Comments on *A Conversation About Fuzzy Logic and Vagueness* by Christian G. Fermüller and Petr Hájek

NICHOLAS J.J. SMITH

Introduction

In the spirit of this enlightening conversation and indeed of this entire volume—promoting mutual understanding and learning between philosophers, logicians and linguists working on vagueness related issues—I shall in these comments consider three points, which if not kept in mind have significant potential to promote misunderstanding between practitioners of these three disciplines. All three points are discussed by Fermüller and Hájek: I do not take myself to be adding points that have been overlooked; I simply focus on these three points and discuss each in a little more detail—at the expense of not mentioning many other interesting issues covered in this fascinating and wide-ranging conversation.

1 Models

On p. 408, Hájek comments:

> the task is not to assign concrete numerical values to given sentences (formulas); rather the task is to study the notion of consequence (deduction) in presence of imprecise predicates. One should not conflate the idea that, in modeling logical consequence and validity, we *interpret* statements over structures where formulas are evaluated in $[0,1]$ with the much stronger claim that we actually single out a particular such interpretation as the "correct" one, by *assigning* concrete values to atomic statements.

When Hájek talks about 'the task' here, he means the task for (mathematical fuzzy) logicians. Philosophers and linguists are often interested in different tasks (still connected with vagueness). Realising that they are (sometimes) interested in different tasks is the first point that must be kept in mind if philosophers, logicians and linguists are not to misunderstand one another.

To help fix ideas, I shall focus here on the different roles that *models* (structures, interpretations) play in relation to different tasks. First, consider the task mentioned by Hájek: modelling logical consequence. When this task is approached from a model-theoretic (rather than purely deductive or proof-theoretic) point of view, models play a role. Indeed, *all* models (of the relevant sort: classical models; fuzzy models of one or another precisely specified kind; and so on) play an equal role: α is a logical consequence of Γ just in case there is *no model at all* on which every $\gamma \in \Gamma$ has degree of truth 1 and α does not, or on which α's degree of truth is lower than the infimum of the degrees of truth of members of Γ, and so on (depending upon the particular notion of consequence in play).

Other tasks also involve models, but not in this even-handed way: some models are more important than others. For example: suppose we want to know whether a particular piece of reasoning (involving vague terms) is *sound*. What we want to know is whether the argument is *valid* (the task considered above) *and* whether the premisses are true. What we mean by 'true' here is really, actually true. To capture this idea, we need a notion of a special, designated model. Real, actual truth is then truth relative to *this* model. Consider a second example. A claim is made, involving vague terms, which is considered by competent speakers to be 'out of order' in some way: not the right thing to say in the circumstances. We want to know whether the claim is false, or true but unassertible. Again, what we mean by 'true' ('false') here is really, actually true (false)—and to capture this idea, we need a notion of a special, designated model.[1]

Yet other tasks involve models in even more subtle ways. For example, suppose we want to know the *truth conditions* of some statement involving vague terms. There are various things that could be meant by this—here is one: we want to know what a *possible world* would have to be like for the statement to come out true relative to that possible world. For concreteness, let's suppose that we are concerned with the statement 'Bob is tall', represented as Tb in a formal language, and that we are working with fuzzy models, each of which comprises a domain (set of objects), an assignment of an individual in that domain to b, and an assignment of a fuzzy subset of that domain (mapping from the domain to $[0,1]$) to T. In this setting, one thing that will serve as a specification of the truth conditions of Tb is an assignment of an *intension* to each of T and b, where the intension of a symbol α is a function from possible worlds to values (extensions): that is, things of the kind α gets assigned in a model. Thus, in our case, the intension of T will be a function from worlds to fuzzy sets, and the intension of b will be a function from worlds to objects. Together, these determine a truth value for Tb relative to each world. Note that an assignment of intensions to T and b does not involve models *directly*: neither all models equally, nor some special models in particular. At the same time, an assignment of intensions to T and b *yields* a model, when combined with a particular possible world. Furthermore, one of these models in particular will often be of special interest: the one obtained by combining the given intensions with the *actual* world.[2]

I have discussed only a small selection of tasks in which philosophers, logicians and linguists working on vagueness related issues might be engaged. Problems facing those working on one task might pose no obstacle to those working on another;[3] results from one investigation may not be (directly) useful in a different investigation. I would say that the first item of business, when a practitioner of one discipline looks at another discipline, is to understand what task is being attempted: it may well be quite different from the tasks typically undertaken in one's home discipline.

[1] For further discussion, see §1 of my 'Fuzzy Logic and Higher-Order Vagueness' (this volume).

[2] For a more detailed discussion of the relationships between models and assignments of intensions, see Smith [6, Ch. 11].

[3] See for example the discussion of the problem of artificial precision in §1 of my 'Fuzzy Logic and Higher-Order Vagueness' (this volume).

2 Propositions, sentences and well formed formulas

On p. 410, Hájek comments:

> I think that the relation of mathematical fuzzy logic to natural language is very similar to that of classical mathematical logic and its relation to natural language: both deal with symbolic sentences (formulas), not with sentences of a natural language.

and on p. 412 Fermüller comments:

> Note that linguists take it for granted that by asserting a declarative sentence S (in usual contexts) a speaker wants to convey that the proposition p_S expressed by S is true in the given context.

Realising that, although they may use typographically similar expressions, they may nevertheless (sometimes) be interested in different kinds of entities—sentences (of natural language), propositions (expressed by the utterance of such sentences in contexts) and well formed formulas (sentences of a formal language)—is the second point that must be kept in mind if philosophers, logicians and linguists are not to misunderstand one another.

For the sake of simplicity and concreteness, let us proceed in terms of the following set of basic distinctions. By producing a *token* of a certain *sentence type* in a particular *context*, a speaker may express a certain *proposition*. Sentence types (and their component expressions) have *meanings*, which play a role in determining which proposition is expressed when a token of the sentence is produced in a certain context. Propositions (and their components, if they are taken to be structured entities) have *contents*, which play a role in determining their *truth values* relative to given *possible worlds*. Now consider some different tasks in which one might be interested:

1. One might want to investigate the meanings of natural language expressions. In this case one might employ a formal language whose symbols represent natural language expression types. (This is, I think, one reasonable construal of what is going on in some of the *linguistics* literature on formal semantics; cf. e.g. Heim and Kratzer [4].)

2. One might want to investigate the truth conditions of propositions expressed by utterances of natural language sentences. In this case one might employ a formal language which looks just like the language employed in case (1), except this time the guiding idea is quite different: the symbols of the formal language represent components of (structured) propositions. (This is, I think, one reasonable construal of what is going on in some of the *philosophical* literature on formal logic; cf. e.g. Smith [6].)

3. One might want to investigate various notions of logical consequence. In this case one might employ a formal language which again looks just like the language employed in cases (1) and (2), except this time the symbols of the formal language are not taken to represent anything else—or at least, whether or not

> they represent anything else (e.g. components of natural language sentences, or of propositions) is beside the point: as far as one's investigations of logical consequence are concerned, the well formed formulas (and sets, multisets, sequences etc. thereof) are themselves the objects of interest—the relata of the consequence relation. (This is, I think, one reasonable construal of what is going on in some of the pure/mathematical *logic* literature; cf. e.g. Hájek's comment quoted above.)

There is one area in particular where the issues just mentioned play an important role—and this is discussed in some detail in the conversation (pp. 7–11). The days are long gone when anyone would be convinced by a simple argument of the form 'Such-and-such a conditional would sound quite wrong when uttered in such-and-such a context; therefore it is incorrect to represent natural language indicative conditionals as material conditionals'. Since Grice [1], it is widely appreciated that there are many moving parts in between judgements of assertibility and assignments of truth values, and that there is no simple, direct argument from the former to the latter. The point seems to be less well remembered when it comes to discussions of fuzzy approaches to vagueness. In this context, surprisingly, arguments of this simple form do seem to have convinced many philosophers, linguists and logicians that truth-functional fuzzy logics cannot provide an adequate account of the phenomena of vague language use. I think that at least part of the problem here may be a tendency to jump freely between seeing (e.g.) $Tb \wedge \neg Tb$ as a representation of the natural language sentence type 'Bob is tall and Bob is not tall', and seeing it as a representation of the proposition expressed by some utterance of this sentence type in some particular context. It may be a commitment of some version of fuzzy logic that the proposition $Tb \wedge \neg Tb$ can have a degree of truth as high as 0.5; but this in no way conflicts with the claim that the sentence 'Bob is tall and Bob is not tall' can be used to express a proposition which cannot be true to a degree greater than 0.[4]

3 'Fuzzy' logic

On p. 414, Hájek comments:

> With hindsight it is hard to understand why Zadeh's proposal to generalize the classical notion of a set ("crisp set") to a fuzzy set by allowing intermediate degrees of membership has been met with so much resistance from traditional mathematics and engineering. Presumably many found it unacceptable to declare that vagueness is not necessarily a defect of language, and that it may be adequate and useful to deal with it mathematically instead of trying to eliminate it. There is a frequently encountered misunderstanding here: fuzzy logic provides precise mathematical means to talk about impreciseness, but it does not advocate imprecise or vague mathematics.

This is the final potential misunderstanding I want to focus on: the idea that fuzzy logic involves making logic or mathematics vague. It is useful to recall that one reason why this idea persists is the multiplicity of uses of the term 'fuzzy logic'. One standard

[4] For related discussion, see Smith [5, §5.5].

use of the term is to pick out logics where the set of truth values is $[0,1]$. There is nothing inherently vague or imprecise about such logics. However, Zadeh himself uses a different terminology, according to which [0,1]-valued logic is *nonfuzzy*, and the term 'fuzzy logic' is reserved for a more elaborate view [7, pp. 409–10]. Now consider the following well-known and influential passages from Haack:

> Fuzzy logic, in brief, is not just a logic for handling arguments in which vague terms occur essentially; it is *itself* imprecise. It is for this reason that I said that Zadeh's proposal is much more radical than anything previously discussed; for it challenges deeply entrenched ideas about the characteristic objectives and methods of logic. For the pioneers of formal logic a large part of the point of formalisation was that only thus could one hope to have *precise* canons of valid reasoning. Zadeh proposes that logic compromise with vagueness. [2, p. 167]
>
> Zadeh offers us not only a radically non-standard logic, but also a radically non-standard conception of the nature of logic. It would scarcely be an exaggeration to say that fuzzy logic lacks every feature that the pioneers of modern logic wanted logic *for* … it is not just a logic of vagueness, it is—what from Frege's point of view would have been a contradiction in terms—a vague logic. [3, p. 441]

Haack notes explicitly that she is concerned with fuzzy logic in Zadeh's elaborate sense and *not* with $[0,1]$-valued logics. Nevertheless, the mistaken view that $[0,1]$-valued logics are inherently vague seems to persist—along with the mistaken attribution of this view to Haack. (For further discussion, see Smith [5, §5.7].)

BIBLIOGRAPHY

[1] Paul Grice. *Studies in the Way of Words*. Harvard University Press, Cambridge MA, 1989.
[2] Susan Haack. *Philosophy of Logics*. Cambridge University Press, Cambridge, 1978.
[3] Susan Haack. Do we need "fuzzy logic"? *International Journal of Man-Machine Studies*, 11:437–45, 1979.
[4] Irene Heim and Angelika Kratzer. *Semantics in Generative Grammar*. Blackwell, Malden MA, 1998.
[5] Nicholas J.J. Smith. *Vagueness and Degrees of Truth*. Oxford University Press, Oxford, 2008.
[6] Nicholas J.J. Smith. *Logic: The Laws of Truth*. Princeton University Press, Princeton, forthcoming.
[7] Lotfi A. Zadeh. Fuzzy logic and approximate reasoning. *Synthese*, 30:407–28, 1975.

Nicholas J.J. Smith
Department of Philosophy
Main Quadrangle A14
The University of Sydney
NSW 2006 Australia
Email: njjsmith@sydney.edu.au