilde{\sigma}}(\alpha_n,r)}(\upsilon) \geq \tilde{\sigma}_{[\omega_n]^c}(\upsilon) > t$ and $\tilde{\varsigma}_{CI^{\theta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)}(\upsilon) \geq \tilde{\varsigma}_{[\omega_n]^c}(\upsilon) > k$. It is a contradiction for Eq. (21).

Conversely, let $\alpha_n \in \mathfrak{R}^{£}_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r) \subset Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k})$. Then by Theorem 4(h), s $> \tilde{\varrho}_{[\alpha_n]^n}(\upsilon) = \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}([\alpha_n]^c,r)}(\upsilon)$, s $> \tilde{\varrho}_{[\alpha_n]^n}(\upsilon) = \tilde{\varrho}_{\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}}([\alpha_n]^c,r)}(\upsilon)$ and $k < \tilde{\sigma}_{[\alpha_n]^n}(\upsilon) = \tilde{\sigma}_{CI^{\delta £}_{\tilde{\tau}\tilde{\sigma}}([\alpha_n]^c,r)}(\upsilon)$. Since, $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r) = \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r)$, $x_{s,t,k}$ is not an r-$\theta\mathcal{J}$-cluster point of $[\alpha_n]^c$. Then there exists $\varepsilon_n \in Q_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $[\alpha_n]^c \overline{q} \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$ implies $\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \alpha_n = \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r), r)$ and by Theorem 7(c), $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is $SVNA£$-regular.

Theorem 10. An SVNITS$(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is $SVNA£$-regular iff for each r-$SVNRIC$ set $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $\in \zeta_0, \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \alpha_n$.

Proof. The proof is similar to Theorem 9; additionally, r-$SVNRIC$ set is r-$\delta£$-closed.

Conversely, let α_n be any r-FRIC set with $x_t \notin \alpha_n$. Then, $x_t \notin \mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$ and hence, x_t is not r-$\theta£$-cluster point of α_n so, there there exists $\varepsilon_n \in Q_{\tau\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(x_{s,t,k}, r)$ such that $\alpha_n \overline{q} \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$. Thus, $\alpha_n \leq [\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)]^c = \omega_n$ and ω_n is r-$SVN\star O$ implies $\omega_n \overline{q} \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r)$. Hence, by Theorem 4(e), $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is SVNA£-regular.

Lemma 1. If $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$, $r \in \zeta_0$ such that $\alpha_n \overline{q} \varepsilon_n$ where ε_n is r-$\delta£$-open, then $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \overline{q} \varepsilon_n$.

Proof. Let $\alpha_n \overline{q} \varepsilon_n$ where ε_n is r-δ£-open. Then, $\alpha_n \leq [\varepsilon_n]^c = \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c$, by Theorem 4(k), $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) \leq \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r), r) = \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^c, r) = [\varepsilon_n]^c$. Hence, $\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\boldsymbol{\mathcal{A}}, r)\overline{q}\varepsilon_n$.

Lemma 2. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS and $\alpha_n \in \zeta^{(\tilde{\mathcal{F}}}$ is δ£-open iff for each $x_{x,t,k} \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ with $x_{s,t,k} q \alpha_n$, there exists r-$SVNRIO$ set $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ such that $x_{x,t,k} q \varepsilon_n \leq \alpha_n$.

Proof. Let $x_{x,t,k} \in \mathrm{P}_{s,t,k}((\tilde{\mathcal{F}})$ with $x_{x,t,k} q \alpha_n$ Then $x_{x,t,k} \notin \alpha_n]^c$. Since α_n is an r-δ£-open set, $x_{x,t,k} \notin [\alpha_n]^c = \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r)$. Thus, $x_{x,t,k}$ is not r-δ£-cluster point of $[\alpha_n]^c$. So, there exists $\omega_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ such that $[\alpha_n]^c \overline{q} \mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\omega_n, r), r)$. Put $\varepsilon_n = \mathrm{int}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\omega_n, r), r)$, so, ε_n is an r-$SVNRIO$ set with $x_{x,t,k} q \varepsilon_n \leq \alpha_n$.

Conversely, let $[\alpha_n]^c \neq \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r)$, then there exist $\upsilon \in \tilde{\mathcal{F}}$ and $s, t, k \in \zeta_0$ such that

$$\left.\begin{aligned} &\tilde{\varrho}_{[\alpha_n]^c}(\upsilon) < s < \tilde{\varrho}_{\mathbf{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}([\alpha_n]^c, r)}(\upsilon) \\ &\tilde{\sigma}_{[\alpha_n]^c}(\upsilon) > t > \tilde{\sigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\sigma}}([\alpha_n]^c, r)}(\upsilon) \\ &\tilde{\varsigma}_{[\alpha_n]^c}(\upsilon) > k > \tilde{\varsigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\varsigma}}([\alpha_n]^c, r)}(\upsilon). \end{aligned}\right\} \tag{22}$$

Because of $x_{x,t,k} q \alpha_n$, then there exists an r-$SVNRIO$ set ε_n such that $x_{x,t,k} q \varepsilon_n \leq \alpha_n$. This implies $[\alpha_n]^c \leq [\varepsilon_n]^n = \mathrm{Cl}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\mathrm{int}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\varepsilon_n]^n, r), r)$. By Theorem 3(d), we have $\tilde{\varrho}_{\mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}}([\alpha_n]^c, r)}(\upsilon)\, \tilde{\varrho}_{\leq([\varepsilon_n]^n)}(\upsilon) < s, \tilde{\sigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\sigma}}([\alpha_n]^c, r)}(\upsilon)\, \tilde{\sigma}_{\leq([\varepsilon_n]^n)}(\upsilon) > t$ and $\tilde{\varsigma}_{CI^{\delta£}_{\tilde{\tau}\tilde{\varsigma}}([\alpha_n]^c, r)}(\upsilon) \tilde{\varsigma}_{\leq([\varepsilon_n]^n)}(\upsilon) > k$. It is a contradiction for Eq. (22). Hence, $[\alpha_n]^c = \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}([\alpha_n]^c, r)$, i.e., α_n is an r-δ£-open set.

Lemma 3. If $\tau^{\tilde{\varrho}}(\alpha_n) \geq r, \tau^{\tilde{\sigma}}(\alpha_n) \leq 1 - r, \tau^{\tilde{\varsigma}}(\alpha_n) \leq 1 - r$, then $\mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}^{\delta£}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$.

Proof. Follows easily by virtue of Theorem 4.

Theorem 11. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. Then the following statements are equivalent:

(a) $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is $SVNA$£-regular,

(b) For each $r - \delta$£-open set $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and each $x_{x,t,k} \in \mathrm{P}_{s,t,k}(\tilde{\mathcal{F}})$ with $x_{s,t,k} q \boldsymbol{\mathcal{A}}$, there exists r-δ£-open set $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ such that $x_{x,t,k} q \varepsilon_n \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \alpha_n$.

Proof. (a)⇒(b): Let α_n be r-fuzzy $\delta\mathcal{J}$-open set such each $x_{s,t,k} q \alpha_n$. Then by Lemma 3, there exists an r-$SVNRIO$ set $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$ such that $x_{s,t,k} q \pi_n \leq \alpha_n$. By $SVNA$£-regularity of X, there exists an r-**FRIO** set ε_n (which is also r-δ£-open such that $x_{s,t,k} q \varepsilon_n \leq \mathrm{Cl}^{\odot}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\varepsilon_n, r) \leq \pi_n \leq \alpha_n$.

Therefore, (b) (a) is clear.

5 Single Valued Neutrosophic θ£-Connected

The aim of this section is to introduce the r-single-valued neutrosophic θ£-separated and r-single-valued neutrosophic δ£-separated. Moreover, we introduce r-single-valued neutrosophic θ£-connected and r-single valued neutrosophic δ£-connected related to the r-single valued neutrosophic operator θ and δ defined on the set $\tilde{\mathcal{F}}$.

Definition 18. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. For $r \in \zeta_0$ and $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Then,

(a) Two non-null SVNSs $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ are said to be r-single-valued neutrosophic θ£-separated if $\alpha_n \overline{q} [\varepsilon_n]_{\theta £}$ and $\varepsilon_n \overline{q} [\alpha_n]_{\theta £}$,

(b) Two non-null SVNSs $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ are said to be r-single-valued neutrosophic δ£-separated if $\alpha_n \overline{q} [\varepsilon_n]_{\delta £}$ and $\varepsilon_n \overline{q} [\alpha_n]_{\delta £}$,

Remark 2. For any two non-null SVNSs$\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$, and by Eq. (8). The following implications hold: r-single-valued neutrosophic θ£-separated $\Rightarrow$ r-single-valued neutrosophic δ£-separated $\Rightarrow$ r-single-valued neutrosophic separated.

The following example shows that the concept of r-single-valued neutrosophic δ£-separated is weaker than that of r-single-valued neutrosophic θ£-separated.

Example 3. Let $\tilde{\mathcal{F}} = \{a, b, c\}$ be a set. Define $[\varepsilon_n]_1, [\varepsilon_n]_2 \in \zeta^{\tilde{\mathcal{F}}}$ as follows:

$$[\varepsilon_n]_1 = \langle (1,1,0), (1,1,0), (1,1,0) \rangle ; [\varepsilon_n]_2 = \langle (0,0,1), (0,0,1), (0,0,1) \rangle .$$

We define an SVNITS$(\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ on $\tilde{\mathcal{F}}$ as follows: for each $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$,

$$\tilde{\tau}^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & if \;\; \alpha_n = \tilde{0}, \\ 1, & if \;\; \alpha_n = \tilde{1}, \\ \frac{1}{3}, & if \;\; \alpha_n = [\varepsilon_n]_1, \\ \frac{1}{2}, & if \;\; \alpha_n = [\varepsilon_n]_2, \\ 0, & otherwise, \end{cases} \quad £^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & if \;\; \alpha_n = \tilde{0}, \\ 0, & otherwise, \end{cases}$$

$$\tilde{\tau}^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & if \;\; \alpha_n = \tilde{0}, \\ 0, & if \;\; \alpha_n = \tilde{1}, \\ \frac{2}{3}, & if \;\; \alpha_n = [\varepsilon_n]_1, \\ \frac{1}{2}, & if \;\; \alpha_n = [\varepsilon_n]_2, \\ 1, & otherwise \end{cases} \quad £^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & if \;\; \alpha_n = \tilde{1}, \\ 1, & otherwise, \end{cases}$$

$$\tilde{\tau}^{\tilde{\varsigma}}(\alpha_n)=\begin{cases}0, & if\, \alpha_n=\tilde{0},\\ 0, & if\ \ \alpha_n=\tilde{0},\\ \dfrac{2}{3}, & if\ \ \alpha_n=[\varepsilon_n]_1\,,\\ \dfrac{1}{2}, & if\ \ \alpha_n=[\varepsilon_n]_2\,,\\ 1, & otherwise,\end{cases}\qquad £^{\tilde{\varsigma}}(\alpha_n)=\begin{cases}0, & if\ \ \alpha_n=\tilde{1},\\ 1, & otherwise.\end{cases}$$

Therefore, we obtain

$$\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varsigma}}(\alpha_n,r)=\begin{cases}\tilde{0}, & if\ \ \alpha_n=\tilde{0}, r\in\zeta_0,\\ \mathcal{E}_2^c, & if\ \ \alpha_n\le[\varepsilon_n]_1\,, r\le\dfrac{1}{2}, 1-r\ge\dfrac{1}{2},\\ \mathcal{E}_1^c, & if\ \ \alpha_n\le[\varepsilon_n]_2\,, r\le\dfrac{1}{3}, 1-r\ge\dfrac{2}{3},\\ \tilde{0}, & otherwise.\end{cases}$$

If $r\le\frac{1}{3}$ and $1-r\ge\frac{2}{3}$, then $[\varepsilon_n]_2^c$ and $[\varepsilon_n]_2$ are not r-single-valued neutrosophic $\theta£$-separated for $r\le\frac{1}{3}$ and $1-r\ge\frac{2}{3}$. If $r>\frac{1}{3}$ and $1-r<\frac{2}{3}$, we have $[\varepsilon_n]_2^c$ and $[\varepsilon_n]_2$ are r-single-valued neutrosophic separated.

Theorem 12. Let $(\tilde{\mathcal{F}},\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}},£^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. For $r\in\zeta_0$ and $\alpha_n,\varepsilon_n\in\zeta^{\tilde{\mathcal{F}}}$.

(a) If α_n and ε_n are single-valued neutrosophic $\theta£$-separated, and $[\alpha_n]_1,[\varepsilon_n]_1\in\zeta^{\tilde{\mathcal{F}}}$ such that $[\alpha_n]_1\le\alpha_n[\varepsilon_n]_1\le\varepsilon_n$, then $[\alpha_n]_1$ and $[\varepsilon_n]_1$ are also single-valued neutrosophic $\theta£$-separated,
(b) If $\alpha_n\overline{q}\varepsilon_n$ either both are r-$\theta£$-open or r-$\delta£$-closed, then α_n and ε_n are single-valued neutrosophic $\theta£$-separated,
(c) If α_n and ε_n either both are r-$\theta£$-open or r-$\delta£$-closed and if $[\omega_n]_1=\alpha_n\cap[\varepsilon_n]^c$ and $\omega_2=\varepsilon_n\cap[\alpha_n]^c$, then $[\omega_n]_1$ and $[\omega_n]_1$ are single-valued neutrosophic $\theta£$-separated.

Proof. (a) Since $[\alpha_n]_1\le\alpha_n$ we have $[[\alpha_n]_1]_{\theta£}\le[\alpha_n]_{\theta£}$. Then, $\varepsilon_n\le[\alpha_n]_{\theta£}\Rightarrow[\varepsilon_n]_1\le[\alpha_n]_{\theta£}\Rightarrow[\varepsilon_n]_1\le[[\alpha_n]_1]_{\theta£}$. Similarly $[\alpha_n]_1\le[[\varepsilon_n]_1]_{\theta£}$. Hence $[\alpha_n]_1$ and $[\varepsilon_n]_1$ are single-valued neutrosophic $\theta£$-separated.

(b) When α_n and ε_n are r-$\delta£$-closed, then $\alpha_n=[\alpha_n]_{\theta£}$ and $\varepsilon_n=[\varepsilon_n]_{\theta£}$. Since $\alpha_n\overline{q}\varepsilon_n$ we have $[\alpha_n]_{\theta£}\overline{q}\varepsilon_n$ and $[\varepsilon_n]_{\theta£}\overline{q}\alpha_n$.

When α_n and ε_n are r-$\theta£$-open, $[\alpha_n]^c$ and $[\varepsilon_n]^c$ are r-$\theta£$-closed. Then $\alpha_n\overline{q}\varepsilon_n\Rightarrow\alpha_n\le[\varepsilon_n]^c\Rightarrow[\alpha_n]_{\theta£}\le[[\varepsilon_n]^c]_{\theta£}=[\varepsilon_n]^c\Rightarrow[\alpha_n]_{\theta£}\overline{q}\varepsilon_n$. Similarly, $[\varepsilon_n]_{\theta£}\overline{q}\alpha_n$. Hence α_n and ε_n are single-valued neutrosophic $\theta£$-separated.

(c) When α_n and ε_n are r-$\theta£$-open, $[\alpha_n]^c$ and $[\varepsilon_n]^c$ are r-$\theta£$-closed. Since $[\omega_n]_1\le[\varepsilon_n]^c$, $[[\omega_n]_1]_{\theta£}\le[[\varepsilon_n]^c]_{\theta£}=[\varepsilon_n]^c$ and so $[[\omega_n]_1]_{\theta£}\overline{q}\varepsilon_n$. Thus $[\omega_n]_2\overline{q}[[\omega_n]_1]_{\theta£}$. Similarly, $[\omega_n]_1\overline{q}[[\omega_n]_2]_{\theta£}$. Hence $[\omega_n]_1$ and $[\omega_n]_1$ are single-valued neutrosophic $\theta£$-separated.

When α_n and ε_n are r-θ£-closed, $\alpha_n = [\alpha_n]_{\theta £}$ and $\varepsilon_n = [\varepsilon_n]_{\theta £}$. Since $[\omega_n]_1 \leq [\varepsilon_n]^c$, $[\varepsilon_n]_{\theta £}\overline{q}[\omega_n]_1$ and hence $[[\omega_n]_2]_{\theta £}\overline{q}[\omega_n]_1$. Similarly, $[[\omega_n]_1]_{\theta £}\overline{q}[\omega_n]_2$. Hence $[\omega_n]_1$ and $[\omega_n]_1$ are single-valued neutrosophic θ£-separated.

Theorem 13. Two non-null $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ are single-valued neutrosophic θ£-separated if and only if there exist two r-θ£-open sets ω_n and π_n such that $\alpha_n \leq \omega_n$, $\varepsilon_n \leq \pi_n$, $\alpha_n\overline{q}\pi_n$ and $\varepsilon_n\overline{q}\omega_n$.

***Proof*.** Let α_n and ε_n be single-valued neutrosophic θ£-separated. Putting $\pi_n = [[\alpha_n]_{\theta £}]^c$ and $\omega_n = [[\varepsilon_n]_{\theta £}]^c$, then ω_n and π_n are r-θ£-open such that $\alpha_n \leq \omega_n$, $\varepsilon_n \leq \pi_n$, $\alpha_n\overline{q}\pi_n$ and $\varepsilon_n\overline{q}\omega_n$.

Conversely, let ω_n and π_n be r-θ£-open sets such that $\alpha_n \leq \omega_n$, $\varepsilon_n \leq \pi_n$, $\alpha_n\overline{q}\pi_n$ and $\varepsilon_n\overline{q}\omega_n$. Since $[\pi_n]^c$ and $[\omega_n]^c$ are r-θ£-closed, we have $[\alpha_n]_{\theta £} \leq [\pi_n]^c \leq [\varepsilon_n]^c$ and $[\varepsilon_n]_{\theta £} \leq [\omega_n]^c \leq [\alpha_n]^c$. Thus $[\alpha_n]_{\theta £}\overline{q}\varepsilon_n$ and $[\varepsilon_n]_{\theta £}\overline{q}\alpha_n$. Hence α_n and ε_n are single-valued neutrosophic θ£-separated.

Definition 19. An SVNS which cannot be expressed as the union of two single-valued neutrosophic θ£-separated is said to be single-valued neutrosophic θ£-connected.

Definition 20. An SVNS α_n in a SVNITS $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is said to be single-valued neutrosophic δ£-connected if α_n cannot be expressed as the union of two single-valued neutrosophic δ£-separated.

For an SVNS α_n in a SVNITS$(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$, the following implications hold: single-valued neutrosophic connected $\Rightarrow$ single-valued neutrosophic δ£-connected $\Rightarrow$ single-valued neutrosophic θ£-connected. If $\tau^{\tilde{\varrho}}(\alpha_n) \geq r, \tau^{\tilde{\sigma}}(\alpha_n) \leq 1-r, \tau^{\tilde{\varsigma}}(\alpha_n) \leq 1-r$, then these three properties are equivalent.

Theorem 14. Let α_n be a non-null single-valued neutrosophic θ£-connected in a SVNITS $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$. If α_n is contained in the union of two single-valued neutrosophic θ£-separated ε_n and ω_n, then exactly one of the following conditions (a) or (b) holds:

(a) $\alpha_n \leq \varepsilon_n$ and $\alpha_n \cap \omega_n = \tilde{0}$,
(b) $\alpha_n \leq \omega_n$ and $\alpha_n \cap \varepsilon_n = \tilde{0}$.

***Proof*.** We first note that when $\alpha_n \cap \omega_n = \tilde{0}$, then $\alpha_n \leq \varepsilon_n$, since $\alpha_n \leq \varepsilon_n \cup \omega_n$. Similarly, when $\alpha_n \cap \varepsilon_n = \tilde{0}$, we have $\alpha_n \leq \omega_n$. Since $\alpha_n \leq \varepsilon_n \cup \omega_n$, both $\alpha_n \cap \varepsilon_n = \tilde{0}$ and $\alpha_n \cap \omega_n = \tilde{0}$ cannot hold simultaneously. Again, if $\alpha_n \cap \varepsilon_n \neq \tilde{0}$ and $\alpha_n \cap \omega_n \neq \tilde{0}$, then, by Theorem 12 (1), $\alpha_n \cap \omega_n$ and $\alpha_n \cap \varepsilon_n$ are single-valued neutrosophic θ£-separated such that $\alpha_n = (\alpha_n \cap \varepsilon_n) \cup (\alpha_n \cap \omega_n)$, contradicting the single-valued neutrosophic θ£-connectedness of α_n. Hence, exactly one of the conditions (1) or (2) above must hold.

Theorem 15. Let $\{[\alpha_n]_j | j \in J\}$ be a collection of single-valued neutrosophic θ£-connected in $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$. If there exists $i \in J$ such that $[\alpha_n]_j \cap [\alpha_n]_i \neq \tilde{0}$ for each $j \in J$, then $\alpha_n = \cup\{[\alpha_n]_j | j \in J\}$ is single-valued neutrosophic θ£-connected.

***Proof*.** Suppose that α_n is not single-valued neutrosophic θ£-connected. Then there exist single-valued neutrosophic θ£-separated ε_n and ω_n such that $\alpha_n = \varepsilon_n \cap \omega_n$. By Theorem 14, we have either (a) $[\alpha_n]_j \leq \varepsilon_n$ with $[\alpha_n]_j \cap \omega_n = \tilde{0}$ or (b) $[\alpha_n]_j \leq \omega_n$ with $[\alpha_n]_j \cap \varepsilon_n = \tilde{0}$ for each $j \in J$. Similarly, either (a$'$)$[\alpha_n]_i \leq \varepsilon_n$ with $[\alpha_n]_i \cap \omega_n = \tilde{0}$ or (b$'$)$[\alpha_n]_i \leq \omega_n$ with $[\alpha_n]_i \cap \varepsilon_n = \tilde{0}$ for each $i \in J$. We may assume, without loss of generality, that $[\alpha_n]_j$ is non-null for each $j \in J$, and hence exactly one of the conditions (a) and (b), and exactly one of (a$'$) and (b$'$) will hold.

Since $[\alpha_n]_j \cap [\alpha_n]_i \neq \tilde{0}$ for each $j \in J$, the conditions (a) and (b$'$) cannot happen, and similarly (b) and (1$'$) cannot hold simultaneously. If (a) and (a$'$) hold, then $[\alpha_n]_j \leq \varepsilon_n$ with $[\alpha_n]_j \cap \omega_n = \tilde{0}$.

Then $\alpha_n \leq \varepsilon_n$ with $\alpha_n \cap \omega_n = \tilde{0}$ and thus $\omega_n = \tilde{0}$ a contradiction. Similarly, if (b) and (b′) hold, then we have $\varepsilon_n = \tilde{0}$ again a contradiction.

Lemma 4. An SVNITS$(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is SVNA£-regular iff $[\alpha_n]_{\delta £} = [\alpha_n]_{\theta £}$ for every $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$.

Proof. Obvious.

Theorem 16. Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS, $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $r \in \zeta_0$. If $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is SVNA£-regular and α_n is single-valued neutrosophic θ£-connected set, then α_n is single-valued neutrosophic δ£-connected set.

Proof. Follows easily by virtue of Lemma 4.

Corollary 1. For a $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ of SVNA£-regular space $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$, the following are equivalent:

(a) α_n is r-single-valued neutrosophic connected,
(b) α_n is r-single-valued neutrosophic δ£-connected,
(c) α_n is r-single-valued neutrosophic θ£-connected.

Proof. Follows easily by virtue of Theorem 16.

6 Conclusion

The neutrosophic set theory has been established and applied extensively to many problems involving uncertainties. Herein, we provided clear definitions of single-valued neutrosophic operators $\mathrm{CI}^{\theta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ and $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ created from an SVNI topological space $\left(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}\right)$ and we established that $\mathrm{CI}^{\delta £}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r) = \mathrm{CI}_{\tilde{\tau}\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}(\alpha_n, r)$ when $£^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}} = £_0^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$. In addition, we presented the idea of r-single-valued neutrosophic θ£-connectedness based on a single-valued neutrosophic ideal $£^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ which has kindred with a preceding r-single-valued neutrosophic connectedness and the relationships among them are inspected. Moreover, we introduced an r-single-valued neutrosophic δ£-connectedness connected to a single-valued neutrosophic δ on the set $\tilde{\mathcal{F}}$ and analyzed some of their properties. This study not only provides a hypothetical basis for additional requests in neutrosophic topology, but also for the expansion of other methodical aspects.

Discussion for further works:

The current concept can be extended by

- Investigating neutrosophic metric topological spaces;
- Investigating the products of connected and Hausdorff spaces for $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$.

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ISBN 978-1-59973-733-1

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