Philosophy Compass 4/4 (2009): 639–665, 10.1111/j.1747-9991.2009.00219.x

Frege's Judgement Stroke and the Conception of Logic as the Study of Inference not Consequence

Nicholas J. J. Smith*
The University of Sydney

Abstract

One of the most striking differences between Frege's Begriffsschrift (logical system) and standard contemporary systems of logic is the inclusion in the former of the judgement stroke: a symbol which marks those propositions which are being *asserted*, that is, which are being used to express *judgements*. There has been considerable controversy regarding both the exact purpose of the judgement stroke, and whether a system of logic should include such a symbol. This paper explains the intended role of the judgement stroke in a way that renders it readily comprehensible why Frege insisted that this symbol was an essential part of his logical system. The key point here is that Frege viewed logic as the study of *inference* relations amongst *acts of judgement*, rather than – as in the typical contemporary view – of *consequence* relations amongst certain *objects* (propositions or well-formed formulae). The paper also explains why Frege's use of the judgement stroke is not in conflict with his avowed anti-psychologism, and why Wittgenstein's criticism of the judgement stroke as 'logically quite meaningless' is unfounded. The key point here is that while the judgement stroke has no *content*, its use in logic and mathematics is subject to a very stringent *norm* of assertion.

A notable feature of Frege's logic is the presence therein of the *judgement stroke* – the vertical line '|': a symbol which marks those propositions which are being asserted. For Frege, *assertion* is the external act corresponding to the inner act of *judgement*: 'we distinguish: (1) the grasp of a thought – thinking, (2) the acknowledgement of the truth of a thought – the act of judgement, (3) the manifestation of this judgement – assertion' ('Logical Investigations' 355–6).

After the two-dimensional graphical nature of his symbolism, the judgement stroke is, to our eyes, the next most striking thing about Frege's logical system(s),[1] because standard current systems of logic employ no analogue of it: that is, they give us no way of *asserting* a proposition – of *putting it forward as being true* – as opposed to presenting or displaying a proposition so that its content may be considered. The judgement stroke was equally noteworthy both for Frege himself, and for his contemporary readers. In a review of Frege's *Grundgesetze*, Peano wrote:

> [Frege] introduces one notation $\vdash a$ to mean '*a* is true', and another notation $— a$ to indicate 'the truth of *a*' (*a* being a proposition). I fail to see the purpose of these conventions, which have nothing corresponding to them in [Peano's work] *Formulaire*. (29)

Frege responded with the counter-accusation that Peano should have a sign corresponding to the judgement stroke ('Letter' 35), and in the later part of his career, in a piece of only ten lines entitled 'What May I Regard as the Result of My Work?', Frege devotes two lines to the comment 'strictly I should have begun by mentioning the judgement-stroke, the dissociation of assertoric force from the predicate' (184).

There has been considerable controversy regarding both the exact purpose of the judgement stroke, and whether a system of logic should include such a symbol. This paper investigates the questions of what purpose Frege intended the judgement stroke to serve, and why he thought it so important that his logical system should have a symbol which serves this purpose. The latter question is particularly intriguing, because not only do standard current systems of logic employ no analogue of the judgement stroke – and seem not to be lacking in any important way as a result – but furthermore the inclusion of a sign for assertion appears to fly in the face of Frege's own vehemently held views about what should and should not find expression in logic – in particular, of his anti-psychologism.

Section 1 presents the data for the discussion to follow: Frege's various introductions of the symbol '|'. Section 2 addresses the question of what the symbol '|' *does* in Frege's logical system(s): what function it performs. Section 3 asks why Frege *wanted* a symbol which performs this function, and addresses the issue of how his employment of such a symbol coheres with his views about what should and should not find expression in logic. Section 4 looks at which aspects of Frege's view have survived in contemporary logic (this section includes a discussion of the differences between Frege's judgement stroke and the contemporary 'turnstile' symbol).

1. Presenting the Judgement Stroke

There are two major occasions on which Frege introduces the judgement-stroke: *Begriffsschrift* and *Grundgesetze*. Associated with each are minor occasions on which Frege more or less repeats what he says about the judgement-stroke on one of the major occasions. Frege first introduces the judgement-stroke in *Begriffsschrift*:

> A judgement will always be expressed by means of the sign
>
> $$\vdash,$$
>
> which stands to the left of the sign, or combination of signs, indicating the content of the judgement. If we *omit* the small vertical stroke at the left end

> of the horizontal one, the judgement will be transformed into a *mere combination of ideas*, of which the writer does not state whether he acknowledges it to be true or not. For example, let
>
> $$\vdash A$$
>
> stand for the judgement 'Opposite magnetic poles attract each other'; then
>
> $$—A$$
>
> will not express this judgement; it is to produce in the reader merely the idea of the mutual attraction of opposite magnetic poles, say in order to derive consequences from it and to test by means of these whether the thought is correct. When the vertical stroke is omitted, we express ourselves *paraphrastically*, using the words 'the circumstance that' or 'the proposition that'. . . . *The horizontal stroke* that is part of the sign $\vdash$ *combines the signs that follow it into a totality, and the affirmation expressed by the vertical stroke at the left end of the horizontal one refers to this totality*. Let us call the horizontal stroke the *content stroke* and the vertical stroke the *judgement stroke*. (11–12)

Later Frege writes, 'If there is no judgement stroke, then here – as in any other place where the ideography is used – no judgement is made. $\top A$ merely calls upon us to form the idea that A does not take place, without expressing whether this idea is true' (18).

Shortly after *Begriffsschrift*, in two papers intended for publication, and in a third, published paper, Frege offers similar formulations: 'The *judgement-stroke* is placed vertically at the left hand end of the content-stroke, it converts the content of possible judgement into a judgement' ('Boole's Logical Calculus' 11, n.***); 'in order to put a content forward as true, I make use of a small vertical stroke, the judgement stroke, as in $\vdash 3^2=9$ whereby the truth of the equation is asserted, whereas in $—3^2=9$ no judgement has been made' ('Boole's Logical Formula-Language' 51);

> If I wish to assert a content as correct, I put the judgement stroke on the left end of the content stroke: $\vdash 2+3=5$ . . . Through this mode of notation I meant to have a very clear distinction between the act of judging and the formation of a mere assertible content. ('On the Aim of the "Conceptual Notation" ' 94)

The next place in which Frege explains the judgement-stroke is 'Function and Concept':

> If we write down an equation or inequality, e.g. 5>4, we ordinarily wish at the same time to express a judgement; in our example, we want to assert that 5 is greater than 4. According to the view I am here presenting, '5>4' and '1+3=5' just give us expressions for truth-values, without making any assertion. This separation of the act from the subject matter of judgement seems to be indispensable; for otherwise we could not express a mere supposition – the putting of a case without a simultaneous judgement as to its arising or not. We thus need a special sign in order to be able to assert something. To this end I make use of a vertical stroke at the left end of the horizontal, so that, e.g., by writing
>
> $$\vdash 2+3=5$$
>
> we assert that 2+3 equals 5. Thus here we are not just writing down a truth-value, as in

> 2+3=5,
> but also at the same time saying that it is the True. (149)

This is very similar to the account that appears two years later in *Grundgesetze*:

> We have already said that in a mere equation there is as yet no assertion; '2+3=5' only designates a truth-value, without its being said which of the two it is. . . . We therefore require another special sign to be able to assert something as true. For this purpose I let the sign '$\vdash$' precede the name of the truth-value, so that for example in
>
> '$\vdash 2^2=4$',
>
> it is asserted that the square of 2 is 4. I distinguish the *judgment* from the *thought* in this way: by a *judgment* I understand the acknowledgment of the truth of a *thought*. The presentation in Begriffsschrift of a judgment by use of the sign '$\vdash$' I call a *proposition of Begriffsschrift* or briefly a *proposition*. I regard this '$\vdash$' as composed of the vertical line, which I call the *judgement-stroke*, and the horizontal line, which I will now simply call the *horizontal*. . . . Of the two signs of which '$\vdash$' is composed, only the judgement-stroke contains the act of assertion. (*Basic Laws of Arithmetic* 37–9)

2. *What Does the Judgement Stroke Do?*

The answer to the question 'What does the judgement stroke do?' that emerges straightforwardly if we take the passages quoted in the previous section at face value is that the judgement stroke *effects assertion*.[2] Placing the judgement stroke before an expression for a content of possible judgement *asserts that content*, that is, puts that content forward as true. Note in particular the following formulations (quoted in context in the previous section, my emphases here):

- clear distinction between the *act of judging* and the formation of a mere assertible content
- separation of the *act* from the subject matter of judgement
- We thus need a special sign *in order to be able to* assert something
- require another special sign *to be able to* assert something as true
- in '$\vdash 2^2=4$', *it is asserted that* the square of 2 is 4
- the judgement-stroke *contains the act of assertion*

The crucial point which emerges here is that the judgement stroke embodies an *action* (the action of assertion). When the judgement stroke is present, something is *being done* (an assertion is being made). An analogy may be useful.[3] Think of a communication situation as being like a game of Scrabble. Each person has a rail on which she can arrange characters (letters, punctuation marks, spaces). Let us suppose that the rail is long and that there are many characters, and that each player makes from these characters not individual words but entire sentences.[4] There are two quite different sorts of thing one can do in this game. First, one can move characters on and off one's rail and change their ordering. In general, such changes will alter the *content* of the sentence on one's rail. Second, there is a completely different kind of move one can make: one can move one's sentence (once it expresses just the content one wants) from one's own rail out onto the board. This does not affect the *content* of the sentence in any way. Rather, let us suppose that this is how, in this set-up, one *says something* – that is, makes a *claim* or *assertion*. It is quite evident here that the act of putting forward one's sentence is not another symbol on a par with the characters one puts together to make up one's sentences. One's act of putting a sentence forward cannot be put onto anyone's rail, that is, made part of the content of a sentence.[5] So there is a clear division in this situation between *acts* and *symbols*. Now for reasons which we shall discuss in section 3, Frege thought it important to have *the act of assertion – the act of making a claim, of putting forward a content as true* – expressed in his logical system. But his system comprises marks on paper. Hence the need to encapsulate *the act of putting a sentence forward* in what looks like just another symbol. But the judgement stroke is *not just another symbol*: it is categorically different from the other symbols in Frege's logic. It stands to them as the act of putting a sentence out onto the board stands to the characters which make up that sentence, in the Scrabble game described above.

To take another analogy, imagine a movie version of *Begriffsschrift* or *Grundgesetze*. There are things written on cards, which are seen propped on an easel long enough for us to read each one. Then at some points Frege picks up a card and thrusts it towards the camera – or taps it with a pointer while giving a meaningful look. In this movie version, we never *see* the judgement stroke as a written symbol on any of the cards. Rather, its role is played by Frege's actions. Now in a written symbolism we do not have available such actions, so everything has to be written as a symbol. But it is crucial that the judgement stroke be thought of not as a symbol alongside the symbols for negation, the conditional and so on, but as embodying or representing an *action*. So wherever you see the judgement stroke, think of it not as just another component of a sentence which sits before you on the page: think instead of the sentence – beginning *after* the judgement stroke – as highlighted, or as jumping out of the page at you. If Frege had written in HTML – the language of Web pages – he might well have used flashing text in place of the judgement stroke!

The *sui generis* nature of the judgement stroke is explicitly confirmed by Frege, a little later on in *Grundgesetze* (i.e. after the initial introduction of the judgement stroke, quoted above): 'The judgement-stroke I reckon neither among the *names* nor among the *marks*; it is a sign of its own special kind' (*Basic Laws of Arithmetic* 82). That prefixing it to an expression for a propositional content does not contribute more *content*, but rather serves to effect the assertion of the original content, is clear from the following remarks (quoted in the previous section, my emphases here):

- A judgement will always be expressed by means of the sign

$$\vdash,$$

which stands to the left of the sign, or combination of signs, indicating the content of the judgement.

It is clear here that the judgment stroke stands to the left of the signs indicating the content of the judgment, and is not itself one of them.

- The judgement-stroke . . . converts the content of possible judgement into a judgement
- in order to put a content forward as true, I make use of a small vertical stroke, the judgement stroke

It is clear here that we have the content, and then wish not to add to it, but to put *it* forward as true.

- If I wish to assert a content as correct, I put the judgement stroke on the left end of the content stroke . . . Through this mode of notation I meant to have *a very clear distinction between the act of judging and the formation of a mere assertible content*
- *separation of the act from the subject matter of judgement*

Here the distinction between content, and the act of asserting a content, is made explicitly.

Note that for Frege, the notion of judgement is primitive, and assertion is elucidated simply as the outer expression of the inner act of judgement. Frege does use phrases such as 'acknowledge to be true', 'present as true' and 'affirm', but he does not regard these as definitions or analyses of judgement or assertion. Rather, the phrases and terms are simply used interchangeably (subject to the distinction between inner and outer acts), and indeed sometimes combined, as in 'assert something as true'.[6] Frege writes: 'Judgements can be regarded as advances from a thought to a truth-value. Naturally this cannot be a definition. Judgement is something quite peculiar and incomparable' ('On *Sinn* and *Bedeutung*' 159). Frege simply takes a distinction between (inwardly) entertaining a proposition (or outwardly expressing it in such a way as simply to offer up its content for consideration, without committing oneself to its truth – running it up the flagpole, so to speak) and judging it to be true (or outwardly presenting it not simply for consideration, but as being the case) to be commonplace and well-understood. He does not think that this distinction needs any further explanation or refinement; what he does think is that it needs to be expressible in logic – hence the judgement stroke.[7]

At this point an important point of clarification is required. To a modern reader, the use of phrases such as 'putting forward as true' might give the impression that the judgement stroke is a *truth predicate*, that '$\vdash A$' is to be read as ' "A" is true' or 'It is true that A', where the latter attribute a property of truth to a sentence or proposition. This was certainly not

Frege's intention, however. Recall his comment, quoted above, about 'the dissociation of assertoric force from the predicate'. Frege was very clear that any thought content or proposition can occur in discourse either asserted or not asserted, without any change in content. Geach ('Assertion') emphasises this and calls it the 'Frege point'. So a predicate, which contributed further *content* to a thought, could never play the role of turning a content merely put forward for consideration into a claim about how things are. That is, no predicate could play the role which, as we have seen, the judgement stroke is supposed to play.[8]

3. Why Have the Judgement Stroke?

3.1. INFERENCE AND CONSEQUENCE

The reason why Frege thought it essential to have a sign in logic which allows us to *assert* things – to express our *judgements* – rests on the fact that for Frege, logic is the science of valid *inference*, where an inference is precisely a sequence of judgements:

> Now the grounds which justify the recognition of a truth often reside in other truths which have already been recognized. But if there are any truths recognized by us at all, this cannot be the only form that justification takes. There must be judgements whose justification rests on something else, if they stand in need of justification at all.
>
> And this is where epistemology comes in. Logic is concerned only with those grounds of judgement which are truths. To make a judgement because we are cognisant of other truths as providing a justification for it is known as *inferring*. There are laws governing this kind of justification, and to set up these laws of valid inference is the goal of logic. ('Logic' [1879–91] 3)[9]

> An inference simply does not belong to the realm of signs; rather, it is the pronouncement of a judgement made in accordance with logical laws on the basis of previously passed judgments. Each of the premises is a determinate Thought recognized as true; and in the conclusion too, a determinate Thought is recognized as true. ('On the Foundations of Geometry' 318)

> We justify a judgement either by going back to truths that have been recognized already or without having recourse to other judgments. Only the first case, inference, is the concern of Logic. ('17 Key Sentences on Logic' 175)

So a system with no means for expressing judgements is a system with no means for expressing the very things – inferences – with which logic is first and foremost concerned.

The view of logic as concerned with inference – where to infer is to *make a judgement* on the basis of previously passed judgements – is very alien to us now. The standard view nowadays is that logic is concerned with *consequence* (aka validity or implication). Where inference is concerned with *judgements*, consequence is concerned with *contents* of judgements –

propositions, as we call them now.[10] Judgements are actions; their contents – propositions – are objects. An inference is a dynamic thing – a sequence of actions (judgements) taking place over time, with later ones made on the basis of earlier ones. That some proposition is a consequence of some others is, by contrast, a static or eternal fact. Logic, as conceived by Frege, is concerned with the *laws of valid inference* – that is, with which ways of *making new judgements on the basis of previous judgements* are correct. Logic as conceived nowadays, by contrast, is not centrally concerned with subjects' judgements at all: it is concerned with eternal relations amongst propositions; that these propositions are possible contents of judgement is, at most, of secondary concern, related only to the possible *applications* of logic to reasoning.[11] (Indeed, on one widely held view of logic, the consequence relation holds not amongst *propositions* – which have truth values, and are possible contents of judgement – but amongst uninterpreted well-formed formulae. This view is even further removed from Frege's.)

The view of logic as concerned with inference has been so thoroughly eclipsed by the view of logic as concerned with consequence that we now tend not even to recognise the former view as a possibility, let alone adopt it. Consider for example Harman's well-known discussion of the relationship between logic and reasoning, which concludes:

> Reasoning in the sense of reasoned change in view should never be identified with proof or argument; inference is not implication. Logic is the theory of implication, not directly the theory of reasoning. (Harman 10)[12]

This is a fair claim about contemporary logic, but it is precisely *not* an accurate view of logic as conceived by Frege. The contrast between the two views of logic comes out most clearly if we consider a rule or law such as modus ponens. Here is Harman:

> Rules of argument are principles of implication, saying that propositions . . . of such and such a sort imply propositions . . . of such and such other sort. Consider the following principle:
>
> *Modus Ponens*: *P* and *if P then* Q taken together imply Q.
>
> Such a rule by itself says nothing at all in particular about belief revision. . . . rules of deduction are rules of deductive argument; they are not rules of inference or reasoning. They are not rules saying how to change one's view. . . . [Modus ponens] does not say that, if one believes *P* and also believes *if P then* Q, then one can infer Q . . . If there is a connection between standard principles of logic and principles of reasoning, it is not immediately obvious. There is a gap. We can't just state principles of logic and suppose that we have said something precise about reasoning. . . . Modus ponens is a principle of argument or implication, not a principle of reasoned revision. (3, 5, 6, 8)

Contrast what Frege says about logical laws in general (my emphasis):

> It will be granted by all at the outset that the laws of logic ought to be guiding principles for thought in the attainment of truth . . . In one sense a law asserts what is; in the other it prescribes what ought to be. Only in the latter sense

> can the laws of logic be called 'laws of thought': *so far as they stipulate the way in which one ought to think*. (*Basic Laws of Arithmetic* 12)

and about modus ponens in particular:

> From the propositions '$\vdash\!\!\lrcorner_{\Delta}^{\Gamma}$' and '$\vdash\!\!\!-\Delta$' we may infer '$\vdash\!\!\!-\Gamma$'; for if Γ were not the True, then since Δ is the True $\lrcorner_{\Delta}^{\Gamma}$ would be the False . . . This is the sole method of inference used in my book *Begriffsschrift*, and one can actually manage with it alone. (*Basic Laws of Arithmetic* 57)

The latter passage needs to be read carefully in order to be understood properly. Frege was very careful about use and mention.[13] When he states the rule of inference, he has quotation marks around *judgements* (i.e. things which include the judgement stroke) – that is, he mentions judgements, but does not make those judgements – and then when he justifies the rule, he presents the *contents* of those judgements – that is, he *uses* signs which have those contents, but does *not* assert them (note the absence of judgement strokes and quotation marks). It is very clear then that for Frege, modus ponens relates *judgements* (which, as I have been at pains to point out, are actions), not their *contents* (which are objects). If he had intended modus ponens in the latter sense, he would have omitted the judgement strokes when presenting it.

That modus ponens relates *judgements* (i.e. actions) in Frege's view is hard for us to notice because of the use of the word 'proposition' in this passage (in Furth's translation). We now use this word to pick out certain *objects* – the *contents* of judgements – so that judgement stands to proposition as act to object (the object being the content of that act). In an older usage, however, the term 'proposition' picked out the verbal expression of a judgement, so that judgement stood to proposition not as act to object, but as inner act to outer act (the outer act being the outward expression of that inner act).[14] When we read closely, it is absolutely clear from the usage of quotation marks and judgement strokes in the above passage that 'proposition' therein means an *action of assertion or judgement*, *not* the content of such an action.[15] Recall also that Frege has already defined 'proposition' as follows: 'The presentation in Begriffsschrift of a judgment by use of the sign "$\vdash\!\!\!-$" I call a *proposition of Begriffsschrift* or briefly a *proposition*' (*Basic Laws of Arithmetic* 38, quoted above); and a little later in *Grundgesetze*, Frege again explicitly defines a *proposition* as consisting of *a judgement stroke* together with a name or mark of a truth value with a horizontal prefixed (82).

Consider the following passage, from a letter to Hugo Dingler, where Frege is commenting on Dingler's statement that 'If we succeed in inferring logically from a group of premises that a certain statement both holds and does not hold for one of the concepts contained in the premises, then I say: *This group of premises is contradictory, or contains a contradiction*':

> Is this case [Dingler's] at all possible? If we derive a proposition from true propositions according to an unexceptionable inference procedure, then the

> proposition is true. Now since at most one of two mutually contradictory propositions can be true, it is impossible to infer mutually contradictory propositions from a group of true propositions in a logically unexceptionable way. On the other hand, we can only infer something from true propositions. Thus if a group of propositions contains a proposition whose truth is not yet known, or which is certainly false, then this proposition cannot be used for making inferences. If we want to draw conclusions from the propositions of a group, we must first exclude all propositions whose truth is doubtful. . . . It is necessary to recognize the truth of the premises. When we infer, we recognize a truth on the basis of other previously recognized truths according to a logical law. Suppose we have arbitrarily formed the propositions
>
> '2<1'
>
> and
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> 'If something is smaller than 1, then it is greater than 2'
>
> without knowing whether these propositions are true. We could derive
>
> '2>2'
>
> from them in a purely formal way; but this would not be an inference because the truth of the premises is lacking. And the truth of the conclusion is no better grounded by means of this pseudo-inference than without it. And this procedure would be useless for the recognition of any truths. So I do not believe that your case . . . could occur at all. (*Philosophical and Mathematical Correspondence* 16–17)

For modern readers, this passage is very difficult to comprehend. In the comments about premises needing to be true, it seems (from the modern perspective) as though Frege is conflating validity and soundness – but even that would not explain the comments about premises needing to be *recognised* as true. Dingler replies to Frege, 'It seems to me that the "truth" of the premises is completely irrelevant to the validity of the inference' (18). Certainly that is the standard view today. But Frege replies to Dingler's reply that we cannot infer anything from a proposition 'as long as we do not know that it is true' (20). If we think that Frege is talking here about consequence – about valid arguments in our sense, where an argument is a sequence (in the mathematical sense) of propositions – his remarks are utterly mystifying.[16] They make perfect sense, however, when we see that he is talking about *inference* – about a temporal (not mathematical) sequence of *judgements*. *Of course* there is no *inference* when the initial propositional contents are not acknowledged to be true: for an *inference* is a sequence of *judgements*, and a judgement is precisely an acknowledging-as-true of a propositional content.[17]

We thus have a distinction between two conceptions of logic. On the view of logic as primarily concerned with inference – Frege's view – the task of logic is to set out the laws of correct *inference*, where an inference is a (temporal) sequence of *judgements* (which are actions). Such laws tell us what we may judge next, given the judgements we have already made. So for Frege, logic was *primarily* concerned with the correctness of inferences – that is, with which ways of *making judgements on the basis of prior judgements* are correct. The consequence relation was for Frege a mere 'pseudo-inference':

the skeleton or husk of a genuine inference that is left when we take away the acts of judgement and leave behind only their contents. By contrast, on the view of logic as primarily concerned with consequence – the view that is now *so* standard as to make expressions of the former view (such as Frege's letter to Dingler, quoted above) seem thoroughly bizarre to many contemporary readers – the task of logic is to delineate the relation of *consequence*, which holds amongst propositions (which are objects). On this view, logical laws such as modus ponens are not concerned with judgements. The claim that for any propositions A and B, B is a consequence of A and $A \rightarrow B$ is no more concerned with judgements than the claim that if A is a well-formed formula (wf) and B is a wf, then $A \rightarrow B$ is a wf. Both simply serve as inductive clauses in a recursive definition of a particular set or relation: the latter in the specification of a certain subset of the set of all sequences of basic symbols – namely, the subset containing the wfs; the former in the specification of a certain relation between sets of propositions and propositions – namely, the classical consequence relation. Furthermore, there is no more a *temporal progression* from A and $A \rightarrow B$ to B in the former principle than there is from A and B to $A \rightarrow B$ in the latter.[18] Now of course if we *do* care to make judgements, then we can *use* the consequence relation amongst their contents as a guide to thought. Once we know that if certain propositions are true, then some other proposition must be true, we can *apply* this knowledge as a guide to reasoning: if we are certain that the former propositions *are* true, we can conclude that the latter is true too. But logic proper does not care about its possible applications in the guidance of judgement. Logic itself is concerned only with the consequence relation amongst propositions; it does not care whether any propositions are ever judged to be true.

Setting aside for a moment the question of which conception of logic we *should* adopt, the point for now is simply that Frege *did* view logic as primarily concerned with *inference* – an inference being a sequence of *judgements* – and this fact goes a long way towards explaining why he wanted a symbol in logic for marking out judgements from non-judgements. Without a means for marking out judgements, we have no means for marking out inferences: that is, no means for distinguishing the things which form the primary subject matter of logic.

I say that the fact that Frege viewed logic as primarily concerned with inference, not consequence, goes *a long way* towards explaining his employment of the judgement stroke. The case is not quite closed. The primary subject matter of pathology is disease. Pathologists need to learn all about the cause and progression of diseases – but they do not need to become infected. They need the means to *talk about*, say, the transmission of influenza, but they do not need a means of *effecting* that transmission. Likewise, we can see now why Frege needs a means for *marking out* inferences, and hence their components: judgements. But the question remains why he actually needs to *make* judgements and inferences in logic. The question remains why he needs a sign of judgement which *effects* assertion. Note for example that when stating modus ponens and the other laws of inference, Frege places the judgement strokes in quotation marks: he does not *make* any inferences when stating the *laws* of inference. So why should logic be equipped with the means actually to make judgements and draw inferences, rather than merely with the means to talk about judgements and inferences – to study them, and the laws governing them, from afar? I turn to this question in the following section.

3.2. THE INTENDED PURPOSE OF FREGE'S BEGRIFFSSCHRIFT

Frege is very clear, throughout his career, on the intended purpose of his Begriffsschrift. In the Preface to *Begriffsschrift* Frege says that he was trying to determine whether the judgements of arithmetic could be proved purely by means of logic, without support from facts of experience:

> To prevent anything intuitive from penetrating here unnoticed, I had to bend every effort to keep the chain of inferences free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography. Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated. . . .
>
> As I remarked at the beginning, arithmetic was the point of departure for the train of thought that led me to my ideography. And that is why I intend to apply it first of all to that science, attempting to provide a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems. For the present I have reported in the third chapter some of the developments in this direction. (5–8)

So, Frege was setting out to *make inferences*, and his Begriffsschrift was to be the vehicle for these inferences. The Begriffsschrift is presented in Parts I and II, and then in Part III it is used for carrying out proofs in arithmetic. As Frege makes clear in the Preface, the Begriffsschrift was designed from the start for the purpose of carrying out inferences. It was not that he later realised that he could apply his logic in actual proofs. Likewise the Introduction to *Grundgesetze* opens 'In this book there are to be found theorems upon which arithmetic is based, proved by the use of symbols, which collectively I call Begriffsschrift' (*Basic Laws of Arithmetic* 1). Here again we see the introduction of the Begriffsschrift bound together with its intended purpose as the vehicle for carrying out inferences. And late in his career, Frege reflects:

> I started out from mathematics. The most pressing need, it seems to me, was to provide this science with a better foundation. . . . The logical imperfections

of language stood in the way of such investigations. I tried to overcome these obstacles with my concept-script. In this way I was led from mathematics to logic. ('Notes for Ludwig Darmstaedter' 253)

Thus logic was never, for Frege, an 'armchair' science in which we study the laws of inference from afar, without actually making any judgements or drawing any inferences: it was always intended by Frege to provide a vehicle for carrying out inferences – in the first instance, proofs in arithmetic.

A question *still* remains about the judgement-stroke, however. Immediately following the passage quoted at the beginning of this paper, Peano continues: 'After all, the particular position a proposition occupies in a given formula shows unequivocally what it is that is being asserted about it in that formula'. Consider an inference of the form '*P*, *if P then Q*; therefore *Q*'. *P* is asserted in the first premise, but not in the second premise. But we do not need a judgement stroke to mark this distinction. We could just as well manage with the convention that all complete propositions – that is, propositions not occurring as parts of other propositions – are asserted. That would simplify the symbolism – something of which Frege is all in favour.[19] And for all we have said so far – that Frege wanted actually to use his Begriffsschrift to carry out proofs – this convention would be sufficient. In a proof in arithmetic, we need actually to draw an inference and hence make judgements: so we need a means of making assertions. But that does not explain why we need the judgement stroke – a means of making assertions which can be present *or absent* – as opposed to the convention considered above: for it does not explain why we would ever need the judgement stroke to be absent. In a reconstruction of arithmetic from axioms by purely logical deductions, every complete proposition, from the axioms on up, *would* be asserted. So while we can now see why Frege wanted a means for making assertions, we cannot yet see why it should take the form of the judgement stroke.

There are two things to be said to complete the explanation of the need for the judgement stroke. First, Frege had reasons for preferring the judgement stroke to the convention considered above even in the case of the reconstruction of arithmetic, in which the judgement strokes would never be left out. Second, Frege envisaged a role for the Begriffsschrift in which one *would* want to express complete propositions without asserting them.

On the first point: Frege was indeed in favour of minimising primitive symbolism, but this desideratum had to be balanced against others. Frege writes of his 'endeavour to have every objective distinction reflected in symbolism' ('On Mr. Peano's Conceptual Notation' 247); he says that 'We need a system of symbols from which every ambiguity is banned' ('On the Scientific Justification' 86); and he says that in his Begriffsschrift '*nothing is left to guesswork*' (*Begriffsschrift* 12). Minimising primitive symbols is important only insofar as it reduces unnecessary clutter, allowing important structure and distinctions to come to light. On the other hand, removing a primitive would be a *bad* thing if it marked 'an objective distinction'. This is the situation regarding the judgement stroke. Supposing that we only wanted to use the Begriffsschrift for the reconstruction of arithmetic, we could omit judgement strokes and adopt the convention mentioned above. But then an objective distinction would not be marked explicitly in logic: it would only be mentioned in the preamble (in which we state the convention). In an assertion there are two quite different elements present: the content, and the act of assertion. Even if we never merely present contents – that is, in the absence of the act of asserting them – still, the act and the content are crucially different, and we should not lose sight of this. So the distinction needs to be reflected *in the symbolism* (not just mentioned in the preamble). Something not explicitly marked is liable to be overlooked, and then confusion is likely to result. As already quoted, in the later part of his career, in a piece of only ten lines entitled 'What may I regard as the Result of my Work?', Frege devotes two lines to the comment 'strictly I should have begun by mentioning the judgement-stroke, the dissociation of assertoric force from the predicate'. Frege thus regards his sharp distinction between content (including subject-predicate structure, or object-function structure) and the act of putting a content forward as true as a key element of his work: so important that it needs to be marked in an explicit way. If we did away with the judgement stroke in favour of the convention that all complete propositions are to be regarded as being asserted, then we would be liable to slip back into thinking that the action of assertion – which we are currently supposing to be always present – is embodied in part of the content: the predicate. Even if this never causes us any practical problems, it is a conceptual confusion and a barrier to clear understanding. Hence the need to make it explicit in the symbolism – there on the page, staring us in the face – that there is a difference between the *act of assertion* and the *content* asserted. A separate, *sui generis* sign – separate from all signs which can be used to make up contents – does this in a way that the convention cannot. The importance of the distinction between act and content justifies the inclusion of a symbol which never lets us forget it – *even if* we never actually present any contents *without* asserting them. This is a point at which the desire for parsimony in the primitive symbolism is trumped by the desire for perspicuity. Frege says precisely this in a response to Peano's review:[20]

> I do not regard the mere counting of primitive symbols as sufficient to substantiate a judgment about the profundity of analysis towards fundamentals. E.g. I have the sign |, the judgment stroke, which serves to assert something as true. You have no corresponding sign; but you do acknowledge the distinction between the case in which a thought is merely expressed without being put forward as true, and that in which it is asserted. Now if, due to the absence of such a sign from your Begriffsschrift, the number of your primitive signs turns out upon close investigation to be the smaller, that would surely not imply that yours is the more profound analysis; for even if it is not reflected in the signs, the objective distinction is still there. ('Letter' 35)

(See also Frege, 'On Mr. Peano's Conceptual Notation' 247: 'I have introduced a special sign with assertoric force, the judgement-stroke. This is a manifestation of my endeavour to have every objective distinction reflected in symbolism'.)

On the second point: Frege writes, in another response to Peano's review:

> Mr. Peano has no such sign [i.e. the judgement stroke]. . . . From this it follows that for Mr. Peano it is impossible to write down a sentence which does not occur as part of another sentence without putting it forward as true. ('On Mr. Peano's Conceptual Notation' 247)

This suggests a quite different point: not that even when we put forward a content as true, we need to mark explicitly in our symbolism the distinction between the content, and the act of putting it forward as true; but that we might want to express a complete proposition without putting it forward as true.[21] Why would we want to do that? The answer is that Frege does not regard his Begriffsschrift simply as a tool for the reconstruction of arithmetic: he regards it as a tool for the working scientist. That is, it is not a tool just for writing up results (all of which are asserted), but a tool for doing science – and in the process of doing science, of arriving at results as opposed to reporting results, one often needs to express contents without asserting them as true. As Frege writes:

> An advance in science usually takes place in this way: first a thought is grasped, and thus may perhaps be expressed in a propositional question; after appropriate investigations, this thought is finally recognised to be true. ('Logical Investigations 356)

That Frege intended for the Begriffsschrift to be usable in this way is reflected in his saying, when he first introduces the judgement stroke in *Begriffsschrift* (quoted above, my emphasis here):

> For example, let
> $$\vdash\!\!-A$$
> stand for the judgement 'Opposite magnetic poles attract each other'; then
> $$-\!\!-A$$
> will not express this judgement; it is to produce in the reader merely the idea of the mutual attraction of opposite magnetic poles, *say in order to derive consequences from it and to test by means of these whether the thought is correct.*

It is also reflected in his comments about the relationship between the Begriffsschrift and ordinary language, which he likens to the relationship between the microscope and the eye: we turn to the microscope when 'scientific goals demand great sharpness of resolution' (*Begriffsschrift* 6). Nothing here suggests that we turn to the Begriffsschrift only when the process of scientific discovery is over and we know exactly what we want to assert: it suggests that we use the Begriffsschrift in place of ordinary language *whenever* scientific goals demand extreme clarity and precision of expression. That

Frege intended the Begriffsschrift to be usable in the process of doing science is also reflected in his claims that the Begriffsschrift was to be 'a *lingua characteristica* in the Leibnizian sense' ('On the Aim of the "Conceptual Notation" ' 91).[22] It is also reflected in Frege's practice in the appendix to *Grundgesetze* in which he considers Russell's Paradox. There Frege writes that 'it will be useful to track down the origin of this contradiction in our signs', and then adds, concerning 'the derivation that follows': 'in consideration of the doubtful truth of it all I shall omit the judgement-stroke'. Then comes a derivation, followed by the claim: 'The propositions (ζ) and (η) contradict one another. The error can be only in our Law (Vb), which must therefore be false' (*Basic Laws of Arithmetic* 130–2). His Begriffsschrift is just as useful in tracking down contradictions as it is in constructing proofs – and Frege gives no indication here that he is misusing the Begriffsschrift, or bending it to a purpose for which it was not designed to be used.

The point that I am making here – that Frege intended the Begriffsschrift to be of potential use not just as a language for the reporting and reconstruction of results in logical order, but also in the process of doing science and discovering truths – should not be confused with the quite different claim that Frege regarded his logical language as something which could only be used for making meaningful claims, and could not be regarded as an object of metalogical investigation. This view, associated primarily with (originally) van Heijenoort and Dreben, (and more recently) Goldfarb and Ricketts,[23] is controversial.[24] Nor should my claim be confused with the much stronger claim that Frege thought that science *had* to be carried out *solely* in Begriffsschrift: a claim which would be highly contentious. My own claim should not be contentious. I am saying only that Frege thought that the Begriffsschrift could potentially be of use in the process of scientific discovery (as opposed to reporting results) – perhaps in conjunction with surrounding use of natural language – just as we see, for example, in his appendix on Russell's paradox.

3.3. FREGE'S ANTI-PSYCHOLOGISM

The account presented above of the purpose of the judgement stroke and the reasons for its inclusion in Begriffsschrift raises a question about the coherence of Frege's overall package of views. Frege's anti-psychologism is famous. How can the inclusion of a symbol which marks (the outward counterpart of) the psychological distinction between merely entertaining a proposition, and judging it to be true, be made to cohere with Frege's vehement opposition to any incursion on the part of psychology into the realm of logic?[25] To feel the tension here, consider the following three claims:

> In logic we must reject all distinctions that are made from a purely psychological point of view. What is referred to as a deepening of logic by psychology is nothing but a falsification of it by psychology. ('Logic' [1897] 142)

> Both grasping a thought and making a judgement are acts of a knowing subject, and are to be assigned to psychology. ('Notes for Ludwig Darmstaedter' 253)

> We . . . require another special sign to be able to assert something as true. For this purpose I let the sign '|—' precede the name of the truth-value . . . I distinguish the *judgement* from the *thought* in this way: by a *judgement* I understand the acknowledgement of the truth of a *thought*. The presentation in Begriffsschrift of a judgement by use of the sign '|—' I call a . . . *proposition*. I regard this '|—' as composed of the vertical line, which I call the *judgement-stroke*, and the horizontal line . . . the *horizontal*. (*Basic Laws of Arithmetic* 38)

The first says that logic must shun psychological distinctions. The second says that the distinction between grasping a thought and making a judgement is a psychological distinction. The third introduces a means of making this distinction in logic.

In fact there is no inconsistency in Frege's views. While Frege does sometimes say simply that psychological distinctions have no place in logic, his considered, fully-articulated position is that logic should give no place to anything which is *irrelevant to inference*. For example:

> I decided to forgo expressing anything that is without significance for the *inferential sequence*. . . . Everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated. (*Begriffsschrift* 6, 12)

> One must always hold fast to the fact that a difference is only logically significant if it has an effect on possible inferences. ('Boole's Logical Calculus' 33, n.*)

> The task of logic being what it is, it follows that we must turn our backs on anything that is not necessary for setting up the laws of inference. In particular we must reject all distinctions in logic that are made from a purely psychological standpoint *and have no bearing on inference*. . . . Therefore let us only distinguish where it serves our purpose. ('Logic' [1879–91] 5, my emphasis)

The enemy, then, is not psychology *per se*, but psychological considerations *that have no bearing on inference*. But of course the distinction marked by the presence or absence of the judgement stroke – the admittedly psychological distinction between considering a proposition and judging it to be true – *does* have a crucial bearing on inference: where we have no judgements, we have no inference!

As Currie stresses, the core of Frege's anti-psychologism consists in anti-relativism and anti-subjectivism. Concerning anti-relativism, think for example of Frege's famous claim that logic is concerned with normative laws of thought, not descriptive ones ('Logical Investigations' 351–2): if they were merely descriptive, the threat would loom that your laws of thought might be different from mine. Concerning anti-subjectivism, think for example of Frege's famous insistence that the contents of statements and thoughts are not private mental items, but objective Thoughts ('Logical Investigations' I). But now think of the judgement stroke. Does its presence not introduce a subjective element into logic? As Wittgenstein put it, the judgement-stroke is 'logically quite meaningless: in the works of Frege (and Russell) it simply indicates that these authors hold the propositions marked with this sign to be true' (§4.442). If logic is to be objective, surely it should not include a symbol which functions simply to indicate whether or not *some particular person* holds some proposition to be true?

In fact, Frege is immune to this objection: although the judgement stroke serves to allow an author of a work written in Begriffsschrift to put forward a proposition as being true, this does *not* mean that the judgement stroke simply serves to indicate that the author of the work in question holds the proposition true. How can this be? Well, as we have seen, the judgement stroke has no *content*. It is a *sui generis* symbol which stands to the symbols which go to make up contents as the action of putting one's rail out onto the Scrabble board stands to the characters lined up on that rail. This means that the judgement stroke certainly does not introduce subjectivism by serving to *say something about* the speaker's mind: about what she holds true. When I use the judgement stroke, I do not *say that* I judge a certain proposition to be true: I *assert* the proposition, I put it forward as being true. So the question of what is indicated by the judgement stroke is not a question of what the judgement stroke *says*. Rather, the question must come down to the issue of what *norms* govern the use of the judgement stroke, that is, the issue of the *conditions of appropriate use* of the judgement stroke. For given that it has no content, the only thing that can be indicated by someone's using the judgement stroke is that she is in a state in which it is appropriate to use it, according to the norms governing its use. Thus the question of whether the inclusion of the judgement stroke threatens the objectivity of logic becomes a question of the stringency of the norms governing its use. Suppose the norm is very relaxed: one may assert any proposition which one thinks is true (no matter how good one's reasons are for thinking it to be true). In this case Wittgenstein's criticism hits home: from the fact that someone prefixes the judgement stroke to a certain proposition, we may infer only that she takes it to be true. But in fact Frege lays down a very stringent norm governing the use of the judgement stroke in logic and mathematics: one may prefix the judgement stroke to a proposition only when one is certain that the proposition is true. This does not mean simply that one must *feel* certain that the proposition is true – that is, give it a credence of 1; it means that one must be *justified with certainty* in thinking that it is true. More specifically, Frege makes it quite clear that the judgement stroke must be prefixed only to propositions which are self-evidently true (basic laws or axioms), or to propositions which follow from self-evidently true propositions according to transparently correct rules of inference. (Recall Frege's comment in his discussion of Russell's paradox that 'in consideration of the doubtful truth of it all I shall omit the judgement-stroke'.) But this means that from the fact that Frege prefixes the judgement stroke to a certain proposition, we may infer much more than simply that Frege takes this proposition to be true. We may infer that Frege is justified with certainty in taking it to be true: whence we may infer that it *is* true. This means, then, that despite the fact that what Frege *does* with the judgement stroke is simply assert a certain proposition himself, what is *indicated* by his use of the judgement stroke is that the proposition to which it is prefixed is true.[26] Thus Wittgenstein's worry is dispelled: the presence of the judgement stroke is perfectly compatible with the objectivity of logic.[27]

When we view logic as concerned with consequence, anti-psychologism comes on the cheap, for on this view, logic proper has nothing *at all* to do with the judgments or other psychological states of subjects. Frege's anti-psychologism is much more sophisticated. He views logic as about inference – as about sequences of *judgements* – and so psychological attitudes are embedded at the heart of logic. Nevertheless, in contrast to the 'psychological logicians' of his day, he manages to give – and argue clearly for the necessity of giving – an account of logic that does not fall into subjectivism or relativism.[28] It is in *this* sense – and *not* in the sense of *having nothing at all to do with psychological attitudes* – that Frege's logic is anti-psychologistic.

4. Contemporary Logic

In Frege's Begriffsschrift, whenever the judgement stroke '|' appears, it is always immediately followed by a horizontal line '—'. The horizontal line may, however, occur without the vertical line before it: namely, when a content is presented for consideration, without being asserted as true.[29] In contemporary logic, the symbol '⊢' is often used. That is not to say that it is found in *every* textbook; but it is a standard symbol whose meaning is known to every logician. Here is a typical account of its meaning:

> We define a (*formal*) *proof* (in the propositional calculus) to be a finite list of (occurrences of) formulas $B_1, \ldots, B_l$ each of which either is an axiom of the propositional calculus or comes by [modus ponens] from a pair of formulas preceding it in the list. A proof is said to be a proof *of* its last formula B_l. If a proof of a given formula B exists, we say B is (*formally*) *provable*, or is a (*formal*) *theorem*, or in symbols ⊢ B. (Kleene, *Mathematical Logic* 34)

> It is to be emphasized that the expression '$D_1, \ldots, D_l \vdash E$', which we use to state briefly that E is deducible from $D_1, \ldots, D_l$, is not a formula of the system, but a brief way of writing a metamathematical statement about the formulas $D_1, \ldots, D_l$, E, namely the statement that there exists a certain kind of a finite sequence of formulas. When $l = 0$, the notation becomes '⊢E', meaning that E is provable. The symbol '⊢' goes back to Frege 1879; the present use of it to Rosser 1935 and Kleene 1934. (Kleene, *Introduction to Metamathematics* 88)

So, we have a system of proof theory for a logical language *L*. The symbol '⊢' is not part of this language *L*, but part of the metalanguage for *L*. '⊢*E*' then states – in the metalanguage – that *E*, which is a proposition of the logical language *L*, is provable in our system of proof theory.

Let us note the key differences between this symbol '$\vdash$' – generally known as the 'turnstile' – and Frege's symbol '$\vdash\!\!-$'.[30] First, the turnstile can occur *between* propositions, as in '$D_1, \ldots, D_l \vdash E$', whereas Frege's symbol '$\vdash\!\!-$' can occur only *in front of* a single proposition.

Second, the turnstile is a simple, indivisible symbol: neither its horizontal nor its vertical part has any meaning in isolation. On the other hand, there is an aspect of modularity in the turnstile symbolism which is not present in Frege's system. It is quite common to add subscripts to the turnstile when several different systems of proof theory are in play; it is also common to write a slash through the turnstile to indicate that a formula is *not* provable in some system. Thus, in a discussion of modal logic, for example, we might write $\vdash_{S5} E$ and $\nvdash_{S4} E$ to indicate that the proposition E is provable in S5 but not in S4. Furthermore, a double turnstile $\vDash$ is used to indicate a semantically defined relation of consequence (as opposed to the proof-theoretically defined notion just considered): so $\vDash E$ means (for example) that E is true in all models of the logical language. Again, where different systems of model theory are in play, $\vDash$ may be subscripted; it may also be slashed. We can then express completeness of proof system P with respect to semantic system S by saying that for all sets of formulae Γ and all formulae E, if $\Gamma \vDash_S E$ then $\Gamma \vdash_P E$, and soundness of P with respect to S by: if $\Gamma \vdash_P E$ then $\Gamma \vDash_S E$.

Third, we saw that Frege's judgement stroke is a *sui generis* symbol, in that unlike the other symbols in his Begriffsschrift, it does not contribute to content: rather, it represents or embodies the *action* of putting forward a content as true. Now given, say, a standard first order logical language, the turnstile is also not of a kind with the symbols of this language, in the sense that it is a symbol of the metalanguage. So when we write, say, $\vdash(\forall x)Fx \rightarrow (\exists x)Fx$, there is a sense in which the first symbol – the turnstile – stands apart from the symbols which follow it. But crucially, the turnstile is still a symbol with *content*. If I write $(\forall x)Fx \rightarrow (\exists x)Fx$, and then add a turnstile at the front, I simply *add more content*. The result is a proposition which may be entertained, doubted, or asserted as I see fit. There would be no more impropriety in writing down $\vdash(\forall x)Fx \rightarrow (\exists x)Fx$ in the course of wondering whether $(\forall x)Fx \rightarrow (\exists x)Fx$ is a theorem than there would be in writing down $(\forall x)Fx \rightarrow (\exists x)Fx$ in the course of wondering whether this proposition is true. There is no more an *action of assertion* in $\vdash(\forall x)Fx \rightarrow (\exists x)Fx$ than in $(\forall x)Fx \rightarrow (\exists x)Fx$: there is just more *content* (or at least, more symbols which contribute to content).

In mainstream classical logic, then, some of Frege's *symbols* have survived, but they have different syntactic properties, and quite different functions. The changes are very natural ones, however, once we move from conceiving of logic as the study of *inference* to conceiving of it as the study of *consequence*. The judgement stroke indicates that we have a judgement: an essential component of an inference. The turnstile states facts about whether propositions stand in the consequence relation to one another.

On the other hand, Frege's distinction between the *action* of assertion and the *content* asserted *is* regarded as of great significance in some contemporary logical circles, outside the classical mainstream. For example:

> We must remember that, even if a logical inference, for instance, a conjunction introduction, is written
>
> $$\frac{A \quad B}{A \,\&\, B}$$
>
> which is the way in which we would normally write it, it does not take us from the propositions A and B to the proposition A & B. Rather, it takes us from the affirmation of A and the affirmation of B to the affirmation of A & B, which we may make explicit, using Frege's notation, by writing it
>
> $$\frac{\vdash A \quad \vdash B}{\vdash A \,\&\, B}$$
>
> instead. It is always made explicit in this way by Frege in his writings, and in Principia, for instance. Thus we have two kinds of entities here: we have the entities that the logical operations operate on, which we call propositions, and we have those that we prove and that appear as premises and conclusion of a logical inference, which we call assertions. It turns out that, in order to clarify the meanings of the logical constants and justify the logical laws, a considerable portion of the philosophical work lies already in clarifying the notion of proposition and the notion of assertion. (Martin-Löf, 'On the Meanings of the Logical Constants' 12)[31]

Yet despite considering Frege's distinction to be of the first importance, Martin-Löf does *not* think that an explicit assertion sign or judgement stroke is required in the formal symbolism ('On the Meanings of the Logical Constants' 16, 24–6).

Thus it would seem that there is no major strand in contemporary logic that agrees with Frege *both* that the distinction between the act of assertion and the content asserted is crucial to logic proper, *and* that it needs to be marked explicitly by a symbol of the formal language.

Acknowledgements

Thanks to the anonymous referees, the section editor and an audience at The University of Sydney for their helpful comments, and to the Australian Research Council for research support.

Short Biography

Nick Smith's main research areas are logic, philosophy of language, metaphysics, and the history of analytic philosophy. He has published in these areas in journals including *The Journal of Philosophy*, *Mind*, *Noûs*, *The Journal of Philosophical Logic*, *The Australasian Journal of Philosophy*, *The British Journal*

for the Philosophy of Science and *The Monist*. His monograph *Vagueness and Degrees of Truth* was published by Oxford University Press in November 2008. Nick is a Senior Lecturer in Philosophy at the University of Sydney; previously he was at Victoria University of Wellington. He did his undergraduate work at Sydney (B.A. Hons 1995) and his graduate work at Princeton (M.A. 1998, Ph.D. 2001).

Notes

* Correspondence address: Department of Philosophy, The University of Sydney, Main Quadrangle A14, New South Wales 2006, Australia. Email: njjsmith@usyd.edu.au.

[1] Frege uses the term 'Begriffsschrift' to denote his logical system. *Begriffsschrift* is the work in which he first presents it; he later presents a somewhat refined and altered Begriffsschrift in *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*; the first volume was originally published in 1893 and the second in 1903).

[2] Accounts of the judgement stroke more or less similar to the one presented in this section can be found, for example, in Geach 'Assertion'; Dudman, 'Frege's Judgement-Stroke' (under the heading 'Geach version' – see parenthetical discussion below); Dummett, *Frege*; Smith; Martin. (According to Dudman, Frege offers – and conflates – two inconsistent accounts of the judgement-stroke. According to the first account, which Dudman calls the 'Geach version' in reference to views put forward by Geach, ('Assertion'), the judgement-stroke is simply an index of assertion: 'it signals by its presence or absence whether or not a given conceptual content (of a kind capable in principle of being put forward as true) is in fact being put forward as true' (Dudman, 'Frege's Judgement-Stroke' 153). According to the second account of the judgement-stroke, which Dudman calls the 'Black version' in reference to views put forward by Black (227), the judgement-stroke converts designations into assertions. Stoothoff and Smith argue, contra Dudman, that Frege never advanced the Black view.) Rather different accounts have recently been proposed by Greimann (who argues that the judgement stroke is a 'truth operator': a logical operator which expresses that something is true) and Green (who argues that the judgement stroke expresses assertoric *commitment*, not actual assertion). I think it can be shown convincingly that where these interpretations diverge from the present account, they also diverge from Frege's texts – but I do not have the space here to present this discussion in detail.

[3] Cf. the somewhat similar analogy presented (independently) by Geach, 'Frege' 153.

[4] There are rules in place which distinguish the sentences – the well-formed strings – from amongst all the possible strings of characters.

[5] One can use characters to talk about someone's act of putting a sentence forward, but that is different: one's sentence about their action does not contain that very action – or any other action – as a constituent.

[6] Cf. Wright, who says that there is a *platitudinous* connection between assertion and truth: 'asserting a proposition – a Fregean thought – is claiming that it is true . . . to assert is to present as true' (23, 34, 37). Wiggins says a similar thing (Wright, 73, n. 2). The view that Frege uses phrases such as 'judge' and 'acknowledge to be true' (and 'assert' and 'present as true') interchangeably – as opposed to regarding the idea that judgement is acknowledgment of truth as a substantive, informative analysis of the notion of judgement – should not be confused with the much stronger view of Ricketts: 'Let us begin with an examination of the interconnections Frege draws among a raft of notions – assertion, judgment, content of judgment or thought, understanding, and inference. None of these notions can be understood apart from the others, and it is by attention to language and our linguistic practices that these notions are to be collectively elucidated' (71).

[7] More recently it has been felt that more needs to be said about assertion than that it is the outer counterpart of the inner act of judgement, and there has been a great deal of work in this area. See Pagin for a survey.

[8] Frege does at one point say that the sign '$\vdash$' can be considered as 'the common predicate for all judgements', but he is very careful to note that 'there cannot be any question here of

subject and predicate in the ordinary sense' (*Begriffsschrift* 12–3). This comment about '|—' being a predicate has tripped up many commentators, who have failed to note the qualification 'there cannot be any question here of subject and predicate in the ordinary sense'; for further discussion see Smith §IV.

[9] This quotation is from an unpublished piece entitled 'Logic', written some time between 1879 (the year of publication of *Begriffsschrift*) and 1891. The editors of *Posthumous Writings* write: 'In this piece . . . we clearly have a fragment of what was intended as a textbook on logic' (1).

[10] However as we shall see, the term 'proposition' has changed its meaning in this connection.

[11] For accounts of the contrast between logic as the study of inference and logic as the study of consequence, and its modern historical development, see e.g. Sundholm, 'Inference versus Consequence'; 'Plea for Logical Atavism'; 'Century of Inference'. The point that Frege did not regard logic as concerned primarily with consequence is made by Currie; Smith; Green.

[12] For more recent statements of the same view see Harman and Kulkarni: 'Deductive logic is a theory of what follows from what, not a theory of inference or reasoning. . . . It is a theory of deductive consequence' ('Problem of Induction' 561–2); or their *Reliable Reasoning* ch. 1.

[13] See for example his warning against 'an inadequate distinction between sign and thing signified' in the Introduction to *Grundgesetze* (*Basic Laws of Arithmetic* 6).

[14] See Martin-Löf 'On the Meanings of the Logical Constants' for a detailed tracing of the history, and changing meanings, of the term 'proposition' (and its various translations in Greek, German and other languages).

[15] For the Frege of *Begriffsschrift*, the content of a judgement is simply that: a 'judgeable content' or 'content of possible judgement'. For the Frege of *Grundgesetze*, content has split into sense and reference (aka denotation, Meaning). In the case of a judgement, the sense is a Thought and the reference is a truth value. See e.g. Frege *Basic Laws of Arithmetic*, 38, n. 14.

[16] Likewise Frege's many other claims to the effect that premises must be true and/or acknowledged as such before they can be used in inference: see e.g. 'Logical Investigations' 375, 402; *Philosophical and Mathematical Correspondence* 79, 182; 'On the Foundations of Geometry' 335; 'On Schoenflies' 180.

[17] One might still wonder why Frege sometimes says that premises have to be *known* to be true, not simply *judged* to be true. For the answer, see the discussion in section 3.3 below of the norm governing Frege's use of the judgement stroke.

[18] This point may not be immediately obvious, as there is a tendency to think of recursive definitions in temporal terms – for example, to think of the recursive specification of wfs as giving us a procedure for 'building up' more complex formulae out of simpler ones which we have 'already' built at an 'earlier' stage. Properly conceived, however, there is nothing temporal about a recursive definition: its inductive aspect is not a 'process' and does not take place in time.

[19] For example: 'I follow the basic principle of introducing as few primitives as possible' ('Boole's Logical Calculus' 36).

[20] The response took the form of a letter to Peano, who was the editor of the journal in which his review of Frege had appeared; Frege's letter was also published in this journal.

[21] Cf. also Frege's introduction of the judgement stroke in 'Function and Concept' (quoted above, my emphasis here): 'This separation of the act from the subject matter of judgement seems to be indispensable; *for otherwise we could not express a mere supposition – the putting of a case without a simultaneous judgement as to its arising or not*. We thus need a special sign in order to be able to assert something'.

[22] Cf. also e.g. *Begriffsschrift* 6–7. Cf. also Frege's remarks that he intends the Begriffsschrift to have application not only in arithmetic, but in calculus and geometry – and physics and philosophy (*Begriffsschrift* 7).

[23] See e.g. van Heijenoort; Dreben and van Heijenoort; Goldfarb, 'Logic in the Twenties', 'Frege's Conception of Logic'; Ricketts; and a number of other works cited by Tappenden, 194–5.

[24] For criticisms of the view, see e.g. Stanley; Tappenden; Heck.

[25] This issue is the focus of Smith. I now see that it is also discussed by Currie; Kenny 34–6. Cf. also Martin §2.

[26] And to repeat, this is *not* done by the judgment stroke *saying that* – i.e. having the content that – the proposition put forward is true, for the judgment stroke has no content. One might think that all we may really infer from the fact that Frege prefixes the judgement stroke to a certain

proposition is that Frege *takes himself* to be justified with certainty in taking the proposition to be true – not that he *is* justified with certainty in taking it to be true. This distinction collapses, however, if we suppose that when it comes to self-evident basic laws of logic, there is no gap between taking oneself to be justified with certainty in thinking that such a proposition is true, and really being justified with certainty in thinking that it is true. Basic Law V does pose a problem here however – as Frege himself acknowledges. In the Introduction to the first volume of *Grundgesetze* he writes: 'If we find everything in order, then we have accurate knowledge of the grounds upon which each individual theorem is based. A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V) . . . I hold that it is a law of pure logic. In any event the place is pointed out where the decision must be made.' (*Basic Laws of Arithmetic* 3–4). In the Appendix to the second volume on Russell's paradox he writes: 'It is a matter of my Basic Law (V). I have never concealed from myself its lack of the self-evidence which the others possess, and which must properly be demanded of a law of logic, and in fact I pointed out this weakness in the Introduction to the first volume' (*Basic Laws of Arithmetic* 127).

[27] A point of clarification is required here concerning the role of the norm governing assertion vis-à-vis the characterisation of assertion. A number of recent accounts of assertion build a norm governing the conditions in which it is appropriate to make assertions into the very characterisation of what an assertion is. For example: 'Assertion in the unique value of X for which the schema "X only what you know!" gives a valid rule' (Pagin §6.2). According to this account, what an assertion *is* is precisely that kind of speech act X for which the rule 'X only what you know!' holds valid. This is *not* the kind of role that Frege gives to his norm of assertion. That is, he does not regard it as *constitutive* of assertion that assertions are subject to the norm that one assert only propositions which one is justified with certainty in believing to be true. As we have seen, Frege regards the notion of judgement as *primitive*, and explains assertion simply as the outward manifestation of judgement. Assertion is then subject to the external (as opposed to constitutive) norm that one must assert only propositions which one is justified with certainty in believing to be true. Note also that outside logic and mathematics, Frege seems to have envisaged the judgement stroke being subject to a weaker norm: 'I am confident that my ideography can be successfully used wherever special value must be placed on the validity of proofs, as for example when the foundations of the differential and integral calculus are established. It seems to me to be easier still to extend the domain of this formula language to include geometry. . . . The transition to the pure theory of motion and then to mechanics and physics could follow at this point. The latter two fields, *in which besides rational necessity empirical necessity asserts itself*, are the first for which we can predict a further development of the notation as knowledge progresses' (*Begriffsschrift* 7, my emphasis). This is understandable: for in empirical science, one cannot demand that only self-evident propositions, or propositions deducible from self-evident propositions, be asserted. See also Frege, 'Sources of Knowledge of Mathematics'.

[28] See in particular the Introduction to *Grundgesetze*, from the footnote 'Mathematicians reluctant to venture into the labyrinths of philosophy are requested to leave off reading the Introduction at this point' (*Basic Laws of Arithmetic* 12, n. 7) onwards. From this point on, the Introduction consists in a diatribe directed against the 'psychological logicians'.

[29] See Smith 164–6 for an account of the change in the role of the horizontal stroke from Frege's early work to his later work – in line with the change from viewing content as simple to distinguishing between sense and reference – and for reasons (contra Dudman, 'Frege's Judgement-Stroke' – cf. also Švob 71–2; Green 202, n. 3) for thinking that Frege did *not* change his view on the role of the *vertical* stroke (the judgement stroke).

[30] I use different symbols for the turnstile and for Frege's judgement-stroke-plus-horizontal simply in order to make my discussion clearer: beyond that, the difference has no significance. The precise shape and size of Frege's symbol differs between various printings and reprintings of his work; likewise, the precise shape and size of the turnstile differs between various logic books. By many standards, Frege's symbol and the turnstile would be regarded as the same symbol from a typographical point of view: that is, as a symbol comprised of a vertical stroke of *more or less* the height of a capital 'I' joined to the left of a horizontal stroke of *more or less* the length of an em dash. Our interest here is not in typography, but in how the *usage* and *significance* of this (one) symbol differs between Frege's works and those of contemporary logicians.

[31] See also Martin-Löf, *Intuitionistic Type Theory*. Thanks to Greg Restall for first pointing me in the direction of Martin-Löf in connection with the distinction between propositions and judgements in contemporary logic.

Works Cited

Black, Max. *A Companion to Wittgenstein's 'Tractatus'*. Ithaca, NY: Cornell University Press, 1964.

Currie, Gregory. 'Remarks on Frege's Conception of Inference'. *Notre Dame Journal of Formal Logic* 28 (1987): 55–68.

Dreben, Burton and Jean van Heijenoort. 'Introductory Note to *1929*, *1930* and *1930a*'. Kurt Gödel, *Collected Works, Vol. 1, Publications 1929–1936*. Eds S. Feferman et al. New York, NY: Oxford UP, 1986. 44–59.

Dudman, V. H. 'Frege's Judgement-Stroke'. *Philosophical Quarterly* 20 (1970): 150–61.

——. 'Peano's Review of Frege's Grundgesetze'. *Southern Journal of Philosophy* 9 (1971): 25–37.

Dummett, Michael. *Frege: Philosophy of Language*. 2nd ed. Cambridge MA: Harvard UP, 1981.

Frege, Gottlob. *The Basic Laws of Arithmetic: Exposition of the System*. Ed. and trans. Montgomery Furth. Berkeley, CA: U of California P, 1964.

——. *Begriffsschrift, a Formula Language, Modeled upon that of Arithmetic, for Pure Thought* (1879). Trans. Stefan Bauer-Mengelberg. Ed. Jean van Heijenoort. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Cambridge MA: Harvard UP, 1967. 1–82.

——. 'Boole's Logical Calculus and the Concept-Script' (1880–81). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 9–46.

——. 'Boole's Logical Formula-Language and My Concept-Script' (1882). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 47–52.

——. *Collected Papers on Mathematics, Logic, and Philosophy*. Ed. Brian McGuiness. Trans. Max Black et al. Oxford: Basil Blackwell, 1984.

——. *Conceptual Notation and Related Articles*. Ed. and trans. Terrell Ward Bynum. Oxford: Clarendon Press, 1972.

——. 'Function and Concept' (1891). *Collected Papers on Mathematics, Logic, and Philosophy*. Ed. Brian McGuiness. Trans. Max Black et al. Oxford: Basil Blackwell, 1984. 137–56.

——. 'Letter from Sig. G. Frege to the Editor' (1896). V. H. Dudman, 'Peano's Review of Frege's Grundgesetze'. *Southern Journal of Philosophy* 9 (1971): 31–6.

——. 'Logic' (1879–91). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 1–8.

——. 'Logic' (1897). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 126–51.

——. 'Logical Investigations: I Thoughts; II Negation; III Compound Thoughts' (1918–26). *Collected Papers on Mathematics, Logic, and Philosophy*. Ed. Brian McGuiness. Trans. Max Black et al. Oxford: Basil Blackwell, 1984. 351–406.

——. 'Notes for Ludwig Darmstaedter' (1919). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 253–7.

——. 'On Mr. Peano's Conceptual Notation and My Own' (1897). *Collected Papers on Mathematics, Logic, and Philosophy*. Ed. Brian McGuiness. Trans. Max Black et al. Oxford: Basil Blackwell, 1984. 234–48.

——. 'On Schoenflies: *Die logischen Paradoxien der Mengenlehre*' (1906). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 177–83.

——. 'On *Sinn* and *Bedeutung*' (1892). *The Frege Reader*. Ed. Michael Beaney. Oxford: Blackwell, 1997. 151–71.

——. 'On the Aim of the "Conceptual Notation" ' (1882–83). *Conceptual Notation and Related Articles*. Ed. and trans. Terrell Ward Bynum. Oxford: Clarendon Press, 1972. 90–100.

——. 'On the Foundations of Geometry: Second Series' (1906). *Collected Papers on Mathematics, Logic, and Philosophy*. Ed. Brian McGuiness. Trans. Max Black et al. Oxford: Basil Blackwell, 1984. 293–340.
——. 'On the Scientific Justification of a Conceptual Notation' (1882). *Conceptual Notation and Related Articles*. Ed. and trans. Terrell Ward Bynum. Oxford: Clarendon Press, 1972. 83–9.
——. *Philosophical and Mathematical Correspondence*. Eds Gottfried Gabriel et al. Trans. Hans Kaal. Chicago, IL: U of Chicago P, 1980.
——. *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979.
——. '17 Key Sentences on Logic' (1906). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 174–5.
——. 'Sources of Knowledge of Mathematics and the Mathematical Natural Sciences' (1924/25). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 267–74.
——. 'What May I Regard as the Result of My Work?' (1906). *Posthumous Writings*. Eds Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach. Trans. Peter Long and Roger White. Chicago, IL: U of Chicago P, 1979. 184.
Geach, P. T. 'Assertion'. *Philosophical Review* 74 (1965): 449–65.
——. 'Frege'. G. E. M. Anscombe and P. T. Geach, *Three Philosophers*. Oxford: Basil Blackwell, 1963. 127–62.
Goldfarb, Warren. 'Frege's Conception of Logic'. *Future Pasts: The Analytic Tradition in Twentieth-Century Philosophy*. Eds Juliet Floyd and Sanford Shieh. Oxford: Oxford UP, 2001. 25–41.
——. 'Logic in the Twenties: The Nature of the Quantifier'. *Journal of Symbolic Logic* 44 (1979): 351–68.
Green, Mitchell S. 'The Inferential Significance of Frege's Assertion Sign'. *Facta Philosophica* 4 (2002): 201–29.
Greimann, Dirk. 'The Judgement-Stroke as a Truth-Operator: A New Interpretation of the Logical Form of Sentences in Frege's Scientific Language'. *Erkenntnis* 52 (2000): 213–38.
Harman, Gilbert. *Change in View: Principles of Reasoning*. Cambridge MA: MIT Press, 1986.
—— and Sanjeev Kulkarni. 'The Problem of Induction'. *Philosophy and Phenomenological Research* 72 (2006): 559–75.
——. *Reliable Reasoning: Induction and Statistical Learning Theory*. Cambridge, MA: MIT Press, 2007.
Heck, Richard G. 'Frege and Semantics'. *Grazer Philosophische Studien* 75 (2007): 27–63.
van Heijenoort, Jean. 'Logic as Calculus and Logic as Language'. *Synthese* 17 (1967): 324–30.
Kenny, Anthony. *Frege: An Introduction to the Founder of Modern Analytic Philosophy*. Oxford: Blackwell, 1995.
Kleene, Stephen Cole. *Introduction to Metamathematics*. Amsterdam: North-Holland, 1952.
——. *Mathematical Logic*. New York, NY: Wiley, 1967.
Martin, Wayne M. 'The Judgment Stroke and the Truth-Predicate: Frege, Heidegger, and the Logical Representation of Judgment'. *The New Yearbook for Phenomenology and Phenomenological Philosophy* 3 (2003): 27–52.
Martin-Löf, Per. *Intuitionistic Type Theory*. Naples: Bibliopolis, 1984.
——. 'On the Meanings of the Logical Constants and the Justifications of the Logical Laws'. *Nordic Journal of Philosophical Logic* 1.1 (1996): 11–60.
Pagin, Peter. 'Assertion'. *The Stanford Encyclopedia of Philosophy*. Ed. Edward N. Zalta. Spring 2007 ed. <http://plato.stanford.edu/archives/spr2007/entries/assertion/>.
Peano, G. 'Review of Frege's *Grundgesetze*' (1895). V. H. Dudman, 'Peano's Review of Frege's Grundgesetze'. *Southern Journal of Philosophy* 9 (1971): 27–31.
Ricketts, Thomas. 'Objectivity and Objecthood: Frege's Metaphysics of Judgment'. *Frege Synthesized*. Eds L. Haaparanta and J. Hintikka. Dordrecht: Reidel, 1986. 65–95.
Smith, Nicholas J. J. 'Frege's Judgement Stroke'. *Australasian Journal of Philosophy* 78 (2000): 153–75.
Stanley, Jason. 'Truth and Metatheory in Frege'. *Pacific Philosophical Quarterly* 77 (1996): 45–70.
Stoothoff, R. H. 'Dudman on Frege's Judgement-Stroke'. *Philosophical Quarterly* 21 (1971): 166–7.

Sundholm, Göran. 'A Century of Inference: 1837–1936'. *In the Scope of Logic, Methodology and Philosophy of Science*. Vol. 2. Eds P. Gärdenfors, J. Wolénski and K. Kijania-Placek. Dordrecht: Kluwer, 2002. 565–80.

——. 'Inference versus Consequence'. *Logica Yearbook 1997*. Ed. Timothy Childers. Prague: Filosofia, 1998. 26–35.

——. 'A Plea for Logical Atavism'. *Logica Yearbook 2000*. Ed. Ondrej Majer. Prague: Filosofia, 2001. 151–62.

Švob, Goran. 'The Heritage of Frege's Begriffsschrift'. *Acta Analytica* 15 (2000): 61–82.

Tappenden, Jamie. 'Metatheory and Mathematical Practice in Frege'. *Gottlob Frege: Critical Assessments of Leading Philosophers*. Vol. 2. Eds M. Beaney and E. H. Reck. London: Routledge, 2005. 190–228.

Wiggins, David. 'What Would Be a Substantial Theory of Truth?' *Philosophical Subjects: Essays Presented to P. F. Strawson*. Ed. Zak van Straaten. Oxford: Clarendon Press, 1980.

Wittgenstein, L. *Tractatus Logico-Philosophicus*. Trans. D. F. Pears and B. F. McGuiness. London: Routledge, 1974.

Wright, Crispin. *Truth and Objectivity*. Cambridge, MA: Harvard UP, 1992.