Can Partial Indexings Be Totalized? *

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Abstract

In examples like the total recursive functions or the computable real numbers the canonical indexings are only partial maps. It is even impossible in these cases to find an equivalent total numbering. We consider effectively given topological T_0-spaces and study the problem in which cases the canonical numberings of such spaces can be totalized, *i. e.*, have an equivalent total indexing. Moreover, we show under very natural assumptions that such spaces can effectively and effectively homeomorphically be embedded into a totally indexed algebraic constructively directed-complete partial order.

Introduction

An *indexed* or *numbered* set is a countable set with a (partial) map of the natural numbers onto the set. Numbered sets appear quite natural in constructive mathematics. The index of an object encodes its construction. Via the indexing notions from computability theory can be transferred to abstract sets.

A prominent example of a numbered set is the set of all partial recursive functions on the natural numbers indexed by a Gödel numbering. As it is well known, this set has also other indexings with very different properties.

First studies of numbered sets have been done by Mal'cev [10]. A comprehensive treatment can be found in Eršov [4, 6, 7]. In this work only total numberings are considered, but there are important cases in which the indexings are only partial maps. The canonical indexing of the computable real numbers defined via normed Cauchy sequences [16] is an example. As has been shown in [15], this numbering cannot be totalized, *i. e.*, there is no equivalent *total* indexing of the computable reals.

In the present paper we are interested in the question in which cases a partial indexing can be totalized. We deal with two aspects of this problem:

- When does a numbered set (X, x) have a total indexing which is equivalent to the given partial numbering x?
- Can (X, x) be embedded into an indexed set $(\hat{X}, \hat{x})$ the indexing $\hat{x}$ of which is total and extends the given indexing x?

In order to treat with these questions we restrict ourselves to effectively given topological T_0-spaces. For computer science this is not a severe limitation. Scott [13] and Smyth [14] pointed out that data types can be thought of as countably based topological T_0-spaces with basic open sets for the finitely describable properties of the data objects. Most structures considered in programming language semantics are equipped with a canonical topology. Prominent examples are metric spaces, Scott domains, A- and f-spaces [2, 3, 5, 6]. As is shown in Stoltenberg-Hansen and Tucker [18] many algebraic structures *e. g.*, all term algebras over a finite signature, can be canonically embedded in complete ultrametric spaces as well as Scott domains.

Topological spaces that satisfy certain natural effectivity requirements have been studied by various authors. We consider countable T_0-spaces (T, τ) with a countable, totally indexed

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basis of the topology. Between the indices of basic open sets a relation of strong inclusion is defined such that the property of being a topological basis holds effectively with respect to this relation instead of to normal set inclusion. The points of the space are numbered in such a way that the collection of all basic open sets containing a given point can uniformly be enumerated. Moreover, from a (normed) enumeration of a base of basic open sets of a neighbourhood filter one can compute the point determined by the filter. Such indexings are called acceptable. As it is shown, for an acceptable numbering an equivalent total indexing of the space can be constructed if and only if all neighbourhood filters can be enumerated in a uniform way and among all such enumerations there is a principal one, which means that there is a uniform enumeration of the neighbourhood filters to which every other such enumeration can be reduced.

The condition that all neighbourhood filters can uniformly be enumerated means that we can effectively list procedures for the generation of bases for the neighbourhood filters, at least one for each such filter. If we can effectively list all such procedures, the space is an algebraic constructive predomain with respect to the specialization order such that all of its points are comparable.

This shows that the condition that all generation procedures for bases of neighbourhood filters are listable is quite strong. Moreover, we see that effectivity requirements may have structural consequences. For the special case of domain-like spaces we study which further implications the existence of a total acceptable numbering has. It turns out that any acceptable indexing of the space has an equivalent total indexing, if and only if the space is constructively directed-complete and the topology is the Scott topology. Here, a space is constructively directed-complete if every enumerable subset has a least upper bound with respect to the specialization order. If the space has a smallest element, the total indexing can be constructed in such a way that it is complete, which implies that the fixed point theorem holds. This result underlines the importance of the Scott topology for the study of computability on abstract structures.

As has been shown by Eršov [4], every indexed set (X, x) with a total indexing x can be embedded in a numbered set $(\hat{X}, \hat{x})$ such that $\hat{x}$ is both total and complete and x is reducible to $\hat{x}$. The set $\hat{X}$ is obtained from X by adjoining a new element $\perp$. The same construction can be carried out, if x is only a partial map. Again the indexing $\hat{x}$ is total and complete. But, in general, if X is an effectively given T_0-space, the numbering $\hat{x}$ need not be acceptable and the embedding need not be effectively homeomorphic. The space $\hat{X}$ does not have enough partial elements. We show in the case that the strong inclusion relation is effectively enumerable that the given space T can be embedded into an algebraic constructive domain $\hat{T}$ with a total, complete, and acceptable numbering. The embedding as well as its partial inverse are effective and effectively continuous. Moreover, under the embedding T is dense in $\hat{T}$. If all basic open sets in the topology τ are closed as well, it is also a total subset of $\hat{T}$ in the sense of Berger [1].

The paper is organized in the following way. Section 1 contains basic definitions and properties. The totalizability problem is studied for the general case of effectively given T_0-spaces in Section 2 and for the special case of domain-like spaces in Section 3. In Section 4, finally, the embedding result is derived.

1 Basic definitions and properties

In what follows, let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a recursive pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2 \rangle) = a_i$. We extend the pairing function in the usual way to an n-tuple encoding. Let $P^{(n)}$ ($R^{(n)}$) denote the set of all n-ary partial (total) recursive functions, and let W_i be the domain of the ith partial recursive function φ_i with respect to some Gödel numbering φ. We let $\varphi_i(a)\downarrow$ mean that the computation of $\varphi_i(a)$ stops, $\varphi_i(a)\downarrow \in C$ that it stops with value in C, and $\varphi_i(a)\downarrow_n$ that it stops within n steps. In the opposite cases we write $\varphi_i(a)\uparrow$ and $\varphi_i(a)\uparrow_n$ respectively. If $A \subseteq \omega$ is not empty and recursively enumerable (r. e.), A_s is the finite subset of A which can be enumerated in s steps with respect to some fixed total enumeration of A, *i. e.*, $A_s = \{f(0), \ldots, f(s \perp 1)\}$, where $f \in R^{(1)}$ is the fixed enumeration.

Let S be a nonempty set. A *(partial) numbering* ν of S is a partial map $\nu : \omega \rightharpoonup S$(onto) with domain $\mathrm{dom}(\nu)$. The value of ν at $n \in \mathrm{dom}(\nu)$ is denoted, interchangeably, by ν_n and $\nu(n)$. Note that instead of numbering we also say *indexing*.

Definition 1.1 Let ν and κ be numberings of the set S.

1. $\nu \leq \kappa$, read ν is *reducible* to κ, if there is a function $g \in P^{(1)}$ such that: $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(g)$, $g(\mathrm{dom}(\nu)) \subseteq \mathrm{dom}(\kappa)$, and $\nu_m = \kappa_{g(m)}$, for all $m \in \mathrm{dom}(\nu)$.

2. $\nu \equiv \kappa$, read ν is *equivalent* to κ, if $\nu \leq \kappa$ and $\kappa \leq \nu$.

If A and C are sets of natural numbers, their characteristic functions χ_A and χ_C, respectively, are numberings of the set $\{0, 1\}$. In this special case the above reducibility and equivalence notions reduce to the corresponding notions for sets known from computability theory [12], which we denote in the same way.

Definition 1.2 A numbering ν of the set S is said to be

1. *precomplete*, if for any function $g \in P^{(1)}$ there is a function $f \in R^{(1)}$ such that $f(n) \in \mathrm{dom}(\nu)$ and $\nu_{f(n)} = \nu_{g(n)}$, for $n \in \mathrm{dom}(g)$ with $g(n) \in \mathrm{dom}(\nu)$

2. *complete*, if there is some element $e \in S$, called the *special element*, such that for any function $g \in P^{(1)}$ there is a function $f \in R^{(1)}$ such that $f(n) \in \mathrm{dom}(\nu)$, for all $n \in \omega$ with either $n \notin \mathrm{dom}(g)$ or $n \in \mathrm{dom}(g)$ and $g(n) \in \mathrm{dom}(\nu)$, and $\nu_{f(n)} = \nu_{g(n)}$, for all $n \in \mathrm{dom}(g)$ with $g(n) \in \mathrm{dom}(\nu)$, and $\nu_{f(n)} = e$, for all $n \in \omega \setminus \mathrm{dom}(g)$

A subset X of S is *completely enumerable*, if there is an r.e. set A such that $\nu_n \in X$ if and only if $n \in A$, for all $n \in \mathrm{dom}(\nu)$. If A is recursive, X is said to be *completely recursive*. X is *enumerable*, if for some r.e. set $A \subseteq \mathrm{dom}(\nu)$ we have that $X = \{\, \nu_n \mid n \in A \,\}$. Thus, X is enumerable if we can enumerate a subset of the index set of X which contains at least one index for every element of X, whereas X is completely enumerable if we can enumerate all indices of elements of X and perhaps some numbers which are not used as indices by the numbering ν. A relation $R \subseteq S \times S$ is *completely enumerable*, if there is an r.e. set A such that $(\nu_i, \nu_j) \in R$ if and only if $\langle i, j \rangle \in A$, for all $i, j \in \mathrm{dom}(\nu)$.

Now, let $\mathcal{T} = (T, \tau)$ be a countable topological T_0-space with a countable basis $\mathcal{B}$. Let B be an indexing of $\mathcal{B}$. In the applications we have in mind the basic open sets can be described in some finite way. The indexing B is then obtained by an encoding of the finite descriptions. Moreover, in these cases there is a canonical relation between the (code numbers of the) descriptions which is stronger than the usual set inclusion between the described sets. This relation is r.e., which in general is not true for set inclusion. It has been turned out in effective topology that one has to work with this stronger relation (*cf. e.g.* [15, 16, 17]).

Definition 1.3 Let $\prec_B$ be a transitive binary relation on ω. We say that:

1. $\prec_B$ is a *strong inclusion*, if for all $m, n \in \mathrm{dom}(B)$, from $m \prec_B n$ it follows that $B_m \subseteq B_n$.

2. $\mathcal{B}$ is a *strong basis*, if $\prec_B$ is a strong inclusion and for all $z \in T$ and $m, n \in \mathrm{dom}(B)$ with $z \in B_m \cap B_n$ there is a number $a \in \mathrm{dom}(B)$ such that $z \in B_a$, $a \prec_B m$ and $a \prec_B n$.

For what follows we assume that $\prec_B$ is a strong inclusion with respect to which $\mathcal{B}$ is a strong basis. Moreover, to simplify matters we will suppose throughout this paper that the empty set is not included in $\mathcal{B}$.

Definition 1.4 Let $\mathcal{T} = (T, \tau)$ be a countable topological T_0-space with a countable basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. We say that:

1. x is *computable* if there is some r.e. set L such that for all $i \in \mathrm{dom}(x)$ and $n \in \mathrm{dom}(B)$, $\langle i, n \rangle \in L$ if and only if $x_i \in B_n$.

2. $\mathcal{T}$ is *effective*, if B is a total indexing and the property of being a strong basis holds effectively, which means that there exists a function $\mathrm{sb} \in P^{(3)}$ such that for $i \in \mathrm{dom}(x)$ and $m, n \in \omega$ with $x_i \in B_m \cap B_n$, $\mathrm{sb}(i, m, n)\!\downarrow$, $x_i \in B_{\mathrm{sb}(i,m,n)}$, $\mathrm{sb}(i, m, n) \prec_B m$, and $\mathrm{sb}(i, m, n) \prec_B n$.

Thus, x is computable if and only if all basic open sets B_n are completely enumerable, uniformly in n.

As it is easily verified, $\mathcal{T}$ is effective if x is computable, B is total and the strong inclusion relation is r. e. Note that very often the totality of B can easily be achieved, if the space is *recursively separable*, which means that it has a dense enumerable subset, called its dense base. In the sequel we always assume that $\mathcal{T}$ is effective.

As it is well-known, each point y of a T_0-space is uniquely determined by its neighbourhood filter $\mathcal{N}(y)$ and/or a base of it. A point y is called *finite*, if $\mathcal{N}(y)$ has a finite and hence a singleton base. Moreover, on T_0-spaces there is a canonical partial order, the specialization order, which we denote by $\leq_\tau$.

Definition 1.5 Let $\mathcal{T} = (T, \tau)$ be a T_0-space, and y, $z \in T$. $y \leq_\tau z$ if $\mathcal{N}(y) \subseteq \mathcal{N}(z)$.

Let us now consider some important standard examples of effective T_0-spaces.

Example 1.6 (Constructive domains) Let $Q = (Q, \sqsubseteq)$ be a partial order. A nonempty subset S of Q is *directed*, if for all y_1, $y_2 \in S$ there is some $u \in S$ with y_1, $y_2 \sqsubseteq u$. The *way-below relation* $\lll$ on Q is defined as follows: $y_1 \lll y_2$ if for every directed subset S of Q the least upper bound of which exists in Q, the relation $y_2 \sqsubseteq \bigsqcup S$ always implies the existence of a $u \in S$ with $y_1 \sqsubseteq u$. Note that $\lll$ is transitive. Elements $y \in Q$ with $y \lll y$ are called *compact*. For points y, $z \in Q$ such that at least one of them is compact, $y \lll z$ if and only if $y \sqsubseteq z$.

A subset Z of Q is a *basis* of Q, if for any $y \in Q$ the set $Z_y = \{ z \in Z \mid z \lll y \}$ is directed and $y = \bigsqcup Z_y$. A partial order that has a basis is called *continuous*. If all elements of Z are compact, Q is said to be *algebraic*.

Now, assume that Q is countable and let x be an indexing of Q. Then Q is *constructively directed-complete*, if each of its enumerable directed subsets has a least upper bound in Q. Let Q be constructively directed-complete and continuous with basis Z. Moreover, let β be a total numbering of Z. Then $(Q, \sqsubseteq, Z, \beta, x)$ is said to be a *constructive predomain*, if the restriction of the way-below relation to Z as well as all sets Z_y, for $y \in Q$, are completely enumerable with respect to the indexing β and $\beta \leq x$. It is called *constructive domain* if, in addition, the partial order has a smallest element.

The numbering x of Q is is said to be *admissible*, if the set $\{ \langle i, j \rangle \mid \beta_i \lll x_j \}$ is r. e. and there is a function $d \in R^{(1)}$ such that for all indices $i \in \omega$ for which $\beta(W_i)$ is directed, $x_{d(i)}$ is the least upper bound of $\beta(W_i)$. In the case of constructive domains it is shown in [19] that such numberings exist. They can even be chosen as total. In what follows we always assume that the numbering x of a constructive predomain is admissible.

It is well known that on constructively directed-complete partial orders there is a canonical topology σ: the *Scott topology*. A subset X of Q is open, if it is upwards closed with respect to $\sqsubseteq$ and intersects each enumerable directed subset of Q of which it contains the least upper bound. In the case of a constructive predomain this topology is generated by the sets $B_n = \{ y \in Q \mid \beta_n \lll y \}$ with $n \in \omega$. It follows that $\mathcal{Q} = (Q, \sigma)$ is a countable T_0-space with a countable basis. Observe that the specialization order on Q coincides with the partial order $\sqsubseteq$ [9]. Moreover, compactness matches with finiteness. Obviously, every admissible numbering is computable. Since Z is dense in $\mathcal{Q}$ we also obtain that $\mathcal{Q}$ is recursively separable.

Define

$$m \prec_B n \Leftrightarrow \beta_n \lll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which the collection of all B_n is a strong basis. Because the restriction of $\lll$ to Z is completely enumerable, $\prec_B$ is r. e. It follows that $\mathcal{Q}$ is effective.

Example 1.7 (Constructive A- and f-spaces) A- and f-spaces have been introduced by Eršov [2, 3, 5, 6, 8] as a more topologically oriented approach to domain theory. They are not required to be complete.

Let $\mathcal{Y} = (Y, \rho)$ be a topological T_0-space. For a subset X of Y, $\mathrm{int}(X)$ is its interior. Moreover, for y, $z \in Y$ define $y \ll z$ if $z \in \mathrm{int}(\{ u \in Y \mid y \leq_\rho u \})$. Then y is finite if and only if $y \ll y$. $\mathcal{Y}$ is an *A-space*, if there is a subset Y_0 of Y satisfying the following three properties:

1. Any two elements of Y_0 which are bounded in Y with respect to the specialization order have a least upper bound in Y_0.

2. The collection of sets $\text{int}(\{ u \in Y \mid y \leq_\rho u \})$, for $y \in Y_0$, is a basis of topology ρ.

3. For any $y \in Y_0$ and $u \in Y$ with $y \ll u$ there is some $z \in Y_0$ such that $y \ll z$ and $z \ll u$.

Any subset Y_0 of Y with these properties is called *basic subspace*.

Let Y be countable and Y_0 have a numbering β. For m, $n \in \text{dom}(\beta)$ set $B_n = \text{int}(\{u \in Y \mid \beta_n \leq_\rho u\})$ and define

$$m \prec_B n \Leftrightarrow \beta_n \ll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which $\{ B_n \mid n \in \text{dom}(\beta) \}$ is a strong basis. The A-space $\mathcal{Y}$ with basic subspace Y_0 is *constructive*, if the numbering β is total, the restriction of $\ll$ to Y_0 is completely enumerable, and the neighbourhood filter of each point has an enumerable strong base of basic open sets. As has been shown in [17], under these assumptions Y has a canonical numbering x such that $\mathcal{Y}$ is effective. Moreover, it is recursively separable with dense basis Y_0.

Let $\mathcal{Y} = (Y, \rho)$ be again an arbitrary topological T_0-space. An open set V is an *f-set*, if there is a some element $z_V \in V$ such that $V = \{ y \in Y \mid z_V \leq_\rho y \}$. The uniquely determined element z_V is called *f-element*. $\mathcal{Y}$ is an *f-space*, if the following two conditions hold:

1. If U and V are f-sets with nonempty intersection, then $U \cap V$ is also an f-set.

2. The collection of all f-sets is a basis of topology ρ.

An f-space is *constructive*, if the set of all f-elements has a total numbering α such that the restriction of the specialization order to this set as well as the boundedness of two f-elements is completely recursive and there is a function $su \in R^{(2)}$ such that in the case that α_n and α_m are bounded, $\alpha_{su(n,m)}$ is their least upper bound, and if the neighbourhood filter of each point has an enumerable base of f-sets.

Every f-space is an A-space with basic subspace the set of all f-elements, which are exactly the finite elements of the space. Moreover, for y, $z \in Y$ with y or z being an f-element, $y \ll z$ if and only if $y \leq_\tau z$. It follows that also every constructive f-space is a constructive A-space.

Example 1.8 (Constructive metric spaces) Let $\mathbb{R}$ denote the set of all real numbers, and let ν be some canonical total indexing of the rational numbers. Then a real number z is said to be *computable*, if there is a function $f \in R^{(1)}$ such that for all m, $n \in \omega$ with $m \leq n$, the inequality $|\nu_{f(m)} \perp \nu_{f(n)}| < 2^{-m}$ holds and $z = \lim_m \nu_{f(m)}$. Any Gödel number of the function f is called an *index* of z. This defines a partial indexing γ of the set $\mathbb{R}_c$ of all computable real numbers.

Now, let $\mathcal{M} = (M, \delta)$ be a separable metric space with $\text{range}(\delta) \subseteq \mathbb{R}_c$, and let β be a total numbering of the dense subset M_0. A sequence $(y_a)_{a \in \omega}$ of elements of M_0 is said to be *normed*, if $\delta(y_m, y_n) < 2^{-m}$, for all $m, n \in \omega$ with $m \leq n$. Moreover, (y_a) is *recursive*, if there is some function $f \in R^{(1)}$ such that $y_a = \beta_{f(a)}$, for all $a \in \omega$. Any Gödel number of f is called an *index* of (y_a).

$\mathcal{M}$ is said to be *constructive*, if the restriction of the distance function δ to M_0 has only rational values and is effective, *i. e.*, if there is some function $d \in R^{(2)}$ such that for all i, $j \in \omega$, $\delta(\beta_i, \beta_j) = \nu_{d(i,j)}$, and each element y of M is the limit of a normed recursive sequence of elements of M_0. If m is the index of such a sequence, set $x_m = y$. Otherwise, let x be undefined. Then x is a numbering of M with respect to which and the indexing γ of the computable real numbers the distance function is effective (*cf.* [15]).

As is well-known, the collection of sets $B_{\langle i,m \rangle} = \{ y \in M \mid \delta(\beta_i, y) < 2^{-m} \}$ $(i,\, m \in \omega)$ is a basis of the canonical Hausdorff topology Δ on M. Because the usual less-than relation on the computable real numbers is completely enumerable [11], it follows that x is computable. As has been shown in [15], a point $y \in M$ is finite if and only if it is isolated.

Define

$$\langle i, m \rangle \prec_B \langle j, n \rangle \Leftrightarrow \delta(\beta_i, \beta_j) + 2^{-m} < 2^{-n}.$$

Using the triangular inequation it is readily verified that $\prec_B$ is a strong inclusion and the collection of all B_a is a strong basis. Moreover, $\prec_B$ is r.e. It follows that $\mathcal{M}$ is effective.

Beside the computable real numbers, well known examples of constructive metric spaces are Baire space, that is, the set $R^{(1)}$ of all total recursive functions with the Baire metric [12], and the set ω with the discrete metric.

Since we work with strong inclusion instead of set inclusion, we had to adjust the notion of a topological basis. In the same way we have to modify that of a filter base.

Definition 1.9 Let $\mathcal{H}$ be a filter. A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called *strong base* of $\mathcal{H}$ if the following two conditions hold:

1. For all m, $n \in \mathrm{dom}(B)$ with B_m, $B_n \in \mathcal{F}$ there is some index $a \in \mathrm{dom}(B)$ such that $B_a \in \mathcal{F}$, $a \prec_B m$, and $a \prec_B n$.

2. For all $m \in \mathrm{dom}(B)$ with $B_m \in \mathcal{H}$ there some index $a \in \mathrm{dom}(B)$ such that $B_a \in \mathcal{F}$ and $a \prec_B m$.

If x is computable, a strong base of basic open sets can effectively be enumerated for each neighbourhood filter. The next result proved in [17] shows that for effective spaces this can be done in a normed way.

Definition 1.10 An enumeration $(B_{f(a)})_{a\in\omega}$ with $f : \omega \to \omega$ such that $\mathrm{range}(f) \subseteq \mathrm{dom}(B)$ is said to be *normed* if f is decreasing with respect to $\prec_B$. If f is recursive, it is also called *recursive* and any Gödel number of f is said to be an *index* of it.

In case $(B_{f(a)})$ enumerates a strong base of the neighbourhood filter of some point, we say it *converges* to that point.

Lemma 1.11 *Let $\mathcal{T}$ be effective and x be computable. Then there are functions $q \in R^{(1)}$ and $p \in R^{(2)}$ such that for all $i \in \mathrm{dom}(x)$ and all $n \in \omega$ with $x_i \in B_n$, $q(i)$ and $p(i,n)$ are indices of normed recursive enumerations of basic open sets which converge to x_i. Moreover, $\varphi_{p(i,n)}(0) \prec_B n$.*

In what follows, we want not only to be able to generate normed recursive enumerations of basic open sets that converge to a given point, but conversely, we need also to be able to pass effectively from such enumerations to the point they converge to.

Definition 1.12 Let x be a numbering of T. We say that:

1. x *allows effective limit passing* if there is a function $\mathrm{pt} \in P^{(1)}$ such that, if m is an index of a normed recursive enumeration of basic open sets which converges to some point $y \in T$, then $\mathrm{pt}(m)\!\downarrow \in \mathrm{dom}(x)$ and $x_{\mathrm{pt}(m)} = y$.

2. x is *acceptable* if it allows effective limit passing and is computable.

If x is computable, each neighbourhood filter $\mathcal{N}(y)$ has a completely enumerable strong base of basic open sets, namely the set of all B_a with $y \in B_a$. As it is shown in [17], T has a precomplete acceptable numbering if, conversely, $\mathcal{N}(y)$ has an enumerable strong base of basic open sets, for all $y \in T$, and $\prec_B$ is r.e. In case that, in addition, the indexing B is total, $\mathcal{T}$ is effective with respect to this numbering. Moreover, indexings which are computable and/or allow effective limit passing are related to each other in the subsequent way.

Lemma 1.13 *Let $\mathcal{T}$ be effective. Then for any two numberings x' and x'' of T the following hold:*

1. *If x' is computable and x'' allows effective limit passing, then $x' \le x''$.*

2. *If x' is computable and $x'' \le x'$, then x'' is also computable.*

3. *If x' allows effective limit passing and $x' \le x''$, then x'' allows effective limit passing too.*

Corollary 1.14 *Let $\mathcal{T}$ be effective and x be acceptable. Then for any numbering x' of T the following hold:*

1. *x' is computable if and only if $x' \le x$.*

2. *x' allows effective limit passing if and only if $x \leq x'$.*

3. *x' is acceptable if and only if $x' \equiv x$.*

As it is easily verified, the acceptable indexings of a constructive predomain are just the admissible ones. In the case of a constructive metric space $\mathcal{M}$ acceptable numberings x allow the computation of limits, which means that there is a function $\mathrm{li} \in P^{(1)}$ such that, if m is an index of a converging normed recursive sequence $(y_a)_{a\in\omega}$ of elements of the dense base of $\mathcal{M}$, then $\mathrm{li}(m)\!\downarrow \in \mathrm{dom}(x)$ and $x_{\mathrm{li}(m)} = \lim_a y_a$ [17]. This shows that acceptable numberings of effective T_0-spaces are well behaved.

2 On totalization

As we have seen in the last section, there are indexed sets like the constructive domains, where the numbering is well behaved and can be chosen as total, whereas in other cases like the computable reals the indexing is also well behaved, but only a partial map. The question we are interested in in this section is the following:

> Given an indexed set (X, x) with a well behaved partial indexing x, is there total well behaved numbering $\hat{x}$ of X, which is equivalent to x, *i. e.*, can x be totalized?

The subsequent lemma is a consequence of a result in [15].

Lemma 2.1 *Let $\mathcal{T}$ be effective without finite points. If the numbering x is acceptable, it cannot be total.*

We have already mentioned that in the case of a Hausdorff space finiteness matches with isolatedness. Thus, it follows that the canonical indexing γ of the computable reals cannot be totalized, which means that the answer to the above question is negative, in general. It is the aim of this section to present a sufficient and necessary condition for totalizability. But note that we do not deal with the problem in the full generality of indexed sets. We restrict ourselves to effective T_0-spaces $\mathcal{T} = (T, \tau)$ with acceptable numberings x.

Let Pt be the collection of all neighbourhood filters of points of T. As has already been mentioned, in the case that $\mathcal{T}$ is effective and T is computably indexed each of these filters has an enumerable strong base of basic open sets. If $\mathcal{H}$ is such a filter and $\{ B_n \mid n \in W_i \}$ is a strong base of $\mathcal{H}$ set $\mathcal{F}_i = \mathcal{H}$, otherwise let $\mathcal{F}_i$ be undefined. Then $\mathcal{F}$ is a numbering of Pt.

Proposition 2.2 *Let $\mathcal{T}$ be effective, x be acceptable, and $\prec_B$ be r. e. Then T has a total numbering $\bar{x}$ with $\bar{x} \leq x$, if and only if* Pt *is enumerable.*

Proof: Let $\bar{x}$ be a total indexing of T with $\bar{x} \leq x$. By Corollary 1.14 a numbering of T is reducible to x, just if it is computable. It follows that there is a function $v \in R^{(1)}$ with $W_{v(i)} = \{ n \mid \bar{x}_i \in B_n \}$, for $i \in \omega$. Since the collection of all B_n with $\bar{x}_i \in B_n$ is a strong base of the neighbourhood filter of $\bar{x}_i$, we have that $\mathcal{F}_{v(i)}$ is this filter, for all $i \in \omega$. This shows that Pt is enumerable.

For the converse implication let $t \in R^{(1)}$ such that $\mathrm{range}(\mathcal{F} \circ t) = \mathrm{Pt}$. Moreover, for $i \in \omega$, define $\bar{x}_i$ to be the uniquely determined point of T with neighbourhood filter $\mathcal{F}_{t(i)}$. Then $\bar{x}$ is a total numbering of T. It remains to show that $\bar{x}$ is computable. We have for $i, n \in \omega$

$$\bar{x}_i \in B_n \Leftrightarrow B_n \in \mathcal{F}_{t(i)} \Leftrightarrow (\exists a \in W_{t(i)}) a \prec_B n,$$

which shows that $\{ \langle i, n \rangle \mid \bar{x}_i \in B_n \}$ is r.e.

Note that the assumption that Pt is enumerable means that Pt has a total numbering which factorizes through $\mathcal{F}$. In [17] it is shown that acceptable numberings of T are maximal among the computable numberings of T with respect to reducibility. As we will show now, an acceptable indexing of T can be totalized, exactly if Pt has a total numbering which is maximal among the numberings of Pt that factorize through $\mathcal{F}$.

Theorem 2.3 *Let $\mathcal{T}$ be effective, x be acceptable, and $\prec_B$ be r. e. Then T has a total numbering $\hat{x}$ which is equivalent to x, if and only if there exists a function $v \in R^{(1)}$ with* $\mathrm{range}(v) \subseteq \mathrm{dom}(\mathcal{F})$ *such that*

1. range$(\mathcal{F} \circ v) = \mathrm{Pt}$ *and*
2. *for all functions* $g \in P^{(1)}$ *with* range$(g) \subseteq$ dom$(\mathcal{F})$ *and* range$(\mathcal{F} \circ g) = \mathrm{Pt}$ *one has* $\mathcal{F} \circ g \leq \mathcal{F} \circ v$.

Proof: Let $\hat{x}$ be a total indexing of T which is equivalent to x. Then $\hat{x}$ is acceptable. Let $v \in R^{(1)}$ be as in the proof of Proposition 2.2. Moreover, let $s \in R^{(1)}$ such that $\varphi_{s(a)}$ is a total enumeration of W_a, if W_a is not empty, and define $f \in R^{(1)}$ by

$$\varphi_{f(i)}(0) = \varphi_{s(i)}(0),$$

$$\varphi_{f(i)}(a+1) = \begin{cases} \text{first } n \text{ enumerated with } n \in W_i, & \\ n \prec_B \varphi_{f(i)}(a), \text{ and } n \prec_B \varphi_{s(i)}(a+1) & \text{if such an } n \text{ exists,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $\varphi_{f(i)}$ is defined on an initial segment of ω. In addition, $\varphi_{f(i)}(a) \prec_B \varphi_{f(i)}(a \perp 1)$, for all $a \in \mathrm{dom}(\varphi_{f(i)})$ with $a > 0$. If $i \in \mathrm{dom}(\mathcal{F})$, the collection of all B_n with $n \in W_i$ is a strong base of some neighbourhood filter $\mathcal{H}$. Thus, $\varphi_{f(i)}$ is a total function in this case and as $\varphi_{s(i)}$ enumerates W_i, we obtain that the set of all B_n with $n \in \mathrm{range}(\varphi_{f(i)})$ is also a strong base of $\mathcal{H}$. Now, let $\hat{\mathrm{pt}} \in P^{(1)}$ witness that $\hat{x}$ allows effective limit passing and let the function $g \in R^{(1)}$ such that range$(g) \subseteq$ dom$(\mathcal{F})$ and range$(\mathcal{F} \circ g) = \mathrm{Pt}$. Then we have for $i \in \mathrm{dom}(g)$ that $f(g(i))$ is an index of a normed recursive enumeration of a base of basic open sets of $\mathcal{F}_{g(i)}$. Moreover, it converges to $\hat{x}_{\hat{\mathrm{pt}}(f(g(i)))}$. Hence $\mathcal{F}_{g(i)} = \mathcal{N}(\hat{x}_{\hat{\mathrm{pt}}(f(g(i)))}) = \mathcal{F}_{v(\hat{\mathrm{pt}}(f(g(i))))}$, for all $i \in \mathrm{dom}(g)$, which shows that $\mathcal{F} \circ g \leq \mathcal{F} \circ v$.

For the converse implication let $\hat{x}$ be the total indexing of T according to Proposition 2.2. We only have to show that $x \leq \hat{x}$. Let $L \subseteq \omega$ witness that x is computable and let $w \in R^{(1)}$ with $W_{w(i)} = \{ n \mid \langle i, n \rangle \in L \}$. Then $\{ B_n \mid n \in W_{w(i)} \}$ is a strong base of $\mathcal{N}(x_i)$, for $i \in \mathrm{dom}(x)$. It follows that range$(\mathcal{F} \circ w) = \mathrm{Pt}$. Let $k \in P^{(1)}$ witness that $\mathcal{F} \circ w \leq \mathcal{F} \circ v$. Then $\hat{x}_{k(i)}$, for $i \in \mathrm{dom}(x)$, is the uniquely determined point of T with neighbourhood filter $\mathcal{F}_{v(k(i))}$. As $\mathcal{F}_{v(k(i))} = \mathcal{F}_{w(i)} = \mathcal{N}(x_i)$, we have that $\hat{x}_{k(i)} = x_i$.

It follows from this proof that under the assumptions of the theorem condition (2) holds just if there is some function $h \in P^{(1)}$ so that for all $i \in \mathrm{dom}(x)$, $h(i)\downarrow$ and $\mathcal{N}(x_i) = \mathcal{F}_{v(h(i))}$, *i. e.*, if one can effectively pass from a point to (an enumeration of a strong base of) its neighbourhood filter, a requirement which reverses the condition of allowing effective limit passing.

The conditions (1) and (2) of the above theorem are obviously satisfied if Pt is completely enumerable. But as we shall see next this requirement is very strong and forces the space T to have a very special structure.

Theorem 2.4 *Let $\mathcal{T}$ be effective, x be acceptable, and* Pt *be completely enumerable. Then $\mathcal{T}$ is an algebraic constructive predomain, in which all elements are comparable.*

Proof: By Theorem 2.3 we can assume that x is a total indexing. Let s, $t \in R^{(1)}$ such that $\varphi_{s(a)}$ is a total enumeration of W_a, if this set is not empty, and $W_{t(a)}$ is the r. e. set enumerated by φ_a. Moreover, let W_c be the index set of Pt.

Claim 1 $(T, \leq_\tau)$ is constructively directed-complete.

Let X be a directed enumerable subset of T and $f \in R^{(1)}$ such that range$(x \circ f) = X$. Furthermore, let $v \in R^{(1)}$ with $W_{v(i)} = \{ n \mid x_i \in B_n \}$, for $i \in \omega$, and $W_e = \bigcup \{ W_{v(f(i))} \mid i \in \omega \}$. Set $g(a) = \mu m : \varphi_c(t(a))\downarrow_m$. By the recursion theorem there is some index b such that for all $n \in \omega$

$$\varphi_b(n) = \begin{cases} \varphi_{s(v(f(b)))}(n) & \text{if } \varphi_c(t(b))\uparrow_n, \\ \varphi_{s(e)}(n \perp g(b)) & \text{otherwise.} \end{cases}$$

Assume that $g(b)\uparrow$. Then φ_b is an enumeration of all indices n with $x_{f(b)} \in B_n$. Hence $t(b) \in W_c$, which contradicts our assumption. Thus $g(b)\downarrow$, *i. e.*, $t(b) \in W_c$.

Since $W_{v(f(b))}$ is included in W_e, it follows that φ_b enumerates W_e and hence that $\{ B_n \mid n \in W_e \}$ is a strong filter base of the neighbourhood filter of some point $z \in T$. For $i \in \omega$, $W_{v(f(i))}$ is the index set of a strong filter base of $\mathcal{N}(x_{f(i)})$. Therefore, z is an upper bound of X. Let x_a be another upper bound of X. Then $W_{v(f(i))}$ is a subset of $W_{v(a)}$, for all $i \in \omega$, which implies that W_e is contained in $W_{v(a)}$ and thus that $z \leq_\tau x_a$. This shows that z is the least upper bound of X.

Claim 2 There are functions h, $k \in R^{(1)}$ such that $x_{h(m)}$ is finite and $B_{k(m)} = \{ y \in T \mid x_{h(m)} \leq_\tau y \}$, for all $m \in \omega$. Moreover, for every $n \in \omega$, B_n is the union of all $B_{k(\langle i,n \rangle)}$ with $x_i \in B_n$,

Let $w \in R^{(1)}$ so that $W_{w(n)}$ is the set of all indices i with $x_i \in B_n$, for all $n \in \omega$. In addition, let the function $\mathrm{pt} \in P^{(1)}$ witness that x allows effective limit passing and let the function $q \in R^{(1)}$ be as in Lemma 1.11. Set $g(b,n) = \mu m : \varphi_c(t(b))\downarrow_m \wedge \varphi_{w(n)}(\mathrm{pt}(b))\downarrow_m$. By the recursion theorem there is then a function $r \in R^{(2)}$ such that

$$\varphi_{r(i,n)}(m) = \begin{cases} \varphi_{q(i)}(m) & \text{if } \varphi_c(t(r(i,n)))\uparrow_m \text{ or } \varphi_{w(n)}(\mathrm{pt}(r(i,n)))\uparrow_m, \\ \varphi_{q(i)}(g(r(i,n),n) \perp 1) & \text{otherwise.} \end{cases}$$

Now, let $x_i \in B_n$ and assume that $g(r(i,n),n)\uparrow$. Then $r(i,n)$ is an index of a normed enumeration of basic open sets that converges to x_i. Hence, $t(r(i,n))$ is an index of $\mathcal{N}(x_i)$, *i. e.*, $t(r(i,n)) \in W_c$. Moreover, we have that $\mathrm{pt}(r(i,n))\downarrow$ and $x_{\mathrm{pt}(r(i,n))} = x_i$. It follows that $x_{\mathrm{pt}(r(i,n))} \in B_n$, which implies that $g(r(i,n),n)\downarrow$ in contradiction to our assumption. Therefore $g(r(i,n),n)\downarrow$, for all i, $n \in \omega$ with $x_i \in B_n$. Set $\hat{g}(i,n) = g(r(i,n),n) \perp 1$.

Since $t(r(i,n)) \in W_c$, the collection of all $B(\varphi_{r(i,n)}(a))$ $(a \in \omega)$, *i. e.*, the set $\{B(\varphi_{q(i)}(0)), \ldots, B(\varphi_{q(i)}(\hat{g}(i,n)))\}$, is a strong base of the neighbourhood filter of some finite point $z \in T$. Then $z \leq_\tau x_i$, as $x_i \in B(\varphi_{q(i)}(a))$, for all $a \in \omega$. Assume next that there is some $y \in B(\varphi_{q(i)}(\hat{g}(i,n)))$ with $z \not\leq_\tau y$. Then there exists a basic open set B_m such that $z \in B_m$, but $y \notin B_m$. Because $\mathcal{N}(z)$ is generated by the basic open set $B(\varphi_{q(i)}(\hat{g}(i,n)))$, we have that this set is included in B_m. It follows that $y \in B_m$. This contradicts our choice of B_m. So, we obtain that

$$B_{\varphi_{q(i)}(\hat{g}(i,n))} = \{ y \in T \mid z \leq_\tau y \}.$$

As $\mathcal{B}$ is a strong basis of the topology, there is some index a such that $a \prec_B \varphi_{q(i)}(\hat{g}(i,n))$ and $z \in B_a$. Moreover, since$\{B(\varphi_{q(i)}(\hat{g}(i,n)))\}$ is a strong filter base of $\mathcal{N}(z)$, we obtain that $\varphi_{q(i)}(\hat{g}(i,n)) \prec_B a$. Thus $\varphi_{q(i)}(\hat{g}(i,n)) \prec_B \varphi_{q(i)}(\hat{g}(i,n))$, which implies that $r(i,n)$ is an index of a normed enumeration of basic open sets converging to z. Therefore $x_{\mathrm{pt}(r(i,n))} = z$.

Because $g(r(i,n),n)\downarrow$, we know that $z \in B_n$. It follows that $B(\varphi_{q(i)}(\hat{g}(i,n)))$ is a subset of B_n. As we have already seen, $x_i \in B(\varphi_{q(i)}(\hat{g}(i,n)))$. Thus, B_n is the union of all sets $B(\varphi_{q(i)}(\hat{g}(i,n)))$ with $i \in \omega$ such that $x_i \in B_n$.

By the computability of the numbering x there is some function $d \in R^{(1)}$ which enumerates all pairs $\langle i,n \rangle$ with $x_i \in B_n$. Define

$$h(m) = \mathrm{pt}(r(\pi_1(d(m)), \pi_2(d(m)))) \quad \text{and} \quad k(m) = \varphi_{q(i)}(\hat{g}(\pi_1(d(m)), \pi_2(d(m)))),$$

then it follows from above that both functions have the desired properties.

Claim 3 For all $m \in \omega$, $x_{h(m)}$ is compact.

Let $m \in \omega$ and X be a directed subset of T with least upper bound z such that $x_{h(m)} \leq_\tau z$. Then $z \in B_{k(m)}$. Since the neighbourhood filter $\mathcal{N}(z)$ of z is the union of all neighbourhood filters $\mathcal{N}(y)$, for $y \in X$, it follows that there is some $y \in X$ so that $y \in B_{k(m)}$, *i. e.*, $x_{h(m)} \leq_\tau y$.

Claim 4 For $i \in \omega$, $x_i = \bigsqcup \{ x_{h(\langle i,n \rangle)} \mid n \in \omega \wedge x_i \in B_n \}$.

As we have already seen, x_i is an upper bound of all $x_{h(\langle i,n \rangle)}$ with $n \in \omega$ such that $x_i \in B_n$. Let $y \in T$ be a further upper bound of these elements and assume that $x_i \in B_a$, for some $a \in \omega$. Then $y \in B_{k(\langle i,a \rangle)}$ and hence $y \in B_a$, as $B_{k(\langle i,a \rangle)}$ is contained in B_a. Thus $x_i \leq_\tau y$, which shows that x_i is the least upper bound of all $x_{h(\langle i,n \rangle)}$, for $n \in \omega$ such that $x_i \in B_n$.

It follows that the set of all $x_{h(m)}$ $(m \in \omega)$ is an algebraic basis of $(T, \leq_\tau)$. Moreover, the collection of all sets $\{ y \in T \mid x_{h(m)} \leq_\tau y \}$ is a basis of topology τ. Since $x_{h(m)} \leq_\tau x_{h(n)}$ if and only if $x_{h(n)} \in B_{k(m)}$, we have that the restriction of the specialization order to the basis $\{ x_{h(a)} \mid a \in \omega \}$ is completely enumerable. Similarly, we obtain for each element x_i that the set of all $x_{h(m)}$ with $m \in \omega$ such that $x_{h(m)} \leq_\tau x_i$ is completely enumerable.

Claim 5 Any two elements of T are comparable.

Let x_i, $x_j \in T$ and assume that both are incomparable. Moreover, let K be the halting set and define $g(n) = \mu a : n \in K_a$. Then there is a function $f \in R^{(1)}$ such that

$$\varphi_{f(n)}(a) = \begin{cases} \varphi_{q(i)}(a) & \text{if } n \notin K_a \text{ or } a \text{ is even,} \\ \varphi_{q(j)}(a \perp g(n)) & \text{otherwise.} \end{cases}$$

If $n \notin K$ it follows that the collection of all B_m with $m \in \text{range}(\varphi_{f(n)})$ is a strong base of $\mathcal{N}(x_i)$. Hence $t(f(n)) \in W_c$.

If $n \in K$ the set of all $B(\varphi_{f(n)}(a))$ such that $a < g(n)$ or a is even is a strong base of $\mathcal{N}(x_i)$ and the set of all $B(\varphi_{f(n)}(a))$ such that $a \geq g(n)$ and a is odd is a strong base of $\mathcal{N}(x_j)$. Since x_i and x_j are incomparable, we obtain that the collection of all B_m with $m \in \text{range}(\varphi_{f(n)})$ is not a filter base. Thus $t(f(n)) \notin W_c$.

This shows that $\overline{K} \leq W_c$, which implies that W_c is not r. e., a contradiction. It follows that x_i and x_j are comparable.

The question arises whether under the above assumptions $\mathcal{T}$ must also have a smallest element. As follows from the next example, this does not hold.

Example 2.5 Let $(Q, \sqsubseteq, Z, \beta, x)$ be an algebraic constructive predomain which is an infinitely descending chain. Then the basic open sets B_i with $B_i = \{\, y \in T \mid \beta_i \sqsubseteq y \,\}$ are comparable. It follows that if W_j is nonempty then the set of all B_i with $i \in W_j$ is a strong filter base of the neighbourhood filter of the least upper bound of all β_i with $i \in W_j$. This shows that that j is the index of a strong base of basic open sets of the neighbourhood filter of some point in Q if and only if W_j is not empty. As a consequence we obtain that Pt is completely enumerable in this case.

We close this section by a necessary and sufficient condition for Pt to be completely enumerable which should be compared with Theorem 2.3.

Proposition 2.6 Pt *is completely enumerable if and only if there are functions* v, $g \in R^{(1)}$ *such that*

1. $\text{range}(\mathcal{F} \circ v) = \text{Pt}$ *and*

2. *for all indices* i *such that* $\text{range}(\varphi_i) \subseteq \text{dom}(\mathcal{F})$ *and* $\text{range}(\mathcal{F} \circ \varphi_i) = \text{Pt}$, $\varphi_i = v \circ \varphi_{g(i)}$.

Proof: Assume that Pt is completely enumerable and let $A \subseteq \omega$ be its index set. Moreover, let $v \in R^{(1)}$ be an enumeration of A. Then there is some function $g \in R^{(1)}$ such that $\varphi_{g(i)}(j) = \mu a : v(a) = \varphi_i(j)$. Obviously, the functions v and g have the desired properties.

For the proof of the converse implication let $i \in \text{range}(v)$ and $j \in \omega$ such that $\mathcal{F}_j = \mathcal{F}_i$. We want to show that also $j \in \text{range}(v)$. Let to this end b, $c \in \omega$ with $v(c) = i$ and

$$\varphi_b(a) = \begin{cases} j & \text{if } a = c, \\ v(a) & \text{otherwise.} \end{cases}$$

Then we have that $\mathcal{F}(\varphi_b(c)) = \mathcal{F}_j = \mathcal{F}_i$ and $\mathcal{F}(\varphi_b(a)) = \mathcal{F}_{v(a)}$, for $a \neq c$. Hence $\text{range}(\mathcal{F} \circ \varphi_b) = \text{Pt}$. It follows that $j = \varphi_b(c) = v(\varphi_{g(b)}(c))$, which shows that $j \in \text{range}(v)$.

3 Domain-like spaces and total numberings

In the last section we derived a necessary and sufficient effectivity requirement for the existence of a total acceptable numbering of a given effective T_0-space. Now, in this section, we consider a more restricted class of spaces, which includes constructive predomains, A- and f-spaces, and present a structural necessary and sufficient condition for the existence of a total acceptable indexing of the space.

An essential property of constructive predomains, A- and f-spaces, is that their canonical topology has a basis with every basic open set B_n being an upper set generated by a point which is not necessarily included in B_n but in $\text{hl}(B_n)$, where

$$\text{hl}(B_n) = \bigcap \{\, B_m \mid n \prec_B m \,\}.$$

Definition 3.1 Let $\mathcal{T} = (T, \tau)$ be a countable T_0-space with a countable strong basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. We say that $\mathcal{T}$ is *effectively pointed*, if there is a function $\mathrm{pd} \in P^{(1)}$ such that for all $n \in \mathrm{dom}(B)$, $\mathrm{pd}(n)\downarrow \in \mathrm{dom}(x)$, $x_{\mathrm{pd}(n)} \in \mathrm{hl}(B_n)$ and $x_{\mathrm{pd}(n)} \leq_\tau z$, for all $z \in B_n$.

Obviously,

$$B_n \subseteq \{\, z \in T \mid x_{pd(n)} \leq_\tau z \,\} \subseteq \mathrm{hl}(B_n).$$

The next result, which is derived in [15], shows that effectively pointed spaces have typical properties of domains.

Lemma 3.2 *Let $\mathcal{T}$ be effective and effectively pointed and let x be computable. Moreover, let $y \in T$ and $n \in \omega$. Then the following hold:*

1. *$\mathcal{T}$ is recursively separable with dense base $\{\, x_a \mid a \in \mathrm{range}(pd) \,\}$.*
2. *The set $\{\, x_{pd(a)} \mid y \in B_a \,\}$ is directed and y is its least upper bound.*
3. *If m is an index of a converging normed recursive enumeration of basic open sets, then the enumeration converges to the least upper bound of $\left(x_{pd(\varphi_m(a))}\right)_{a\in\omega}$.*
4. *If y is finite, then $y \in \{\, x_a \mid a \in \mathrm{range}(pd) \,\}$.*
5. *If $x_{pd(n)}$ is finite, then $\mathrm{hl}(B_n) = \{\, z \in T \mid x_{pd(n)} \leq_\tau z \,\}$.*

As we will see first, for effectively pointed spaces the existence of a total acceptable numbering has very strong consequences: the space is constructively directed-complete and the topology is the Scott topology.

Proposition 3.3 *Let $\mathcal{T}$ be effective and effectively pointed. Moreover, let x be total and acceptable. Then $(T, \leq_\tau)$ is constructively directed-complete and τ is the Scott topology.*

Proof: Let X be an enumerable directed subset of T. Then

$$C = \{\, n \in \omega \mid X \cap B_n \neq \emptyset \,\}$$

is r.e. Since X is not empty, the same is true for C. Let $f \in R^{(1)}$ be an enumeration of C. Moreover, let $v \in R^{(1)}$ such that $W_{v(n)} = \{\, i \in \omega \mid x_i \in B_n \,\}$ and let $A \subseteq \omega$ be r.e. so that $X = \{\, x_i \mid i \in A \,\}$. Finally, let $\mathrm{sb} \in P^{(3)}$ witness that $\mathcal{T}$ is effective. Define

$$g(0) = f(0),$$
$$g(a+1) = \mathrm{sb}(\text{first } i \text{ enumerated with } i \in A \cap W_{v(g(a))} \cap W_{v(f(a+1))}, g(a), f(a+1)).$$

Claim 1 For all $a \in \omega$, $g(a)\downarrow \in C$.

By definition $g(0) \in C$. Assume that $g(a) \in C$. Then X intersects both $B_{g(a)}$ and $B_{f(a+1)}$. Let this be witnessed by y, $y' \in X$. Since X is directed, there is some $y'' \in X$ such that y, $y' \leq_\tau y''$. The point y'' witnesses that $X \cap B_{g(a)} \cap B_{f(a+1)}$ is not empty. It follows that $g(a+1)\downarrow$. Let $\hat{\imath}$ be the first i enumerated with $i \in A \cap W_{v(g(a))} \cap W_{v(f(a+1))}$. Then $x_{\hat{\imath}} \in B_{g(a+1)}$ and $x_{\hat{\imath}} \in X \cap B_{g(a)} \cap B_{f(a+1)}$. Hence X intersects $B_{g(a+1)}$, *i. e.*, $g(a+1) \in C$.

It follows that $g \in R^{(1)}$. Now, let $p \in R^{(2)}$ be as in Lemma 1.11 and let pt, $\mathrm{pd} \in P^{(1)}$, respectively, witness that x allows effective limit passing and $\mathcal{T}$ is effectively pointed. Moreover, let $h \in P^{(2)}$ and r, $s \in R^{(2)}$ be defined by

$$h(a, j) = \mu c : \varphi_{v(g(a))}(j)\downarrow_c,$$

$$r(0, j) = 0,$$

$$r(a+1, j) = \begin{cases} r(a, j) & \text{if } h(r(a,j), j)\uparrow_a \text{ or there is some } n \leq a \text{ such that } \varphi_{v(n)}(j)\downarrow_a, \\ & \text{and for all } m \leq r(a,j)+1,\ \varphi_{v(n)}(\mathrm{pd}(g(m)))\uparrow_a, \\ r(a,j)+1 & \text{otherwise,} \end{cases}$$

and

$$s(a,j) = \begin{cases} \max\{\, c \le a \mid r(c,j) \neq r(c+1,j)\,\} & \text{if for some } c \le a,\ r(c+1,j) \neq r(c,j), \\ 0 & \text{otherwise.} \end{cases}$$

By the recursion theorem there is some index c such that

$$\varphi_c(0) = \varphi_{p(\mathrm{pd}(g(1)),g(0))}(0)$$

$$\varphi_c(a+1) = \begin{cases} \varphi_{p(\mathrm{pd}(g(r(a,\mathrm{pt}(c))+1)),\varphi_c(s(a,\mathrm{pt}(c))))}(a \perp s(a,\mathrm{pt}(c))) & \text{if } r(a+1,\mathrm{pt}(c)) = \\ & r(a,\mathrm{pt}(c)), \\ \varphi_{p(\mathrm{pd}(g(r(a,\mathrm{pt}(c))+2)),\varphi_c(a))}(0) & \text{otherwise.} \end{cases}$$

Then c is an index of a normed recursive enumeration of basic open sets which starts to converge to $x_{\mathrm{pd}(g(1))}$, until $x_{\mathrm{pt}(c)}$ has been found in $B_{g(0)}$, say in a steps, and for all $n \le a$ such that $x_{\mathrm{pt}(c)}$ is found in B_n there is some $e \le 1$ such that also $x_{\mathrm{pd}(g(e))}$ is found in B_n, then goes on converging to $x_{\mathrm{pd}(g(2))}$, until $x_{\mathrm{pt}(c)}$ has been found in $B_{g(1)}$, say in a' steps, and for all $n \le a'$ such that $x_{\mathrm{pt}(c)}$ is found in B_n there is some $e' \le 2$ such that also $x_{\mathrm{pd}(g(e'))}$ is found in B_n, and so on.

Claim 2 The function k with $k(a) = r(a,\mathrm{pt}(c))$ is of unrestricted growth.

Obviously, the function k is monotone. Assume that there is some $a \in \omega$ such that $k(\bar a) = k(a)$, for all $\bar a \ge a$. Then c is an index of a normed recursive enumeration of basic open sets converging to $v_{\mathrm{pd}(g(k(a)+1))}$. Thus $\mathrm{pt}(c)\!\downarrow$ and $x_{\mathrm{pt}(c)} = x_{\mathrm{pd}(g(k(a)+1))}$. Since $g(k(a)+1) \prec_B g(k(a))$, we have that $x_{\mathrm{pd}(g(k(a)+1))} \in B_{g(k(a))}$. It follows that $h(k(a),\mathrm{pt}(c))\!\downarrow$. Let $\bar a = \max\{a, \mu j : h(k(a),\mathrm{pt}(c))\!\downarrow_j\}$. Then $h(k(a),\mathrm{pt}(c))\!\downarrow_{\bar a}$. By our assumption on k and a there is therefore some $n \le \bar a$ with $\varphi_{v(n)}(\mathrm{pt}(c))\!\downarrow_{\bar a}$ such that $\varphi_{v(n)}(\mathrm{pd}(g(m)))\!\uparrow$, for all $m \le k(a)+1$. It follows that $x_{\mathrm{pt}(c)} \in B_n$ and hence $x_{\mathrm{pd}(g(k(a)+1))} \in B_n$, which means that $\varphi_{v(n)}(\mathrm{pd}(g(k(a)+1))\!\downarrow$, in contradiction to what we have seen before.

It follows that for all $a \in \omega$, $h(a,\mathrm{pt}(c))\!\downarrow$, which implies that $\mathrm{pt}(c)\!\downarrow$. Let $z = x_{\mathrm{pt}(c)}$.

Claim 3 The point z is an upper bound of X.

By the construction of the function φ_c we have that $z \in B_{g(a)}$, for all all $a \in \omega$. Thus, z is an upper bound of all $x_{\mathrm{pd}(g(a))}$. Now, let $y \in X$. By Lemma 3.2, y is the least upper bound of all points $x_{\mathrm{pd}(n)}$ such that $y \in B_n$. If $y \in B_n$, it follows that $n \in C$. As a consequence of the construction of the function g, there exists a number a_n, for each $n \in C$, such that $g(a_n) \prec_B n$. Hence $x_{\mathrm{pd}(n)} \le_\tau x_{\mathrm{pd}(g(a_n))} \le_\tau z$, which implies that $y \le_\tau z$.

Claim 4 Let z' be another upper bound of X. Then $z \le_\tau z'$.

As a further consequence of the unrestricted growth of the function k we have that for every $n \in \omega$ with $z \in B_n$ there is some $m \in \omega$ such that $x_{\mathrm{pd}(g(m))} \in B_n$. By Claim 1 we know that X intersects $B_{g(a)}$, for all $a \in \omega$. It follows that X intersects B_n, for all $n \in \omega$ with $z \in B_n$. Now, let $n \in \omega$ so that $z \in B_n$. Then there is some $y \in X \cap B_n$. Hence $y \le_\tau z'$. It follows that also $z' \in B_n$, which shows that $z \le_\tau z'$.

We obtain that z is the least upper bound of X. Moreover, we have seen that X intersects each basic open set B_n with $z \in B_n$, which means that τ is the Scott topology.

The next result shows that the above condition is not only necessary but also sufficient for the existence of a total acceptable numbering of space T.

Proposition 3.4 *Let $\mathcal{T}$ be effective, effectively pointed, and constructively directed-complete. Moreover, let x be computable. Then T has a total numbering $\hat x$ which allows effective limit passing. If, in addition, τ is the Scott topology the indexing $\hat x$ is acceptable.*

Proof: Let $L \subseteq \omega$ witness that x is computable and let $v \in R^{(1)}$ such that $W_{v(n)} = \{\, i \mid \langle i,n\rangle \in L\,\}$. Moreover, let $s \in R^{(1)}$ be an enumeration of all indices i such that W_i is not empty and let $r \in R^{(1)}$ such that for those $i \in \omega$ for which W_i is not empty, $\varphi_{r(i)}$ enumerates all indices n for which $W_{v(n)}$ intersects $\mathrm{pd}(W_i)$. Here, the function $\mathrm{pd} \in P^{(1)}$ witnesses that $\mathcal{T}$ is effectively pointed. Set $f = r \circ s$ and let the function $\text{sb} \in P^{(3)}$ witness that $\mathcal{T}$ is effective. Furthermore, define

$$\begin{aligned} g(i,0) &= \varphi_{f(i)}(0), \\ g(i,a+1) &= \text{sb}(\text{first } n \text{ enumerated with } n \in \text{pd}(W_{s(i)}) \cap W_{v(g(i,a))} \cap W_{v(\varphi_{f(i)}(a+1))}), \\ &\qquad g(i,a), \varphi_{f(i)}(a+1)). \end{aligned}$$

Claim 1 If $\{ x_{\text{pd}(n)} \mid n \in W_{s(i)} \}$ is directed, then $g(i, a+1)\downarrow \in \text{range}(\varphi_{f(i)})$.

By definition $g(i,0) \in \text{range}(\varphi_{f(i)})$. Assume that $g(i,a)\downarrow \in \text{range}(\varphi_{f(i)})$. Then we have for $X = \{ x_{\text{pd}(n)} \mid n \in W_{s(i)} \}$ that X intersects both $B_{g(i,a)}$ and $B(\varphi_{f(i)}(a+1))$. Since X is directed, it follows as in the above proof that the common intersection of these three sets is also not empty and hence that $g(i,a+1)\downarrow$. Let $\hat{n}$ be the first n enumerated with $n \in \text{pd}(W_{s(i)}) \cap W_{v(g(i,a))} \cap W_{v(\varphi_{f(i)}(a+1))}$. Then $x_{\hat{n}} \in X \cap B_{g(i,a+1)}$, *i. e.*, $g(i,a+1) \in \text{range}(\varphi_{f(i)})$.

Let $h \in R^{(1)}$ with $\varphi_{h(i)}(a) = g(i,a)$. As follows from the construction, for every $i \in \omega$, $(\varphi_{h(i)}(a))$ is a nonempty finite or infinite sequence that is decreasing with respect to the strong inclusion relation $\prec_B$. Thus, the set $\{ x_a \mid a \in \text{range}(\text{pd} \circ \varphi_{h(i)}) \}$ is directed. Since $\mathcal{T}$ is constructively directed-complete, it has a least upper in T. Set

$$\hat{x}_i = \bigsqcup \{ x_a \mid a \in \text{range}(\text{pd} \circ \varphi_{h(i)}) \}.$$

Then $\hat{x}$ is total.

Claim 2 If the set $\{ x_{\text{pd}(a)} \mid a \in W_{s(i)} \}$ is directed, the point $\hat{x}_i$ is its least upper bound.

Let $X = \{ x_{\text{pd}(n)} \mid n \in W_{s(i)} \}$ and let z be its least upper bound. As we have seen in the proof of Claim 1, there is some $y \in X \cap B_{g(i,a)}$, for every $a \in \omega$. It follows that $x(\varphi_{h(i)}(a)) \leq_\tau y \leq_\tau z$, for each a. Hence $\hat{x}_i \leq_\tau z$. For the verification of the converse inequality let $n \in W_{s(i)}$ and $x_{\text{pd}(n)} \in B_m$. By the construction of the function g there is then some number a_m such that $g(i,a_m) \prec_B m$. We obtain that $x_{\text{pd}(m)} \leq_\tau x_{\text{pd}(g(i,a_m))} \leq_\tau \hat{x}_i$ and thus, by Lemma 3.2, that $x_{\text{pd}(n)} \leq_\tau \hat{x}_i$, which implies that also $z \leq_\tau \hat{x}_i$.

Since every point $y \in T$ is the least upper bound of the enumerable directed set of all $x_{\text{pd}(n)}$ with $y \in B_n$, it follows that the map $\hat{x}$ is onto. Thus, it is a numbering of T.

Claim 3 The indexing $\hat{x}$ allows effective limit passing.

Let m be an index of a normed recursive enumeration of basic open sets which converges to a point $y \in T$. Then y is the least upper bound of the set of all $x(\text{pd}(\varphi_m(a)))$ with $a \in \omega$, by Lemma 3.2. As this set is directed, it follows with Claim 2 that $y = \hat{x}_{\hat{t}(m)}$. Here $\hat{t}(n) = \mu i : t(n) = s(i)$, where $t \in R^{(1)}$ such that $W_{t(n)} = \text{range}(\varphi_n)$, for $n \in \omega$.

Now, in addition, assume that τ is the Scott topology.

Claim 4 The indexing $\hat{x}$ is computable.

Since $\hat{x}_i$ is the least upper bound of a directed set and τ is the Scott topology, we have

$$\begin{aligned} \hat{x}_i \in B_n \quad &\Leftrightarrow \quad \bigsqcup \{ x_a \mid a \in \text{range}(\text{pd} \circ \varphi_{h(i)}) \} \in B_n \\ &\Leftrightarrow \quad (\exists a) a \in \text{range}(\text{pd} \circ \varphi_{h(i)}) \wedge x_a \in B_n. \end{aligned}$$

Thus, the set $\hat{L} = \{ \langle i,n \rangle \mid (\exists a) a \in \text{range}(\text{pd} \circ \varphi_{h(i)}) \wedge \langle a,n \rangle \in L \}$ witnesses that $\hat{x}$ is computable.

Summing up what we have proved so far, we obtain the subsequent result.

Theorem 3.5 *Let $\mathcal{T}$ be effective and effectively pointed. Moreover, let x be computable. Then T has a total acceptable numbering, if and only if $\mathcal{T}$ is constructively directed-complete and τ is the Scott topology.*

The next result extends the necessary and sufficient condition which ensures the existence of a total acceptable numbering of the space so that the indexing can be constructed in such a way that it is even complete.

Theorem 3.6 *Let $\mathcal{T}$ be effective and effectively pointed. Moreover, let x be computable. Then T has a total acceptable complete numbering with special element $\perp \in T$, if and only if $\mathcal{T}$ is constructively directed-complete, τ is the Scott topology, and T has a smallest element $\perp$.*

Proof: Assume that $\mathcal{T}$ is constructively directed-complete, τ is the Scott topology, and T has a smallest element $\perp$. Then T is obviously basic open. Let $\bar{n}$ be an index of T with respect to the indexing B. Moreover, let $s \in R^{(1)}$ such that $W_{s(i)} = W_i \cup \{\bar{n}\}$. Let the function $r \in R^{(1)}$ be as in the proof of Proposition 3.4 and set $f = r \circ s$. Define the function $g \in P^{(2)}$ with this function f as in the proof of Proposition 3.4. All other functions and sets which are not defined here will be as in that proof. As there it follows that the set $\{ x_a \mid a \in \operatorname{range}(\operatorname{pd} \circ \varphi_{h(i)}) \}$ is directed.

Set

$$\hat{x}_i = \bigsqcup \{ x_a \mid a \in \operatorname{range}(\operatorname{pd} \circ \varphi_{h(i)}) \}.$$

Then we obtain that $\hat{x}$ is a total acceptable numbering of T. Note that in the proof that $\hat{x}$ allows effective limit passing one has now to use t as witnessing function.

It remains to show that $\hat{x}$ is complete with special element $\perp$. Let $k \in P^{(1)}$ and define $v \in R^{(1)}$ such that $W_{v(i)} = \{ n \mid k(i)\downarrow \wedge \langle k(i), n \rangle \in \hat{L} \}$. By Lemma 3.2 we obtain for $i \in \operatorname{dom}(k)$ that the set of all points $x_{\operatorname{pd}(a)}$ with $a \in W_{v(i)}$ is directed with least upper bound $\hat{x}_{k(i)}$. Therefore, it follows with Proposition 3.4, Claim 2 that $\hat{x}_{v(i)} = \hat{x}_{k(i)}$, for $i \in \operatorname{dom}(k)$. If $i \notin \operatorname{dom}(k)$, the set $W_{s(v(i))}$ contains only the index $\bar{n}$. Hence $\hat{x}_{v(i)} = \hat{x}_{\operatorname{pd}(\bar{n})} = \perp$.

Now, conversely, let T have a total acceptable complete numbering with special element $\perp$. It remains to show that $\perp$ is the smallest element of T. Let $n \in \omega$ with $\perp \in B_n$ and $\hat{x}_a \in T$. Moreover, assume that $\hat{x}_a \notin B_n$. Since $\hat{x}$ is computable, B_n is completely enumerable. Thus, the function d which is defined by $d(i) = a$, if $\hat{x}_i \in B_n$, and is undefined, otherwise, is partial recursive. By the completeness of $\hat{x}$ there is a total recursive function $\bar{d}$ so that

$$\hat{x}_{\bar{d}(i)} = \begin{cases} \hat{x}_a & \text{if } \hat{x}_i \in B_n, \\ \perp & \text{otherwise.} \end{cases}$$

Moreover, the fixed point theorem holds with respect to $\hat{x}$ [4]. Hence, there is some index e such that $\hat{x}_e = \hat{x}_{\bar{d}(e)}$. If we assume that $\hat{x}_e \in B_n$, we obtain that $\hat{x}_e = \hat{x}_a$ and thus that $\hat{x}_e \notin B_n$. On the other hand, if we assume that $\hat{x}_e \notin B_n$, we have that $\hat{x}_e = \perp$, which implies that $\hat{x}_e \in B_n$. It follows that $\hat{x}_a \in B_n$. This shows that $\perp \leq_\tau \hat{x}_a$.

4 Embedding into totally indexed spaces

In the Section 2 we have seen that there are examples of effective T_0-spaces with an acceptable partial numbering such that the numbering cannot be totalized. As followed from a result in [15], this is always the case if the space contains no finite points. The question which we want to investigate in this section is the following:

> Given an effective T_0-space $\mathcal{T}$ with a partial acceptable numbering x. Can $\mathcal{T}$ be embedded into another effective space $\hat{\mathcal{T}}$ with a *total* acceptable indexing $\hat{x}$ such that $\hat{x}$ extends x?

Since we are dealing with effective topological spaces, the embedding should of course be effective and preserve the topological structure.

Definition 4.1 Let $\mathcal{T} = (T, \tau)$ and $\mathcal{T}' = (T', \tau')$, respectively, be countable topological spaces with countable topological bases $\mathcal{B}$ and $\mathcal{B}'$, and let x, x', B, and B', respectively, be numberings of T, T', $\mathcal{B}$, and $\mathcal{B}'$. Then a map $F: T \to T'$ is is said to be

1. *effective* if there is a function $f \in P^{(1)}$ such that $f(i)\downarrow \in \operatorname{dom}(x')$ and $F(x_i) = x'_{f(i)}$, for all $i \in \operatorname{dom}(x)$.

2. *effectively continuous* if there is a function $h \in R^{(1)}$ such that for all $n \in \operatorname{dom}(B')$, $W_{h(n)} \subseteq \operatorname{dom}(B)$ and $F^{-1}(B'_n) = \bigcup \{ B_m \mid m \in W_{h(n)} \}$.

In case F is an embedding, *i. e.*, one-to-one, it is called *effectively homeomorphic*, if both F and its partial inverse $F^{-1}: F(T) \to T$ are effectively continuous.

In the last section it was shown that effectively pointed effective spaces do have a total acceptable indexing, if the space is constructively directed-complete and the topology is the Scott topology. In the applications such spaces contain many finite points. An obvious idea is therefore to construct $\hat{T}$ by adding sufficiently many finite elements to T.

Eršov [4] showed for total numberings that every indexed set (X, x) can be effectively embedded into an indexed set $(\hat{X}, \hat{x})$ with a complete indexing $\hat{x}$. The set $\hat{X}$ is obtained from X by adding a new element, say $\perp$, to X which is the special element of the numbering $\hat{x}$. This construction can also be applied to the case of partial indexings. The result is again a superset with a total complete numbering. But note that the embedding is only effective. In general it is not effectively homeomorphic. Moreover, the indexing $\hat{x}$ need not be computable.

To see this assume that $\hat{\mathcal{T}}$ is an effective space with $\hat{T} = T \cup \{\perp\}$ and the embedding of T into $\hat{T}$ is effectively homeomorphic. Moreover, assume that $\hat{x}$ is computable and T is not basic open. Let $\hat{B}_n$ be a basic open set in topology $\hat{\tau}$. Then it is completely enumerable. As we have seen in the proof of Theorem 3.6, $\hat{B}_n$ is already the whole space $\hat{T}$ if it contains the special element $\perp$. Thus, if $\hat{B}_n$ is not $\hat{T}$, it is contained in T, which means that is also open in the induced topology on T. Now, let F be the embedding of T into $\hat{T}$ and let $G: F(T) \to T$ be its partial inverse. Moreover, let $g \in R^{(1)}$ witness the effective continuity of G. Then we have that

$$\begin{aligned} F(T) &= G^{-1}(T) = \bigcup \{\, G^{-1}(B_n) \mid n \in \omega \,\} = \bigcup \{\, \hat{B}_m \cap F(T) \mid (\exists n \in \omega) m \in W_{g(n)} \,\} \\ &= \bigcup \{\, \hat{B}_m \mid (\exists n \in \omega) m \in W_{g(n)} \,\}. \end{aligned}$$

Note here that $G^{-1}(B_n)$ and hence $\hat{B}_m$ are strictly included in $F(T)$, for $m \in W_{g(n)}$ and $n \in \omega$. As the numbering $\hat{x}$ is both total and computable and the embedding F is effective, it follows that $\mathrm{dom}(x)$ is r. e., which is not the case, in general. Consider for example the metric space of all total recursive functions with the Baire metric and let it be indexed by the restriction of some Gödel numbering. Then it is an effective space with an acceptable numbering. But, as is well known, the domain of the indexing is a Π^0_2-complete set.

In what follows, by starting from a given effective space with an r. e. strong inclusion relation and a partial acceptable indexing we construct a new space which has a total acceptable numbering and in which the given space can be embedded in an effective and effectively homeomorphic way. The points of the new space will be r. e. filters of the reflexive hull of the given strong inclusion relation extended by a greatest element.

Theorem 4.2 *Let $\mathcal{T}$ be effective, x be acceptable, and $\prec_B$ be r. e. Then there is an algebraic constructive domain $\hat{T}$ with a total acceptable complete numbering $\hat{x}$ and an effectively homeomorphic embedding $F: T \to \hat{T}$ such that both F and its partial inverse are effective and $F(T)$ is an enumerable dense subset of $\hat{T}$.*

Proof: For $m, n \in \omega$ define

$$m \preccurlyeq n \quad \Leftrightarrow \quad n = 0 \vee m = n \vee [m \neq 0 \wedge n \neq 0 \wedge m \perp 1 \prec_B n \perp 1].$$

Then the relation $\preccurlyeq$ is obviously r. e., reflexive, and transitive with 0 as greatest element. Define $\hat{T}$ to be the set of all r. e. filters of $\preccurlyeq$, *i. e.*, the collection of all nonempty r. e. subsets of ω which are upwards closed with respect to $\preccurlyeq$ and which with any two elements m and n contain an element a such that $a \preccurlyeq m, n$, and order it by set inclusion. Then $(\hat{T}, \subseteq)$ is a partial order with the filter $\{0\}$ as smallest element.

Our first aim is to construct a total numbering $\hat{x}$ of $\hat{T}$. Let to this end $s \in R^{(1)}$ be an enumeration of all indices i such that W_i is not empty and let $r \in R^{(1)}$ such that for those $i \in \omega$ for which W_i is not empty, $\varphi_{r(i)}$ enumerates W_i. Set $f = r \circ s$. Moreover, let A be the set of all coded pairs $\langle m, n \rangle$ with $m \preccurlyeq n$ and E be the set of all $\langle m, n, n' \rangle$ such that $\langle m, n \rangle$, $\langle m, n' \rangle \in A$. Then there are functions $h, k \in R^{(1)}$ such that $\varphi_{h(i)}(0) = \varphi_{f(i)}(0)$, $\varphi_{k(i)}(0) = 1$, and the following conditions hold:

1. If $\langle \varphi_{h(i)}(a), \varphi_{f(i)}(\varphi_{k(i)}(a)) \rangle \in A_{a+1}$, then $\varphi_{h(i)}(a+1) = \varphi_{h(i)}(a)$ and $\varphi_{k(i)}(a+1) = \varphi_{k(i)}(a) + 1$.

2. If (1) does not hold and $\langle \varphi_{f(i)}(\varphi_{k(i)}(a)), \varphi_{h(i)}(a)\rangle \in A_{a+1}$, then $\varphi_{h(i)}(a+1) = \varphi_{f(i)}(\varphi_{k(i)}(a))$ and $\varphi_{k(i)}(a+1) = \varphi_{k(i)}(a)+1$.
3. If (1) and (2) do not hold and there is some $m \leq a+1$ such that $m \in W_{s(i),a+1}$ and $\langle m, \varphi_{h(i)}(a), \varphi_{f(i)}(\varphi_{k(i)}(a))\rangle \in E_{a+1}$, then $\varphi_{h(i)}(a+1) = \mu m \leq a+1 : m \in W_{s(i),a+1} \wedge \langle m, \varphi_{h(i)}(a), \varphi_{f(i)}(\varphi_{k(i)}(a))\rangle \in E_{a+1}$ and $\varphi_{k(i)}(a+1) = \varphi_{k(i)}(a)+1$.
4. If (1)-(3) do not hold then $\varphi_{h(i)}(a+1) = \varphi_{h(i)}(a)$ and $\varphi_{k(i)}(a+1) = \varphi_{k(i)}(a)$.

Obviously $\varphi_{h(i)}(a) \in W_{s(i)}$, for all $a \in \omega$.

Claim 1 If $W_{s(i)}$ is a filter, then the function $\varphi_{k(i)}$ is of unrestricted growth.

Assume that there is some $a \in \omega$ such that $\varphi_{k(i)}(\bar{a}) = \varphi_{k(i)}(a)$, for all $\bar{a} \geq a$. Then $\varphi_{k(i)}(a)$ and $\varphi_{f(i)}(\varphi_{k(i)}(a))$ are not comparable with respect to $\preccurlyeq$ because of conditions (1) and (2). Since $W_{s(i)}$ is a filter, there is some $m \in W_{s(i)}$ such that $m \preccurlyeq \varphi_{k(i)}(a), \varphi_{f(i)}(\varphi_{k(i)}(a))$. It follows that condition (3) holds with respect to some argument $\bar{a} \geq a$. This implies that $\varphi_{k(i)}(\bar{a}) > \varphi_{k(i)}(a)$, which contradicts our assumption.

Now, define $\hat{x}$ by

$$\hat{x}_i = \{ m \in \omega \mid (\exists a)\varphi_{h(i)}(a) \preccurlyeq m \}.$$

Then $\hat{x}_i \in \hat{T}$, for every $i \in \omega$.

Claim 2 The map $\hat{x}$ is onto.

Let $\hat{y} \in \hat{T}$. Since $\hat{y}$ is nonempty and r.e., there is some index i such that $\hat{y} = W_{s(i)}$. Hence $\hat{x}_i \subseteq \hat{y}$, as $\mathrm{range}(\varphi_{h(i)}) \subseteq W_{s(i)}$. For the converse inclusion let $c \in \hat{y}$. Then there exists some $a \in \omega$ with $c = \varphi_{f(i)}(a)$. As we have just seen, the function $\varphi_{k(i)}$ ranges over all nonzero natural numbers. Moreover, $\varphi_{h(i)}(0) = \varphi_{f(i)}(0)$. Thus, it follows from the above construction that there is some $\bar{a} \geq a$ such that $\varphi_{h(i)}(\bar{a}) \preccurlyeq c$, which means that $c \in \hat{x}_i$.

This shows that $\hat{x}$ is a total numbering of $\hat{T}$. As it is readily verified, $(\hat{T}, \subseteq)$ is constructively directed-complete with respect to this numbering. Note that the least upper bound of a directed enumerable subset of $\hat{T}$ is the union of all filters in this set.

As next step we construct a basis for $\hat{T}$. For $n \in \omega$ set

$$\hat{z}_n = \{ m \in \omega \mid n \preccurlyeq m \}.$$

Then $\hat{z}_n$ is an r.e. filter.

Claim 3 For all $n \in \omega$, $\hat{z}_n$ is compact.

Let $\hat{X}$ be an directed subset of $\hat{T}$ such that $\hat{z}_n \subseteq \bigcup \hat{X}$. Then $n \in \bigcup \hat{X}$, which means that there is some $\hat{y}$ in $\hat{X}$ with $n \in \hat{y}$. Since $\hat{y}$ is a filter, it follows that $\hat{z}_n \subseteq \hat{y}$.

Claim 4 For all $\hat{y} \in \hat{T}$ the set $\{ \hat{z}_n \mid \hat{z}_n \subseteq \hat{y} \}$ is directed and $\hat{y}$ is its least upper bound.

Let $\hat{z}_m, \hat{z}_n \subseteq \hat{y}$. Then $m, n \in \hat{y}$. Hence there is some $a \in \hat{y}$ such that $a \preccurlyeq m, n$. It follows that $\hat{z}_m, \hat{z}_n \subseteq \hat{z}_a$. Thus, the set of all $\hat{z}_n$ with $\hat{z}_n \subseteq \hat{y}$ is directed. In order to see that $\hat{y}$ is its least upper bound, note for $a \in \bigcup \{ \hat{z}_n \mid \hat{z}_n \subseteq \hat{y} \}$ that $a \in \hat{z}_m$, for some $m \in \omega$ such that $\hat{z}_m \subseteq \hat{y}$, and hence that $a \in \hat{y}$. If, conversely, $a \in \hat{y}$ then $\hat{z}_a \subseteq \hat{y}$, which shows that $a \in \bigcup \{ \hat{z}_n \mid \hat{z}_n \subseteq \hat{y} \}$.

As $\hat{y}$ is r.e., for $\hat{y} \in \hat{T}$, and $\hat{z}_n \subseteq \hat{y}$ exactly if $n \in \hat{y}$, the set of all $n \in \omega$ with $\hat{z}_n \subseteq \hat{y}$ is r.e. Moreover, since $\hat{z}_m \subseteq \hat{z}_n$ if and only if $n \preccurlyeq m$, we also have that $\{ \langle m, n\rangle \mid \hat{z}_m \subseteq \hat{z}_n \}$ is r.e. Finally, with $t \in R^{(1)}$ such that $W_{s(t(n))} = \{n\}$ it is $\hat{x}_{t(n)} = \hat{z}_n$. Thus, $\hat{T}$ is an algebraic constructive domain.

The Scott topology on $\hat{T}$ has as canonical basis the collection of all sets

$$\hat{B}_n = \{ \hat{y} \in \hat{T} \mid \hat{z}_n \subseteq \hat{y} \}.$$

As

$$\hat{x}_i \in \hat{B}_n \Leftrightarrow \hat{z}_n \subseteq \hat{x}_i \Leftrightarrow n \in \hat{x}_i \Leftrightarrow (\exists a)\varphi_{h(i)}(a) \preccurlyeq n,$$

we obtain that the indexing $\hat{x}$ is computable.

Claim 5 The numbering $\hat{x}$ allows effective limit passing.

Let $\hat{\text{pt}} \in P^{(1)}$ such that $W_{s(\hat{\text{pt}}(a))} = \{ n \in \omega \mid (\exists c)\varphi_a(c) \preccurlyeq n \}$. Moreover, let m be an index of a converging normed recursive enumeration of basic open sets. We have to show that $\hat{\text{pt}}(m)\downarrow$ in this case and that the enumeration converges to $\hat{x}_{\hat{\text{pt}}(m)}$. As

$$\hat{B}_a \subseteq \hat{B}_c \Leftrightarrow \hat{z}_c \subseteq \hat{z}_a \Leftrightarrow a \preccurlyeq c,$$

it follows that the set $\{ n \in \omega \mid (\exists c)\varphi_m(c) \preccurlyeq n \}$ is an r.e. filter. Hence, it is not empty, which means that $\hat{\text{pt}}(m)\downarrow$. Moreover, $\hat{x}_{\hat{\text{pt}}(m)} = W_{s(\hat{\text{pt}}(m))}$. Since $\hat{x}_{\hat{\text{pt}}(m)} \in \hat{B}_n$ if and only if $n \in \hat{x}_{\hat{\text{pt}}(m)}$, *i.e.*, if and only if $\varphi_m(a) \preccurlyeq n$, for some $a \in \omega$, we have that $\hat{x}_{\hat{\text{pt}}(m)} \in \hat{B}(\varphi_m(c))$, for all $c \in \omega$. Furthermore , for all $n \in \omega$, if $\hat{x}_{\hat{\text{pt}}(m)} \in \hat{B}_n$ then there is some $a \in \omega$ such that $\hat{B}(\varphi_m(a)) \subseteq \hat{B}_n$. Thus, the collection of all $\hat{B}(\varphi_m(c))$, for $c \in \omega$, is a strong base of the neighbourhood filter of $\hat{x}_{\hat{\text{pt}}(m)}$.

It follows that the numbering $\hat{x}$ is acceptable, which by Theorem 3.6 implies that it is also complete.

Since for any $y \in T$ the set of all B_n such that $y \in B_n$ is a strong base of the neighbourhood filter of y, it follows that $\{ n+1 \mid y \in B_n \} \cup \{0\}$ is a filter with respect to $\preccurlyeq$. We define $F: T \to \hat{T}$ by

$$F(y) \quad = \quad \{ n+1 \mid y \in B_n \} \cup \{0\}.$$

As every point in T is uniquely determined by its neighbourhood filter and/or a base of it, the map F is one-to-one. Moreover, since x is computable, we have that it is also effective. Note that $x_i \in B_n$ if and only if $F(x_i) \in \hat{B}_{n+1}$. Hence, both F and F^{-1} are effectively continuous. In addition, we obtain that $F(T)$ is dense in $\hat{T}$.

Claim 6 F^{-1} is effective.

Let $w \in R^{(1)}$ such that $\varphi_{w(a)}$ is a total enumeration of all numbers n with $n+1 \in W_a$, if there are such numbers, and define $g \in R^{(1)}$ by

$$\varphi_{g(i)}(0) = \varphi_{w(i)}(0),$$
$$\varphi_{g(i)}(a+1) = \begin{cases} \text{first } n \text{ enumerated with } n+1 \in W_i, & \\ n \prec_B \varphi_{g(i)}(a), \text{ and } n \prec_B \varphi_{w(i)}(a+1) & \text{if such an } n \text{ exists,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

If $\hat{x}_i \in F(T)$ there is some point $y \in T$ such that $\hat{x}_i = \{ n+1 \mid y \in B_n \} \cup \{0\}$. Moreover, $\hat{x}_i = W_{s(i)}$ in this case. It follows that $\{ B_n \mid n+1 \in W_{s(i)} \}$ is a strong base of the neighbourhood filter of y. Hence, $g(s(i))$ is an index of a normed recursive enumeration of basic open set which converges to y. Now, let the function pt $\in P^{(1)}$ witness that the numbering x allows effective limit passing. Then $F^{-1}(\hat{x}_i) = x_{\text{pt}(g(s(i)))}$.

The idea to obtain a total acceptable extension of an acceptable numbering x of an effective space $\mathcal{T}$ was to enlarge T by sufficiently many finite elements. By definition an element $y \in T$ is finite if it has a singleton strong base, say $\{B_n\}$. This means that for all B_m with $y \in B_m$ one has that $n \prec_B m$. It follows that $F(y) = \hat{z}_{n+1}$. This shows that the embedding F preserves finiteness. If one thinks of a basic open set as a finitely describable property, the finite elements are characterized by only a finite part of the (infinite) information used to describe the total elements of the space.

The following definitions are essentially due to Berger [1], who gave them in the context of Scott domains.

Definition 4.3 Let (T, τ) be a T_0-space.

1. A finite subset $\{y_0, \ldots, y_a\}$ of T is called *separable*, if there are open sets $O_0, \ldots, O_a$ such that $y_0 \in O_0, \ldots, y_a \in O_a$ and the intersection $O_0 \cap \cdots \cap O_a$ is empty. We say in this case that $O_0, \ldots, O_a$ *separate* $y_0, \ldots, y_a$.

2. A system $\mathcal{U}$ of disjunct pairs of open sets *separates* $\{y_0, \ldots, y_a\}$, if there are $O_0, \ldots, O_a$ among the first components of the pairs in $\mathcal{U}$ which separate $y_0, \ldots, y_a$.

3. $\mathcal{U}$ is *separating* if it separates every separable finite subset of T.

For a subset M of T, let

$$\mathcal{E}(M) \quad = \quad \{ (O_1, O_2) \in \tau \times \tau \mid M \subseteq O_1 \cup O_2 \}.$$

M is a *total* set if the system $\mathcal{E}(M)$ is separating and an element y of T is *total* if the singleton $\{y\}$ is a total set. Note that all elements of a total set are total.

Of course, one would like that the embedding F also preserves totality. But this is not true, in general. The next result should be compared with [1, Lemma 5].

Proposition 4.4 *Let $\mathcal{T}$ be effective such that all basic open sets are also closed. Moreover, let x be acceptable, $\prec_B$ be r. e., $\hat{T}$ be the algebraic constructive domain constructed in Theorem 4.2, and $F: T \to \hat{T}$ be the embedding of T in $\hat{T}$. Then $F(T)$ is total.*

Proof: Since the inverse image of B_n under F^{-1} is $\hat{B}_{n+1} \cap F(T)$, we have that $\hat{B}_{n+1} \cap F(T)$ is both open and closed in the induced topology on $F(T)$, for all $n > 0$. It follows that $\mathcal{E}(F(T))$ contains all pairs $(\hat{B}_n, \mathrm{ext}(\hat{B}_n))$, where $\mathrm{ext}(\hat{B}_n)$ is the exterior of $\hat{B}_n$. Note here that $\hat{B}_0$ is $\hat{T}$. As a finite set of points is separable if and only if it can be separated by basic open sets, we obtain that $\mathcal{E}(F(T))$ is separating, which means that $F(T)$ is total.

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