Possible World Semantics without Modal Logic

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Abstract

Possible worlds are commonly seen as an interpretation of modal operators such as “possible” and “necessary”. Here, we develop possible world semantics (PWS) which can be expressed in basic set theory and first-order logic, thus offering a reductionist account of modality. Specifically, worlds are understood as complete sets of statements and possible worlds are sets whose statements are consistent with a set of conceptual laws. We introduce the construction calculus (CC), a set of axioms and rules for truth, possibility, worldness and consistency. We show that CC allows to prove fundamental theorems about necessity, possibility, impossibility and contigency, thus demonstrating prima-facie plausibility of PWS. Finally, we discuss the explanatory power of our approach and draw connections between our account and established philosophical conceptions of possible worlds.

Contents

1 Introduction

Possible worlds have become a common thought device in modal logic, metaphysics and philosophy of mind. In such contexts, possible worlds are typically used in an informal way to demonstrate that some statement in question – e.g. "Pain is identical with the firing of C-fibers"[1] – is either necessarily true (because true in all possible worlds) or not necessarily true (because false in some possible world).

Besides their use in philosophy of mind, possible worlds are used to clarify the meaning of modal operators such as "possible" and "necessary"[2]:

(1a) It is possible that p, if there is a possible world in which p is true.
(1b) It is necessary that p, if the statement p is true in all possible worlds.

The present work attempts to draw a fundamental connection between such quantified statements about possible worlds and logical consistency:

(2a) There is a possible world in which p is true, if there exists a set of conceptual laws C, such that p is consistent with C.
(2b) The statement p is true in all possible worlds, if there exists a set of conceptual laws C, such that $\neg p$ is inconsistent with C.

It is clear that, if one accepts statements of the form (1) and statements of the form (2), this allows to infer the following relationships:

(3a) If p is consistent with a set of conceptual laws C, then p is possible.
(3b) If $\neg p$ is inconsistent with a set of conceptual laws C, then p is necessary.

While there is consensus that assertions like "It is possible that ..." and "It is necessary that ..." can be equated with assertions about possible worlds as in (1a) or (1b), there is ongoing debate (i) how the concept of worlds should be construed, (ii) what makes a world a possible world and (iii) what limits the space of possible worlds[3].
In this paper, our goal is to provide a formal treatment of possible worlds – including definitions of "worldness" and "possibility" – in order to formally spell out conditions for something to be a (possible) world. This formal treatment will be presented as an axiomatization of possible world semantics which is only based on first-order logic and basic set theory. One can then formally prove relationships between logical (in-)consistency and quantified statements about possible worlds as in (2a) and (2b).

The present work is inspired by two features typically associated with possible worlds. The first of these features is *completeness* – any possible world must be complete picture of what the world could have looked like. For example, Kripke writes:

> "What do we mean when we say 'In some other possible world I would not have given this lecture today?' We just imagine the situation where I didn't

[1] Cf. Kripke (1980), pp. 148ff.
[2] Cf. Lewis (1986).
[3] Cf. Menzel (2016).

> decide to give this lecture or decided to give it on some other day. Of course, we don't imagine everything that is true or false, but only those things relevant to my giving the lecture; but, in theory, everything needs to be decided to make a total description of the world."[4]

The second of these features is *consistency* – any possible world must be compatible with what is conceived to be possible, not allowing for impossible ways of reality. Again quoting Saul A. Kripke from his seminal work *Naming and Necessity*:

> "What's the difference between asking whether it's necessary that 9 is greater than 7 or whether it's necessary that the number of planets is greater than 7? (...) The answer (...) might be intuitively 'Well, look, the number of planets might have been different from what it in fact is. It doesn't make any sense, though, to say that nine might have been different from what it in fact is'."[5]

These features reflect aspects that are most often associated with possible worlds[6] and defines two requirements for something to be a possible world:

(R1) Possible worlds are seen as *complete* ways the things could have been.
(R2) Any conception of reality which *obeys* the laws of logic is a possible world.

The conception which will be laid out here unifies these two aspects and, consequently, will be based on the following working definition of "possible world":

(MW) A possible world is a complete and consistent way the world is or could have been.

Based on this understanding, we will first spell out possible world semantics (PWS), a set of definitions to capture the concept of possible worlds in first-order logical terms (see Section 2). Specifically, a possible world will be understood as a complete set of statements that is consistent with some set of conceptual laws.
Then, we will recall definitions of logical and modal-logical terms in terms of possible world semantics[7] (see Section 3). Next, we will introduce the construction calculus (CC), a set of rules that will allow to "calculate" with the predicates of PWS (see Section 4). Using this calculus, we will finally derive relationships between possibility or necessity and consistency or inconsistency of statements[8] (see Section 5).
Concluding, we relate our account to existing philosophical conceptions of possible worlds (see Section 6), namely concretism, abstractionism and combinatioralism[9]. We will see that PWS exhibits properties from all three theories of possible worlds.

[4] Kripke (1980), p. 44.
[5] Kripke (1980), p. 48
[6] Cf. Beckermann (2001), pp. 129-130.
[7] This will include relationships such as (1a) and (1b).
[8] This will prove statements such as (2a) and (2b).
[9] Cf. Menzel (2016), ch. 2.

2 Possible World Semantics

In this section, we will introduce a theory of possible worlds which is based on formal definitions of terms such as “world” and “consistency” and thus allows to formally distinguish between possible worlds and impossible worlds.
Specifically, we take a language-based perspective which understands worlds as sets of statements and possible worlds as sets of statements which are consistent with a set of conceptual laws. Our aim will be to express the entire theory in terms of first-order logic (PL1) and basic set theory. To establish such a linguistic account of possible world semantics (PWS), we need the following prerequisites:

> **Definition 1:** (*Sprache*) Let L_I^S be a language with signature S in PL1 where the signature consists of a set of constant symbols (“constants”), a set of function symbols (“functions”) and a set of relation symbols (“predicates”).

> **Definition 2:** (*Menge begrifflicher Gesetze*) Let C be a set of conceptual laws, formulated in L_I^S using the symbols contained in S.

> **Definition 3:** (*Menge der Sachverhalte*) Let P be a consistent set of propositions, formulated in L_I^S using the symbols contained in S.

The conceptual laws C in Definition 2 are intended to reflect the fact that the language in which we describe possible worlds is governed by rules which permit certain types of statements and prohibit other types of statements, such that statements inconsistent with C are logical contradictions (cf. example below).
The requirement that the propositions P shall be consistent entails that any P obeying Definition 3 is composed of statements which can all be true at the same time – but they may not be consistent with C. Specifically, P may not contain a statement p and its negation $\neg p$; and ideally, P only contains “atomic” statements such as Pa, Qb etc. where P and Q are predicates and a and b are constants.

> **Example:** As our running example, consider the two constants “a” (referring to Bertrand Russell) and “b” (referring to Ludwig Wittgenstein) and the two predicates “B” and “U” defined as follows:
>
> $$\begin{aligned} \mathrm{B}x: &\quad x \text{ is a bachelor.} \\ \mathrm{U}x: &\quad x \text{ is unmarried.} \end{aligned} \tag{1}$$
>
> Because by definition, bachelors are unmarried – but not vice versa –, the set of conceptual laws may contain a logical expression of this rule:
>
> $$C = \{\forall x\, (\mathrm{B}x \rightarrow \mathrm{U}x)\} \ . \tag{2}$$
>
> Finally, the presence of two predicates and two constants allows to formulate four atomic statements which make up the set of propositions:
>
> $$P = \{\mathrm{B}a,\, \mathrm{B}b,\, \mathrm{U}a,\, \mathrm{U}b\} \ . \tag{3}$$

Using these prerequisties, we can now start to approach the core concepts of PWS. For example, it becomes very natural to conceive a "world" as some combination of the atomic statements, in the sense that in each world, some statements from P are true, some statements from P are false and none of the statements in P is undetermined.

Definition 4: (*Welt*) Consider a set of statements w. Then, this set is refered to as a world relative to P, if for each statement in P, either the statement or its negation is contained in w. Put differently, w is a world, if there exists a subset of P, such that w is the union of all statements in this subset and the negation of all statements from P not in this subset:

$$\mathrm{W}_P w \leftrightarrow \exists x \left[x \subseteq P \wedge (w = \{p \,|\, p \in x\} \cup \{\neg p \,|\, p \in P \backslash x\}) \right] \ . \tag{4}$$

Remark: To avoid complicated notation, we will simply write $\mathrm{W}w$ or $\mathrm{W}(w)$ instead of $\mathrm{W}_P w$.

Example: Given the set of propositions P specified in (3),

$$\begin{array}{ll} w' = \{\neg \mathrm{B}a,\, \neg \mathrm{B}b,\, \mathrm{U}a,\, \mathrm{U}b\} & \text{is a world whereas} \\ w'' = \{\neg \mathrm{B}a,\, \neg \mathrm{U}a\} & \text{is not a world.} \end{array} \tag{5}$$

As we will see later (see Definition 8), if w is a world, this does not yet imply that w is a possible world. Consequently, when we say that "there exists a world such that ...", this is not taken to mean that this world exists as a possible world. The only thing that is entailed by Definition 4 is completeness: Each world is a complete description of reality and no statement in P is left undecided regarding its truth value.

Corollary: Given a set of statements P, the number of worlds composable from P is 2^n where n is the number of statements in P.

Proof: This is because each subset of P corresponds to a world and the number of subsets of P – or, equivalently: the number of elements in the power set of P – is $2^{|P|}$, where $|P|$ is the cardinality of P.

How else can a world according to Definition 4 be described? Obviously, a world w is completely described by the conjunction of all statements which hold in w.

Definition 5: (*Weltsatz*) Consider a world w. Then, the single statement $S(w)$ describing this world is given by the conjunction of all statements in w:

$$S(w) = \bigwedge_{p \in w} p \ . \tag{6}$$

Remark: Although the function $S(w)$ is also defined for non-worlds, it will in most cases only be applied to sets of statements satisfying Definition 4.

Example: For the world w' specified in (5), the world sentence is

$$S(w') = \neg \mathrm{B}a \wedge \neg \mathrm{B}b \wedge \mathrm{U}a \wedge \mathrm{U}b \ . \tag{7}$$

This definition mirrors the famous dictum that the "world is the totality of facts, not of things"[10] or that the "world is everything that is the case"[11]. Thus, a world sentence is nothing else than a reiteration of the states of affair[12], such that $S(w)$ can be seen as the complete statement of affairs in w.
There is a special world which is referred to as "the actual world" or "the real world"[13]. The actual world differs from other worlds by the fact that every statement which holds in this world, is actually true.

Definition 6: (*aktuale Welt*) The world @ for which $S(@)$ is a true statement is called the actual world:

$$\mathrm{W}w \wedge S(w) \rightarrow w = @ \ . \tag{8}$$

Remark: We here employ the symbol "@" to refer to the actual world, in order not to confuse it with the constant "a" also used in the present context (refering to Bertrand Russell, see example below) and in line with common notation in possible world theory[14]. To avoid complicated notation, we later replace "@" by "a" (see Section 3ff.).

Example: For simplicity, let us assume that statements made by plugging a and b into Bx and Ux describe the world in the year 1925. Then, the actual world would be given by

$$@ = \{\neg\mathrm{B}a,\ \mathrm{B}b,\ \neg\mathrm{U}a,\ \mathrm{U}b\} \ , \tag{9}$$

because Bertrand Russell was married to Dora Black in 1925 and Wittgenstein was unmarried throughout his life.

The concepts which we have defined so far provide no means of labeling a world as "possible" or "impossible". To do this, one requires an understanding of what makes a world a possible world. According to the proposal made here, this amounts to the world sentence of a possible world being consistent with other statements in some way. Therefore, before we define "possibility", we provide a definition of "consistency".

Definition 7: (*Vereinbarkeit*) Let C be a set of conceptual laws and let p be a proposition, formulated in L_I^S. Then, p is consistent with C, if the negation of p is not a logical consequence of C in PL1:

$$\mathrm{V}(C, p) \leftrightarrow \neg\left[C \models_{\mathrm{PL}} \neg p\right] \ . \tag{10}$$

[10] Wittgenstein (1922), 1.1.
[11] Wittgenstein (1922), 1.
[12] Cf. Wittgenstein (1922), 2.01.
[13] Cf. Beckermann (2001), p. 206
[14] Cf. Menzel (2016), §1.3

Example: Given the conceptual laws C specified in (2),

$$\begin{array}{ll} \mathrm{V}(C,\ \mathrm{B}a \wedge \mathrm{U}a) & \text{is true and} \\ \mathrm{V}(C,\ \mathrm{B}b \wedge \neg\mathrm{U}b) & \text{is false,} \end{array} \tag{11}$$

because the negation of $(\mathrm{B}a \wedge \mathrm{U}a)$ is not a logical consequence from the conceptual law $\forall x\,(\mathrm{B}x \rightarrow \mathrm{U}x)$, but the negation of $(\mathrm{B}b \wedge \neg\mathrm{U}b)$ logically follows from this rule, as implied by the meaning of '$\forall$' and '$\rightarrow$'.

Note that we here define consistency with recourse to "logical consequence" ($\models_{\mathrm{PL}}$). If consistency is conceptualized in first-order logic, it might equally well be defined with recourse to the provability relation ($\vdash_{\mathrm{PL}}$), because the completeness of first-order logic implies that everything which can be proven in PL1 is also a logical consequence in PL1.[15] When possible worlds are defined using the consistency relation (see Definition 8) and the laws of logic are intended to extend that which can be expressed in PL1, the logical consequence relation in (10) may also be replaced with the provability relation of another logical system (i.e. $\vdash_{\mathrm{XYZ}}$).
Furthermore, if consistency is defined via logical consequence in PL1, this has the nice consequence that, even if C is chosen to be the empty set ($C = \emptyset$), all logical truths are still logical consequences of C. Thus, contradictory statements are automatically inconsistent with any set C, because the negation of a contradiction p is a logical truth and logical truths are logical consequences of any set C, making $V(C, p)$ false by Definition 7. With our next Definition 8, this also implies that there are no possible worlds in which statements are true that are in conflict with logical laws such as the law of non-contradiction (LNC) or the law of the excluded middle (LEM), because LNC and LEM follow in PL1 without premises.
With that, we have finally gathered the conceptual material necessary to formally define the term "possible world".

Definition 8: (*mögliche Welt*) Let C be a set of conceptual laws (see Definition 2) and consider a set of statements w. Then, w is a possible world relative to C, if w is a world (see Definition 4) and the world sentence of w (see Definition 5) is consistent with C (see Definition 7):

$$\mathrm{M}_C w \leftrightarrow \mathrm{W}w \wedge \mathrm{V}\,[C,\ S(w)]\ . \tag{12}$$

Remark: To avoid complicated notation, we will also write $\mathrm{M}w$ or $\mathrm{M}(w)$ instead of $\mathrm{M}_C w$.

[15]Cf. Beckermann (2001), Beckermann, chs. 26, 27.2; Barwise & Etchemendy (2005), vol. II, ch. 19.1.

Example: Continuing the example from (2) and (3),

$$\begin{aligned} w' &= \{\neg\mathrm{B}a,\ \neg\mathrm{B}b,\ \mathrm{U}a,\ \mathrm{U}b\} \quad \text{is a possible world whereas} \\ w'' &= \{\neg\mathrm{B}a,\ \mathrm{B}b,\ \mathrm{U}a,\ \neg\mathrm{U}b\} \quad \text{is an impossible world,} \end{aligned} \tag{13}$$

because as stated in (11), $(\mathrm{B}b \wedge \neg\mathrm{U}b)$ violates the general law that $\forall x\,(\mathrm{B}x \rightarrow \mathrm{U}x)$, such that $S(w'')$ is inconsistent with C.[16]

Note that the negation of Mw does not imply that w is an impossible world, because $\neg$Mw is also the case, if $\neg$Ww is the case. Therefore, $\neg$Mw is not to be understood as "impossible world", but "not a world or not possible"; rather, "impossible world" should be expressed as $\mathrm{W}w \wedge \neg\mathrm{V}\,[C,\, S(w)]$.
Taken together, the definitions of PWS imply the following ontology of worlds:

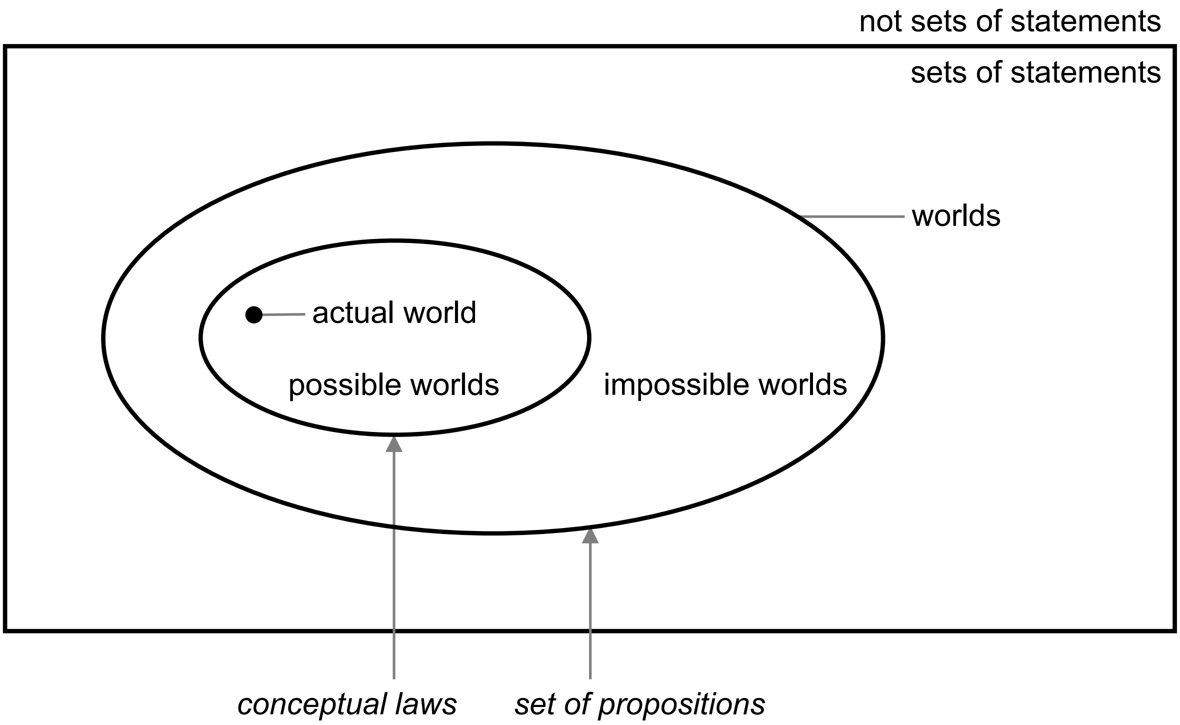

Figure 1: *Ontology of possible world semantics.* The theory considers all sets of statements (large box). A set of propositions P defines some of these sets as "worlds" (larger ellipse). A set of conceptual laws C defines some of these worlds as "possible worlds" (smaller ellipse). One of these possible worlds is the "actual world" (black dot).

[16]By convention, a statement of the form Pa is false in a world w, if the constant a does not refer in w. Thus, in a world in which Bertrand Russell and Ludwig Wittgenstein do not exist, the statements Ba, Bb, Ua, Ub would all be false. This avoids that statements are neither true nor false which would violate the definition of worldness (see Definition 4). It also ensures that the conceptual law $\forall x\,(\mathrm{B}x \rightarrow \mathrm{U}x)$ is not invalidated by the existence of a possible world in which no B's and U's exist. For a discussion of the conditions that an individual a exists in world w, cf. Menzel (2016), ch. 2.

3 Modal-Logical Definitions

Although possible worlds are used for a variety of purposes, esp. in philosophy of mind[17], their most natural application is to provide for an interpretation of terms such as "possible" or "necessary". Specifically, PWS allows to clarify the meaning of several logical or modal-logical terms as follows[18]:

> **Definition 9:** Let a set of statements P define a set of worlds and let a set of possible worlds be constrained by conceptual laws C. Consider a statement p which is contained in P. Then,
>
> - p is called "true", if p is the case in the actual world;
> - p is called "false", if $\neg p$ is the case in the actual world;
> - p is called "possible", if p is true in at least one possible world;
> - p is called "unncessary", if p is false in at least one possible world;
> - p is called "necessary", if p is true in all possible worlds;
> - p is called "impossible", if p is false in all possible worlds;
> - p is called "contingent", if p is true in at least one possible world and false in at least one possible world;
> - C is called "inconsistent", if a contradiction can be derived from C.
>
> **Remark:** Note that sometimes, contingent statements are understood as statements which are true in the actual world, but not necessarily true. The present terminology also allows statements which are false in the actual world, but not necessarily false, to be contingent statements.

After introducing the necessary formal apparatus, we will proof sufficient conditions for each of these statements (see Section 5) which are summarized in the following table[19]:

Logical statement	**PWS expression**	**Sufficient condition**
"p is true"	$p \in a$	p
"p is false"	$\neg p \in a$	$\neg p$
"p is possible"	$\exists x\,(\mathrm{M}x \wedge p \in x)$	$\mathrm{V}(C,\, p)$
"p is unnecessary"	$\exists x\,(\mathrm{M}x \wedge \neg p \in x)$	$\mathrm{V}(C,\, \neg p)$
"p is necessary"	$\forall x\,(\mathrm{M}x \rightarrow p \in x)$	$\neg\mathrm{V}(C,\, \neg p)$
"p is impossible"	$\forall x\,(\mathrm{M}x \rightarrow \neg p \in x)$	$\neg\mathrm{V}(C,\, p)$
"p is contingent"	$\exists x\,(\mathrm{M}x \wedge p \in x) \wedge \exists x\,(\mathrm{M}x \wedge \neg p \in x)$	$\mathrm{V}(C,\, p) \wedge \mathrm{V}(C,\, \neg p)$
"C is inconsistent"	$\mathrm{V}(C,\, \bot)$	$\neg\mathrm{V}(C,\, \neg p) \wedge \neg\mathrm{V}(C,\, p)$

Table 1: *(Modal-)logical concepts and their PWS analogues.* A list of assertions that can be made about single statements, along with formulations in PWS (see Section 2) and sufficient conditions in CC (see Section 5).

[17] Cf. Beckermann (2001), pp. 129ff., 206ff., 354ff.
[18] Cf. Lewis (1986); Menzel (2016), §1.2; Baehr (2003), ch. 3.
[19] In order to avoid complicated notation, we refer to the actual world as "a" from here on.

4 Construction Calculus

In this section, we will introduce a calculus for possible world semantics which consists of laws for consistency and possibility and thus allows to derive statements about possible worlds from statements about consistent propositions. The goal of this is to link quantified statements about possible worlds – e.g. $\exists x \, (\mathrm{M}x \wedge p \in x)$ – with their sufficient conditions in PWS – in this case: $\mathrm{V}(C,\, p)$ (see Table 1).

We will introduce this calculus in the form of axioms and rules. *Axioms* are statements which can be introduced without precondition at any time within an argument. *Rules* are statements which allow transitioning from respective antecedents (statements before '$\rightarrow$') to consequences (statements after '$\rightarrow$').
In axiomatization of a theory, the axioms are intended to be *self-evident* statements neither capable of nor requiring justification. Therefore, in what follows, the substantiation of those axioms will be kept rather short.

First, we introduce two axioms dealing with the function $S(w)$:

Definition: (*Axiome der Wahrheit*) The axioms of truth are:

- S1: $S(w) \leftrightarrow \bigwedge_{q \in w} q$
- S2: $S(a)$

S1 is equivalent to Definition 5. S2 reflects the fact that everything in the actual world is the case, so everything in the actual world can be assumed without precondition.

Second, we introduce two axioms dealing with the predicate $\mathrm{M}w$:

Definition: (*Axiome der Möglichkeit*) The axioms of possibility are:

- M1: $\mathrm{M}w \leftrightarrow \mathrm{W}w \wedge \mathrm{V}\,[C,\, S(w)]$
- M2: $\mathrm{M}a$

M1 is identical to Definition 8. M2 reflects the common assumption that reality implies possibility, such that the actual world is also a possible world without precondition.

Next, we introduce two rules dealing with the predicate $\mathrm{W}w$:

Definition: (*Regeln der Weltheit*) The rules of worldness are:

- W1: $\mathrm{W}w \wedge p \in P \leftrightarrow [p \in w \leftrightarrow \neg\,(\neg p \in w)]$
- W2: $Q \subseteq P \rightarrow \mathrm{W}\left(\bigcup_{q \in Q} q \;\cup\; \bigcup_{q \in P \setminus Q} \neg q\right)$

W1 says that, if w is a world relative to P and p is a statement from P, then either p or $\neg p$ must be the case in w. This follows from the completeness requirement of the worldness property. W2 says that, if Q is a subset of P, then the union of q for all statements in this subset and $\neg q$ for all remaining statements is a world. In principle, this is a reformulation of Definition 4.

Finally, we introduce two rules dealing with the predicate $\mathrm{V}[C,\, p]$:

Definition: (*Regeln der Vereinbarkeit*) The rules of consistency are:

- V1: $\mathrm{V}[C,\, p] \wedge [p \rightarrow q] \rightarrow \mathrm{V}[C,\, q]$
- V2: $\mathrm{V}[C,\, p \vee q] \rightarrow \mathrm{V}[C,\, p] \vee \mathrm{V}[C,\, q]$

V1 expresses a fundamental property of logical consistency: If a statement is consistent with C and this statement implies another statement, then this second statement is also consistent with C. V2 expresses another another fundamental property of logical consistency: If a disjunction is consistent with C, then this implies that at least one statement of the disjunction must be consistent with C.

When we refer to CC as a "calculus", this does not entail that it is the smallest possible set from which statements in PWS can be derived. For example, W1 and W2 may be deduced from a single first principle (likewise, V1 and V2). Rather, we intend to provide a system of natural deduction for PWS in which intuitive modal statements can be "recovered" by proving them along straightforward lines of reasoning.
This system is referred to as "construction calculus", because – given suitable premises, e.g. $\mathrm{V}(C,\, p)$ – it allows to *construct* (the existence of) possible worlds (see Section 5 and Appendix C) satisfying constraints entailed by those premises.

Generally, axioms and rules are chosen in a way, such that proofs using CC can be conducted solely based on first-order logic (see Section 5), without any further set-theoretic theorems. Within those proofs, axioms can be introduced without precondition while rules allow transitioning from some statements to others.
As a simple example of the calculus in action, we here prove the intuitive statement "If something is the case in the actual world, then it must be consistent with the conceptual laws". This proceeds by transitioning from the PWS expression of "p holds in a" to the PWS expression of "p is consistent with C".

Theorem: Let there be a statement p that is contained in the actual world a.
Then, p is consistent with the conceptual laws C that define possible worlds.

Proof:

1	$p \in a$	
2	$\mathrm{M}a \leftrightarrow \mathrm{W}a \wedge \mathrm{V}[C,\, S(a)]$	M1
3	$\mathrm{M}a$	M2
4	$\mathrm{W}a \wedge \mathrm{V}[C,\, S(a)]$	$\leftrightarrow$Elim: 2,3
5	$\mathrm{V}[C,\, S(a)]$	$\wedge$Elim: 4
6	$\quad S(a)$	
7	$\quad S(a) \leftrightarrow \bigwedge_{q\in a} q$	S1
8	$\quad \bigwedge_{q\in a} q$	$\leftrightarrow$Elim: 7,6
9	$\quad p$	$\wedge$Elim: 8,1
10	$\mathrm{V}[C,\, p]$	V1: 5,6,9

5 Modal-Logical Theorems

Based on the construction calculus, we will now prove modal-logical theorems, making connections between statements about possible worlds and statements about consistency as their underlying preconditions. These theorems fall into four categories: (i) non-modal theorems (Theorem 1 & 2); (ii) existence theorems (Theorem 5 & 6); (iii) universal theorems (Theorem 3 & 4); and (iv) C theorems (Theorem 7 & 8).

First, we will prove the trivial results that, if p is true, then p holds in the actual world, and that, if p is false, then $\neg p$ holds in the actual world.

Theorem 1: If $p \in P$ is a true statement, then p holds in the actual world.

Informal Proof: By combining S1 and S2, we can deduce that the conjunction $\bigwedge_{q \in a} q$ is true. Let us assume that $\neg p$ holds in a. Then, we could deduce $\neg p$ from the conjunction. This contradicts with the premise, therefore $\neg p$ does not hold in a. Because a is a world (from M1 and M2), this implies, together with W1, that p must hold in a.

Formal Proof: see Appendix A.

Theorem 2: If $p \in P$ is a false statement, then $\neg p$ holds in the actual world.

Informal Proof: By combining S1 and S2, we can deduce that the conjunction $\bigwedge_{q \in a} q$ is true. Let us assume that p holds in a. Then, we could deduce p from the conjunction. This contradicts with the premise, therefore p does not hold in a. Because a is a world (from M1 and M2), this implies, together with W1, that $\neg p$ must hold in a.

Formal Proof: see Appendix B.

Next, consider the cases in which either $\mathrm{V}(C,\, p)$ or $\mathrm{V}(C,\, \neg p)$. This allows to demonstrate that, if p is consistent with C, then p is possible under C, and that, if $\neg p$ is consistent with C, then p is unnecessary under C.

Theorem 3: Let there be a statement $p \in P$, such that p is consistent with C. Then, there exists a possible world, such that p holds in this world.

Informal Proof: Let us assume that p and consider all statements in P except for p. By the extended law of the excluded middle[20] (ELEM), some combination of those statements and their negations must be true. Therefore, the conjunction of some combination of those statements and their negations and p must be true. Due to W2, each of those conjunctions is the world sentence $S(w)$ of a world for which $p \in w$. By applying V1, the disjunction of all those world sentences is consistent with C. Thus, following V2, at least one of these world sentences must be consistent with C. Thus, at least one of those sentences describes a possible world. Because p holds in all of these worlds, there exists at least one possible world in which p is true.

Formal Proof: see Appendix C.

[20] The extended law of the excluded middle states that some consistent combination from a set of statements P and their negations $\overline{P} = \{\neg p \mid p \in P\}$ must be true (see Appendix 0).

Theorem 4: Let there be a statement $p \in P$, such that $\neg p$ is consistent with C. Then, there exists a possible world, such that $\neg p$ holds in this world.

Informal Proof: Let us assume that $\neg p$ and consider all statements in P except for p. By the extended law of the excluded middle (ELEM), some combination of those statements and their negations must be true. Therefore, the conjunction of some combination of those statements and their negations and $\neg p$ must be true. Due to W2, each of those conjunctions is the world sentence $S(w)$ of a world for which $\neg p \in w$. By applying V1, the disjunction of all those world sentences is consistent with C. Thus, following V2, at least one of these world sentences must be consistent with C. Thus, at least one of those sentences describes a possible world. Because $\neg p$ holds in all of these worlds, there exists at least one possible world in which p is false.

Formal Proof: see Appendix D.

Then, consider the cases in which either $\neg \mathrm{V}(C, \neg p)$ or $\neg \mathrm{V}(C, p)$. This allows to demonstrate that, if $\neg p$ is not consistent with C, then p is necessary under C, and that, if p is not consistent with C, then p is impossible under C.

Theorem 5: Let there be a statement $p \in P$, such that $\neg p$ is inconsistent with C. Then, p holds in all possible worlds.

Informal Proof: Consider a possible world w. According to M1, w is a world and its world sentence $S(w)$ is compatible with C. According to S1, $S(w)$ is equivalent to $\bigwedge_{q \in w} q$. Let us assume that $\neg p$ holds in w. Then, $S(w)$ would imply $\neg p$. Applying V1, it would then follow that $\neg p$ is consistent with C. This contradicts with the premise, therefore $\neg p$ does not holds in w. Because w is a world, this implies, together with W1, that p must hold in w. Thus, p must be true in all possible worlds.

Formal Proof: see Appendix E.

Theorem 6: Let there be a statement $p \in P$, such that p is inconsistent with C. Then, $\neg p$ holds in all possible worlds.

Informal Proof: Consider a possible world w. According to M1, w is a world and its world sentence $S(w)$ is compatible with C. According to S1, $S(w)$ is equivalent to $\bigwedge_{q \in w} q$. Let us assume that p holds in w. Then, $S(w)$ would imply p. Applying V1, it would then follow that p is consistent with C. This contradicts with the premise, therefore p does not holds in w. Because w is a world, this implies, together with W1, that $\neg p$ must hold in w. Thus, p must be false in all possible worlds.

Formal Proof: see Appendix F.

Finally, we turn to the remaining combinations: If p and $\neg p$ are both consistent with C, then p is contingent under C. And if p and $\neg p$ are both not consistent with C, then C may be called inconsistent.

Theorem 7: Let there be a statement $p \in P$, such that both p and $\neg p$ are consistent with C. Then, there is a possible world in which p holds and there is a possible world in which $\neg p$ worlds.

Informal Proof: This follows Theorem 3 and Theorem 4.

Formal Proof: see Appendix G.

Theorem 8: Let there be a statement $p \in P$, such that both p and $\neg p$ are inconsistent with C. Then, a contradiction can be derived from C.

Informal Proof: Based on the premises, Theorem 5 implies that p holds in all possible worlds and Theorem 6 implies that $\neg p$ holds in all possible worlds. The actual world is a possible world (M2). Thus, its world sentence $S(a)$ is consistent with C (M1). Because p and $\neg p$ are both part of the actual world, $S(a)$ implies p and $\neg p$ (S1). Thus, the contradition is consistent with C (V1).

Formal Proof: see Appendix H.

Taken together, the results obtained in this section can be visualized as follows:

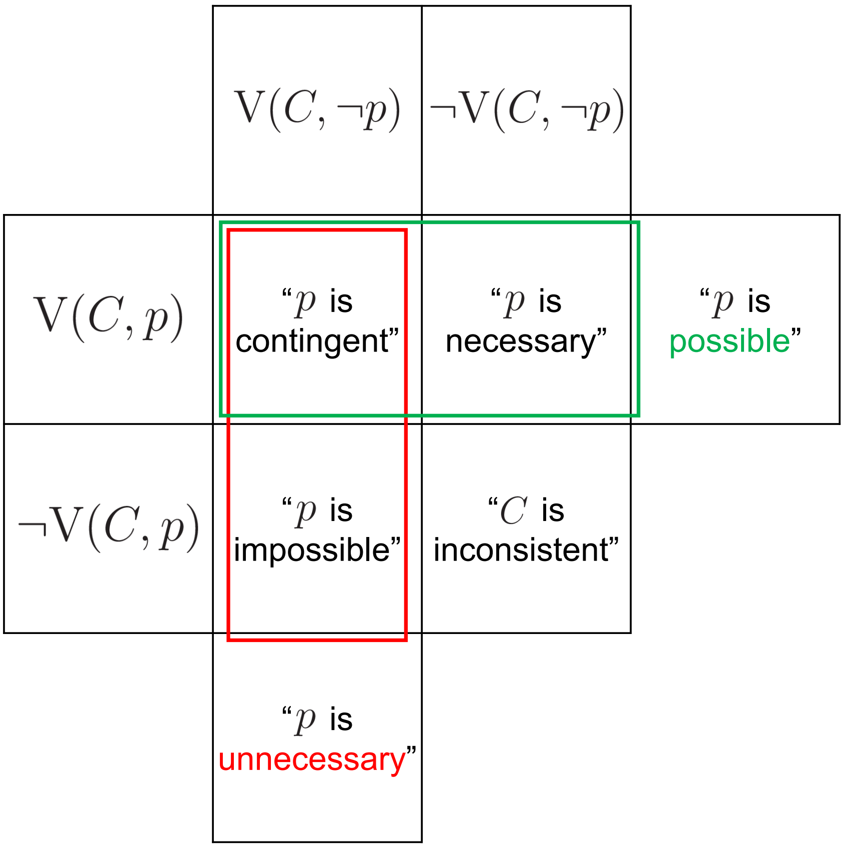

Figure 2: *Summary of modal-logical theorems.* The diagram clarifies relationships between consistency, represented by the predicate $\mathrm{V}(C, p)$, and modal-logical properties such as "possible" and "necessary". A statement is possible, if it is contingent or necessary (green box), and a statement is unnecessary, if it is contingent or impossible (red box). If p and $\neg p$ are both not consistent with C, then C must be inconsistent (bottom right).

6 Discussion

We have outlined possible world semantics (PWS) and the construction calculus (CC), a theory and a calculus of possible worlds that attempts to ground the concepts of possibility and necessity in terms of logical consistency. In this theory, worlds are seen as sets of statements and one world is a possible world, if the conjunction of all its statements is consistent with a predefined set of conceptual laws. Since logical consistency can be well captured in predicate-logical terms, this allows to formulate and formally prove non-modal presuppositions of modal assertions.

6.1 On the explanatory power of PWS

What does it mean to say that something is "possible"? In philosophical deliberation, something being possible is often equated with it being "conceivable", "thinkable" or "sayable". However, this only replaces one dubious concept with another dubious concept – these notions are, although not in conflict with the present proposal, mere interpretaions of what is really behind the concept of possibility.

According to the proposal made here, something is "possible" (or "conceivable"), if it does not violate conceptual laws that express themselves in our way of talking, e.g. the conceptual law "All bachelors are unmarried". It is *possible* that Ludwig Wittgenstein is unmarried and not a bachelor (e.g. because he is divorced). It is *not possible* that Bertrand Russell is not unmarried and a bachelor (i.e. a married bachelor). It is *necessary* that all those who are bachelors are unmarried. All these modal-logical truths are implied by the rules for applying the words "bachelor" and "married".[21]

PWS captures these intuitions about the inherent relationship between consistency and possibility. There is no other way possible ("conceivable") than a bachelor being unmarried, because if they were not unmarried, they would not count as bachelors anymore. All the ways that are possible are possible worlds.

PWS, in conjunction with CC, is not only capable of proving the relationships between consistency and possibility discussed in this manuscript (see Appendix), but is powerful enough to also derive, among others, the following statements:

- If a statement is necessary, it is also true (necessity implies truth).
- If a statement is true, it is also possible (truth implies possibility).
- If a statement is impossible, it also false (impossibility implies falsehood).
- If a statement is false, it is also not necessary (falsehood implies non-necessity).
- If there is a statement p consistent with C which does not hold in a, then there is at least one possible world other than the actual world.
- If there is a statement $\neg p$ consistent with C, but p holds in a, then there is at least one possible world other than the actual world.

Proofs of these theorems are left to the interested reader, as are the inverse versions of the relationships considered in the main part[22] (see Section 5).

[21] It is, of course, true that, in some other possible world, the *word* "bachelor" may be used for married people and married men may thus be *called* "bachelors". This would, however, not change the fact that – from *our* perspective – these married men would – according to *our* conceptual laws – still be "non-bachelors"; cf. Kripke (1980), e.g. p. 62, fn. 25.

[22] Note that the inverse version of the theorem on p. 10 does not hold true.

6.2 Primacy of propositions over worlds

Concretism is the view that all possible worlds are equally concrete and real. The term "actual world" is simply an indexical term whose extension is specified by the context in which it is used. Modal talk can be fully understood in terms of a non-modal first-order logic by explicit quantification over worlds. Then, a proposition p can be understood as the set of worlds in which it holds, $\{w \mid p \in w\}$.[23,24]

PWS agrees with concretism that "actual world" is an indexical term. If the word is uttered by an individual in a possible world other than the actual world, it refers to the world in which it is uttered. It also agrees with concretism in that modality is a reducible category in our language, to be replaced by quantification over (possible) worlds, as this is very much in line with how possible worlds are used to clarify the meaning of the modal operators "possible" and "necessary" (cf. Definition 9).

However, PWS disagrees with concretism about how this reduction should take place and especially what the more fundamental category should be – worlds or propositions. Put simply, while concretism understands propositions as sets of worlds[25], PWS understands worlds as sets of propositions (see Definition 4). PWS considers propositions more fundamental and assumes that what is given, are atomic statements that can be combined into worlds as well as conceptual laws that define the space of possible worlds. Contrarily, concretism considers worlds more fundamental and assumes that what is given, are possible worlds, sets of which can be construed as propositions.

This however comes at the cost of "not [having] a complete mechanism for proving which particular sets do or do not exist"[26] PWS explains this: It delineates possible from non-possible (i.e. impossible) worlds by rooting possibility in consistency. Given a set of conceptual laws, the set of possible worls is uniquely specified.

6.3 Possible worlds as thought devices

Abstractionalism is the view that possible worlds are abstract entities, e.g. states of affair or conditions of the world, reflecting ways how things are or could be. Possible worlds are alternatively defined as total and consistent sets that, for every proposition p, either contain p or $\neg p$ [27], or as maximal propositions that, for every proposition p, either imply p or $\neg p$ [28]. The actual world is simply the world for which this proposition is true. States of affair in the actual world actually happen to obtain. In contrast, a possible world is a set of states of affair that possibly obtains.[29,30]

PWS agrees with abstractionalism that possible worlds "are abstract entities of a certain sort"[31], i.e. not located in space or time, rather than concretely existing realities. The above "consistent sets" can be equated with "worlds" (cf. Definition 4) and the above

[23]Cf. Menzel (2016), §§ 2.1.1-2.1.3.
[24]See e.g. Lewis (1968) and Lewis (1986).
[25]Cf. Simons (2005), p. 6; Kratzer (2012), p. 10.
[26]Menzel (2016), § 2.1.4.
[27]Cf. Adams (1974).
[28]Cf. Fine (1977) and Prior(1977).
[29]Cf. Menzel (2016), §§ 2.2.1-2.2.2.
[30]See e.g. Plantinga (1974) and Plantinga (1976).
[31]Menzel (2016), § 2.2.

“maximal propositions” can be equated with “world statements” (cf. Definition 5). Possible worlds should be considered as mere “thought devices” – purely linguistic objects, used in a technical way to clarify statements as (not) violating conceptual laws. For example, saying that “bachelors are unmarried in all possible worlds” expresses that bachelors are conceptually identically to unmarried men; and saying that “there is no possible world in which pain is not identical with the firing of C-fibers” expresses that instances of pain are conceptually identical with instances of C-fiber firing.
PWS also agrees with abstractionalism on *actualism*: The actual world is everything that exists, because something can only be real as a concrete entity, if it can be located in space and time. This is not in contradiction with our statements about the “existence” of possible worlds, as possible worlds are abstract entities and thus do not require to be locatable in space and time in order to exist. Consequently, when we say that “there exists a world” (cf. Definition 4) or “there exists a possible world” (cf. Definition 8), this is to be purely understood in the abstract sense, similar to saying “there is a prime number larger than 100” or “there are solutions to $0 = x^2 - 5x + 6$”.
PWS however diagrees with abstractionalism in that the latter is a non-reductive account of modality. It presupposes modal assertions and offers no way to formally derive them from more basic properties. PWS does this: It defines possibility by consistency, offering the possibility of ultimately replacing modal operators such as $\Box$ and $\Diamond$.

6.4 The world as the totality of facts

Combinatorialism is the view that possible worlds are recombinations of metaphysically simple entities such as atomic facts, singular states of affair or elementary propositions. Combinatorialism explicitly draws inspiration from Wittgenstein’s *Tractatus Logico-Philosophicus*[32] and defines the (actual) world as the (logical) sum of all atomic facts and a possible world as a recombination of the actual world.[33,34]
PWS agrees with combinatorialism that worlds (but not possible worlds!) should be construed as recombinations from a set of atomic statements (cf. Definition 4). In fact, we are alluding to Wittgensteinian ontology (see p. 5), especially by considering the world as the totality of facts, not the totality of individuals or objects.
PWS also agrees with combinatiorialism in the attempt to provide a reduction of modality to non-modal notions. However, combinationatorialism requires a worked-out ontology of metaphysically simple entities and its reference to recombinations of the actual world fails “to account for (...) the intuition that there could have been other things”[35] than those which actually are. PWS achieves this: By using logically atomic statements as the basis for world construction, there is no need for ontological committments towards a fixed set of physical entities that happen to exist in the actual world. Instead, PWS argues for a purely linguistic account in which possible worlds acquire their possibility through logical consistency with the conceptual laws that govern our ordinary language.

[32]Cf. Skyrms (1981).
[33]Cf. Menzel (2016), §§ 2.3.1-2.3.2.
[34]See e.g. Armstrong (1986) and Armstrong (1989).
[35]Menzel (2016), § 2.3.7.

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8 Appendix

In this appendix, we provide formal proofs for the theorems demonstrated informally in the main text (see Section 5). These proofs use Fitch notation[36,37] where each step either refers to an introduction and elimination rule for logical connectives and quantifier symbols[38] or to an axiom or rule from the construction calculus (see Section 4).

0 The extended law of the excluded middle

The law of the excluded middle states that, for any statement p, we have[39]

$$\text{(LEM)} \quad p \vee \neg p \,.$$

This can be proven using the Fitch system as follows:

1		
2	$\neg(p \vee \neg p)$	
3	p	
4	$p \vee \neg p$	$\vee$Intro: 3
5	$\perp$	$\perp$Intro: 4,2
6	$\neg p$	$\neg$Intro: 3,5
7	$p \vee \neg p$	$\vee$Intro: 6
8	$\perp$	$\perp$Intro: 7,2
9	$\neg\neg(p \vee \neg p)$	$\neg$Intro: 2,8
10	$p \vee \neg p$	$\neg$Elim: 9

The extended law of excluded middle states that, for any set of statements P, we have

$$\text{(ELEM)} \quad \bigvee_{Q \subseteq P} \left(\bigwedge_{q \in Q} q \;\wedge \bigwedge_{q \in P \setminus Q} \neg q \right) \,.$$

[36]Cf. Barwise & Etchemendy (2005), vol. I, ch. 6ff.
[37]Cf. Kocurek (2019) for writing down formal proofs in LaTeX documents.
[38]Cf. Barwise & Etchemendy (2005), vol. I, pp. 375-378; vol. II, pp. 207-210.
[39]Cf. Cooper (1978), p. 161.

Based on LEM, this can be derived as follows:

1	$P = \{p_1, \ldots, p_n\}$	
2	$p_1 \vee \neg p_1$	LEM
3	p_1	
4	$\vdots$	
5	$p_n \vee \neg p_n$	LEM
6	p_n	
7	$p_1 \wedge \ldots \wedge p_n$	$\wedge$Intro: 3,. . .,6
8	$\ldots$ $p_1 \wedge \ldots \wedge p_n \vee \bigvee_{Q \subseteq P,\, Q \neq \{p_1, \ldots, p_n\}} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Intro: 7
9	$\neg p_n$	
10	$p_1 \wedge \ldots \wedge \neg p_n$	$\wedge$Intro: 3,. . .,9
11	$p_1 \wedge \ldots \wedge \neg p_n \vee \bigvee_{Q \subseteq P,\, Q \neq \{p_1, \ldots\}} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Intro: 10
12	$\bigvee_{Q \subseteq P} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Elim: 5,8,11
13	$\bigvee_{Q \subseteq P} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Elim: 4,. . .
14	$\neg p_1$	
15	$\vdots$	
16	$p_n \vee \neg p_n$	LEM
17	p_n	
18	$\neg p_1 \wedge \ldots \wedge p_n$	$\wedge$Intro: 14,. . .,17
19	$\ldots$ $\neg p_1 \wedge \ldots \wedge p_n \vee \bigvee_{Q \subseteq P,\, Q \neq \{\ldots, p_n\}} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Intro: 18
20	$\neg p_n$	
21	$\neg p_1 \wedge \ldots \wedge \neg p_n$	$\wedge$Intro: 14,. . .,20
22	$\neg p_1 \wedge \ldots \wedge \neg p_n \vee \bigvee_{Q \subseteq P,\, Q \neq \{\ldots\}} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Intro: 21
23	$\bigvee_{Q \subseteq P} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Elim: 16,19,22
24	$\bigvee_{Q \subseteq P} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Elim: 15,. . .
25	$\bigvee_{Q \subseteq P} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in P \setminus Q} \neg q \right)$	$\vee$Elim: 2,13,24

A Proof of Theorem 1

1	$p \in P$	
2	p	
3	$S(a) \leftrightarrow \bigwedge_{q \in a} q$	S1
4	$S(a)$	S2
5	$\bigwedge_{q \in a} q$	$\leftrightarrow$Elim: 3,4
6	$\quad \neg p \in a$	
7	$\quad \neg p$	$\wedge$Elim: 5,6
8	$\quad \bot$	$\bot$Intro: 2,7
9	$\neg(\neg p \in a)$	$\neg$Intro: 6,8
10	$\mathrm{M}a \leftrightarrow \mathrm{W}a \wedge \mathrm{V}[C, S(a)]$	M1
11	$\mathrm{M}a$	M2
12	$\mathrm{W}a \wedge \mathrm{V}[C, S(a)]$	$\leftrightarrow$Elim: 10,11
13	$\mathrm{W}a$	$\wedge$Elim: 12
14	$p \in a \leftrightarrow \neg(\neg p \in a)$	W1: 13,1
15	$p \in a$	$\leftrightarrow$Elim: 14,9

B Proof of Theorem 2

1	$p \in P$	
2	$\neg p$	
3	$S(a) \leftrightarrow \bigwedge_{q \in a} q$	S1
4	$S(a)$	S2
5	$\bigwedge_{q \in a} q$	$\leftrightarrow$Elim: 3,4
6	$\quad p \in a$	
7	$\quad p$	$\wedge$Elim: 5,6
8	$\quad \bot$	$\bot$Intro: 7,2
9	$\neg\, (p \in a)$	$\neg$Intro: 6,8
10	$\mathrm{M}a \leftrightarrow \mathrm{W}a \wedge \mathrm{V}\,[C,\, S(a)]$	M1
11	$\mathrm{M}a$	M2
12	$\mathrm{W}a \wedge \mathrm{V}\,[C,\, S(a)]$	$\leftrightarrow$Elim: 10,11
13	$\mathrm{W}a$	$\wedge$Elim: 12
14	$p \in a \leftrightarrow \neg\, (\neg p \in a)$	W1: 13,1
15	$\quad \neg\, (\neg p \in a)$	
16	$\quad p \in a$	$\leftrightarrow$Elim: 14,15
17	$\quad \bot$	$\bot$Intro: 16,9
18	$\neg\neg\, (\neg p \in a)$	$\neg$Intro: 15,17
19	$\neg p \in a$	$\neg$Elim: 18

C Proof of Theorem 3

1	$p \in P$	
2	$\mathrm{V}[C,\, p]$	
3	$\quad p$	
4	$\quad \bigvee_{Q \subseteq P\setminus\{p\}} \left(\bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in (P\setminus\{p\})\setminus Q} \neg q \right)$	ELEM
5	$\quad\quad \boxed{Q} \subseteq P\setminus\{p\} :\; \bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in (P\setminus\{p\})\setminus Q} \neg q$	
6	$\quad\quad p \;\wedge\; \bigwedge_{q \in Q} q \;\wedge\; \bigwedge_{q \in (P\setminus\{p\})\setminus Q} \neg q$	$\wedge$Intro: 3,5
7	$\quad\quad \bigwedge_{q \in \{p\} \cup Q} q \;\wedge\; \bigwedge_{q \in (P\setminus\{p\})\setminus Q} \neg q$	Reit: 6
8	$\quad\quad S\left(\{q \mid q \in \{p\} \cup Q\} \;\cup\; \{\neg q \mid q \in (P\setminus\{p\})\setminus Q\}\right) \leftrightarrow \bigwedge_{q \in \{p\} \cup Q} q \;\wedge\; \bigwedge_{q \in (P\setminus\{p\})\setminus Q} \neg q$	S1
9	$\quad\quad S\left(\{q \mid q \in \{p\} \cup Q\} \;\cup\; \{\neg q \mid q \in (P\setminus\{p\})\setminus Q\}\right)$	$\leftrightarrow$Elim: 8,7
10	$\quad\quad S\left(\{q \mid q \in \{p\} \cup Q\} \;\cup\; \{\neg q \mid q \in (P\setminus\{p\})\setminus Q\}\right) \vee \bigvee_{R \subseteq P\setminus\{p\},\, R \neq Q} S\left(\{r \mid r \in \{p\} \cup R\} \;\cup\; \{\neg r \mid r \in (P\setminus\{p\})\setminus R\}\right)$	$\vee$Intro: 9
11	$\quad\quad \bigvee_{\mathrm{W}w,\, p \in w} S(w)$	Reit, W1: 10
12	$\quad \bigvee_{\mathrm{W}w,\, p \in w} S(w)$	$\vee$Elim: 4,5,11
13	$\mathrm{V}\left[C,\, \bigvee_{\mathrm{W}w,\, p \in w} S(w)\right]$	V1: 2,3,12
14	$\bigvee_{\mathrm{W}w,\, p \in w} \mathrm{V}\left[C,\, S(w)\right]$	V2: 13
15	$\quad \mathrm{W}\boxed{w},\; p \in \boxed{w} :\; \mathrm{V}\left[C,\, S(w)\right]$	
16	$\quad p \in w$	Reit: 15
17	$\quad \mathrm{W}w$	Reit: 15
18	$\quad \mathrm{W}w \wedge \mathrm{V}\left[C,\, S(w)\right]$	$\wedge$Intro: 17,15
19	$\quad \mathrm{M}w \leftrightarrow \mathrm{W}w \wedge \mathrm{V}\left[C,\, S(w)\right]$	M1
20	$\quad \mathrm{M}w$	$\leftrightarrow$Elim: 19,18
21	$\quad \mathrm{M}w \wedge p \in w$	$\wedge$Intro: 20,16
22	$\quad \exists x\, (\mathrm{M}x \wedge p \in x)$	$\exists$Intro: 21
23	$\exists x\, (\mathrm{M}x \wedge p \in x)$	$\vee$Elim: 14,15,22

D Proof of Theorem 4

1	$p \in P$	
2	$\mathrm{V}[C,\, \neg p]$	
3	$\neg p$	
4	$\bigvee_{Q \subseteq P \setminus \{p\}} \left(\bigwedge_{q \in Q} q \ \wedge\ \bigwedge_{q \in (P \setminus \{p\}) \setminus Q} \neg q \right)$	ELEM
5	$\boxed{Q} \subseteq P \setminus \{p\} :\ \bigwedge_{q \in Q} q \ \wedge\ \bigwedge_{q \in (P \setminus \{p\}) \setminus Q} \neg q$	
6	$\neg p \ \wedge\ \bigwedge_{q \in Q} q \ \wedge\ \bigwedge_{q \in (P \setminus \{p\}) \setminus Q} \neg q$	$\wedge$Intro: 3,5
7	$\bigwedge_{q \in Q} q \ \wedge\ \bigwedge_{q \in P \setminus Q} \neg q$	Reit: 6
8	$S\left(\{q \mid q \in Q\} \ \cup\ \{\neg q \mid q \in P \setminus Q\}\right) \leftrightarrow$ $\bigwedge_{q \in Q} q \ \wedge\ \bigwedge_{q \in P \setminus Q} \neg q$	S1
9	$S\left(\{q \mid q \in Q\} \ \cup\ \{\neg q \mid q \in P \setminus Q\}\right)$	$\leftrightarrow$Elim: 8,7
10	$S\left(\{q \mid q \in Q\} \ \cup\ \{\neg q \mid q \in P \setminus Q\}\right) \vee$ $\bigvee_{R \subseteq P \setminus \{p\},\, R \neq Q} S\left(\{r \mid r \in R\} \ \cup\ \{\neg r \mid r \in P \setminus R\}\right)$	$\vee$Intro: 9
11	$\bigvee_{\mathrm{W}w,\, \neg p \in w} S(w)$	Reit, W1: 10
12	$\bigvee_{\mathrm{W}w,\, \neg p \in w} S(w)$	$\vee$Elim: 4,5,11
13	$\mathrm{V}\left[C,\, \bigvee_{\mathrm{W}w,\, \neg p \in w} S(w)\right]$	V1: 2,3,12
14	$\bigvee_{\mathrm{W}w,\, \neg p \in w} \mathrm{V}\left[C,\, S(w)\right]$	V2: 13
15	$\mathrm{W}\boxed{w},\ \neg p \in \boxed{w} :\ \mathrm{V}\left[C,\, S(w)\right]$	
16	$\neg p \in w$	Reit: 15
17	$\mathrm{W}w$	Reit: 15
18	$\mathrm{W}w \wedge \mathrm{V}\left[C,\, S(w)\right]$	$\wedge$Intro: 17,15
19	$\mathrm{M}w \leftrightarrow \mathrm{W}w \wedge \mathrm{V}\left[C,\, S(w)\right]$	M1
20	$\mathrm{M}w$	$\leftrightarrow$Elim: 19,18
21	$\mathrm{M}w \wedge \neg p \in w$	$\wedge$Intro: 20,16
22	$\exists x\, (\mathrm{M}x \wedge \neg p \in x)$	$\exists$Intro: 21
23	$\exists x\, (\mathrm{M}x \wedge \neg p \in x)$	$\vee$Elim: 14,15,22

E Proof of Theorem 5

1	$p \in P$	
2	$\neg \mathrm{V}[C, \neg p]$	
3	$\boxed{w}\ \mathrm{M}w$	
4	$\mathrm{M}w \leftrightarrow \mathrm{W}w \wedge \mathrm{V}\,[C,\, S(w)]$	M1
5	$\mathrm{W}w \wedge \mathrm{V}\,[C,\, S(w)]$	$\leftrightarrow$Elim: 4,3
6	$\mathrm{W}w$	$\wedge$Elim: 5
7	$\mathrm{V}\,[C,\, S(w)]$	$\wedge$Elim: 5
8	$S(w) \leftrightarrow \bigwedge_{q \in w} q$	S1
9	$\neg p \in w$	
10	$S(w)$	
11	$\bigwedge_{q \in w} q$	$\leftrightarrow$Elim: 8,10
12	$\neg p$	$\wedge$Elim: 11,9
13	$\mathrm{V}\,[C,\, \neg p]$	V1: 7,10,12
14	$\perp$	$\perp$Intro: 13,2
15	$\neg\,(\neg p \in w)$	$\neg$Intro: 9,14
16	$p \in w \leftrightarrow \neg\,(\neg p \in w)$	W1: 6,1
17	$p \in w$	$\leftrightarrow$Elim: 16,15
18	$\forall x\,(\mathrm{M}x \rightarrow p \in x)$	$\forall$Intro: 3,17

F Proof of Theorem 6

1	$p \in P$	
2	$\neg \mathrm{V}[C,\, p]$	
3	$\boxed{w}\ \mathrm{M}w$	
4	$\mathrm{M}w \leftrightarrow \mathrm{W}w \wedge \mathrm{V}\left[C,\, S(w)\right]$	M1
5	$\mathrm{W}w \wedge \mathrm{V}\left[C,\, S(w)\right]$	$\leftrightarrow$Elim: 4,3
6	$\mathrm{W}w$	$\wedge$Elim: 5
7	$\mathrm{V}\left[C,\, S(w)\right]$	$\wedge$Elim: 5
8	$S(w) \leftrightarrow \bigwedge_{q \in w} q$	S1
9	$p \in w$	
10	$S(w)$	
11	$\bigwedge_{q \in w} q$	$\leftrightarrow$Elim: 8,10
12	p	$\wedge$Elim: 11,9
13	$\mathrm{V}\left[C,\, p\right]$	V1: 7,10,12
14	$\bot$	$\bot$Intro: 13,2
15	$\neg\left(p \in w\right)$	$\neg$Intro: 9,14
16	$p \in w \leftrightarrow \neg\left(\neg p \in w\right)$	W1: 6,1
17	$\neg\left(\neg p \in w\right)$	
18	$p \in w$	$\leftrightarrow$Elim: 16,17
19	$\bot$	$\bot$Intro: 18,15
20	$\neg\neg\left(\neg p \in w\right)$	$\neg$Intro: 17,19
21	$\neg p \in w$	$\neg$Elim: 20
22	$\forall x\left(\mathrm{M}x \rightarrow \neg p \in x\right)$	$\forall$Intro: 3,21

G Proof of Theorem 7

1	$p \in P$	
2	$\mathrm{V}[C,\, p]$	
3	$\mathrm{V}[C,\, \neg p]$	
4	$\exists x\, (\mathrm{M}x \wedge p \in x)$	Theorem 3: 1,2
5	$\exists x\, (\mathrm{M}x \wedge \neg p \in x)$	Theorem 4: 1,3
6	$\exists x\, (\mathrm{M}x \wedge p \in x) \wedge \exists x\, (\mathrm{M}x \wedge \neg p \in x)$	$\wedge$Intro: 4,5

H Proof of Theorem 8

1	$p \in P$	
2	$\neg\mathrm{V}[C,\, \neg p]$	
3	$\neg\mathrm{V}[C,\, p]$	
4	$\forall x\, (\mathrm{M}x \to p \in x)$	Theorem 5: 1,2
5	$\forall x\, (\mathrm{M}x \to \neg p \in x)$	Theorem 6: 1,3
6	$\mathrm{M}a \to p \in a$	$\forall$Elim: 4
7	$\mathrm{M}a \to \neg p \in a$	$\forall$Elim: 5
8	$\mathrm{M}a \leftrightarrow \mathrm{W}a \wedge \mathrm{V}\,[C,\, S(a)]$	M1
9	$\mathrm{M}a$	M2
10	$p \in a$	$\to$Elim: 6,9
11	$\neg p \in a$	$\to$Elim: 7,9
12	$\mathrm{W}a \wedge \mathrm{V}\,[C,\, S(a)]$	$\leftrightarrow$Elim: 8,9
13	$\mathrm{V}\,[C,\, S(a)]$	$\wedge$Elim: 12
14	$S(a) \leftrightarrow \bigwedge_{q \in a} q$	S1
15	$\quad S(a)$	
16	$\quad \bigwedge_{q \in a} q$	$\leftrightarrow$Elim: 14,15
17	$\quad p$	$\wedge$Elim: 16,10
18	$\quad \neg p$	$\wedge$Elim: 16,11
19	$\quad \bot$	$\bot$Intro: 17,18
20	$\mathrm{V}\,[C,\, \bot]$	V1: 13,15,19