Cont Philos Rev (2010) 42:555–572
DOI 10.1007/s11007-009-9120-2

Hegel and Deleuze on the metaphysical interpretation of the calculus

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Published online: 11 December 2009

Abstract The aim of this paper is to explore the uses made of the calculus by Gilles Deleuze and G. W. F. Hegel. I show how both Deleuze and Hegel see the calculus as providing a way of thinking outside of finite representation. For Hegel, this involves attempting to show that the foundations of the calculus cannot be thought by the finite understanding, and necessitate a move to the standpoint of infinite reason. I analyse Hegel's justification for this introduction of dialectical reason by looking at his responses to Berkeley's criticisms of the calculus. For Deleuze, instead, I show that the differential must be understood as escaping from both finite and infinite representation. By highlighting the sub-representational character of the differential in his system, I show how the differential is a key moment in Deleuze's formulation of a transcendental empiricism. I conclude by dealing with some of the common misunderstandings that occur when Deleuze is read as endorsing a modern mathematical interpretation of the calculus.

Keywords Hegel · Deleuze · Calculus · Mathematics · Representation

1 Introduction

In this paper, I want to look at the role of the differential calculus in the development of two different philosophical systems. I want to show how, prior to the development of a consistent interpretation of the foundations of the calculus in the nineteenth century by Weierstrass, differential calculus seemed to offer the possibility, and also show the necessity, of a new way of thinking about the fundamental logical principles we use to describe the world. In particular, I want to look at the way in which Hegel drew support for the dialectical method from the

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paradoxes which seemed to result from these foundational issues. Indeed, while by Hegel's time, the foundations of the calculus had been frequently attacked, most notably by Berkeley, Hegel argues that "it is the inability to justify the object [i.e. the differential coefficient *dy*/*dx*] as *Notion* which is mainly responsible for these attacks."[1] Thus for Hegel, if we are to understand the meaning of the contradictions which seemed inherent in the foundations of the differential calculus, we need to move to a dialectical understanding of these foundations. After having shown why Hegel believes this to be the case, I want to discuss how this metaphysical interpretation sheds light on a conflict between Hegel and the French philosopher, Gilles Deleuze, about the role of the transcendental in philosophy. In particular, I want to look at the way in which Hegel argues for the need to incorporate the infinite into the finite, while Deleuze instead argues that the calculus requires us to move to a transcendental account. Thus, while Hegel argues that the antinomical nature of the calculus entails the need for a logic capable of incorporating contradiction, Deleuze follows Kant in arguing that antinomy pushes us towards a transcendental account of the world. For Deleuze, the differential provides the model for an element that is not sensible, which he uses to ground his transcendental account of the genesis of the sensible. I will do this by looking at his own use of the calculus, where he also looks back to what he calls a "barbaric" interpretation of the calculus.

We can see the importance of the calculus for metaphysics in Bertrand Russell's early adherence to Hegelian philosophy. While the young Russell "found comfort for a time"[2] in a form of Hegelianism, this was premised on what appeared at the time to be the fundamentally contradictory nature of mathematics, and in particular, the theory of infinite numbers, and the foundations of the calculus. It was only after "Weierstrass, soon after the middle of the nineteenth century, showed how to establish the calculus without infinitesimals, and thus at last made it secure"[3] that Russell was able to apply the method of logical analysis to philosophy. This allowed Russell to reject the synthetic method of Hegel in favour of his own formal analytic method. Thus, for Russell, a dialectical understanding, not only of mathematics, but also of the world in general, was the only approach open in philosophy until the development of set-theoretical foundations of mathematics.[4] Russell's approach, with its basis in classical logic, is representative of the model that both Hegel and Deleuze try and overcome with their metaphysical interpretations of the calculus. For Hegel, the kind of approach exemplified by Russell would be classified as "finite thinking," which Hegel opposes by attempting to bring the infinite into the finite. For Deleuze, Russell's approach would be called "representation." Deleuze argues that if we are to understand how representation comes into being, the account we give must use terms that fall outside of representation. Deleuze thinks that the material for such an account can be found in the calculus. Furthermore, he believes that the calculus gives us the means to provide an alternative to Hegelian

[1] Hegel (1989, p. 254).

[2] Russell (1956, p. 21).

[3] Russell (1946, p. 783).

[4] For more on this point, and Russell's early Hegelianism, see Monk (1997).

philosophy. The dialectical approach to the calculus did not merely allow Hegel to propose a resolution to the problems of the calculus, but also allowed Hegel to provide further support for his own dialectical method. Before turning to Hegel's interpretation, however, we will first look at how the calculus itself works. We will then look at the logical difficulties with this approach, particularly as expounded by Berkeley. After we have an idea of how these difficulties occur, we will move on to the positive accounts of Hegel and Deleuze in the second half of the paper.

2 The calculus

The calculus presents one of the greatest achievements of mathematics. In this first part of the paper, I want to give a brief account of its operations. The calculus allows us to characterise the relations between quantities whose ratios to one another vary. Although it can relate any variables, I will mostly refer to the relation between time and distance travelled by a hypothetical body. In order to understand the issues resulting from the calculus' foundations, however, we will only need to look at a simple case, that is, the application of the calculus to polynomials. The basic approach of the differential calculus was first used by the fourteenth century mathematician Oresme,[5] and came with the recognition that the velocity of a body could be represented by a line on a graph. If we represent on a graph the relation between distance travelled and time taken to travel that distance by an object, it becomes possible to determine the velocity of the object by dividing the distance travelled by the time, hence velocity is measured in terms such as miles per hour, metres per second, etc. If the velocity is constant, the relation between distance and time will be proportional. This means that the line representing the moving object will be straight. If, therefore, we wish to determine the velocity of the object, we simply need to take a section of the line, and divide the distance travelled over that time, which will be represented by the change in the value on the distance axis over the length of the section, by the time, which will be represented by the change in the value of the time axis over the length of the section. As the two values are directly proportional, a section of any arbitrary length will provide the same result. Moving on from Oresme, if we are dealing with an object moving at a velocity that is not constant, then this procedure cannot be used, as we were able to determine the velocity at any point using the previous method only because the velocity was the same at every point (the average velocity is the same as the velocity at every instant). Instead, however, we can measure the velocity at any point of a system with a changing velocity by drawing a graph of the function[6] of this change, that is, of the relation of distance to time, and drawing a line which runs parallel to a particular point on the curve. This produces a vector of the velocity of the system at

[5] Boyer (1959, p. 82), although it was Archimedes who provided the first recognised anticipation of the calculus in the method of exhaustion used to find the area of a circle. Archimedes' use was concerned with static figures, rather than rates of change, however.

[6] A function consists in a domain (or range) of numbers, and a rule which associates each of these numbers with a value of another variable. Thus, $f(x) = x^2$ associates any real number, positive or negative, with a value equal to x^2.

this particular moment. The difficulty with this approach is that it can only be approximate, as we are attempting to draw a line through a point, which in itself can seemingly have no direction. The alternative, to draw a line through two points of the curve, is equally flawed, as although it gives us an accurate line, we are dealing with a curve, and so the tangent we are now drawing will not represent the velocity at one particular moment, but the average between the two points. Leibniz's solution to this difficulty was to draw a line between the point whose velocity we wish to measure and another arbitrary point on the curve, and then to imagine the distance between these two points decreasing towards zero. As we now have a straight line between these two points, we can treat the case in the same manner as the case of constant velocity described above, measuring the change in values of both axes along a length of the line. Thus mathematically, we end up with two lines, one representing the change over the section in terms of distance, and one in terms of time, neither of which on its own will have any determinate value, as the lines are infinitely short, but when divided, one by the other, will give a vector at the particular point. Since the axes of the graph can represent more than simply time and distance, these values are referred to more generally as *dy* and *dx*. Further, using Descartes' insight that a function can be represented by a graph, and a graph by a function, we can define this procedure as a function of the original formula, dispensing with the graphical method altogether.

In giving a brief outline of the method of the calculus, I shall follow Hegel's statement that "the whole method of the differential calculus is complete in the proposition that $dx^n = nx^{n-1}dx$, or $\frac{f(x+i)-fx}{i} = P$."[7] While these statements may appear intimidating, they are in fact simply algebraic representations of geometrical functions. The second equation simply represents the gradient function of a curve. Gradient, as we saw in the case of straight lines, is defined in terms of the change in the value on the y axis divided by the corresponding change on the x axis, thereby giving a ratio. Now, if we take i to be a length along the x axis, assuming that $y = f(x)$, that is, that the value of y is dependent on the value of x, the top half of the equation will give us the difference between the values of y when $y = x$ and $y = x + i$. That is, the top half of the equation will give us the corresponding value of the rise on the y axis for a run of length, i, on the x axis, which begins at any arbitrary point on that axis. We thus have a function which will provide us with the value of the tangent at any point on the line. The first equation follows from this second equation through the substitution of an infinitesimal value for i. This substitution is the equivalent of making *dy* and *dx* infinitesimal values in Leibniz's work. Although showing that the first equation is true for all values of n involves some complex mathematics, it is an easy result to prove for any specific function. We will therefore focus on one particular function, that given the equation, $y = x^2$, then $\frac{dy}{dx} = 2x$.

The first step is to rewrite the equation, $\frac{f(x+i)-fx}{i} = P$, replacing the value i with dx, the Leibnizian infinitesimal, and replacing $f(x)$ with $(x)^2$, as in this case, $y = x^2$. We thus arrive at the function:

[7] Hegel (1989, p. 274).

$$\frac{(x+dx)^2 - x^2}{dx} = \frac{dy}{dx}$$

We can now multiply out the brackets on the top half of the equation to give:

$$\frac{x^2 + 2xdx + dx^2 - x^2}{dx} = \frac{dy}{dx}$$

Which in turn reduces to:

$$\frac{2xdx + dx^2}{dx} = \frac{dy}{dx}$$

Cancelling out *dx* therefore gives us:

$$\frac{dy}{dx} = 2x + dx$$

Thus the differential of x^2 turns out to be $2x + dx$, where, given that the value *dx* is infinitesimal with respect to *x*, we can just write $\frac{dy}{dx} = 2x$ (*dx* retains a value in relation to *dy*, which is also infinitesimal, and so is retained on the left side of the equation). Thus, it is easy to show that for the value $n = 2$, Hegel's first equation follows from his second. For higher values, a similar result can be obtained with more time consuming applications of such elementary algebra. The procedure that we have just carried out is called differentiation. If we reverse this process, however, we have a procedure which allows us to find what is known as the primitive function of the derivative, i.e. the function from which the derivative is differentiated.[8] This process is known as integration. For modern set theory, the interpretation given above has given way to an interpretation in terms of the concept of limit, in order to avoid certain difficulties which emerge from the paradoxical qualities of the infinitesimal, which we will come back to below in our discussion of Hegel's theory of the calculus. In order to escape these difficulties, the modern set theoretic interpretation of the differential calculus largely ignores the individual values of *dy* and *dx*, instead giving a value simply to the ratio, *dy*/*dx*, as a whole, called a differential. This is accomplished by admitting the concept of a limit of an infinite series. An infinite series such as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\ldots$ may be said to approach the value, 1, and this allows us on the modern interpretation to equate the derivative, not with the final, infinitesimal value of the series, but instead with the limit of the series. We do not in the case of the series need to worry about the fact that any finite summation will always be less than one. In a similar way, the differential is conceived of under the set-theoretic interpretation as precisely equalling the value of the gradient at any particular point. By defining the derivative to be the limit of the ratio, rather than the values of *dy* and *dx*, questions about what happens when or whether the ratio actually reaches this limit are put out of play. This is the method of Weierstrass, which allowed Russell to move away from an antinomic interpretation

[8] The two operations are not strictly inverse, as differentiation removes any constant values from the initial function, so, for instance, if $y = f(x)$ is $y = x^2 + 1$, the first stage of the differentiation will give us: $\frac{dy}{dx} = \frac{(x+dx)^2+1-(x^2+1)}{dx}$, which is equivalent to the function derived from $y = x^2$. In integration, the constant values are therefore normally represented by the value *c*. The value of *c* can be determined if any value of the function is known, however.

of mathematics, and therefore also from his early Hegelianism. We will now turn, however, to Hegel's theory of the calculus.

3 Hegel and the calculus

The basic function of the differential calculus is to find the gradient of a curve at a particular point. The difficulty with a curve is that as the gradient varies, we need to find the gradient of a point. Gradient, however, seems to involve a difference. For Leibniz, at least in his informal account, the solution was to resort to the notion of the infinitesimal, as an infinitely small difference between two points, *dy*. As this difference was infinitely small, it could be discounted for the purposes of calculation, but, as it retained a magnitude relative to *dx*, it could be used to form a ratio, *dy*/*dx* which had a determinate value. For Deleuze, this approach forms the foundation for his interpretation of the differential calculus, although he takes advantage of later developments which finesse the paradoxes resulting from it. Hegel rejects this Leibnizian approach since the idea of neglecting infinitesimal values, whilst it may lead us to the correct results, does not give us the rigour essential to a proper mathematical proof.

> "Even if ordinary common sense in fairness allows such inexactitude, all geometricians reject this conception. It is quite obvious that in the science of mathematics there cannot be any question of such empirical accuracy; mathematical measuring by operations of the calculus or by geometrical constructions and proofs is altogether different from land-surveying, from the measuring of empirical lines, etc".[9]

In speaking of Leibniz in particular, Hegel asserts that "it is chiefly this call to neglect which, along with a gain in facility, has given the calculus the appearance of inexactitude and express inexactness in its method of procedure."[10] Although Leibniz's method gives a result which is out by only an infinitesimal amount (the gradient is given as $2x + dx$, rather than the correct value of $2x$), this error is not congruent with the standards of a formal discipline such as mathematics.[11] The target of Hegel's criticisms is especially clear in the first postulate of the calculus given in L'Hôpital's *Analyse* (1696), which provided the first general introduction to the method:

> *Postulate I.* Grant that two quantities, whose difference is an infinitely small quantity, may be taken (or used) indifferently for each other: or (which is the

[9] Hegel (1989, p. 258).

[10] Hegel (1989, p. 258).

[11] In fact, Leibniz was aware of these limitations of the infinitesimal method, and in his first published account of the calculus in 1684, he attempted to provide a proof of the calculus based on finite line segments (see OU Mathematics Team 1974, book III p. 50). It turned out, however, that this definition could not be applied to differentials occurring in the integral calculus, and so Leibniz eventually resorted to considering the differential as a useful fiction. This was at odds with the general approach of the Leibnizian school as a whole, however, and thinkers such as L'Hôpital, who attempted to secure the foundations of the calculus on the existence of infinitesimals.

> same thing) that a quantity, which is increased or decreased only by an infinitely smaller quantity, may be considered as remaining the same.[12]

It is this postulate which allows us to discount the effects of dx in the results of our derivation of the differential function, but merely by incorporating a lack of rigor into the foundations of the calculus themselves. Leibniz did not provide the only account of the foundations of the calculus, however, and so we shall instead look at Newton's interpretation.

While Leibniz was the first to publish his theory of the calculus, Newton had independently developed a form of the calculus called the method of fluxions. Newton tried to distinguish his own method of fluxions from the calculus, which were practically equivalent, through his attempt to determine a more rigorous foundation. This was the method of prime and ultimate ratios which Newton first published in the *Principia* (1687). The key to this approach was to incorporate the idea of time into his demonstration. As he writes in the *Quadrature of Curves*:

> I don't here consider Mathematical Quantities as composed of parts extreamly small, but as *generated by a continual motion*. Lines are described, and by describing are generated, not by the apposition of Parts, but by a continual motion of points. Surfaces are generated by the motion of Lines, Solids by the motion of Surfaces, Angles by the Rotation of their Legs, Time by the continual flux, and so on in the rest. These *Geneses* are founded upon Nature, and are every Day seen in the motion of Bodies.[13]

Newton uses this dynamic conception of the generation of the curve itself to posit the idea that the curve could be understood as the movement of a point with a certain velocity. With this, it is a small step to see the gradient of the point on the curve as representing the instantaneous velocity of the point itself, which we find by noting that the time it takes the point to move can be reduced to zero. Under this interpretation, the differentials, called fluxions in Newton's system, do not have to be seen as being too small to affect the result of the calculation, but can actually be seen to vanish at the limit point, when $dt = 0$:

> Quantities, and the ratios of quantities, which may in any finite time converge continually to equality, and before the end of time approach nearer to each other than by any given difference, become ultimately equal.
>
> If you deny it, suppose them to be ultimately unequal, and let *D* be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference *D;* which is contrary to the supposition.[14]

The ultimate ratio therefore functions much as the idea of a limit does in modern mathematics. If we take a normal ratio, such as $\frac{8}{64}$, if we reduce the values of the

[12] This quotation of L'Hôpital's text is taken from Struik (1986, p. 314). See OU Mathematics Team (1974, book IV, p. 10) for a fuller analysis.

[13] Newton (1964, book I, p. 141). See OU Mathematics Team (1974, book III, p. 26), for a fuller discussion of Newton's argument in the *Quadrature of Curves*.

[14] Newton (1934, p. 29).

denominator and numerator proportionately, for instance, to $\frac{4}{32}$, or $\frac{2}{16}$, we find that the ratio of terms itself remains unchanged. Newton uses the idea of an ultimate ratio to argue that at the limit case, where the numerator and denominator are reduced to $\frac{0}{0}$ (where the lines become a point), we can still interpret the point as having a determinate ratio: "And, in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of quantities, not before they vanish, nor after, but with which they vanish."[15] It is this conception of the calculus, as grounded in the idea of an ultimate ratio, which Hegel takes to characterise it most adequately, albeit with the elimination of "those determinations which belong to the idea of motion and velocity (from which, mainly, he took the name, *fluxions*) because in them the thought does not appear in its proper abstraction but as concrete and mixed with non-essential forms."[16] To see how Newton's understanding of the calculus is integrated into Hegel's dialectic, as well as why Hegel believed it to require a dialectical foundation, we will now turn to the idea of the ratio as it occurs in Hegel's remarks on the mathematical infinite in order to trace the genetic development of Hegel's own understanding of the ultimate ratio.

4 The ratio in Hegel's discussion of quantitative infinity

Hegel's analysis of the differential calculus occurs in his discussion of quantitative infinity in the *Science of Logic*. Here, he uses his two different conceptions of infinity, the true (dialectical) infinite and the spurious infinite. As we shall see, in this, the true infinite will share many of the properties of the more commonly discussed qualitative true infinite. For Hegel, the true infinite exhibits the negation and preservation of both the finite and the infinite within one whole. This overcomes the spurious infinite, which is merely the perpetual repetition of the finite. Within the ratio, both these moments appear simultaneously. If we take Hegel's example of the fraction $\frac{2}{7}$, we can represent it in the two following ways:

$$\frac{2}{7} = 0.2857142\ldots$$

On the right, we have the decimal expansion of the fraction, which in this case forms an infinite series.[17] For Hegel, this infinite series cannot be construed as a true infinity, as it just represents a bare repetition of finite values. "What the series is meant to express remains an *ought-to-be* and what it does express is burdened with a beyond which does not vanish and *differs* from what was meant to be expressed."[18] When we assert an equality between $\frac{2}{7}$ and its expansion, we have to note that while continuing the expansion of the series allows the value on the right of the equals sign to be made as accurate as required, it will always preserve a difference from the

[15] Newton (1934, p. 39).

[16] Hegel (1989, p. 255). See also Eves (1990, p. 400) on the redundancy of time in Newton's method.

[17] Hegel notes (1989, p. 247) that fractions such as $\frac{1}{10}$, which do not normally lead to an infinite sequence can be made to do so by interpreting them in another number base (i.e. instead of base 10). Thus, for instance, $\frac{1}{10}$, when interpreted in the octal system (base 8), also leads to an infinite sequence of 0.0631463….

[18] Hegel (1989, p. 248).

value on the left. This difference is the beyond which would have to be incorporated into the series to make it a representation of the true infinite, and therefore the source of the *ought*. For Hegel, the contradiction implied by the sign of equality combined with the necessary difference between the two sides of the equation emerges from the fact that what we have on the left of the equation is no longer a quantitative notion, but has passed on into a qualitative determinateness, as the ratio is not tied to any specific value, but instead to their relation. This ratio cannot be captured by the expansion of the purely quantitative series of numbers on the right, no matter how closely it may approach the fraction. The fractional representation, as including that which is unobtainable from the quantitative determination (the moment of difference), therefore becomes Hegel's first approximation of the qualitative mathematical infinite.

The fractional understanding of the calculus is ultimately not an adequate representation of true infinity for Hegel, however. What is important about the fraction $\frac{2}{7}$ is that it surpasses the purely quantitative determination of the series. As such, the two apparent quanta of the ratio show the property of variability, to the extent that the values of 2 and 7 can be replaced with other values without changing the overall value of the fraction. Thus, $\frac{2}{7}$ is equivalent to $\frac{4}{14}$ or $\frac{6}{21}$. In all of these cases, however, we are still dealing with quanta for which the relationship is not essential. Thus, while the unity of the different terms is essential for the fraction itself, it is not essential for the terms which make up the fraction. Similarly, Hegel notes that moving to an algebraic description of the fraction does not overcome this limitation, as an algebraic formulation such as $y/x = a$ can equally be written in a form which does not contain a ratio, such as $y = ax$, the equation of a straight line. Variability, therefore is shown not to be the defining characteristic of the qualitative mathematical infinite, as even in the case of the algebraic variable, we still have a symbol standing in for an arbitrary quantum, that is, we have variability, but variability still conceived of as magnitude. It should be clear now why Hegel moves to the Newtonian interpretation of the calculus. In Newton's ultimate ratio, as we are dealing with the ratio of values at a point, neither of the terms in the ratio, dy/dx, can have any meaning outside of the ratio itself. "Apart from their relation they are pure nullities, but they are intended to be taken only as moments of the relation, as the *determinations* of the differential co-efficient dx/dy."[19] In the differential relation, we therefore have a situation whereby both the ratio itself as well as the terms can only be understood as a totality. For Hegel, this represents the true dialectical relation of the quantum and the infinite. In the differential ratio, each of the quanta can only exist in relation to the other. That is, the dy and dx only have meaning through their reciprocal relation. What is important to recognise is that here we have an example of the Hegelian synthesis of the finite and the infinite. That is, the finite determinations, the elements in the ratio, can only exist in relation to one another. Going back to Hegel's initial discussion of the ratio, the ratio in its totality therefore represents the infinite, as in this case, neither the infinite nor the finite can therefore exist without the other. We should note further that this structure resembles the structure of contradiction found in the doctrine of essence. As Hegel moves to a logic of

[19] Hegel (1989, p. 253).

contradiction in order to overcome the limitations of the finite understanding, however, we must now look at why Hegel believes that this understanding of the infinite cannot be arrived at purely through an analytic deduction.

5 Berkeley and the foundations of the calculus

Hegel writes that “the operations which [mathematics] allows itself to perform in the differential and integral calculus are in complete contradiction with the finite determinations and their relations and would therefore have to be justified solely by the *Notion*.”[20] We have seen that Hegel rejected Leibniz’s interpretation as lacking rigour. Newton’s interpretation of prime ratios in terms of instantaneous velocity also presents problems, however, as velocity is a rate of change. For Newton, these ratios were also described as nascent ratios, or the ratios at the point where the fluxions start to be. What is problematic in Newton’s model, therefore, is whether such a thing as an ultimate ratio can actually be said to exist, since it also appears implicitly to rely on the dual properties of the ratio not yet having, but at the same time already having a determinate quantity. The problems of the foundations of the calculus, although known to Leibniz and Newton, were made notorious in Berkeley’s treatise, “The Analyst; or, a Discourse Addressed to an Infidel Mathematician wherein it is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.” The essence of Berkeley’s criticism lies in the statement that:

> If with a view to demonstrating any proposition a certain point is supposed, by virtue of which certain other points are attained; and such supposed point be itself afterwards destroyed or rejected by a contrary supposition; in that case, all other points, attained thereby and consequent thereupon, must also be destroyed and rejected, so as from thence forward to be no more supposed or applied in the demonstration. This is so plain as to need no proof.[21]

Thus, Berkeley attacks Newton’s notion of the ultimate ratio for both appearing to be unequal to zero (as the terms forming the ratio can be divided by one another), but also equal to zero (in order for the ratio to be applied to an instant). Going on to question the fluxions which make up the elements of both Newton’s ultimate ratio as well as Hegel’s mathematical infinite, Berkeley asks: “What are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small nor yet nothing. May we not call them the ghosts of departed quantities?”[22]

The ground of Berkeley’s critique therefore is that the idea of the fluxion is grounded in “direct impossibilities and contradictions”[23] which result from “the

[20] Hegel (1989, p. 254).

[21] Berkeley (1992, paragraph 12).

[22] Berkeley (1992, paragraph 35).

[23] Berkeley (1992, paragraph 8).

most incomprehensible metaphysics."[24] From a Hegelian perspective, however, Berkeley's criticisms can be seen as resting on a form of picture thinking which is based on taking a purely static view of the grounds of the calculus.[25] Hegel writes:

> Although the mathematicians of the infinite maintained that these quantitative determinations are vanishing magnitudes, i.e. magnitudes which are not any particular quantum and yet are not nothing but are still a *determinateness* relative to an other, it seemed perfectly clear that such an *indeterminate state*, as it were called, between being and nothing does not exist.[26]

The fact that there cannot be a state which contains both being and nothingness does not trouble Hegel, however. "The unity of being and nothing is, of course, not a *state*; … on the contrary, this mean and unity, the vanishing or equally the becoming is alone their truth."[27] Thus, the true foundation of the calculus, according to Hegel, is to be found in the results obtained in the dialectic of being and nothing which opens the *Science of Logic*. The differential calculus is therefore seen as being grounded in the fundamental dialectical moment of transition, within which the two moments of the fluxion, being and nothing, are to be taken as immanently related. We thus have a ratio of two terms, both of which are on the brink of vanishing, but which, when related to one another, give a determinate value. It is the two transitions of the fluxions, therefore, when related to one another through the form of the ratio, which generate the determinacy of the calculus. Berkeley's attempt to show the contradiction of the calculus reduces it to "a diseased condition externally induced through erroneous thinking,"[28] essentially the thought of the finite understanding which eschews any consideration of movement. Berkeley's criticism shows that the calculus is the expression of a contradiction, but according to Hegel, without a proper understanding of the nature of contradiction itself, this truth separates the differential calculus from speculative science, rather than being the principle of its unity with it. Ultimately, Hegel's characterisation of the differential calculus will therefore be premised on what is almost an axiom of dialectical thinking, that "there is nothing which is not an intermediate state between being and nothing."[29] When these fluxions are incorporated into the ultimate ratio itself, we have a structure which is isomorphic with the structure of both contradiction and the infinite:

> The truth is rather that that which has being *solely* in the ratio is not a quantum; the nature of quantum is such that it is supposed to have a

[24] Berkeley (1992, paragraph 48).

[25] Cf. Berkeley (1992, paragraph 4): "Now as our sense is strained and puzzled with the perception of objects extremely minute, even so the imagination, which faculty derives from sense, is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein: and much more so to comprehend the moments, or those increments of the flowing quantities in *statu nascenti*, in their very first origin or beginning to exist, before they become finite particles".

[26] Hegel (1989, p. 254).

[27] Hegel (1989, p. 254).

[28] Hegel (1989, p. 254).

[29] Hegel (1989, p. 105).

> completely indifferent existence apart from its ratio, and its difference from another quantum is not supposed to concern its own determination; on the other hand, the qualitative is what it is only in its distinction from an other. The said infinite magnitudes, therefore, are not merely comparable, but they exist only as moments of comparison.[30]

We thus have the unity of moments that can only exist in their difference through this unity. Further, we can see that Hegel's approach is not to move to anything like a transcendental account in justifying the calculus. Instead, Hegel supplements representation by incorporating a moment of movement, in the form of vanishing, into representation itself. Everything therefore happens on the same ontological plane for Hegel. What this analysis has attempted to show is how, for Hegel, the differential calculus both requires a move to a dialectical understanding of mathematics, and also, in its dialectical development, comes to represent the structure of the system as a whole as it incorporates the movement from being to nothing. Whilst Hegel's analysis of the calculus goes on to engage with a variety of his contemporaries' interpretations of the calculus, we will now turn to Deleuze, having shown the necessity of this dialectical moment in Hegel's account.[31] For Deleuze also, the calculus represents a founding metaphor. In the next section, therefore, we will move on to explore how Deleuze's understanding of the calculus differs from that of Hegel in order, through the large number of commonalities which they share in their interpretations, to highlight in precisely which ways they differ in their responses to the problems to be found in representation.

6 Deleuze and the calculus

We have an opposition between two conceptions of the calculus; on the one hand, we have the finite representational schema of the classical interpretation, and on the other, the infinite representational schema of the Hegelian dialectic. In the light of this, it has become relatively standard in reading Deleuze to treat him as using the tools of modern mathematics to cut off the path to Hegelian dialectic by resolving the antinomies at the base of the calculus. This view is clearly implicit in Delanda's interpretation, and is most clearly expressed by Simon Duffy in "The Logic of Expression: Quality, Quantity and Intensity in Spinoza, Hegel and Deleuze," where he writes that, "Deleuze … establishes a historical continuity between Leibniz's differential point of view of the infinitesimal calculus and the differential calculus of contemporary mathematics thanks to the axioms of non-standard analysis which allow the inclusion of the infinitesimal in its arithmetisation; a continuity which effectively bypasses the methods of the differential calculus which Hegel uses in the *Science of Logic* to support the development of the dialectical logic."[32] I want to argue, contrary to this view, that Deleuze in fact wants to reject both positions in

[30] Hegel (1989, p. 255).

[31] See Duffy (2006) for a discussion of Hegel's engagements in these contemporary debates.

[32] Duffy (2006, pp. 74–75).

order to develop a theory of the calculus which escapes completely from the dichotomy of the finite and infinite. Deleuze's aim will be to use the calculus to foster an understanding of the transcendental free from all resemblance to the empirical, as it is only once this resemblance has been removed that the transcendental can be seen as the grounds of the generation of the empirical, rather than simply as a conditioning factor. Thus he will reject both finite interpretations, which understand the differential quantitatively, as a determinate, if infinitesimal magnitude, and the infinite interpretation of Hegel, which only gets as far as the vanishing of the quantum, and therefore leaves its status as vanished (from the realm of quanta at least) untouched. This will allow him construe the transcendental field as constituted entirely non-objectively, opening up the possibility of a generative account of the empirical.

Deleuze's own engagement with the calculus is intertwined with his critique of Hegel. Historically, Deleuze situates his own project in relation to what he calls a "generalised anti-Hegelianism"[33] that pervaded intellectual culture in 1960s France. In looking at Deleuze's own interpretation of the calculus, I therefore want to show that he also retains a pre-Weierstrassian interpretation of the calculus, and that he uses this interpretation to differentiate his own position from that of Hegel. Deleuze's exposition of the calculus begins with the pronouncement that "just as we oppose difference in itself to negativity, so we oppose *dx* to not-A, the symbol of difference to that of contradiction."[34] In spite of this, in his interpretation of the calculus, there are a great many similarities with the Hegelian model. One of Deleuze's main aims in giving his exposition of the calculus is to provide a metaphysic which will "take *dx* seriously."[35] For Deleuze, this approach can only be achieved by providing an interpretation that will leave it "separated from its infinitesimal matrix."[36] What is driving Deleuze in this matter is, in parallel with Hegel, an attempt to provide an interpretation of the calculus which does not give the differentials the status of infinitely small quanta. Mirroring Hegel's argument about the representation of the infinite in the ratio, Deleuze writes, "quanta as objects of intuition always have particular values; and even when they are united in a fractional relation, each maintains a value independently of the relation."[37] For this reason, the Leibnizian interpretation of the calculus, which relies on infinitely small quantities is rejected by Deleuze. Similarly, the calculus cannot be conceived of as operating with algebraic terms, since in this case "there must always be a particular value charged with representing the others and with standing in for them."[38] Thus, the equation of the circle, $x^2 + y^2 - R^2 = 0$, is opposed to its differential, $ydy + xdx = 0$, in which such a substitution of arbitrary values cannot be made. In this example, however, we can already see the difference between Hegel and Deleuze becoming clear. While the differential of the circle can be

[33] Deleuze (1994, p. xix).

[34] Deleuze (1994, p. 170).

[35] Deleuze (1994, p. 170).

[36] Deleuze (1994, p. 170).

[37] Deleuze (1994, p. 171).

[38] Deleuze (1994, p. 171).

represented as a ratio, Deleuze instead chooses the representation which maintains the separability of the differentials. His rhetorical device here signifies his intention to present an interpretation of the differential which falls outside of the positions of the infinitesimal and the ultimate ratio.

For Deleuze, "the symbol, *dx* appears as simultaneously undetermined, determinable, and determination."[39] To each of these three moments, we have a principle, which together form the basis of a transcendental interpretation of the calculus. The first stage, the undetermined as such, corresponds to the differentials themselves (*dx*, *dy*). Here, in what is his most important break with Hegel, Deleuze argues that while *dx* is strictly nothing in relation to *x*, this is not because the differential is not in a sense real, but rather because it cannot be captured by either (Kantian) intuition or the categories of quantity. This characteristic of falling outside of both quantity and intuition is what Deleuze calls "continuousness,"[40] presumably to highlight its difference from the discrete nature of quantitative determinations. In this sense, rather confusingly, continuousness is the "ideal cause of continuity,"[41] continuity here understood as the kind of continuity discovered in the sensible realm. Just as with Hegel, Deleuze is opposed to thinking of the differential ratio in terms of variability, and continuousness instead represents the transcendental condition for variability, since it is through the reciprocal determination of differential elements, which themselves escape determination as quantities, that determinate quantities are generated by the differential function. In this sense, the differential, *dx*, as a symbol of difference, is "completely undetermined,"[42] that is, as the representation of the "closest noumenon,"[43] difference, it escapes the symbolic order. The symbols, *dy* and *dx*, and their values of 0 in respect to *y* and *x*, therefore represent the annihilation of the quantitative within them in favour of what Deleuze calls the sub-representational, or extra-propositional.

It should be clear from what we have said of Leibniz's interpretation of the calculus that once the differential is not seen as an infinitesimal quantity, we can no longer conceive of the process of differentiation as entailing the infinitesimal approximation of the tangent of a point. Instead, Deleuze introduces the notion of limit, but not the limit of the modern interpretation, whereby the differential is the value an infinite series converges on. Instead, "the limit must be conceived not as the limit of a function but as a genuine cut"[44] Deleuze's point is that whereas *dy* and *dx* are completely undetermined in relation to *x* and *y*, they are completely determinable in relation to one another. We can relate this back to our initial description of Leibniz's interpretation of the calculus by noting that Leibniz held differentials to be merely quantitatively distinct from the values of *x* and *y*, as infinitesimally small values. This led to the difficulty that on the one hand they

[39] Deleuze (1994, p. 171).

[40] Deleuze (1994, p. 171).

[41] Deleuze (1994, p. 171).

[42] Deleuze (1994, p. 171).

[43] Deleuze (1994, p. 286).

[44] Deleuze (1994, p. 172).

relied on this quantitative moment to form a ratio and thereby determine the value of x, but on the other hand, this moment had to be ignored in order that we could determine the value of x itself, rather than $x + dx$. For Deleuze, dy and dx are not to be characterised quantitatively, and so, when we determine the quantitative answer, they fall out of the equation, as they cannot be captured by the categories of quantity. That is to say, the differentials themselves escape representation, but, through their reciprocal interaction, are able to generate determinate representations. For Hegel, neither dy or dx could be understood outside of the ultimate ratio, precisely because each could only become determinate in relation to the other. For Deleuze, it is also true that dy and dx only become *determinate* in relation to one another, but this does not imply that the differentials are not *determinable* prior to their determination. They thus stand outside of the Hegelian notion of the ultimate ratio. This determinable level is for Deleuze a transcendental condition of the ratio which forms the basis of Hegel's account, what he calls elsewhere the virtual. This brings us on to the status of the reciprocally determinable, as dy/dx.

For Deleuze, as for Hegel, the differential relation is not to be understood in terms of quantities or variable magnitudes, but against Hegel, the value of the differential equation is to be understood according to its difference from the primitive function (the function which is differentiated). In the example which we used at the beginning of this chapter, the primitive function was the initial equation, $y = x^2$, which was differentiated to give, $dy/dx = 2x$. As we saw, differentiating a function gives us the tangent to that function, enabling us to determine the gradient at any point. To be more precise, however, we can say that the differential of a function is itself a function which will give us the tangent at any point on the primitive curve. The function, $dy/dx = 2x$ gives the gradient at any point on the curve, as we can insert any value of x into the equation. There is thus a change in quality between the differential equation and the primitive function. The primitive function, which deals with the relations of actual magnitudes, is tied to representation, whereas the differential function, which specifies values in terms of dy and dx falls outside of representation. The movement from the differential to the primitive function is therefore seen by Deleuze to be a movement of generation, and solves a difficulty he sees in standard transcendental accounts. Kant, according to Deleuze, merely repeats the structure of the understanding at a higher level. This means that ultimately, his account of the transcendental categories appears empty (a criticism similar to one Hegel also levels at Kant). Deleuze hopes, by using the difference between the primitive function and the differential, to give a model of the transcendental that does not simply repeat the empirical. There is also a third stage of Deleuze's analysis, complete determination, which corresponds to the actual values of dy/dx. The importance of this stage is that in specifying the tangent to various points on the curve, we can also determine the points, known as singularities, where the nature of the curve changes. These points, where dy/dx becomes null, infinite, or equal to zero, represent points of transition within the structure of the curve itself, as for instance, when the gradient of a tangent is equal to zero, we have a local maximum or minimum as the direction of the curve changes. This means, for Deleuze that the differential calculus allows us both to specify the general, in the form of the curve as a whole, and as the singular, in the form of the singularity. Reciprocal determination gives us the characteristic of generality, in the form of the line which all of the singularities, as partaking in it, together determine.

We have dealt with two conceptions of the foundations of the calculus apart from that of Deleuze. The first approach is Hegel's, which posits that the calculus is fundamentally dialectical, and as such can only be understood in terms of infinite thought. Central to this interpretation was the idea that the ultimate ratio, as a vanishing, fell outside of the procedures which a thinker of finite representation, such as Berkeley, was able to countenance. The differential, as a vanishing, could not be understood in the static terms today exemplified by set theory. The second approach was that of set theory itself, which proceeded by rejecting Berkeley's interpretation of the foundations of the calculus, and thereby seeking additional resources within finite representation itself with which to lend consistency to its operation. This led to the theory of limits. For Deleuze, "it is precisely this alternative between infinite and finite representation which is at issue when we speak of the 'metaphysics' of calculus."[45] We saw this question of representation played out in terms of the ontological status of the differential, with Hegel preserving it, and modern interpretations removing it. Deleuze attempts instead to finesse this problem by granting the differential a status outside of both forms of representation. In this sense, the differential is "neither real nor fictive, [instead] differentials express the nature of the problematic as such."[46] The question of the differential therefore bears on the question of the two forms of representation: "Perhaps the other alternative collapses as well, that between finite and infinite representation", as "what is still missing is the extra-propositional or sub-representative element expressed in the Idea of the differential, precisely in the form of a problem."[47] This sub-representational element cannot be provided by the move to a modern interpretation, as in order to bring the differential into the system of representation, it must conceive of it as the limit of a series, and a limit that has, "lost its phoronomic character and involves only static considerations."[48] It is for this reason that Deleuze rejects the kind of interpretation of the calculus which would simply remove its metaphysical character.

7 Conclusion

We can summarise the differences found in the interpretations of the differential calculus by noting that Hegel takes a broadly Newtonian line in his interpretation of the calculus, whereas Deleuze takes his inspiration from the work of Leibniz, albeit with a non-intuitional interpretation of the differential. For Deleuze, the difficulty of differentials appearing in the resultant formulae is resolved through the belief that we are dealing with two different ontological planes, for Hegel once again through

[45] Deleuze (1994, p. 176).

[46] Deleuze (1994, p. 178).

[47] Deleuze (1994, p. 178).

[48] Deleuze (1994, p. 176).

recognising that the status of the nascent ratio differs from that of normal numbers. Aside from this difference, we should note that what Hegel discovers in the calculus is "the infinity of 'relation'," [49] meaning that what is important in the calculus is the relation between the two fluxions, to such a degree that this relation not only governs the determination of them, but also their existence. The questions of the determination and existence of the differential remain separate on Deleuze's interpretation, as the differential is given the characteristic of "determinability." The centrality of these differences is clear in that Hegel first refers to the differential calculus in his discussion of the notion of vanishing in the fourth remark to the dialectic of becoming, and it is this moment of vanishing which Deleuze uses to characterise Hegel's philosophy as a whole as one of infinite representation. Hegel adds movement to the dialectic by considering finite representation "not as having vanished and disappeared but as vanishing and on the point of disappearing." [50] That is to say, Hegel does not renounce finite representation, but instead calls for it to be reintegrated into the interiority of the infinite. "Opposition remains abstract so long as it does not extend to the infinite, and the infinite remains abstract every time it is posed outside of finite oppositions." [51] In this sense, for Deleuze, Hegel's criticism of those who went before him is that they had not taken difference to the level of absolute difference, the contradiction. In making this final move, Hegel goes from finite, organic representation to what Deleuze calls infinite, orgiastic representation. Organic representation is given content by participating in orgiastic representation, just as both being and the differential are maintained as being just on the point of vanishing.

Whereas Hegel focuses on the structural elements of the ratio *dy*/*dx*, Deleuze is instead interested in the difference in kind between the primitive function and the differential, and the fact that by integration, the primitive function can be *generated* from the differential. In recognising the difference in kind between them, Deleuzian ontology returns to the Bergsonian idea that the absence of order is in fact the presence of a different kind of order, in Deleuze's terms a distinction which is played out between sedentary (representational) distributions and nomadic (differential) distributions. The differential provides the tools for this, presenting a transcendental logic which is capable of explaining the genesis of objectivity through a non-objectival (differential) transcendental field. Everything, however, takes place in the middle for Deleuze, and negation and opposition appear in the world "only insofar as these are cut off from their virtuality which they actualise, and from the movement of their actualisation." [52] The fact that differential relations have a structure of their own, and, through Deleuze's analysis of the differential as escaping representation, a structure which itself escapes representation, means that Deleuze can attempt to provide an account of the genesis of determination which does not reduce the absence of determination to indifference.

[49] Deleuze (1994, p. 310, n. 9).

[50] Deleuze (1994, pp. 43–44).

[51] Deleuze (1994, p. 44).

[52] Deleuze (1994, p. 207).

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