# NeutroAlgebra Theory
# Volume I

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# NeutroAlgebra Theory
# Volume I

# Aims and Scope

Neutrosophic theory and its applications have been expanding in all directions at an astonishing rate especially after of the introduction the journal entitled "Neutrosophic Sets and Systems". New theories, techniques, algorithms have been rapidly developed. One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, hesitant fuzzy set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic hesitant fuzzy set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been an important tool in the application of various areas such as data mining, decision making, e-learning, engineering, medicine, social science, and some more.

*Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay & Abdullah Kargın*

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# Preface

Neutrosophic set has been derived from a new branch of philosophy, namely Neutrosophy. Neutrosophic set is capable of dealing with uncertainty, indeterminacy and inconsistent information. Neutrosophic set approaches are suitable to modeling problems with uncertainty, indeterminacy and inconsistent information in which human knowledge is necessary, and human evaluation is needed.

Neutrosophic set theory firstly proposed in 1998 by Florentin Smarandache, who also developed the concept of single valued neutrosophic set, oriented towards real world scientific and engineering applications. Since then, the single valued neutrosophic set theory has been extensively studied in books and monographs introducing neutrosophic sets and its applications, by many authors around the world. Also, an international journal - Neutrosophic Sets and Systems started its journey in 2013.

Smarandache in 2020, let <A> be an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, his splited the nonempty space we work on into three regions {two opposite ones corresponding to <A> and <antiA>, and one corresponding to neutral (indeterminate) <neutA> (also denoted <neutroA>) between the opposites}, which may or may not be disjoint – depending on the application, but they are exhaustive (their union equals the whole space). A NeutroAlgebra is an algebra which has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A Partial Algebra is an algebra that has at least one Partial Operation, and all its Axioms are classical (i.e. axioms true for all elements). Through a theorem Smarandache has proved that NeutroAlgebra is a generalization of Partial Algebra, and he gives examples of NeutroAlgebras that are not Partial Algebras. His also has introduced the NeutroFunction (and NeutroOperation).

This first volume collects original research and applications from different perspectives covering different areas of neutrosophic studies, such as decision making, neutroalgebra, metric, and some theoretical papers.

This volume contains two sections: NEUTROALGEBRAS, NEUTROSOPHIC RELATED OTHER PAPERS.

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# SECTION ONE

# NEUTROALGEBRAS

# Chapter One

# Universal NeutroAlgebra and Universal AntiAlgebra

## Florentin Smarandache

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## ABSTRACT

This paper introduces the Universal NeutroAlgebra that studies the common properties of the NeutroAlgebra structures, and the Universal AntiAlgebra that studies the common properties of the AntiAlgebraic structures.

**Keywords:** NeutroAlgebra, AntiAlgebra, Universal NeutroAlgebra, Universal AntiAlgebra

## INTRODUCTION

In 2019 and 2020 Smarandache [1, 2, 3, 4] generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or *NeutroAlgebra*) {whose operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or *AntiAlgebra*) {whose operations and axioms are totally false}.

The NeutroAlgebras & AntiAlgebras are a *new field of research*, which is inspired from our real world.

In classical algebraic structures, all axioms are 100%, and all operations are 100% well-defined,

but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some laws or some operations.

Using the process of *NeutroSophication* of a classical algebraic structure we produce a NeutroAlgebra, while the process of *AntiSophication* of a classical algebraic structure produces an AntiAlgebra.

## BACKGROUND

1. **(Operation, NeutroOperation, AntiOperation)**

01. A classical Operation ($*_m$) is an operation that is well-defined (inner-defined) for all elements of the set S, i.e. $*_m(x_1, x_2, \ldots ,x_m) \in S$ for all $x_1, x_2, \ldots, x_m \in S$.

02. An AntiOperation ($*_m$) is an operation that is not well-defined (i.e. it is outer-defined) for all elements for the set S; or $*_m(x_1, x_2, \ldots,x_m) \in U \setminus S$ for all $x_1, x_2, \ldots, x_m \in S$.

03. A NeutroOperation ($*_m$) is an operation that is partially well-defined (the degree of well-defined is T), partially indeterminate (the degree of indeterminacy is I), and partially outer-defined (the degree of outer-defined is F); where $(T, I, F) \neq (1, 0, 0)$ that represents the classical Operation, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiOperation.

An operation ($*_m$) is indeterminate if there exist some elements $a_1, a_2, \ldots, a_n \in S$ such that $*_m(a_1, a_2, \ldots, a_m)$ = undefined, or unknown, or unclear, etc.

2. **(Axiom, NeutroAxiom, AntiAxiom)**

A1. A classical Axiom is an axiom that is true for all elements of the set S.

A2. An AntiAxiom is an axiom that is false for all elements of the set S.

A3. A NeutroAxiom is an axiom that is partially true (the degree of truth is T), partially indeterminate (the degree of indeterminacy is I), and partially false (the degree of falsehood is F), where $(T, I, F) \neq (1, 0, 0)$ that represents the classical Axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.

3. **(Algebra, NeutroAlgebra, AntiAalgebra)**

S1. A classical Algebra (or Algebraic Structure) is a set S endowed only with classical Operations and classical Axioms.

S2. An AntiAlgebra (or AntiAlgebraic Structure) is a set S endowed with at least one AntiOperation or one AntiAxiom

S3. A NeutroAlgebra (or NeutroAlgebraic Structure) is a set S endowed with at least one NeutroOperation or one NeutroAxiom, and no AntiOperation and no AntiAxiom.

# UNIVERSAL NEUTROALGEBRA AND UNIVERSAL ANTIALGEBRA

1. **A Universe of Discourse, a Set, some Operations, and some Axioms**

Let's consider a non-empty set S included in a universe of discourse U, or $S \subset U$.

The set S is endowed with *n* operations, $1 \leq n \leq \infty$, $*_1$, $*_2$, …, $*_n$.

Each operation $*_i$, for $i \in \{1, 2, ..., \infty\}$, is an $m_i$-ary operation, where $0 \leq m_i \leq \infty$. {A o-ary operation, where "0" stands for zero (or null-ary operation), simply denotes a constant.}

Then a number of α axioms, $0 \leq \alpha \leq \infty$, is defined on S.

The axioms may take the form of identities (or equational laws), quantifications {universal quantification ($\forall$) except before an identity, existential quantification ($\exists$)}, inequalities, inequations, and other relations.

With the condition that there exist at least one m-ary operation, with $m \geq 1$, or at least one axiom.

We have taken into consideration the possibility of infinitary operations, as well as infinite number of axioms.

## 2. The Structures, almost all, are NeutroStructures

A classical Structure, in any field of knowledge, is composed of: a non-empty <u>space</u>, populated by some <u>elements</u>, and both (the space and all elements) are characterized by some <u>relations</u> among themselves, and by some <u>attributes</u>.

Classical Structures are mostly in theoretical, abstract, imaginary spaces.

Of course, when analysing a structure, it counts with respect to what relations and attributes we analyse it.

In our everyday life almost all structures are NeutroStructures, governed by Universal NeutroAlgebras and Universal AntiAlgebras, since they are neither perfect nor uniform, and not all elements of the structure's space have the same relations and same attributes in the same degree (not all elements behave in the same way).

# Conclusions

Since our world is full of indeterminacies, uncertainties, vagueness, contradictory information almost all existing structures are NeutroStructures, since either their spaces, or their elements or their relationships between elements or between are characterized by such indeterminacies.

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# Chapter Two

# Neutro-Topological Space and Anti-Topological Space

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## ABSTRACT

In this chapter, neutro-topological spaces and anti-topological spaces are obtained. Also, corresponding basic properties and examples for neutro-topological spaces and anti-topological spaces are given and proved. Furthermore, classical topological space and neutro-topological space are compared to each other. Neutro-topological spaces are shown to have a more general structure according to neutro-topological spaces. Thus, (T, I, F) components which constitute the neutrosophic theory are added to classical topological space (without using neutrosophic sets) and a new structure is obtained. In addition, it is shown that a neutro-topological space can be obtained from every classical topological space and a neutro-topolagical space can be obtained from every anti-topological space.

**Keywords:** Topological Space, Neutro-algebra, Anti-algebra, Neutro-Topological Space, Anti-Topological space.

## INTRODUCTION

Topology; it is a science of mathematics that deals with specific definitions for spatial structure concepts, compares different definitions and investigates the connections between the structure and properties defined on the set. The first thing to do in topology is to give a general definition of the fit, and then to investigate the connections between topological structures given by different methods. Topology has a wide range of applications, from analysis to geometry. Because of these features, many researchers have worked on topology [1 - 10]. Recently, Şahin, et al. studied neutrosophic triplet metric topology [11]; Şahin et al. obtained neutrosophic triplet topology [12]; Chandran et al. introduced on product of smooth neutrosophic topological spaces [13]; Mohammed et al. studied continuity and contra continuity via preopen sets in new construction fuzzy neutrosophic topology [14].

We encounter many uncertainties in every moment of our lives. Many times, classical mathematical logic is insufficient to get rid of these uncertainties. The reason is that when explaining a situation or a

problem, it is not possible to say that it is correct or certain. Smarandache defined neutrosophic logic and the concept of neutrosophic set in 1998 [1]. In the concept of neutrosophic logic and neutrosophic sets, there is the degree of membership T, degree of uncertainty I and degree of non-membership F. These degrees are defined independently from each other. It has the form of a neutrosophic value (T, I, F). In other words, a situation is handled in neutrosophy according to its accuracy, its falsehood, and its uncertainty. Thus, neutrosophic sets are the more general form of fuzzy sets [2] and intuitionistic fuzzy sets [3]. For this reason, many researchers have conducted studies on neutrosophic set theory [4 - 25]. Recently, Bakbak et al. studied neutrosophic soft expert multiset and their application [26]; Uluçay et al. obtained generalized neutrosophic soft expert set for multiple-criteria decision-making [27]; Bal et al. introduced soft neutrosophic modules [28]; Şahin et al. studied centroid single valued triangular neutrosophic numbers and their applications [29]; Uluçay et al. obtained neutrosophic soft lattices [30]. The theories have studied in various areas such as [41-55].

Florentin Smarandache defined neutro-structures and anti-structures in 2019 [31] and in 2020 [32]. Similar to neutrosophic logic, an algebraic structure divides into three regions: A, the set of elements that satisfy the conditions of the algebraic structure, the truth region; Neutro-A, the set of elements that do not meet the conditions of the algebraic structure, the uncertainty region and the anti-A, the set of elements that do not meet the conditions of the algebraic structure, the inaccuracy region. Thus, the structure of neutrosophic logic has been transferred to the structure of classical algebras, without using neutrosophic sets and neutrosophic numbers. In recent years, the academic community has witnessed growing research interests in neutrosophic set theory [56-82]. Therefore, neutro-algebraic structures, which have a more general structure than classical algebras, can be obtained. In addition, the region of the elements that do not satisfy any of the classical algebras are also taken as anti-algebraic structures. For this reason, many researchers have conducted studies on neutro-algebraic structures and anti-algebraic structures [4 - 7]. Recently, Smarandache studied neutro-algebra [36]; Rezaei et al. obtained the neutrosophic triplet of BI-algebras [37]; Smarandache et al. introduced neutro-bck-algebra [38]; Ibrahim et al. studied neutro-vector spaces [39]; Ibrahim et al. studied neutro-hypergroups [40].

In the second section, basic definitions on classical topology [1]; basic definitions of neutro-structure are given [36]. In the third section, the neutro-topology is defined and its basic properties are given. Similarities and differences between the classical topology and the neutro-topology are given. It is shown that a neutro-topology can be obtained from every classic topology. In the fourth section, anti-topology is defined and its basic features are given. Similarities and differences between the classic topology and the anti-topology are given. Also, it is shown that a neutro-topological space can be obtained from every anti-topological space. In the last part, results and suggestions are given.

## BACKGROUND

**Definition 1. [36] The Neutro-sophication of the Law** (degree of well-defined, degree of indeterminacy, degree of outer-defined)

Let X be a non-empty set. * be binary operation. For at least a double element $(x, y) \in (X, X)$, $x * y \in X$ (degree of well defined (T)) and for at least two double elements $(a, b)$, $(c, d) \in (X, X)$, $[a * b =$ indeterminate (degree of indeterminacy (I)) or $c * d \notin X$ (degree of outer-defined (F))].

**Property 2. [36]** In neutro-algebra, the classical well-defined for * binary operation is divided into three regions: degree of well-defined (T), degree of indeterminacy (I) and degree of outer-defined (F) similar to neutrosophic set and neutrosophic logic.

**Definition 3**. **[36] The Anti-sophication of the Law** (totally outer-defined)

Let X be a non-empty set. * be binary operation. For all double element $(x, y) \in (X, X)$, $x * y \notin X$ (totally outer-defined)

**Definition 4. [1]** Let X be a non-empty set and $\mathcal{T}$ be a collection of subsets of X. If $\mathcal{T}$ satisfies the following conditions, then $\mathcal{T}$ is called a topology on X and $(X, \mathcal{T})$ is called a topological space.

G-1) $\emptyset$ and X belongs to $\mathcal{T}$.

G-2) Any union of elements of $\mathcal{T}$ belongs to $\mathcal{T}$,

G-3) Any finite intersection of elements of $\mathcal{T}$ belongs of $\mathcal{T}$,

## NEUTRO-TOPOLOGICAL SPACE

**Note 5:** In this chapter, the symbol "$=_I$" will be used for situations where equality is uncertain. For example, if it is not certain whether "a" and "b" are equal, then it is denoted by $a =_I b$.

**Note 6:** In this chapter, the symbol "$\in_I$" will be used for situations where it is unclear to be an element. For example, if it is not certain whether "a" is a member of the set B, then it is denoted by $a \in_I B$.

**Definition 7.** Let X be a non-empty set, $\mathcal{T}$ be a collection of subset of X. If at least one of the following conditions {i, ii, iii} is satisfied, then $\mathcal{T}$ is called a neutro-topology on X and $(X, \mathcal{T})$ is called a neutro-topological space.

**i)** $[\emptyset \in \mathcal{T}, X \notin \mathcal{T}$ or $X \in \mathcal{T}, \emptyset \notin \mathcal{T}]$ or $[\emptyset, X \in_I \mathcal{T}]$

**ii)** For at least n elements $p_1, p_2, \ldots p_n \in \mathcal{T}$, $\cap_{i=1}^{n} p_i \in \mathcal{T}$ and for at least n elements $q_1, q_2, \ldots q_n \in \mathcal{T}$, $r_1, r_2, \ldots r_n \in \mathcal{T}$; $[(\cap_{i=1}^{n} q_i \notin \mathcal{T}$ or $(\cap_{i=1}^{n} r_i \in_I \mathcal{T})]$. Where n is finite.

**iii)** For at least n elements $p_1, p_2, \ldots p_n \in \mathcal{T}$, $\cup_{i \in I} p_i \in \mathcal{T}$ and for at least n elements $q_1, q_2, \ldots q_n \in \mathcal{T}$, $r_1, r_2, \ldots r_n \in \mathcal{T}$; $[(\cup_{i \in I} q_i \notin \mathcal{T}$ or $(\cup_{i \in I} r_i \notin \mathcal{T})]$.

Where, we obtain the Definition 7 using Definition 1 and Property 2.

**Corollary 8:** From Definition 7, the neutro-topological spaces are different from the classical topological spaces. Also, the neutro-topological spaces are given as an alternative to the classical topological spaces. However, for a neutro-topological space, instead of the ones that are not met in Definition 7, classical topological space conditions are valid.

**Example 9.** Let $X = \{a, b, c, d\}$ be a set and $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{b, c\}\}$ be a collection of subsets of X. Then,

i) It is clear that $X \notin \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

ii) Let $p_1 = \{a\}, p_2 = \{a, b\}, q_1 = \{c, d\}, q_2 = \{b, c\}$. We obtain that $(\cap_{i=1}^{2} p_i \in \mathcal{T})$ and $(\cap_{i=1}^{n} q_i \notin \mathcal{T})$.

iii) Let $p_1 = \{a, b\}, p_2 = \{c, d\}, q_1 = \{a\}, q_2 = \{b, c\}$. We obtain that $(\cup_{i \in 1,2} p_i \in \mathcal{T})$ and $(\cup_{i \in I} q_i \notin \mathcal{T})$.

Thus, $(X, \mathcal{T})$ satisfies the conditions {i, ii, iii} in Definition 7. Hence, $(X, \mathcal{T})$ is a neutro-topological space.

**Example 10.** Let $X = \{0, 1, 2, 3\}$ be a set and $\mathcal{T} = \{\emptyset, X, \{0\}, \{1\}, \{2\}, \{2, 3\}, \{1, 3\}\}$ be a collection of subsets of X. Then,

ii) Let $p_1 = X, p_2 = \{0\}, q_1 = \{2, 3\}, q_2 = \{1, 3\}$. We obtain that $(\cap_{i=1}^{2} p_i \in \mathcal{T})$ and $(\cap_{i=1}^{n} q_i \notin \mathcal{T})$.

iii) Let $p_1 = \{1\}, p_2 = X, q_1 = \{1\}, q_2 = \{2\}$. We obtain that $(\cup_{i \in 1,2} p_i \in \mathcal{T})$ and $(\cup_{i \in I} q_i \notin \mathcal{T})$.

Thus, $(X, \mathcal{T})$ satisfies the conditions {ii, iii} in Definition 7. Hence $(X, \mathcal{T})$ is a neutro-topological space.

Where, $(X, \mathcal{T})$ is satisfies the classical topological condition i.

**Corollary 11.** In Example 9, $(X, \mathcal{T})$ is a neutro-topological space, but $(X, \mathcal{T})$ is not a classical topological space. Also, in Example 10, $(X, \mathcal{T})$ is a neutro-topological space, but $(X, \mathcal{T})$ is not a classical topological space. Thus, neutro-topological space have a more general structure than classical topological spaces.

**Theorem 12.** Let $(X, \mathcal{T})$ be a classical topological space. Then, $(X, \mathcal{T} - \emptyset)$ is a neutro – topological space.

**Proof:** Since $(X, \mathcal{T})$ is a classical topological space, it is clear that $X \in \mathcal{T} - \emptyset$ and $\emptyset \notin \mathcal{T} - \emptyset$. Thus, $(X, \mathcal{T} - \emptyset)$ satisfies the conditions {i} in Definition 7. Hence $(X, \mathcal{T})$ is a neutro-topological space.

**Theorem 13.** Let $(X, \mathcal{T})$ be a classical topological space. Then, $(X, \mathcal{T} - X)$ is a neutro – topological space.

**Proof:** Since $(X, \mathcal{T})$ is a classical topological space, it is clear that $\emptyset \in \mathcal{T} - X$ and $X \notin \mathcal{T} - X$. Thus, $(X, \mathcal{T} - X)$ satisfies the conditions {i} in Definition 7. Hence $(X, \mathcal{T})$ is a neutro-topological space.

**Corollary 14.** From Theorem 12 and Theorem 13, we obtain that a neutro-topological space can be obtained from every classical topological space.

**Theorem 15.** Let $(\mathcal{T}_i)$ be a non-empty family of neutro-topologies $\mathcal{T}_i$ such that $\emptyset \in \mathcal{T}_i$, $X \notin \mathcal{T}_i$ and $(i = 1, 2, ..., n)$. Then, $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ is a neutro-topology on X.

**Proof:**

**i)** Since $X \notin \mathcal{T}_i$ and $\emptyset \in \mathcal{T}_i$, it is clear that

$$(\emptyset \in \bigcup_{i=1}^{n} \mathcal{T}_i) \text{ and } (X \notin \bigcup_{i=1}^{n} \mathcal{T}_i).$$

Thus, $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ satisfies the condition {i} in Definition 7. Hence $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ is a neutro-topological space. Also, $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ satisfies the conditions {ii, iii} in Definition 7.

**ii)** It is clear that for $\emptyset, q_i \in \bigcup_{i \in I} \mathcal{T}_i$, $\emptyset \cup q_i \in \bigcup_{i=1}^{n}(\mathcal{T}_i)$. Also, we assume that $p_i, q_i \in \bigcup_{i \in I} \mathcal{T}_i$ such that $p_i \cup q_i = X$. Since $X \notin \mathcal{T}_i$, we obtain $p_i \cup q_i \notin \bigcup_{i=1}^{n}(\mathcal{T}_i)$.

**iii)** It is clear that for $\emptyset, q_i \in \bigcup_{i \in I} \mathcal{T}_i$, $\emptyset \cap q_i \in \bigcup_{i=1}^{n}(\mathcal{T}_i)$. Also, we assume that $p_i, q_i \in \bigcup_{i \in I} \mathcal{T}_i$ such that $p_i \cap q_i = r_i \notin \mathcal{T}_i$. Thus, we obtain $p_i \cap q_i \notin \bigcup_{i=1}^{n}(\mathcal{T}_i)$.

**Example 16.** Let $X = \{x | 1 \leq x \leq 3, \ x \in \mathbb{Z}\}$, $\mathcal{T}_1 = \{\emptyset, \{1\}, \{1, 3\}, \{2, 3\}\}$, $\mathcal{T}_2 = \{\emptyset, \{3\}, \{2, 3\}, \{1, 2\}, \{1, 3\}\}$. Then,

a) $\emptyset \in \mathcal{T}_1, \mathcal{T}_2$ and $X \notin \mathcal{T}_1, \mathcal{T}_2$.

b) $\{1\} \cup \{1,3\} \in \mathcal{T}_1$ and $\{1,3\} \cup \{2,3\} \notin \mathcal{T}_1$, $\{3\} \cup \{2,3\} \in \mathcal{T}_2$ and $\{1,3\} \cup \{2,3\} \notin \mathcal{T}_2$.

c) $\{1\} \cap \{1,3\} \in \mathcal{T}_1$ and $\{1,3\} \cap \{2,3\} \notin \mathcal{T}_1$, $\{3\} \cap \{2,3\} \in \mathcal{T}_2$ and $\{1,2\} \cap \{2,3\} \notin \mathcal{T}_2$.

Thus, $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ two neutro-topological spaces since $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ satisfy the conditions {i, ii, iii} in Definition 7.

Now, we show that

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{1\}, \{3\}, \{1,3\} \{1,2\}\{2,3\}\}$$

is a neutro-topology since $\mathcal{T}_1 \cup \mathcal{T}_2$ satisfies the conditions {i, ii, iii} in Definition 7.

d) $\emptyset \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $X \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

e) $\{1\} \cup \{1,3\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $\{1,3\} \cup \{2,3\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

f) $\{1\} \cap \{1,3\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $\{1,3\} \cap \{2,3\} \notin \mathcal{T}_1 \cup \mathcal{T}_{2_1}$.

Thus, $\mathcal{T}_1 \cup \mathcal{T}_2$ satisfies the conditions {i, ii, iii} in Definition 7.

**Theorem 17.** Let $(\mathcal{T}_i)$ be a non-empty family of neutro-topologies $\mathcal{T}_i$ such that $\emptyset \notin \mathcal{T}_i$, $X \in \mathcal{T}_i$ and (i = 1, 2, …, n). Then, $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ is a neutro-topology on X.

**Proof:**

**i)** Since $\emptyset \notin \mathcal{T}_i$, $X \in \mathcal{T}_i$, it is clear that

$$(\emptyset \notin \bigcup_{i=1}^{n} \mathcal{T}_i) \text{ and } (X \in \bigcup_{i=1}^{n} \mathcal{T}_i).$$

Thus, $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ satisfies the condition {i} in Definition 7. Hence $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ is a neutro-topological space. Also, $\bigcup_{i=1}^{n}(\mathcal{T}_i)$ satisfies the conditions {ii, iii} in Definition 7.

**ii)** It is clear that for $X, q_i \in \bigcup_{i \in I} \mathcal{T}_i$, $X \cap q_i \in \bigcup_{i=1}^{n}(\mathcal{T}_i)$. Also, we assume that $p_i, q_i \in \bigcup_{i \in I} \mathcal{T}_i$ such that $p_i \cap q_i = \emptyset$. Since $\emptyset \notin \mathcal{T}_i$, we obtain $p_i \cap q_i \notin \bigcup_{i=1}^{n}(\mathcal{T}_i)$.

**iii)** It is clear that for $X, q_i \in \bigcup_{i \in I} \mathcal{T}_i$, $X \cup q_i \in \bigcup_{i=1}^{n}(\mathcal{T}_i)$. Also, we assume that $p_i, q_i \in \bigcup_{i \in I} \mathcal{T}_i$ such that $p_i \cup q_i = r_i \notin \mathcal{T}_i$. Thus, we obtain $p_i \cup q_i \notin \bigcup_{i=1}^{n}(\mathcal{T}_i)$.

**Example 18.** Let $X = \{n, e, u, t, r, o\}$ be a set. We assume that $\mathcal{T}_1 = \{X, \{n, e\}, \{n, t\}, \{n, e, r\}, \{e, u, r, o\}\}$, $\mathcal{T}_2 = \{X, \{n, u\}, \{n, u, t\}, \{e, u, t, r\}, \{n, e, t, r, o\}\}$. Then,

a) $\emptyset \notin \mathcal{T}_1, \mathcal{T}_2$ and $X \in \mathcal{T}_1, \mathcal{T}_2$.

b) $X \cup \{n, e\} \in \mathcal{T}_1$ and $\{n, e\} \cup \{n, t\} \notin \mathcal{T}_1$, $X \cup \{n, u\} \in \mathcal{T}_2$ and $\{n, u\} \cup \{e, u, t, r\} \notin \mathcal{T}_2$.

c) $\{n, e\} \cap \{n, e, r\} \in \mathcal{T}_1$ and $\{n, e\} \cap \{n, t\} \notin \mathcal{T}_1$, $\{n, u\} \cap \{n, u, t\} \in \mathcal{T}_2$ and $\{n, u\} \cap \{e, u, t, r\} \notin \mathcal{T}_2$.

Thus, $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ two neutro-topological spaces since $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ satisfy the conditions {i, ii, iii} in Definition 7.

Now, we show that

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{X, \{n, e\}, \{n, t\}, \{n, u\}, \{n, e, r\}, \{n, u, t\}, \{e, u, r, o\}, \{e, u, t, r\}, \{n, e, t, r, o\}\}$$

is a neutro-topology since $\mathcal{T}_1 \cup \mathcal{T}_2$ satisfies the conditions {i, ii, iii} in Definition 7.

d) $\emptyset \notin \mathcal{T}_1 \cup \mathcal{T}_2$ and $X \in \mathcal{T}_1 \cup \mathcal{T}_2$.

e) $X \cup \{n, e\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $\{n, e\} \cup \{n, t\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

f) $\{n, e\} \cap \{n, e, r\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $\{n, t\} \cap \{n, u\} \notin \mathcal{T}_1 \cup \mathcal{T}_{2_1}$.

Thus, $\mathcal{T}_1 \cup \mathcal{T}_2$ satisfies the conditions {i, ii, iii} in Definition 7.

**Corollary 19.** The classical topological spaces do not satisfy the Theorem 15 and Theorem 17. However, neutro-topological spaces satisfy the Theorem 15 and Theorem 17.

# ANTI-TOPOLOGICAL SPACE

**Definition 20.** Let X be a non-empty set, $\mathcal{T}$ be a collection of subset of X. If the following conditions {Ai, Aii, Aiii} is satisfied, then $\mathcal{T}$ is called an anti-topology on X and $(X, \mathcal{T})$ is called an anti-topological space.

**Ai)** $\emptyset, X \notin \mathcal{T}$

**Aii)** For all $\forall q_1, q_2, \dots q_n \in \mathcal{T}, (\cap_{i=1}^n q_i \notin \mathcal{T})$. Where, n is finite.

**Aiii)** For all $\forall q_1, q_2, \dots q_n \in \mathcal{T}, (\cup_{i \in I} q_i \notin \mathcal{T})$

**Example 21.** Let $X = \{1, 2, 3, 4\}$ be a set and $\mathcal{T} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ be a collection of subset of X. Then,

Ai) $\emptyset, X \notin \mathcal{T}$

Aii) We assume that

$$q_1 = \{1, 2\}, q_2 = \{2, 3\}, q_3 = \{3, 4\}.$$

Thus, we obtain that

$$q_1 \cap q_2 = \{1, 2\} \cap \{2, 3\} = \{2\} \notin \mathcal{T}$$

$$q_1 \cap q_3 = \{1, 2\} \cap \{3, 4\} = \emptyset \notin \mathcal{T}$$

$$q_2 \cap q_3 = \{2, 3\} \cap \{3, 4\} = \{3\} \notin \mathcal{T}$$

Aiii) We assume that

$$q_1 = \{1, 2\}, q_2 = \{2, 3\}, q_3 = \{3, 4\}.$$

Thus, we obtain that

$$q_1 \cup q_2 = \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\} \notin \mathcal{T}$$

$$q_1 \cup q_3 = \{1, 2\} \cup \{3, 4\} = \{1, 2, 3, 4\} \notin \mathcal{T}$$

$$q_2 \cup q_3 = \{2, 3\} \cup \{3, 4\} = \{2, 3, 4\} \notin \mathcal{T}$$

Hence, $(X, \mathcal{T})$ satisfies the {Ai, Aii, Aiii} conditions in Definition 20. Therefore, $(X, \mathcal{T})$ is an anti-topological space.

**Corollary 22:** In Example 21, $(X, \mathcal{T})$ is an anti-topological space. But $(X, \mathcal{T})$ is not a neutro-topological space and classical topological space. Thus, anti-topological spaces is different from neutro-topological spaces and classical topological spaces.

**Theorem 23.** Let $(X, \mathcal{T})$ be an anti-topological space. Then, $(X, \mathcal{T} \cup \emptyset)$ is a neutro-topological space.

**Proof:** We obtan $\emptyset, X \notin \mathcal{T}$ since $(X, \mathcal{T})$ is an anti-topological space. Thus, we obtain that

$$\emptyset \in \mathcal{T} \cup \emptyset \text{ and } X \notin \mathcal{T} \cup \emptyset.$$

Hence, $(X, \mathcal{T} \cup \emptyset)$ satisfies the {i} condition in Definition 7. Then, $(X, \mathcal{T} \cup \emptyset)$ is a neutro-topological space. Also, it is clear that $(X, \mathcal{T} \cup \emptyset)$ satisfies the {ii, iii} condition in Definition 7.

**Theorem 24.** Let $(X, \mathcal{T})$ be an anti-topological space. Then, $(X, \mathcal{T} \cup X)$ is a neutro-topological space.

**Proof:** We obtain $\emptyset, X \notin \mathcal{T}$ since $(X, \mathcal{T})$ is an anti-topological space. Thus, we obtain that

$$X \in \mathcal{T} \cup X \text{ and } \emptyset \notin \mathcal{T} \cup X.$$

Hence, $(X, \mathcal{T} \cup X)$ satisfies the {i} condition in Definition 7. Then, $(X, \mathcal{T} \cup X)$ is a neutro-topological space. Also, it is clear that $(X, \mathcal{T} \cup X)$ satisfies the {ii, iii} condition in Definition 7.

**Example 25:** In Example 21, $\mathcal{T} = \{\{1,2\},\{2,3\},\{3,4\}\}$ is an anti-topology. From Theorem 23, $(X, \mathcal{T} \cup \emptyset)$ is a neutro-topological space.

**Example 26:** In Example 21, $\mathcal{T} = \{\{1,2\},\{2,3\},\{3,4\}\}$ is an anti-topology. From Theorem 23, $(X, \mathcal{T} \cup X)$ is a neutro-topological space.

**Corollary 27:** From Theorem 23 and Theorem 24, we obtain that a neutro-topological space can be obtained from every anti-topological space.

# Conclusions

In this chapter, neutro-topological space is defined and relevant basic properties are given. Similarities and differences between the classical topological space and neutro-topological spaces are discussed. It is shown that a neutro-topological space can be obtained from every classical topological space. In addition, anti- topological spaces is defined and corresponding basic properties are given. Similarities and differences between the classical topological space and anti- topological spaces are discussed. Also, it is shown that a neutro-topological space can be obtained from every anti- topological spaces. Thus, we add new structures to neutro-algebra.

Thanks to definition of neutro-topological spaces and anti-topological spaces researchers can define neutro-metric topological space, anti-metric topological space, neutro-continuous function, anti-continuous function, neutro-convergent function, anti-convergent function, neutro-topological geometry, anti-topological geometry …

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# Chapter Three

# On Neutrosophic RM-algebras

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## ABSTRACT

In this chapter, we introduced the notion of neutrosophic filters in RM-algebras. Moreover, implicative neutrosophic filters on RM-algebras is defined and the relation between implicative neutrosophic filters and neutrosophic filters in investigated. Further, some necessary and sufficient conditions for a neutrosophic filter to be implicative neutrosophic filter are investigated.

**Keywords:** RM-algebra, filter, neutrosophic filter, implicative neutrosophic filter, Neutro-RM-algebra, Anti-RM-algebra.

## INTRODUCTION

The notion of a fuzzy subset of a set was introduced by Zadeh [49]. Then, Atanassov introduced an intuitionistic Fuzzy set (IFS) as a generalization of the Fuzzy set [4-5]. The IFS represents the uncertainty with respect to both membership and non-membership. However, it can only handle incomplete information but not the indeterminate and inconsistent information which exists commonly in real situations. In 1999, Smarandache firstly proposed the theory of neutrosophic sets, which is the generalization of the classical sets, fuzzy sets and intuitionistic fuzzy sets [28]. It is known that the neutrosophic set theory is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Then neutrosophic sets has been applied to many fields such as: control theory [1], database [3], medical diagnosis problem [2, 9, 27, 36, 46], decision making problem [6, 8, 10, 15, 35, 37-41, 48], topology [17] and algebraic structures [25, 29-30, 34,42-44, 50], and so on.

In 1966, K. Iseki introduced BCI-algebras as algebraic models of BCI-logic [14]. There are some generalizations of BCI-algebras have been extensively investigated in many papers. The algebras $(X, \rightarrow, 1)$ of type $(2,0)$ satisfying the axioms: $x \rightarrow x = 1$ and $1 \rightarrow x = x$, for all $x \in X$, were named RM-algebras (see, [11-13, 45-47]). Kim et al. introduced the notion of BE-algebras as a generalization of dual BCK-algebras [16]. A. Rezaei et al. investigated the relationship between Hilbert algebras and BE-algebras [23]. Meng introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCI/BCK-algebras, and studied some relations with BE-algebras [18]. Then he defined the notion of atoms in CI-algebras and

singular CI-algebras and investigated their properties [19]. Filters and upper sets were studied in detail by Piekart et al. [20].

In 2015, neutrosophic set theory was applied to BE-algebras, and the notion of neutrosophic filter was discussed [21]. In 2020, further contribute to the properties of neutrosophic filters in BE-algebras was redefined the notion of neutrosophic filters in BE-algebras some mistakes and definitions of orginal definition of neutrosophic filter was investigated by some examples and a new definition was introduced [51]. In 2020, A. Rezaei et al. porposed the concepts of a Neutro-BE-algebra and Anti-BE-algebra, and they discussed several properties in the new algebraic structrues [25].

The chapter is organized as follows: In section 1, introduces some concepts and basic operations are reviewed. In section 2, the notion of a neutrosophic filter in RM-algebras is established, some basic properties are presented, and the relationships between fuzzy filters and neutrosophic filters are discussed. In section 3, the concept of implicative neutrosophic filter in RM-algebra is introduced, and some necessary and sufficient conditions for a neutrosophic filter to be implicative are presented. Finally, in section 4, the concepts of a Neutro-RM-algebra and Anti-RM-algebra are introduced, and gave a triplet <RM-algebra, Neutro-RM-algebra, Anti-RM-algebra>.

## 1. Preliminaries

In this section, we recall the fundamental definitions that will be used in the sequel:

For more details, the reader could refer to [7, 13, 16, 18-19, 22, 24-25, 32].

**Definition 1.1.** A RM-*algebra* is an algebra $(X, \rightarrow, 1)$ of type $(2, 0)$ (i.e. $X$ is a non-empty set, $\rightarrow$ is a binary operation and 1 is a constant element) satisfying the following axioms, for all $x \in X$:

(RM1) $x \rightarrow x = 1$;

(RM2) $1 \rightarrow x = x$.

A CI-algebra $(X, \rightarrow, 1)$ is said to be a CI-*algebra* if

(CI3) $(\forall x, y, z \in X, \text{ with } x \neq y)(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z))$.

Define a binary relation $\leq$ on $X$ by $x \leq y$ if and only if $x \rightarrow y = 1$.

By (RM1), $\leq$ is only reflexive.

**Example 1.2.** Consider the real interval [0, 1], and let $\rightarrow$ be the binary operation on [0, 1] defined by

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y; \\ \max\{1 - x, y\} & \text{otherwise.} \end{cases}$$

Then, $([0,1], \rightarrow, 1)$ is a RM-algebra.

**Definition 1.3.** A RM-algebra $X$ is said to be *self-distributive* if for all $x, y, z \in X$,

$$x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z).$$

**Definition 1.4.** A RM-algebra X is said to be *transitive* if for all $x, y, z \in X$,

$$(y \rightarrow z) \rightarrow \big((x \rightarrow y) \rightarrow (x \rightarrow z)\big) = 1.$$

**Definition 1.5.** A non-empty subset $S$ of a RM-algebra $X$ is said to be a *subalgebra* of $X$ if $x \rightarrow y \in S$, whenever $x, y \in S$.

**Definition 1.6.** A non-empty subset $F$ of a RM-algebra $X$ is said to be a *filter* of $X$ if

(F1) $1 \in F$;

(F2) $x \to y \in F$ and $x \in F$ imply $y \in F$.

**Definition 1.7.** Let $X$ be a RM-algebra and $A$ a nonempty subset of $X$. If $B$ is the least filter containing $A$ in $X$, then $B$ is said to be the filter generated by $A$ and is denoted by $(A]$. If $A$ is a finite set of $X$, then $(A]$ is said to be finitely generated.

For convenience, let $(\emptyset] = \{1\}$ and $(\{a_1, \dots, a_n\}]$ is simply denoted by $(a_1, \dots, a_n]$.

For any $a_1, \dots, a_n, x \in X$, we denote

$$\prod_{i=1}^{n} a_i \to x = a_n \to (\cdots \to (a_1 \to x) \cdots).$$

**Definition 1.8.** A non-empty subset $F$ of $X$ is called an *implicative filter* of $X$ if

(IF1) $1 \in F$;

(IF2) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$, for all $x, y, z \in X$.

**Definition 1.9.** A nonempty subset $F$ of $X$ is called a *positive implicative filter* of $X$ if

(PIF1) $1 \in F$;

(PIF2) $z \to ((x \to y) \to x) \in F$ and $z \in F$ imply $x \in F$, for all $x, y, z \in X$.

**Definition 1.10.** A fuzzy set $\mu$ of $X$ is called a *fuzzy filter* of X if it satisfies:

(FF1) $\mu(1) \geq \mu(x)$;

(FF2) $\mu(y) \geq \min\{\mu(x), \mu(x \to y)\}$, for all $x, y \in X$.

**Proposition 1.11.** Let $\mu$ be a fuzzy set of X. Then the following conditions are equivalent:

(i)  $\mu$ is a fuzzy filter of $X$;
(ii)  for all $x \in X$, $\mu(1) \geq \mu(x)$ and for all $x, y, z \in X$, $x \to (y \to z) = 1$ implies $\mu(z) \geq \min\{\mu(x), \mu(y)\}$;
(iii) for each $\alpha \in [0,1]$, the level subset $U(\mu, \alpha) = \{x \in X : \mu(x) \geq \alpha\}$ is a filter of $X$, where $U(\mu, \alpha) \neq \emptyset$.

Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$. A neutrosophic set $A$ in $X$ is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. A simple valued neutrosophic set $A$ can be denoted by $A = \{(x, T_A(x), I_A(x)$ and $F_A(x) : x \in X\}$, where $T_A(x), I_A(x), F_A(x) \in [0,1]$ for each point $x$ in $X$. Therefore, the sum of $T_A(x), I_A(x)$ and $F_A(x)$, satisfies the condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

A simple valued neutrosophic set $A$ is contained in the other simple valued neutrosophic set $B$, denote $A \subseteq B$, if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$ and $F_A(x) \geq F_B(x)$ for any $x \in X$.

Two simple valued neutrosophic sets $A$ and $B$ are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

For convenience, simple valued neutrosophic set is abbreviated to neutrosophic set later.

The union of two neutrosophic sets $A$ and $B$ is a neutrosophic set $C$, written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of $A$ and $B$ by, for all $x \in X$

$$T_C(x) = \max(T_A(x), T_B(x));$$

$$I_C(x) = \min(I_A(x), I_B(x));$$

$$F_C(x) = \min\big(F_A(x), F_B(x)\big).$$

The intersection of two neutrosophic sets $A$ and $B$ is a neutrosophic set $C$, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity- membership functions are related to those of $A$ and $B$ by, for all $x \in X$

$$T_C(x) = \min\big(T_A(x), T_B(x)\big);$$

$$I_C(x) = \max\big(I_A(x), I_B(x)\big);$$

$$F_C(x) = \max\big(F_A(x), F_B(x)\big).$$

**Theorem 1.12.** [25] Let $U$ be a nonempty finite or infinite universe of discourse, and $S$ a nonempty finite or infinite subset of $U$. A classical Algebra is defined on $S$.

In general, for a given classical Algebra, having $n$ operations (laws) and axioms altogether, for integer $n \geq 1$, there are $3^n$ total number of Algebra / NeutroAlgebras / AntiAlgebras as below:

1 (classical) Algebra, $(2^n - 1)$ Neutro-Algebras, and $(3^n - 2^n)$ Anti-Algebras.

The finite or infinite cardinal of set the classical algebra is defined upon, does not influence the numbers of Neutro-BE-algebras and Anti-BE-algebras.

## 2. Neutrosophic filters of RM-algebras

In this section, we introduce the notion of neutrosophic filters in RM-algebras, some basic properties are investigated.

**Definition 2.1.** A neutrosophic set $A$ in a RM-algebra $X$ is called a neutrosophic filter in $X$ if it satisfies: for all $x, y \in X$,

(NF1) $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$;

(NF2) $\min\{T_A(x), T_A(x \to y)\} \leq T_A(y)$, $\max\{I_A(x), I_A(x \to y)\} \geq I_A(y)$ and $\max\{F_A(x), F_A(x \to y)\} \geq F_A(y)$.

**Example 2.2.** Let $X = \{1, a, b, c\}$, and let $\to$ be the binary operation on $X$ defined by the following Cayley table.

Table 1 RM-algebra $(X, \to, 1)$

| $\to$ | 1 | $a$ | $b$ | $c$ |
|-------|---|-----|-----|-----|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $b$ | c |
| $b$ | 1 | $a$ | 1 | b |
| $c$ | 1 | $a$ | 1 | 1 |

Then, $(X, \to, 1)$ is a RM-algebra. Define a neutrosophic set $A$ of $X$ as follows:

$$T_A(x) = \begin{cases} 0.90 & \text{if } x = 1, a; \\ 0.12 & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.16 & \text{if } x = 1, a; \\ 0.72 & \text{otherwise.} \end{cases}$$

And

$$F_A(x) = \begin{cases} 0.14 & \text{if } x = 1, a; \\ 0.62 & \text{otherwise.} \end{cases}$$

Then, $A = \{(x, T_A(x), I_A(x), F_A(x): x \in X\}$ is a neutrosophic filter of $X$.

**Proposition 2.3.** Let $A$ be a neutrosophic filter of $X$ and $x, y \in X$. If $x \leq y$, then $T_A(x) \leq T_A(y)$, $I_A(x) \geq I_A(y)$ and $F_A(x) \geq F_A(y)$.

**Proof.** Assume that $x \leq y$. Then $x \to y = 1$. It follows that $T_A(x \to y) = T_A(1)$. From this, using Definition 4.1 (NF1) and (NF2) we get

$$T_A(x) = \min\{T_A(x), T_A(1)\} = \min\{T_A(x), T_A(x \to y)\} \geq T_A(y).$$

Similarly, we can get $I_A(x) \geq I_A(x)$ and $F_A(x) \geq F_A(x)$.

**Proposition 2.4.** Let $A$ be a neutrosophic filter of CI-algebra $X$. Then for all $a_1, a_2, \dots, a_k, x, y, z \in X$ and $n \in \mathbb{N}$

(i)   $T_A(x) \leq T_A(y \to x)$, $I_A(x) \geq I_A(y \to x)$ and $F_A(x) \geq F_A(y \to x)$;

(ii)   $\min\{T_A(x), T_A(y)\} \leq T_A(x \to y)$, $\max\{I_A(x), I_A(y)\} \geq I_A(x \to y)$ and $\max\{F_A(x), F_A(y)\} \geq F_A(x \to y)$;

(iii) $T_A(x) \leq T_A((x \to y) \to y)$, $I_A(x) \geq I_A((x \to y) \to y)$ and $F_A(x) \geq F_A((x \to y) \to y)$;

(iv) $\min\{T_A(x), T_A(y)\} \leq T_A((x \to (y \to z)) \to z)$, $\max\{I_A(x), I_A(y)\} \geq I_A((x \to (y \to z)) \to z)$ and $\max\{F_A(x), F_A(y)\} \geq F_A\left((x \to (y \to z)) \to z\right)$;

(v)   if $\min\{T_A(x), T_A((x \to y) \to z)\} \leq T_A(z \to x)$, then $T_A$ is order reversing and $I_A$, $F_A$ are order (i.e. if $x \leq y$, then $T_A(y) \leq T_A(x)$, $I_A(y) \geq I_A(x)$ and $F_A(y) \geq F_A(x)$;

(vi) if $z \in A(x, y)$, then $\min\{T_A(x), T_A(y)\} \leq T_A(z)$, $\max\{I_A(x), I_A(y)\} \geq I_A(z)$ and $\max\{F_A(x), F_A(y)\} \geq F_A(z)$;

(vii) if $\prod_{i=1}^{n} a_i \to x = 1$, then $\bigwedge_{i=1}^{n} T_A(a_i) \leq T_A(x)$, $\bigvee_{i=1}^{n} I_A(a_i) \geq I_A(x)$ and $\bigvee_{i=1}^{n} F_A(a_i) \geq F_A(x)$.

**Proof.** (i) Since $x \leq y \to x$, by Proposition 2.3 the proof is clear.

(ii) By using (ii) we have $\min\{T_A(x), T_A(y)\} \leq T_A(x) \leq T_A(x \to y)$, $\max\{I_A(x), I_A(y)\} \geq I_A(x) \geq I_A(x \to y)$ and $\max\{F_A(x), F_A(y)\} \geq F_A(x) \geq F_A(x \to y)$.

(iii) It follows from Definition 4.1,

$$T_A(x) = \min\{T_A(x), T_A(1)\} = \min\{T_A(x), T_A((x \to y) \to (x \to y))\}$$
$$= \min\left\{T_A(x), T_A\left(x \to ((x \to y) \to y)\right)\right\} \leq T_A((x \to y) \to y);$$

$$I_A(x) = \max\{I_A(x), I_A(1)\} = \max\{I_A(x), I_A((x \to y) \to (x \to y))\} = \max\left\{I_A(x), I_A\left(x \to ((x \to y) \to y)\right)\right\}$$
$$\geq I_A((x \to y) \to y).$$

Similarly, $F_A(x) \geq F_A((x \to y) \to y)$.

(iv) From (iv) we have

$$\min\{T_A(x), T_A(y)\} \leq \min\left\{T_A(x), T_A\left((y \to (x \to z)) \to (x \to z)\right)\right\}$$
$$= \min\left\{T_A(x), T_A\left((x \to (y \to z)) \to (x \to z)\right)\right\}$$
$$= \min\left\{T_A(x), T_A\left(x \to ((x \to (y \to z)) \to z)\right)\right\} \leq T_A((x \to (y \to z)) \to z);$$

$$\max\{I_A(x), I_A(y)\} \geq \max\left\{I_A(x), I_A\left((y \to (x \to z)) \to (x \to z)\right)\right\}$$
$$= \max\left\{I_A(x), I_A\left((x \to (y \to z)) \to (x \to z)\right)\right\}$$
$$= \max\left\{I_A(x), I_A\left(x \to ((x \to (y \to z)) \to z)\right)\right\} \geq I_A((x \to (y \to z)) \to z).$$

Similarly, $\max\{F_A(x), F_A(y)\} \geq F_A\left((x \to (y \to z)) \to z\right)$.

(v) Assume that $x \leq y$. Then $x \to y = 1$, and so

$$T_A(y) = \min\{T_A(y), T_A(1 \to 1)\} = \min\{T_A(y), T_A((x \to y) \to 1)\} \leq T_A(1 \to x) = T_A(x);$$

$$I_A(y) = \max\{I_A(y), I_A(1 \to 1)\} = \max\{I_A(y), I_A((x \to y) \to 1)\} \geq I_A(1 \to x) = I_A(x).$$

Similarly, if $x \leq y$, then $F_A(y) \geq F_A(x)$.

(vi) Assume that $z \in A(x, y)$. Then $x \to (y \to z) = 1$, and so

$$\min\{T_A(x), T_A(y)\} = \min\{T_A(x), T_A(y), T_A(1)\} = \min\{T_A(x), T_A(y), T_A(x \to (y \to z))\}$$
$$\leq \min\{T_A(y), T_A(y \to z)\} \leq T_A(z);$$

$$\max\{I_A(x), I_A(y)\} = \max\{I_A(x), I_A(y), I_A(1)\} = \max\{I_A(x), I_A(y), I_A(x \to (y \to z))\}$$
$$\geq \max\{I_A(y), I_A(y \to z)\} \geq I_A(z).$$

Similarly, if $z \in A(x, y)$, then $\max\{F_A(x), F_A(y)\} \geq F_A(z)$.

(vii) The proof is by induction on $n$. By (

ii) it is true for $n = 1,2$. Assume that it satisfies for $n = k$, that is,if $\prod_{i=1}^{k} a_i \to x = 1$, then $\bigwedge_{i=1}^{k} T_A(a_i) \leq T_A(x)$, $\bigvee_{i=1}^{k} I_A(a_i) \geq I_A(x)$ and $\bigvee_{i=1}^{k} F_A(a_i) \geq F_A(x)$, for all $a_1, a_2, \dots, a_k, x \in X$.

Suppose $\prod_{i=1}^{k+1} a_i \to x = 1$, for all $a_1, a_2, \dots, a_k, a_{k+1}, x \in X$. Then

$$\bigwedge_{i=1}^{k+1} T_A(a_i) \leq T_A(a_1 \to x), \bigvee_{i=1}^{k+1} I_A(a_i) \geq I_A(a_1 \to x) \text{ and } \bigvee_{i=1}^{k+1} F_A(a_i) \geq F_A(a_1 \to x).$$

Since $A$ is a neutrosophice filter of $X$, we get

$$\bigwedge_{i=1}^{k+1} T_A(a_i) = \min\left\{\bigwedge_{i=1}^{k+1} T_A(a_i), T_A(a_1)\right\} \leq \min\{T_A(a_1 \to x), T_A(a_1)\} \leq T_A(x);$$

$$\bigvee_{i=1}^{k+1} I_A(a_i) = \max\left\{\bigvee_{i=1}^{k+1} I_A(a_i), I_A(a_1)\right\} \geq \max\{I_A(a_1 \to x), I_A(a_1)\} \geq I_A(x)$$

and

$$\bigvee_{i=1}^{k+1} F_A(a_i) = \max\left\{\bigvee_{i=1}^{k+1} F_A(a_i), F_A(a_1)\right\} \geq \max\{F_A(a_1 \to x), F_A(a_1)\} \geq F_A(x).$$

**Theorem 2.5.** If $\{A_i : i \in \Lambda\}$ is a family of neutrosophic filters of $X$, then $\bigcap_{i \in \Lambda} A_i$, is too.

**Proposition 2.6.** Let $A$ be a neutrosophic filter of $X$, and let $t \in [0,1]$. If

$U(A, t) = \{x \in X : t \leq T_A(x), I_A(x) \leq t \text{ and } F_A(x) \leq t\} \neq \emptyset$, then $U(A, t)$ is a filter of $X$.

**Proof.** Assume that $A$ is neutrosophic filter of $X$, and let $t \in [0,1]$ such that $U(A, t) \neq \emptyset$. Then there exists $x_0 \in X$ such that $t \leq T_A(x_0), I_A(x_0) \leq t$ and $F_A(x_0) \leq t$. By applying Definition 2.1 (NF1) we have

$$t \leq T_A(x_0) \leq T_A(1), I_A(1) \leq I_A(x_0) \leq t \text{ and } F_A(1) \leq F_A(x_0) \leq t.$$

This means that $1 \in U(A, t)$. Let $x, y \in X, x \to y \in U(A, t)$ and $x \in U(A, t)$.

Then $t \leq T_A(x \to y)$, $I_A(x \to y) \leq t$ and $F_A(x \to y) \leq t$, $t \leq T_A(x)$, $I_A(x) \leq t$ and $F_A(x) \leq t$. From these, using Definition 2.1 (NF2) we have

$$t \leq \min\{T_A(x), T_A(x \to y)\} \leq T_A(y), \quad I_A(y) \leq \max\{I_A(y), I_A(x \to y) \leq t.$$

And $F_A(y) \leq \max\{F_A(y), F_A(x \to y) \leq t$, and so $y \in U(A, t)$. Thus, $U(A, t)$ is a filter of $X$.

The following example shows that the converse of Proposition 2.6 may not be true in general.

**Example 2.7.** Let $X = \{1, a, b, c, d\}$, and let $\to$ be the binary operation on $X$ defined by the following Cayley table.

Table 1 RM-algebra $(X, \to, 1)$

| $\to$ | 1 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ |
| $a$ | 1 | 1 | $b$ | c | $b$ |
| $b$ | 1 | $a$ | 1 | b | $a$ |
| $c$ | 1 | $a$ | 1 | 1 | $a$ |
| $d$ | 1 | 1 | 1 | b | 1 |

Then, $(X, \to, 1)$ is a RM-algebra. Define a neutrosophic set $A$ of $X$ as follows:

$$T_A(x) = \begin{cases} 0.62 & \text{if } x = 1, a; \\ 0.12 & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.12 & \text{if } x = 1, a; \\ 0.62 & \text{otherwise.} \end{cases}$$

And

$$F_A(x) = \begin{cases} 0.08 & \text{if } x = 1, a, b; \\ 0.62 & \text{otherwise.} \end{cases}$$

If $t > 0.62$, then $U(A, t) = \emptyset$;

If $0.62 \geq t > 0.12$, then $U(A, t) = \{1, a\}$;

If $0.12 \geq t > 0.08$, then $U(A, t) = \emptyset$;

If $0.08 \geq t$, then $U(A, t) = \emptyset$.

Then, for all $t \in [0,1]$, $U(A, t) = \{x \in X : t \leq T_A(x), I_A(x) \leq t \text{ and } F_A(x) \leq t\} \neq \emptyset$, imply $U(A, t)$ is a filter of $X$, but $A$ is not a neutrosophic filter of $X$, since

$$\max\{F_A(b), F_A(b \to d) = \max\{F_A(b), F_A(a)\} = \max\{0.08, 0.08\} = 0.08 \ngeq 0.62 = F_A(d).$$

**Theorem 2.8.** Let $A$ be a neutrosophic set of $X$. Then $A$ is a neutrosophic filter in $X$ if and only if $A$ satisfies: for all $x \in X$

(i) $T_A$ is a fuzzy filter of $X$;

(ii) $1 - I_A$ is a fuzzy filter of X, where $(1 - I_A)(x) = 1 - I_A(x)$;

(iii) $1 - F_A$ is a fuzzy filter of X, where $(1 - F_A)(x) = 1 - F_A(x)$.

Applying Theorem 2.8 and Proposition 1.11 (iii) we can get:

**Corollary 2.9.** Let $A$ be a neutrosophic filter of $X$, and let $t \in [0,1]$. Then:

(i)  $U(T_A, t) = \{x \in X : T_A(x) \geq t\}$ is a filter of $X$ when $U(T_A, t) \neq \emptyset$;

(ii) $U(1 - I_A, t) = \{x \in X : 1 - I_A(x) \geq t\}$ is a filter of $X$ when $U(1 - I_A, t) \neq \emptyset$;

(iii) $U(1 - F_A, t) = \{x \in X : 1 - F_A(x) \geq t\}$ is a filter of $X$ when $U(1 - F_A, t) \neq \emptyset$.

Let $\alpha, \beta, \gamma \in [0,1]$, where $\leq \alpha + \beta + \gamma \leq 3$. For a neutrosophic set $A$ of $X$, $(\alpha, \beta, \gamma)$-level set of $A$ denoted by $A^{(\alpha, \beta, \gamma)}$ is defined as follows:

$$A^{(\alpha, \beta, \gamma)} = \{x \in X : T_A(x) \geq \alpha, I_A(x) \leq \beta \text{ and } F_A(x) \leq \gamma\}.$$

**Theorem 2.10.** Let $A$ be a neutrosophic set of $X$. Then $A$ is a neutrosophic filter of $X$ if and only if all of $(\alpha, \beta, \gamma)$-level set $A^{(\alpha, \beta, \gamma)}$ are filters of $X$ when $\alpha, \beta, \gamma \in [0,1]$, such that $A^{(\alpha, \beta, \gamma)} \neq \emptyset$.

**Proof.** Assume that $A$ is a neutrosophic filter in $X$, and let $\alpha, \beta, \gamma \in [0,1]$ such that $A^{(\alpha, \beta, \gamma)} \neq \emptyset$. Then $U(T_A, \alpha) \neq \emptyset$, $U(1 - I_A, 1 - \beta) \neq \emptyset$ and $U(1 - F_A, 1 - \gamma) \neq \emptyset$. Applying Theorem 2.8 and Proposition 1.11 (iii), we get that $U(T_A, \alpha)$, $U(1 - I_A, 1 - \beta)$ and $U(1 - F_A, 1 - \gamma)$. are filters of $X$.

Thus, $U(T_A, \alpha) \cap U(1 - I_A, 1 - \beta) \cap U(1 - F_A, 1 - \gamma)$ is also a filter of $X$. Moreover, it is easy to verify that $A^{(\alpha, \beta, \gamma)} = U(T_A, \alpha) \cap U(1 - I_A, 1 - \beta) \cap U(1 - F_A, 1 - \gamma)$. Hence, $(\alpha, \beta, \gamma)$-level set $A^{(\alpha, \beta, \gamma)}$ is a filter of $X$.

Conversely, assume that all of $(\alpha, \beta, \gamma)$-level sets $A^{(\alpha, \beta, \gamma)}$ are filters of $X$ when $\alpha, \beta, \gamma \in [0,1]$ such that $A^{(\alpha, \beta, \gamma)} \neq \emptyset$. For any $t \in [0,1]$, if $U(T_A, t) = \{x \in X : T_A(x) \geq t\} = \emptyset$, then there exists $x_0 \in X$ such that $T_A(x_0) \geq t$. Obviously, $I_A(x_0) \leq 1$ and $F_A(x_0) \leq 1$. Then $x_0 \in A^{(t,1,1)}$, that is,

$$A^{(t,1,1)} = \{x \in X : t \leq T_A(x), I_A(x) \leq 1 \text{ and } F_A(x) \leq 1\} = \{x \in X : t \leq T_A(x)\} = U(T_A, \alpha) \neq \emptyset.$$

Hence, by the assumption $U(T_A, t) = A^{(t,1,1)}$ is a filter of $X$. Applying Proposition 1.11 we know that $T_A$ is a fuzzy filter of $X$. Moreover, for any $t \in [0,1]$, if

$$U(1 - I_A, t) = \{x \in X : 1 - I_A(x) \geq t\} \neq \emptyset,$$

then there exists $x_0 \in X$ such that $1 - I_A(x_0) \geq t$, that is, $I_A(x_0) \leq 1 - t$. Obviously, $T_A(x_0) \geq 0, F_A(x_0) \leq 1$. It follows that $x \in A^{(0, 1-t, 1)}$, that is,

$$A^{(1, 1-t, 1)} = \{x \in X : 0 \leq T_A(x), I_A(x) \leq 1 - t \text{ and } F_A(x) \leq 1\} = \{x \in X : I_A(x) \leq 1 - t\} = U(1 - I_A, t) \neq \emptyset.$$

Hence, by the assumption $U(1 - I_A, t) = A^{(1, 1-t, 1)}$ is a filter of $X$. Applying Proposition 1.11 we know that $1 - I_A$ is a fuzzy filter of $X$.

Similarly, we can get that $1 - F_A$ is a fuzzy filter of $X$. Combining the above results, using Theorem 2.8, we know that $A$ is a neutrosophic filter of $X$.

Applying Theorem 2.10 we can get:

**Corollary 2.11.** Let $A$ be a neutrosophic filter of $X$, we denote

(i)  $A_T = \{x \in X : T_A(x) = T_A(1)\}$;

(ii) $A_I = \{x \in X : I_A(x) = I_A(1)\}$;

(iii) $A_F = \{x \in X : F_A(x) = F_A(1)\}$.

Then $A_T$, $A_I$ and $A_F$ are filters of $X$.

**Corollary 2.12.** Let $A$ be a neutrosophic filter of $X$, and let $b \in X$. Define

$$A_b = \{x \in X : T_A(x) \geq T_A(b), I_A(x) \leq I_A(b) \text{ and } F_A(x) \leq F_A(b)\}.$$

Then $A_b$ is a filter of $X$.

**Theorem 2.13.** Let $A$ be a neutrosophic set of $X$. Then $A$ is a neutrosophic filter of $X$ if and only if it satisfies (NF1) and if for all $x, y, z \in X$, $x \to (y \to z) = 1$, then $\min\{T_A(x), T_A(y)\} \leq T_A(z)$, $\max\{I_A(x), I_A(y)\} \geq I_A(z)$ and $\max\{F_A(x), F_A(y)\} \geq F_A(z)$.

**Proof.** Assume that $A$ be a neutrosophic filter of $X$, and let $x, y, z \in X$. Suppose $x \to (y \to z) = 1$. Applying Definition 2.1 we have

$$T_A(y \to z) \geq \min\{T_A(x), T_A(x \to (y \to z))\} = \min\{T_A(x), T_A(1)\} = T_A(x);$$

$$T_A(z) \geq \min\{T_A(y \to z), T_A(y)\} \geq \min\{T_A(x), T_A(y)\};$$

$$I_A(y \to z) \leq \max\{I_A(x), I_A(x \to (y \to z))\} = \max\{I_A(x), I_A(1))\} = I_A(x);$$

$$I_A(z) \leq \max\{I_A(y \to z), I_A(y)\} \leq \max\{I_A(x), I_A(y))\}$$

and

$$F_A(y \to z) \leq \max\{F_A(x), F_A(x \to (y \to z))\} = \max\{F_A(x), F_A(1))\} = F_A(x);$$

$$F_A(z) \leq \max\{F_A(y \to z), F_A(y)\} \leq \max\{F_A(x), F_A(y))\}.$$

Conversely, using (CI1) we have $(x \to y) \to (x \to y) = 1$.

Thus, $\min\{T_A(x \to y), T_A(x)\} \leq T_A(y)$, $\max\{I_A(x \to y), I_A(x)\} \geq I_A(y)$ and $\max\{F_A(x \to y), F_A(x)\} \geq F_A(y)$.

Thus, $A$ is a neutrosophic filter of $X$.

# 3. Implicative neutrosophic filters of RM-algebras

In this section, we introduce the notion of implicative neutrosophic filters in RM-algebras. The relation between implicative neutrosophic filters and neutrosophic filters is investigated and we show that in self distributive CI-algebras these notions are equivalent.

**Definition 3.1.** A neutrosophic set $A$ of $X$ is called an implicative neutrosophic filter if it satisfies: for all $x, y, z \in X$

(INF1) $T_A(x) \leq T_A(1)$; $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$;

(INF2) $T_A(x \to z) \geq \min\{T_A(x \to y), T_A(x \to (y \to z))\}$; $I_A(x \to z) \leq \max\{I_A(x \to y), I_A(x \to (y \to z))\}$ and $F_A(x \to z) \leq \max\{F_A(x \to y), F_A(x \to (y \to z))\}$.

**Example 3.2.** Let $X = \{1, a, b, c\}$, and let $\to$ be the binary operation on $X$ defined by the following Cayley table.

Table 2 RM-algebra $(X, \to, 1)$

| $\to$ | a | b | c | 1 |
|---|---|---|---|---|

| a | 1 | b | c | 1 |
|---|---|---|---|---|
| b | 1 | 1 | b | 1 |
| c | a | 1 | 1 | 1 |
| 1 | a | b | c | 1 |

Then, $(X, \to, 1)$ is a RM-algebra. Define a neutrosophic set $A$ of $X$ as follows:

$$T_A(x) = \begin{cases} 0.87 & \text{if } x = 1; \\ 0.69 & \text{if } x = b; \\ 0.11 & \text{otherwise.} \end{cases}$$

$$I_A(x) = \begin{cases} 0.09 & \text{if } x = 1; \\ 0.15 & \text{if } x = b; \\ 0.84 & \text{otherwise.} \end{cases}$$

And

$$F_A(x) = \begin{cases} 0.05 & \text{if } x = 1; \\ 0.14 & \text{if } x = b; \\ 0.79 & \text{otherwise.} \end{cases}$$

Then, $A = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\}$ is an implicative neutrosophic filter of $X$.

If we replace $x$ of the condition (INF2) by the element 1, then it can be easily observed that every implicative neutrosophic filter is a neutrosophic filter. However, every neutrosophic filter is not an implicative neutrosophic filter as shown in the following example.

**Example 3.3.** Consider Example 3.2. Define a neutrosophic set $A$ of $X$ as follows:

$$T_A(x) = \begin{cases} 0.90 & \text{if } x = 1, a; \\ 0.10 & \text{otherwise.} \end{cases}$$

$$I_A(x) = F_A(x) = \begin{cases} 0.12 & \text{if } x = 1, a; \\ 0.85 & \text{otherwise.} \end{cases}$$

Then $A$ is not an implicative neutrosophic filter of $X$, since

$$\min\{T_A(b \to (d \to c)), T_A(b \to d)\} = \min\{T_A(1), T_A(a)\} = \min\{0.90, 0.90\} = 0.90 \nleq 0.10 = T_A(b)$$
$$= T_A(b \to c).$$

**Theorem 3.4.** Let $X$ be a self distributive CI-algebra. Then every neutrosophic filter of $X$ is an implicative neutrosophic filter of $X$.

**Proof.** Assume that $A$ is a neutrosophic filter of $X$ and $x \in X$. Obviously, $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$. By definition of self distributivity and (NF2), we have

$$\min\{T_A(x \to (y \to z)), T_A(x \to y)\} = \min\{T_A((x \to y) \to (x \to z)), T_A(x \to y)\} \leq T_A(x \to z);$$

$$\max\{I_A(x \to (y \to z)), I_A(x \to y)\} = \max\{I_A((x \to y) \to (x \to z)), I_A(x \to y)\} \geq T_A(x \to z)$$

and

$$\max\{F_A(x \to (y \to z)), F_A(x \to y)\} = \max\{F_A((x \to y) \to (x \to z)), F_A(x \to y)\} \geq F_A(x \to z).$$

Thus, $A$ is an implicative neutrosophic filter of $X$.

**Proposition 3.5.** Let $A$ be an implicative neutrosophic of CI-algebra $X$. Then $A$ satisfies the following conditions: for all $x, y \in X$

(i) $\quad T_A(x \to y) = T_A(x \to (x \to y))$;

(ii) $\quad I_A(x \to y) = I_A(x \to (x \to y))$;

(iii) $F_A(x \to y) = F_A\big(x \to (x \to y)\big)$.

**Proof.** Putting $y = x$ and $z = y$ in (INF2), we can get

$$\min\{T_A\big(x \to (x \to y)\big), T_A(x \to x)\} \le T_A(x \to y);$$

$$\max\{I_A\big(x \to (x \to y)\big), I_A(x \to x)\} \ge T_A(x \to y) \text{ and}$$

$$\max\{F_A\big(x \to (x \to y)\big), F_A(x \to x)\} \ge F_A(x \to y).$$

Applying (RM1) and (INF1) we have

$$T_A(x \to (x \to y)) = \min\{T_A\big(x \to (x \to y)\big), \; T_A(1)\} = \min\{T_A\big(x \to (x \to y)\big), T_A(x \to x)\} \le T_A(x \to y);$$

$$I_A(x \to (x \to y)) = \max\{I_A\big(x \to (x \to y)\big), \; I_A(1)\} = \max\{I_A\big(x \to (x \to y)\big), I_A(x \to x)\} \ge I_A(x \to y)$$

and

$$F_A(x \to (x \to y)) = \max\{F_A\big(x \to (x \to y)\big), \; F_A(1)\} = \max\{F_A\big(x \to (x \to y)\big), F_A(x \to x)\} \ge F_A(x \to y)$$

On the other hand, since $x \to y \le x \to (x \to y)$, we get

$$T_A(x \to y) \le T_A\big(x \to (x \to y)\big), \; I_A(x \to y) \ge I_A\big(x \to (x \to y)\big) \text{ and } F_A(x \to y) \ge F_A\big(x \to (x \to y)\big).$$

Combining the above two hands, we get

$$T_A(x \to y) = T_A(x \to (x \to y)), \; I_A(x \to y) = I_A(x \to (x \to y)) \text{ and } F_A(x \to y) = F_A(x \to (x \to y)).$$

**Theorem 3.6.** Let $A$ be a neutrosophic filter in a transitive CI-algebra $X$. Then $A$ is an implicative neutrosophic filter of $X$ if and only if it satisfies: for all $x, y \in X$

(i) $\quad T_A(x \to y) = T_A(x \to (x \to y))$;

(ii) $\quad I_A(x \to y) = I_A(x \to (x \to y))$;

(iii) $F_A(x \to y) = F_A\big(x \to (x \to y)\big)$.

**Proof.** If $A$ is an implicative neutrosophic filter of $X$, then by Proposition 1.10 we know that the conditions (i), (ii) and (iii) hold.

Conversely, suppose $A$ satisfies the conditions (i), (ii) and (iii). For any $x, y, z \in X$, by the definition of a transitive CI-algebra we have

$$x \to y \le (y \to z) \to (x \to z) \le (x \to (y \to z)) \to (x \to (x \to z));$$

$$(x \to y) \to ((x \to (y \to z)) \to (x \to (x \to z))) = 1$$

and

$$(x \to (y \to z)) \to ((x \to y) \to (x \to (x \to z))) = 1.$$

Applying Theorem 2.13 we have

$$\min\{T_A\big(x \to (y \to z)\big), T_A(x \to y)\} \le T_A(x \to (x \to z));$$

$$\max\{I_A(x \to (y \to z)), I_A(x \to y)\} \geq I_A(x \to (x \to z))$$

and

$$\max\{F_A(x \to (y \to z)), F_A(x \to y)\} \geq F_A(x \to (x \to z)).$$

From these, by (i), (ii) and (iii) we get

$$\min\{T_A(x \to (y \to z)), T_A(x \to y)\} \leq T_A(x \to (x \to z)) = T_A(x \to z);$$

$$\max\{I_A(x \to (y \to z)), I_A(x \to y)\} \geq I_A(x \to (x \to z)) = I_A(x \to z)$$

and

$$\max\{F_A(x \to (y \to z)), F_A(x \to y)\} \geq F_A(x \to (x \to z)) = F_A(x \to z).$$

Thus, $A$ is an implicative neutrosophic filter of $X$.

**Theorem 3.7.** Let $X$ be a self distributive CI-algebra, and let $A$ be a neutrosophic filter of $X$. Then the following conditions are equivalent: for all $x, y, z \in X$

(i)  $A$ is an implicative neutrosophic filter of $X$;

(ii) $T_A(x \to (x \to y)) \leq T_A(x \to y); I_A(x \to (x \to y)) \geq I_A(x \to y)$ and $F_A(x \to (x \to y)) \geq F_A(x \to y);$

(iii) $\min\{T_A(z \to (x \to (x \to y))), T_A(z)\} \leq T_A(x \to y);$

$\max\{I_A(z \to (x \to (x \to y))), I_A(z)\} \geq I_A(x \to y)$ and $\max\{F_A(z \to (x \to (x \to y))), F_A(z)\} \geq F_A(x \to y).$

**Proof.** $(i \to ii)$. Let $x, y \in X$. Using (INF2), we have

$$\min\{T_A(x \to (x \to y)), T_A(x \to x)\} \leq T_A(x \to y);$$

$$\max\{I_A(x \to (x \to y)), I_A(x \to x)\} \geq I_A(x \to y)$$

and

$$\max\{F_A(x \to (x \to y)), F_A(x \to x)\} \geq F_A(x \to y).$$

Applying (CI1) and (INF1), we have

$$T_A(x \to (x \to y)) = \min\{T_A(x \to (x \to y)), T_A(1)\} = \min\{T_A(x \to (x \to y)), T_A(x \to x)\} \leq T_A(x \to y);$$

$$I_A(x \to (x \to y)) = \max\{I_A(x \to (x \to y)), I_A(1)\} = \max\{I_A(x \to (x \to y)), I_A(x \to x)\} \geq I_A(x \to y)$$

and

$$F_A(x \to (x \to y)) = \max\{F_A(x \to (x \to y)), F_A(1)\} = \max\{F_A(x \to (x \to y)), F_A(x \to x)\} \geq F_A(x \to y).$$

Hence, (2) holds.

$(ii \to iii)$. Assume that $x, y, z \in X$. By (NSF2), we have:

$$\min\{T_A(x \to (x \to y)), T_A(z)\} \leq T_A(x \to (x \to y));$$

$$\max\{I_A(x \to (x \to y)), I_A(z)\} \geq I_A(x \to (x \to y))$$

and

$$\max\{F_A\big(x \to (x \to y)\big), F_A(z)\} \geq F_A(x \to (x \to y)).$$

From these, using (ii) we get

$$\min\{T_A\big(x \to (x \to y)\big), T_A(z)\} \leq T_A(x \to y);$$

$$\max\{I_A\big(x \to (x \to y)\big), I_A(z)\} \geq I_A(x \to y)$$

and

$$\max\{F_A\big(x \to (x \to y)\big), F_A(z)\} \geq F_A(x \to y).$$

Therefore, (iii) holds.

$(iii \to ii)$. Assume that $x, y, z \in X$, by the definition of a self distributive CI-algebra, we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to \big(x \to (x \to z)\big).$$

By applying Proposition 2.3, we get

$$T_A(x \to (y \to z)) \leq T_A\Big((x \to y) \to \big(x \to (x \to z)\big)\Big);$$

$$I_A(x \to (y \to z)) \geq I_A\Big((x \to y) \to \big(x \to (x \to z)\big)\Big)$$

and

$$F_A\big(x \to (y \to z)\big) \geq F_A\Big((x \to y) \to \big(x \to (x \to z)\big)\Big).$$

From these, using (iii) we get

$$\min\{T_A\big(x \to (y \to z)\big), T_A(x \to y)\} \leq \min\Big\{T_A\Big((x \to y) \to \big(x \to (x \to z)\big)\Big), T_A(x \to y)\Big\} \leq T_A(x \to z);$$

$$\max\{I_A\big(x \to (y \to z)\big), I_A(x \to y)\} \geq \max\Big\{I_A\Big((x \to y) \to \big(x \to (x \to z)\big)\Big), I_A(x \to y)\Big\} \geq I_A(x \to z)$$

and

$$\max\{F_A\big(x \to (y \to z)\big), F_A(x \to y)\} \geq \max\Big\{F_A\Big((x \to y) \to \big(x \to (x \to z)\big)\Big), F_A(x \to y)\Big\} \geq F_A(x \to z).$$

Thus, $A$ is an implicative neutrosophic filter of $X$.

**Theorem 3.8.** Let $A$ be a neutrosophic set of $X$. Then $A$ is an implicative neutrosophic filter in $X$ if and only if $A$ satisfies: for all $x \in X$

(i)   $T_A$ is a fuzzy implicative filter of $X$;

(ii)  $1 - I_A$ is a fuzzy implicative filter of X, where $(1 - I_A)(x) = 1 - I_A(x)$;

(iii) $1 - F_A$ is a fuzzy implicative filter of X, where $(1 - F_A)(x) = 1 - F_A(x)$.

**Theorem 3.9.** Let $A$ be a neutrosophic set of $X$. Then $A$ is an implicative  neutrosophic filter of $X$ if and only if all of $(\alpha, \beta, \gamma)$-level set $A^{(\alpha,\beta,\gamma)}$ are implicative filters of $X$ when $\alpha, \beta, \gamma \in [0,1]$, such that $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.

## 4. On Neutro-RM-algebras and Anti-RM-algebras

The Neutrosophy's Triplet is (A, Neutro-A, Anti-A), where A may be an item (concept, idea, proposition, theory, structure, algebra, etc.), Anti-A the opposite of A, while Neutro-A {also the notation Neutro-A was employed before} the neutral between these opposites. Based on the above triplet the following Neutrosophic Principle one has: a law of composition defined on a given set may be true (T) for some set elements,

indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroLaw. A law of composition defined on a given sets, such that the law is false (F) for all set's elements is called AntiLaw. Similarly, an operation defined on a given set may be well-defined for some set elements, indeterminate for other set's elements, and undefined for the remainder of the set's elements; we call it NeutroOperation. While, an operation defined on a given set that is undefined for all set's elements is called AntiOperation.

In classical algebraic structures, the laws of compositions or operations defined on a given set are automatically well-defined [i.e. true (T) for all set's elements], but this is idealistic. Consequently, an axiom (let's say Commutativity, or Associativity, etc.) defined on a given set, may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroAxiom. In classical algebraic structures, similarly an axiom defined on a given set is automatically true (T) for all set's elements, but this is idealistic too. A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or some NeutroAxiom. The NeutroLaw, NeutroOperation, NeutroAxiom, NeutroAlgebra and respectively AntiLaw, AntiOperation, AntiAxiom and AntiAlgebra were introduced by Smarandache in 2019-2020 [29-31]. Recently, the concept of a Neutrosophic Triplet of BI-algebra was defined [26].

**Definition 4.1. (Neutro-sophications)**

**The Neutro-sophication of the Law** (degree of well-defined, degree of indeterminacy, degree of outer-defined)

(NL) $(\exists x, y \in X)(x \to y \in X)$ and $(\exists x, y \in X)(x \to y = \text{ indeterminate or} \to y \notin X)$.

**The Neutro-sophication of the Axioms** (degree of truth, degree of indeterminacy, degree of falsehood)

(NRM1) $(\exists x \in X)(x \to x = 1)$ and $(\exists x \in X)(x \to x = \text{indeterminate or } x \to x \neq 0)$;

(NRM2) $(\exists x \in X)(x \to 1 = 1)$ and $(\exists x \in X)(x \to 1 = \text{indeterminate or } x \to 0 \neq 0)$;

(NCI) $(\exists x, y, z \in X, \text{with } x \neq y)(x \to (y \to z) = y \to (x \to z))$ and

$$(\exists x, y, z \in X, \text{with } x \neq y)\big(x \to (y \to z) = \text{indeterminate  or } x \to (y \to z) \neq y \to (x \to z)\big).$$

**Definition 4.2. (Anti-sophications)**

**The Anti-sophication of the Law** (totally outer-defined)

(AL) $(\forall x, y \in X)(x \to y \notin X)$.

**The Anti-sophication of the Axioms** (totally false)

(ARM1) $(\forall x \in X)(x \to x \neq 1)$;

(ARM2) $(\forall x \in X)(x \to 1 \neq 1)$;

(ACI) $(\forall x, y, z \in X, \text{with } x \neq y)\big(x \to (y \to z) \neq y \to (x \to z)\big)$.

**Definition 4.3. (Neutro-RM-algebras)**

A Neutro-RM-algebra is an alternative of RM-algebra that has at least a (NL) or at least one (NRMi), $i \in \{1, 2\}$, with no anti-law and no anti-axiom.

**Definition 4.4. (Neutro-CI-algebras)**

A Neutro-CI-algebra is an alternative of CI-algebra that is a Neutro-RM-algebra or has at least (NCI), but not satisfying (ACI).

**Example 4.5.** [25] (i) Let $\mathbb{N}$ be the set of all natural numbers and $\rightarrow$ be the Neutro-sophication of the Law $\rightarrow$ on $\mathbb{N}$ defined by

$$x \rightarrow y = \begin{cases} y & \text{if } x = 1; \\ \dfrac{1}{4} & \text{if } x \in \{3,5,7\}; \\ 1 & \text{otherwise.} \end{cases}$$

Then $(\mathbb{N}, \rightarrow, 1)$ is a Neutro-CI-algebra. Since

(NL) if $x \in \{3,5,7\}$, then $x \rightarrow y = \frac{1}{4} \notin \mathbb{N}$, for all $y \in \mathbb{N}$, while if $x \notin \{3,5,7\}$ and $x \in \mathbb{N}$, then $x \rightarrow y \in \{1, y\} \subseteq \mathbb{N}$, for all $y \in \mathbb{N}$.

(NCI1) $1 \rightarrow 1 = 1 \in \mathbb{N}$ and $3 \rightarrow 3 = \frac{1}{4} \notin \mathbb{N}$;

(NCI2) $5 \rightarrow 1 = \frac{1}{2} \neq 1$ and if $x \notin \{3,5,7\}$, then $x \rightarrow 1 = 1$;

(NCI3) $5 \rightarrow (3 \rightarrow 4) = 5 \rightarrow \frac{1}{4} = ?$ (indeterminate) and $3 \rightarrow (5 \rightarrow 4) = 3 \rightarrow \frac{1}{4} = ?$ (indeterminate).

Also, $2 \rightarrow (3 \rightarrow 4) = 2 \rightarrow \frac{1}{4} = ?$ (indeterminate), but $3 \rightarrow (2 \rightarrow 4) = 3 \rightarrow 1 = \frac{1}{4}$.

Further, $4 \rightarrow (8 \rightarrow 2) = 4 \rightarrow 1 = 1 = 8 \rightarrow (4 \rightarrow 2)$.

(ii) Let $S$ be a nonempty set and $\mathcal{P}(S)$ be the power set of $S$. Then $(\mathcal{P}(S), \cap, \emptyset)$ is a Neutro-$BE$-algebra. $\cap$ is the binary set intersection operation, but (NCI1) is valid, since $\emptyset \cap \emptyset = \emptyset$ and for all $\emptyset \neq A \in \mathcal{P}(S)$, $A \cap A = A \neq \emptyset$.

(CI2) holds, since $A \cap \emptyset = \emptyset$;

(CI3) holds, since $A \cap (B \cap C) = B \cap (A \cap C)$.

(iii) Similarly, $(\mathcal{P}(S), \cup, \emptyset)$, $(\mathcal{P}(S), \cap, S)$, $(\mathcal{P}(S), \cup, S)$, where $\cup$ is the binary set union operation, are Neutro-$BE$-algebras.

(iv) Let $\mathbb{R}$ be the set of all real numbers and $\rightarrow$ be a binary operation on $\mathbb{R}$ defined by $x \rightarrow y = |x - y|$. Then $(\mathbb{R}, \rightarrow, 0)$ is a Neutro-CI-algebra.

(CI1) holds, since $x \rightarrow x = |x - x| = 0$, for all $x \in \mathbb{R}$;

(NCI2) is valid, since if $x \neq 0$, then $0 \rightarrow x = |0 - x| = |-x| \neq 0$, and if $x = 0$, then $0 \rightarrow 0 = 0$.

(NCI3) holds, if $x = 2$, $y = 3$, $z = 4$ we get $|2 - |3 - 4|| = |2 - 1| = 1$ and $|3 - |2 - 4|| = |3 - 2| = 1$;

while for $x = 4$, $y = 8$, $z = 3$ we get $|4 - |8 - 3|| = |4 - 5| = 1$ and $|8 - |4 - 3|| = |8 - 1| = 7 \neq 1$.

**Theorem 4.6.** The total number of Neutro-RM-algebras is 7.

**Proof.** The classical RM-algebra has 1 classical Law and 2 classical Axioms. Then has $1 + 2 = 3$ classical mathematical propositions. Using Theorem 1.12, we get $2^3 - 1 = 7$.

**Theorem 4.7.** The total number of Neutro-CI-algebras is 15.

**Proof.** The classical CI-algebra has 1 classical Law and 3 classical Axioms. Then has $1 + 4 = 4$ classical mathematical propositions. Using Theorem 1.12, we get $2^4 - 1 = 15$.

**Definition 4.8. (Anti-RM-algebras)**

An Anti-RM-algebra is an alternative of RM-algebra that has at least an (AL) or at least one (ARMi), $i \in \{1, 2\}$.

**Definition 4.9. (Anti-CI-algebras)**

An Anti-RM-algebra is an alternative of RM-algebra that is an Anti-RM-algebra or satisfying (ACI).

**Example 4.10.** [25] (i) Let $\mathbb{N}$ be the natural number set and $X := \mathbb{N} \cup \{0\}$. Define a binary operation $\rightarrow$ on $X$ by $x \rightarrow y = x^2 + y^2 + 1$. Then $(X, \rightarrow, 0)$ is not a CI-algebra, nor a Neutro-CI-algebra, but an Anti-CI-algebra. Since $x \rightarrow x = x^2 + x^2 + 1 \neq 0$, for all $x \in X$, and so (ACI1) holds.

For all $x \in \mathbb{N}$, we have $x \rightarrow 0 = x^2 + 1 \neq 0$, so (ACI2) is valid.

Since for $x \neq y$, one has $x \rightarrow (y \rightarrow z) = x^2 + (y^2 + z^2 + 1)^2 + 1 \neq y \rightarrow (x \rightarrow z) = y^2 + (x^2 + z^2 + 1)^2 + 1$, thus, (ACI3) is valid.

(ii) Let $S$ be a nonempty set and $\mathcal{P}(S)$ be the power set of $S$. Define the binary operation $\Delta$ (i.e. symmetric difference) by $A\Delta B = (A \cup B) - (A \cap B)$ for every $A, B \in \mathcal{P}(S)$. Then $(\mathcal{P}(S), \Delta, S)$ is not a CI-algebra, nor Neutro-CI-algebra, but it is an Anti-CI-algebra.

Since $A\Delta A = \emptyset \neq S$ for every $A \in \mathcal{P}(S)$ we get (ACI1) holds, and so (CI1) and (NCI1) are not valid.

Also, for all $A, B, C \in \mathcal{P}(S)$ one has $A\Delta(B\Delta C) = B\Delta(A\Delta C)$. Thus, (CI3) is valid.

Since there is at least one anti-axiom (ACI1), then $(\mathcal{P}(S), \Delta, S)$ is an Anti-CI-algebra.

(iii) Let $\mathcal{U} = \{1, a, b, c, d\}$ be a universe of discourse, and a subset $X = \{1, c\}$, and the below binary well-defined Law $\rightarrow$ with the following Cayley table.

Table 3 Anti-CI-algebra $(X, \rightarrow, 1)$

| $\rightarrow$ | 1 | c |
|---|---|---|
| 1 | c | 1 |
| c | c | c |

Then $(X, \rightarrow, 1)$ is an Anti-CI-algebra, since (ACI1) is valid, because: $0 \rightarrow 0 = c \neq 0$ and $c \rightarrow c = c \neq 0$, and it is sufficient to have a single anti-axiom.

**Theorem 4.11.** The total number of Anti-RM-algebras is 19.

**Proof.** The classical RM-algebra has 1 classical Law and 2 classical Axioms. Then has $1 + 2 = 3$ classical mathematical propositions. Using Theorem 1.12, we get $3^3 - 2^3 = 19$.

**Theorem 4.12.** The total number of Anti-CI-algebras is 65.

**Proof.** The classical CI-algebra has 1 classical Law and 3 classical Axioms. Then has $1 + 3 = 4$ classical mathematical propositions. Using Theorem 1.12, we get $3^4 - 2^4 = 65$.

**Corollary 4.13.** Let X be a RM-algebras. Then

1 (classical) RM-algebra $+$ 7 Neutro-RM-algebras $+$ 19 Anti-BE-algebras $= 27 = 3^3$ algebras. Where, $7 = 2^3 - 1$, and $19 = 3^3 - 2^3$.

**Proof.** It results from the previous Theorems 4.6 and 4.11.

**Corollary 4.14.** Let X be a CI-algebras. Then

1 (classical) CI-algebra $+$ 15 Neutro-CI-algebras $+$ 65 Anti-CI-algebras $= 81 = 3^4$ algebras. Where, $15 = 2^4 - 1$, and $65 = 3^4 - 2^4$.

**Proof.** It results from the previous Theorems 4.7 and 4.12.

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# Chapter Four

# Neutro–G Modules and Anti–G Modules

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## ABSTRACT

In this chapter, neutro-G modules and anti-G modules are defined, corresponding basic properties and examples for neutro-G modules and anti-G modules are given and proved. Moreover, classical G modules and neutro-G modules are compared to each other. Neutro-G modules are shown to have a more general structure according to neutro-G modules. Thus, (T, I, F) components which constitute the neutrosophic theory are added to classical G modules (without using neutrosophic sets) and a new structure is obtained. In addition, it is shown that a neutro-G module can be obtained from every classical G module and a neutro-G module can be obtained from every anti-G module.

**Keywords:** G-Modules, Neutrosophic Theory, Neutro-algebraic structures, Neutro-G Modules, Anti-G Modules

## INTRODUCTION

Curties defined G Modules [1] in 1972. G Modules built on classical group and classical vector spaces are especially important in the representation of finite groups and thus they are frequently used in representation theory. Many researchers have worked on this topic [2 - 5]. Recently, Şahin et al. worked on isomorphism theorems for soft G Modules [6]; Smarandache et al. studied the neutrosophic triplet G Modules [7]; Şahin et al. worked on isomorphism theorems for the neutrosophic triplet G Module [8].

Smarandache defined neutrosophic logic and the concept of neutrosophic set in 1998 [9]. In the concept of neutrosophic logic and neutrosophic sets, there is a degree of membership T, a degree of uncertainty I and a degree of falsity F. These degrees are defined independently from each other. A neutrosophic value has the form (T, I, F). In other words, in neutrosophy, a situation is handled according to its accuracy, its falsehood, and its uncertainty. Therefore, neutrosophic logic and neutrosophic clusters help

us explain many uncertainties in our lives. So, many researchers have made studies on this subject [10 -14]. Recently, Bakbak et al. worked on neutrosophic soft expert multiset and their application [15]; Olgun et al. studied the neutrosophic logic on the decision tree [16]; Uluçay worked on interval-valued refined neutrosophic sets and their application [17]; Aslan et al. examined neutrosophic modeling of Talcott Parsons's action and decision-making applications [18]. The theories have studied in various areas such as [30-55].

Florentin Smarandache introduced new research areas in neutrosophy, which he called neutro-structures and anti-structures, respectively, in 2019 [19, 20]. When evaluating <A> as an element (concept, attribute, idea, proposition, theory, etc.), during the neutrosification process, he worked on three regions; two opposites corresponding to <A> and <antiA> and also a neutral (indeterminate) <neutA> (also called <neutralA>). A neutro-algebra consists of at least one neutro-operation (indeterminate for other items and false for other items) or it is an algebra well-defined for some items (also called internally defined), indeterminate for others, and externally defined for others. In recent years, the academic community has witnessed growing research interests in neutrosophic set theory [56-71]. Therefore, the subject attracted the attention of many researchers [21–25]. Recently, Rezaei et al. worked on the neutrosophic triplet of BI-algebras [26]; Smarandache et al. studied neutro-bck algebra [27], Smarandache et al. studied neutro-BCK algebras [28]; İbrahim et al. has obtained neutro-vector spaces [29].

In the second section, basic definitions on classical G Module [1]; definitions of neutro-group, neutro-field and neutro-vector spaces are given [29]. In the third chapter, the neutro-G Module is defined and its basic properties are given. Similarities and differences between the classical G Module and the neutro-G Module are given. It is shown that a neutro-G Module can be obtained from every classic G Module. In the fourth chapter, anti-G Module is defined and its basic features are given. Similarities and differences between the classic G Module and the anti-G Module are given. Also, it is shown that a neutro-G Module can be obtained from every anti-G Module. In the last section, results and suggestions are given.

## BACKGROUND

**Definition1. [1]** Let G be a finite group. A vector space M over a field K is called a G- module if for every g $\epsilon$ G and m $\epsilon$ M, there exists a product (called the action of G on M)   m.g $\epsilon$ M satisfying the following axioms:

i) $m.1_G = m$, $\forall$ m $\epsilon$ M ($1_G$ being the identity element in G)

ii) m.(g.h) = (m.g).h, $\forall$ m $\epsilon$ M; g, h $\epsilon$ G

iii) $(k_1m_1 + k_2m_2).g = k_1(m_1.g) + k_2(m_2.g)$, $\forall k_1, k_2$ $\epsilon$ K; $m_1, m_2 \epsilon$ M; g$\epsilon$ G.

**Definition2. [1]** Let M be a G-module. A vector subspace N of M is a G-submodule if N is also a G-module under the same action of G.

**Definition3. [1]** LetM and $M^*$ be G-modules. A mapping φ: $M \to M^*$ is a G-module homomorphism if

    i)        $\varphi(k_1.m_1 + k_2.m_2) = k_1. \varphi(m_1) + k_2.\varphi(m_2)$ and,

    ii)       $\varphi(m.g) = \varphi(m).g, \quad \forall k_1, k_2 \in K ; m, m_1, m_2 \in M ; g \in G$

**Definition4. [29]**

i) [Law of neutro-well defined]

There exists a double (b, n) ∈ (G, G) such that b # n ∈ G [degree of well-defined (or inner-defined) T] and there exist a double (u, v) ∈ (G, G) such that u # v = indeterminate [degree of indeterminacy I], or there exist a double (p, q) ∈ (G, G) such that p # q ∉ G [degree of outer-defined F], where (T, I, F) is different from (1,0,0) and (0,0,1). Because (1,0,0) represents the classical well-defined law (100% well-defined law; T =1, I = 0, F = 0), while (0,0,1) represents the outer-defined law (i.e. 100% outer-defined law, or T=0, I=0, F =1).

ii) [Axiom of neutro-associativity]

There exists a triplet (b, n, m) ∈ (G, G, G) such that b # (n # m) = (b # n) # m [degree of truth T], and there exist two triplets (p, q, r) ∈ (G, G, G) such that p # (q # r) or (p # q) # r = indeterminate [degree of indeterminacy I], or there exist (u, v, w) ∈ (G, G, G) or u # (v # w) ≠ ( u # v) # w [degree of falsehood F], where (T, I, F) is different from (1,0,0) and (0,0,1). Because (1,0,0) represents the classical law (100% true law; T =1, I = 0, F = 0), while (0,0,1) represents the anti- law (i.e. 100% false law, or T=0, I=0, F =1).

iii) [Axiom of existence of the neutro-identity element]

For an element a ∈ G, there exists e ∈ G such that a # e = e # a = a [degree of truth T], and for two elements b, c ∈ G, there exists an e ∈ G such that [b # e or e # b = indeterminate ( degree of indeterminacy I) or c # e ≠ c ≠ e # c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) [Axiom of existence of the neutro-inverse element]

For an element a ∈ G, there exists u ∈ G such that a # u = u # a = a (degree of truth T), and for two elements b, c ∈ G, there exists u ∈ G such that [b # u or u # b = indeterminate (degree of indeterminacy I) or c # u ≠ c ≠ u # c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

v) [Axiom of neutro-commutativity]

There exists a double (b, n) ∈ (G, G) such that b # n = n # b (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (G, G) such that [u # v or v # u = indeterminate (degree of indeterminacy I) or p # q ≠ q # p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Definition 5. [29]** A neutro-group is a neutro-algebraic structure which possesses at least one of the axioms {i – iv} of Definition 4 and is an alternative to classical group.

**Definition 6. [29]** A neutro-commutative group is a neutro – algebraic structure which possesses at least one of the axioms {i – v} of Definition 4 and is an alternative to classical commutative group.

**Definition 7. [29]**

Let C be a nonempty set and let + : C × C → C and . : C × C → C be two binary operations on C.

i) [Law of neutro-well defined with respect to addition]

There exists a double (b, n) ∈ (C, C) such that b + n ∈ R (degree of truth T) and there exist two doubles (u, v) and (p, q) ∈ (C, C) such that [u + v = indeterminate (degree of indeterminacy I) or p + q ∉ R (degree of falsity (F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) [Axiom of neutro-Associativity with respect to addition]

There exists a triplet (b, n, m) ∈ (C, C, C) such that  b + (n + m) = (b + n) + m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (C, C, C) such that [[p + (q + r)] or [(p + q) + r] = indeterminate (degree of indeterminacy I) or u + (v + w) ≠ ( u + v) + w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iii) [Axiom of existence of the neutro-identity element with respect to addition]

For an element a ∈ C, there exists e ∈ C such that a + e = e + a = a (degree of truth T) and for two elements b, c ∈ C, there exists an e ∈ C such that [b + e or e + b = indeterminate (degree of indeterminacy I) or c + e ≠  c ≠ e + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) [Axiom of existence of the neutro-inverse element with respect to addition]

For an element a ∈ C, there exists u ∈ C such that a + u = u + a = a (degree of truth T) and for two elements b, c ∈ C, there exists u ∈ C such that [b + u or u + b = indeterminate (degree of indeterminacy I) or c + u ≠  c ≠ u + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

v) [Axiom of neutro-commutativity with respect to addition]

There exists a double (b, n) ∈ (C, C) such that b + n = n + b (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (C, C) such that [u + v or v + u = indeterminate (degree of indeterminacy I) or p + q ≠ q + p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vi) [Law of neutro-well defined with respect to multiplication]

There exists a double (b, n) ∈ (C, C) such that b . n ∈ (C, C) (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (C, C) such that [u . v = indeterminate (degree of indeterminacy I) or p . q ∉ C (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vii) [Axiom of neutro-associativity with respect to multiplication]

There exists a triplet (b, n, m) ∈ (C, C, C) such that b . (n . m) = (b . n) . m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (C, C, C) such that [[p . (q . r)] or [(p . q) . r] = indeterminate (degree of indeterminacy I) or u . (v . w) ≠( u . v) . w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

viii) [Axiom of neutro-left distribution]

There exists a triplet (b, n, m) ∈ (C, C, C) such that b . (n + m) = b .n + b .m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (C, C, C) such that [p . (q + r) or p .q + p . r = indeterminate (degree of indeterminacy I) or u . (v + w) ≠u . v + u .w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ix) [Axiom of neutro-right distribution]

There exists a triplet (b, n, m) ∈ (C, C, C) such that (n + m) . b = n . b + m .b (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (C, C, C) such that [ (q + r) . p or q . p + r . p = indeterminate (degree of indeterminacy I) or (v + w) .u≠ v . u + w .u (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

x) [Axiom of existence of the neutro-identity element with respect to multiplication]

For an element a ∈ C, there exists e ∈ C such that a . e = e . a = a (degree of truth T) and for two elements b, c ∈ C, there exists an e ∈ C such that [b . e or e . b = indeterminate (degree of indeterminacy I) or c . e ≠ c ≠ e . c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

xi) [Axiom of existence of the Neutro - inverse element with respect to Multiplication]

For an element a ∈ C, there exists u ∈ C such that a . u = u . a = a (degree of truth T) and for two element b, c ∈ C, there exists u ∈ C such that [b . u or u . b = indeterminate (degree of indeterminacy I) or c . u ≠ c ≠ u . c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Definition 8. [29]** A neutro-field is a neutro – algebraic structure which possesses at least one of the axioms {i – xi} of Definition 7 and is an alternative to classical field.

**Definition 9. [29]** [Neutrofication of the axioms of law and classical vector space]

Let V be a nonempty set, let (C, +₁, .₁) be a neutro-field and let +: V × V → V and . : C × V → V be two binary operations.

i) [Law of neutro-well defined with respect to addition]

There exists a double (b, n) ∈ (V, V) such that b + n ∈ V (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (V, V) such that [u + v = indeterminate (degree of indeterminacy I) or p + q ∉ V (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) [Axiom of neutro-associativity with respect to addition]

There exists a triplet (b, n, m) ∈ (V, V, V) such that b + (n + m) = (b + n) + m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (V, V, V) such that [[p + (q + r)] or [(p + q) + r] = indeterminate (degree of indeterminacy I) or u + (v + w) ≠( u + v) + w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iii) [Axiom of existence of the neutro-identity element with respect to addition]

For an element a ∈ V, there exists e ∈ V such that a + e = e + a = a (degree of truth T) and for two elements b, c ∈ C, there exists an e ∈ V such that [b + e or e + b = indeterminate (degree of indeterminacy I) or c + e ≠ c ≠ e + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) [Axiom of existence of the neutro-inverse element with respect to addition]

For an element a ∈ V, there exists u ∈ V such that a + u = u + a = a (degree of truth T) and for two elements b, c ∈ V, there exists u ∈ V such that [b + u or u + b = indeterminate (degree of indeterminacy I) or c + u ≠ c ≠ u + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

v) [Axiom of neutro-commutativity with respect to addition]

There exists a double (b, n) ∈ (V, V) such that b + n = n + b (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (V, V) such that [u + v or v + u = indeterminate (degree of indeterminacy I) or p + q ≠ q + p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vi) [Law of neutro-well defined with respect to multiplication]

Let f, g, h ∈ C and v, u, b ∈ V. There exists a double (f, v) such that f .n ∈ V (degree of truth T) and there exist two doubles (g, u) and (c, b) such that [g . u = indeterminate (degree of indeterminacy I) or c . b ∉ V (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vii) [Axiom of neutro-left distribution]

Let p, q, r ∈ C and x, y, u, v, w ∈ V. There exist a triplet (p, x, y) such that p . (x + y) = p .x + p .y (degree of truth T) and there exist two triplets (y, u, q) and (v, w, r) such that [y . (u + q) or y .u + y . q = indeterminate (degree of indeterminacy I) or v . (w + r) ≠ v . w + v . r (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

viii) [Axiom of  neutro-right distribution]

Let k, m, p, q, r, s ∈ C and u, v, w ∈ V. There exists a triplet (k, m, u) such that (k $+_1$ m) . u = k .u $+_1$ m . u (degree of truth T) and there exist two doubles (p, q, v) and (r, s, w) such that [(p $+_1$ q) . v or p . v $+_1$ q . v = indeterminate (degree of indeterminacy I) or (r $+_1$ s) . w ≠ r . w $+_1$ s . w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ix) [Axiom of neutro-associativity with respect to multiplication]

Let k, m, p, q, r, s ∈ C and u, v, w ∈ V. There exists a triplet (k, m, u) such that  k . (m . u) = (k $._1$ m) .y (degree of truth T) and there exist two triplets (p, q, v) and (r, s, w) such that [[p . (q . v)] or [(p $._1$ q) . v] = indeterminate (degree of indeterminacy I)  or  r . (s . z) ≠ ( r $._1$ s) . w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

x) [Axiom of neutro – identity with respect to multiplication]

Let k, m, p ∈  C and v, u, w ∈ V. For an element v, there exists a element k ∈ C such that k . v = v . k = v (degree of truth T) and for a double element (u, w), there exists a double element (m, p) such that [u . m or  m . u = indeterminate (degree of indeterminacy I)  or  p . w ≠  w ≠ w . p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Definition 10. [29]** Let $(C, +_1, \cdot_1)$ be a neutro-field. A strong neutro-vector space is a neutro-algebraic structure which possesses at least one of the axioms $\{i - x\}$ of Definition 9 and is an alternative to classical vector space.

**Definition 11. [29]** Let $(C, +_1, \cdot_1)$ be a classical field. A weak neutro-vector space is a neutro-algebraic structure which possesses at least one of the axioms $\{i - x\}$ of Definition 9 and is an alternative to classical vector space.

**Definition 12.[29]** Let C be a nonempty set and let $+ : C \times C \rightarrow C$ and $. : C \times C \rightarrow C$ be two binary operations on C.

i) For each double $(b, n) \in C$, $b + n \notin C$.

ii) For each triplet $(b, n, m) \in C$, $b + (n + m) \neq (b + n) + m$.

iii) For each element $b \in C$, there exists an element $e \in C$ such that $b + e \neq e + b \neq b$.

iv) For each element $b \in C$, there exists an element $u \in C$ such that $b + u \neq u + b \neq e$.

v) For each $(b, n) \in C$, $b + n \neq n + b$.

vi) For each double $(b, n) \in C$, $b . n \notin C$.

vii) For each triplet $(b, n, m) \in C$, $b . (n . m) \neq (b . n) . m$

viii) For each triplet $(b, n, m) \in C$, $b . (n + m) \neq b .n + b .m$

ix) For each triplet $(b, n, m) \in C$, $(n + m) .b \neq n . b + m .b$

x) For each $b \in C$, there exists an $e \in C$ such that $b .e \neq e . b \neq b$.

xi) For each $b \in C$, there exists a $u \in C$ such that $b .u \neq u . b \neq e$.

**Definition 13. [29]** An anti-field is a structure which satisfies the anti-axioms $\{i) - xi)\}$ of Definition 12.

**Definition 14. [29]** Let V be a nonempty set, let $(C, +_1, \cdot_1)$ be an anti-field and let $+: V \times V \rightarrow V$ and $. : C \times V \rightarrow V$ be binary operations, respectively.

i) For each double $(b, n) \in V$, $b + n \notin V$.

ii) For each triplet $(b, n, m) \in V$, $b + (n + m) \neq (b + n) + m$.

iii) For each element b ∈ V, there exists an element e ∈ V such that b + e ≠ e + b ≠ b.

iv) For each element b ∈ V, there exists an element u ∈ V such that b + u ≠ u + b ≠ e.

v) For each double (b, n) ∈ V, b + n ≠ n + b.

vi) For each double (f, n) ∈ (C, V), f .n ∉ V.

vii) For each triplet (p, x, y)∈ (C, V, V), p . (x + y) ≠ p .x + p .y

viii) For each triplet (k, m, u) ∈ (C, C, V), (k +₁ m) .u≠ k . u +₁ m . u

ix) For each triplet (k, m, u) ∈ (C, C, V), k . (m . u) ≠ (k .₁ m) . y

x) For each double (k, v)∈ (C, V), there exists a k ∈ C such that k .v≠v . k≠ v.

**Definition 15. [29]** Let (C, +₁, .₁) be an anti-field. An anti-vector space is a structure which satifies the anti-axioms {i – x} of Definition 14.

# NEUTRO-G MODULES

In this section,

the symbol "=$_U$" will be used for situations where equality is uncertain. For example, if it is not certain whether "a" and "b" are equal, then it is denoted by a =$_U$ b.

the symbol "∈$_U$" will be used for situations where it is unclear to be an element. For example, if it is not certain whether "a" is a member of the set B, then it is denoted by a ∈$_U$ B.

**Definition 16.** Let (G, #) be a neutro-group, (F, +₁, .₁) be a neutro-field and (V, #₁, *₁) be a strong neutro-vector space. If at least one of the following {i, ii, iii, iv} conditions is satisfied, then (V, #₁, #₂) is called a neutro-G Module.

i) There exists a double (b, n) ∈ (V, G) such that b # n ∈ V (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (V, G) such that [p # q ∉ G (degree of falsehood F) or u # v ∈$_U$ V (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) For a double (a, e) ∈ (V, G), there exist an element e ∈ G such that a # e = a (degree of truth T) and for two doubles (b, e), (c, e) ∈ (V, G), there exists e ∈ G such that [b # e ≠ b (degree of falsehood F) or c # e =$_U$ c (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iii) There exists a triplet (k, m, u) $\in$ (V, G, G) such that k # (m # u) = (k $\#_1$ m) # y (degree of truth T) and there exist two triplets (p, q, v), (r, s, w)$\in$ (V, G, G) such that [r # (s # w) $\neq$ ( r $\#_1$ s) # w (degree of falsehood F) or      p # (q # v) $=_U$ (p $\#_1$ q) # v (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) There exis a quintet (t, l, k, m, u) $\in$ (F, F, V, V, G) such that [(t $*_1$k)$\#_1$(l $*_1$m)] # u = [(t $*_1$ k) # u]$\#_1$ [ (l $*_1$ m) # u] (degree of truth T) and there exists two quintets (p, q, v, w, y), (r, s, t, k, m) $\in$ (F, F, V, G, G) such that [[(p $*_1$v) $\#_1$ (q$*_1$w)] # y $\neq$ [(p $*_1$v) # y] $\#_1$ [ (q $*_1$w) # y] (degree of falsehood F) or [(r $*_1$t) $\#_1$ (s$*_1$k)] # m $=_U$ [(r $*_1$t) # m] $\#_1$ [ (s $*_1$k) # m] (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Note 17.** From Definition 16, the neutro-G Modules differrent from the classical G-modules. Neutro-G Modules are given as an alternative to classical G-Modules. But, for a neutro-G Module, instead of the ones that are not met in Definition 16, classical G module conditions are valid.

**Definition 18.** Let (G, #) be a classical group, (F, $+_1$, $._1$) be a classical field and (V, $\#_1$, $*_1$)  be a classical vector space.   If at least one of the {i, ii, iii, iv} conditions in Definition 16 is satisfied, then (V, $\#_1$, $\#_2$) is called a weak neutro-G Module.

**Example 19.**

 a) Let a$\in\mathbb{Z}$ and . be multiplication. As a.1 =1.a = a, 1 is the identity element. Since 1.1 = 1for 1 $\in\mathbb{Z}$, there exists an inverse element. But, for a $\in\mathbb{Z} \setminus \{1\}$, there is no inverse. Thus, ($\mathbb{Z}$, .) is a neutro-group since ($\mathbb{Z}$, .) satisfies the condition iv) in Definition 4. (Classical group conditions are valid for the remaining ones in Definiton 4)

b) Let a $\in \mathbb{N}$ and + be addition. As a + 0 = 0 + a = a, 0 is the identity for the addition. For 0 $\in$ $\mathbb{N}$, since 0 + 0 = 0, there exists an inverse. But, for a $\in \mathbb{N} \setminus \{0\}$, there is no inverse. Also, Let a $\in \mathbb{N}$. As a.1 = 1.a = a, 1 is the identity element for ($\mathbb{N}$, .). For 1 $\in$ $\mathbb{N}$, as 1.1 = 1, 1 has an inverse. But, for a $\in \mathbb{N} \setminus \{1\}$, there is no inverse Thus, ($\mathbb{N}$, +, .) is a neutro-field since it satisfies the conditions iv) and xi) in Definition 7. (Classical field conditions are valid for the remaining ones in Definition 7)

c) Let a $\in \mathbb{N}$. As a + 0 = 0 + a = a, 0 is the identity for the addition. For 0 $\in$ $\mathbb{N}$, since 0 + 0 = 0, there exists an inverse. But, for a $\in \mathbb{N} \setminus \{0\}$, there is no inverse. Thus, ($\mathbb{N}$, +, .) is a strong neutro-vector space since it satisfies the condition iv) in Definition 9. (Classical vector space conditions are valid for the remaining ones in Definition 9.)

Now, we show that the neutro-vector space ($\mathbb{N}$, +, .) is a neutro-G module with respect to the neutro -group ($\mathbb{Z}$, .).

d) For all n ∈ ℕ and g ∈ ℤ \ {-1, -2, -3, …}, we have g.n ∈ ℕ. But, for all n ∈ ℕ  and g ∈ℤ \ {0, 1, 2, 3, …}, we have g.n ∉ ℕ. Thus, the neutro-vector space (ℕ, +, .) and the neutro-group (ℤ, .) satisfies the condition i) in Definiton 16. (Classical G - module conditions are valid for the remaining ones in Definition 16.)

Therefore, the strong neutro-vector space (ℕ, +, .) is a neutro-G module with respect to the neutro-group (ℤ, .).

**Corollary 20.** In example 19,

(ℤ, .) is a neutro-group but not a classical group.

(ℕ, +, .) is a neutro-field but not not a classical field.

(ℕ, +, .) is a strong neutro-vector space but not a classical vector space.

(ℕ, +, .) is a neutro-G module with respect to the neutro-group (ℤ, .) but not a classical G-module.

This is the reason for the fact that neutro-G modules have a more general structure than classical G-Modules.

**Theorem 21.** A neutro-G module can be obtained from every classical neutro-G module.

**Proof** Let (G, #) be a classical group, (F, $+_1$, $._1$) be a classical field, (V, $\#_1$, $*_1$) be a classical vector space and let (V, $\#_1$, $*_1$) be a classical G Module.

i) We show that a neutro-group can be obtained from a classical group (G, #).

Let a∉G and b ∈ G and let a # b ∉ G. Now, as (G ∪ {a}, #) satisfies condition i) of Definition 5, it is a neutro-group.

Let c∉ G and b, d∈ G and let c # (b # d) ≠ (c # b) # d. Here, as (G ∪ {c}, #) satisfies condition ii) of Definition 5, it is a neutro-group.

Let f∉ G and b ∈ G and let e # f≠e ≠f # e. Now, as (G ∪ {f}, #) satisfies condition iii) of Definition 5, it is a neutro-group (e is the classical identity element).

Let c∉ G and b ∈ G and let c # b ≠ e ≠b # c. Hence, as (G ∪ {e}, #) satisfies condition iv) of Definition 5, it is a neutro-group.

Thus, if we take A = {a, c, f, e} and let P(A) denote the power set of A. Then, (G ∪ A, #) is a neutro-group for            B ∈P(A) \ ∅. As a result, a neutro-group can be obtained from the classical group (G, #).

ii) Similar to (i), for the classical field $(F, +_1, \cdot_1)$, by the above ten conditions of a neutro – field, $(D, +_1, \cdot_1)$ is a neutro-field where $C = \{f_1, f_2, \ldots, f_{10}\}$ and $D \in P(C) \setminus \emptyset$. Here, $f_1, f_2, \ldots, f_{10}$ are the elements which do not satisfy the conditions of classical field, respectively. So, a neutro-field can be obtained from a classical field $(F, +_1, \cdot_1)$.

iii) Again similar to (i), for the classical vector space $(V, \#_1, *_1)$, by the above ten conditions of a strong neutro-vector space, $(V \cup K, \#_1, *_1)$ is a strong neutro-vector space where $E = \{v_1, v_2, \ldots, v_{10}\}$ and $\in P(E) \setminus \emptyset$. $v_1, v_2, \ldots, v_{10}$ are the elements which do not satisfy the conditions of classical vector space, respectively. Therefore, a strong neutro-vector space can be obtained from a classical vector space $(V, \#_1, *_1)$.

iv) Using similar operations, by adding new elements to the neutro-group $(G \cup A, \#)$ or to the strong neutro-vector space $(V \cup K, \#_1, *_1)$, a neutro-G Module can be obtained from $(V \cup K, \#_1, *_1)$.

**Theorem 22.** A neutro-G Module can be obtained from a weak neutro-G Module.

**Proof** Let $(G, \#)$ be a classical group, $(F, +_1, \cdot_1)$ be a classical field, $(V, \#_1, *_1)$ be a classical vector space and let $(V, \#_1, *_1)$ be a neutro-G Module. By similar arguments to (i), (ii) and (iii) of Theorem 21, the proof is straightforward. (iv) need not be proven, since $(V, \#_1, *_1)$ is a neutro - G Module.

**Definition 23.** Let $(G, \#)$ be a neutro-group, $(F, +_1, \cdot_1)$ be a neutro-field and $(V, \#_1, *_1)$ be a neutro-G Module. A vector subspace M of V is a neutro-G Submodule if M is a neutro-G Module which satisfies at least one of the conditions of neutro-G Module in Definition 16.

**Example 24.** In Example 19, the strong neutro-vector space $(\mathbb{N}, +, .)$ is a neutro – G Module with respect to the neutro-group $(\mathbb{Z}, .)$. Let $A = \{1, 2, 3, 4, 5\} \subset \mathbb{N}$. The strong neutro-subvector space $(A, +, .)$ is a neutro-G submodule with respect to the neutro-group $(\mathbb{Z}, .)$.

**Definition 25.** Let $(V_1, \#_1, *_1)$ and $(V_2, \#_2, *_2)$ be two neutro-G Modules on neutro-field and $\varphi$ be a mapping such that $\varphi : V_1 \rightarrow V_2$. If at least one of the following conditions {i, ii} is satisfied then $\varphi$ is called a neutro-G Module homomorphism.

i) There exist two doubles $(p, q), (v, r) \in (F, V)$ such that $\varphi(p.q + v.r) = p.\varphi(q) + v.\varphi(r)$ (degree of truth T) and there exist four doubles $(s, t), (k, m), (w, n), (b, c) \in (F, V)$ such that $[\varphi(s.k + t.m) \neq s.\varphi(k) + t.\varphi(m)$ (degree of falsehood F) or $\varphi(w.b + n.c) =_U w.\varphi(b) + n.\varphi(c)$ (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) There exists a double $(p, q) \in (V, G)$ such that $\varphi(p.q) = p.\varphi(q)$ (degree of truth T) and there exist two doubles $(s, t), (k, m) \in (V, G)$ such that $[\varphi(s.k) \neq s.\varphi(k)$ (degree of falsehood F) or $\varphi(k.m) =_U k.\varphi(m)$ (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Example 26.** In Example 19, the strong neutro-vector space ($\mathbb{N}$, +, .) is a neutro-G Module with respect to the neutro-group ($\mathbb{Z}$, .). Define the mapping $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\varphi(x) = |x|$. $\varphi$ is a neutro-G module homemorphism since $\varphi$ satisfies the condition i) in Definition 25. In other words, For 0 $\in \mathbb{Z}$ and for all n $\in \mathbb{N}$,

we have $\varphi(0.n) = |0.n| = 0 = 0.|n|$. But, for all a $\in \{-1, -2, -3, \dots\}$ and for all n $\in \mathbb{N}$, $\varphi(a.n) = |a.n| \neq a.|n|$.

**Corollary 27.** The mapping $\varphi$ in Example 26 is a neutro-G Module homomorphism but not a classical G- Module homeomorphism. This is the reason for the fact that neutro-G Module homeomorphisms are more general structures than classical G-Module homomorphisms.

# ANTI – G MODULES

**Definition 28.** Let (G, #) be an anti-group, (F, $+_1$, $._1$) be an anti-field, (V, $\#_1$, $*_1$) be an anti-vector space. If the following {i, ii, iii, iv} conditions are satisfied, then (V, $\#_1$, $*_1$) is called a anti-G Module.

i) For every double (p, q) $\in$ (V, G), p # q $\notin$ G (F)

ii) For every double (b, e) $\in$ (V, G), there exists an e $\in$ G such that b # e $\neq$ b. (F)

iii) For every triplet (r, s, w) $\in$ (V, G, G), r # (s # w) $\neq$ ( r $\#_1$ s) . w (F)

iv) For every quintet (p, q, v, w, y) $\in$ (F, F, V, G, G), [(p $*_1$ v) $\#_1$ (q$*_1$ w)] # y $\neq$ [(p $*_1$ v) # y] $\#_1$ [ (q $*_1$ w) # y] (F)

**Example 29.** Let A = {1, -2, 4} and B = {3, 5, 7} be two sets and let * and # be two binary operations defined as below.

$$a*b = \begin{cases} -2a - b, & \text{if } a + b > 0 \text{ and } a \neq b \\ -2a - b, & \text{if } a + b \leq 0 \text{ and } a \neq b \\ 0, & \text{if } a = b \end{cases}$$

$$a\#b = \begin{cases} -\frac{b}{a}, & \text{if } a.b > 0 \text{ and } a \neq b \\ -\frac{a}{b}, & \text{if } a.b \leq 0 \text{ and } a \neq b \\ 1, & \text{if } a = b \end{cases}$$

Thus, below tables can be generated for the sets A and B according to * and #.

| *  | 1  | -2 | 4  |
|----|----|----|----|
| 1  | 0  | -4 | -6 |
| -2 | 5  | 0  | 0  |
| 4  | -9 | -6 | 0  |

| #  | 1        | -2      | 4       |
|----|----------|---------|---------|
| 1  | 0        | $1/2$   | $-4$    |
| -2 | $1/2$    | 0       | $1/2$   |
| 4  | $-1/4$   | 2       | 0       |

| #  | 3        | 5        | 7        |
|----|----------|----------|----------|
| 3  | 0        | $-5/3$   | $-7/3$   |
| 5  | $-3/5$   | 0        | $-7/5$   |
| 7  | $-3/7$   | $-5/7$   | 0        |

Using the above tables and with respect to the operations * and #,

(B, *) is an anti-group.

(A, *, #) is an anti-field.

(A, *, #) is an anti-vector space.

In addition, with respect to the anti-group (B, *), the anti-vector space (A, *, #) is an anti-G Module.

**Theorem 30.** A neutro-G Module can be obtained from every anti-G Module.

**Proof:** Let (G, #) be an anti-group, (F, $+_1$, $._1$) be an anti-field, (V, $\#_1$, $*_1$) be an anti-vector space and let (V, $\#_1$, $*_1$) be an anti-G Module. By using a similar argument of Theorem 21 and adding new elements to the anti-algebraic structures (G, #), (F, $+_1$, $._1$) and (V, $\#_1$, $*_1$) to transform these into neutro-algebraic structures, one can obtain a neutro-G Module.

**Theorem 31.** An anti-G Module can be obtained from every neutro-G Module

**Proof:** Let (G, #) be a neutro-group, (F, $+_1$, $._1$) be a neutro-field, (V, $\#_1$, $*_1$) be a strong neutro-vector space and let (V, $\#_1$, $*_1$) be a neutro-G Module. By taking out the elements that satisfy classical conditions and contain uncertainty from (G, #), (F, $+_1$, $._1$) and (V, $\#_1$, $*_1$), one can obtain an anti-G Module.

**Definition 32.** Let (G, #) be a classical group, (F, $+_1$, $._1$) be a classical field and (V, $\#_1$, $*_1$) be a classical vector space. If at least one of the {i, ii, iii, iv} conditions in Definition 28 is satisfied, then (V, $\#_1$, $\#_2$) is called a weak anti-G Module.

**Definition 33.** Let (V, $\#_1$, $*_1$) be an anti-G Module. A anti-vector subspace M of V is an anti-G Submodule if (M, $\#_1$, $*_1$) is an anti-G Mmodule which satisfies the conditions of anti-G Module in Definition 28.

**Example 34.** In Example 29, (B, *) is an anti-group, (A, *, #) is an anti-field, (A, *, #) is an anti-vector space and the anti-vector space (A, *, #) is an anti-G Module with respect to the anti-group (B, *). Also, for any nonempty C ⊂ A, the anti-subvector space (C, *, #) is an anti-G submodule with respect to the anti-group (B, *).

**Definition 35.** Let (V$_1$, $\#_1$, $*_1$), (V$_2$, $\#_2$, $*_2$) be two anti-G Modules on anti-field (F, +, .) and φ be a mapping such that φ : V$_1$ → V$_2$ . If the following conditions {i, ii} is satisfied then φis called an anti-G Module homomorphism.

i) For every two doubles (s, t), (k, m) ∈ (F, V$_1$), we have $\varphi(s*_1 k + t*_1 m) \neq s*_2 \varphi(k) + t*_2 \varphi(m)$ (F)

ii) For every double (s, k) ∈ (V$_1$, G), we have $\varphi(s._1 k) \neq s._1 \varphi(k)$ (F)

**Example 36.** In Example 29, the anti-vector space (A, *, #) is an anti-G Module with respect to the anti-group (B, *). Define the mapping φ: A → A such that φ(x) = |x| . Then, φ is an anti-G Module homeomorphism.

# CONCLUSIONS

In this chapter, the neutro-G Module is defined and relevant basic properties are given. Similarities and differences between the classical and neutro-G Modules are discussed. It is shown that a neutro-G Module can be obtained from every classical G Module. In addition, anti-G Module is defined and corresponding basic properties are given. Similarities and differences between the classical and anti-G Modules are discussed. Also, it is shown that a neutro-G Module can be obtained from every anti-G Module.

Researchers can make use of this chapter to define neutro-reducible G Modules, neutro-irreducible G Modules, neutro-totally reducible G Modules, anti-irreducible G Modules, anti-totally reducible G Modules. Also, since the classical G modules constitute a large role in the theory of group presentations, neutro-group representations and anti-group representations can be defined using this section.

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# Chapter Five

# Neutro-Metric Spaces

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## ABSTRACT

In this chapter, neutro-metric spaces are obtained. Also, corresponding basic properties and examples for neutro-metric spaces are given and proved. Furthermore, classical metric spaces and neutro-metric spaces are compared to each other. Neutro-metric spaces are shown to have a more general structure according to classical metric spaces. Thus, (T, I, F) components which constitute the neutrosophic theory are added to classical metric space (without using neutrosophic sets) and a new structure is obtained. In addition, it is shown that a neutro-metric spaces can be obtained from every classical metric space.

**Keywords:** Metric Space, Neutro-algebra, Neutro-metric space

## INTRODUCTION

In many branches of mathematics, as in analysis and geometry, there was a need for a concept of distance applicable to the elements of an abstract sentence. This distance is defined as the metric space. Metric space; It is the concept that emerged to examine the convergence of sequences, continuity of functions and basic analytical concepts. The necessary tool for this is just a distance or a metric. The concept of metric space dating back to the 1900s has been explored by many researches for these reasons [1 - 6]. Recently, Şahin et al. studied neutrosophic triplet g - metric space [7]; Kirişci et al. obtained neutrosophic metric spaces [8]; Yadav et al. introduced fixed-point theorems in intuitionistic fuzzy metric space [9]; Şahin et al. studied neutrosophic triplet partial bipolar metric spaces [10]; Cook et al. obtained the topology of a quantale valued metric space [11]; Şahin et al. studied neutrosophic triplet partial g-metric spaces [12]; Uluçay et al. obtained soft expert metric spaces [13].

We encounter many uncertainties in every moment of our lives. Many times, classical mathematical logic is insufficient to get rid of these uncertainties. The reason is that when explaining a situation or a problem, it is not possible to say that it is correct or certain. Smarandache defined neutrosophic logic and the concept of neutrosophic set in 1998 [14]. In the concept of neutrosophic logic and neutrosophic sets, there is the degree of membership T, degree of uncertainty I and degree of non-membership F. These degrees are defined independently from each other. It has the form of a neutrosophic value (T, I, F). In other words, a situation is handled in neutrosophy according to its accuracy, its falsehood, and its uncertainty. Thus, neutrosophic sets are the more general form of fuzzy sets [15] and intuitionistic fuzzy sets [16]. For this reason, many researchers have conducted studies on neutrosophic set theory [17 - 22]. Recently, Uluçay et al. studied decision making problem based on time-neutrosophic soft expert sets [23]; Hassan et al. obtained Q-neutrosophic soft expert set and its application [24]; Jeyaraman et al. obtain fixed point results for neutrosophic metric spaces [25]; Das et al. studied neutrosophic multiset topological space [26]; Kargın et al. obtained neutrosophic triplet m-Banach spaces [27]; Aslan et al. studied neutrosophic modeling of Talcott Parsons's action [28]; Kargın et al. obtained decision making application for law science based on generalized single valued neutrosophic quadruple numbers [29]; Şahin et al. studied decision making applications for adequacy of online education based on neutrosophic quadruple numbers [30].

Florentin Smarandache defined neutro-structures and anti-structures in 2019 [31] and in 2020 [32]. Similar to neutrosophic logic, an algebraic structure divides into three regions: A, the set of elements that satisfy the conditions of the algebraic structure, the truth region; Neutro-A, the set of elements that do not meet the conditions of the algebraic structure, the uncertainty region and the anti-A, the set of elements that do not meet the conditions of the algebraic structure, the inaccuracy region. Thus, the structure of neutrosophic logic has been transferred to the structure of classical algebras, without using neutrosophic sets and neutrosophic numbers. Therefore, neutro-algebraic structures, which have a more general structure than classical algebras, can be obtained. In addition, the region of the elements that do not satisfy any of the classical algebras are also taken as anti-algebraic structures. For this reason, many researchers have conducted studies on neutro-algebraic structures and anti-algebraic structures [33 - 35]. The theories have studied in various areas such as [43-68]. Recently, Smarandache studied neutro-algebra [36]; Rezaei et al. obtained the neutrosophic triplet of BI-algebras [37]; Smarandache et al. introduced neutro-bck-algebra [38]; Ibrahim et al. studied neutro-vector spaces [39]; Ibrahim et al. studied neutro-hypergroups [40]; Jiménez et al. studied neutroalgebra for the evaluation of barriers to migrants' access [41]; Al-Tahan et al. obtained NeutroOrderedAlgebra: Applications to Semigroups [42].

In the second section, basic definitions on classical metric [1]; basic definitions of neutro-structure are given [36]. In the third section, the neutro-metric space is defined and its basic properties are given. Similarities and differences between the classical metric spaces and the neutro-metric spaces are given. It is shown that a neutro-metric space can be obtained from every classic metric spaces. Also; we defined

neutro-converges sequence, neutro-Cauchy sequence for neutro-metric spaces, neutro-complete metric spaces and neutro-complete neutro-metric spaces. In the last part, results and suggestions are given.

## BACKGROUND

**Definition 1. [36] The Neutro-sophication of the Law** (degree of well-defined, degree of indeterminacy, degree of outer-defined)

Let X be a non-empty set. * be binary operation. For at least a double element (x, y) ∈ (X, X), x ∗ y ∈ X (degree of well defined (T)) and for at least two double elements (a, b), (c, d) ∈ (X, X), [a ∗ b = indeterminate (degree of indeterminacy (I)) or c ∗ d ∉ X (degree of outer-defined (F)]. Where; (T, I, F) is different from (1, 0, 0) and (0, 0, 1). Because (1, 0, 0) represents the classical well-defined law (100% well-defined law; T = 1, I = 0, F = 0), while (0, 0, 1) represents the outer-defined law (i.e. 100% outer-defined law, or T = 0, I = 0, F = 1).

**Property 2. [36]** In neutro-algebra, the classical well-defined for * binary operation is divided into three regions: degree of well-defined (T), degree of indeterminacy (I) and degree of outer-defined (F) similar to neutrosophic set and neutrosophic logic. Also, (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**Definition 3**. **[36] The Anti-sophication of the Law** (totally outer-defined)

Let X be a non-empty set. * be binary operation. For all double element (x, y) ∈ (X, X), x ∗ y ∉ X (totally outer-defined). Where, (T, I, F) = (0, 0, 1).

**Definition 4. [1]** Let A be a non-empty set, $d_m$: A x A → $\mathbb{R}^+$ ∪ {0} be a function. If the following properties are satisfied, then $d_m$ is called a metric.

For every k, l, m ∈ A,

D1)  $d_m(k, l) \geq 0$

D2)  $d_m(k, l) = 0 \Leftrightarrow k = l$

D3)  $d_m(k, l) = d_m(l, k)$

D4)  $d_m(k, l) \leq d_m(k, m) + d_m(m, l)$

Also, (A, $d_m$) is called a metric space.

## NEUTRO-METRIC SPACE

**Note 5:** In this chapter, the symbol "$=_u$" will be used for situations where equality is uncertain. For example, if it is not certain that "a" and "b" are equal, then it is denoted by a $=_u$ b.

**Note 6:** In this chapter, the symbol "$\leq_u$" will be used for situations where it is unclear to be an element. For example, If it is not clear that "a" is less than or equal to "b", then it is denoted by a $\leq_u$ b.

**Note 7:** In this chapter, the symbol "$\geq_u$" will be used for situations where it is unclear to be an element. For example, If it is not clear whether "a" is greater than or equal to "b", then it is denoted by a $\geq_u$ b.

**Definition 8.** Let A be a non-empty set, $d_{nm}$: A x A → ℝ be a function. If at least one of the following conditions {i, ii, iii, iv} is satisfied, then $d_{nm}$ is called a neutro-metric.

**i)** There exists a pair (k, l) ∈ A such that $d_{nm}(k, l) \geq 0$ (degree of truth T) and (there exist two pair (m, n), (p, r) ∈ A such that $d_{nm}(m, n) < 0$ (degree of falsehood F) or $d_{nm}(p, r) \geq_u 0$ (degree of indeterminacy I)). Where;    (T, I, F) is different from (1, 0, 0) and (0, 0, 1). Because (1, 0, 0) represents the classical well-defined law (100% well-defined law; T = 1, I = 0, F = 0), while (0, 0, 1) represents the outer-defined law (i.e. 100% outer-defined law, or T = 0, I = 0, F = 1).

**ii)** If k = l, m = n or p = r, then there exists a pair (k, l) ∈ A such that $d_{nm}(k, l) = 0$ (degree of truth T) and (there exist two pair (m, n), (p, r) ∈ A such that $d_{nm}(m, n) \neq 0$ (degree of falsehood F) or $d_{nm}(p, r) =_u 0$ (degree of indeterminacy I)). Where; (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

If $d_{nm}(k, l) = 0$, $d_{nm}(m, n) = 0$ or $d_{nm}(p, r) = 0$, then then there exists a pair (k, l) ∈ A such that k = l (degree of truth T) and (there exist two pair (m, n), (p, r) ∈ A such that m ≠ n (degree of falsehood F) or p =_u r (degree of indeterminacy I)). Where; (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**iii)** There exists a pair (k, l) ∈ A such that $d_{nm}(k, l) = d_{nm}(l, k)$ (degree of truth T) and (there exist two pair (m, n), (p, r) ∈ A such that $d_{nm}(m, n) \neq d_{nm}(n, m)$ (degree of falsehood F) or $d_{nm}(p, r) =_u d_{nm}(r, p)$ (degree of indeterminacy I)). Where; (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**iv)** There exists a triplet (k, l, m) ∈ A such that $d_{nm}(k, l) \leq d_{nm}(k, m) + d_{nm}(m, l)$ (degree of truth T) and (there exist two triplet (n, p, r), (s, t, v) ∈ A such that $d_{nm}(n, p) > d_{nm}(n, r) + d_{nm}(r, p)$ (degree of falsehood F) or d(s, t) $\leq_u$ d(s, v) + d(v, t) (degree of indeterminacy I)). Where; (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

Also, (A, $d_{nm}$) is called a neutro-metric space.

**Corollary 9:** Definition 8 is different from the definition of a classical metric space and a classical partial metric space. But, for a neutro-metric space, classical metric space's conditions are valid if neutro-metric space is not satisfied as in Definition 8.

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**Example 10:** Let $d_{nm}: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that $d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3}$ for $k = (k_1, k_2), l = (l_1, l_2) \in \mathbb{R}^2$.

i) If $k_1 - l_1 \geq 0$ and $k_2 - l_2 \geq 0$, then

$$d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3} \geq 0.$$

Also, if $k_1 - l_1 < 0$ and $k_2 - l_2 < 0$, then

$$d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3} < 0.$$

Thus, $d_{nm}$ satisfies the condition {i} in Definition 8. Therefore, $d_{nm}$ is a neutro-metric and $(\mathbb{R}^2, d_{nm})$ is a neutro-metric space. Furthermore, $d_{nm}$ satisfies the conditions {iii, iv} in Definition 8. Now, we show that $d_{nm}$ satisfies the conditions {iii, iv} in Definition 8.

iii) If $k = l$, then

$$d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3}$$

$$= \sqrt[3]{(l_1 - k_1)^3 + (l_2 - k_2)^3}$$

$$= d_{nm}(l, k).$$

Also, If $k \neq l$, then

$$d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3}$$

$$\neq \sqrt[3]{(l_1 - k_1)^3 + (l_2 - k_2)^3}$$

$$= d_{nm}(l, k).$$

iv) We assume that $k = (k_1, k_2), l = (l_1, l_2), m = (m_1, m_2) \in \mathbb{R}^2$. If $k_1 - l_1 \geq 0, k_2 - l_2 \geq 0, k_1 - m_1 \geq 0, k_2 - m_2 \geq 0, m_1 - l_1 \geq 0, m_2 - l_2 \geq 0$, then

$$d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3}$$

$$\leq \sqrt[3]{(k_1 - m_1)^3 + (k_2 - m_2)^3} + \sqrt[3]{(m_1 - l_1)^3 + (m_2 - l_2)^3}$$

$$= d_{nm}(k, m) + d_{nm}(m, l).$$

Also, If $k_1 - l_1 < 0$, $k_2 - l_2 < 0$, $k_1 - m_1 < 0$, $k_2 - m_2 < 0$, $m_1 - l_1 < 0$, $m_2 - l_2 < 0$, then

$$d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3}$$

$$> \sqrt[3]{(k_1 - m_1)^3 + (k_2 - m_2)^3} + \sqrt[3]{(m_1 - l_1)^3 + (m_2 - l_2)^3}$$

$$= d_{nm}(k,m) + d_{nm}(m,l).$$

**Corollary 11:**

**a)** In Example 10, $d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3}$ do not satisfy the condition {ii}. Thus, $d_{nm}$ is satisfies the condition {ii} for classical metric.

**b)** In Example 10, $d_{nm}(k,l) = \sqrt[3]{(k_1 - l_1)^3 + (k_2 - l_2)^3}$ is a neutro-metric. However, $d_{nm}$ is not a classical metric and a classical partial metric. Thus, a neutro-metric space is a more general structure compared to a classical metric space and classical partial metric space.

**Theorem 12:** Let A be a non-empty set, $d_m$: A x A → $\mathbb{R}^+ \cup \{0\}$ be a metric and m ∈ $\mathbb{R}^+$. Then,

$$d_{nm}(k,l) = d_m(k,l) - m$$

is a neutro-metric. (for every k, l ∈ A, $d_m(k,l) \neq m$).

**Proof:**

i) We assume that for k, l ∈ A, $d_m(k,l) \geq m$. Thus,

$$d_{nm}(k,l) \geq 0.$$

Also, We assume that for m, n ∈ A, $d_m(m,n) < m$. Thus,

$$d_{nm}(k,l) < 0.$$

Hence, $d_{nm}$ satisfies the condition {i} in Definition 8. Therefore, $d_{nm}(k,l) = d_m(k,l) - m$

is a neutro-metric.

**Corollary 13:**

**a)** In Theorem 12, $d_{nm}$ satisfies the condition {i} in Definition 8. But, $d_{nm}$ do not satisfy the conditions {ii, iii, iv} in Definition 8. Thus, $d_{nm}$ satisfies the conditions {ii, iii, iv} of classical metric.

**b)** From Theorem 12, a neutro-metric space can be obtained from every classical metric space.

**Theorem 14:** Let A be a non-empty set, f: A → ℝ be a function such that f is not one to one. Then,

$d_{nm}$: A x A → $\mathbb{R}^+ \cup \{0\}$ is a neutro-metric such that

$$d_{nm}(k, l) = |f(k) - f(l)|$$

**Proof:** We assume that n≠m for n, m ∈ A. Then, we can take f(m) = a = f(n) since f is not one to one.

For k = l ∈ A, then

$$d_{nm}(k, l) = |f(k) - f(l)| = 0.$$

Also, for n, m ∈ A, n≠m and

$$d_{nm}(n, m) = |f(n) - f(m)| = |a - a| = 0.$$

Furthermore, it is clear that

if $d_{nm}(k, l) = |f(k) - f(l)| = 0$, then k =l

and

if $d_{nm}(n, m) = |f(n) - f(m)| = |a - a| = 0$, then n≠m.

Thus, $d_{nm}$ satisfies the condition {ii} in Definition 8. Then, $d_{nm}(k, l) = |f(k) - f(l)|$ is a neutro-metric.

**Corollary 15:** In Theorem 12, $d_{nm}$ satisfies the condition {ii} in Definition 8. But, $d_{nm}$ do not satisfy the conditions{i, iii, iv} in Definition 8. Thus, $d_{nm}$ satisfies the conditions {i, iii, iv} of classical metric.

**Example 16:** Let A be a non-empty set, P(A) be power set of A, s(A) be number of elements of A. Then, from Theorem 14, $d_{nm}$: P(A) x P(A) → $\mathbb{R}^+ \cup \{0\}$ is a neutro-metric such that

$$d_{nm}(K, L) = |s(K) - s(L)|.$$

Because, s(A) is not one to one.

**Example 17:** Let A be a non-empty set, Then, $d_{nm}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a neutro-metric such that

$$d_{nm}(k, l) = k^2 - l^2.$$

i) There are $k, l \in \mathbb{R}$ such that

$$d_{nm}(k, l) = k^2 - l^2 \geq 0.$$

Also, for $m, n \in \mathbb{R}^+$ and m<n, it is clear that

$$d_{nm}(m, n) = m^2 - n^2 < 0.$$

Thus, $d_{nm}$ satisfies the condition {i} in Definition 8. Therefore, $d_{nm}$ is a neutro-metric and $(\mathbb{R}, d_{nm})$ is a neutro-metric space. Furthermore, $d_{nm}$ satisfies the conditions {ii, iii} in Definition 8. Now, we show that $d_{nm}$ satisfies the conditions {ii, iii} in Definition 8.

ii) If $k = l$, then

$$d_{nm}(k, l) = k^2 - l^2 = 0$$

If $m = -n$, then $m \neq n$ and

$$d_{nm}(m, n) = m^2 - n^2 = 0.$$

Also, it is clear that

If $d_{nm}(k, l) = k^2 - l^2 = 0$, then $k = l$ and if $d_{nm}(m, n) = m^2 - n^2 = 0$, then $m \neq n$ (for $m = -n$).

iii) If $m = -n$, and it is clear that

$$d_{nm}(m, n) = d_{nm}(n, m).$$

If $k, l \in \mathbb{R}^+$ and $k \neq l$, then

$$d_{nm}(m, n) \neq d_{nm}(n, m).$$

**Corollary 18:** In Example 17, $d_{nm}$ satisfies the conditions {i, ii, iii} in Definition 8. However, $d_{nm}$ do not satisfy the condition {iv} in Definition 8. Thus, $d_{nm}$ satisfies the condition {iv} of classical metric.

**Definition 19:** Let $(A, d_{nm})$ be a neutro-metric space and $\{x_n\}$ be a sequence in $(A, d_{nm})$ and $k \in A$.

If there exists $p \in \mathbb{N}$ for every $\varepsilon > 0$ and for some elements of $\{x_n\}$ such that

$$d_{nm}(k, \{x_p\}) < \varepsilon \text{ (degree of truth T)}$$

and

If there exist $r, s \in \mathbb{N}$ for every $\varepsilon > 0$ and for some elements of $\{x_n\}$ such that

$$d_{nm}(k, \{x_s\}) \geq \varepsilon \text{ (degree of falsehood F) or } d_{nm}(k, \{x_s\}) <_u \varepsilon \text{ (degree of indeterminacy I)}$$

then, $\{x_n\}$ is neutro-converges to k. Where; (T, I, F) is different from (1, 0, 0) and (0, 0, 1). Also, $\{x_n\}$ is called neutro-convergent.

**Definition 20:** Let $(A, d_{nm})$ be a neutro-metric space and $\{x_n\}$ be a sequence in $(A, d_{nm})$ and $t \in \mathbb{N}$.

If there exists $p \in \mathbb{N}$ for every $\varepsilon > 0$ and for some elements of $\{x_n\}$ such that

$$d_{nm}(\{x_p\}, \{x_n\}) < \varepsilon \text{ (degree of truth T)}$$

and

If there exist $r, s \in \mathbb{N}$ for every $\varepsilon > 0$ and for some elements of $\{x_n\}$ such that

$$d_{nm}(\{x_r\}, \{x_n\}) \geq \varepsilon \text{ (degree of falsehood F) or } d_{nm}(\{x_s\}, \{x_s\}) <_u \varepsilon \text{ (degree of indeterminacy I)}$$

then, $\{x_n\}$ is called a neutro-Cauchy sequence. Where; (T, I, F) is different from (1, 0, 0) and $n > p \geq t$, $n > r \geq t$, $n > s \geq t$.

**Definition 21:** Let $(A, d_{nm})$ be a neutro-metric space and $\{x_n\}$ be a Cauchy sequence in $(A, d_{nm})$. Then, $(A, d_{nm})$ is called a complete neutro-metric space if and only if every $\{x_n\}$ Cauchy sequence converges in $(A, d_{nm})$.

**Definition 22:** Let $(A, d_{nm})$ be a neutro-metric space and $\{x_n\}$ be a neutro-Cauchy sequence in $(A, d_{nm})$. Then, $(A, d_{nm})$ is called a neutro-complete neutro-metric space if and only if every $\{x_n\}$ neutro-Cauchy sequence converges in $(A, d_{nm})$.

# Conclusions

In this chapter, neutro-metric space is defined and relevant basic properties are given. Similarities and differences between the classical metric space and neutro-metric spaces are discussed. It is shown that a neutro-metric space can be obtained from every classical metric space. Also, we defined neutro-converges sequence, neutro-Cauchy sequence for neutro-metric spaces and neutro-complete metric spaces. Thus, we add new structures to neutro-algebra.

In this chapter, we take $d_{nm}$ as a classical function. Thus, researchers can obtain a new (more generalized) neutro-metric by taking the neutro function instead of $d_{nm}$ classical function. Also, thanks to definition of neutro-metric spaces researchers can define anti-metric spaces, neutro-metric topological space, anti-metric topological space, neutro-normed space, and other specific neutro-metric spaces.

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# Chapter Six

# Neutro-R Modules

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## ABSTRACT

In this chapter, neutro - R modules are obtained, basic properties and examples related to these structures are given. Furthermore, classical R modules and neutro-R modules are compared. It is shown that the neutro-R modules have a more general structure. Thus, a new structure is obtained by adding the (T, I, F) components, which form the structure of the neutrosophic theory, to the classical R modules (without using neutrosophic sets). Also, it is shown that a neutro-R module can be obtained from every classical R module.

**Keywords:** R Modules, Neutrosophic Theory, Neutro – Algebraic Structures, Neutro-R Modules

## INTRODUCTION

An R module is one of the basic algebraic structures used in abstract algebra. An R module on a ring is a generalization of the concept of a vector space on a field. Here the corresponding scalars are the (unit) elements of an arbitrary given ring R and a (left and / or right) multiplication is defined on the elements of the ring and the elements of the module. A module that takes its scalars from a ring R is called an R module. Therefore, a module is an additive abelian group, like a vector space. An operation, distributed over every parameter on addition and is compatible with the multiplication of the ring, is defined on the elements of the ring and the elements of the module. R modules are very closely related to the representation theory of groups. It is also one of the basic concepts of commutative algebra and homological algebra and is widely used in algebraic geometry and algebraic topology. For this reason, many researchers worked on the R modules [1 - 5]. Recently, Abobala et al. studied AH-Substructures In Strong Refined Neutrosophic Modules [6]; Veliyeva et al. worked on derivative functor of inverse limit functor in the category of neutrosophic soft modules [7]; Cai et al. studied classification of simple Harish-Chandra modules for map (super) algebras related to the Virasoro algebra [8].

Smarandache defined neutrosophic logic and the concept of a neutrosophic set in 1998 [9]. In the concept of neutrosophic logic and neutrosophic sets, there is a degree of membership T, a degree of uncertainty I and a degree of non-membership F. These are defined independently from each other. A neutrosophic value has the form (T, I, F). In other words, in neutrosophy, a situation is handled according to its trueness, its falsity, and its uncertainty. Therefore, neutrosophic logic and neutrosophic sets help us explain many uncertainties in our lives. Therefore, many researchers have made studies on this subject [10-12, 35-78]. Recently, Şahin et al. obtain some operations for interval valued neutrosophic sets [13]; Uluçay et al. studied neutrosophic multigroups and Applications [14]; Hassan et al. introduced Q-neutrosophic soft expert set and its application [15]; Sahin et al. obtained neutrosophic soft expert sets [16]; Uluçay studied interval-valued refined neutrosophic sets and their applications [17]; Khalifa et al. obtained neutrosophic set significance on deep transfer learning models [18]; Kargın et al. studied generalized Hamming similarity measure aased on neutrosophic quadruple numbers and its applications [19]; Şahin et al. obtain Hausdorff Measures on generalized set valued neutrosophic quadruple numbers and decision making applications for adequacy of online education [20].

In 2019, Florentin Smarandache introduced new research areas in neutrosophy, which he called neutro-structures and anti-structures [21, 22]. When evaluating <A> as an element (concept, attribute, idea, proposition, theory, etc.), during the neutrosification process, he worked on three regions; two opposites corresponding to <A> and <antiA> and also a neutral (indeterminate) <neutA> (also called <neutralA>). A neutro-algebra consists of at least one neutro-operation (indeterminate for other items and false for other items) or it is an algebra well-defined for some items (also called internally defined), indeterminate for others, and externally defined for others. Therefore, the subject attracted the attention of many researchers [23–29]. Recently, Smarandache et al. studied neutro-BCK algebras [30]; İbrahim et al. obtained neutro - vector spaces [31]; Al-Tahan et al. studied NeutroOrderedAlgebra and applications [32]: Jiménez et al. indroduced neutroalgebra for the evaluation of barriers to migrants' access in Primary Health Care in Chile [33]; Smarandache studied generalizations and alternatives of classical algebraic structures to neutroalgebraic structures and antialgebraic structures [34].

In this chapter; in the second section, basic definitions of a classical R module [1], and the definitions of neutro-group and neutro-ring are given [31]. In the third section, neutro-R module is defined and its basic properties are given. The similarities and differences between the classical R module and the neutro - R module are given. It is shown that a neutro-R module can be obtained from every classical R module. In the last part, results and suggestions were given.

# BACKGROUND

**Definition1. [1]** Let (G, #) be an abelian group, (R, +, .) be a commutative ring and let *: RxG→R be a binary operation. If the following conditions are satisfied, then G is called an R module. For p, r ∈ R and s, t ∈ G,

i) p* (s # t) = (p*s) # (p*t)

ii) (p + r) *s = (p*s) + (r*s)

iii) s*(p.r) = (s*p).r

iv) s*1 = s (1 being the identity element in G)

**Definition2. [1]** Let G be an R module. A subring N of R is an R submodule if N is also an R module under the same action of G.

**Definition3. [1]** Let (R, +, .) be a commutative ring; $*_1$: RxG$_1$→ R, $*_2$: RxG$_2$→ R be two binary operations, and let (G$_1$, #$_1$) and (G$_2$, #$_2$) be two R modules. If the following conditions are satisfied, then φ: G$_1$ → G$_2$ is called an R module homomorphism.

For p, r ∈ G$_1$ and t ∈ R,

    i)        φ(p#$_1$r) = φ( p) #$_2$φ(r)

    ii)       φ(t*$_1$p) = t*$_2$φ(p) .

**Definition 4. [31]**

i) [Law of neutro-closedeness]

There exists a double (b, n) ∈ G such that b # n ∈ G (degree of truth T) and there exist two doubles (u, v) and (p, q) ∈ G such that [u # v = indeterminate (I) or p # q ∉ G (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1). Because (1, 0, 0) represents the classical well-defined law (100% well-defined law; T = 1, I = 0, F = 0), while (0, 0, 1) represents the outer-defined law (i.e. 100% outer-defined law, or T = 0, I = 0, F = 1).

ii) [Axiom of neutro-associativity]

There exists a triplet (b, n, m) ∈ G such that b # (n # m) = (b # n) # m (degree of truth T) and there exist two triplets (p, q, r) and (u, v, w) ∈ G such that [[p # (q # r)] or [(p # q) # r] = indeterminate (degree of indeterminacy I) or u # (v # w) ≠ ( u # v) # w (degree of falsehood F)]; where (T, I, F) is different from (1, 0,

0) and (0, 0, 1). Because (1, 0, 0) represents the classical well-defined law (100% well-defined law; T = 1, I = 0, F = 0), while (0, 0, 1) represents the outer-defined law (i.e. 100% outer-defined law, or T = 0, I = 0, F = 1).

iii) [Axiom of existence of the neutro-identity element]

For at least one b ∈ G, there exists e ∈ G such that b # e = e # b = b (degree of truth T) and there exists e ∈ G such that [b # e or e # b = indeterminate (degree of indeterminacy I) or b # e ≠ b ≠ e # b (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iv) [Axiom of existence of the neutro-inverse element]

For at least one b ∈ G, there exists u ∈ G such that b # u = u # b = b (degree of truth T) and there exists u ∈ G such that [b # u or u # b = indeterminate (degree of indeterminacy I) or b # u ≠ b ≠ u # b (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

v) [Axiom of neutro-commutativity]

There exists a double (b, n) ∈ G such that b # n = n # b (degree of truth T) and there exists two doubles (u, v) and (p, q) ∈ G such that [u # v or v # u = indeterminate (degree of indeterminacy I) or p # q ≠ q # p (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**Definition 5. [31]** A neutro-group is a neutro-algebraic structure which possesses at least one of the axioms {i) – iv)} of Definition 4 and is an alternative to classical group.

**Definition 6. [31]** A neutro-commutative group is a neutro-algebraic structure which possesses at least one of the axioms {i) – iv)} of Definition 4 and is an alternative to classical commutative group.

**Definition 7. [31]**

Let R be a nonempty set and let + : R × R → R and . : R × R → R be two binary operations on R.

i) [Law of neutro-Closedness with respect to addition]

There exists a double (b, n) ∈ R such that b + n ∈ R (degree of truth T) and there exists two doubles (u, v) and (p, q) ∈ R such that [u + v = indeterminate (degree of indeterminacy I) or p + q ∉ R (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

ii) [Axiom of neutro-associativity with respect to addition]

There exists a triplet (b, n, m) ∈ R such that  b + (n + m) = (b + n) + m (degree of truth T) and there exists two triplets  (p, q, r), (u, v, w) ∈ R such that  [[p + (q + r)] or [(p + q) + r] = indeterminate (degree of indeterminacy I) or u + (v + w) ≠ ( u + v) + w (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iii) [Axiom of existence of the neutro-identity element with respect to addition]

For at least one b ∈ R, there exists an e ∈ R such that b + e = e + b = b (degree of truth T) and for at least one b ∈ R, there exists an e ∈ R such that [b + e or e + b = indeterminate (degree of indeterminacy I) or b + e ≠  b ≠ e + b (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iv) [Axiom of existence of the neutro-inverse element with respect to addition]

For at least one b ∈ R, there exists a u ∈ R such that b + u = u + b = b (degree of truth T) and for at least one b ∈ R, there exists a u ∈ R such [b + u or u + b indeterminate (degree of indeterminacy I) or b + u ≠  b ≠ u + b (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

v) [Axiom of neutro-commutativity with respect to addition]

There exists a double (b, n) ∈ R such that b+ n = n + b (degree of truth T) and there exists two doubles (u, v) and (p, q) ∈ R such that [u + v or v + u = indeterminate (degree of indeterminacy I) or p + q ≠ q + p (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

vi) [Law of  neutro-closedness with respect to multiplication]

There exists a double (b, n) ∈ R such that b . n∈ R (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ R such that [u . v = indeterminate (degree of indeterminacy I) or p . q ∉ R (degree of falsehood F)]; where    (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

vii) [Axiom of  neutro-associativity with respect to multiplication]

There exists a triplet (b, n, m) ∈ R such that  b . (n . m) = (b . n) . m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ R such that [[p . (q . r)] or [(p . q) . r] = indeterminate (degree of indeterminacy I) or                         u . (v . w) ≠( u . v) . w  (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

viii) [Axiom of neutro-left distribution]

There exists a triplet (b, n, m) ∈ R such that  b . (n + m) = b .n + b .m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ R such that [p . (q + r) or p .q + p . r = indeterminate (degree of indeterminacy I)

or  u . (v + w) $\neq$ u . v + u .w (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

ix) [Axiom of neutro-Right distribution]

There exists a triplet (b, n, m) $\in$ R such that  (n + m) . b = n . b + m .b (degree of truth T) and there exists two triplets (p, q, r), (u, v, w) $\in$ R such that [ (q + r) . p or q . p + r . p = indeterminate (degree of indeterminacy I) or  (v + w) .u$\neq$ v . u + w .u (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

x) [Axiom of neutro-commutativity with respect to multiplication]

There exists a double (b, n) $\in$ R such that b . n = n .b (degree of truth T) and  there exist two doubles (u, v) and (p, q) $\in$ R such that [ u . v or v . u = indeterminate (degree of indeterminacy I) or p . q $\neq$ q . p (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**Definition 8. [31]** A neutro-ring is a neutro-algebraic structure which possesses at least one of the axioms {i) – ix)} of Definition 7 and is an alternative to classical ring.

**Definition 9. [31]** A commutative neutro-ring is a neutro-algebraic structure which possesses at least one of the axioms {i) – x)} of Definition 7 and is an alternative to classical commutative ring.

# NEUTRO-R MODULES

Throughout this section, "$=^U$" is used to symbolize the conditions where an equality is indeterminate. For example, "a $=^U$ b" means that it is not certain whether a is equal to b.

The symbol " $\in^U$ ", similarly, is used to indicate that the corresponding membership is indeterminate; i.e. if it is not known whether the element a is in the set B, we write a $\in^U$ B.

**Definition 10.** Let (G, #) be an abelian neutro-group, (R, $+_1$, $._1$) a commutative neutro-ring and let *: RxG $\rightarrow$ R be a binary operation. If at least one of the following conditions {i, ii, iii, iv, v} is satisfied, then (G, #) is called a neutro-R module.

i) There exists a double (b, n) $\in$ (R, G) such that b * n $\in$ G   (degree of truth T) and there exist two doubles (u, v) and (p, q) $\in$ (R, G) such that [p * q $\notin$ R (degree of falsehood F) or u * v $\in^U$V (indeterminacy (I))]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

ii) There exists a triplet (b, n, m) $\in$(R, G, G) such that b * (n # m) = (b *n)#(b * m) (degree of truth T) and there exist two triplets (p, q, r) and (u, v, w) $\in$ (R, G, G) such that [p * (q # r)$=^U$(p * q)#(p * r) (degree of

indeterminacy I) or [u*(v # w)≠(u *v) # (u* w) (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iii) There exists a triplet (b, n, m) ∈(R, G, G) such that (b +$_1$ n) * m = (b *m) +$_1$ (n * m) (degree of truth T) and there exist two triplets (p, q, r) and (u, v, w) ∈ (R, R, G) such that [(p +$_1$ q) * r =$^U$ (p * r) +$_1$ (q* r) (degree of indeterminacy I) or [(u +$_1$ v) * w ≠(u * w) +$_1$ (v * w) (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iv) There exists a triplet (b, n, m) ∈(R, G, G) such that *(n .$_1$ m) = (b * n) .$_1$ m (degree of truth T) and there exist two triplets (p, q, r) and (u, v, w) ∈ (R, R, G) such that [p * (q .$_1$ r) =$^U$(p * q) .$_1$ r (degree of indeterminacy I) or u * (v .$_1$ w) ≠ ( u * v) .$_1$ w (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

v) For a double (a, e) ∈ (R, G), there exists an e ∈ G such that a * e = a (degree of truth T) and (for two doubles (b, e), (c, e) ∈ (R, G), there exists e ∈ G such that b * e ≠ b (degree of falsehood F) or c * e =$^U$ c (degree of indeterminacy I)); where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**Note 11.** Definition 10 is different from the definition of a classical R module and neutro-R modules are given as an alternative for classical R module. But, for a neutro-R Module, classical R module conditions are valid if neutro-R module is not satisfied as in Definition 10.

**Definition 12.** Let (G, #) be an abelian group, (R, +$_1$, .$_1$) a commutative ring and let *: RxG → R be a binary operation. If at least one of the conditions {i, ii, iii, iv, v} in Definition 10 is satisfied, then (G, #) is called a weak neutro-R module.

**Example 13.** (ℕ, .) is a neutro-group since the condition iv) of Definition 4 is satisfied. (Classical group conditions are valid for the remaining conditions in Definition 1), where "." is the known multiplication.

As

$$\text{For all } a \in \mathbb{N}, \ a.1 = 1.a = a ,$$

1 is the unit element for (ℕ, .) . Since 1.1 = 1, 1∈ ℕ has an inverse. But, for a ∈ ℕ \ {1}, there is no b ∈ ℕ \ {1} such that a.b = b.a = 1. So, every element does not have an inverse.

As it satisfies condition iv) of Definition 7, (ℕ, +, .) is a neutro-ring, where "." and "+" are the known multiplication and addition, respectively. (Again, classical ring conditions are valid for the remaining ones.) For all a∈ ℕ, 0 is the unit element for addition as a + 0 = 0 + a = a. For 0∈ ℕ, 0 + 0 = 0 and hence 0 has an inverse. On the other hand, for all a ∈ ℕ \ {0}, there does not exist any b ∈ ℕ \ {0} such that a + b = b + a = 0. So, every element does not have an inverse.

Now, define the operation *: ℕ x ℕ → ℕ such that a*b = a/b. Therefore, (ℕ, .) is a neutro-R module as it satisfies conditions i), ii) and iv) of Definition 10. For instance,

i) 1∈ ℕ and 1*1 = 1/1 = 1∈ ℕ. However, 1∈ ℕ, 2∈ ℕ and 1*2 = 1/2 ∉ ℕ.

ii) 1∈ ℕ and 1*(1.1) = 1 = (1*1).1 . Yet, for 1∈ ℕ and 2∈ ℕ, 1*(2.2) = 1/4 ≠ (1*2).2 = 1 .

iv) 1∈ ℕ  and 1*(1.1) = 1 = (1*1).(1*1) . But, for 1∈ ℕ, 2∈ ℕ and 3∈ ℕ, 2*(2.3) = 1/3≠ (2*2).(2*3) = $^2/_3$ .

**Corollary 14.** In Example 13,

(ℕ, .) is a neutro-group but not a classical group.

(ℕ, +, .) is a neutro-ring but not a classical ring.

(ℕ, .) is a neutro-R module but not a classical R module.

Thus, a neutro-R module is a more general structure compared to a classical R module.

**Theorem 15.** A neutro R-module can be obtained from every classical R module.

**Proof:** Let (G, #) be a classical abelian group, (R, $+_1$, $._1$) be a classical commutative ring and *: RxG → R be a binary operation. We assume that (G, #) be a classical R module.

i) First, we show that a commutative neutro-group can be obtained from a classical commutative group (G, #).

Let a∉G, b ∈ G and let a # b ∉ G. Here, (G ∪ {a}, #) satisfies the condition i) of Definition 4 and therefore it is a neutro-group.

Let c∉ G, b∈ G, d∈ G and let c # (b # d) ≠ (c # b) # d. Now, (G ∪ {c}, #) satisfies condition ii) of Definition 5 and hence it is a neutro-group.

Let f∉ G, b ∈ G and let e # f≠e ≠f # e. Clearly, (G ∪ {f}, #) satisfies condition iii) of Definition 5. So,     (G ∪ {f}, #) is a neutro-group. (e is the classical unit element.)

Let g∉ G, b ∈ G and let c # b ≠ e ≠b # c. As (G ∪ {g}, #) satisfies condition iv) of Definition 5, it is a neutro-group.

Let h ∉ G,  b ∈ G and let h # b≠b # h. Hence, (G ∪ {h}, #) is a commutative neutro-group as it satisfies condition iv) of Definition 5.

Thus, take A = {a, c, f, g, h} and let P(A) be the power set of A. For B ∈P(A) \ Ø, (G ∪ B, #) is a commutative neutro-group. Hence, a commutative neutro-group is obtained from the commutative classical group (G, #).

ii) Similar to i), take the classical commutative ring (R, $+_1$, $._1$). Now, by the 10 conditions of the commutative neutro-ring and letting C = $\{f_1, f_2, ..., f_{10}\}$, for D ∈P(C) \ Ø, (R ∪D, $+_1$, $._1$) is a commutative neutro-ring. Here, $f_1, f_2, ..., f_{10}$ are the elements which do not satisfy the conditions of the classical commutative ring. As a result, a commutative neutro-ring can be obtained from the classical commutative ring (R, $+_1$, $._1$) .

iii) Consider the commutative neutro-group (G ∪ A, #) or the commutative neutro-ring (R∪D ,$+_1$, $._1$) obtained above. By adding new elements which do not satisfy classical R – module conditions to the above, one can obtain a neutro-R module from the commutative neutro-group (G ∪ A, #).

**Theorem 16.** A neutro-R module can be obtained from every weak neutro-R module.

**Proof:** Let (G, #) be a classical abelian group, (R, $+_1$, $._1$) a classical commutative ring and let *: RxG → R be a binary operation. Assume that (G, #) is a neutro-R module. If the conditions i) and ii) of Theorem 15 are satisfied, then the theorem is proved. Condition iii) need not be satisfied, since (R, $+_1$, $._1$) is already a neutro-G module.

**Definition 17.** Let (G, #) be a classical abelian group, (R, $+_1$, $._1$) be a classical commutative ring and let *: R x G → R be a binary operation. Assume that (G, #) is a neutro-R module. A subgroup M of G is a neutro-R submodule if M  is a neutro-R module which satisfies at least one of the conditions of neutro-R module in Definition 10.

**Example 18.** In Example 13, (ℕ, +, .) is a commutative neutro-ring, a*b = a/b is a binary operation and (ℕ, .) is a neutro-R module. Taking A = {1, 2, 3, 4, 5}⊂ ℕ, the neutro-submodule (A, .) is a neutro-R submodule since   (A, .) satisfies the condition i) of Definition 10.

**Definition 19.**Let (R, $+_1$, $._1$) be a commutative neutro – ring; $*_1$: Rx$G_1$→ R and $*_2$: Rx$G_2$→ R be two binary operations; ($G_1$, $\#_1$) and ($G_2$, $\#_2$) be two neutro – R modules and let φ be a mapping such that φ: $G_1$ → $G_2$ . If at least one of the following conditions {i, ii} is satisfied then φ is called a neutro-R module homomorphism:

i) There exists a double (p, q)∈ (G, G) such that φ(p$\#_1$ q) = φ(p)$\#_2$φ(q) (degree of truth T) and there exist two doubles (s, t), (k, m) ∈ (F, V) such that [φ(s $\#_1$ t) ≠φ(s) $\#_1$φ(t) (degree of falsehood F) or φ(k $\#_1$ m) =$^U$φ(k) $\#_1$ φ(m) (degree of indeterminacy I)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

ii) There exists a double (p, q) ∈ (R, G₁) such that φ(p*₁q) = p*₂φ(q) (degree of truth T) and there exist two doubles (s, t), (k, m) ∈ (R, G₁) such that [φ(s*₁t) = s*₂φ(t) (degree of falsehood F) or φ(k*₁m) =$^U$k*₂φ(m) (degree of indeterminacy I)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**Example 20.** In Example 13, (ℕ, +, .) is a commutative neutro-ring, a*b = a/b is a binary operation and (ℕ, .) is a neutro-R module. Define the mapping φ :ℕ → ℕ such that φ(x) = 1/|x| . So, φ is a neutro-R module homeomorphism as it satisfies the conditions i) and ii) of Definition 19. Namely, for a ∈ℕ - {0} and b∈ℕ - {0}, if      a = b, then φ(a.b) = φ(a).φ(b) . Also, for a ∈ℕ - {0} and b∈ℕ - {0}, if a ≠ b, then φ(a.b)≠φ(a).φ(b) . If a = 0 or  b = 0, then it is indeterminate. In addition, for a ∈ℕ - {0} and b∈ℕ - {0}, if a = b, then φ(a.b) = a.φ(b) . But, for  a ∈ℕ - {0}  and b∈ℕ - {0}, if  a ≠ b,  then  φ(a.b) ≠a.φ(b). If a = 0 or  b = 0, then it is indeterminate.

**Corollary 21.** The mapping φ in Example 20 is a neutro-R module homomorphism but not a classical R module homeomorphism. Thus, a neutro-R module homeomorphism is a more general structure compared to a classical R module homeomorphism.

# CONCLUSIONS

In this chapter, the neutro-R module is defined and its basic properties are given. The similarities and differences between the classical R module and the neutro-R module are given. It is shown that a neutro-R module can be obtained from every classical R module. Thus, a new structure is obtained by adding the (T, I, F) components, which form the structure of the neutrosophic theory, to the classical R modules (without using neutrosophic sets). Also, researchers can use this section to obtain new structures of neutro-algebra by redefining the classical R module structures. For instance, neutro-group representations, neutro-commutative algebras, neutro-homological algebras, neutro-algebraic geometry and neutro-algebraic topology can be defined.

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# Chapter Seven

# Neutro-Lie Algebras and Anti-Lie Algebras

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## ABSTRACT

In this chapter, we obtain neutro-Lie algebras and anti- Lie algebras. We give corresponding basic properties and examples for neutro-Lie algebra and anti-Lie algebra and we proved them. Also, we compare classical Lie algebra with neutro-Lie algebra. We show that Neutro-Lie algebra have a more general structure according to neutro- Lie algebra. Hence, (T, I, F) components which constitute the neutrosophic theory are added to classical Lie Algebra (without using neutrosophic sets) and we obtain a new structure. In addition, we show that a neutro- Lie algebra can be obtained from every classical Lie algebra and a neutro-Lie algebra can be obtained from every anti-Lie algebra.

## INTRODUCTION

Lie groups are named after the Norwegian mathematician Sophus Lie, who laid the foundations of the theory of continuous transformation groups. Hermann Weyl [16] used Sophus Lie's Lie study on group theory in newspapers in 1922 and 1923, and Lie groups play a role in many places such as quantum mechanics today. Lie algebra was used in the preface of Hermann Weyl (in memory of Sophus Lie) in the 1930s as algebraic structures introduced to examine the concept of infinitesimal transformations. Lie algebra, which is widely used in mathematics and physics, is called a vector space defined by this transformation on a body. It is also linear algebra (ie linear algebra). Lie algebra, one of the most discussed topics of the twenty-first century, is related to many fields of mathematics such as differential geometry and harmonic analysis. Thus, a lot of researchers studied Lie algebras [1- 5]. Recently, Kurniadi et al. obtained the construction of real Frobenius Lie algebras [6]; Şahin et al. studied neutrosophic triplet Lie algebras [7]; Rasuli introduced the norms over fuzzy Lie algebra [8]; Estaji et al. obtained the category of soft Lie algebra [9].

Smarandache obtained neutrosophic logic and the concept of neutrosophic set in 1998 [10]. In the concept of neutrosophic logic and neutrosophic sets, there is a degree of membership T, a degree of uncertainty I and a degree of falsity F. These degrees are defined independently from each other. A neutrosophic value has the form (T, I, F). In other words, in neutrosophy, a situation is handled according to its accuracy, its falsehood, and its uncertainty. Thus, neutrosophic sets are the more general form of fuzzy sets [11] and intuitionistic fuzzy sets [12]. Therefore, neutrosophic logic and neutrosophic clusters help us explain many uncertainties in our lives. Hence, a lot of researchers have made studies on this subject [12 -24]. Recently, Jeyaraman et al. studied fixed point theorems based on neutrosophic metric spaces [25]; Uluçay et al. obtained decision making problems based on neutrosophic multi-sets [26]; Kargın et al. introduced neutrosophic triplet m-Banach Spaces [27]; Kargın et al. obtained Law Sciences decision making applications based on generalized neutrosophic quadruple numbers [28]; Şahin et al. introduced decision making applications for Adequacy of Online Education based on generalized neutrosophic quadruple numbers [29]; Aslan et al. obtained Neutrosophic modeling of Talcott Parsons's action and decision-making applications for it [30]; Şahin et al. studied neutrosophic triplet group based on set valued neutrosophic quadruple numbers [31]. The theories have studied in various areas such as [45-71].

Smarandache obtained new research areas in neutrosophy, which he called neutro-structures and anti-structures, respectively, in 2019 [31, 33]. When evaluating <A> as an element (concept, attribute, idea, proposition, theory, etc.), during the neutrosification process, he worked on three regions; two opposites corresponding to <A> and <antiA> and also a neutral (indeterminate) <neutA> (also called <neutralA>). A neutro-algebra consists of at least one neutro-operation (indeterminate for other items and false for other items) or it is an algebra well-defined for some items (also called internally defined), indeterminate for others, and externally defined for others. Therefore, the subject attracted the attention of many researchers [34–37]. Recently, Smarandache studied neutro-algebra [38]; Rezaei et al. obtained the neutrosophic triplet of BI-algebras [39]; Smarandache et al. introduced neutro-bck-algebra [40]; Ibrahim et al. studied neutro-vector spaces [41]; Ibrahim et al. studied neutro-hypergroups [42]; Jiménez et al. studied neutroalgebra for the evaluation of barriers to migrants' access [43]; Al-Tahan et al. obtained NeutroOrderedAlgebra: Applications to Semigroups [44].

In this chapter; in the second section, we give basic definitions on classical Lie algebra [1]; definitions of neutro-group, neutro-field and neutro-vector spaces [41]. In the third chapter, we define the neutro-Lie algebra and we give its basic properties. Also, we obtain similarities and differences between the classical Lie algebra and the neutro- Lie algebra. Furthermore, we show that a neutro-Lie algebra can be obtained from every classic Lie algebra. In the fourth chapter, we obtain anti-Lie algebra and we give its basic features. Also, we obtain similarities and differences between the classic Lie algebra and the anti- Lie algebra. Furthermore, we show that a neutro- Lie algebra can be obtained from every anti- Lie algebra. In the last section, results and suggestions are given.

## BACKGROUND

**Definition 1: [1]** Let M be an field and V be a finite-dimensional vector space on F. V is called a Lie algebra if $[.,.]: h \times h \to h$ is defined as a binary function that satisfies the following conditions on V.

i) For every $m \in V$, $[m, m] = 0$

ii) For every $m, n, r \in V$ and $\alpha, \beta \in M$,

$$[m+\alpha n, r] = [m, r] + \alpha[n, r] \text{ and } [m, n+\beta r] = [m, n] + \beta[m, r]$$

iii) For every $m, n, r \in V$,

$$\big[m, [n, r]\big] + \big[n, [r, m]\big] + \big[r, [m, n]\big] = 0 \text{ or } \big[[m, n], r\big] + \big[[n, r], m\big] + \big[[r, m], n\big] = 0$$

**Definition 2: [1]** Let $V_1$ and $V_2$ be two Lie algebras and $\Psi: V_1 \to V_2$ a linear function. If

$$\Psi([m, n]) = [\Psi(m), \Psi(n)]$$

then $\Psi$ is called a Lie algebra homomorphism.

**Definition 3: [38] The Neutro-sophication of the Law** (degree of well-defined, degree of indeterminacy, degree of outer-defined)

Let X be a non-empty set. * be binary operation. For at least a double element $(x, y) \in (X, X)$, $x * y \in X$ (degree of well defined (T)) and for at least two double elements $(a, b)$, $(c, d) \in (X, X)$, $[a * b =$ indeterminate (degree of indeterminacy (I)) or $c * d \notin X$ (degree of outer-defined (F))]. Where; (T, I, F) is different from (1, 0, 0) and (0, 0, 1). Because (1, 0, 0) represents the classical well-defined law (100% well-defined law; T = 1, I = 0, F = 0), while (0, 0, 1) represents the outer-defined law (i.e. 100% outer-defined law, or T = 0, I = 0, F = 1).

**Definition 4. [41]**

Let C be a nonempty set and let $+ : C \times C \to C$ and $. : C \times C \to C$ be two binary operations on C.

i) [Law of neutro-well defined with respect to addition]

There exists a double $(b, n) \in (C, C)$ such that $b + n \in R$ (degree of truth T) and there exist two doubles $(u, v)$ and $(p, q) \in (C, C)$ such that $[u + v =$ indeterminate (degree of indeterminacy I) or $p + q \notin R$ (degree of falsity (F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) [Axiom of neutro-Associativity with respect to addition]

There exists a triplet $(b, n, m) \in (C, C, C)$ such that $b + (n + m) = (b + n) + m$ (degree of truth T) and there exist two triplets $(p, q, r)$, $(u, v, w) \in (C, C, C)$ such that $[[p + (q + r)]$ or $[(p + q) + r] =$ indeterminate (degree

of indeterminacy I) or u + (v + w) ≠ ( u + v) + w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iii) [Axiom of existence of the neutro-identity element with respect to addition]

For an element a ∈ C, there exists e ∈ C such that a + e = e + a = a (degree of truth T) and for two elements b, c ∈ C, there exists an e ∈ C such that [b + e or e + b = indeterminate (degree of indeterminacy I) or c + e ≠ c ≠ e + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) [Axiom of existence of the neutro-inverse element with respect to addition]

For an element a ∈ C, there exists u ∈ C such that a + u = u + a = a (degree of truth T) and for two elements b, c ∈ C, there exists u ∈ C such that [b + u or u + b = indeterminate (degree of indeterminacy I) or c + u ≠ c ≠ u + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

v) [Axiom of neutro-commutativity with respect to addition]

There exists a double (b, n) ∈ (C, C) such that b + n = n + b (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (C, C) such that [u + v or v + u = indeterminate (degree of indeterminacy I) or p + q ≠ q + p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vi) [Law of neutro-well defined with respect to multiplication]

There exists a double (b, n) ∈ (C, C) such that b . n ∈ (C, C) (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (C, C) such that [u . v = indeterminate (degree of indeterminacy I) or p . q ∉ C (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vii) [Axiom of neutro-associativity with respect to multiplication]

There exists a triplet (b, n, m) ∈ (C, C, C) such that b . (n . m) = (b . n) . m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (C, C, C) such that [[p . (q . r)] or [(p . q) . r] = indeterminate (degree of indeterminacy I) or u . (v . w) ≠( u . v) . w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

viii) [Axiom of neutro-left distribution]

There exists a triplet (b, n, m) ∈ (C, C, C) such that b . (n + m) = b .n + b .m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (C, C, C) such that [p . (q + r) or p .q + p . r = indeterminate (degree of indeterminacy I) or u . (v + w) ≠u . v + u .w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ix) [Axiom of neutro-right distribution]

There exists a triplet (b, n, m) ∈ (C, C, C) such that (n + m) . b = n . b + m .b (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (C, C, C) such that [ (q + r) . p or q . p + r . p = indeterminate (degree of indeterminacy I) or (v + w) .u≠ v . u + w .u (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

x) [Axiom of existence of the neutro-identity element with respect to multiplication]

For an element a ∈ C, there exists e ∈ C such that a . e = e . a = a (degree of truth T) and for two elements b, c ∈ C, there exists an e ∈ C such that [b . e or e . b = indeterminate (degree of indeterminacy I) or c . e ≠ c ≠ e . c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

xi) [Axiom of existence of the Neutro - inverse element with respect to Multiplication]

For an element a ∈ C, there exists u ∈ C such that a . u = u . a = a (degree of truth T) and for two element b, c ∈ C, there exists u ∈ C such that [b . u or u . b = indeterminate (degree of indeterminacy I) or c . u ≠ c ≠ u . c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Definition 5. [41]** A neutro-field is a neutro – algebraic structure which possesses at least one of the axioms {i – xi} of Definition 4 and is an alternative to classical field.

**Definition 6. [41]** [Neutrofication of the axioms of law and classical vector space]

Let V be a nonempty set, let (C, $+_1$, $._1$) be a neutro-field and let +: V × V → V and . : C × V → V be two binary operations

i) [Law of neutro-well defined with respect to addition]

There exists a double (b, n) ∈ (V, V) such that b + n ∈ V (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (V, V) such that [u + v = indeterminate (degree of indeterminacy I) or p + q ∉ V (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) [Axiom of neutro-associativity with respect to addition]

There exists a triplet (b, n, m) ∈ (V, V, V) such that b + (n + m) = (b + n) + m (degree of truth T) and there exist two triplets (p, q, r), (u, v, w) ∈ (V, V, V) such that [[p + (q + r)] or [(p + q) + r] = indeterminate (degree of indeterminacy I) or u + (v + w) ≠( u + v) + w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iii) [Axiom of existence of the neutro-identity element with respect to addition]

For an element a ∈ V, there exists e ∈ V such that a + e = e + a = a (degree of truth T) and for two elements b, c ∈ C, there exists an e ∈ V such that [b + e or e + b = indeterminate (degree of indeterminacy I) or c + e ≠ c ≠ e + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) [Axiom of existence of the neutro-inverse element with respect to addition]

For an element a ∈ V, there exists u ∈ V such that a + u = u + a = a (degree of truth T) and for two elements b, c ∈ V, there exists u ∈ V such that [b + u or u + b = indeterminate (degree of indeterminacy I) or c + u ≠ c ≠ u + c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

v) [Axiom of neutro-commutativity with respect to addition]

There exists a double (b, n) ∈ (V, V) such that b + n = n + b (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (V, V) such that [u + v or v + u = indeterminate (degree of indeterminacy I) or p + q ≠ q + p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vi) [Law of neutro-well defined with respect to multiplication]

Let f, g, h ∈ C and v, u, b ∈ V. There exists a double (f, v) such that f .n ∈ V (degree of truth T) and there exist two doubles    (g, u) and (c, b) such that [g . u = indeterminate (degree of indeterminacy I) or c . b ∉ V (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

vii) [Axiom of neutro-left distribution]

Let p, q, r ∈ C and x, y, u, v, w ∈ V. There exist a triplet (p, x, y) such that p . (x + y) = p .x + p .y (degree of truth T) and there exist two triplets (y, u, q) and (v, w, r) such that [y . (u + q) or y .u + y . q = indeterminate (degree of indeterminacy I) or v . (w + r) ≠ v . w + v . r (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

viii) [Axiom of  neutro-right distribution]

Let k, m, p, q, r, s ∈ C and u, v, w ∈ V. There exists a triplet (k, m, u) such that $(k +_1 m)$ . u = k .u $+_1$ m . u (degree of truth T) and there exist two doubles (p, q, v) and (r, s, w) such that [$(p +_1 q)$ . v or p . v $+_1$ q . v = indeterminate (degree of indeterminacy I) or $(r +_1 s)$ . w ≠ r . w $+_1$ s . w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ix) [Axiom of neutro-associativity with respect to multiplication]

Let k, m, p, q, r, s ∈ C and u, v, w ∈ V. There exists a triplet (k, m, u) such that  k . (m . u) = (k $._1$ m) .y (degree of truth T) and there exist two triplets (p, q, v) and (r, s, w) such that [[p . (q . v)] or [(p $._1$ q) . v] =

indeterminate (degree of indeterminacy I)  or  r . (s . z) $\neq$( r .$_1$ s) . w (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

x) [Axiom of neutro – identity with respect to multiplication]

Let k, m, p $\in$ C and v, u, w $\in$ V. For an element v, there exists a element k $\in$ C such that k . v = v . k = v (degree of truth T) and for a double element (u, w), there exists a double element (m, p) such that [u . m or  m . u = indeterminate (degree of indeterminacy I)  or  p . w $\neq$  w $\neq$ w . p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Definition 7. [41]** Let (C, +$_1$, .$_1$) be a neutro-field. A strong neutro-vector space is a neutro-algebraic structure which possesses at least one of the axioms {i – x} of Definition 6 and is an alternative to classical vector space.

**Definition 8. [41]** Let (C, +$_1$, .$_1$) be a classical field. A weak neutro-vector space is a neutro-algebraic structure which possesses at least one of the axioms {i – x} of Definition 6 and is an alternative to classical vector space.

**Definition 9. [41]** Let C be a nonempty set and let + : C × C → C and . : C × C → C be two binary operations on C.

i) For each double (b, n) $\in$ C, b + n $\notin$ C.

ii) For each triplet (b, n, m) $\in$ C, b + (n + m) $\neq$ (b + n) + m.

iii) For each element b $\in$ C, there exists an element e $\in$ C such that b + e $\neq$ e + b $\neq$ b.

iv) For each element b $\in$ C, there exists an element u$\in$ C such that b + u $\neq$ u + b$\neq$e.

v) For each (b, n) $\in$ C, b + n $\neq$ n + b.

vi) For each double (b, n) $\in$ C, b . n $\notin$ C.

vii) For each triplet (b, n, m) $\in$ C, b . (n . m) $\neq$ (b . n) . m

viii) For each triplet (b, n, m) $\in$ C, b . (n + m) $\neq$ b .n + b .m

ix) For each triplet (b, n, m) $\in$ C, (n + m) .b$\neq$ n . b + m .b

x) For each b $\in$ C, there exists an e $\in$ C such that b .e$\neq$e . b$\neq$b.

xi) For each b ∈ C, there exists a u ∈ C such that b .u ≠ u . b ≠ e.

**Definition 10. [41]** An anti-field is a structure which satisfies the anti-axioms {i) – xi)} of Definition 9.

**Definition 11. [41]** Let V be a nonempty set, let (C, $+_1$, $._1$) be an anti-field and let +: V × V → V and . : C × V → V be binary operations, respectively.

i) For each double (b, n) ∈ V, b + n ∉V.

ii) For each triplet (b, n, m) ∈ V, b + (n + m) ≠ (b + n) + m.

iii) For each element b ∈ V, there exists an element e ∈ V such that b + e ≠ e + b ≠ b.

iv) For each element b ∈ V, there exists an element u ∈ V such that b + u ≠ u + b ≠ e.

v) For each double (b, n) ∈ V, b + n ≠ n + b.

vi) For each double (f, n) ∈ (C, V), f .n ∉ V.

vii) For each triplet (p, x, y)∈ (C, V, V), p . (x + y) ≠ p .x + p .y

viii) For each triplet (k, m, u) ∈ (C, C, V), (k $+_1$ m) .u≠ k . u $+_1$ m . u

ix) For each triplet (k, m, u) ∈ (C, C, V), k . (m . u) ≠ (k $._1$ m) . y

x) For each double (k, v)∈ (C, V), there exists a k ∈ C such that k .v≠v . k≠ v.

**Definition 12. [29]** Let (C, $+_1$, $._1$) be an anti-field. An anti-vector space is a structure which satifies the anti-axioms {i – x} of Definition 11.

# NEUTRO-LIE ALGEBRAS

In this chapter,

the symbol "$=_U$" will be used for situations where equality is uncertain. For example, if it is not certain whether "a" and "b" are equal, then it is denoted by a $=_U$ b.

**Definition 13.** Let (F, $+_1$, $._1$) be a neutro-field, (V, $\#_1$, $*_1$) be a strong neutro-vector space and [. , .]: VxV → V be a function. If at least one of the following {i, ii, iii} conditions is satisfied, then (V, $\#_1$, $\#_2$) is called a neutro-Lie algebra.

i) There exists a $n \in V$ such that $[m, m] = 0$ (degree of truth T) and there exists a double $(n, r) \in (V, V)$ such that $[[n, n] \neq 0$ (degree of falsehood F) or $[r, r] =_U 0$ (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) There exists a triplet $(m, n, r) \in (V, V, V)$ and a pair $(\alpha, \beta) \in (F, F)$ such that $([m \#_1 (\alpha *_1 n), r] = [m, r] \#_1 (\alpha *_1 [n, r])$ and $[m, n \#_1 (\beta *_1 r)] = [m, n] \#_1 (\beta *_1 [m, r])$ (degree of truth T)) and there exists two triplets $(p, s, t)$, $(k, l, q) \in (V, V, V)$ and two pair $(\gamma, \delta), (\varepsilon, \theta) \in (F, F)$ such that $(([p + (\gamma *_1 s), t] \neq [p, t] \#_1 (\gamma *_1 [s, t])$ and $[p, s \#_1 (\delta *_1 t)] \neq [p, s] \#_1 (\delta *_1 [p, t])$ (degree of falsehood T)) or $([k \#_1 (\varepsilon *_1 l), q] =_U [k, q] \#_1 (\varepsilon *_1 [l, q])$ and $[k, l \#_1 (\theta *_1 q)] =_U [k, l] \#_1 (\theta *_1 [k, q])$ (degree of indeterminacy I))). Where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iii) There exists a triplet $(m, n, r) \in (V, V, V)$ such that $([m, [n, r]] *_1 [n, [r, m]] *_1 [r, [m, n]] = 0$ or $[[m, n], r] *_1 [[n, r], m] *_1 [[r, m], n] = 0$ (degree of truth T)) and there exists two triplets $(p, s, t)$, $(k, l, q) \in (V, V, V)$ such that $(([p, [s, t]] *_1 [s, [t, p]] *_1 [t, [p, s]] \neq 0$ or $[[p, s], t] *_1 [[s, t], p] *_1 [[t, p], s] \neq 0$ (degree of falsehood T)) or $([k, [l, q]] *_1 [l, [q, k]] *_1 [q, [k, l]] =_U 0$ or $[[k, l], q] *_1 [[l, q], k] *_1 [[q, k], l] =_U 0$ (degree of indeterminacy I))). Where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

**Note 14.** From Definition 13, the neutro-Lie Algebras differrent from the classical Lie Algebras. Neutro- Lie Algebras are given as an alternative to classical Lie Algebras. But, for a neutro-Lie Algebras, instead of the ones that are not met in Definition 13, classical Lie Algebras conditions are valid.

**Definition 15.** Let $(F, +_1, \cdot_1)$ be a classical field, $(V, \#_1, *_1)$ be a classical vector space and $[. , .]: V \times V \to V$ be a function. If at least one of the {i, ii, iii, iv} conditions in Definition 13 is satisfied, then $(V, \#_1, *_1)$ is called a weak neutro-Lie algebra.

**Example 16.** It is celar that $\mathbb{R}$ (real numbers) be a vector space on $\mathbb{Q}$ (quotient numbers). Let $[.,.]: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $[n, m] = \frac{n.m}{3}$. Now, we show that $\mathbb{R}$ is a weak neutro-Lie Algebra.

i) For n = 0, it is clear that

$$[n, n] = \frac{n.n}{3} = \frac{0.0}{3} = 0.$$

Also, for $n \neq 0$; it is clear that

$$[n, n] \neq \frac{n.n}{3} \neq 0.$$

Thus, $\mathbb{R}$ satisfies the condition {i} in Definition 13. Hence, $\mathbb{R}$ is a weak neutro-Lie Algebra according to

$$[n, m] = \frac{n.m}{3}.$$

Also, $\mathbb{R}$ satisfies the condition {ii} in Definition 13. Because,

for n = m = r = 0, it is clear that

$$\big[m, [n, r]\big] + \big[n, [r, m]\big] + \big[r, [m, n]\big] = 0 \text{ or } \big[[m, n], r\big] + \big[[n, r], m\big] + \big[[r, m], n\big] = 0.$$

Also, for n ≠ m ≠ r ≠ 0; it is clear that

$$\big[m, [n, r]\big] + \big[n, [r, m]\big] + \big[r, [m, n]\big] \neq 0 \text{ or } \big[[m, n], r\big] + \big[[n, r], m\big] + \big[[r, m], n\big] \neq 0.$$

**Note 17.** In Example 16, $\mathbb{R}$ do not satisfy the condition {ii} in Definition 13. Thus, $\mathbb{R}$ satisfies the condition {ii} in Definition 1 (classical Lie algebra).

**Note 18.** In Example 16, $\mathbb{R}$ do not satisfy the conditions {i, iii} Definition 1. Thus, $\mathbb{R}$ is not a classical Lie algebra according to $[n, m] = \frac{n.m}{3}$.

**Example 19.** Let a ∈ $\mathbb{N}$ and + be addition operation. As a + 0 = 0 + a = a, 0 is the identity for the addition. For 0 ∈ $\mathbb{N}$, since 0 + 0 = 0, there exists an inverse. But, for a ∈ $\mathbb{N} \setminus \{0\}$, there is no inverse. Also, Let a ∈ $\mathbb{N}$. As a.1 = 1.a = a, 1 is the identity element for ($\mathbb{N}$, .). For 1 ∈ $\mathbb{N}$, as 1.1 = 1, 1 has an inverse. But, for a ∈ $\mathbb{N} \setminus \{1\}$, there is no inverse Thus, ($\mathbb{N}$, +, .) is a neutro-field since it satisfies the conditions iv) and xi) in Definition 4. (Classical field conditions are valid for the remaining ones in Definition 4)

Let a ∈ $\mathbb{N}$. As a + 0 = 0 + a = a, 0 is the identity for the addition. For 0 ∈ $\mathbb{N}$, since 0 + 0 = 0, there exists an inverse. But, for a ∈ $\mathbb{N} \setminus \{0\}$, there is no inverse. Thus, ($\mathbb{N}$, +, .) is a strong neutro-vector space since it satisfies the condition iv) in Definition 6. (Classical vector space conditions are valid for the remaining ones in Definition 6).

For all n ∈ $\mathbb{N}$ and g ∈ $\mathbb{Z} \setminus \{-1, -2, -3, \ldots\}$, we have g.n ∈ $\mathbb{N}$. But, for all n ∈ $\mathbb{N}$ and g ∈ $\mathbb{Z} \setminus \{0, 1, 2, 3, \ldots\}$, we have g.n ∉ $\mathbb{N}$. Thus, the neutro-vector space ($\mathbb{N}$, +, .) and the neutro-group ($\mathbb{Z}$, .) satisfies the condition i) in Definiton 16. (Classical G - module conditions are valid for the remaining ones in Definition 16.)

Let $[.,.]: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that [n, m] = 5m + 4n.

Now, we show that the strong neutro-vector space ($\mathbb{N}$, +, .) is a neutro-Lie algebra according to [n, m] = 5m + 4n.

i) For n = 0, it is clear that

$$[n, n] = 5n + 4n = 9n = 0.$$

Also, for n ≠ 0; it is clear that

$$[n, n] = 5n + 4n \neq 9n \neq 0.$$

$$[n, n] \neq \frac{n.n}{3} \neq 0.$$

Thus, (ℕ, +, .) satisfies the condition {i} in Definition 13. Hence, (ℕ, +, .) is a neutro-Lie Algebra according to            [n, m] = 5m + 4n.

Also, (ℕ, +, .) satisfies the condition {ii} in Definition 13. Because,

for n = m = r = 0, it is clear that

$$\big[m, [n, r]\big] + \big[n, [r, m]\big] + \big[r, [m, n]\big] = 0 \text{ or } \big[[m, n], r\big] + \big[[n, r], m\big] + \big[[r, m], n\big] = 0.$$

Also, for n ≠ m ≠ r ≠ 0; it is clear that

$$\big[m, [n, r]\big] + \big[n, [r, m]\big] + \big[r, [m, n]\big] \neq 0 \text{ or } \big[[m, n], r\big] + \big[[n, r], m\big] + \big[[r, m], n\big] \neq 0.$$

**Note 20.** In Example 19, (ℕ, +, .) do not satisfy the condition {ii} in Definition 13. Thus, ℝ satisfies the condition {ii} in Definition 1 (classical Lie algebra).

**Corollary 21**. In Example 19, (ℕ, +, .) do not satisfy the conditions {i, iii} Definition 1. Thus, ℝ is not a classical Lie algebra according to [n, m] = 5m + 4n. But, (ℕ, +, .) is a neutro-Lie Algebra according to [n, m] = 5m + 4n. This is the reason for the fact that neutro-Lie algebras are more general structures than classical Lie algebras.

**Theorem 22.** A weak neutro-Lie algebra can be obtained from every classical Lie algebra.

**Proof** Let $(F, +_1, ._1)$ be a classical field, $(V, \#_1, *_1)$ be a classical vector space, and $[. , .]: V \times V \to V$ be a function. We assume that $(V, \#_1, *_1)$ is a classical Lie algebra. Now, we show that a weak neutro-Lie algebra can be obtained from classical Lie algebra $(V, \#_1, *_1)$.

i) Let n ∉ V such that

$$[n, n] \neq 0.$$

Thus, $(V \cup \{n\}, \#_1, *_1)$ satisfies condition i) of Definition 15. Hence $(V \cup \{n\}, \#_1, *_1)$ is a weak neutro-Lie algebra.

ii) Let $(p, s, t) \notin (V, V, V)$ and $(\gamma, \delta) \notin (F, F)$ such that

$$(([p + (\gamma *_1 s), t] \neq [p, t] \#_1 (\gamma *_1 [s, t]) \text{ and } [p, s \#_1 (\delta *_1 t)] \neq [p, s] \#_1 (\delta *_1 [p, t]).$$

Thus, $(V \cup \{p, s, t\}, \#_1, *_1)$ satisfies condition i) of Definition 15. Hence $(V \cup \{p, s, t\}, \#_1, *_1)$ is a weak neutro-Lie algebra.

iii) Let $(k, l, q) \notin (V, V, V)$ such that

$$([k, [l, q]] *_1 [l, [q, k]] *_1 [q, [k, l]] \neq 0 \text{ or } [[k, l], q] *_1 [[l, q], k] *_1 [[q, k], l] \neq 0.$$

Thus, $(V \cup \{k, l, q\}, \#_1, *_1)$ satisfies condition i) of Definition 15. Hence $(V \cup \{k, l, q\}, \#_1, *_1)$ is a weak neutro-Lie algebra.

Hence, from i, ii and iii, if we take $A = \{n, p, s, t, k, l, q\}$ and let $P(A)$ denote the power set of A. Then, $(V \cup B, \#)$ is a weak neutro-Lie algebra for $B \in P(A) \setminus \emptyset$. As a result, a weak neutro-Lie algebra can be obtained from the classical Lie algebra.

**Corollary 23.** From Theorem 22, a neutro-Lie algebra can be obtained from a classical Lie algebra.

**Definition 24.** Let $[. , .]_1: V_1 x V_1 \to V_1$ and $[. , .]_2: V_2 x V_2 \to V_2$ be two functions, $(V_1, \#_1, *_1)$, $(V_2, \#_2, *_2)$ be two neutro-Lie algebra according to $[. , .]_1$ and $[. , .]_2$, respectively. Let $\varphi$ be a mapping such that $\varphi: V_1 \to V_2$. If the following conditions is satisfied then $\varphi$ is called a neutro-Lie algebra homomorphism.

i) There exists a double $(n, m) \in (V_1, V_1)$ such that $\varphi([n, m]_1) = [\varphi(n), \varphi(m)]_2$ (degree of truth T) and there exist two doubles $(r, p), (s, t) \in (V_1, V_1)$ such that $[\varphi([r, p]_1) \neq [\varphi(r), \varphi(p)]_2$ (degree of falsehood F) or $\varphi([s, t]_1) =_U [\varphi(s), \varphi(t)]_2$ (degree of indeterminacy I)], where (T, I, F) is different from (1,0,0) and (0,0,1).

**Example 25.** In Example 19, the strong neutro-vector space $(\mathbb{N}, +, .)$ is a neutro-Lie algebra according to $[n, m] = 5m + 4n$. Similarly, the strong neutro-vector space $(\mathbb{N}, +, .)$ is a neutro-Lie algebra according to $[n, m] = 5m - 4n$.

Let $\varphi: \mathbb{N} \to \mathbb{N}$ be a function such that

$\varphi(n) = |n|$.

Now we show that $\varphi$ satisfies the condition in Definition 24.

For $5m - 4n \geq 0$, it is clear that

$$\varphi([n, m]) = |5m - 4n|$$

$$= 5m - 4n$$

$$= |5m| - |4n|$$

$$= 5|m| - 4|n|$$

$$= [\varphi(n), \varphi(m)].$$

Also, for $5m - 4n < 0$, it is clear that

$$\varphi([n, m]) = |5m - 4n|$$

$$\neq 5m - 4n$$

$$= |5m| - |4n|$$

$$= 5|m| - 4|n|$$

$$= [\varphi(n), \varphi(m)].$$

Thus, $\varphi$ is a neutro-Lie algebra homomorphism.

**Corollary 26.** The mapping $\varphi$ in Example 25 is a neutro-Lie homomorphism but not a classical Lie algbera homeomorphism. This is the reason for the fact that neutro-Lie algebra homomorphisms are more general structures than classical Lie algebra homomorphisms.

# ANTI-LIE ALGEBRA

**Definition 27.** Let $(F, +_1, \cdot_1)$ be an anti-field, $(V, \#_1, *_1)$ be an anti-vector space and $[., .]: VxV \to V$ be afunction. If the following {i, ii, iii} conditions are satisfied, then $(V, \#_1, *_1)$ is called an anti-Lie algebra.

i) For every $(n, r) \in (V, V)$, $[n, n] \neq 0$ (degree of falsehood F). Where $(T, I, F) = (0, 0, 1)$.

ii) For every $(p, s, t) \in (V, V, V)$ and $(\gamma, \delta) \in (F, F)$,

$[p + (\gamma *_1 s), t] \neq [p, t] \#_1 (\gamma *_1 [s, t])$ and $[p, s \#_1 (\delta *_1 t)] \neq [p, s] \#_1 (\delta *_1 [p, t])$ (degree of falsehood T). Where $(T, I, F) = (0, 0, 1)$.

iii) For every $(p, s, t)$ $(V, V, V)$,

$[p, [s, t]] *_1 [s, [t, p]] *_1 [t, [p, s]] \neq 0$ or $[[p, s], t] *_1 [[s, t], p] *_1 [[t, p], s] \neq 0$ (degree of falsehood T). Where $(T, I, F) = (0, 0, 1)$.

**Corollary 28.** From Definition 27, anti-Lie Algebras are different from neutro-Lie algebra and classical Lie algebra.

**Definition 29.** Let $(F, +_1, ._1)$ be a classical field, $(V, \#_1, *_1)$ be a classical vector space and $[. , .]: V \times V \to V$ be a function. If the following {i, ii, iii} conditions are satisfied, then $(V, \#_1, *_1)$ is called a weak anti-Lie algebra.

**Theorem 30.** A weak neutro-Lie algebra can be obtained from every weak anti-Lie algebra.

**Proof** Let $(F, +_1, ._1)$ be a classical field, $(V, \#_1, *_1)$ be a classical vector space, and $[. , .]: V \times V \to V$ be a function. We assume that $(V, \#_1, *_1)$ is a weak anti-Lie algebra. Now, we show that a weak neutro-Lie algebra can be obtained from the weak anti-Lie algebra $(V, \#_1, *_1)$.

i) Let $n \notin V$ such that

$$[n, n] = 0.$$

Thus, $(V \cup \{n\}, \#_1, *_1)$ satisfies condition i) of Definition 15. Hence $(V \cup \{n\}, \#_1, *_1)$ is a weak neutro-Lie algebra.

ii) Let $(p, s, t) \notin (V, V, V)$ and $(\gamma, \delta) \notin (F, F)$ such that

$$(([p + (\gamma *_1 s), t] = [p, t] \#_1 (\gamma *_1 [s, t]) \text{ and } [p, s \#_1 (\delta *_1 t)] = [p, s] \#_1 (\delta *_1 [p, t]).$$

Thus, $(V \cup \{p, s, t\}, \#_1, *_1)$ satisfies condition i) of Definition 15. Hence $(V \cup \{p, s, t\}, \#_1, *_1)$ is a weak neutro-Lie algebra.

iii) Let $(k, l, q) \notin (V, V, V)$ such that

$$([k, [l, q]] *_1 [l, [q, k]] *_1 [q, [k, l]] = 0 \text{ or } [[k, l], q] *_1 [[l, q], k] *_1 [[q, k], l] = 0.$$

Thus, $(V \cup \{k, l, q\}, \#_1, *_1)$ satisfies condition i) of Definition 15. Hence $(V \cup \{k, l, q\}, \#_1, *_1)$ is a weak neutro-Lie algebra.

Hence, from i, ii and iii, if we take $A = \{n, p, s, t, k, l, q\}$ and let $P(A)$ denote the power set of A. Then, $(V \cup B, \#)$ is a weak neutro-Lie algebra for $B \in P(A) \setminus \emptyset$. As a result, a weak neutro-Lie algebra can be obtained from the classical Lie algebra.

**Corollary 31.** From Theorem 30, a neutro-Lie algebra can be obtained from an anti-Lie algebra.

**Definition 32.** Let $[.,.]_1: V_1 \times V_1 \to V_1$ and $[.,.]_2: V_2 \times V_2 \to V_2$ be two functions, $(V_1, \#_1, *_1)$, $(V_2, \#_2, *_2)$ be two anti-Lie algebra according to $[.,.]_1$ and $[.,.]_2$, respectively. Let $\varphi$ be a mapping such that $\varphi: V_1 \to V_2$. If the following conditions is satisfied then $\varphi$ is called an anti-Lie algebra homomorphism.

i) For every $(r, p) \in (V_1, V_1)$, $[\varphi([r, p]_1) \neq [\varphi(r), \varphi(p)]_2$ (degree of falsehood F). Where $(T, I, F) = (0, 0, 1)$.

**Corollary 33.** From Definition 32, anti-Lie algebra homomorphisms are different from neutro-Lie algebra homomorphisms and classical Lie algebra homomorphisms.

# CONCLUSIONS

In this chapter, the neutro-Lie algebra is defined and relevant basic properties are given. Similarities and differences between the classical and neutro-Lie algebra are discussed. It is shown that a neutro-Lie algebra can be obtained from every classical Lie algebra. In addition, anti-Lie algebra is defined and corresponding basic properties are given. Similarities and differences between the classical and anti-Lie algebra are discussed. Also, it is shown that a neutro-Lie algebra can be obtained from every anti-Lie algebra.

Researchers can make use of this chapter to define neutro-Frobenius Lie algebra, neutro-category of Lie algebra, anti-Frobenius Lie algebra, anti-category of Lie algebra. Also, since the classical Lie algebra constitute a large role in the theory of group theory and quantum mechanics, neutro-group theory, anti-group theory, neutro- quantum mechanics and anti-quantum mechanics can be defined using this chapter.

Also, in this chapter; we take $[.,.]$ and $\varphi$ as a classical functions. Thus, researchers can obtain a new (more generalized) neutro-Lie algebra and anti-Lie algebra by taking the neutro function instead of $[.,.]$ and $\varphi$ classical functions.

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# SECTION TWO

# NEUTROSOPHIC RELATED OTHER PAPERS

# Chapter Eight

## Hierarchical Clustering Methods in Architecture

## Based On Refined Q-Single-Valued Neutrosophic Sets

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**Abstract:** Information measures such as distance measures, similarity measures and entropy measures are very useful tools to be used in many applications such as multi-criteria decision making (MCDM), medical diagnosis, pattern recognition and clustering problems. A lot of such information measures have been proposed for the SVNS model. However, many of these measures have inherent problems that prevent them from producing reasonable or consistent results to the decision makers. Clustering plays an important role in data mining, pattern recognition, and machine learning. In this section, a new clustering algorithm is proposed based on refined Q- single-valued neutrosophic sets. We proposed a hierarchical clustering method using distance-based similarity measures on refined Q-single-valued neutrosophic sets. Then, we present a clustering algorithm based on the similarity measures of refined Q- single-valued neutrosophic sets to cluster refined Q-single-valued neutrosophic data. We illustrate the feasibility of the new method by an example in architecture. Finally, a comparison of the proposed method to existing methods is furnished to verify the effectiveness of our novel concept.

**Keywords:** Neutrosophic sets; refined neutrosophic set; refined Q-single-valued neutrosophic set, architecture.

# 1. Introduction

Housing throughout history; Interpersonal communication has been a bond that preserves the integrity of interaction, time, space and meaning. It reflects not only the lifestyle, culture and preference of the family, group or community to which it belongs, but also the essence of its user. It is a reflection of the individual's personality and worldview with its material, shape and equipment. Housing types, construction type, number of rooms per person, spaces that allow different uses increase the uncertainties of the users. In this respect, the decision-making process in architecture is very important. Intuitionistic fuzzy sets were introduced by Atanassov [1], followed by Molodtsov [2] on soft set and neutrosophy logic [3] and neutrosophic sets [4] by Smarandache. The term neutrosophy means knowledge of neutral thought and this neutral represents the main distinction between fuzzy and intuitionistic fuzzy logic and set. Presently, work on neutrosophic set theory is progressing rapidly. Ashour et al. [5] proposed a new skin lesion detection approach called HBCENCM using a histogram-based cluster prediction (HBCE) algorithm to determine the required number of clusters in the neutrosophic-averaged clustering (NCM) method. Zhang et al.[6] presented a new neutrosophic clustering algorithm with the help of regularization. Rashno et al. [7] a new clustering algorithm is proposed based on handle boundary and outlier points as challenging points of clustering methods neutrosophic set (NS) theory. Chai et al.[8] proposed measures have been verified and proven to comply with the axiomatic definition of the distance and similarity measure for the SVNS model. Vandhana and Anuradha [9] proposed a new algorithm called Neutrosophic-Fuzzy Hierarchical Clustering algorithm (NFHC) that includes indeterminacy. In recent years, the academic community has witnessed growing research interests in neutrosophic set theory; neutrosophic triplet v-generalized metric space [10], neutrosophic triplet partial bipolar metric spaces [11], neutrosophic triplet topology [12]. There are extensive literature about Q-fuzzy sets theory, for example; Q-fuzzy soft sets [13–15], and multi Q-fuzzy sets [16–18], thereby opening avenues to many applications [19-24] and multi Q-fuzzy soft expert set [25]. Şahin et al. [26] introduced neutrosophic soft expert sets, while Hassan et al. [27] extended it further to Q-neutrosophic soft expert

set, Broumi et al. [28] defined neutrosophic parametrized soft set theory and its decision making, Deli [29] introduced refined neutrosophic sets and refined neutrosophic soft sets. Since membership values are inadequate for providing complete informationin some real problems which has different membership values for each element, different generalization of fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets have been introduced is called multi fuzzy set [30], intuitionistic fuzzy multiset [31] and neutrosophic multiset [32,33], respectively. In the multisets an element of a universe can be constructed more than once with possibly the same or different membership values. Some work on the multi fuzzy set [34,35], on intuitionistic fuzzy multiset [36-39] and on neutrosophic multiset [40-43] have been studied. The above set theories have been applied to many different areas including real decision making problems [44-80].

The aim of this chapter, besides the objective evaluation, a decision making model that can be effective in expressing the subjective evaluations within the structure of architecture (mass, spatial, semantic, form and experience) has been developed. Finally, we apply this new concept to solve a decision-making problem in architecture and compare it with other existing methods.

## 2. Preliminaries

**Definition 1** ([4]) Let $U$ be a universe of discourse, with a generic element in $U$ denoted by $u$, then a  neutrosophic (NS) set $A$ is an object having the form

$$A = \{< u: \mu_A(u), \vartheta_A(u), w_A(u) >, u \in U\}$$

where the functions $\mu, \vartheta, w : V \rightarrow ]^-0, 1^+[$  define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $v \in V$ to the set $A$ with the condition.

$$^-0 \leq \mu_A(u) + \vartheta_A(u) + w_A(u) \leq 3^+$$

**Definition 2** [55] A single-valued neutrosophic set A on universe set E is given by

$$A = \{\langle u, \mu_A(u), \vartheta_A(u), w_A(u) \rangle : u \in E\}$$

where $\mu_A : E \rightarrow [0,1]$, $\vartheta_A : E \rightarrow [0,1]$ and $w_A : E \rightarrow [0,1]$  satisfy the condition $0 \leq \mu_A(u) + \vartheta_A(u) + w_A(u) \leq 3$, for every $u \in$ E. The function  $\mu_A$, $\vartheta_A$ and $w_A$ defines the degree of

truth-membership function, indeterminacy-membership function and falsity-membership function, respectively.

**Definition 3** ([29]) Let $\mathcal{U}$ be a universe. A neutrosophic multiset set (Nms) $A$ on $\mathcal{U}$ can be defined as follows:

$$A = \{< u, \left(\mu_A^1(u), \mu_A^2(u), \dots, \mu_A^p(u)\right), \left(v_A^1(u), v_A^2(u), \dots, v_A^p(u)\right),$$

$$\left(w_A^1(u), w_A^2(u), \dots, w_A^p(u)\right) >: u \in \mathcal{U}\}$$

where,

$$\mu_A^1(u), \mu_A^2(u), \dots, \mu_A^p(u): \mathcal{U} \to [0,1],$$

$$v_A^1(u), v_A^2(u), \dots, v_A^p(u): \mathcal{U} \to [0,1],$$

and

$$w_A^1(u), w_A^2(u), \dots, w_A^p(u): \mathcal{U} \to [0,1],$$

such that

$$0 \leq sup\mu_A^i(u) + supv_A^i(u) + supw_A^i(u) \leq 3$$

$(i = 1,2, \dots, P)$ and

$$\left(\mu_A^1(u), \mu_A^2(u), \dots, \mu_A^p(u)\right), \left(v_A^1(u), v_A^2(u), \dots, v_A^p(u)\right) and \left(w_A^1(u), w_A^2(u), \dots, w_A^p(u)\right)$$

Is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element $u$, respectively. Also, P is called the dimension (cardinality) of Nms $A$, denoted $d(A)$. We arrange the truth- membership sequence in decreasing order but the corresponding indeterminacy- membership and falsity-membership sequence may not be in decreasing or increasing order.

The set of all Neutrosophic multisets on $\mathcal{U}$ is denoted by $\text{NMS}(\mathcal{U})$.

**Definition 4** [29,33] Let $A, B \in NMS(\mathcal{U})$. Then,

   (1) $\mathcal{A}$ is said to be a Nm-subset of $\mathcal{B}$ and is denoted by $\mathcal{A} \widetilde{\subseteq} \mathcal{B}$ if $\mu_{\mathcal{A}}^i(u) \leq \mu_{\mathcal{B}}^i(u)$, $v_{\mathcal{A}}^i(u) \geq v_{\mathcal{B}}^i(u)$, $w_{\mathcal{A}}^i(u) \geq w_{\mathcal{B}}^i(u)$, $\forall u \in \mathcal{U}$ and $i = 1,2, \dots P$.

   (2) $\mathcal{A}$ is said to be neutrosophically equal of to $\mathcal{B}$ and is denoted by $\mathcal{A} = \mathcal{B}$ if $\mu_{\mathcal{A}}^i(u) = \mu_{\mathcal{B}}^i(u)$, $v_{\mathcal{A}}^i(u) = v_{\mathcal{B}}^i(u)$, $w_{\mathcal{A}}^i(u) = w_{\mathcal{B}}^i(u)$, $\forall u \in \mathcal{U}$ and $i = 1,2, \dots P$.

   (3) The complement of $\mathcal{A}$ is denoted by $\mathcal{A}^{\tilde{c}}$ and is defined by

$$\mathcal{A}^{\tilde{c}} =$$

$$\prec u, (w_{\mathcal{A}}^1(u), w_{\mathcal{A}}^2(u), \dots w_{\mathcal{A}}^p(u)), \left(v_{\mathcal{A}}^1(u), v_{\mathcal{A}}^2(u), \dots v_{\mathcal{A}}^p(u)\right), \left(\mu_{\mathcal{A}}^1(u), \mu_{\mathcal{A}}^2(u), \dots \mu_{\mathcal{A}}^p(u)\right)$$

$$\succ : u \in \mathcal{U}\}$$

(4) If $\mu_{\mathcal{A}}^i(u) = 0$ and $v_{\mathcal{A}}^i(u) = w_{\mathcal{A}}^i(u) = 1$ for all $u \in \mathcal{U}$ and $i = 1,2, \dots P$, then $\mathcal{A}$ is called null ns-set and is denoted by $\Phi$.

(5) If $\mu_{\mathcal{A}}^i(u) = 1$ and $v_{\mathcal{A}}^i(u) = w_{\mathcal{A}}^i(u) = 0$ for all $u \in \mathcal{U}$ and $i = 1,2, \dots P$, then $\mathcal{A}$ is called universal ns-set and is denoted by $\tilde{\mathcal{U}}$.

(6) The union of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \tilde{\cup} \mathcal{B} = \mathcal{C}$ and is defined by

$$\mathcal{C} = \{\prec u, \left(\mu_{\mathcal{C}}^1(u), \mu_{\mathcal{C}}^2(u), \dots \mu_{\mathcal{C}}^p(u)\right), \left(v_{\mathcal{C}}^1(u), v_{\mathcal{C}}^2(u), \dots v_{\mathcal{C}}^p(u)\right), (w_{\mathcal{C}}^1(u), w_{\mathcal{C}}^2(u), \dots w_{\mathcal{C}}^p(u))$$

$$\succ : u \in \mathcal{U}\}$$

where $\mu_{\mathcal{C}}^i = \mu_{\mathcal{A}}^i(u) \vee \mu_{\mathcal{B}}^i(u), \ v_{\mathcal{C}}^i = v_{\mathcal{A}}^i(u) \wedge v_{\mathcal{B}}^i(u), w_{\mathcal{C}}^i = w_{\mathcal{A}}^i(u) \wedge w_{\mathcal{B}}^i(u), \forall u \in \mathcal{U}$ and $i = 1,2, \dots P$.

(7) The intersection of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \tilde{\cap} \mathcal{B} = \mathcal{D}$ and is defined by

$$\mathcal{D} = \{$$

$$\prec u, \left(\mu_{\mathcal{D}}^1(u), \mu_{\mathcal{D}}^2(u), \dots \mu_{\mathcal{D}}^p(u)\right), \left(v_{\mathcal{D}}^1(u), v_{\mathcal{D}}^2(u), \dots v_{\mathcal{D}}^p(u)\right), (w_{\mathcal{D}}^1(u), w_{\mathcal{D}}^2(u), \dots w_{\mathcal{D}}^p(u)) \succ : u$$

$$\in \mathcal{U}\}$$

where $\mu_{\mathcal{D}}^i = \mu_{\mathcal{A}}^i(u) \vee \mu_{\mathcal{B}}^i(u), \ v_{\mathcal{D}}^i = v_{\mathcal{A}}^i(u) \wedge v_{\mathcal{B}}^i(u), w_{\mathcal{D}}^i = w_{\mathcal{A}}^i(u) \wedge w_{\mathcal{B}}^i(u), \forall u \in \mathcal{U}$ and $i = 1,2, \dots, P$.

(8) The addition of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \tilde{+} \mathcal{B} = \mathcal{U}_1$ and is defined by

$$\mathcal{U}_1 = \{$$

$$\prec u, \left(\mu_{\mathcal{U}_1}^1(u), \mu_{\mathcal{U}_1}^2(u), \dots \mu_{\mathcal{U}_1}^p(u)\right), \left(v_{\mathcal{U}_1}^1(u), v_{\mathcal{U}_1}^2(u), \dots v_{\mathcal{U}_1}^p(u)\right), (w_{\mathcal{U}_1}^1(u), w_{\mathcal{U}_1}^2(u), \dots w_{\mathcal{U}_1}^p(u))$$

$$\succ : u \in \mathcal{U}\}$$

where $\mu_{\mathcal{U}_1}^i = \mu_{\mathcal{A}}^i(u) + \mu_{\mathcal{B}}^i(u) - \mu_{\mathcal{A}}^i(u).\mu_{\mathcal{B}}^i(u),$ $\quad v_{\mathcal{U}_1}^i = v_{\mathcal{A}}^i(u).v_{\mathcal{B}}^i(u)$ ,

$w_{\mathcal{U}_1}^i = w_{\mathcal{A}}^i(u).w_{\mathcal{B}}^i(u)$

$\forall u \in \mathcal{U}$ and $i = 1,2, \dots P$.

(9) The multiplication of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \tilde{x} \mathcal{B} = \mathcal{U}_2$ and is defined by

$$\mathcal{U}_2 = \{$$

$$\prec u, \left(\mu_{\mathcal{U}_2}^1(u), \mu_{\mathcal{U}_2}^2(u), \dots \mu_{\mathcal{U}_2}^p(u)\right), \left(v_{\mathcal{U}_2}^1(u), v_{\mathcal{U}_2}^2(u), \dots v_{\mathcal{U}_2}^p(u)\right), (w_{\mathcal{U}_2}^1(u), w_{\mathcal{U}_2}^2(u), \dots w_{\mathcal{U}_2}^p(u))$$

$$\succ : u \in \mathcal{U}\}$$

where

$$\mu_{\mathcal{U}_2}^i = \mu_{\mathcal{A}}^i(u).\mu_{\mathcal{B}}^i(u)\,, \qquad v_{\mathcal{U}_2}^i = v_{\mathcal{A}}^i(u) + v_{\mathcal{B}}^i(u) - v_{\mathcal{A}}^i(u).v_{\mathcal{B}}^i(u)\,, \qquad w_{\mathcal{U}_2}^i = w_{\mathcal{A}}^i(u) +$$

$w_{\mathcal{B}}^i(u) - w_{\mathcal{A}}^i(u).w_{\mathcal{B}}^i(u)\ \forall\ u \in \mathcal{U}$ and $i = 1,2,\dots P$.

Here $\vee,\ \wedge, +, ., -$ denotes maximum, minimum, addition, multiplication, subtraction of real numbers respectively.

**Definition 5** [25]  Let $I$ be unit interval and $k$ be a positive integer. A multi $Q$ -fuzzy set $\tilde{A}_Q$ in $V$ and a non-empty set $Q$ is a set of ordered sequences $\tilde{A}_Q = \{(v.q), \mu_i(v,q): v \in V, q \in Q\}$ where

$$\mu_i: V \times Q \to I^k, \quad i = 1,2,\dots,k.$$

The function $\big(\mu_1(v,q), \mu_2(v,q), \dots, \mu_k(v,q)\big)$ is called the membership function of multi $Q$- fuzzy set $\tilde{A}_Q$: and $\mu_1(v,q) + \mu_2(v,q) + \dots + \mu_k(v,q) \le 1, k$ is called the dimension of $\tilde{A}_Q$. The set of all multi $Q$- fuzzy sets of dimension $k$ in $V$ and $Q$ is denoted by $M^k QF(V)$.

**Definition 6** [53] Let

$\mathcal{A} = \{$

$< u, \big(\mu_{\mathcal{A}}^1(u), \mu_{\mathcal{A}}^2(u), \dots \mu_{\mathcal{A}}^p(u)\big), \big(v_{\mathcal{A}}^1(u), v_{\mathcal{A}}^2(u), \dots v_{\mathcal{A}}^p(u)\big), (w_{\mathcal{A}}^1(u), w_{\mathcal{A}}^2(u), \dots w_{\mathcal{A}}^p(u))$

$>: u \in \mathcal{U}\}$

and

$\mathcal{B} = \{< u, \big(\mu_{\mathcal{B}}^1(u), \mu_{\mathcal{B}}^2(u), \dots \mu_{\mathcal{B}}^p(u)\big), \big(v_{\mathcal{B}}^1(u), v_{\mathcal{B}}^2(u), \dots v_{\mathcal{B}}^p(u)\big), (w_{\mathcal{B}}^1(u), w_{\mathcal{B}}^2(u), \dots w_{\mathcal{B}}^p(u))$

$>: u \in \mathcal{U}\}$

And be two NMSs, then the normalized hamming distance between $\mathcal{A}$ and $\mathcal{B}$ can be defined as follows:

$$d_\lambda(A_1, A_2) = \left[ \sum_{j=1}^n \omega_j \left( \sum_{i=1}^4 \beta_i\, \varphi_i(x_j) \right)^\lambda \right]^{\frac{1}{\lambda}} \tag{1}$$

where

$$\lambda > 0, \beta_i \in [0,1] \text{ and } \sum_{i=1}^4 \beta_i = 1, \quad \omega_j \in [0,1] \text{ and } \sum_{j=1}^n \omega_j = 1.$$

## 3. A New Approach Distance Measure of Refined Q-Single-Valued Neutrosophic Sets

In this section, we defined new distance measure two refined Q- single-valued neutrosophic sets that are based on clustering method by extending the studies in [54].

**Definition 7** For two refined Q- single-valued neutrosophic sets $\mathcal{A}$ and $\mathcal{B}$ in a universe of discourse which are denoted by

$$\mathcal{A} = \{< (u,q), \left(\mu_{\mathcal{A}}^1(u_1,q_1), \mu_{\mathcal{A}}^2(u_2,q_2), \dots \mu_{\mathcal{A}}^p(u_p,q_p)\right), \left(v_{\mathcal{A}}^1(u_1,q_1), v_{\mathcal{A}}^2(u_2,q_2), \dots v_{\mathcal{A}}^p(u_p,q_p)\right),$$
$$(w_{\mathcal{A}}^1(u_1,q_1), w_{\mathcal{A}}^2(u_2,q_2), \dots, w_{\mathcal{A}}^p(u_p,q_p)) >: u \in \mathcal{U}, q \in Q\}$$

and

$$\mathcal{B} = \{< (u,q), \left(\mu_{\mathcal{B}}^1(u_1,q_1), \mu_{\mathcal{B}}^2(u_2,q_2), \dots, \mu_{\mathcal{B}}^p(u_p,q_p)\right), \left(v_{\mathcal{B}}^1(u_1,q_1), v_{\mathcal{B}}^2(u_2,q_2), \dots, v_{\mathcal{B}}^p(u_p,q_p)\right),$$
$$(w_{\mathcal{B}}^1(u_1,q_1), w_{\mathcal{B}}^2(u_2,q_2), \dots, w_{\mathcal{B}}^p(u_p,q_p)) >: u \in \mathcal{U}, q \in Q\}$$

The refined Q- single-valued neutrosophic weighted distance measure are defined by

$$d(\mathcal{A}, \mathcal{B}) = \left[\sum_{i=1}^{n} \omega_i \left(\sum_{j=1}^{6} \beta_j \, \varphi_j(u_i, q_i)\right)^{\lambda}\right]^{\frac{1}{\lambda}} \tag{2}$$

where

$$\lambda > 0, \beta_j \in [0,1] \text{ and } \sum_{j=1}^{6} \beta_j = 1, \omega_i \in [0,1] \text{ and } \sum_{i=1}^{n} \omega_i = 1.$$

$$\varphi_1(u_1, q_1) = \left(\frac{\left|\mu_{\mathcal{A}}^i(u_1,q_1) - \mu_{\mathcal{B}}^i(u_1,q_1)\right|}{9} + \frac{\left|v_{\mathcal{A}}^i(u_1,q_1) - v_{\mathcal{B}}^i(u_1,q_1)\right|}{9}\right.$$
$$\left. + \frac{\left|w_{\mathcal{A}}^i(u_1,q_1) - w_{\mathcal{B}}^i(u_1,q_1)\right|}{9}\right)$$

$$\varphi_2(u_2, q_2) = \left(\begin{array}{l} max\left\{\frac{3+\mu_{\mathcal{A}}^i(u_2,q_2)-v_{\mathcal{A}}^i(u_2,q_2)-w_{\mathcal{A}}^i(u_2,q_2)}{9}, \frac{3+\mu_{\mathcal{B}}^i(u_2,q_2)-v_{\mathcal{B}}^i(u_2,q_2)-w_{\mathcal{B}}^i(u_2,q_2)}{9}\right\} \\ -min\left\{\frac{3+\mu_{\mathcal{A}}^i(u_2,q_2)-v_{\mathcal{A}}^i(u_2,q_2)-w_{\mathcal{A}}^i(u_2,q_2)}{9}, \frac{3+\mu_{\mathcal{B}}^i(u_2,q_2)-v_{\mathcal{B}}^i(u_2,q_2)-w_{\mathcal{B}}^i(u_2,q_2)}{9}\right\} \end{array}\right)$$

$$\varphi_3(u_3, q_3) = \frac{\left|\mu_{\mathcal{A}}^i(u_3, q_3) - \mu_{\mathcal{B}}^i(u_3, q_3) + v_{\mathcal{B}}^i(u_3, q_3) - v_{\mathcal{A}}^i(u_3, q_3)\right|}{3}$$

$$\varphi_4(u_4, q_4) = \frac{\left|\mu_{\mathcal{A}}^i(u_4, q_4) - \mu_{\mathcal{B}}^i(u_4, q_4) + w_{\mathcal{B}}^i(u_4, q_4) - w_{\mathcal{A}}^i(u_4, q_4)\right|}{3}$$

$$\varphi_5(u_5, q_5) = \frac{1}{3P} \sum_{i=1}^{P} \left[ \left|\mu_{\mathcal{A}}^i(u_5, q_5) - \mu_{\mathcal{B}}^i(u_5, q_5)\right| + \left|v_{\mathcal{A}}^i(u_5, q_5) - v_{\mathcal{B}}^i(u_5, q_5)\right| \right.$$

$$\left. + \left|w_{\mathcal{A}}^i(u_5, q_5) - w_{\mathcal{B}}^i(u_5, q_5)\right| \right]$$

$$\varphi_6(u_6, q_6) = \left\{ \frac{1}{3P} \sum_{i=1}^{P} \left[ \left|\mu_{\mathcal{A}}^i(u_6, q_6) - \mu_{\mathcal{B}}^i(u_6, q_6)\right|^2 + \left|v_{\mathcal{A}}^i(u_6, q_6) - v_{\mathcal{B}}^i(u_6, q_6)\right|^2 \right. \right.$$

$$\left. \left. + \left|w_{\mathcal{A}}^i(u_6, q_6) - w_{\mathcal{B}}^i(u_6, q_6)\right|^2 \right] \right\}^{1/2}$$

**Proposition 8** The distance measure $d_\lambda(\mathcal{A}, \mathcal{B})$ for $\lambda > 0$ satisfies the following properties:

(1) $0 \le d_\lambda(\mathcal{A}, \mathcal{B}) \le 1$;

(2) $d_\lambda(\mathcal{A}, \mathcal{B}) = 0 \iff \mathcal{A} = \mathcal{B}$;

(3) $d_\lambda(\mathcal{A}, \mathcal{B}) = d_\lambda(\mathcal{B}, \mathcal{A})$;

(4) If $\mathcal{A} \cong \mathcal{B} \cong \mathcal{C}$, $\mathcal{C}$ is an Refined Q- single-valued neutrosophic in $\mathcal{U}$, then $d_\lambda(\mathcal{A}, \mathcal{B}) \le d_\lambda(\mathcal{A}, \mathcal{C})$ and $d_\lambda(\mathcal{B}, \mathcal{C}) \le d_\lambda(\mathcal{A}, \mathcal{C})$.

**Proof:** It is easy to see that $d_\lambda(\mathcal{A}, \mathcal{B})$ satisfies the properties $(1) - (3)$. Therefore, we only prove (4).

When $\mathcal{A} \cong \mathcal{B} \cong \mathcal{C}$ if $\mu_{\mathcal{A}}^i(u, q) \le \mu_{\mathcal{B}}^i(u, q) \le \mu_{\mathcal{C}}^i(u, q), v_{\mathcal{A}}^i(u, q) \ge v_{\mathcal{B}}^i(u, q) \ge v_{\mathcal{C}}^i(u, q),$ $w_{\mathcal{A}}^i(u, q) \ge w_{\mathcal{B}}^i(u, q) \ge w_{\mathcal{C}}^i(u, q), \forall u \in \mathcal{U}, q \in Q$ and $i = 1, 2, \dots P$. Then we have

$\left|\mu_{\mathcal{A}}^i(u, q) - \mu_{\mathcal{B}}^i(u, q)\right| \le \left|\mu_{\mathcal{A}}^i(u, q) - \mu_{\mathcal{C}}^i(u, q)\right|; \left|\mu_{\mathcal{B}}^i(u, q) - \mu_{\mathcal{C}}^i(u, q)\right| \le \left|\mu_{\mathcal{A}}^i(u, q) - \mu_{\mathcal{C}}^i(u, q)\right|,$

$\left|v_{\mathcal{A}}^i(u, q) - v_{\mathcal{B}}^i(u, q)\right| \le \left|v_{\mathcal{A}}^i(u, q) - v_{\mathcal{C}}^i(u, q)\right|; \left|v_{\mathcal{B}}^i(u, q) - v_{\mathcal{C}}^i(u, q)\right| \le \left|v_{\mathcal{A}}^i(u, q) - v_{\mathcal{C}}^i(u, q)\right|,$

$\left|w_{\mathcal{A}}^i(u, q) - w_{\mathcal{B}}^i(u, q)\right| \le \left|w_{\mathcal{A}}^i(u, q) - w_{\mathcal{C}}^i(u, q)\right|; \left|w_{\mathcal{B}}^i(u, q) - w_{\mathcal{C}}^i(u, q)\right| \le \left|w_{\mathcal{A}}^i(u, q) - w_{\mathcal{C}}^i(u, q)\right|,$

$$\frac{2 + \mu_{\mathcal{A}}^i(u,q) - v_{\mathcal{A}}^i(u,q) - w_{\mathcal{A}}^i(u,q)}{3} \leq \frac{2 + \mu_{\mathcal{B}}^i(u,q) - v_{\mathcal{B}}^i(u,q) - w_{\mathcal{B}}^i(u,q)}{3}$$

$$\leq \frac{2 + \mu_{\mathcal{C}}^i(u,q) - v_{\mathcal{C}}^i(u,q) - w_{\mathcal{C}}^i(u,q)}{3}$$

$$0 \leq \frac{\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{B}}^i(u,q) + v_{\mathcal{B}}^i(u,q) - v_{\mathcal{A}}^i(u,q)\right|}{2}$$

$$\leq \frac{\left|\mu_{\mathcal{C}}^i(u,q) - \mu_{\mathcal{A}}^i(u,q) + v_{\mathcal{A}}^i(u,q) - v_{\mathcal{C}}^i(u,q)\right|}{2}$$

$$0 \leq \frac{\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{B}}^i(u,q) + w_{\mathcal{B}}^i(u,q) - w_{\mathcal{A}}^i(u,q)\right|}{2}$$

$$\leq \frac{\left|\mu_{\mathcal{C}}^i(u,q) - \mu_{\mathcal{A}}^i(u,q) + w_{\mathcal{A}}^i(u,q) - w_{\mathcal{C}}^i(u,q)\right|}{2}$$

$$\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{B}}^i(u,q)\right| + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{B}}^i(u,q)\right| + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{B}}^i(u,q)\right|$$

$$\leq \left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{C}}^i(u,q)\right| + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{C}}^i(u,q)\right| + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{C}}^i(u,q)\right|;$$

$$\frac{1}{3P} \sum_{i=1}^{P} \left[\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{B}}^i(u,q)\right| + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{B}}^i(u,q)\right| + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{B}}^i(u,q)\right|\right]$$

$$\leq \frac{1}{3P} \sum_{i=1}^{P} \left[\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{C}}^i(u,q)\right| + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{C}}^i(u,q)\right| + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{C}}^i(u,q)\right|\right]$$

$$\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{B}}^i(u,q)\right|^2 + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{B}}^i(u,q)\right|^2 + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{B}}^i(u,q)\right|^2$$

$$\leq \left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{C}}^i(u,q)\right|^2 + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{C}}^i(u,q)\right|^2 + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{C}}^i(u,q)\right|^2;$$

$$\left\{\frac{1}{3P} \sum_{i=1}^{P} \left[\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{B}}^i(u,q)\right|^2 + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{B}}^i(u,q)\right|^2 + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{B}}^i(u,q)\right|^2\right]\right\}^{1/2}$$

$$\leq \left\{\frac{1}{3P} \sum_{i=1}^{P} \left[\left|\mu_{\mathcal{A}}^i(u,q) - \mu_{\mathcal{C}}^i(u,q)\right|^2 + \left|v_{\mathcal{A}}^i(u,q) - v_{\mathcal{C}}^i(u,q)\right|^2\right.\right.$$

$$\left.\left. + \left|w_{\mathcal{A}}^i(u,q) - w_{\mathcal{C}}^i(u,q)\right|^2\right]\right\}^{1/2}$$

$d_\lambda(\mathcal{A}, \mathcal{B}) \leq d_\lambda(\mathcal{A}, \mathcal{C})$ and $d_\lambda(\mathcal{B}, \mathcal{C}) \leq d_\lambda(\mathcal{A}, \mathcal{C})$ for $\lambda > 0$.


**Definition 9** Let

$$\mathcal{A} = \{< (u,q), \left(\mu_{\mathcal{A}}^1(u,q), \mu_{\mathcal{A}}^2(u,q), \dots, \mu_{\mathcal{A}}^p(u,q)\right), \left(v_{\mathcal{A}}^1(u,q), v_{\mathcal{A}}^2(u,q), \dots, v_{\mathcal{A}}^p(u,q)\right),$$

$$\left(w_{\mathcal{A}}^1(u,q), w_{\mathcal{A}}^2(u,q), \dots, w_{\mathcal{A}}^p(u,q)\right) >: u \in \mathcal{U}, q \in Q\}$$

and

$$\mathcal{B} = \{< (u,q), \left(\mu_{\mathcal{B}}^1(u,q), \mu_{\mathcal{B}}^2(u,q), \dots, \mu_{\mathcal{B}}^p(u,q)\right), \left(v_{\mathcal{B}}^1(u,q), v_{\mathcal{B}}^2(u,q), \dots, v_{\mathcal{B}}^p(u,q)\right),$$

$$\left(w_{\mathcal{B}}^1(u,q), w_{\mathcal{B}}^2(u,q), \dots, w_{\mathcal{B}}^p(u,q)\right) >: u \in \mathcal{U}, q \in Q\}$$

be two refined Q-single-valued neutrosophic sets. Then, hybrid similarity measure between refined Q-single-valued neutrosophic $\mathcal{A}$ and $\mathcal{B}$, denoted

$$\vartheta_\lambda(\mathcal{A}, \mathcal{B}) = 1 - d_\lambda(\mathcal{A}, \mathcal{B})$$

**Proposition 10** The similarity measure $\vartheta_\lambda(\mathcal{A}, \mathcal{B})$ for $\lambda > 0$ satisfies the following properties;

(s1) $0 \le \vartheta_\lambda(\mathcal{A}, \mathcal{B}) \le 1$;

(s2) $\vartheta_\lambda(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$;

(s3) $\vartheta_\lambda(\mathcal{A}, \mathcal{B}) = \vartheta_\lambda(\mathcal{B}, \mathcal{A})$;

(s4) If $\mathcal{A} \cong \mathcal{B} \cong \mathcal{C}$, $\mathcal{C}$ is a refined Q- single-valued neutrosophic in $\mathcal{U}$, then $\vartheta_\lambda(\mathcal{A}, \mathcal{B}) \ge \vartheta_\lambda(\mathcal{A}, \mathcal{C})$ and $\vartheta_\lambda(\mathcal{B}, \mathcal{C}) \ge \vartheta_\lambda(\mathcal{A}, \mathcal{C})$.

**Definition 11** Let $\phi$ be a set-to-point mapping: $\phi$: refined Q- single-valued neutrosophic $\to [0,1]$, then $\phi$ is an entropy measure if it satisfies the following conditions:

(E1) $\phi(\mathcal{A})=0$(minimum) if and only if $\mathcal{A}$ or $\mathcal{A}^c$ is a crisp set;

(E2) $\phi(\mathcal{A})=1$(maximum) if and only if $\mathcal{A}=\mathcal{A}^c$, i.e. $\mu_{\mathcal{A}}^i(u,q) = w_{\mathcal{A}}^i(u,q), v_{\mathcal{A}}^i(u,q) = 0.5$ for all $u \in \mathcal{U}, q \in Q$

(E3) $\phi(\mathcal{A}) \le E(\mathcal{B})$ if $\mathcal{A}$ is less neutrosophic than $\mathcal{B}$,

$\mu_{\mathcal{A}}^i(u,q) \le \mu_{\mathcal{B}}^i(u,q), \quad w_{\mathcal{B}}^i(u,q) \le w_{\mathcal{A}}^i(u,q)$, for $\mu_{\mathcal{B}}^i(u,q) \le w_{\mathcal{B}}^i(u,q)$ and $v_{\mathcal{A}}^i(u,q) = v_{\mathcal{B}}^i(u,q) = 0.5$

or

$\mu_{\mathcal{A}}^i(u,q) \ge \mu_{\mathcal{B}}^i(u,q), \quad w_{\mathcal{B}}^i(u,q) \ge w_{\mathcal{A}}^i(u,q)$, for $\mu_{\mathcal{B}}^i(u,q) \ge w_{\mathcal{B}}^i(u,q)$ and $v_{\mathcal{B}}^i(u,q) = v_{\mathcal{B}}^i(u,q) = 0.5$

(E4) $\phi(\mathcal{A}) = \phi(\mathcal{A}^c)$.

**Remark 12** In some cases, we not only think about the distance between $\mathcal{A}$ *and* $\mathcal{B}$, but also need to consider the distance between $\mathcal{A}$ *and* $\mathcal{B}^c$. So we can define the index of distance for two refined Q- single-valued neutrosophic sets $\mathcal{A}$ *and* $\mathcal{B}$ as follows.

**Definition 13** For two refined Q-single-valued neutrosophic sets $\mathcal{A}$ and $\mathcal{B}$, the index of distance is defined by

$$I_\lambda(\mathcal{A}, \mathcal{B}) = \frac{d_\lambda(\mathcal{A}, \mathcal{B})}{d_\lambda(\mathcal{A}, \mathcal{B}^c)}.$$

**Definition 14**

$$\mathcal{A} = \{< (u, q), \left(\mu_\mathcal{A}^1(u, q), \mu_\mathcal{A}^2(u, q), \dots, \mu_\mathcal{A}^p(u, q)\right), \left(v_\mathcal{A}^1(u, q), v_\mathcal{A}^2(u, q), \dots, v_\mathcal{A}^p(u, q)\right),$$

$$\left(w_\mathcal{A}^1(u, q), w_\mathcal{A}^2(u, q), \dots, w_\mathcal{A}^p(u, q)\right) >: u \in \mathcal{U}, q \in Q\}$$

be refined Q-single-valued neutrosophic sets. A refined Q- single-valued neutrosophic algebraic geometric averaging (RQ-SVNAGA) operator;

$$RQ - SVNAGA(A_1, A_2, \dots, A_p) = \left|\left(\left(\prod_{i=1}^{p} \mu_{\mathcal{A}_s}^i(u, q)\right), \left(\sum_{i=1}^{p} v_{\mathcal{A}_s}^i(u, q) - \prod_{i=1}^{p} v_{\mathcal{A}_s}^i(u, q)\right)\right.\right.$$

$$\left.\left.\left(\sum_{i=1}^{p} w_{\mathcal{A}_s}^i(u, q) - \prod_{i=1}^{p} w_{\mathcal{A}_s}^i(u, q)\right)\right)\right| \qquad (3)$$

$(s = 1, 2, \dots p)$.

**Proof:** The first result follows quickly from definition 3. In the following, we prove the second result by usıng mathematical induction on $p$. We firs prove that equation (3) holds for $p = 2$.

$$RQ - SVNAGA(A_1, A_2)$$

$$= \left|\left(\left(\prod_{i=1}^{2} \mu_{\mathcal{A}_s}^i(u, q)\right), \left(\sum_{i=1}^{2} v_{\mathcal{A}_s}^i(u, q) - \prod_{i=1}^{2} v_{\mathcal{A}_s}^i(u, q)\right), \left(\sum_{i=1}^{2} w_{\mathcal{A}_s}^i(u, q)\right.\right.\right.$$

$$\left.\left.\left. - \prod_{i=1}^{2} w_{\mathcal{A}_s}^i(u, q)\right)\right)\right|$$

$$= \left|\begin{array}{c} \left(\mu_{\mathcal{A}_1}^i(u, q). \mu_{\mathcal{A}_2}^i(u, q)\right), \left(\left(v_{\mathcal{A}_1}^i(u, q) + v_{\mathcal{A}_2}^i(u, q)\right) - \left(v_{\mathcal{A}_1}^i(u, q). v_{\mathcal{A}_2}^i(u, q)\right)\right), \\ \left(\left(w_{\mathcal{A}_1}^i(u, q) + w_{\mathcal{A}_2}^i(u, q)\right) - \left(w_{\mathcal{A}_1}^i(u, q). w_{\mathcal{A}_2}^i(u, q)\right)\right) \end{array}\right|$$

if that equation (3) hold for $p = k$, that is

$RQ - SVNAGA(A_1, A_2, \dots, A_k)$

$$= \left\langle \left( \left( \prod_{i=1}^{k} \mu_{\mathcal{A}_s}^i(u,q) \right), \left( \sum_{i=1}^{k} v_{\mathcal{A}_s}^i(u,q) - \prod_{i=1}^{k} v_{\mathcal{A}_s}^i(u,q) \right), \right. \right. \\ \left. \left. \left( \sum_{i=1}^{k} w_{\mathcal{A}_s}^i(u,q) - \prod_{i=1}^{k} w_{\mathcal{A}_s}^i(u,q) \right) \right) \right\rangle \quad (4)$$

$$= \left\langle \begin{array}{c} \left( \mu_{\mathcal{A}_s}^1(u,q). \mu_{\mathcal{A}_s}^2(u,q). \dots . \mu_{\mathcal{A}_s}^k(u,q) \right), \\ \left( \left( v_{\mathcal{A}_s}^1(u,q) + v_{\mathcal{A}_s}^2(u,q) + \cdots + v_{\mathcal{A}_s}^k(u,q) \right) - \left( v_{\mathcal{A}_s}^1(u,q). v_{\mathcal{A}_s}^2(u,q). \dots . v_{\mathcal{A}_s}^k(u,q) \right) \right), \\ \left( \left( w_{\mathcal{A}_s}^1(u,q) + w_{\mathcal{A}_s}^2(u,q) + \cdots + w_{\mathcal{A}_s}^k(u,q) \right) - \left( w_{\mathcal{A}_s}^1(u,q). w_{\mathcal{A}_s}^2(u,q) \right). \dots . w_{\mathcal{A}_s}^k(u,q) \right) \end{array} \right\rangle \quad (5)$$

Then, both sides of the equation is added by $A_{k+1}$ by the**:**

$$RQ - SVNAGA(A_1, A_2, \dots, A_k, A_{k+1}) = \left\langle \left( \mu_{\mathcal{A}_s}^1(u,q). \mu_{\mathcal{A}_s}^2(u,q). \dots . \mu_{\mathcal{A}_s}^k(u). \mu_{\mathcal{A}_s}^{k+1}(u,q) \right), \right.$$

$$\left( v_{\mathcal{A}_s}^1(u,q) + v_{\mathcal{A}_s}^2(u,q) + \cdots + v_{\mathcal{A}_s}^k(u,q) + v_{\mathcal{A}_s}^{k+1}(u,q) \right)$$

$$- \left( v_{\mathcal{A}_s}^1(u,q). v_{\mathcal{A}_s}^2(u,q). \dots . v_{\mathcal{A}_s}^k(u,q). v_{\mathcal{A}_s}^{k+1}(u,q) \right),$$

$$\left( \left( w_{\mathcal{A}_s}^1(u,q) + w_{\mathcal{A}_s}^2(u,q) + \cdots + w_{\mathcal{A}_s}^k(u,q) + w_{\mathcal{A}_s}^{k+1}(u,q) \right) \right.$$

$$\left. - \left( w_{\mathcal{A}_s}^1(u,q). w_{\mathcal{A}_s}^2(u,q) \right). \dots . w_{\mathcal{A}_s}^k(u). w_{\mathcal{A}_s}^{k+1}(u,q) \right) \right\rangle.$$

$RQ - SVNAGA(A_1, A_2, \dots, A_k, A_{k+1})$

$$= \left\langle \left( \left( \prod_{i=1}^{k+1} \mu_{\mathcal{A}_s}^i(u,q) \right), \left( \sum_{i=1}^{k+1} v_{\mathcal{A}_s}^i(u,q) - \prod_{i=1}^{k+1} v_{\mathcal{A}_s}^i(u,q) \right), \left( \sum_{i=1}^{k+1} w_{\mathcal{A}_s}^i(u,q) \right. \right. \right.$$

$$\left. \left. \left. - \prod_{i=1}^{k+1} w_{\mathcal{A}_s}^i(u,q) \right) \right) \right\rangle$$

i.e. that equation (3) hold for $p = k + 1$.

**Proposition 15** The index of distance $I_\lambda(\mathcal{A}, \mathcal{B})$ for two refined Q-single-valued neutrosophic sets $\mathcal{A}$ and $\mathcal{B}$ satisfies the following properties:

(I1) $I_\lambda(\mathcal{A}, \mathcal{B}) = 0 \Leftrightarrow \mathcal{A} = \mathcal{B}$;

(I2) $I_\lambda(\mathcal{A}, \mathcal{B}) = 0 \Leftrightarrow d_\lambda(\mathcal{A}, \mathcal{B}) = d_\lambda(\mathcal{A}, \mathcal{B}^c)$;

(I3) $I_\lambda(\mathcal{A}, \mathcal{B}) \to +\infty, \mathcal{A} = \mathcal{B}^c$, these means $\mathcal{A}$ and $\mathcal{B}$ are completely different;

(I4) *When* $\mathcal{A} = \mathcal{B} = \mathcal{B}^c$, the entropy measure of $\mathcal{A}$ and $\mathcal{B}$ reaches its maximum value;

(*I5*) $I_\lambda(\mathcal{A}, \mathcal{B}) < 1$ means compare with $\mathcal{B}^c\mathcal{A}$ is more similar to $\mathcal{B}$;

(*I6*) $I_\lambda(\mathcal{A}, \mathcal{B}) > 1$ means compare with $\mathcal{B}^c\mathcal{A}$ is less similar to $\mathcal{B}$.

# 4. Hierarchical Clustering Algorithm Based On The Similarity Measures of Refined Q- Single-Valued Neutrosophic Sets

Then, we can give the algorithm of clustering refined Q- single-valued neutrosophic sets as follows:

**Step1** Let each of the objects $A_k (k = 1,2, \dots, p)$ be considered as unique clusters $\{A_1\}, \{A_2\}, \dots, \{A_p\}$, and calculate the distance $d_{ij} = d(A_i, A_j)$ by equations (2) and get the refined Q- single-valued neutrosophic distance matrix

$$D = (d_{ij})_{pxp}$$

**Step2** In the refined Q- single-valued neutrosophic distance matrix $D = (d_{ij})_{pxp}$, search the minimal distance

$d(A_i, A_j) = min_{1 \le l, m \le p} d(A_l, A_m)(l \ne m),$

and combine clusters $A_i$ and $A_j$ to from a new cluster $A_{ij}$; meanwhile calculate the centre of $A_{ij}$ by using equation (3).

**Step3** Update the refined Q- single-valued neutrosophic distance matrix by computing the distance between the new cluster $A_{ij}$ and the other clusters.

**Step4** Repeat steps 2 and 3 until all objects are in the one cluster.

**Example 16**

People both in line with their own actions and needs and the conditions of the living space and the environment; It meets the main act of housing in different types of housing. The planning and design of the immediate surroundings of the house depends on what is expected of it, how it will be used and how it will serve the house in it. Answers to these questions should be sought before starting design. As everywhere in the world's climate is the main factor that determines the types of housing conditions in Turkey. In addition, natural natural conditions such as geological structure and vegetation determine the housing types. Economic and cultural development in our country reduces the impact of the natural environment on housing types. Therefore Ezgi Construction Company wants to

build housing in four different regions in the site planning. There are four alternatives $A_m$ (m=1, 2, 3, 4), with two types of building material $Q = \{q_1, q_2 \}$. Every regions have two evaluation factors: $u_1$: transportation; $u_2$: location. The characteristics of each region under the two attributes are represented by the form of refined neutrosophic data are as follows;

**Step1.** Construct the decision matrix provided by the customer as**;**

**Table 4:** Decision matrix

|  | $(u_1, q_1)$ | $(u_1, q_2)$ |
|---|---|---|
| $A_1$ | $\langle(0.5,0.8,0.2),(0.1,0.4,0.6),(0.7,0.5,0.2)\rangle$ | $\langle(0.7,0.9,0.2),(0.3,0.5,0.6),(0.1,0.2,0.2)\rangle$ |
| $A_2$ | $\langle(0.8,0.8,0.2),(0.8,0.4,0.6),(0.1,0.5,0.2)\rangle$ | $\langle(0.3,0.7,0.2),(0.1,0.6,0.8),(0.7,0.7,0.7)\rangle$ |
| $A_3$ | $\langle(0.5,0.6,0.2),(0.1,0.8,0.6),(0.7,0.9,0.2)\rangle$ | $\langle(0.1,0.8,0.8),(0.7,0.4,0.3),(0.4,0.5,0.9)\rangle$ |
| $A_4$ | $\langle(0.5,0.8,0.4),(0.1,0.4,0.6),(0.7,0.5,0.1)\rangle$ | $\langle(0.8,0.8,0.7),(0.2,0.4,0.9),(0.7,0.5,0.3)\rangle$ |

| $(u_2, q_1)$ | $(u_2, q_2)$ |
|---|---|
| $\langle(0.8,0.8,0.8),(0.8,0.4,0.6),(0.2,0.5,0.2)\rangle$ | $\langle(0.5,0.5,0.5),(0.1,0.4,0.5),(0.5,0.5,0.2)\rangle$ |
| $\langle(0.1,0.9,0.2),(0.6,0.5,0.7),(0.1,0.2,0.1)\rangle$ | $\langle(0.1,0.8,0.5),(0.7,0.5,0.3),(0.4,0.5,0.1)\rangle$ |
| $\langle(0.6,0.7,0.5),(0.4,0.4,04),(0.8,0.5,0.5)\rangle$ | $\langle(0.8,0.7,0.5),(0.4,0.4,0.5),(0.8,0.5,0.5)\rangle$ |
| $\langle(0.5,0.5,0.6),(0.5,0.4,0.6),(0.9,0.5,0.2)\rangle$ | $\langle(0.1,0.1,0.2),(0.6,0.5,0.9),(0.1,0.2,0.1)\rangle$ |

Let $\lambda = 2$ choosing the weight vectors $w_j = \frac{1}{4}$ $(j = 1,2,3,4)$ and $\beta_i = \frac{1}{6}$ $(i = 1,2,3,4,5,6)$, then we use similarity measure to classify the four different regions of $A_m (m = 1,2,3,4)$ by the refined Q- single-valued neutrosophic hierarchical clustering algorithms.

First, we utilize the distance measure to calculate the distance measures between each pair of refined Q- single-valued neutrosophic $A_m (m = 1,2,3,4)$. The result are as follows;

$$D_1 = \begin{bmatrix} 1 & 0,033382 & 0,027595 & 0,023055 \\ 0,033382 & 1 & 0,04068 & 0,038205 \\ 0,027595 & 0,04068 & 1 & 0,033135 \\ 0,023055 & 0,038205 & 0,033135 & 1 \end{bmatrix}$$

**Step2.** In the refined Q- neutrosophic distance matrix $D_1$, we search the smallest distance $d_{min}(A_i, A_j) = d_1(A_1, A_4) = 0,023055$ then combine $\{A_1\}$ and $\{A_4\}$ to from a new cluster $\{A_1, A_4\}$; so the region alternatives $A_j(j = 1,2,3,4)$ can be clustered into the following four clusters: $\{A_2\}, \{A_3\}, \{A_1, A_4\}$, and compute the centre of each new cluster by using Definition 3.8:

| | $(u_1, q_1)$ | $(u_1, q_2)$ |
|---|---|---|
| $A_1, A_4$ | $\langle(0.25,0.64,0.08), (0.19,0.64,0.84), (0.91,0.75,0.28)\rangle$ | $\langle(0.56,0.72,0.14), (0.44,0.7,0.96), (0.73,0.6,0.44)\rangle$ |
| | $(u_2, q_1)$ | $(u_2, q_2)$ |
| | $\langle(0.4,0.4,0.48), (0.9,0.64,0.84), (0.92,0.75,0.36)\rangle$ | $\langle(0.05,0.05,0.1), (0.64,0.7,0.95), (0.55,0.6,0.28)\rangle$ |

**Step3.**

| | $(u_1, q_1)$ | $(u_1, q_2)$ |
|---|---|---|
| $A_2$ | $\langle(0.8,0.8,0.2), (0.8,0.4,0.6), (0.1,0.5,0.2)\rangle$ | $\langle(0.3,0.7,0.2), (0.1,0.6,0.8), (0.7,0.7,0.7)\rangle$ |
| $A_3$ | $\langle(0.5,0.6,0.2), (0.1,0.8,0.6), (0.7,0.9,0.2)\rangle$ | $\langle(0.1,0.8,0.8), (0.7,0.4,0.3), (0.4,0.5,0.9)\rangle$ |
| $A_1, A_4$ | $\langle(0.25,0.64,0.08), (0.19,0.64,0.84), (0.91,0.75,0.28)\rangle$ | $\langle(0.56,0.72,0.14), (0.44,0.7,0.96), (0.73,0.6,0.44)\rangle$ |
| | $(u_2, q_1)$ | $(u_2, q_2)$ |
| | $\langle(0.1,0.9,0.2), (0.6,0.5,0.7), (0.1,0.2,0.1)\rangle$ | $\langle(0.1,0.8,0.5), (0.7,0.5,0.3), (0.4,0.5,0.1)\rangle$ |
| | $\langle(0.6,0.7,0.5), (0.4,0.4,04), (0.8,0.5,0.5)\rangle$ | $\langle(0.8,0.7,0.5), (0.4,0.4,0.5), (0.8,0.5,0.5)\rangle$ |
| | $\langle(0.4,0.4,0.48), (0.9,0.64,0.84), (0.92,0.75,0.36)\rangle$ | $\langle(0.05,0.05,0.1), (0.64,0.7,0.95), (0.55,0.6,0.28)\rangle$ |

$$D_2 = \begin{bmatrix} 1 & 0,04068 & 0,045896 \\ 0,04068 & 1 & 0,051046 \\ 0,045896 & 0,051046 & 1 \end{bmatrix}$$

**Step4** Check whether all objects is in the one cluster; if not, then repeat Steps 2 and 3. Since there are still three clusters $\{A_2\}, \{A_3\}, \{A_1, A_4\}$, we repeat Steps 2 and 3 as follows: In the refined Q-single-valued neutrosophic distance matrix $D_2$, we find the smallest distance $d_{min}(A_i, A_j) = d_1(A_2, A_3) = 0,04068$,

$A_1, A_2, A_3, A_4$

$A_2, A_3$     $A_1, A_4$

$A_3$     $A_1$

$A_2$     $A_4$
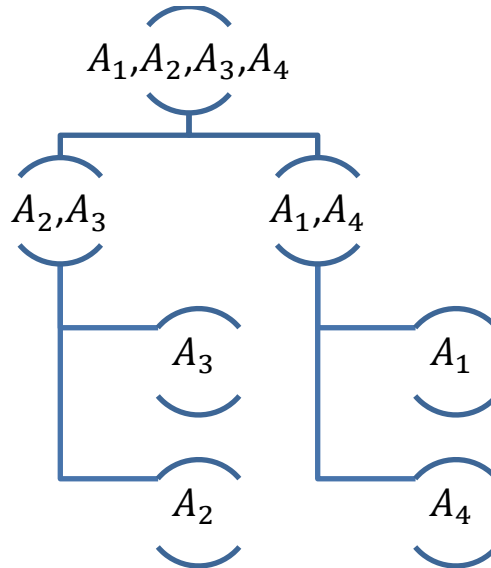
## 5. Conclusion

In this chapter, a hierarchical clustering method using distance-based similarity measures on refined Q-single-valued neutrosophic sets are presented and some of its basic properties are discussed. This measure greatly reduces the influence of imprecise measures and provides an extremely intuitive quantification. Finally, an illustrative example was given to demonstrate the application and effectiveness of the refined Q-single-valued neutrosophic clustering methods. The clustering results have shown that the refined Q-single-valued neutrosophic clustering algorithm is more general than the intuitionistic fuzzy clustering algorithm and the fuzzy clustering algorithm. The authors hope that the proposed concept can be applied in solving realistic multi-criteria decision making problems.

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# Chapter Nine

# Comparing the Social Justice Leadership Behaviors of School Administrators According to Teacher Perceptions Using Classical and Fuzzy Logic

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## ABSTRACT

In this study, it was tried to examine the level of social justice leadership behaviours exhibited by school administrators according to teachers' perceptions. The study was carried out with 114 teachers working in different secondary schools. The pattern of the research was designed with the classical survey. The sample was determined by the typical case sampling method. Data were collected with the Social Justice Leadership Scale. Statistical analysis including arithmetic mean and standard deviation were used in the analysis of the data. As a result of the research, it was found that teachers perceived social justice leadership behaviours of their administrators at the level of agree and completely agree in different schools. In addition, in this study, together with classical survey, a fuzzy survey was administered to on the same sample and the data obtained from fuzzy survey were evaluated in the fuzzy matlab application. The results obtained from fuzzy matlab application, which is used in many decision-making problems, and the results obtained from the classical application were compared and, different results were found. Since the fuzzy survey and the fuzzy matlab in decision makıng processes provide more accurate and sensitive results, it is concluded that the results from the fuzzy survey are more valid. By taking advantage of this study, other researchers can use fuzzy survey and fuzzy matlab for appropriate problems.

**Keywords:** Social Justice Leadership, School Administrators, Classical Logic, Fuzzy Logic, Fuzzy Survey, Fuzzy Matlab.

# INTRODUCTION

Educational institutions not only are influenced by the society because of being open social systems, they also have a mutual interaction that affects the society. Educational institutions have a social structure including different ideas, ideologies, beliefs, economic conditions, individual characteristics etc. According to Cunliffe and Erikson, studying the relationship and interaction between these social structures in organizations constitutes the core activities of leadership [1]. According to Reyes and Wagstaff, the effectiveness of leadership directly affects schools [2]. Especially after 2000, when considering the type of leadership in the field of education, many school-based social justice leadership theory and practice in terms of social diversity and difference in educational administration began to be dealt with many researchers intensively [3-7]In school there are individuals havıng different ideology, cultural / ethnical background, different learning style and disability. Therefore, school administrators, as social justice leaders, should observe these differences and have awareness of the formation of a fair environment [8]. Although bureaucratic education policies can cause to ignore these differences in schools, school administrators are expected to eliminate these injustices and inequalities [4]. For this reason, social justice leadership skills of school administrators have been studied by different researchers [8-11]. In this context, in line with the aim of the research, social justice leadership behaviours of administrators in schools were tried to be examined according to the perceptions of teachers.

Many uncertainties arise in daily life. Most of the time, Aristotelian logic (classical logic) is insufficient to deal with these uncertainties. Because in Aristotle logic, an element is either an element of a set or not. That is, the membership value of an element belongs to the set $\{0, 1\}$. If we explain this situation with examples in daily life, for example, according to classical logic, the weather is either cold or hot. Whether the weather is cool or warm cannot be explained by classical logic. Again, according to classical logic, a bottle is either full or empty with water. Situations such as half full, less full, and quarter full bottles cannot be explained with classical logic. Because of these shortcomings, classical logic is insufficient to explain the uncertainties. Zadeh [27] defined fuzzy logic in 1965 to explain uncertainties more precisely mathematically. In fuzzy logic, the membership degree of each element of a set can be in the range of $[0, 1]$. Thus, different from the classical logic, the membership of each element is rated. For example, the weather can be hot, cold, warm, cool, too hot, too cold, etc. and with different degrees of membership. Thus, a more sensitive type of logic has been obtained, including classical logic in explaining uncertainties. Fuzzy logic is one of the most used logic types in decision-making applications in almost every field of science, especially in artificial intelligence applications, including the present day.

In this study, we developed a fuzzy survey. Unlike the classical survey, we wanted the answer of each item as a value in the range of $[0, 100]$. Thus, instead of 5 conformity statements (strongly disagree, disagree, undecided, agree, strongly agree) in the classic survey, we actually used 100 different conformity statements (as numbers) by requesting a score between 0 and 100. Therefore, we will have made a more objective and sensitive assessment. We evaluated the data in this fuzzy survey using the fuzzy matlab

program used in many artificial intelligence applications and decision-making applications, and compared the success of school administrators in the leadership of social justice. In order to compare the classical survey results with the fuzzy survey results, we applied two different surveys to the same sample. Although the items in these surveys are the same, only the answers given to the items were taken in a different category. In other words, for the classical survey, one of the answers: For the fuzzy survey, a value between 0 and 100 was requested. As a result of this comparison, we saw that the results obtained from the classical survey and the fuzzy survey were different. Since the data in the fuzzy survey is obtained in a more sensitive and objective way, the result of the fuzzy survey will be a more consistent and acceptable result than the classical survey result.

## BACKGROUND

### 2.1 Social Justice Leadership

Although many studies have been conducted in different areas related to social justice by different researchers, it cannot be said that a common conclusion has been reached regarding the definition of the concept [12-13]. For example Gümüş et al. conceptualized social justice as  individuals' having equal access to resources and opportunities [14]. Furman defines social justice leadership as the recognition, resolution, and transformation of differences and unfair practices through egalitarian practices [6]. Arar and Oplatka describe social justice leadership as leadership that identifies disadvantaged elements in schools, rejects discriminatory practices, and promotes othered groups as inclusive practices and change [15]. Wasonga defines it as a leadership approach that participates in decision-making, builds relationships between different groups, and supports justice-based educational practices As with defining social justice, it is not possible to mention a general definition adopted in the field summer in defining social justice leadership[16]. Freire states that such a definition would be restrictive [17]. But it is possible to reach definitions that deal with social justice leadership from different dimensions Although different definitions of social justice leadership can be seen to be made, it can be said that the recognition of differences in general, the emphasis on equality and justice-based practices, the understanding that does not otherize, is inclusive, participatory and prioritizes change, is adopted.

In the classification of social justice leadership Furman classified it as personal, interpersonal, societal, systematic, and ecologically diverse [6].  Studies on the classification of social justice leadership are examined and similar and different perspectives are exhibited in the summer [6, 8, 17, 18]. In his work on the nature and conceptualization of social justice leadership, Bozkurt defined four dimensions: critical consciousness, stakeholder support, participation, and distributive justice in social justice leadership behaviors [8].

**Figure 1.** Social Justice Leadership Dimensions [8].

*Critical Consciousness:* Social justice leaders should be able to freely express the ideas and thoughts of those who follow in the context of critical consciousness, not exclude differences, avoid putting pressure on criticism openly. Bates states that the social structures of organizations lead to prejudices in their nature, and that a social critical structure should be developed [19]. Hay and Reedy state that disadvantaged groups in schools are othered and excluded [20].

*Participation:* In the context of participation, it is necessary to gather differences for a common purpose, see differences as wealth, and encourage the participation of all stakeholders in decisions related to organizational goals. Social justice leadership is also closely related to the concept of democratic education. According to Dantley and Tillman, social justice lives in a democratic environment because at the core of democracy is embracing multiple identities, different voices, different perspectives [21].

*Stakeholder Support:* The support dimension is associated with social justice leaders providing a fair environment for disadvantaged individuals, supporting individuals with disabilities, supporting teachers, and supporting inclusive practices. According to Furman and shields, as a social justice leader, school administrators must demonstrate leadership that encourages change in the formation of a supportive and inclusive school culture and climate in their management [22]. Social justice leaders should take an approach that combines position, sexual preferences, ideology, cultural differences, race and language differences among the school's stakeholders in a common denominator [23].

*Distributive Justice:* In the context of distributive justice, as a social justice leader, school administrators need to monitor the principles of fairness and equality of institutional resources and develop a school environment on this basis. According to Blackmore, removing disparities between different groups in education can contribute to building a fair society [24].

In the context of social justice leadership, it can be said that critical consciousness prevails in schools, inequality and injustices are minimized, differences are not othered and excluded, and school administrators have important tasks as social justice leaders to create a democratic, participatory, inclusive and supportive corporate culture. Due to the bureaucratic and decentralized nature of the regulations in Turkey, it makes it difficult for school administrators to demonstrate their social justice leadership skills. However, in different

studies, it is seen that managers take initiative and take responsibility for social justice leadership and try to demonstrate their leadership skills [25,8]. Therefore, it can be said that education politicians should work and support the development of social justice leadership skills of school administrators. In this context, the aim of the study is to demonstrate the levels of social justice leadership behaviors where school administrators are exhibited in teacher perceptions.

## 2.2 Fuzzy Sets

**Definition 1: [27]** Let $\mathcal{B}$ be the universal set. A fuzzy set $\mathcal{A}$ on $\mathcal{B}$ is defined by

$$\mathcal{A} = \{\langle a, \mu_{\mathcal{A}}(a)\rangle : a \in \mathcal{B}\}.$$

Here, $\mu_{\mathcal{A}}$ is membership function such that $\mu_{\mathcal{A}} : \mathcal{B} \to [0,1]$.

**Definition 2: [28]** A triangular fuzzy number $\tilde{n} = [k_1, \ l_1, \ m_1]$ is a special fuzzy set on the real number set $\mathbb{R}$, whose membership function is defined as follows

$$\mu_{\tilde{n}}(a) = \begin{cases} (a-k_1)/(l_1-k_1), & if \ (k_1 \leq a < l_1) \\ 1, & if \ (a = l_1) \\ (m_1-a)/(m_1-l_1), & if \ (l_1 < a \leq m_1) \\ 0, & if \ otherwise \end{cases}$$
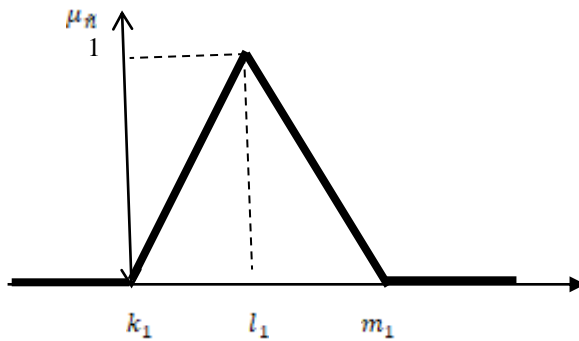


**Figure 2.** $\tilde{n} = [k_1, \ l_1, \ m_1]$ triangular fuzzy membership function

# CLASSICAL METHOD

## 3.1 Research Design

This study is designed with survey method in order to determine the social justice leadership behaviours of school administrators according to teachers' perceptions. The survey method is a research design to determine the opinions and thoughts of a specific group [26].

## 3.2 Sampling

The sample of the study consists of 114 teachers working in different public secondary schools. The teachers participating in the study were determined by the typical case sampling method, one of the purposeful sampling methods that are not random. Typical case sampling is created with individuals typical of many situations in the universe depending on the research problem [26].

## 3.3 Data Collection

Social Justice Leadership Scale developed by Bozkurt to determine social justice leadership behaviors of school admınıstrators was used [8]. In the process of developing the scale, the items of social justice leadership behavior were formed depending on detailed literature review. To determine the validity of the sale, the confirmatory and explanatory factor analysis were performed. As a result, 34 –item social justice leadership scale including critical consciousness (7 items), stakeholder support (11 items), participation (10 items) and distributive justice (6 items) was developed. In order to determine the internal consistency coefficient of the scale, Cronbach alpha reliability analysis was calculated and it was found for critical consciousness 94; for stakeholder support 95; for participation 59; for distributor justice 95 respectively. Five- Likert type rating strongly disagree (1), disagree (2), partly agree (3), agree (4), strongly agree (5) was used.

## 3.4 Data Analysis

The data was analyzed with SPSS 21.0 package program. In accordance with the purpose of the study, descriptive statistical analysis, including the mean and standard deviation value were calculated. Suitability, score and arithmetic mean levels used in the interpretation of the descriptive statistics of teachers' social justice leadership were given in Table 1.

**Table 1.** Score Ranges And Classifications Used in İnterpretation of Teachers' Social Justice Leadership Perceptions

| Suitability | Score | Limits (arithmetic mean) |
|---|---|---|
| Strongly Disagree | 1 | 1.00-1.79 |
| Disagree | 2 | 1.80-2.59 |
| Partly Agree | 3 | 2.60-3.39 |
| Agree | 4 | 3.40-4.19 |
| Strongly Agree | 5 | 4.20-5.00 |

## 3.5 Classical Findings

**Table 2.** *Mean of Social Justice Leadership Behaviours of administrators in five Schools According to Teacher's Perceptions*

|        | N  | Mean | Sd   |
|--------|----|------|------|
| $SA_1$ | 25 | 3,82 | 0,86 |
| $SA_2$ | 21 | 3,67 | 0,43 |
| $SA_3$ | 23 | 3,59 | 0,69 |
| $SA_4$ | 23 | 3,44 | 0,86 |
| $SA_5$ | 22 | 3,26 | 0,73 |

Table 2 when examined, social justice leadership levels are shown in five different schools according to teacher views according to the average scores of school administrators. Accordingly mean of $SA_5$ (X = 3,26) is at the school and partially agree. $SA_4$, $SA_2$, $SA_1$ and $SA_3$ schools are shown in the social justice leadership levels where administrators are exhibited (X = 3,40 - 4,19). However, as compared to the average scores, social justice leadership levels were compared to the lowest in order to the lowest, $SA_1$ (X = 3,82), $SA_2$ (X = 3,67), $SA_3$ (3,59), $SA_4$ (X = 3,44), $SA_5$ (X = 3.26) was observed. Where, SA is indicated that school administrator. For example, $SA_1$ is first school administrator.

# FUZZY METHOD

In this section, we will evaluate the data obtained with the fuzzy survey using the fuzzy matlab application and compare the social justice leadership of school administrators. The difference of this method from the method in Section 3 is that the item answers in the survey are requested as a value between 0 and 100 and the fuzzy matlab application is used in the evaluation and comparison part. In both methods, the sample, frequency, and the dimensions of the conceptual classification of social justice leadership are the same. Now let's give the fuzzy set and the triangular fuzzy number and the triangular fuzzy number membership functions that we will use in this section.

### 4.1 Fuzzy Matlab Application

In the fuzzy matlab application, the process is given at Figure 3.
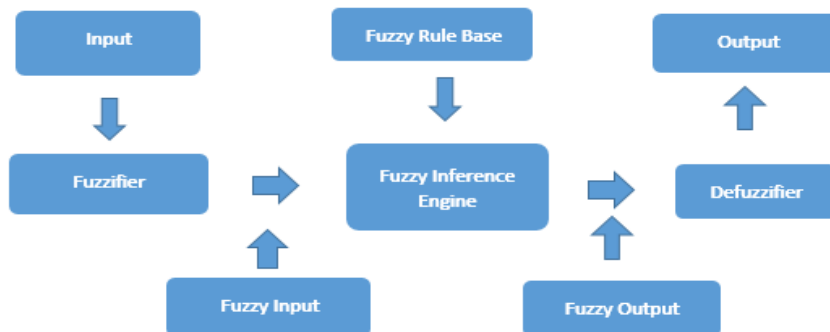
**Figure 3.** Fuzzy Matlab Algorithm

Now let's give the inputs for this fuzzy matlab application in Table 3.

**Table 3.** Inputs for this fuzzy matlab application

| Input | Abbreviation |
|---|---|
| Critical Consciousness | C.C |
| Support | S.U |
| Participation | P |
| Distributive Justice | D.J |

Now, let's give the triangular fuzzy membership functions of these inputs and the representation of these functions as triangular fuzzy numbers in Table 4.

**Table 4.** *Triangular Fuzzy Membership Functions of Inputs*

| Triangular Fuzzy Membership Functions | Abbreviation | Triangular Fuzzy Number |
|---|---|---|
| I do not agree | N.A | [0, 20, 45] |
| I am indecisive | I | [40, 45, 65] |
| I am agree | A | [60, 100, 100] |
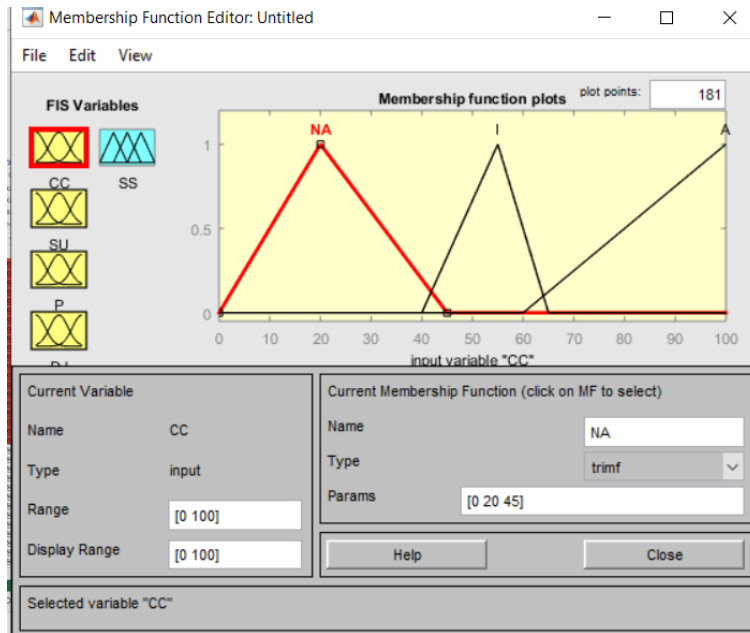
**Figure 4.** Triangular Fuzzy Membership Functions of Fuzzy Matlab Inputs

Now let's give the output for this fuzzy matlab application in Table 5.

**Table 5**. Outputs for this fuzzy matlab application

| Output | Abbreviation |
|---|---|
| Success Score | S.S |

Now let's give the triangular fuzzy membership functions of this output and the representation of these functions as triangular fuzzy numbers in Table 6.

**Table 6.** *Triangular Fuzzy Membership Functions of Outputs*

| Triangular Fuzzy Membership Functions | Abbreviation | Triangular Fuzzy Number |
|---|---|---|
| Very Little Successful | V.L.S | [0, 0, 20] |
| Little Successful | L.S | [15, 25, 45] |
| Medium Successful | M.S | [40, 55, 65] |
| Successful | S | [60, 70, 85] |
| Very Successful | V.S | [80, 100, 100] |
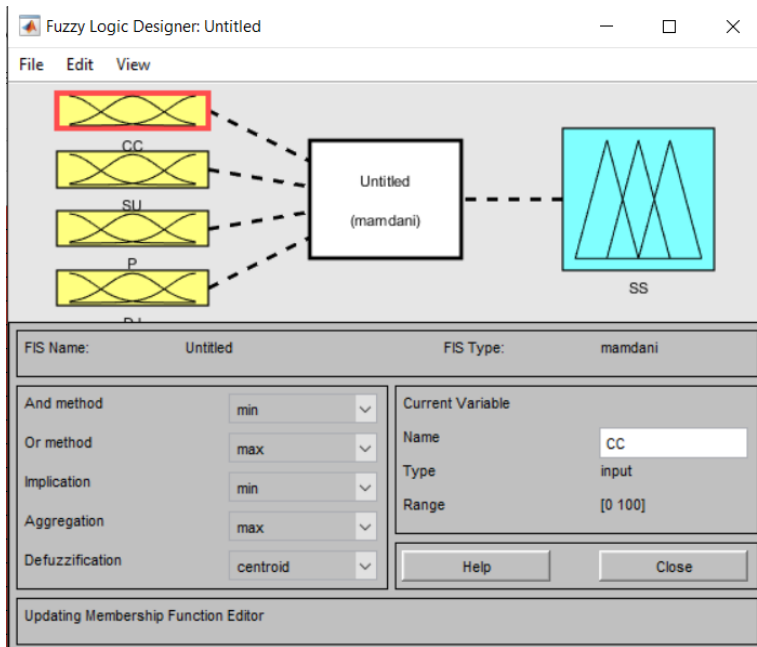
**Figure 5.** Representation of Inputs and Outputs in Fuzzy Matlab

Now let's give the fuzzy rules for the fuzzy inference engine in Table 7. In this study, Mandami fuzzy inference engine was used. The centroid method was used for defuzzifier.

**Table 7.** *Fuzzy Rules for the Fuzzy Inference Engine*

|     | 1. Input C.C | 2. Input S.U | 3. Input P | 4. Input D.J | Output S.S |
|-----|-----|-----|-----|-----|-----|
| 1   | N.A | N.A | N.A | N.A | V.L.S |
| 2   | N.A | N.A | N.A | I   | V.L.S |
| 3   | N.A | N.A | I   | I   | V.L.S |
| 4   | N.A | I   | I   | I   | V.L.S |
| 5   | I   | I   | I   | I   | L.S |
| 6   | I   | I   | I   | N.A | V.L.S |
| 7   | I   | I   | N.A | N.A | V.L.S |
| 8   | I   | N.A | N.A | N.A | V.L.S |
| 9   | N.A | I   | N.A | N.A | V.L.S |
| 10  | N.A | N.A | I   | N.A | V.L.S |
| 11  | N.A | I   | I   | N.A | V.L.S |
| 12  | N.A | I   | N.A | I   | V.L.S |
| 13  | I   | N.A | I   | I   | V.L.S |
| 14  | I   | I   | N.A | I   | V.L.S |
| 15  | I   | N.A | N.A | I   | V.L.S |
| 16  | I   | N.A | I   | N.A | V.L.S |
| 17  | A   | A   | A   | A   | S |
| 18  | A   | A   | A   | N.A | V.S |
| 19  | A   | A   | N.A | N.A | M.S |

| 20 | A | N.A | N.A | N.A | L.S |
|---|---|---|---|---|---|
| 21 | N.A | N.A | N.A | A | L.S |
| 22 | N.A | N.A | A | A | M.S |
| 23 | N.A | A | N.A | A | L.S |
| 24 | N.A | A | A | A | S |
| 25 | A | N.A | A | A | S |
| 26 | A | A | N.A | A | S |
| 27 | A | N.A | N.A | A | M.S |
| 28 | A | N.A | A | N.A | L.S |
| 29 | N.A | A | N.A | N.A | L.S |
| 30 | N.A | N.A | A | N.A | L.S |
| 31 | N.A | A | A | N.A | M.S |
| 32 | I | I | I | A | L.S |
| 33 | I | I | A | A | S |
| 34 | I | A | I | I | L.S |
| 35 | I | A | A | A | V.S |
| 36 | A | I | A | A | V.S |
| 37 | A | A | I | A | V.S |
| 38 | A | I | I | A | S |
| 39 | A | I | A | I | S |
| 40 | I | A | I | A | S |
| 41 | I | I | A | I | L.S |
| 42 | I | A | A | I | M.S |
| 43 | A | A | A | I | V.S |
| 44 | A | A | I | I | M.S |
| 45 | A | I | I | I | L.S |
| 46 | N.A | N.A | I | A | M.S |
| 47 | N.A | N.A | A | I | M.S |
| 48 | I | I | N.A | A | M.S |
| 49 | I | I | A | N.A | M.S |
| 50 | A | A | I | N.A | S |
| 51 | A | A | N.A | I | S |
| 52 | A | I | N.A | N.A | M.S |
| 53 | I | A | N.A | N.A | M.S |
| 54 | A | N.A | I | I | M.S |
| 55 | N.A | A | I | I | M.S |
| 56 | N.A | I | A | A | S |
| 57 | I | N.A | A | A | S |
| 58 | I | N.A | N.A | A | M.S |
| 59 | A | N.A | N.A | I | M.S |
| 60 | N.A | I | I | A | M.S |
| 61 | A | I | I | N.A | M.S |
| 62 | N.A | A | A | I | S |
| 63 | I | A | A | N.A | S |
| 64 | N.A | I | N.A | A | M.S |
| 65 | N.A | A | N.A | I | M.S |

| 66 | I | N.A | I | A | M.S |
|----|-----|-----|-----|-----|-----|
| 67 | I | A | I | N.A | M.S |
| 68 | A | I | A | N.A | S |
| 69 | A | N.A | A | I | S |
| 70 | N.A | I | A | I | M.S |
| 71 | A | I | N.A | I | M.S |
| 72 | N.A | A | I | A | S |
| 73 | I | A | N.A | A | S |
| 74 | I | N.A | A | N.A | M.S |
| 75 | A | N.A | I | N.A | M.S |
| 76 | I | N.A | A | I | M.S |
| 77 | I | A | N.A | I | M.S |
| 78 | N.A | I | A | N.A | M.S |
| 79 | N.A | A | I | N.A | M.S |
| 70 | A | I | N.A | A | S |
| 81 | A | N.A | I | A | S |
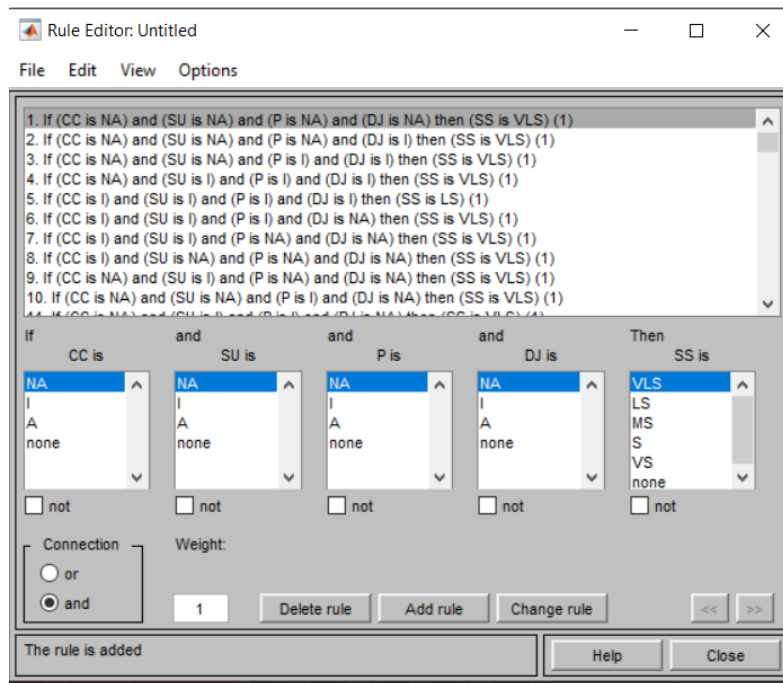


**Figure 6.** Representation of Fuzzy Rules in Fuzzy Matlab
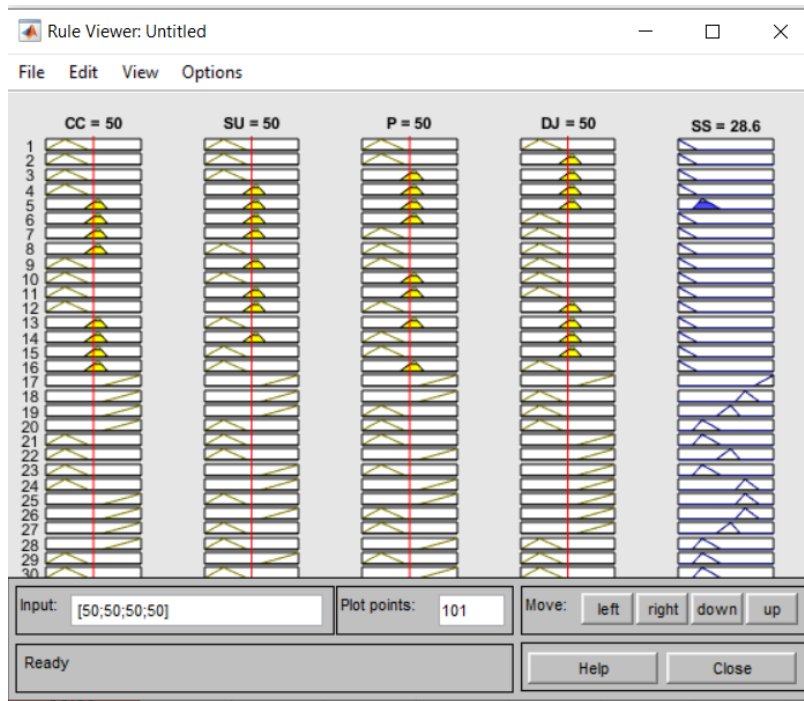
**Figure 7.** Getting Results with Fuzzy Matlab Rules

### 4.3 Fuzzy Findings

Now, let's give Table 8 which shows the success of fuzzy matlab, school administrators in the leadership of social justice by using the average of the dimensions from the data obtained from fuzzy survey.

**Table 8.** *Social Justice Leadership Average Scores Exhibited in Different Schools According to Teachers' Views*

|        | Mean (out of 5) | Mean (out of 100) |
|--------|-----------------|-------------------|
| $SA_1$ | 4,19            | 83,982            |
| $SA_2$ | 4,29            | 85,841            |
| $SA_3$ | 2,33            | 46,600            |
| $SA_4$ | 4,55            | 91,104            |
| $SA_5$ | 4,39            | 87,825            |

When Table 8 is examined, five different schools were evaluated according to the opinions of teachers in the fuzzy survey. In other words, the social justice leadership levels that these school administrators exhibited according to the results in the fuzzy matlab can be seen. Accordingly, the lowest average is in $SA_3$ (X = 2.33) school. The highest average was obtained in $SA_4$ (X = 4.55) school. When the social justice leadership levels are compared according to the average scores from the highest to the lowest $SA_4$ (X = 4.55), $SA_5$ (X = 4.39), $SA_2$ (X = 4.29), $SA_1$ (X = 4.19), , $SA_3$ (X = 2.33); respectively.

# CONCLUSIONS

If we compare the classical survey results obtained in Table 2 in Section 2.5 with the fuzzy survey results obtained in Table 8 in Section 4.3, we obtain Table 9.

**Table 9.** *Comparison of Classical Survey and Fuzzy Survey Results*

| Survey Type | Rank of School Administrators' Success |
|---|---|
| Clasical  Survey | $SA_1$, $SA_2$, $SA_3$, $SA_4$, $SA_5$ |
| Fuzzy Survey | $SA_4$, $SA_5$, $SA_2$, $SA_1$, $SA_3$ |

When Table 9 is examined, it is found that the results obtained from the classical survey and the fuzzy survey are different. Since the fuzzy survey and the fuzzy matlab provide a more comprehensive and more objective assessment, the result of the fuzzy survey is more consistent than the result of the classical survey. Because for the classical survey, one of the answers: strongly disagree, disagree, partly agree, agree, strongly agree. For the fuzzy survey, a value between 0 and 100 was requested. In other words, while the classical survey evaluates in 5 categories, the fuzzy survey evaluates in 100 categories. For example; while in the classical survey interval between 4.20 - 5.00 is accepted as absolutely agree, in the fuzzy survey and fuzzy matlab there is a separate membership value for each real number between 4.20 - 5.00. In other words, instead of adding some values to a category in the fuzzy survey, it accepts each value as a different category. For this reason, the result obtained from the fuzzy survey is more acceptable and more realistic than the classical survey result.

In this study, the average success level of each school administrator under the social justice leadership was obtained using the fuzzy matlab application by taking the average of the dimensions in the data obtained from the fuzzy survey. In addition, each school administrator can be compared with the fuzzy matlab application in separate dimensions. In the fuzzy matlab application, we used 3 different triangular fuzzy membership functions for each dimension, and 5 different triangular fuzzy membership functions for the output. As the number of these triangular fuzzy membership functions is increased, more precise results can be obtained. In addition, the triangular fuzzy membership function was used for inputs and outputs in the fuzzy matlab application. Researchers can also use other membership functions (trapezoidal fuzzy membership function, Gaussian fuzzy membership function, etc.) suitable for their problems. In addition, centroid rinsing function was used in fuzzy matlab with Mandami method. Researchers can use other rinse functions (bisector, mom, lom, etc.) or the Sugeno method to suit their problem.

| Survey Items | Please write a score between 0 and 100 for each item. |
|---|---|
| 1. | |
| 2. | |
| .<br><br>.<br><br>. | |

**Figure 8.** Example for a Fuzzy Survey

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# Chapter Ten

# A new decision-making method for architecture based on the Jaccard similarity measure of intuitionistic trapezoidal fuzzy multi-numbers

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## ABSTRACT

Recently, intuitionistic fuzzy multisets, applied to various areas, such as architecture, engineering, image segmentation, and decision-making, have prominent among similar concepts. ITFM-numbers are special multi-sets on a real number set based on intuitionistic fuzzy multisets. In the intuitionistic trapezoidal fuzzy multi-numbers (ITFM-numbers), the occurrences are more than one with the possibility of the same or the different membership function and non-membership functions. The intuitionistic fuzzy multi sets are one of the most important concepts to accommodate more uncertainties than the fuzzy sets, intuitionistic fuzzy sets and hence its applications are more extensive. The well-known vector similarity measures function satisfies the expectations of decision-makers over the parameters of the multi-time process. In this chapter, the authors develop a Jaccard similarity measure on the ITFM-numbers. For application of architecture, a comparative analysis is presented with a numerical example at the end of this chapter.

**Keywords:** Architecture, Intuitionistic Trapezoidal Fuzzy Multi-Numbers, Jaccard measure, Vector Similarity Measure, Decision-making.

## INTRODUCTION

Increasing material variety, construction systems, production methods, scientific and technological advances in architectural practice, especially in the intervention of problems arising in the design and implementation process, making the right decision based on personal ability, experience or some examined

examples is not always sufficient. It is considered that these problems should be handled with an analytical approach, the problem should be defined at a comprehensible level of relationship, and the process of determining the problem and deciding on the intervention should be carried out with systematic methods. Therefore, Multi-Criteria Decision-Making is based upon formation and designing decision and outlining problems composed of complex multi-pattern. The whole purpose is to give decision makers a feasible solution to such problems. Predictably, there does not exist an exclusive optimal answer for such matter and it is mandatory to utilize the choice maker's performance to evaluate and characterize between solutions. MCDM is a dynamic region of research since the 1960's. A different approach has been proposed by distinct scholars to solve the MCDM problems. The concept of fuzzy sets was first initiated by Zadeh [1] to manage uncertainty in real life. It has emerged that one component is insufficient to represent some special types of information. In this situation, a component namely non-membership value is invited to illustrate the information properly and in addition to this new component Atanassov [2] first defined the intuitionistic fuzzy set. Because of its ability to measure the fuzziness in a quite precise and comprehensive manner, intuitionistic fuzzy set theory has achieved a great deal. In some ambiguous circumstances, however, the sum of the grades of positive membership and negative membership can exceed 1, which is not suitable for intuitionistic fuzzy set. Yager [3] conducted the first study on the fuzzy multisets. They defined the concept of fuzzy multisets and basic operations including desired properties. Then, Shinoj and John [4] introduced intuitionistic fuzzy multisets based on fuzzy multisets and intuitionistic fuzzy sets. As a result, the multisets have been gradually drawn attention by the scholars [5-7]. Although the fuzzy multi-number and intuitionistic fuzzy multi-number are important tools to model problems involving uncertainty, these theories are inadequate to model some uncertainties. Therefore, many extended forms of the theories have been studied on fuzzy numbers [8-14], intuitionistic fuzzy numbers [15-18], fuzzy multi-numbers [19-24], and other fuzzy sets [25-28], but very few methods consider value of the uncertainty in the occurrences are more than one. The theories have studied in various areas such as [32-54].

This chapter intends to provide a multi-criteria decision making method based on the weighted Jaccard similarity measure with intuitionistic trapezoidal fuzzy multi-numbers, and the innovative contributions of this paper are mainly reflected in the following aspects:

(I)  To order to eliminate the situations of uncertainty in the occurrences are more than one a developed Jaccard similarity measure is described.

(II) Using the described Jaccard similarity measure for ITFM-numbers to present weighted Jaccard similarity measure. Furthermore, to illustrate its importance, the several fundamental relationships between the proposed weighted Jaccard similarity measures are examined.

(III) Presenting a new MCDM framework based on the developed weighted jaccard similarity measure to address ITFM-numbers.

Additionally we also show the application of these suggested similarity measures. Here in this chapter, Section 2 introduces some basic definitions and properties deal with Jaccard similarity measures. Section 3 proposed a Jaccard similarity measure for ITFMNs. Section 4, illustrates the effectiveness and applicability of a new Jaccard vector similarity measures based on ITFM-number. After that, a comparative analysis is presented with illustrative examples in Section 5. The conclusions and innovations are summarized in Section 6.

## BACKGROUND

**Definition 1** [1] The fuzzy sets defined on a non-empty $Y$ as objects having the form $F = \{\langle y, \varphi_F(y) \rangle : y \in Y\}$ where the functions $\varphi_F : Y \to [0,1],$ for $y \in Y.$

**Definition 2** [6] Let $Y$ be a non-empty set. A multi-fuzzy set $G$ over $Y$ is defined as $G = \left\{ \langle y, \varphi_G^1(y), \varphi_G^2(y), ..., \varphi_G^i(y), ... \rangle : y \in Y \right\}$ where $\varphi_G^i : Y \to [0,1]$ and $\mathrm{i} \in \{1, 2, ..., p\}.$

**Definition 3** [19] An ITFM number $A = \left\langle [a,b,c,d]; \left(\varphi_A^1, \varphi_A^2, ..., \varphi_A^p\right), \left(\sigma_A^1, \sigma_A^2, ..., \sigma_A^p\right) \right\rangle$ over $\mathbb{R}$ (The set of all ITFM-number on $\mathbb{R}$ will be denoted by $\Omega$.) is characterized by membership functions and non-membership functions are defined as, respectively:

$$
\mu_A^i(y) = \begin{cases}
(y-a)\varphi_A^i / (b-a), & a \leq y \leq b \\
\varphi_A^i, & b \leq y \leq c \\
(d-y)\varphi_A^i / (d-c), & c \leq y \leq d \\
0, & \text{otherwise}
\end{cases}
$$

$$
v_A^i(y) = \begin{cases}
\dfrac{(b-y)+\sigma_A^i(y-a)}{(b-a)}, & a \leq y \leq b \\
\sigma_A^i, & b \leq y \leq c \\
\dfrac{(y-c)+\sigma_A^i(d-y)}{(d-c)}, & c \leq y \leq d \\
1, & \text{otherwise.}
\end{cases}
$$

**Definition 4** [19] Let $A = \left\langle [a_1,b_1,c_1,d_1]; \left(\varphi_A^1, \varphi_A^2, ..., \varphi_A^p\right), \left(\sigma_A^1, \sigma_A^2, ..., \sigma_A^p\right) \right\rangle$,

$B = \left\langle [a_2,b_2,c_2,d_2]; \left(\varphi_B^1, \varphi_B^2, ..., \varphi_B^p\right), \left(\sigma_B^1, \sigma_B^2, ..., \sigma_B^p\right) \right\rangle \in \Omega$ and $\gamma \neq 0$ be any real number. Then,

1. $A+B = \left\langle [a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2]; \right.$
$\left. \left(s(\varphi_A^1, \varphi_B^1), s(\varphi_A^2, \varphi_B^2), ..., s(\varphi_A^P, \varphi_B^P)\right), \left(t(\sigma_A^1, \sigma_B^1), t(\sigma_A^2, \sigma_B^2), ..., t(\sigma_A^P, \sigma_B^P)\right) \right\rangle.$

2. $A-B = \left\langle [a_1-a_2,\ b_1-b_2,\ c_1-c_2,\ d_1-d_2]; \right.$
$\left. \left(s(\varphi_A^1, \varphi_B^1), s(\varphi_A^2, \varphi_B^2), ..., s(\varphi_A^P, \varphi_B^P)\right), \left(t(\sigma_A^1, \sigma_B^1), t(\sigma_A^2, \sigma_B^2), ..., t(\sigma_A^P, \sigma_B^P)\right) \right\rangle.$

3. $A.B = \begin{cases}
\left\langle [a_1a_2, b_1b_2, c_1c_2, d_1d_2]; \left(t(\varphi_A^1, \varphi_B^1), t(\varphi_A^2, \varphi_B^2), ..., t(\varphi_A^p, \varphi_B^p)\right), \left(s(\sigma_A^1, \sigma_B^1), s(\sigma_A^2, \sigma_B^2), ..., s(\sigma_A^P, \sigma_B^P)\right) \right\rangle, (d_1>0, d_2>0) \\[12pt]
\left\langle [a_1d_2, b_1c_2, c_1b_2, d_1a_2]; \left(t(\varphi_A^1, \varphi_B^1), t(\varphi_A^2, \varphi_B^2), ..., t(\varphi_A^p, \varphi_B^p)\right), \left(s(\sigma_A^1, \sigma_B^1), s(\sigma_A^2, \sigma_B^2), ..., s(\sigma_A^P, \sigma_B^P)\right) \right\rangle, (d_1<0, d_2>0) \\[12pt]
\left\langle [d_1d_2, c_1c_2, b_1b_2, a_1a_2]; \left(t(\varphi_A^1, \varphi_B^1), t(\varphi_A^2, \varphi_B^2), ..., t(\varphi_A^p, \varphi_B^p)\right), \left(s(\sigma_A^1, \sigma_B^1), s(\sigma_A^2, \sigma_B^2), ..., s(\sigma_A^P, \sigma_B^P)\right) \right\rangle, (d_1<0, d_2<0)
\end{cases}$

4.

$$A/B = \begin{cases} \left\langle \left[\dfrac{a_1}{d_2},\dfrac{b_1}{c_2},\dfrac{c_1}{b_2},\dfrac{d_1}{a_2}\right]; \left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle, (d_1>0,d_2>0) \\[1.5em] \left\langle \left[\dfrac{d_1}{d_2},\dfrac{c_1}{c_2},\dfrac{b_1}{b_2},\dfrac{a_1}{a_2}\right]; \left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle, (d_1<0,d_2>0) \\[1.5em] \left\langle \left[\dfrac{d_1}{a_2},\dfrac{c_1}{b_2},\dfrac{b_1}{c_2},\dfrac{a_1}{d_2}\right]; \left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle, (d_1<0,d_2<0) \end{cases}$$

5. $\gamma A = \left\langle [\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1]; \left(1-(1-\varphi_A^1)^\gamma, 1-(1-\varphi_A^2)^\gamma,...,1-(1-\varphi_A^p)^\gamma\right), \left((\sigma_A^1)^\gamma,(\sigma_A^2)^\gamma,...,(\sigma_A^P)^\gamma\right)\right\rangle (\gamma \geq 0).$

6.
$$A^\gamma = \left\langle [a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma]; \left((\varphi_A^1)^\gamma,(\varphi_A^2)^\gamma,...,(\varphi_A^P)^\gamma\right), \left((1-(1-\sigma_A^1)^\gamma, 1-(1-\sigma_A^2)^\gamma,...,1-(1-\sigma_A^p)^\gamma)\right)\right\rangle (\gamma \geq 0).$$

**Definition 5** [19] Let $A = \left\langle [a_1,b_1,c_1,d_1]; \left(\varphi_A^1,\varphi_A^2,...,\varphi_A^p\right), \left(\sigma_A^1,\sigma_A^2,...,\sigma_A^p\right)\right\rangle \in \Omega$. Then, the normalized ITFM-number of A is given by

$$\bar{A} = \left\langle \left[\frac{a_1}{a_1+b_1+c_1+d_1}, \frac{b_1}{a_1+b_1+c_1+d_1}, \frac{c_1}{a_1+b_1+c_1+d_1}, \frac{d_1}{a_1+b_1+c_1+d_1}\right]; \left(\varphi_A^1,\varphi_A^2,...,\varphi_A^p\right), \left(\sigma_A^1,\sigma_A^2,...,\sigma_A^p\right)\right\rangle.$$

**Definition 6** [15] Let $A = \langle (a_1, a_2, a_3, a_4); \mu_A, \nu_A\rangle$ and $B = \langle (b_1, b_2, b_3, b_4); \mu_B, \nu_B\rangle$ be two TIFN over $\mathbb{R}$. Then, JSM between normalized TIFN $A$ and $B$, denoted is defined as $(\bar{A}, \bar{B})$;

$$Jac(\bar{A}, \bar{B}) = \left(1 - \frac{\sum_{i=1}^4 |a_i - b_i|}{4}\right).$$

$$\frac{\mu_{\bar{A}}(x).\mu_{\bar{B}}(x) + \nu_{\bar{A}}(x).\nu_{\bar{B}}(x)}{\left(\mu_{\bar{A}}{}^2(x) + \nu_{\bar{A}}{}^2(x)\right) + \left(\mu_{\bar{B}}{}^2(x) + \nu_{\bar{B}}{}^2(x)\right) - \left(\mu_{\bar{A}}(x) \times \mu_{\bar{B}}(x) + \nu_{\bar{A}}(x) \times \nu_{\bar{B}}(x)\right)}$$

**Proposition 7** [15] Let $Jac(\bar{A}, \bar{B})$ be JSM between normalized TIFNs $\bar{A}$ and $\bar{B}$. Then, we have

i.    $0 \leq Jac(\bar{A}, \bar{B}) \leq 1$,

ii.   $Jac(\bar{B}, \bar{A}) = Jac(\bar{A}, \bar{B})$,

iii.  $Jac(\bar{A}, \bar{B}) = 1$ for  $\bar{A} = \bar{B}$  i.e., $\mu_{\bar{A}} = \mu_{\bar{B}}$ and $\nu_{\bar{A}} = \nu_{\bar{B}}$.

## 3. Jaccard Similarity Measure (JSM)

**Definition 8** Let $A = \left\langle (a_1, a_2, a_3, a_4); \left( \varphi_A^1, \varphi_A^2, ..., \varphi_A^p \right), \left( \sigma_A^1, \sigma_A^2, ..., \sigma_A^p \right) \right\rangle$,

$B = \left\langle (b_1, b_2, b_3, b_4); \left( \varphi_B^1, \varphi_B^2, ..., \varphi_B^p \right), \left( \sigma_B^1, \sigma_B^2, ..., \sigma_B^p \right) \right\rangle$ in $\Omega$. Then; JSM between $A$ and $B$ denoted

$Jac(A,B)$ is defined as follows;

$$Jac(A,B) = \left( 1 - \frac{\sum_{j=1}^{4} |a_j - b_j|}{4} \right) \cdot \frac{\sum_{i=1}^{p} (\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)}{\sum_{i=1}^{p} (\varphi_A^i)^2 + (\sigma_A^i)^2 + \sum_{i=1}^{p} (\varphi_B^i)^2 + (\sigma_B^i)^2 - \sum_{i=1}^{p} \left( (\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i) \right)}$$

**Note:** Let $A = \left\langle (a_1, a_2, a_3, a_4); \left( \varphi_A^1, \varphi_A^2, ..., \varphi_A^p \right), \left( \sigma_A^1, \sigma_A^2, ..., \sigma_A^p \right) \right\rangle \in \Omega$. $a_1 < a_2 < a_3 < $

and $a_1, a_2, a_3, a_4 \in \mathbb{R}$ if $a_2 = a_3$ then this intuitionistic trapezoidal fuzzy multi-number turns to intuitionistic triangular fuzzy multi-number.

**Proposition 9** $A = \left\langle (a_1, a_2, a_3, a_4); \left( \varphi_A^1, \varphi_A^2, ..., \varphi_A^p \right), \left( \sigma_A^1, \sigma_A^2, ..., \sigma_A^p \right) \right\rangle$,

$B = \left\langle (b_1, b_2, b_3, b_4); \left( \varphi_B^1, \varphi_B^2, ..., \varphi_B^p \right), \left( \sigma_B^1, \sigma_B^2, ..., \sigma_B^p \right) \right\rangle$ in $\Omega$. Then $Jac(A,B)$ satisfies the following properties;

i. $0 \le Jac(A,B) \le 1$

ii. $Jac(B,A) = Jac(A,B)$

iii. $Jac(A,B) = 1$ for $B = A$, i.e., $\varphi_A^i = \varphi_B^i$ and $\sigma_A^i = \sigma_B^i, (i = 1, 2..., p)$.

**Proof:**

i. Since $0 \le \frac{2a_1 a_2}{a_1^2 + a_2^2} \le 1$, for all $a_1, a_2 \in [0,1]$, it is clear from equation

$$0 \le 1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \le 1,$$

$$0 \le \frac{\sum_{i=1}^{p} (\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)}{\sum_{i=1}^{p} (\varphi_A^i)^2 + (\sigma_A^i)^2 + \sum_{i=1}^{p} (\varphi_B^i)^2 + (\sigma_B^i)^2 - \sum_{i=1}^{p} \left( (\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i) \right)} \le 1,$$

$$0 \le Jac(A,B) \le 1$$

**ii.**

$$Jac(A,B) = \left(1 - \frac{\sum_{j=1}^{4}|a_j - b_j|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)}{\sum_{i=1}^{p}(\varphi_A^i)^2 + (\sigma_A^i)^2 + \sum_{i=1}^{p}(\varphi_B^i)^2 + (\sigma_B^i)^2 - \sum_{i=1}^{p}\left((\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)\right)}$$

$$= \left(1 - \frac{\sum_{j=1}^{4}|b_j - a_j|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\varphi_B^i).(\varphi_A^i) + (\sigma_B^i).(\sigma_A^i)}{\sum_{i=1}^{p}(\varphi_B^i)^2 + (\sigma_B^i)^2 + \sum_{i=1}^{p}(\varphi_A^i)^2 + (\sigma_A^i)^2 - \sum_{i=1}^{p}\left((\varphi_B^i).(\varphi_A^i) + (\sigma_B^i).(\sigma_A^i)\right)}$$

$$= Jac(B,A)$$

**iii.**

$$Jac(A,B) = \left(1 - \frac{\sum_{j=1}^{4}|a_j - b_j|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)}{\sum_{i=1}^{p}(\varphi_A^i)^2 + (\sigma_A^i)^2 + \sum_{i=1}^{p}(\varphi_B^i)^2 + (\sigma_B^i)^2 - \sum_{i=1}^{p}\left((\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)\right)}$$

$$= \left(1 - \frac{|a_1-b_1|+|a_2-b_2|+|a_3-b_3|+|a_4-b_4|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)}{\sum_{i=1}^{p}(\varphi_A^i)^2 + (\sigma_A^i)^2 + \sum_{i=1}^{p}(\varphi_B^i)^2 + (\sigma_B^i)^2 - \sum_{i=1}^{p}\left((\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)\right)}$$

$$= \left(1 - \frac{|a_1-a_1|+|a_2-a_2|+|a_3-a_3|+|a_4-a_4|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\varphi_A^i).(\varphi_A^i) + (\varphi_A^i).(\varphi_A^i)}{\sum_{i=1}^{p}(\varphi_A^i)^2 + (\varphi_A^i)^2 + \sum_{i=1}^{p}(\varphi_A^i)^2 + (\varphi_A^i)^2 - \sum_{i=1}^{p}\left((\varphi_A^i).(\varphi_A^i) + (\varphi_A^i).(\varphi_A^i)\right)}$$

$$= 1$$

**Example 10** Assume that $A = \left\langle(0.1,0.2,0.3,0.5);(0.2,0.3,0.6,0.8),(0.8,0.7,0.4,0.2)\right\rangle$ and $B = \left\langle(0.3,0.5,0.6,0.9);(0.2,0.5,0.6,0.7),(0.8,0.5,0.4,0.3)\right\rangle$ in $\Omega$. Then

$$Jac(A,B) = \left(1 - \frac{\sum_{j=1}^{4}|a_j - b_j|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)}{\sum_{i=1}^{p}(\varphi_A^i)^2 + (\sigma_A^i)^2 + \sum_{i=1}^{p}(\varphi_B^i)^2 + (\sigma_B^i)^2 - \sum_{i=1}^{p}\left((\varphi_A^i).(\varphi_B^i) + (\sigma_A^i).(\sigma_B^i)\right)}$$

$$= \left[\left(1 - \frac{|0.1-0.3|+|0.2-0.5|+|0.3-0.6|+|0.5-0.9|}{4}\right) \cdot \frac{(0,2.0,2+0,3.0,5+0,6.0,6+0,8.0,7)+}{(0,8.0,8+0,7.0,5+0,4.0,4+0,2.0,3)}\right/$$

$$
\begin{aligned}
= \quad & \left(0,2^2+0,3^2+0,6^2+0,8^2\right)+\left(0,8^2+0,7^2+0,4^2+0,2^2\right)+\left(0,2^2+0,5^2+0,6^2+0,7^2\right) \\
& +\left(0,8^2+0,5^2+0,4^2+0,3^2\right)-(0,2.0,2+0,3.0,5+0,6.0,6+0,8.0,7)-(0,8.0,8+0,7.0,5+0,4.0,4+0,2.0,3)] \\
= \quad & (0,7).\left(\frac{2,32}{2,48}\right) \\
= \quad & 0,6548.
\end{aligned}
$$

**Definition 11** Let $A=\left\langle\left(a_1,a_2,a_3,a_4\right);\left(\varphi_A^1,\varphi_A^2,...,\varphi_A^p\right),\left(\sigma_A^1,\sigma_A^2,...,\sigma_A^p\right)\right\rangle$,

$B=\left\langle\left(b_1,b_2,b_3,b_4\right);\left(\varphi_B^1,\varphi_B^2,...,\varphi_B^p\right),\left(\sigma_B^1,\sigma_B^2,...,\sigma_B^p\right)\right\rangle$ in $\Omega$ and the unknown weight vector of criteria is denoted by $w=(w_1,w_2,...,w_n)^T$ with subject to $w_j \in [0,1]$ such that $\sum_{j=1}^{n} w_j =1$. Then; weighted JSM between ITFMN $\overline{A}$ and $\overline{B}$ denoted $Jac_w(\overline{A},\overline{B})$ is defined as;

$$
Jac_w(\overline{A},\overline{B})=\sum_{r=1}^{n}\left[\left(1-\frac{\sum_{j=1}^{4}\left|a_j-b_j\right|}{4}\right).\frac{w_r.\sum_{i=1}^{p}(\varphi_{\overline{A}}^i).(\varphi_{\overline{B}}^i)+(\sigma_{\overline{A}}^i).(\sigma_{\overline{B}}^i)}{\sum_{i=1}^{p}(\varphi_{\overline{A}}^i)^2+(\sigma_{\overline{A}}^i)^2+\sum_{i=1}^{p}(\varphi_{\overline{B}}^i)^2+(\sigma_{\overline{B}}^i)^2-\sum_{i=1}^{p}\left((\varphi_{\overline{A}}^i).(\varphi_{\overline{B}}^i)+(\sigma_{\overline{A}}^i).(\sigma_{\overline{B}}^i)\right)}\right]
$$

**Proposition 12** Let $A=\left\langle\left(a_1,a_2,a_3,a_4\right);\left(\varphi_A^1,\varphi_A^2,...,\varphi_A^p\right),\left(\sigma_A^1,\sigma_A^2,...,\sigma_A^p\right)\right\rangle$,

$B=\left\langle\left(b_1,b_2,b_3,b_4\right);\left(\varphi_B^1,\varphi_B^2,...,\varphi_B^p\right),\left(\sigma_B^1,\sigma_B^2,...,\sigma_B^p\right)\right\rangle$ in $\Omega$ and $w_r \in[0,1]$ be the weight of for all element for $r=(1,2,...,n)$ such that $\sum_{r=1}^{n} w_r =1.$ Then $Jac_w(\overline{A},\overline{B})$ satisfies the following properties;

i. $0\leq Jac_w(\overline{A},\overline{B})\leq1$,

ii. $Jac_w(\overline{B},\overline{A})=Jac_w(\overline{A},\overline{B})$,

iii. $Jac_w(\overline{A},\overline{B})=1$ *for* $A=B$ i.e. $\varphi_A^i=\varphi_B^i$ and $\sigma_A^i=\sigma_B^i,(i=1,2...,p).$

**Proof :**

It can be done in a similar way as proof of Proposition 9.

**Example 13** Let $A=\left\langle(0.1,0.2,0.3,0.5);(0.2,0.3,0.6,0.8),(0.8,0.7,0.4,0.2)\right\rangle$ and

$B=\left\langle(0.3,0.5,0.6,0.9);(0.2,0.5,0.6,0.7),(0.8,0.5,0.4,0.3)\right\rangle$ in $\Omega$ and $w_1=0.3,w_2=0.7$. Then;

$$Jac_w(\overline{A},\overline{B}) = \sum_{r=1}^{n}\left[\left(1-\frac{\sum_{j=1}^{4}|a_j-b_j|}{4}\right)\cdot\frac{w_r\cdot\sum_{i=1}^{p}(\varphi_{\overline{A}}^i).(\varphi_{\overline{B}}^i)+(\sigma_{\overline{A}}^i).(\sigma_{\overline{B}}^i)}{\sum_{i=1}^{p}(\varphi_{\overline{A}}^i)^2+(\sigma_{\overline{A}}^i)^2+\sum_{i=1}^{p}(\varphi_{\overline{B}}^i)^2+(\sigma_{\overline{B}}^i)^2-\sum_{i=1}^{p}\left((\varphi_{\overline{A}}^i).(\varphi_{\overline{B}}^i)+(\sigma_{\overline{A}}^i).(\sigma_{\overline{B}}^i)\right)}\right]$$

$$= \left[1-\frac{|0.1-0.3|+|0.2-0.5|+|0.3-0.6|+|0.5-0.9|}{4}\times\frac{0.3\left[(0,2.0,2+0,3.0,5+0,6.0,6+0,8.0,7)+(0,1.0,3+0,4.0,4+0,6.0,8+0,7.0,9)\right]}{\phantom{x}}\right.$$

$$= \frac{\left(0,2^2+0,3^2+0,6^2+0,8^2\right)+\left(0,8^2+0,7^2+0,4^2+0,2^2\right)+\left(0,2^2+0,5^2+0,6^2+0,7^2\right)}{+\left(0,8^2+0,5^2+0,4^2+0,3^2\right)-(0,2.0,2+0,3.0,5+0,6.0,6+0,8.0,7)-(0,8.0,8+0,7.0,5+0,6.0,6+0,8.0,7)]}$$

$$= \left[1-\frac{|0.1-0.3|+|0.2-0.5|+|0.3-0.6|+|0.5-0.9|}{4}\cdot\frac{0.7\left[(0,2.0,2+0,3.0,5+0,6.0,6+0,8.0,7)+(0,1.0,3+0,4.0,4+0,6.0,8+0,7.0,9)\right]}{\phantom{x}}\right.$$

$$= \frac{\left(0,2^2+0,3^2+0,6^2+0,8^2\right)+\left(0,8^2+0,7^2+0,4^2+0,2^2\right)+\left(0,2^2+0,5^2+0,6^2+0,7^2\right)}{+\left(0,8^2+0,5^2+0,4^2+0,3^2\right)-(0,2.0,2+0,3.0,5+0,6.0,6+0,8.0,7)-(0,8.0,8+0,7.0,5+0,6.0,6+0,8.0,7)]}$$

$$= 0,7.\frac{0,3.2,41}{4,99-2,41}+0,7.\frac{0,7.2,41}{4,99-2,41}$$

$$= 0,6548.$$

## 4. MCDM Method with ITFM-number

**Definition 14** A matrix $\left[b_{ij}\right]_{mxn}$, consider set of $\mathcal{U}$ alternatives $(u_1,u_2,...,u_m)$ and $\mathcal{C}$ criteria $(c_1,c_2,...,c_n)$. The unknown weight vector of $\mathcal{C}$ is denoted by $w=(w_1,w_2,...,w_n)^T$ with subject to $w_j\in[0,1]$ such that $\sum_{j=1}^{n}w_j=1$. List of the effects of each alternative on the attribute in decision matrix

$$\left[b_{ij}\right]_{mxn}=\left\langle\left[a_{ij},b_{ij},c_{ij},d_{ij}\right];\left(\varphi_{ij}^1,\varphi_{ij}^2,...,\varphi_{ij}^p\right),\left(\sigma_{ij}^1,\sigma_{ij}^2,...,\sigma_{ij}^p\right)\right\rangle$$

are summarize based on ITFM-number.

**Algorithm of the proposed method:**

**Step1.** Construction of the decision matrix with ITFM-numbers,

**Step2.** Calculate of the weighted JSM between each alternative $u_i$ and the positive ideal;

$$u_i=\left\langle\left[a_{ij},b_{ij},c_{ij},d_{ij}\right];\left(\varphi_{ij}^1,\varphi_{ij}^2,...,\varphi_{ij}^p\right),\left(\sigma_{ij}^1,\sigma_{ij}^2,...,\sigma_{ij}^p\right)\right\rangle$$

$$r^+=\left\langle[1,1,1,1],(1,1,...,1),(0,0,...,0)\right\rangle,$$

and $(i=1,2,...,m)$ as;

$$Jac_w(\overline{A},\overline{B}) = \sum_{r=1}^{n}\left[\left(1-\frac{\sum_{j=1}^{4}|a_j^k-b_j^k|}{4}\right)\cdot\frac{w_r\cdot\sum_{i=1}^{p}(\varphi_A^i).(\varphi_B^i)+(\sigma_A^i).(\sigma_B^i)}{\sum_{i=1}^{p}(\varphi_A^i)^2+(\sigma_A^i)^2+\sum_{i=1}^{p}(\varphi_B^i)^2+(\sigma_B^i)^2-\sum_{i=1}^{p}\left((\varphi_A^i).(\varphi_B^i)+(\sigma_A^i).(\sigma_B^i)\right)}\right]$$

**Step3.** Pick the best alternative depending on the maximum score of $Jac_{w_i}(u_i, r^+)$,

**Step4.** Determine the best alternative.

**Example 15** Architecture means the design of structures. It means designing and shaping structures in a way. It requires great imagination. Then it should be transferred to paper. At this stage, there may be some difficulties, and in terms of time and design, it will be difficult to put the design literally on paper. So it would be best to use computer-aided programs. Deniz architecture firm wants to choose the computer aided programs for drawing the entrance gate of AVM. Therefore, there are four computer programs indicated $u_i(i = 1,2,3,4)$ are available. For this computer aided programs have a criteria set $C = \{c_1 = \text{RAM}; c_2 = \text{SSD}\}$. Using the computer data, the proposed algorithm will select the best computer aided program for the Deniz architecture firm. In addition, criteria weight vector is computed using JSM as follows:

$$\omega = (\omega_1 = 0.3, \omega_2 = 0.4, \omega_3 = 0.2, \omega_4 = 0.1)^{\text{T}}.$$

**Step1.** Build a decision matrix;

$$c_1$$
$$\begin{array}{l}u_1 \\ u_2 \\ u_3 \\ u_4\end{array}\left(\begin{array}{l}\langle[0.3,0.5,0.7,0.9];(0.4,0.5,0.6),(0.6,0.5,0.4)\rangle \\ \langle[0.2,0.3,0.5,0.6];(0.2,0.3,0.8),(0.8,0.7,0.2)\rangle \\ \langle[0.1,0.4,0.5,0.6];(0.3,0.4,0.5),(0.7,0.6,0.5)\rangle \\ \langle[0.2,0.5,0.6,0.7];(0.3,0.4,0.6),(0.7,0.6,0.4)\rangle\end{array}\right.$$

$$c_2$$
$$\left.\begin{array}{l}\langle[0.6,0.7,0.8,0.9];(0.1,0.4,0.7),(0.9,0.6,0.3)\rangle \\ \langle[0.1,0.3,0.4,0.5];(0.2,0.4,0.7),(0.8,0.6,0.3)\rangle \\ \langle[0.3,0.4,0.5,0.6];(0.1,0.4,0.7),(0.9,0.6,0.3)\rangle \\ \langle[0.2,0.3,0.5,0.6];(0.1,0.2,0.5),(0.9,0.8,0.5)\rangle\end{array}\right)$$

**Step2.** Calculated for $r^+$

**Step3.**

**Table 1**: Calculated for $Jac_{w_i}(u_i, r^+)$

| The Proposed | Measure value | Ranking order |
| --- | --- | --- |

| method | | | |
|---|---|---|---|
| $Jac_{w_i}(u_i, r^+)$ | $Jac_{w_i}(u_1, r^+) = 0,1302$ | | |
| | $Jac_{w_i}(u_2, r^+) = 0,3341$ | | |
| | $Jac_{w_i}(u_3, r^+) = 0,0685$ | $u_2 > u_1 > u_4 > u_3$ | |
| | $Jac_{w_i}(u_4, r^+) = 0,1197$ | | |

**Step4.** So the best alternative $u_2$.

# 5. Comparison Analysis

In this section, numerical comparisons of the proposed Jaccard similarity measures with some existing ones developed by Ramli et al. [29], Mohomed et al. [30] and Ye [31] are given.

**Table 2**: Comparison with existing methods

| | Ranking methods | Ranking | Characteristic |
|---|---|---|---|
| Existing Method | Ramli et al. [29] | $u_4 > u_2 > u_3 > u_1$ | Fuzzy Number |
| | Mohomed et al.[30] | $u_2 > u_4 > u_3 > u_1$ | Generalized Fuzzy Number |
| | Ye [31] | $u_2 > u_4 > u_1 > u_3$ | Intuitionistic fuzzy set |
| Proposed Method | | $u_2 > u_1 > u_4 > u_3$ | ITFM-number |

# 6. Conclusions

In this chapter, we have suggested a new Jaccard similarity measures on ITFM-numbers and weighted Jaccard similarity measure based on the ITFM-numbers. Finally, a numerical example is introduced to illustrate the availability and practicability of the proposed method. The proposed Jaccard similarity measure is able to evaluate the similarity value of a similar ITFM-number while the existing methods cannot discriminate clearly the similarity value of the similar ITFM-numbers. The approach proposed in this chapter will be extended in future research to other ambiguous fields, such as neutrosophic sets, probabilistic linguistic term sets, hesitant fuzzy sets etc.

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# Chapter Eleven

# Prospector Neutro Function and Their Application

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## ABSTRACT

The migrant population has beliefs, values and health practices different from Aboriginal people. The purpose of this research is to investigate which are the access barriers to health that the international migrant population faces in primary health care. To accomplish this objective, a group of migrants was surveyed on an evaluation scale between $-5$ $to$ $+5$ in various aspects of access to health. A generalization of the well-known Prospector function was used as aggregator, where the aggregation between the two extreme values is undefined. We preferred to keep the indeterminacy to take into account totally contradictory opinions. This turns the generalization of the PROSPECTOR function into a Neutro-Function, and this problem into an application of Neutro-Algebra.

**Keywords:** Access barriers, access to health, international migrant, primary care, Prospector Neutro-Function, Neutro-Algebra, Neutro Associativity, Neutro-Commutativity**.**

## INTRODUCTION

Vagueness or uncertainty is a critical issue in the representation of incomplete knowledge in the fields of Computer Science and artificial intelligence. To deal with the uncertainty, the fuzzy set introduced by Zadeh [20] allows the uncertainty of a set with a membership degree between 0 and 1. Then, Atanassov [1] introduced an intuitionistic Fuzzy set ($IFS$) as a generalization of the Fuzzy set. The $IFS$ represents the uncertainty with respect to both membership and non-membership. However, it can only handle incomplete information but not the indeterminate and inconsistent information which exists commonly in real situations. Therefore, Smarandache [12] proposed a neutrosophic set. It can independently express truth-membership degree $T$, indeterminacy-membership degree $I$, and false membership degree $F$ and deal with incomplete, indeterminate, and inconsistent information. The indeterminate element $I$ is such that ordinary multiplication $I. I = I^2 = I$ , $I^{-1}$ the inverse of $I$ is not defined and hence does not exist. Moreover $I + I + \cdots I = nI : n \in N$. Also, several generalizations of the set theories made such as fuzzy multi-set theory [15, 16], intuitionistic fuzzy multi-set theory [10, 11] and refined neutrosophic set theory [3, 4, 6, 8, 13, 18, 27, 28, 39- 41]. Many research treating imprecision and uncertainty have been developed and studied. Since then, it is applied to various areas, such as decision-making problems [2, 5, 7, 9, 14, 17, 19, 26, 29] machine learning [30, 31], intelligent disease diagnosis [32, 33] communication services [34] pattern recognition [35] social network

analysis and e-learning systems [36] physics [37, 38], … etc. The theories have studied in various areas such as [44-82].

Smarandache [22] recently introduced new fields of research in neutrosophy called Neutro-Structures and Anti-Structures respectively. In,[23] Smarandache introduced the concepts of Neutro-Algebras and Anti-Algebras and in,[21] he revisited the concept of Neutro-Algebras and Anti-Algebras where he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that Neutro-Algebras are generalization of Partial Algebras. Agboola [21] introduced the concept of Neutro-Group. In continuation of this work the present research is devoted to the presentation of the concept of Neutro $R - module$ by considering Neutro-Axioms (Neutro-Abelian Group, Neutro-Ring, Neutro-Distributivity (multiplication over addition) and Neutro-Associative) Several interesting results and examples on Neutro $R - modules$, Neutro-Sub $R - modules$, and Neutro-Ring Homomorphisms are presented.

## BACKGROUND

In this section, we will give some definitions, examples and results that will be useful in other sections of the research.

### 2.1. Neutrosophic Sets [12]

Let $\mathcal{U}$ be a universe. $\mathcal{A}$ neutrosophic sets $\mathcal{A}$ over $\mathcal{U}$ is defined by

$$\mathcal{A} = \{< u, (T_{\mathcal{A}}(u), I_{\mathcal{A}}(u), F_{\mathcal{A}}(u)) >: u \in \mathcal{U}\}$$

where, $T_{\mathcal{A}}(u)$, $I_{\mathcal{A}}(u)$ and $F_{\mathcal{A}}(u)$ are called truth-membership function, indeterminacy-membership function and falsity- membership function, respectively. They are respectively defined by

$$T_{\mathcal{A}}: \mathcal{U} \rightarrow ]^-0, 1^+[\,, \qquad I_{\mathcal{A}}: \mathcal{U} \rightarrow ]^-0, 1^+[\,, \qquad F_{\mathcal{A}}: \mathcal{U} \rightarrow ]^-0, 1^+[$$

such that $0^- \leq T_{\mathcal{A}}(u) + I_{\mathcal{A}}(u) + F_{\mathcal{A}}(u) \leq 3^+$.

### 2.2. Single Valued Neutrosophic Set [18]

Let $\mathcal{U}$ be a universe. A single valued neutrosophic set (SVN-set) over $\mathcal{U}$ is a neutrosophic set over $\mathcal{U}$, but the truth-membership function $T$, indeterminacy-membership function $I$ and falsity- membership function $F$ are respectively defined by

$$T_{\mathcal{A}}: \mathcal{U} \rightarrow ]^-0, 1^+[\,, \qquad I_{\mathcal{A}}: \mathcal{U} \rightarrow ]^-0, 1^+[\,, \qquad F_{\mathcal{A}}: \mathcal{U} \rightarrow ]^-0, 1^+[$$

Such that $0 \leq T_{\mathcal{A}}(u) + I_{\mathcal{A}}(u) + F_{\mathcal{A}}(u) \leq 3$.

### 2.3. Neutro-Axiom, Anti-Axiom [21]

i-    A classical axiom defined on a nonempty set is an axiom that is totally true (i.e., true for all set's elements).

ii-    A Neutro-Axiom (or Neutrosophic Axiom) defined on a nonempty set is an axiom that is true for some set's elements

[degree of truth $(T)$], indeterminate for other set's elements.

[degree of indeterminacy $(I)$], or false for the other set's elements.

[degree of falsehood $(F)$], where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the Anti-Axiom.

iii-    An Anti-Axiom defined on a nonempty set is an axiom that is false for all set's elements. Therefore, we have the neutrosophic triplet:

$$< \ Axiom, Neutro - Axiom, Anti - Axiom \ >.$$

## 2.4. Neutro Associativity & Anti Associativity [21]

If $\langle A \rangle$= (classical) Associativity, then $\langle nonA \rangle$=(classical) Non-Associativity. We refine/split $\langle nonA \rangle$ into two parts, like this

$\langle neutA \rangle$=Neutro Associativity;

The Associativity's neutrosophic triplet is: $\langle \text{Associativity}, \text{NeutroAssociativity}, \text{Non} - \text{Associativity} \rangle$

## 2.5. Neutro-Commutativity & Anti-Commutativity [21]

If $\langle A \rangle$= (classical) Commutativity, then$\langle nonA \rangle$=(classical) Non-Commutativity. We refine/split$\langle nonA \rangle$ into two parts, like this

$\langle neutA \rangle$=Neutro-Commutativity;

$\langle antiA \rangle$=Anti-Commutativity.

Therefore, Non-Commutativity = Neutro-Commutativity ∪ AntiCommutativity.

The Commutativity's neutrosophic triplet is:

$$\langle Commutativity, NeutroCommutativity, AntiCommutativity \rangle$$

## 2.10. Neutro-Defined Binary Law [21]

there exist at least two elements (that could be equal) $a, b \in S$ such that $a * b \in S$ . And there exist at least other two elements (that could be equal too) $a, b \in S$ such that $a * b \notin s$.

## 2.11. Partial Function [21]

A function f: X → Y is called a Partial Function if it is well-defined for some elements in X, and undefined for all the other elements in X. Therefore, there exist some elements $a \in X$ such that $f(a) \in Y$ (well-defined), and for all other element $b \in X$ we have $f(b)$ is undefined.

## 2.12. Neutro-Function [21]

A function $f : X \rightarrow Y$ is called a Neutro-Function if it has elements in $X$ for which the function is well-defined {degree of truth $(T)$ }, elements in $X$ for which the function is indeterminate {degree of indeterminacy $(I)$}, and elements in $X$ for which the function is outer-defined {degree of falsehood $(F)$}, where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1,0,0)$ that represents the (Total) Function, and $(T, I, F) \neq (0,0,1)$ that represents the Anti-Function.

Classification of Functions

   i)      (Classical) Function, which is a function well-defined for all the elements in its domain of definition.
   ii)     Neutro-Function, which is a function partially well-defined, partially indeterminate, and partially outer-defined on its domain of definition.
   iii)    Anti-Function, which is a function outer-defined for all the elements in its domain of definition.

# Prospector Neutro-Function and Their Application

In this section, we define the Prospector Neutro-Function, the method we will use to achieved its applications.

### 3.1. Prospector Function [42-43]

is defined in the following way; it is a mapping from $[-1,1]^2$ *into* $[-1,1]$ with formula:

$$f(x,y) = \frac{x+y}{1+x \cdot y}$$

This function with $f(-1,1)$ and $f(1,-1)$ are undefined.

### 3.2. The Extended Prospector Function [77]

we extend $f(x,y)$ to $g(x,y)$ such that:

1- $g(x,y) = \begin{cases} f(x,y), & if\ (x,y) \in [1,-1]^2 \setminus \{(-1,1),(1,-1)\} \\ undefined, & if\ (x,y) = (-1,1) = (1,-1) \end{cases}$

2- $g(-1,1) = g(1,-1) = undefined,$
3- $g(undefined, undefined) = undefined.$

4- $g(undefined, x) = g(x, undefined) = \begin{cases} undefined, & if\ x > 0 \\ x, & if\ x \leq 0 \end{cases}$

### 3.3. Neutro- Prospector Binary Law [77]:

Let $A$ be a finite set defined as $A = \{(x,y): x,y \in \{k, undefined\}.$The **Binary Law** $\Theta$ is defined for every

1- If $g(x,y)$ is not undefined, then $x\Theta y = \frac{round(g(x,y)*10)}{10}$, where round is the function that output the integer nearest to the argument.
2- If $g(x,y)$ is undefined then $x\ \Theta y = undefined.$

Then $\Theta$ is a finite **Neutro Binary Law**. This is because $\Theta$ is commutative and associative for the

subset of elements of $A$ without any undefined component, but it is not associative otherwise.

$E.g., if\ a = -0.9, b = 0.8, c = undefined, then\ a\ \Theta\ (b\ \Theta\ c) = a\ and\ (a\ \Theta\ b)\ \Theta\ c = -0.4 \neq a,$
therefore, associativity is a **Neutro Binary Law**.

The following tables summarize the Cayley table of the **Neutro Binary Law** $\Theta$ which is not associative when we included the undefined value and it generates a **Neutro Binary Law** $\Theta$. We preferred to maintain the undefinition of the Prospector function because this indicates there is contradiction.

| $x\Theta y$ | -1 | -0.9 | -0.8 | -0.7 | -0.6 | -0.5 | -0.4 | -0.3 | -0.2 | -0.1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **-1** | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | undef |
| **-0.9** | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -0.9 | -0.9 | -0.9 | -0.9 |
| **-0.8** | -1 | -1 | -1 | -1 | -0.9 | -0.9 | -0.9 | -0.9 | -0.9 | -0.8 | -0.8 |
| **-0.7** | -1 | -1 | -1 | -0.9 | -0.9 | -0.9 | -0.9 | -0.8 | -0.8 | -0.7 | -0.7 |
| **-0.6** | -1 | -1 | -0.9 | -0.9 | -0.9 | -0.8 | -0.8 | -0.8 | -0.7 | -0.7 | -0.6 |
| **-0.5** | -1 | -1 | -0.9 | -0.9 | -0.8 | -0.8 | -0.8 | -0.7 | -0.6 | -0.6 | -0.5 |
| **-0.4** | -1 | -1 | -0.9 | -0.9 | -0.8 | -0.8 | -0.7 | -0.6 | -0.5 | -0.5 | -0.4 |
| **-0.3** | -1 | -0.9 | -0.9 | -0.8 | -0.8 | -0.7 | -0.6 | -0.6 | -0.4 | -0.4 | -0.3 |
| **-0.2** | -1 | -0.9 | -0.9 | -0.8 | -0.7 | -0.6 | -0.6 | -0.5 | -0.3 | -0.3 | -0.2 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **-0.1** | -1 | -0.9 | -0.8 | -0.7 | -0.7 | -0.6 | -0.5 | -0.4 | -0.2 | -0.2 | -0.1 |
| **undef** | -1 | -0.9 | -0.8 | -0.7 | -0.6 | -0.5 | -0.4 | -0.3 | -0.2 | -0.1 | 0 |
| **0** | -1 | -0.9 | -0.8 | -0.7 | -0.6 | -0.5 | -0.4 | -0.3 | -0.1 | -0.1 | 0 |
| **0.1** | -1 | -0.9 | -0.8 | -0.6 | -0.5 | -0.4 | -0.3 | -0.2 | 0 | 0 | 0.1 |
| **0.2** | -1 | -0.9 | -0.7 | -0.6 | -0.5 | -0.3 | -0.2 | -0.1 | 0.1 | 0.1 | 0.2 |
| **0.3** | -1 | -0.8 | -0.7 | -0.5 | -0.4 | -0.2 | -0.1 | 0 | 0.2 | 0.2 | 0.3 |
| **0.4** | -1 | -0.8 | -0.6 | -0.4 | -0.3 | -0.1 | 0 | 0.1 | 0.3 | 0.3 | 0.4 |
| **0.5** | -1 | -0.7 | -0.5 | -0.3 | -0.1 | 0 | 0.1 | 0.2 | 0.4 | 0.4 | 0.5 |
| **0.6** | -1 | -0.7 | -0.4 | -0.2 | 0 | 0.1 | 0.3 | 0.4 | 0.5 | 0.5 | 0.6 |
| **0.7** | -1 | -0.5 | -0.2 | 0 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.6 | 0.7 |
| **0.8** | -1 | -0.4 | 0 | 0.2 | 0.4 | 0.5 | 0.6 | 0.7 | 0.7 | 0.8 | 0.8 |
| **0.9** | -1 | 0 | 0.4 | 0.5 | 0.7 | 0.7 | 0.8 | 0.8 | 0.9 | 0.9 | 0.9 |
| **1** | undef | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 1

| $x\Theta y$ | **undef** | **0.1** | **0.2** | **0.3** | **0.4** | **0.5** | **0.6** | **0.7** | **0.8** | **0.9** | **1** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **-1** | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | undef |
| **-0.9** | -0.9 | -0.9 | -0.9 | -0.8 | -0.8 | -0.7 | -0.7 | -0.5 | -0.4 | 0 | 1 |
| **-0.8** | -0.8 | -0.8 | -0.7 | -0.7 | -0.6 | -0.5 | -0.4 | -0.2 | 0 | 0.4 | 1 |
| **-0.7** | -0.7 | -0.6 | -0.6 | -0.5 | -0.4 | -0.3 | -0.2 | 0 | 0.2 | 0.5 | 1 |
| **-0.6** | -0.6 | -0.5 | -0.5 | -0.4 | -0.3 | -0.1 | 0 | 0.2 | 0.4 | 0.7 | 1 |
| **-0.5** | -0.5 | -0.4 | -0.3 | -0.2 | -0.1 | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 1 |
| **-0.4** | -0.4 | -0.3 | -0.2 | -0.1 | 0 | 0.1 | 0.3 | 0.4 | 0.6 | 0.8 | 1 |
| **-0.3** | -0.3 | -0.2 | -0.1 | 0 | 0.1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.8 | 1 |
| **-0.2** | -0.2 | -0.1 | 0 | 0.1 | 0.2 | 0.3 | 0.5 | 0.6 | 0.7 | 0.9 | 1 |
| **-0.1** | -0.1 | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.8 | 0.9 | 1 |
| **undef** | undef | undef | undef | undef | undef | undef | undef | undef | undef | undef | undef |
| **0** | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| **0.1** | undef | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.7 | 0.8 | 0.9 | 1 |
| **0.2** | undef | 0.3 | 0.4 | 0.5 | 0.6 | 0.6 | 0.7 | 0.8 | 0.9 | 0.9 | 1 |
| **0.3** | undef | 0.4 | 0.5 | 0.6 | 0.6 | 0.7 | 0.8 | 0.8 | 0.9 | 0.9 | 1 |
| **0.4** | undef | 0.5 | 0.6 | 0.6 | 0.7 | 0.8 | 0.8 | 0.9 | 0.9 | 1 | 1 |
| **0.5** | undef | 0.6 | 0.6 | 0.7 | 0.8 | 0.8 | 0.8 | 0.9 | 0.9 | 1 | 1 |
| **0.6** | undef | 0.7 | 0.7 | 0.8 | 0.8 | 0.8 | 0.9 | 0.9 | 0.9 | 1 | 1 |
| **0.7** | undef | 0.7 | 0.8 | 0.8 | 0.9 | 0.9 | 0.9 | 0.9 | 1 | 1 | 1 |
| **0.8** | undef | 0.8 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 1 | 1 | 1 | 1 |
| **0.9** | undef | 0.9 | 0.9 | 0.9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **1** | undef | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 2

A group of Syrian migrants was surveyed on an evaluation scale between -5 to 5 in various aspects of access to health.

The World Health Organization (WHO) states that a health system brings together all the institutions and organizations whose primary objective is to maintain and improve the health of the population. Most health systems are made up of different sectors, public, private, traditional and informal, and must provide good treatments and services that respond to the needs of the population and are fair from a financial point of view.

Access to health services is the ability to get care when it is needed. This can be determined by various factors and variables such as the location of health centers and the availability of medical or health providers (geographical or physical barriers), up to health insurance and health care costs, also can be influenced by cultural barriers or language.

This research aims to evaluating the access barriers to health that the international migrant population faces in primary health care in Turkey. To achieve this objective, a group of 20 Syrian migrants of different sexes are surveyed. Respondents evaluated different relevant aspects in health care on a numerical scale with a maximum of 5 for approval and a minimum of $-5$ for disapproval.

Variables that have been used are the following:

1. Location access barriers

2. language access barriers

3. Financial access barriers

4. Legal access barrier.

The assessments provided by the interviewed on the four barriers were as follows:

| Assessments | Location access barriers | language access barriers | Financial access barriers | Legal access barrier |
|---|---|---|---|---|
| $v_{ij} = -5, \bar{v}_{ij} = -1$ | 0 | 1 | 0 | 0 |
| $v_{ij} = -4, \bar{v}_{ij} = -0.8$ | 0 | 4 | 0 | 0 |
| $v_{ij} = -3, \bar{v}_{ij} = -0.6$ | 0 | 2 | 0 | 2 |
| $v_{ij} = -2, \bar{v}_{ij} = -0.4$ | 0 | 6 | 0 | 7 |
| $v_{ij} = -1, \bar{v}_{ij} = -0.2$ | 0 | 3 | 0 | 5 |
| $v_{ij} = 0, \bar{v}_{ij} = 0$ | 0 | 0 | 1 | 1 |

| $v_{ij} = 1, \bar{v}_{ij} = 0.2$ | 0 | 2 | 2 | 2 |
|---|---|---|---|---|
| $v_{ij} = 2, \bar{v}_{ij} = 0.4$ | 3 | 2 | 5 | 2 |
| $v_{ij} = 3, \bar{v}_{ij} = 0.6$ | 6 | 0 | 8 | 0 |
| $v_{ij} = 4, \bar{v}_{ij} = 0.8$ | 10 | 0 | 3 | 0 |
| $v_{ij} = 5, \bar{v}_{ij} = 1$ | 1 | 0 | 1 | 1 |

Table 3

1- The value obtained in the evaluation of each aspect for each migrant is rescaled to the interval $[-1,1]$, dividing by 5. That is, $\bar{v}_{ij} = \frac{v_{ij}}{5}$, we denote by $v_{ij}, (i = 1,2, \dots, 20; \quad j = 1,2,3,4)$ the evaluation of the ith migrant on the jth aspect.

2- It is decided on two different situations:

1) If less than 33.333% of the respondents show contradictory results for each fixed *j*, that is, if there are 4 pairs or less of values $(-1,1)$ or $(1,-1)$, these values are eliminated for aggregating.

2) Otherwise, the *jth* aspect is evaluated as "undefined" and it should be reviewed in more detail because there is such a contradiction.

3- When we have the case (1) the aggregation of the remaining values is calculated by using $\Theta$. The results obtained from applying this method were as follows:

Aggregating the data of Table 3 using $\Theta$ we have the following results based on Tables 1 and 2:

1- $\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \dots \Theta \bar{v}_{201} = 1$ which means there is sufficient evidence that "Location access barriers" is good.

2- $\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \dots \Theta \bar{v}_{201} = -1$ which means there is sufficient evidence that "language access barriers" is bad.

3- $\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \dots \Theta \bar{v}_{201} = 1$ which means there is sufficient evidence that "Financial access barriers" is good.

4- $\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \dots \Theta \bar{v}_{201} =$ undefined which means there is no sufficient evidence that means "Legal access barrier" it should be reviewed in more detail why there is such a contradiction.

## Conclusions

This paper is dedicated to evaluate the barriers to Syrian migrants' access in primary health care in Turkey. We evaluated four types of barriers, which are "location access barriers", "language access barriers", "Financial access barriers", and "Legal access barrier". Twenty Syrian migrants of a Family Health Center

provided their opinions in a scale from -5 to 5. This is not a statistical study; thus, we defined an operator based on the Prospector function for determining if there is sufficient evidence to evaluate every aspect. We concluded that "location access barriers" and "Financial access barriers" are good, whereas "language access barriers" is bad and "Legal access barrier" is undefined it should be reviewed in more detail because there is such a contradiction. We provided the Cayley table of $\theta$, which is not associative when we included the undefined value and it generates a **Neutro Binary Law**.

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# Chapter Twelve

# Decision-Making Problem With Neutrosophic Values

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## ABSTRACT

In this study, we will expand the data to cover indeterminacy situations in addition to classic decision-making conditions. In this way, we will move classic decision-making events to neutrosopic decision-making events. In our study, we will evaluate with the decision-making tree to make the most accurate investment in 3 different regions. In this way, we will solve the decision-making problem in making the best investment decision. One of the important limitations affecting decision-making is the lack of information. It will use the decision-making tree, one of the most powerful methods used in mathematics, to minimize the lack of knowledge and analyze many decision-making problems. Neutrosophic logic will be separated from classical logic by adding only ambiguous data to classical logic. As a result, many of us may have difficulty making decisions to invest in any region. In this study, we obtain the data necessary for a person to make the right decision from investors living in the region and help them go to the most useful region. as a result, the risk situation will be minimized because when we approach decision-making problems with neutrosophic logic, we see that we get more general and real data than classical logic. So we'll make the best decision with the neutrosophic decision-making tree.

**Keywords:** Neutrosophic logic, classical logic, decision tree model

## INTRODUCTION

We see 3 different types of logic that we can encounter in our lives. First, what we call classical logic is a type of logic that gives values the form "true or false, 0 or 1". The second is fuzzy logic, first developed by Zadeh in 1960 [2]. Unlike classical logic, it recognizes more than right and wrong values. In other words, with fuzzy logic, propositions can be represented by degrees of accuracy and inaccuracy. And third, Neutrosophic contemporary research has included indeterminacy in fuzzy logic and neutrosophic logic has been formed. Neutrosophic science, Neutrosophic logic / Set / measure / Integral / probability, etc.means development and applications. We can define neutrosophic measure, integral and different aspects of the possibility. The reason for this is that a problem will be solved. The concept of uncertainty is completely different from randomly possible situation. Uncertainty; The physical field can be explained as the effect of many factors such as materials and structure type and either the effect of unexpected conditions. Florentin Smarandanche introduced the concept of neutrosophy which is a new philosophy branch by adding this concept in 1995. After he introduced the concept of neutrosophic logic and neutrosophic set where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the

percentage of indeterminacy in subset I and the percentage of falsity in a subset F where T, I, F are a subset of $]0^-, 1^+[$ [3] so that this neutrosophic logic is called an extension of fuzzy logic especially to intuitionistic fuzzy logic. There are many indeterminancies in the world. Classical math logic is usually insufficient to explain indeterminancies. In this way, we may not always be able to say that an event in our lives is just right or wrong. For example, when we ask people about the weather, we may not get clear answers from everyone. some can say hot, some can say cold, some can say cool. Therefore, Smarandache obtained the neutrosophic logic and neutrosophic set to deal with indeterminancies more objectively in 1998 [1]. 'T' is the membership degree, 'I' is the indeterminacy degree and 'F' is the non-membership degree in the neutrosophic logic and neutrosophic sets. "T, I, F" are defined independently. In addition, a neutrosophic number has the form (T, I, F). Furthermore, neutrosophic logic is a generalization of fuzzy logic [2] and intuitionistic fuzzy logic [3] since fuzzy and intuitionistic fuzzy logic's membership, non–membership degrees are defined dependently. Smarandache, F. searched Introduction to Neutrosophic Sociology (Neutrosociology) in [4]. Many researchers have studied the concept neutrosophic theory, its application to issue multi-criteria decision analysis and decision making in [5-25]. Recently many researchers [26-32] introduced several similarity measures, single-valued neutrosophic sets, neutrosophic numbers, neutrosophic geometric programming, neutrosophic multi-sets, neutrosophic soft sets in data analysis, neutrosophic graphs and pattern recognition. Finally Broumi, S., & Smarandache, F. searched Several similarity measures of neutrosophic sets in [33]. The theories have studied in various areas such as [34-64].

In this research, which is an extension of classical decision-making process and the fact that decision-making is ignored by the classic rationale underpinning the problems of indeterminancy conditions to include expanding the data, we will provide neutrosophic decision-making process. Lack of information is a major constraint affecting the effectiveness of the decision-making process. Restrictions will use decision tree model, which is one of the most powerful mathematical methods used to analyze the many decisions to minimize problems. Neutrosophic logic, indeterminate adding data classic logic will be separated from classical logic. This extended model, neutrosophic decide between the alternatives for reaching the best decisions based on the data available for the more general and true than conventional tree will be used alternatively.

On the other hand, good by some experts, decided to get worse as we can see whether or not the same idea of some of the expected value of the benefit. Therefore, the confrontation with a problem affecting the quality of the decisions taken absolutely for the best solution (k) (0) between the need to add value and reduce range. (0) represents the minimum value of this range, or decision makers among experts means that any disagreement concerning the expected values. (K) represents the maximum value between expected values and either the state of indeterminacy between experts and decision makers. (k)means that the estimated maximum value is.

Therefore, all the different ideas about the expected value (0, k) by note will take place in the intervals (0, k) to add to or reduce the range and will discuss the expected value of the benefit. In this way, it will present us with the most useful expected values, which include all views.

## BACKGROUND

In this section, we present the basic definitions that are important for the development of the paper.

**Definition 1 (Neutrosophic Set): [5]** Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A$, a indeterminacy-membership function $I_A$ and a falsity-membership function $F_A$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$. That is

$$T_A : X \rightarrow ]0^-, 1^+[$$

$$I_A : X \rightarrow ]0^-, 1^+[$$

$$F_A : X \rightarrow ]0^-, 1^+[$$

For all x∈ X.

**Definition 2: [5]** The complement of a neutrosophic set И is denoted by R̂ and is defined by

$$T_{\hat{R}}(x) = \{1 +\} \ominus T_{И}(x),$$

$$I_{\hat{R}}(x) = \{1 +\} \ominus I_{И}(x),$$

$$,F_{\hat{R}}(x) = \{1 +\} \ominus F_{И}(x),$$

for all x∈ X.

**Definition 3: [5]** A neutrosophic set И is contained in the other neutrosophic set R̂, И ⊆ R̂, if and only if

$$inf\ T_{И}(x) \leq inf\ T_{\hat{R}}(x) ,\ sup\ T_{И}(x) \leq sup\ T_{\hat{R}}(x)$$

$$inf\ I_{И}(x) \geq inf\ I_{\hat{R}}(x) ,\ sup\ I_{И}(x) \geq sup\ I_{\hat{R}}(x)$$

$$inf\ F_{И}(x) \geq inf\ F_{\hat{R}}(x) ,\ supF_{И}(x) \geq supF_{\hat{R}}(x)$$

for all x ∈ X.

**Definition 4: [24]** A logic in which each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F, where T, I, F are defined above, is called Neutrosophic Logic.

**Definition 5:[33]** Let X be the universal set and R be the set of attributes. We consider the non-empty set A ⊆ R. Let R̂(X) denotes the assembling of all fuzzy neutrosophic sets of X. The aggregation $\Omega_A$ is called the fuzzy neutrosophic soft set (fns-set) over X, where $\Omega_A$ is a mathematical function given by $\Omega_A : A \to \hat{R}(X)$. We can write it as

$\Omega_A = \{(\delta\{\langle \phi, T_{A(\delta)}(\phi), I_{A(\delta)}(\phi),\ F_{A(\delta)}(\phi) \rangle : \phi \in X\}) : \delta \in A \}.$

**Definition 6:[33]** We have a matrix, where rows represent the person names $p_1$, $p_2$, $p_3$, ..., $p_n$ and columns represent the parameters $q_1$, $q_2$, $q_3$,..., $q_m$. The entries $\varepsilon_{\alpha\beta}$ are designed by $\varepsilon_{\alpha\beta} = a+b–c$, where a is the number premeditated as how many times $T_{p\alpha}(q_\beta)$ exceeds or equals $T_{p\gamma}(q_\beta)$ for $p_\alpha \neq p_\gamma\ \forall p_\gamma \in X$, b is the number premeditated as how many times $I_{p\alpha}(q_\beta)$ exceeds or equals $I_{p\gamma}(q_\beta)$ for $p_\alpha \neq p_\gamma\ \forall p_\gamma \in X$ and c is the number premeditated as how many times $F_{p\alpha}(q_\beta)$ exceeds or equals $F_{p\gamma}(q_\beta)$ for $p_\alpha \neq p_\gamma\ \forall p_\gamma \in X$.

**Definition7: [33]** Let A be a hesitant fuzzy set on a universe of discourse $X=\{x_1,x_2,\ldots,x_n\}X=\{x_1,x_2,\ldots,x_n\}$ denoted by $A=\{x_i,h_A(x_i):x_i \in X\}.A=\{x_i,h_A(x_i):x_i \in X\}$. Then, the informational energy of A is defined as

$$E_{HFS}(A) = \sum_{i=1}^{n}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} \zeta_{A\sigma(j)}^2(x_i)\right)$$

where $li=l(h_A(x_i))l_i=l(h_A(x_i))$ is the number of values in $h_A(x_i)$ and $\zeta_{A\sigma}(x_i)$ the jth value in $h_A(x_i)$, $x_i \in X$.

**Definition 8:[33]** Let И be a universal space of points (objects), with a generic element of И denoted by n. A single-valued neutrosophic set (SVNS) $\mathcal{M} \subset$ И is characterized by a truth-membership function $T_{\mathcal{M}}(n)$, an indeterminacy-membership function $I_{\mathcal{M}}(n)$, and a falsity-membership function $F_{\mathcal{M}}(n)$ with $T_{\mathcal{M}}(n)$, $I_{\mathcal{M}}(n)$, $F_{\mathcal{M}}(n) \in [0,1]\ \forall n \in$ И.

The sum of three membership functions of a *SVNS* $\mathcal{M}$, the relation

$$0 \leq T_{\mathcal{M}}(n)+I_{\mathcal{M}}(n)+F_{\mathcal{M}}(n) \leq 3\ \ \forall n \in И$$

holds good. When И is continuous, a SVNS $\mathcal{M}$ can be written as

$$\mathcal{M} = \int_n \langle\, T_{\mathcal{M}}(n),\, I_{\mathcal{M}}(n),\, F_{\mathcal{M}}(n)\rangle | n,\ \forall n \in \mathbb{N}.$$

When $\mathbb{N}$ is discrete, a SVNS $\mathcal{M}$ can be written as

$$\mathcal{M} = \sum_n \langle\, T_{\mathcal{M}}(n),\, I_{\mathcal{M}}(n),\, F_{\mathcal{M}}(n)\rangle | n,\ \forall n \in \mathbb{N}.$$

SVNS can be represented with the notation ,

$$\mathcal{M} = \{(n|\langle\, T_{\mathcal{M}}(n),\, I_{\mathcal{M}}(n),\, F_{\mathcal{M}}(n)\rangle)|n,\ n \in \mathbb{N}\ \}$$

Thus, finite SVNS can be presented by the ordered tetrads:

$$\mathcal{M} = \{(n_1|\langle T_{\mathcal{M}}(n_1), I_{\mathcal{M}}(n_1), F_{\mathcal{M}}(n_1)\rangle), \ldots, (n_n|\langle T_{\mathcal{M}}(n_n), I_{\mathcal{M}}(n_n), F_{\mathcal{M}}(n_n)\rangle)\}$$

for $\forall n_i \in \mathbb{N}$  (i=1,2,…,k).For convenience, a SVNS

$\mathcal{M} = \{(n|\langle\, T_{\mathcal{M}}(n),\, I_{\mathcal{M}}(n),\, F_{\mathcal{M}}(n)\rangle)|n,\ n \in \mathbb{N}\ \}$ is denoted by the simplified symbol $\mathcal{M} = \langle\, T_{\mathcal{M}}(n),\, I_{\mathcal{M}}(n),\, F_{\mathcal{M}}(n)\rangle$ for $\forall n \in \mathbb{N}$.

# PROBLEM

S1, S2 and S3 represent the regions to be selected and we can make an assessment with high and low turnout in the table below:

|  | High participation | Low participation |
| --- | --- | --- |
| S 1 | $A \pm k_1$ | $B \pm k_2$ |
| S 2 | $C \pm k_3$ | $D \pm k_4$ |
| S 3 | $E \pm k_5$ | $F \pm k_6$ |

*A, B, C, D, E,* F represents the characterizing part of the expected value. $k_1$ $k_2$ $k_3$ $k_4$ $k_5$ $k_6$ represents the indeterminate portion of the expected fragment.

Therefore, by examining the optimistic and pessimistic viewpoints on the neutrosophic analyzing decision trees and upbeat classical decision tree to select the best alternative to stay and will consider the possibility of pessimistic thinking. To clarify this, the decision-makers to invest in their holiday resort we talk about a sample faced with three options. These elections (S1) springs in the region (S2) and in the maritime region (S3) cultural district dr. In every election, two normal state based on the following data (high gain) and (Low Gain) have, benefits will vary depending on two variables.

Experts, in the case of a high plateau participation plans to achieve earnings 250,000 pounds, but by participation in indeterminacy situations account (0.50000) between predictions will provide a benefit to us in our changing everything to be better prepared. Low participation in plans to achieve 80,000 pounds but gain has been calculated by the indeterminacy situations (0.15000) will provide us with benefits ranging from an estimate again in our vigilance.

Moreover, if the high participation plans to achieve earnings 280 000 pounds in the sea, but by participation in indeterminacy situations account (0.15000) between predictions will provide a benefit to us in our

changing everything to be better prepared. Low participation in plans to achieve 70,000 pounds but gain has been calculated by the indeterminancy situations (0.5000) will provide us with benefits ranging from an estimate again in our vigilance.

Moreover, if the high participation in the cultural district plans to achieve earnings 230 000 pounds but by participation in indeterminancy situations account (0.20000) between predictions will provide a benefit to us in our changing everything to be better prepared. Low turnout of 65000 pounds but plans to achieve earnings has been calculated by the indeterminancy situations (0.35000) will provide us with benefits ranging from an estimate again in our vigilance.

|  | High participation | Low participation |
|---|---|---|
| Highland region | 250000±[0,50000] | 80000±[0,15000] |
| Sea region | 280000±[0,15000] | 70000±[0,5000] |
| Cultural region | 230000±[0,20000] | 65000±[0,35000] |

Table 2

|  | High participation | Low participation |
|---|---|---|
| Highland region | [200000,300000] | [65000,95000] |
| Sea region | [265000,295000] | [65000,75000] |
| Cultural region | [210000,250000] | [30000,100000] |

Table 3

**Examination of approaches:**

**Definition 9: (Optimistic Approach)** This approach, decision-making for under an optimistic point of view without taking into account the angle pessimistic view, taking into account the best odds possible evaluation yapılacaktır.b case (Max Max), with assessment refers to us the highest monetary value and optimistic outlook.

|  | Max Max |
|---|---|
| Highland region | max[200000,300000]=300000 |
| Sea region | max[265000,295000]=295000 |
| Cultural region | max[210000,250000]=250000 |

According to the optimistic approach, investing in the Plateau region is the best choice, because it allows the maximum possible gain. According to the table below $i_1=i_3=i_5=0$ if that would be the best choice if we'd decided according to the classic case of the sea.

|  | High participation |
| --- | --- |
| Highland region | 250000 |
| Sea region | 280000 |
| Cultural region | 230000 |

As a result, we found that the differentiation neutrosophic as we expand the data in making decisions. Considering the available data, including all the classic form opinions based on results from future results will help in the form of neutrosophic better decision making by investors.

**Definition 10: (Pessimistic approach)** This approach, under a pessimistic point of view without considering the optimistic outlook for decision-making will be assessed taking into account the best odds possible. In this case (Max Max), assessment refers to us with the highest monetary value and a pessimistic outlook.

|  | Max Max |
| --- | --- |
| Highland region | max[65000,95000]=95000 |
| Sea region | max[65000,75000]=75000 |
| Cultural region | max[30000,100000]=100000 |

According to the pessimistic approach, investing in the cultural district is the best choice, because it allows the maximum possible gain. The following table based $i_2 = i_4 = i_6 = 0$ if it was decided that we should give according to the classic situation would be the best choice of springs.

|  | Low participation |
| --- | --- |
| Highland region | 80000 |
| Sea region | 70000 |
| Cultural region | 65000 |

We see that the decision to choose classic forms changed compared with neutrosophic form. According Neutrosophic form, this approach leads to invest in the cultural area, but based on classic form, leading to invest in the plateau region. When the data are correctly defined, certainly neutrosophic values will lead us to the right and best choice.

**Definition 11: (Unstable Approach)** This approach is not optimistic or pessimistic. A possible opportunity / chance is dependent on the setting of a moderate approach to make the best choice without losing the elections. According to this approach and make the most appropriate choice, taking into account the highest monetary value of lost opportunities if the selection down to zero requires to build a new matrix as follows.

|  | High participation | Low participation |
|---|---|---|
| Highland region | [200000,300000]-[200000,300000] | [30000,100000]-[65000,95000] |
| Sea region | [200000,300000]-[265000,295000] | [30000,100000]-[65000,75000] |
| Cultural region | [200000,300000]-[210000,250000] | [30000,100000]-[30000,100000] |

|  | High participation | Low participation |
|---|---|---|
| Highland region | [0,0] | [-35000,5000] |
| Sea region | [-65000,5000] | [-35000,25000] |
| Cultural region | [-10000,50000] | [0,0] |

We have lowered the highest monetary value in case of high participation from other current monetary value. Also, the highest monetary value in the case of low turnout, we reduced the monetary value of the other.

Now, with the highest value of the lost opportunity for each option we are preparing a short matrix:

|  | Opportunities to lose |
|---|---|
| Highland region | [-35000,5000] |
| Sea region | [-35000,25000] |
| Cultural region | [-10000,50000] |

As a result, according to this approach, the plateau region is the best choice, because it is the place where monetary loss is least.

$i_1=i_2=i_3=i_4=i_5=i_6=0$  When we get the following table:

|  | High participation | Low participation |
|---|---|---|
| Highland region | 250000 | 80000 |
| Sea region | 280000 | 70000 |
| Cultural region | 230000 | 65000 |

We regret matrix form:

|  | High participation | Low participation |
|---|---|---|
| Highland region | 30000 | 0 |
| Sea region | 0 | 10000 |
| Cultural region | 50000 | 15000 |

|  | Opportunities to lose |
|---|---|
| Highland region | 30000 |
| Sea region | 10000 |
| Cultural region | 15000 |

Considering that this approach has fewer lost opportunities, it is the most appropriate option Sea region. However, the decision will be different from the classical form, we see that the decision will be taken neutrosophic form. Therefore, it is better for us to be tied to this method having the right data to make the best choice.

When we examine three different options in the form of uniform approach by Neutrosophic classical logic we saw that we had obtained.

When we look at another point of view, the investor can obtain data according to the percentages of weight in high participation, low participation or indeterminancy participation. In such a case, we will be able to decide according to the neutrocopic decision-making tree.
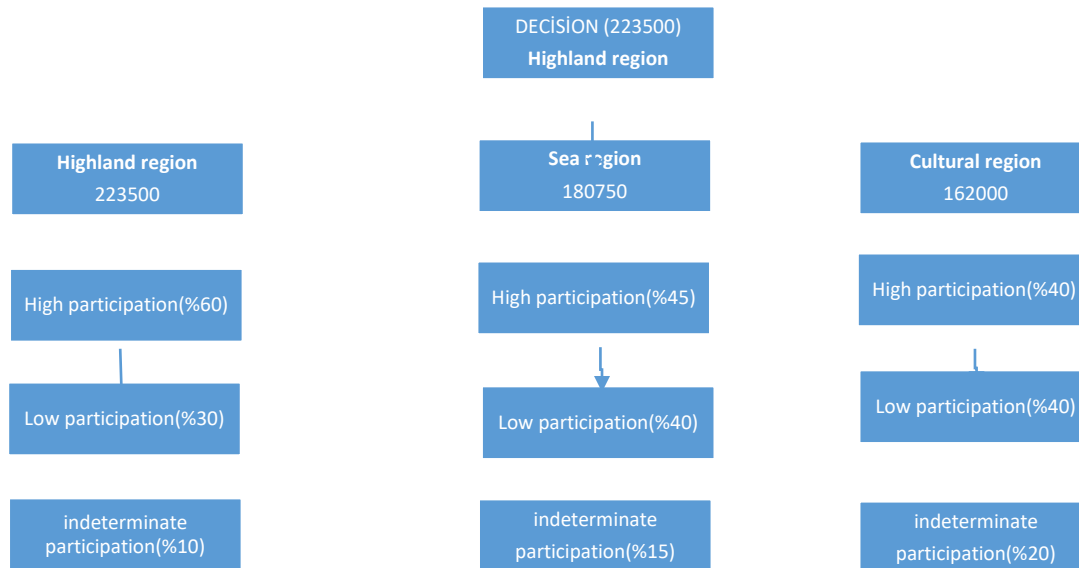
Highland region

(300000).(0,60)+(95000).(0,30)+(150000).(0,10)=223500

Sea region

(295000).(0,45)+(75000).(0,40)+(120000).(0,15)=180750

Cultural region

 (250000).(0,40)+(100000).(0,40)+(110000).(0,20)=162000

# CONCLUSIONS

In this study, we examined three approaches of classical logic and found that we took different options. Therefore, situations of indeterminancy associated with our decision making in investments should also be added..Most people make the wrong decisions for investment without considering indeterminancies.In this study, data are obtained according to the high and low possibilities of a person who wants to invest in the highland region, sea region and cultural region. Then, after obtaining indeterminancy data for the regions, the most accurate decision can be made with the help of the decision tree. Different results from classical logic are obtained by adding indeterminancies. People should consider indeterminancies together with the necessary data before making a decision for investment. Therefore, the best decision can be made according to positive, negative and indeterminancy situations. From here, we bring our results to the decision-making tree. So we find it much easier to invest with the help of the decision tree.

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# Chapter Thirteen

# Neutro-Law

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## ABSTRACT

In this chapter, we define neutro-law (neutrosophic law) and we give basic properties for neutro-law. Thus, we used the neutrosophic theory to explain the uncertainties and indecisive situations that we encounter in law, as in other sciences or other disciplines. Neutro-law will be particularly useful in studies aiming to use artificial intelligence applications in judgments. Because neutro-law is one of the most suitable systems for artificial intelligence applications aiming at an objective and consistent judgment, where uncertainties and indecision situations are fully examined and considered. In addition, thanks to neutro-law, which offers an objective method in revealing the uncertainty and indecision situations in the law, more consistent results will be obtained in law and judgment.

**Keywords:** Law, Neutrosophic Theory, Neutro-Law (neutrosophic law)

## INTRODUCTION

We encounter many uncertainties in every moment of our lives. Many times, classical mathematical logic is insufficient to get rid of these uncertainties. The reason is that when explaining a situation or a problem, it is not possible to say that it is correct or certain. Smarandache defined neutrosophic logic and the concept of neutrosophic set in 1998 [1]. In the concept of neutrosophic logic and neutrosophic sets, there is the degree of membership T, degree of uncertainty I and degree of non-membership F. These degrees are defined independently from each other. It has the form of a neutrosophic value (T, I, F). In other words, a situation is handled in neutrosophy according to its accuracy, its falsehood, and its uncertainty. Thus,

neutrosophic sets are the more general form of fuzzy sets [2] and intuitionistic fuzzy sets [3]. For this reason, many researchers have conducted studies on neutrosophic set theory [4 – 8, 21 - 52]. Recently, Das et al. studied neutrosophic multiset topological space [9]; Kargın et al. obtained neutrosophic triplet m-Banach spaces [10]; Aslan et al. studied neutrosophic modeling of Talcott Parsons's action [11]; Kargın et al. obtained decision making application for law science based on generalized single valued neutrosophic quadruple numbers [12]; Şahin et al. studied decision making applications for adequacy of online education based on neutrosophic quadruple numbers [13]. Also, Florentin Smarandache defined Neutrosociology [14] and Neutropsychic Personality [15]. In these studies, Smarandache explain the uncertainties and indecisive situations about sociology and psychic personality.

In law, as in other sciences and disciplines, there are many uncertainties and difficulties in making decisions. This uncertainty and the inability to make a decision may arise from the legal rules created, the legal gaps, the judge's decisions, and the contradictions in the expressions. Coping with these uncertainties has often been the most important problem of law. Many new theories have been put forward to resolve these uncertainties, and it is clear that many new ideas will emerge. Recently, it is aimed to use artificial intelligence applications in judgment in order to cope with these uncertainties and some trials have been made [16 - 19].

In this chapter, we have obtained the neutro-law theory to deal with the uncertainties in the law and to overcome the decision-making difficulties in law. Thus, the uncertainties in law will be dealt with systematically, the structure of the neutrosophic theory will be brought to law and more objective and correct decisions will be made. In addition, the system that will facilitate the use of artificial intelligence applications, which are used in many fields in our age and whose usage area is constantly increasing, in law, more precisely, will form the basis of this use will be neutro-law. The definitions and features we have obtained in this section are only the basic and most necessary definitions and features for neutro-law theory. The aim of this section is to give the basic definition and characteristics of the new theory, neutro-law, which is necessary for law, which has many sub-branches and fields of application, to make better decisions and is a preliminary preparation for researchers who will work in this field. It will also be a useful system for artificial intelligence applications in law.

In the second section, basic definitions on neutrosophic sets [1], [4], [6] are given. In the third section, the neutro-law is defined and its basic properties are given. In the last part, results and suggestions are given.

## BACKGROUND

**Definition 1:** [1] Let $E$ be the universal set. For $\forall x \in E, 0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$, by the help of the functions $T_A : E \rightarrow ]^-0, 1^+[$, $I_A : E \rightarrow ]^-0, 1^+[$ and $F_A : E \rightarrow ]^-0, 1^+[$ a neutrosophic set $A$ on $E$ is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\}.$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of trueness, indeterminacy and falsity of $x \in E$ respectively.

**Definition 2: [4]** Let $E$ be the universal set. For $\forall x \in E, 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, using the functions $T_A : E \to [0,1]$, $I_A : E \to [0,1]$ and $F_A : E \to [0,1]$, a SvNs $A$ on $E$ is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\}.$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of trueness, indeterminacy and falsity of $x \in E$ respectively.

**Definition 3: [6]:** Let $\mu$ be a binary operation. A neutrosophic triplet set (L, $\mu$) is a set such that for $l \in L$,

i) There is neutral of "l" such that $l \mu$ neut(l) = neut(l)$\mu l = l$,

ii) There is anti of "l" such that $l\mu$anti(l) = anti(l)$\mu l$ = neut(l).

Also, a neutrosophic triplet "l" is showed with (l, neut(l), anti(l)).

Furthermore, neut(l) must different from classical unit element.

**Definition 4: [14]** Let $\mu$ be a binary operation. An neutrosophic refined triplet set (L, $\mu$) is a set such that for $l_n \in L$, (n = 1, 2, …, k)

i) There is neutral of "$l_n$" such that $l_n \mu$ neut($l_n$) = neut($l_n$)$\mu$ $l_n = l_n$,

ii) There is anti of "$l_n$" such that $l_n\mu$anti($l_n$) = anti($l_n$)$\mu$ $l_n$ = neut($l_n$).

Also, a neutrosophic refined triplet "$l_n$" is showed with

($l_1, l_2$, …, $l_k$; neut($l_1$), neut($l_2$), …, neut($l_k$);  anti($l_1$), anti($l_2$), …, anti($l_k$)).

# NEUTRO-LAW

Neutro-law (Neutrosophic law) is a law study using neutrosophic scientific methods. The biggest factor in our definition of Neutro-law is that there are many uncertainties in law and legal practice. For example, uncertainties arising in the preparation of laws, uncertainties arising from gaps in the law, uncertainties arising from contradictions in expressions, uncertainty in the absence of a law suitable for the crime, etc. Therefore, neutrosophic theory that deals with uncertainties independently from other situations (truth and falsehood) should be used in the field of law. Because it is not always possible to say 100% guilty or 100% innocent for a suspect. It is almost impossible to explain other situations with classical logic.

**Neutro-law**

Classical law is defined as the set of social rules that regulate the relations of individuals with each other and society in social life and whose compliance is supported by public power [20]. We can say to neutro-law that it is the body of neutrosophic social rules that regulates the relations of individuals with each other and

society in community life and whose adherence is supported by public power. We will have obtained the neutro-law by taking into account many uncertainties in classical law independently from other factors.

Now, let' s give some specific types of neutrosophic sets that can be used in neutro-law theory with examples.

**Neutrosophic Set and Single-Valued Neutrosophic Sets in Neutro-Law**

For a suspect $x$, neutrosophic value $(T_x, I_x, F_x)$ has following components,

- $T_x$: degree to which suspect $x$ is not guilty or right
- $I_x$: degree to which suspect $x$ is uncertainty of not guilty or uncertainty of right
- $F_x$: degree to which suspect $x$ is guilty or wrong.

Where, $T_x, I_x$ and $F_x$ components are obtained independently of each other. If $0 \leq T_x + I_x + F_x \leq 3$, then the neutrosophic number is a single-valued neutrosophic number. Also, $T_x, I_x$ and $F_x$ are neutro-law's degrees.

For example, if there is a monovalent neutrosophic number (0.5, 0.2, 0.4) for a suspect;

- **0.5** (% 50)  suspect a is not guilty or justified
- 0.2 (%20) uncertain whether the suspect a is not guilty or wrong
- 0.4 (%40) suspect a is guilty or wrong

Also, situations in classical law can be opened with neutro-law. For example, the suspect is represented by (0, 0, 1) if he is 100% guilty and the suspect is represented by (1, 0, 0) if 100% not guilty.

**Neutrosophic Triplet Sets in Neutro-Law**

For a law $x$, neutrosophic triplet value $(x, \text{neut}(x), \text{anti}(x))$ has following components,

- $x$: positively affected by the law x
- neut($x$): unaffected by the law $x$
- anti($x$): negatively affected by the law $x$

Also, we can represent a neutro-law with (law, neutral-law, anti-law) neutrosophic triplet value.

**Neutrosophic Refined Triplet Sets in Neutro-Law**

We can represent neutro-laws with

(<law$_1$>, <law$_2$>, …; <neutral-law$_1$>, <neutral-law$_2$>, …; <anti-law$_1$>, <anti-law$_2$>, …)

neutrosophic refined triplet value.

**Rules of Neutro-Law**

In classical law, a law or a rule is either enforceable (100%) or unenforceable (0%). However, this is not always the case. Because in some cases a law or rule may be partially enforceable (T), partially uncertain (I), partially unenforceable (F). Therefore, neutro-law rules are more useful than classical legal rules.

**Neutro-Presumption**

Presumption is a sign, symptom that enables us to draw conclusions about a phenomenon whose existence is uncertainty [20]. Legal presumptions, on the other hand, are the signs and symptoms stipulated in the law that enable us to draw conclusions about a phenomenon whose existence is uncertainty [20]. However, classical prescriptions fall short of explaining most uncertainties. Therefore, neutro-presumption (or neutrosophic presumption) will be more useful. We can represent the neutro-presumption with

(<presumption >, <neutral-presumption>, <anti-presumption>) neutrosophic triplet value.

Where,

-   <presumption>: positively affected by the presumption
-   <neutral-presumption>: unaffected by the presumption
-   <anti-presumption>: negatively affected by the presumption

# Conclusions

In this chapter, we obtain the neutro-law theory and we give basic properties for the neutro-law. Also, we study some specific neutrosophic sets (single-valued neutrosophic set for neutro-law, neutrosophic triplet set for    neutro-law, neutrosophic refined triplet set for neutro-law)for neutro-law and we defined neutro-law's degrees, rules of neutro-law and neutro-presumption. Actually, neutro-law (Neutrosophic law) is a law study using neutrosophic scientific methods. The biggest factor in our definition of Neutro-law is that there are many uncertainties in classical law and classical legal practice. Also, We can say to neutro-law that it is the body of neutrosophic social rules that regulates the relations of individuals with each other and society in community life and whose adherence is supported by public power. Recently, artificial intelligence applications have begun to be used in law and judgment and these applications are being developed. Fuzzy logic, one of the principles of creating artificial intelligence, has an important place in these applications. The neutrosophic logic theory used in neutro-law has a more general and useful structure than fuzzy logic. For

this reason, neutro-law can be used in artificial intelligence applications developed for judgment and more objective and consistent results can be obtained. In this chapter, we obtain only basic properties for neuro-law. There are situations involving many more uncertainties in law and in many sub-branches of law. With the help of neutro-law, researchers can conduct more comprehensive studies by addressing other law uncertainties more objectively. Especially since the gaps in the law contain many uncertainties, studies can be carried out on this subject.

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# Chapter Fourteen

# Neutrosophic Triplet R-Module Chain Conditions

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## ABSTRACT

This section has been studied on Commutative Algebra structures. However, non-Abelian and identity rings have been studied. First, basic definition and properties of chain conditions in classical structures are given. It was then incorporated into the NT R-module structure of chain conditions. The definition, theorem and results of NT Artinian and NT Noetherian structures are given. In addition, the definitions of ascending chain condition(ACC) and descending chain condition(DCC) used in neutrosophic triple R-module structure chain conditions are given. Consequently, where classical chain conditions and NT chain conditions different from each other.

**Keywords:** NT submodule, NT R – module, NT quotient Module, NT- homomorphism, NT-isomorphism

## INTRODUCTION

F. Smarandache presented a new branch of philosophy, neutrosophy, in 1980, working on the state of uncertainty. Neutrosophy is a type of logic that depends on the probability set. [1]. Neutrosophic logic is the logic of some general concepts such as fuzzy logic presented by Zadeh in [2] and Provided by Atanassov intuitionistic fuzzy logic in [3].Fuzzy sets membership function but has an intuitionistic fuzzy set membership function and non-function and does not define membership indeterminancy. But; neutrosophic set includes all the functions. Many researchers have studied the concept neutrosophic theory and its application to issue multiple-criteria decision analysis in [4-11]. Lately, Olgun et al. introduced the neutrosophic module in [12]; Şahin et al. presented Neutrosophic soft lattices, soft normed rings and

neutrosophic soft lattices  in [13-15]; Ji et al. searched multi – valued neutrosophic environments and its applications in [16]. Also, Smarandache et al. searched NT theory in [17] and NT groups in [18]. A NT has a form <m, neut(m), anti(m)> where; neut(m) is neutral of "m" and anti(m) is opposite of "m". Moreover, neut(m) is different from the classical unitary element and NT group is different from the classical group as well. Many researchers have studied investigated the NT field, the NT ring, NT inner product, NT cosets and quotient groups,NT partial metric space and NT v – generalized metric space, its application to issue multi-criteria decision analysis and decision making and fundamental homomorphism theorems for NETGs in [19-49]; Lately, Çelik et al. Searched NT R-module in [50]. Creutzig at al. introduced Braided tensor categories of admissible modules for affine lie algebras in [51] and finally Yiqiang Z. Searched Relative chain conditions and module classes in [52]. The theories have studied in various areas such as [53-59].

In this study, This section is examined in Commutative algebra. However, non-Abelian and identity rings have been studied. First, basic definition and properties of chain conditions in classical structures are given. It was then incorporated into the NT R-module structure of chain conditions. The definition, theorem and results of NT Artinian and NT Noetherian structures are given.  In addition, the definitions of ascending chain condition (ACC) and descending chain condition (DCC) used in NT R-module structure chain conditions are given. Consequently, where classical chain conditions and NT chain conditions different from each other. Finally, in Chapter 4, we give some results.

## BACKGROUND

In this section, we present the basic definitions that are important for the development of the paper.

**Definition 1:[17]** Let (N,*) be a NT set. Then, N is called a NT group if the following conditions are satisfied.

(1) If (N,*) is well-defined, i.e., for any a, b ∈ N, one has a*b ∈ N.

(2) If (N,*) is associative, i.e., (a*b)*c = a*(b*c) for all a, b, c ∈ N.

**Definition 2: [20]** Let $N$ be a set together with a binary operation $\nabla$. Then, $N$ is called a NT set if for any $k \in N$ there exists a neutral of "$k$" called $neut(k)$ that is different from the classical algebraic unitary element and an opposite of "$k$" called $anti(k)$ with $neut(k)$ and $anti(k)$ belonging to $N$, such that

$$k \nabla neut(k) = neut(k) \nabla k = k,$$

and

$$k \nabla anti(k) = anti(k) \nabla k = neut(k).$$

**Definition 3: [17]** Let $(N, \nabla)$ be a NT set. Then, $N$ is called a NT group if the following conditions hold.

(1) If $(N, \nabla)$ is well-defined, i.e., for any $k, l \in N$, one has $\quad k \nabla l \in N$.

(2) If $(N, \nabla)$ is associative, i.e.,

$(k \nabla l)\nabla m = k \nabla (l \nabla m)$ for all $k, l, m \in N$.

**Definition 4:[19]** Let $(NTF, \nabla_1, \blacksquare_1)$ be a NT field, and let $(NTV, \nabla_2, \blacksquare_2)$ be a NT set together with binary operations "$\nabla_2$" and "$\blacksquare_2$". Then $(NTV, \nabla_2, \blacksquare_2)$ is called a NT vector space if the following conditions hold. For all $p, r \in NTV$, and for all $t \in NTF$, such that $p\nabla_2 r \in NTV$ and $p\blacksquare_2 t \in NTV$;

(1) $(p\nabla_2 r) \nabla_2 s = p\nabla_2 (r\nabla_2 s); p, r, s \in NTV$;

(2) $p\nabla_2 r = r\nabla_2 p; p, r \in NTV$;

(3) $(r\nabla_2 p)\blacksquare_2 t = (r\blacksquare_2 t)\nabla_2 (p\blacksquare_2 t); t \in NTF \ and \ p, r \in NTV$;

(4) $(t\nabla_1 c)\blacksquare_2 p = (t\blacksquare_2 p)\nabla_1 (c\blacksquare_2 p); t, c \in NTF, \ p \in NTV$;

(5) $(t\blacksquare_1 c) \blacksquare_2 p = t\blacksquare_1 (c\blacksquare_2 p); t, c \in NTF \ and \ p \in NTV$;

(6) There exists any

$$t \in NTF \ni p\blacksquare_2 neut(t) = neut(t) \blacksquare_2 p = p; p \in NTV.$$

**Definition 5: [20]** The NT ring is a set endowed with two binary laws $(M, *, \#)$ such that,

a) $(M, *)$ is a abelian NT group; which means that:

- $(M, *)$ is a commutative NT with respect to the law * (i.e. if x belongs to M, then $neut(x)$ and $anti(x)$, defined with respect to the law *, also belong to M)

- The law * is well – defined, associative, and commutative on $M$ (as in the classical sense);

b) $(M, *)$ is a set such that the law # on $M$ is well-defined and associative (as in the classical sense);

c) The law # is distributive with respect to the law * (as in the classical sense)

**Definition 6: [50]** Let (NTR,$\nabla$, $\blacksquare$) be a commutative NT ring and let (NTM, *) be a NT abelian group and ° be a binary operation such that °: NTR x NTM→ NTM. Then (NTM, *, °) is called a NT R-Module on (NTR,$\nabla$,$\blacksquare$) if the following conditions are satisfied. Where,

1) °($\alpha * \beta$) =( $\sigma$ °$\alpha$)* ($\sigma$ °$\beta$), $\forall$ $\alpha$, $\beta$ $\in$ NTM and $\sigma$ $\in$ NTR.

2) $(\sigma\nabla\rho)°\alpha = (\sigma\nabla\alpha)°(\rho\nabla\alpha)$, $\forall$ $\sigma$, $\rho$ $\in$ NTR and $\forall$ $\alpha$ $\in$ NTM

3) $(\sigma\blacksquare\rho)°\alpha = \sigma\blacksquare(\rho°\alpha)$, $\forall$ $\alpha$, $\beta$ $\in$ NTR and $\forall$ m $\in$ NTM

4) For all m ∈ NTM; there exists at least a ɾ ∈ NTR such that m°neut(ɾ)= neut(ɾ)°m = m. Where, neut(ɾ) is neutral element of ɾ for ■.

**Definition 7: [50]** Let (NTM, ∗, °) be a NT R-Module on NT ring (NTR,∇,■) and NTSM ⊂ NTM. Then (NTSM, ∗, °) is called NT R - submodule of (NTM, ∗, °), if (NTSM, ∗, °) is a NT R – module on NT ring (NTR,∇,■).

**Definition 8: [50]** (NTM₁, , °₁) be a NT R-module on NT ring (NTR,∇,■) and (NTM₂, ∗₂, °₂) be a NT R-module on NT ring (NTR,∇,■). A mapping f: NTM₁ → NTM₂ is said to be NT R-module homomorphism when

$f((r°_1m) *_1 (s°_1n)) = (r°_2f(m)) *_2 (s°_2f(n))$, for all r, s ∈ NTR and     m, n ∈ NTM₁.

**Definition 9: [52]** Let $\mathscr{M}$ be the set of submodules of a module M . Regarding $\mathscr{M}$ as a poset with respect to ⊆ , we refer to

(i) as the ascending chain condition (ACC)

and

(ii) as the maximal condition. Any module satisfying the ACC or equivalently the maximal condition is called Noetherian.

**Definition 10: [52]** On the other hand regarding $\mathscr{M}$ as a poset with respect to ⊇ , we refer to

(i) as the descending chain condition (DCC)

and

(ii) as the miminal condition. Any module satisfying the DCC or equivalently the minimal condition is called Artinian.

**Definition 11: [52]** A ring $\hat{R}$ is left (respectively, right) Noetherian if $\hat{R}$ satisfies the ACC on left (respectively, right) ideals. $\hat{R}$ is Noetherian if $\hat{R}$ is both left and right Noetherian. A ring is left (respectively, right) Artinian if $\hat{R}$ satisfies the DCC on left (respectively, right) ideals. $\hat{R}$ is Artinian if $\hat{R}$ is both left and right Artinian.

*Let A be a ring and let* $0 \to \mathscr{H} \overset{\phi}{\to} \mathbb{N} \overset{\sigma}{\to} \mathscr{M} \to 0$ *be an exact sequence of A-modules. Then:*

(1) *The module* ℕ *is noetherian if and only if the modules* $\mathscr{H}$ *and* $\mathscr{M}$ *are noetherian.*

(2) *The module* $\mathcal{N}$ *is artinian if and only if the modules* $\mathscr{H}$ *and* $\mathscr{M}$ *are artinian.*

**Definition 12: [52]** Assume that *(N₁, ∗)* and *(N₂,∘)* be two *NETG's*. If a mapping *f : N₁ → N₂* of *NETG* is only one to one (injective) *f* is called neutro-monomorphism.

**Definition 13: [52]** Let *(N₁, ∗)* and *(N₂, ∘)* be two *NETG's*. If a mapping $f : N_1 \rightarrow N_2$ is only onto (surjective) *f* is called neutro-epimorphism.

**Definition 14: [52]** Let *(N₁, ∗)* and *(N₂, ∘)* be two *NETGs*. If a mapping $f : N_1 \rightarrow N_2$ neutro-homomorphism is one to one and onto *f* is called neutro-isomorphism. Here, $N_1$ and $N_2$ are called neutro-isomorphic and denoted as $N_1 \cong N_2$.

# NEUTROSOPHIC TRIPLET R-MODULE CHAIN CONDITIONS

This section is examined in commutative algebra. However, non-Abelian and identity rings have been studied. In addition, chain conditions were included in the NT R-module structure. Definition, theorem and results of NT-Artinian and NT-Noetherian structures that constitute chain conditions were given.

**Definition 15:** A NT *R*-module $N$ satisfies the NT *ACC* on NT submodules if for all chain $N_1 \subset N_2 \subset N_3 \subset \cdots$ of NT submodules of $N$, there is $m \in \mathbb{N}$ such that $N_m = N_k$ for $\forall\, m \geq k$. Such a *NT R*-module is called NT-Noetherian. A module *P* satisfies the NT *DCC* on NT submodules if for every chain $P_1 \supset P_2 \supset P_3 \supset \cdots$ of NT submodules of *P*, there is $k \in P$ such that $P_i = P_k$ for all i ≥ k. Such a *NT R*-module is also called NT-Artinian .

**Note 16:** The NT-*ACC* and NT-*DCC* definitions for *NT R*-modules are similar to the *NT-ACC* and *NT-DCC* definitions for *NTG*, and chains for *NTG* are not only *NTG*, but also NT normal subgroups.

**Example 17:** Notice that the multiplicative NT group $Z_{15}$ is a *NT $Z_{15}$*-module (where the scalars are form *NTR $Z_{15}$* and the NT vectors are from additive NT group $Z_{15}$). $Z_{15}$ satisfies the NT *ACC* but not the *DCC*. That is, *Z* is NT-Noetherian but not NT-Artinian .

**Example 18:** The multiplicative NT abelian group $Z(p^{\infty}) = Q/Z$, the NT group of *NT Q*-modulo. This is also a NT *Z*-module (where the scalars are from NT ring *Z* the NT vectors are from multiplicative NT abelain group $Z(p^{\infty})$). $Z(p^{\infty})$ satisfies the NT *DCC* but not be NT *ACC* That is, $Z(p^{\infty})$ is NT-Artinian but not NT-Noetherian.

**Note 19:** We have seen the chain conditions in the *NT R*-modules. Let's move to chain conditions in the NT rings. To do so, we consider NT ring $\mathcal{R}$ as a left (or right) *NT R*-module over itself. So both the NT vectors and scalars come from R. Then if $\mathcal{R}$ is a NT submodule of NT R-module then $\mathcal{R}$ must be an NT R-module itself. That is, for any scalar r ∈ R and any vector s ∈ $\mathcal{R}$ we must have rs ∈ $\mathcal{R}$ in the case of R as a left NT R-module, or sr ∈ $\mathcal{R}$ in the case of $\mathcal{R}$ as a right NT R-module. Therefore, we see that the NT submodules are in fact NT ideals of $\mathcal{R}$ (for left NT ideal and NT right ideal). That NT ideals are to NT rings as NT normal

subgroups are to groups. Given the *ACC* and *DCC* definitions and theorems given for groups as a result, we can make the following definition.

**Definition 20:** A NT ring $\hat{R}$ is left (in order of, right) NT-Noetherian if $\hat{R}$ provides the ACC on left (in order of, right) NT ideals. $\hat{R}$ is NT-Noetherian

if $\hat{R}$ is both left and right NT-Noetherian. A NT ring is left (in order of, right) NT-Artinian if $\hat{R}$ satisfies the DCC on left (in order of, right) NT ideals. $\hat{R}$ is NT-Artinian if $\hat{R}$ is both left and right NT-Artinian .

**Note 21:** Most of the results related to NT *ACC* and NT *DCC* in this section are given for *NT R*-modules, but with the previous definition, these results also apply to NT rings.

**Example 22:** $\mathscr{R}$ NT ring with an identity is a NT division ring if and only if $\mathscr{R}$ has no proper left (or right) NT ideals. So the only ascending chain for NT division ring $\mathscr{B}$ is {neut(0)} $\subset \mathscr{B}$ and the only descending chain is $\mathscr{B} \supset$ {neut(0)}. So a NT division ring is both NT-Noetherian and NT-Artinian .

**Example 23:** if $\mathscr{R}$ is NT set, NT ring and $(\rho_1,\text{neut}(\rho_1),\text{anti}(\rho_1)) \subset (\rho_2,\text{neut}(\rho_2),\text{anti}(\rho_2)) \subset (\rho_3,\text{neut}(\rho_3),\text{anti}(\rho_3) \subset \cdots$ is a chain of NT sets in $\mathscr{R}$, then for $k \in$ N, $(\rho_i,\text{neut}(\rho_i),\text{anti}(\rho_i)) = (\rho_k,\text{neut}(\rho_k),\text{anti}(\rho_k))$ for all $i \geq k$. If $\mathscr{R}$ is a commutative NT ring then each left NTset is also a right NTset (or simply an ideal) and so for commutative NT ideal NTring $\mathscr{R}$ we see that $\mathscr{R}$ satisfies the NT-*ACC* and so is NT-Noetherian. Now Z and $Z_m$ are commutative NTset NTrings so Z and $Z_m$ are NT-Noetherian. if $F$ is a NTfield then $F[\text{neut}(x)]$ is a NTset NTdomain and so $F[\text{neut}(x)]$ is NT-Noetherian.

**Example 24:** We'll see that if $\mathscr{B}$ is a NTdivision ring, then the NT ring Matn($\mathscr{B}$) of all n × n matrices over $\mathscr{B}$ is both NT-Artinian and NT-Noetherian.

**Note 25:** It is to be shown that a NT subring of mat$_2$(Q) is right NT-Noetherian but not left NT-Noetherian. It is to be shown that a NT subring of Mat$_2$(R) is right NT-Artinian but not left NT Artinian. Z is NT-Noetherian but not NT-Artinian . I t is to be shown that every left (respectively, right) NT-Artinian NT ring with identity is left (respectively right) NT-Noetherian.

**Definition 26:** A pair of NT R-module NT-homomorphisms, $M \xrightarrow{\phi} N \xrightarrow{\sigma} P$ is precise at N provided NT-*Im(ϕ)=NT-Ker(σ)*.

A finite sequence of NT R-module NT-homomorphisms,

$$M_0 \xrightarrow{\phi_1} M_1 \xrightarrow{\phi_2} M_2 \xrightarrow{\phi_3} \dots \xrightarrow{\phi_{n-1}} M_{k-1} \xrightarrow{\phi_n} M_n$$

is exact provided NT-*Im(ϕ$_i$)=NT-Ker(ϕ$_{i+1}$)* for *i=1,2,3,...,k-1*

The sequence in $\{neut(0)\} \rightarrow M \xrightarrow{f} N \xrightarrow{g} \{neut(0)\}$ form is also a short full sequence. Notice that f is a NT-monomorphism (one to one) and g is an NT-epimorphism (onto).

**Lemma 27:** Let $S$ be a NTSM of a NTM $\mathcal{M}$.

(a)       $\mathcal{M}$ is NT-Noetherian $\Leftrightarrow$ $S$ , $\mathcal{M}/S$ are NT-Noetherian.

(b)       $\mathcal{M}$ is NT-Artinian $\Leftrightarrow$ $N$, $\mathcal{M}/N$ are NT-Artinian .

**Proof.** It is only to be able to prove (a), because (b) all inclusions can be proved easily by taking the opposite.

"$\Rightarrow$" Let $\mathcal{M}$ be NT-Noetherian. Firstly, some chain

$$\mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots$$

of NTSM of $\mathcal{H}$ is   furthermore a chain of NTSM of $\mathcal{M}$, and So it is seen to be constant. Therefore $\mathcal{H}$ is NT-Noetherian. Like this, let

$$B_0 \subset B_1 \subset B_2 \subset \ldots$$

be a chain of NT submodules of $\mathcal{M}/\mathcal{H}$. If we set $\mathcal{M}_k = q^{-1}(P_j)$ for every $j \in \mathbb{N}$, where  $\sigma : \mathcal{M} \to \mathcal{M}/\mathcal{H}$ is the NT quotient map, at that case

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$$

is a chain of NTSM of $\mathcal{M}$. Like $\mathcal{M}$ is NT-Noetherian, we have $\mathcal{M}_j = \mathcal{M}_n$ for all $j \geq r$ with some $r \in \mathbb{N}$. But since $\sigma$ is onto we at that case have $B_j = (\mathcal{M}_j) = (\mathcal{M}_n) = P_n$ for all $j \geq r$. Therefore $\mathcal{M}/\mathcal{H}$ is NT-Noetherian.

"$\Leftarrow$" let's consider

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$$

as an increasing chain of NTSM in $\mathcal{M}$. If we set $\mathcal{H}_j := \mathcal{M}_j \cap \mathcal{H}$ and $B_j = (\mathcal{M}_j + \mathcal{H})/\mathcal{H}$ for all    $j \in \mathbb{N}$, then

$$\mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots \quad \text{and} \quad B_0 \subset B_1 \subset B_2 \subset \ldots$$

are chains of NTSM of $\mathcal{H}$ and $\mathcal{M}/\mathcal{H}$, in order of.  Assuming that both will be standing, and therefore there is an element $r \in \mathbb{N}$ such that $\mathcal{H}_j = \mathcal{H}_r$ and $B_j = B_r$ for every $j \geq r$.

Here we get a commutative diagram for every $j \geq r$

$$Neut(0) \to \quad \mathcal{H}_r \to \quad \mathcal{M}_r \to \quad B_r \to Neut(0)$$

$$\downarrow \quad\quad \downarrow \quad\quad \downarrow$$

$$Neut(0) \to \quad \mathcal{H}_j \to \quad \mathcal{M}_j \to \quad B_j \to Neut(0)$$

whose rows are exact and whose columns are induced by the inclusions $\mathcal{M}_r \to \mathcal{M}_j$. As the left and right vertical map are NT-isomorphisms. Thus we have $\mathcal{M}_j = \mathcal{M}_r$ for $j \geq r$ as well. Therefore $\mathcal{M}$ is NT-Noetherian.

**Corollary 28:** Let $\mathscr{M}$ and $\mathscr{H}$ be NTM.

(a) The NT direct sum $\mathscr{M} \oplus \mathscr{H}$ is NT-Noetherian $\Leftrightarrow$ $\mathscr{M}$ and $\mathscr{H}$ are NT-Noetherian.

(b) If $R$ is NT-Noetherian and $\mathscr{M}$ is NT finitely generated, at that case $\mathscr{M}$ is furthermore NT-Noetherian.

The proposition used for NT-Noetherian also apply to NT-Artinian in the same way

***Proof.*** we will show only proposition for proof for NT-Noetherian modules, because of the NT-Artinian counterpart follows in exactly the same way.

i. This is followed by the full series

$neut(0) \rightarrow \mathscr{M} \rightarrow \mathscr{M} \oplus N \rightarrow N \rightarrow neut(0)$.

ii. Let

$\mathscr{M} = ((m_1, neut(m_1), anti(m_1)), \ldots, (m_k, neut(m_k), anti(m_k))$ for some $m_1, \ldots, m_k \in \mathscr{M}$. Then the NT ring NT-homomorphism

$\phi$ : $R^k$ $\rightarrow$ $\mathscr{M}$, $((\rho_1, neut(\rho_1), anti(\rho_1)),$ . . . , $(\rho_k, neut(\rho_k), anti(\rho_k))$ $\rightarrow$ $(\rho_1 m_1, neut(\rho_1) neut(m_1), anti(\rho_1) anti(m_1)) + \cdots + (\rho_k m_k, neut(\rho_k) neut(m_k), anti(\rho_k) anti(m_k))$

is onto, therefore we have an full series $neut(0) \rightarrow ker\varphi \rightarrow R^k \overset{\varphi}{\rightarrow} M \rightarrow neut(0)$ . Now as R is NT-Noetherian, so is $R^k$ by (a), and therefore too $\mathscr{M}$.

**Lemma 29:** A NT $R$-module $\mathscr{M}$ is of finite length $\Leftrightarrow$ it is both NT-Noetherian and NT-Artinian.

**Proof.** If the length of $\mathscr{M}$ is finite, then all strict chains of NTSM of $\mathscr{M}$ are finite. Therefore, $\mathscr{M}$ is clearly both NT-Noetherian and NT-Artinian.

Now let's think about the opposite. Let $\mathscr{M}$ is both NT-Noetherian and NT-Artinian . Starting from $\mathscr{M}_{neut(0)}$ =neut(0), let's try to create a chain $\mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset ...$ of NTSM of $\mathscr{M}$ as follows: for $k \in \mathbb{N}$ let $\mathscr{M}_{k+1}$ be a minimal NTSM of $\mathscr{M}$ that strictly contains $\mathscr{M}_k$ on condition that $\mathscr{M}_k \neq \mathscr{M}$ this works since $\mathscr{M}$ is NT-Artinian . However, since NT-Noetherian we cannot get the finite ascending chain of NTSM. In this way, we will get the result we want $\mathscr{M}_k = \mathscr{M}$ for some $k \in \mathbb{N}$. The resulting chain

$$neut(0) = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset ... \subset \mathscr{M}_k = \mathscr{M}$$

series is also a series of composition for $\mathscr{M}$, We can see that NTSM is not in this structure between $\mathscr{M}_{i-1}$ and $\mathscr{M}_i$ for all $i = 1,2,3,...,k$.

# CONCLUSIONS

This section mainly examines the chain conditions for NT R-modules. These structures have been studied through non-Abelian and identity rings. Classical R-module chain conditions were incorporated into the NT

R-module structure. In addition, NT-Artinian and NT-Noetherian definitions, theorems and results that constitute the chain conditions in module structures were given. In addition, the definitions of ACC and DCC used in neutrosophic triple R-module structure chain conditions are given. As a result of the studies, it was observed that the R-module chain conditions in classical structures different from NT R-module structures. So we moved the classical structure to the NT algebraic structure and established the relationship between the classical structure and the NT structures.

# ABBREVIATIONS

NT: Neutrosophic triplet

NTG: Neutrosophic triplet group

NTR: Neutrosophic triplet ring

NTM: Neutrosophic triplet R-module

NTSM: Neutrosophic triplet R-submodule

NT-ACC: Neutrosophic triplet ascending chain condition

NT-DCC: Neutrosophic triplet descending chain condition

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# <Algebra, NeutroAlgebra, AntiAlgebra>

**From classical Algebraic Structures to NeutroAlgebraic (NeutroAlgebra) and AntiAlgebraic (AntiAlgebra) Structures**

In 2019 and 2020 Smarandache generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) {whose operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) {whose operations and axioms are totally false}.
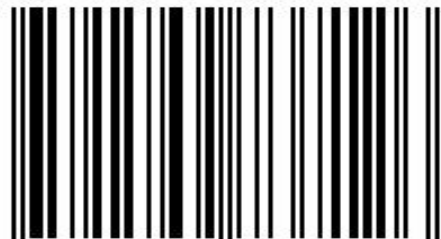
The NeutroAlgebras & AntiAlgebras are a *new field of research*, which is inspired from our real world.

In classical algebraic structures, all axioms are 100%, and all operations are 100% well-defined, but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some laws or some operations.

Using the process of *NeutroSophication* of a classical algebraic structure we produce a NeutroAlgebra, while the process of *AntiSophication* of a classical algebraic structure produces an AntiAlgebra.
See: http://fs.unm.edu/NA/NeutroAlgebra.htm