Effectivity and Effective Continuity of Multifunctions*

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Abstract

If one wants to compute with infinite objects like real numbers or data streams, continuity is a necessary requirement: better and better (finite) approximations of the input are transformed in better and better (finite) approximations of the output. In case the objects are constructively generated, they can be represented by a finite description of the generating procedure. By effectively transforming such descriptions for the generation of the input (respectively, their codes) in (the code of) a description for the generation of the output another type of computable operation is obtained. Such operations are also called effective. The relationship of both classes of operations has always been a question of great interest and well known theorems such as those of Myhill and Shepherdson, Kreisel, Lacombe and Shoenfield, Ceĭtin, and/or Moschovakis present answers for important special cases. A general, unifying approach has been developed by the present author in [36].

In this paper the approach is extended to the case of multifunctions. Such functions appear very naturally in applied mathematics, logic and theoretical computer science. Various ways of coding (indexing) sets are discussed and their relationship is investigated. Moreover, effective versions of several continuity notions for multifunctions are introduced. For each of these notions an indexing system for sets is exhibited so that the multifunctions that are effective with respect to this indexing system are exactly the multifunction which are effectively continuous with respect to the continuity notion under consideration. Mostly, in addition to being effective the multifunctions need also possess certain witnessing functions. Important special cases are discussed where such witnessing functions always exist.

1 Introduction

As is well known [23, 42], the test whether for two given real numbers the first is smaller than the second is not computable as a map from the reals to, say, $\{0, 1\}$. This fact creates a serious problem to the design of programming languages for real number computations, as tests of this kind abundantly appear in algorithms. The problem can be solved by using the relaxed tests $<_k$ (k a natural number) instead, which compare two reals with a given uncertainty of 2^{-k}:

$$x <_k y = \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x + 2^{-k} > y. \end{cases}$$

However, these test functions are over-defined: for real numbers x and y with $y - 2^{-k} < x < y$ both values 0 and 1 are possible.

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Consider the equation

$$f(x) = u.$$

Among others one would be interested in knowing whether the solutions behave well under small perturbations of the right hand side. One will have to study $f^{-1}(u)$ as a function of u in this case. But this is a *set-valued* or *multifunction* in general.

There are many more examples showing that multifunctions occur very naturally in mathematical practice. They have indeed been used with great success in various branches of mathematics, logic and computer science and there is already a vast literature (cf. e.g. [2, 3, 30, 33, 40]).

In this paper we will study multifunctions in the setting of effective topological spaces [36]. These are second-countable T_0 spaces where we assume that there has already been a way to define what are their computable points and it is only these elements that our spaces contain. We moreover expect the space to come with a canonical numbering of its elements as well as an indexing of its topological basis. Here, we follow M. B. Smyth's approach [34] and think of the basic open sets as easy to encode observations that can be made about the computational process determining the elements. So, we let the indexing of the basic open sets be total. As has been shown in [35] however, in general we cannot expect canonical numberings of the points to be total as well. By a canonical numbering we mean a numbering that is obtained from a coding of the computational process determining the elements in such a way that we can enumerate all basic open sets containing a given point, uniformly in any of its indices.

By doing better and better observations we want finally be able to determine every element. (A second requirement for a numbering to be canonical or, as we will later say, *acceptable* is that this can be done in an effective way.) Thus, we need a relation of definite refinement between the basic open sets which in many cases will be stronger than set inclusion. In most applications it will be recursively enumerable. As it turns out in these cases, the refinement relation is a relation between the codes of the basic open sets rather than the sets itself.

Therefore, we assume that the indexing of the basic open sets is such that there is a transitive relation on the indices so that the property of being a topological basis holds with respect to this relation instead of just set inclusion. The property of being a base of the topology is a $\forall\exists$ statement. We require it to be realised by a computable function on the involved indices. This leads us to the notion of an *effective space*.

Note that we think of the topological basis with its numbering and the associated refinement relation as being part of the structure under consideration. This seems to be a typical feature of constructive approaches: constructive notions may depend on how objects are represented.

A well known prerequisite for a (single-valued) function to be computable is its continuity. It allows to transform converging approximations of the argument in converging approximations of the function value. All one has to ensure in addition is that this can be done in an effective way. In the framework of effective spaces, however, there is also another kind of functions that could be called computable. Since our spaces contain only points that can be approximated in an effective way, each point can be represented by a program that computes such an approximation, or a code of it. This is the way the already mentioned numbering of the points is obtained. What this other kind of functions do is simply to effectively map (the codes of) programs generating an approximation of the argument to (the codes of) programs generating an approximation of the function value. We call such functions *effective*.

Functions computable in the first way are also computable in the second way, i.e., they are also effective. The converse is not true in general, but it is true in some important special cases such as constructive domains and recursive metric spaces [28, 22, 11, 27, 13, 41]. In [36] this situation has been analysed in detail and a further condition has been presented that forces any effective map between effective spaces to be computable. As was shown, in the case of effective maps between constructive domains or recursive metric spaces the extra condition is always satisfied.

It is the aim of this paper to study the analogous question for set-valued maps. To do so we first have to look for a suitable coding system by which we can represent the values of such maps. In the point case the codes were obtained by effectively enumerating sufficiently many basic properties of a point, uniquely determining it. In the set case there are too many subsets to be uniquely representable by codes. So, we no longer require the objects under consideration to be uniquely determined by the properties we are listing. The coding system will induce an equivalence relation among the subsets of a given space and what is actually listed are properties of certain canonical members of the respective equivalence classes. In computations only information that does not distinguish between members of a class is used. Our attitude is that the objects we are dealing with are given by other means. We will examine several coding systems for sets of different strength.

A function is computable in the above way if it is effectively continuous. For multifunctions several continuity notions are in use. We consider at least three of them and discuss effective versions. For each of them the question of when an effective multifunction is effectively continuous is studied and sufficient conditions are exhibited. Note that in two cases the outcome is a consequence of the central result in [36]. Finally it is shown that the extra conditions are satisfied in the case of constructive domains and effectively given metric spaces. In all these investigations the choice of the coding system for sets we are using will be important.

The paper is organised as follows: Section 2 contains basic definitions and properties. The notion of an effective space as well as results that are needed in later sections are recalled in Section 3. Moreover, important, standard examples of such spaces are discussed. In Section 4 various subspace indexings are introduced and their interrelation is studied.

Notions of effective continuity for multifunctions are defined in Sections 5 and 6. By applying one of the central results in [36] it is shown in Section 5 that multifunctions are effectively lower semi-continuous just if they are effective with respect to hit indices and possess certain witness functions, and that compact-valued multifunctions are effectively upper semi-continuous exactly if they are effective with respect to covering indices and also possess certain witness functions. Hit indices allow the generation of all basic open sets that meet the indexed set, and from a covering index one can compute all finite covers of the indexed set, where in this case only compact sets are considered. As is shown in [36], the witnessing condition is always satisfied if the domain space of the multifunction is a constructive domain. In the present paper we will show that it is always satisfied as well, if the range space of the multifunction is an effectively given metric space. However, the multifunction has to satisfy stronger effectivity requirements in this case. In order to obtain lower semi-continuity e.g., we must, for each value of the multifunction, uniformly be able to list all basic open sets missing the value set. An example given in Section 7 will show that the theorem does not hold without this extra requirement.

In Section 6 the effective outer semi-continuity of multifunctions is studied. Outer semi-continuous multifunctions have been considered by Rockafellar and Witts [30]. A multifunction is effectively outer semi-continuous if, and only if, it is jointly effective with respect to density and closedness indices. A density index of a set codes a procedure generating a dense subset of the given set and a closedness index witnesses that the complement of the set is effectively open.

The research reported on here has been started in the 1990's. Preliminary versions of some results have been presented at the workshop "Computability and Models", Almaty, Kazakhstan, June 24-28, 2002, the "Second Irish Conference on the Mathematical Foundations of Computer Science and Information Technology", Galway, Ireland, July 18-19, 2002, and the workshop "From Sets and Types to Topology and Analysis: Towards Practical Foundations for Constructive Mathematics", Venice, Italy, May 12-16, 2003. It has been taken up again when the author was visiting the Universities of Cape Town and Stellenbosch, South Africa, in 2006, and results have been presented at the workshop "Trends in Constructive Mathematics", Frauenwörth, Germany, June 19-23, 2006 and the joint workshop "Domains VIII" and "Computability over Continuous Data Types", Novosibirsk, Russia, November 11-15, 2007.

2 Basic definitions and properties

In what follows, let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a recursive pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2\rangle) = a_i$, and let D be a standard coding of all finite subsets of natural numbers. Moreover, let $P^{(n)}$ ($R^{(n)}$) denote the set of all n-ary partial (total) recursive functions, and let W_i be the domain of the ith partial recursive function φ_i with respect to some Gödel numbering φ. We let $\varphi_i(a){\downarrow}$ mean that the computation of $\varphi_i(a)$ stops, $\varphi_i(a){\downarrow} \in C$ that it stops with value in C, and $\varphi_i(a){\downarrow_n}$ that it stops within n steps. In the opposite cases we write $\varphi_i(a){\uparrow}$ and $\varphi_i(a){\uparrow_n}$ respectively.

Let S be a nonempty set. If X is a subset of S, then its complement $S \setminus X$ will be denoted by $\overline{X}$. A *(partial) numbering* ν of S is a partial map $\nu\colon \omega \rightharpoonup S$ (onto) with domain $\mathrm{dom}(\nu)$. The value of ν at $n \in \mathrm{dom}(\nu)$ is denoted, interchangeably, by ν_n and $\nu(n)$. Note that instead of numbering we also say *indexing*.

Definition 2.1 For numberings ν and κ of set S, ν is *reducible* to κ, written $\nu \leq \kappa$, if there is a function $g \in P^{(1)}$ such that $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(g)$, $g(\mathrm{dom}(\nu)) \subseteq \mathrm{dom}(\kappa)$, and $\nu_m = \kappa_{g(m)}$, for all $m \in \mathrm{dom}(\nu)$.

Definition 2.2 Let S, S' be nonempty sets with numberings ν and ν', respectively. A map $F\colon S^n \to S'$ is *effective*, if there is a function $f \in P^{(n)}$ such that $f(m_1, \ldots, m_n){\downarrow} \in \mathrm{dom}(\nu')$ and $F(\nu_{m_1}, \ldots, \nu_{m_n}) = \nu'_{f(m_1,\ldots,m_n)}$, for all $m_1, \ldots, m_n \in \mathrm{dom}(\nu)$.

A subset X of S is *completely enumerable (c.e.)*, if there is a recursively enumerable (r.e.) set W_n such that $\nu_i \in X$ if and only if $i \in W_n$, for all $i \in \mathrm{dom}(\nu)$. Set $M_n = X$, for any such n and X, and let M_n be undefined, otherwise. Then M is a numbering of the class *CE* of completely enumerable subsets of S. Every index of X with respect to M is called a *c.e. index* of X. As is easily verified, the collection *CE* is closed under set union and intersection and both operations are effective.

A relation $R \subseteq S \times S$ is *completely enumerable*, if there is an r.e. set A so that $(\nu_i, \nu_j) \in R$ if and only if $\langle i, j\rangle \in A$, for all $i, j \in \mathrm{dom}(\nu)$.

X is *enumerable*, if there is an r.e. set $A \subseteq \mathrm{dom}(\nu)$ such that $X = \{\, \nu_i \mid i \in A \,\}$. Thus, X is enumerable if we can enumerate a subset of the index set of X which contains at least one index for every element of X, whereas X is completely enumerable if we can enumerate all indices of elements of X and perhaps some numbers which are not used as indices by the numbering ν. Any r.e. index of A, that is any index of A with respect to W, is said to be an *enumeration index* of X.

Lemma 2.3 *Let S be a nonempty set with numbering ν. Then the following two statements hold:*

1. *The collection of all enumerable subsets of S is closed under set union and this operation is effective with respect to enumeration indices.*

2. *There is a function $f \in P^{(1)}$ such that for any enumerable subset X of S and any enumeration index n of X, $f(n)\downarrow \in \mathrm{dom}(\nu)$ and $\nu_{f(n)} \in X$.*

In the latter case we say that the collection of all enumerable subsets of S has an *intensional selection function.* As follows from the next example, the intersection of two enumerable sets need not be enumerable again.

Example 2.4 Let $I \subseteq \omega$ be an immune set, i.e. an infinite set without infinite r.e. subset (cf. [31]). Set $S = \{a, b\} \cup I$. Moreover, for $X \subseteq \omega$ let $2X = \{ 2i \mid i \in X \}$. Similarly for $2X + 1$. Now, define the indexing $\nu \colon \omega \to S$ of S by

$$\nu_i = \begin{cases} a & \text{if } i \in 2\overline{I}, \\ b & \text{if } i \in 2\overline{I} + 1, \\ \lfloor \frac{i}{2} \rfloor & \text{otherwise,} \end{cases}$$

and set $X = I \cup \{a\}$ and $X' = I \cup \{b\}$. Obviously, both sets are enumerable: $X = \{ \nu_i \mid i \text{ even} \}$ and $X' = \{ \nu_i \mid i \text{ odd} \}$. Moreover, $X \cap X' = I$. If I were enumerable, there were an r.e. set $A \subseteq \omega$ with $I = \nu(A)$, which means that $A \subseteq 2I \cup (2I + 1)$. Since I is infinite, the same would be true for A. It followed that either the set of all even numbers in A or the set of all odd numbers in A would be infinite as well. Since $\{ i \mid 2i \in A \}, \{ i \mid 2i + 1 \in A \} \subseteq I$, it would follow that I had an infinite r.e. subset, contrary to its choice. So, I is not enumerable.

Every indexing ν of S induces a family of natural topologies on this set. A topology η on S is a *Mal'cev topology* [24], if it has a subbasis $\mathcal{C}$ of completely enumerable subsets of T. Any such subbasis is called a *Mal'cev subbasis.* All Mal'cev subbases on S can be indexed in a uniform canonical way. Let $M^{\eta}_n = M_n$, if $M_n \in \mathcal{C}$, and let it be undefined, otherwise.

Now, let $\mathcal{T} = (T, \tau)$ be a topological T_0 space with countable basis $\mathcal{B}$. We also write $\tau = \langle \mathcal{B} \rangle$ to express that $\mathcal{B}$ is a countable basis and $\tau = \langle\langle \mathcal{B} \rangle\rangle$ in case that $\mathcal{B}$ is a countable subbasis of τ. For any subset X of T, $\mathrm{int}_\tau(X)$, $\mathrm{cl}_\tau(X)$ and $\mathrm{ext}_\tau(X)$, respectively, are the interior, the closure and the exterior of X.

As is well known, each point y of a T_0 space is uniquely determined by its neighbourhood filter $\mathcal{N}(y)$ and/or a base of it. Moreover, on T_0 spaces there is a canonical partial order, the *specialisation order*, which we denote by $\leq_\tau$.

Definition 2.5 Let $\mathcal{T} = (T, \tau)$ be a T_0 space, and $y, z \in T$. $y \leq_\tau z$ if $\mathcal{N}(y) \subseteq \mathcal{N}(z)$.

Let B be a numbering of $\mathcal{B}$. By definition each open set is the union of certain basic open sets. In the context of effective topology one is only interested in enumerable unions. We call an open set $O \in \tau$ *Lacombe*-open or a *Lacombe* set, if there is an r.e. set $A \subseteq \mathrm{dom}(B)$ such that

$$O = \bigcup \{ B_a \mid a \in A \}.$$

Set $L^\tau_n = \bigcup \{ B_a \mid a \in W_n \}$, if $W_n \subseteq \mathrm{dom}(B)$, and let L^τ_n be undefined, otherwise. Then L^τ is a numbering of the Lacombe sets of τ. The indices are called *Lacombe* indices. Obviously, $B \leq L^\tau$.

If we want to deal with the points and open sets of space $\mathcal{T}$ in an effective way, then the interplay between both should at least be such that we can effectively list the points of each basic open set, uniformly in its index. To this end we restrict ourselves to countable spaces.

Definition 2.6 Let $\mathcal{T} = (T, \tau)$ be a countable topological T_0 space with countable basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. We say that x is *computable* if there is some r.e. set $L \subseteq \omega$ such that for all $i \in \mathrm{dom}(x)$ and all $n \in \mathrm{dom}(B)$,

$$\langle i, n \rangle \in L \Leftrightarrow x_i \in B_n.$$

Clearly, if x is computable then every Lacombe set is completely enumerable, uniformly in its Lacombe index, i.e. $L^\tau \leq M$.

Now, we can effectively compare second-countable topologies.

Definition 2.7 Let $\tau = \langle \mathcal{B} \rangle$ and $\eta = \langle\langle \mathcal{C} \rangle\rangle$ be a topologies on T, and B and C, respectively, be numberings of $\mathcal{B}$ and $\mathcal{C}$.

1. $\eta \subseteq_p \tau$, read η is *effectively pointwise coarser* than τ, if there is some function $h \in P^{(2)}$ such that $h(i,m)\downarrow \in \mathrm{dom}(B)$ and $x_i \in B_{h(i,m)} \subseteq C_m$, for all $i \in \mathrm{dom}(x)$ and $m \in \mathrm{dom}(C)$ with $x_i \in C_m$.

2. $\eta \subseteq_e \tau$, read η is *effectively coarser* than τ, if $C \leq L^\tau$.

Lemma 2.8 ([36][1]) *Let x be computable. Then, if η is effectively coarser than τ, it is also effectively pointwise coarser that τ.*

For Mal'cev topologies η the converse holds as well, in case $\mathcal{T}$ and x satisfy certain stronger requirements.

3 Effective spaces

In this section, let $\mathcal{T} = (T, \tau)$ be a countable topological T_0 space with countable basis $\mathcal{B}$.

At first sight the requirement that $\mathcal{T}$ is countable seems quite restrictive. We think of $\mathcal{T}$ as being the subspace of computable elements of some larger space. There are several approaches to topology that come with natural computability notions for points and maps (cf. e.g. [32, 38, 7, 42]). It allows to assign indices to the computable points in a canonical way so that important properties become computable. In general the notion of computable point is rather complex, mainly harder than Σ^0_1. Consequently, the indexings of the computable points thus obtained are only partial maps.

Contrary to this, in most applications the basic open sets have a simple finite description. By coding the descriptions one obtains a total numbering of the topological basis. For us basic open sets are predicates. Each point is uniquely determined by the collection of all predicates it satisfies, thus the T_0 requirement.

Usually, set inclusion between basic open sets is not completely enumerable. But in the applications we have in mind there is a canonical relation between the descriptions of the basic open sets (respectively, their codes), which in many cases is stronger than set inclusion. This relation is r.e. We assume that the topological basis $\mathcal{B}$ comes with a numbering B of its elements and such a relation between the codes.

[1]Note that in earlier papers we always assumed the spaces under consideration to have an indexed basis. In certain cases, however, it is sufficient to require only the existence of an indexed subbasis.

Definition 3.1 Let $\prec_B$ be a transitive binary relation on ω. We say that:

1. $\prec_B$ is a *strong inclusion*, if for all m, $n \in \mathrm{dom}(B)$, from $m \prec_B n$ it follows that $B_m \subseteq B_n$.

2. $\mathcal{B}$ is a *strong basis*, if $\prec_B$ is a strong inclusion and for all $z \in T$ and m, $n \in \mathrm{dom}(B)$ with $z \in B_m \cap B_n$ there is a number $a \in \mathrm{dom}(B)$ such that $z \in B_a$, $a \prec_B m$ and $a \prec_B n$.

For what follows we assume that $\prec_B$ is a strong inclusion with respect to which $\mathcal{B}$ is a strong basis.

Definition 3.2 Let $\mathcal{T} = (T, \tau)$ be a countable topological T_0 space with countable basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. Then $\mathcal{T}$ is *effective*, if B is total and the property of being a strong basis holds effectively, which means that there exists a function $sb \in P^{(3)}$ such that for $i \in \mathrm{dom}(x)$ and m, $n \in \omega$ with $x_i \in B_m \cap B_n$, $sb(i, m, n){\downarrow}$, $x_i \in B_{sb(i,m,n)}$, $sb(i, m, n) \prec_B m$, and $sb(i, m, n) \prec_B n$.

Note that very often the totality of B can easily be achieved, if the space is *recursively separable*, which means that it has a dense enumerable subset, called its *dense base*.

As is readily verified, $\mathcal{T}$ is effective if x is computable, B is total and the strong inclusion relation is r.e.

Since we work with strong inclusion instead of set inclusion, we had to adjust the notion of a topological basis. In the same way we need to modify that of a filter base.

Definition 3.3 Let $\mathcal{H}$ be a filter. A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called *strong base* of $\mathcal{H}$ if the following two conditions hold:

1. For all m, $n \in \mathrm{dom}(B)$ with B_m, $B_n \in \mathcal{F}$ there is some index $a \in \mathrm{dom}(B)$ such that $B_a \in \mathcal{F}$, $a \prec_B m$, and $a \prec_B n$.

2. For all $m \in \mathrm{dom}(B)$ with $B_m \in \mathcal{H}$ there some index $a \in \mathrm{dom}(B)$ such that $B_a \in \mathcal{F}$ and $a \prec_B m$.

If x is computable, a strong base of basic open sets can effectively be enumerated for each neighbourhood filter. For effective spaces this can always be done in a normed way [36].

Definition 3.4 An enumeration $(B_{f(a)})_{a\in\omega}$ with $f : \omega \to \omega$ such that $\mathrm{range}(f) \subseteq \mathrm{dom}(B)$ is said to be *normed* if f is decreasing with respect to $\prec_B$. If f is recursive, it is also called *recursive* and any Gödel number of f is said to be an *index* of it.

In case $(B_{f(a)})$ enumerates a strong base of the neighbourhood filter of some point, we say it *converges* to that point.

Lemma 3.5 ([36]) *Let $\mathcal{T}$ be effective and x be computable. Then there are functions $q \in R^{(1)}$ and $p \in R^{(2)}$ such that for all $i \in \mathrm{dom}(x)$ and all $n \in \omega$ with $x_i \in B_n$, $q(i)$ and $p(i, n)$ are indices of normed recursive enumerations of basic open sets which converge to x_i. Moreover, $\varphi_{p(i,n)}(0) \prec_B n$.*

We want not only to be able to generate normed recursive enumerations of basic open sets that converge to a given point, but conversely, we need also be able to pass effectively from such enumerations to the point they converge to.

Definition 3.6 Let x be a numbering of T. We say that:

1. x *allows effective limit passing* if there is a function $\mathrm{pt} \in P^{(1)}$ such that, if m is an index of a normed recursive enumeration of basic open sets which converges to some point $y \in T$, then $\mathrm{pt}(m)\downarrow \in \mathrm{dom}(x)$ and $x_{\mathrm{pt}(m)} = y$.
2. x is *acceptable* if it allows effective limit passing and is computable.

For neighbourhood filters of points having an enumerable strong base, we can always construct a normed enumeration of a strong base of the same filter. But not every such enumeration needs to converge. This gives rise to the following completeness notion.

Definition 3.7 A T_0 space $\mathcal{T} = (T, \tau, \mathcal{B}, B, \prec_B)$ with a countable strong basis is *constructively complete*, if each normed recursive enumeration of nonempty basic open sets converges.

Note that the constructive completeness of a space may depend on the choice of the topological basis $\mathcal{B}$ (as well as the numbering B and the strong inclusion relation $\prec_B$ belonging to it) (cf. [37]).

Proposition 3.8 ([37]) *Let $\mathcal{T}$ be effective and constructively complete such that all basic open sets are nonempty. Let $\prec_B$ be r.e. and x allow effective limit passing. Then $\mathcal{T}$ is recursively separable.*

As we have already seen, every T_0 space comes equipped with a canonical partial order, the specialisation order. For indexed ordered structures another completeness notion is of importance.

Definition 3.9 Let $(Q, \sqsubseteq)$ be a countable partial order and x be an indexing of Q.

1. A nonempty subset S of Q is *directed*, if for all $y_1, y_2 \in S$ there is some $u \in S$ with $y_1, y_2 \sqsubseteq u$.
2. Q is *constructively d-complete*, if each of its enumerable directed subsets has a least upper bound in Q.

Proposition 3.10 *Let $\mathcal{T}$ be constructively complete. Moreover, let B be total, x be computable and $\prec_B$ be r.e. Then $(T, \leq_\tau)$ is constructively d-complete.*

Proof: Let $L \subseteq \omega$ witness that x is computable and $v \in R^{(1)}$ such

$$W_{v(i)} = \{\, n \in \omega \mid (\exists a \in W_i)\langle a, n\rangle \in L \,\}.$$

Moreover, let $s \in R^{(1)}$ such that $\varphi_{s(i)}$ is a total enumeration of W_i, if this set is not empty. Define $h \in R^{(1)}$ by

$$\varphi_{h(i)}(0) = \varphi_{s(v(i))}(0),$$

$$\varphi_{h(i)}(m+1) = \begin{cases} \text{first } c \text{ enumerated with } c \in W_{v(i)}, & \\ c \prec_B \varphi_{h(i)}(m) \text{ and } c \prec_B \varphi_{s(v(i))}(m+1) & \text{if such a } c \text{ exists,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now, let $i \in \omega$ such that $\{\, x_a \mid a \in W_i \,\}$ is directed. We will show that $\varphi_{h(i)}$ is total in this case.

Since $\{ x_a \mid a \in W_i \}$ is directed, this set is not empty and hence $\varphi_{s(v(i))}$ is a total function. Thus, $\varphi_{h(i)}(0)$ is defined. Assume that $\varphi_{h(i)}(n)$ is defined for all $n \leq m$. By definition of $W_{v(i)}$ there is then some $a_m \in W_i$ so that $x_{a_m} \in B(\varphi_{h(i)}(m))$. Moreover, there is some $b \in W_i$ with $x_b \in B(\varphi_{s(v(i))}(m+1))$. Since $\{ x_a \mid a \in W_i \}$ is directed, we have that there exists some $e \in W_i$ such that $x_{a_m}, x_b \leq_\tau x_e$. Hence $x_e \in B(\varphi_{h(i)}(m)) \cap B(\varphi_{s(v(i))}(m+1))$. Because $\mathcal{B}$ is a strong basis of the topology, we obtain that there is some $c \prec_B \varphi_{h(i)}(m), \varphi_{s(v(i))}(m+1)$ with $x_e \in B_c$. This shows that $\varphi_{h(i)}(m+1)$ is defined as well.

It follows that for $i \in \omega$ such that $\{ x_a \mid a \in W_i \}$ is directed, $h(i)$ is an index of a normed recursive enumeration of nonempty basic open sets. Let the enumeration converge to $y \in T$. If $a \in W_i$ and $m \in \omega$ so that $x_a \in B_m$, then $B(\varphi_{h(i)}(c)) \subseteq B_m$, for some $c \in \omega$. Since $\{ B(\varphi_{h(i)}(\nu)) \mid \nu \in \omega \}$ is a base of the neighbourhood filter of y, we obtain that $y \in B_m$. Thus, y is an upper bound of $\{ x_a \mid a \in W_i \}$.

Let $z \in T$ be a further upper bound of $\{ x_a \mid a \in W_i \}$. Then $z \in B_n$, for all $n \in W_{v(i)}$. If $m \in \omega$ such that $y \in B_m$, then there is some $c \in \omega$ with $B(\varphi_{h(i)}(c)) \subseteq B_m$. Thus, $z \in B_m$, which shows that $y \leq_\tau z$. Therefore, y is the least upper bound of $\{ x_a \mid a \in W_i \}$.

The following result will be needed later.

Lemma 3.11 ([36]) *Let $\mathcal{T}$ be effective and recursively separable with dense base DB. Moreover, let x be acceptable. Then, for any completely enumerable subset X of T and any basic open set B_n, if B_n intersects X, then it also intersects $X \cap DB$.*

At the end of the last section we mentioned that for Mal'cev topologies the converse of Lemma 2.8 is true as well.

Lemma 3.12 ([36]) *Let $\mathcal{T}$ be effective and recursively separable and let x be acceptable. Then any Mal'cev topology on T that is effectively pointwise coarser than τ is also effectively coarser than τ.*

We have already seen, if x is computable, all basic open sets are completely enumerable, which means that τ is a Mal'cev topology. The next condition helps classifying those Mal'cev topologies which are effectively coarser than τ. Let to this end for $n \in \mathrm{dom}(B)$,

$$\mathrm{hl}(B_n) = \bigcap \{ B_m \mid m \in \mathrm{dom}(B) \wedge n \prec_B m \}.$$

Definition 3.13 Let $\eta = \langle\!\langle \mathcal{C} \rangle\!\rangle$ be a topology on T, and C a numbering of $\mathcal{C}$. We say that a pair of functions (s, r) with $s \in P^{(2)}$ and $r \in P^{(3)}$ is a *realiser for noninclusion* of τ with respect to η, if for all $i \in \mathrm{dom}(x)$, $n \in \mathrm{dom}(B)$ and $m \in \mathrm{dom}(C)$ the following hold:

1. If $x_i \in C_m$, then $s(i,m)\downarrow \in \mathrm{dom}(M)$ and $x_i \in M_{s(i,m)} \subseteq C_m$.
2. If moreover $B_n \not\subseteq C_m$, then also $r(i,n,m)\downarrow \in \mathrm{dom}(x)$ and $x_{r(i,n,m)} \in \mathrm{hl}(B_n) \setminus M_{s(i,m)}$.

Theorem 3.14 ([36]) *Let $\mathcal{T}$ be effective and x be acceptable. Then any Mal'cev topology on T with respect to which τ has a realiser for noninclusion is effectively pointwise coarser than τ. If $\mathcal{T}$ is also recursively separable, then any such topology is even effectively coarser than τ.*

Proposition 3.15 *Let $\eta = \langle\!\langle \mathcal{C} \rangle\!\rangle$ be a topology on T with a numbering C of $\mathcal{C}$. If τ has a realiser for noninclusion with respect to itself and η is effectively pointwise coarser than τ, then τ also has a realiser for noninclusion with respect to η.*

Proof: Let $h \in P^{(2)}$ witness that η is effectively pointwise coarser than τ and (s, r) be a realiser for noninclusion of τ with respect to itself. Furthermore, assume that $x_i \in C_m$. Then $h(i,m)\downarrow \in \mathrm{dom}(B)$ and $x_i \in B_{h(i,m)} \subseteq C_m$. It follows that $s(i, h(i,m))\downarrow \in \mathrm{dom}(M)$ and $x_i \in M_{s(i,h(i,m))} \subseteq B_{h(i,m)}$. Define $s' \in P^{(2)}$ by $s'(j,a) = s(j, h(j,a))$.

Suppose in addition that $B_n \not\subseteq C_m$. Then $B_n \not\subseteq B_{h(i,m)}$. So, $r(i, n, h(i,m))\downarrow \in \mathrm{dom}(x)$ and $x_{r(i,n,h(i,m))} \in \mathrm{hl}(B_n) \setminus M_{s(i,h(i,m))}$. Define $r' \in P^{(3)}$ by $r'(j,c,a) = r(j,c,h(j,a))$. Then (s', r') is a realiser for noninclusion of τ with respect to η.

Corollary 3.16 *Let $\mathcal{T}$ be effective, x be acceptable and let τ have a realiser for noninclusion with respect to itself. Then any Mal'cev topology η on T is effectively pointwise coarser than τ if, and only if, τ has a realiser for noninclusion with respect to η.*

If the strong inclusion relation $\prec_B$ is r.e., we effectively obtain positive information about set inclusion between basic open sets: pairs (m, n) are listed such that $B_m \subseteq B_n$. However, no information is obtained, if $B_n \not\subseteq B_m$. Such knowledge is provided, if topology τ has a realiser for noninclusion with respect to itself.

Let us next consider some important standard examples of effective T_0 spaces.

Example 3.17 (Constructive metric spaces). Let $\mathbb{R}$ denote the set of all real numbers, and let ν be some canonical total indexing of the rational numbers. Then a real number z is said to be *computable*, if there is a function $f \in R^{(1)}$ such that for all m, $n \in \omega$ with $m \leq n$, the inequality $|\nu_{f(m)} - \nu_{f(n)}| < 2^{-m}$ holds and $z = \lim_m \nu_{f(m)}$. Any Gödel number of the function f is called an *index* of z. This defines a partial indexing γ of the set $\mathbb{R}_c$ of all computable real numbers.

Now, let $\mathcal{M} = (M, \delta)$ be a separable metric space, and let β be a total numbering of the dense subset M_0. As is well-known, the collection of sets $B_{\langle i,m\rangle} = \{\, y \in M \mid \delta(\beta_i, y) < 2^{-m} \,\}$ (i, $m \in \omega$) is a basis of the canonical Hausdorff topology Δ on M. Define

$$\langle i,m\rangle \prec_B \langle j,n\rangle \Leftrightarrow \delta(\beta_i, \beta_j) + 2^{-m} < 2^{-n}.$$

Using the triangle inequality it is readily verified that $\prec_B$ is a strong inclusion and the collection of all B_a is a strong basis.

$\mathcal{M}$ is said to be *effectively given*, if the distance function δ maps $M_0 \times M_0$ into $\mathbb{R}_c$ and the restriction of δ to this set is effective. Since the usual less-than relation on the computable real numbers is completely enumerable [26], the strong inclusion relation $\prec_B$ is r.e. in this case.

A sequence $(y_a)_{a\in\omega}$ of elements of M_0 is said to be *fast*, if $\delta(y_m, y_n) < 2^{-m}$, for all $m, n \in \omega$ with $m \leq n$. Moreover, (y_a) is *recursive*, if there is some function $f \in R^{(1)}$ such that $y_a = \beta_{f(a)}$, for all $a \in \omega$. Any Gödel number of f is called an *index* of (y_a).

$\mathcal{M}$ is called *constructive*, if it is effectively given and, in addition, each element y of M is the limit of a fast recursive sequence of elements of M_0. If m is the index of such a sequence, set $x_m = y$. Otherwise, let x be undefined. Then x is a numbering of M with respect to which and the indexing γ of the computable real numbers the distance function is effective (cf. [35]). Moreover, x is computable. It follows that $\mathcal{M}$ is effective.

In [36, Lemma 3.2] a function $h \in R^{(1)}$ is constructed such that, if m is an index of a normed recursive enumeration of basic open sets converging to some point $y \in M$, then $h(m)$ is an index of a fast recursive sequence of elements of the dense subset M_0 converging to y as well. Thus, $y = x_{h(m)}$, which shows that x allows effective limit passing. So, x is acceptable.

Theorem 3.18 ([35]) *Let $\mathcal{M}$ be a constructive metric space. Then $\mathcal{M}$ is constructively complete if, and only if, every fast recursive sequence of elements of the dense subset converges.*

Well-known examples of constructive metric spaces include $\mathbb{R}_c^n$ with the Euclidean or the maximum norm, Baire space, that is, the set $R^{(1)}$ of all total recursive functions with the Baire metric [31], and the set ω with the discrete metric. By using an effective version of Weierstraß's Approximation Theorem [29] and Sturm's Theorem [39] it can be shown that $C_c[0,1]$, the space of all computable functions from $[0,1]$ to $\mathbb{R}$ [29] with the supremum norm, is a constructive metric space. A proof of this result and further examples can be found in Blanck [6].

Example 3.19 (Constructive domains). Let $Q = (Q, \sqsubseteq)$ be a partial order. The *way-below relation* $\ll$ on Q is defined as follows: $y_1 \ll y_2$ if for every directed subset S of Q the least upper bound of which exists in Q, the relation $y_2 \sqsubseteq \bigsqcup S$ implies the existence of an element $u \in S$ with $y_1 \sqsubseteq u$. Note that $\ll$ is transitive. Elements $y \in Q$ with $y \ll y$ are called *compact*.

A subset Z of Q is a *basis* of Q, if for any $y \in Q$ the set $Z_y = \{\, z \in Z \mid z \ll y \,\}$ is directed and $y = \bigsqcup Z_y$. A partial order that has a basis is called *continuous*. If all elements of Z are compact, Q is said to be *algebraic* and Z is called *algebraic basis*.

Now, assume that Q is countable and let x be an indexing of Q. Let Q be constructively d-complete and continuous with basis Z. Moreover, let β be a total numbering of Z. Then $(Q, \sqsubseteq, Z, \beta, x)$ is said to be a *constructive pre-domain*, if the restriction of the way-below relation to Z as well as all sets Z_y, for $y \in Q$, are completely enumerable with respect to the indexing β and $\beta \leq x$. In case Q also has a smallest element the structure is called *constructive domain*.

The numbering x of Q is said to be *admissible*, if the set $\{\, \langle i, j \rangle \mid \beta_i \ll x_j \,\}$ is r.e. and there is a function $d \in R^{(1)}$ such that for all indices $i \in \omega$ for which $\beta(W_i)$ is directed, $x_{d(i)}$ is the least upper bound of $\beta(W_i)$. As shown in [41], such numberings always exist. They can even be chosen as total.

Partial orders come with several natural topologies. In the applications we have in mind, one is mainly interested in the *Scott topology* σ: a subset X of Q is open in σ, if it is upwards closed with respect to the partial order and intersects each enumerable directed subset of Q of which it contains the least upper bound. In the case of a constructive domain this topology is generated by the sets $B_n = \{\, y \in Q \mid \beta_n \ll y \,\}$ with $n \in \omega$. It follows that $\mathcal{Q} = (Q, \sigma)$ is a countable T_0-space with countable basis. Observe that the partial order on Q coincides with the specialisation order defined by the Scott topology [21]. Obviously, every admissible numbering is computable. Since Z is dense in $\mathcal{Q}$ we also obtain that $\mathcal{Q}$ is recursively separable.

Define

$$m \prec_B n \Leftrightarrow \beta_n \ll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which the collection of all B_n is a strong basis. Because the restriction of $\ll$ to Z is completely enumerable, $\prec_B$ is r.e. It follows that $\mathcal{Q}$ is effective. Moreover, it is constructively complete and each admissible indexing allows effective limit passing, i.e., it is acceptable. Conversely, every acceptable numbering of Q is admissible.

Note here that since we have to make use of the effectivity characteristics of the basis, these properties can only be verified if we choose the strong inclusion relation as above and do not use simple set inclusion instead.

Examples of constructive domains include the set $P^{(1)}$ of all partial recursive functions with the extension ordering, i.e., $f \sqsubseteq g$ if $\mathrm{graph}(f) \subseteq \mathrm{graph}(g)$, and the flat domain $\omega_\perp$ ($= \omega \cup \{\perp\}$) of the natural numbers, where for $u, v \in \omega_\perp$, $u \sqsubseteq v$ if $u = \perp$. Both domains are algebraic. In the first case the compact elements are just the finite functions indexed in some canonical way. In the second case all domain elements are compact, indexed by $\beta_0 = \perp$ and $\beta_{n+1} = n$, for $n \in \omega$.

Example 3.20 (Constructive A- and f-spaces). A- and f-spaces have been introduced by Eršov [15, 16, 17, 18, 19] as a more topologically oriented approach to domain theory. They are not required to be complete.

Let $\mathcal{Y} = (Y, \rho)$ be a topological T_0 space. For elements $y,\ z \in Y$ define $y \ll z$ if $z \in \mathrm{int}_\rho(\{\, u \in Y \mid y \leq_\rho u \,\})$. $\mathcal{Y}$ is an *A-space*, if there is a subset Y_0 of Y satisfying the following three properties:

1. Any two elements of Y_0 which are bounded in Y with respect to the specialisation order have a least upper bound in Y_0.

2. The collection of sets $\mathrm{int}_\rho(\{\, u \in Y \mid y \leq_\rho u \,\})$, for $y \in Y_0$, is a basis of topology ρ.

3. For any $y \in Y_0$ and $u \in Y$ with $y \ll u$ there is some $z \in Y_0$ such that $y \ll z$ and $z \ll u$.

Any subset Y_0 of Y with these properties is called *basic subspace*.

Let Y be countable and Y_0 have a numbering β. For $m,\ n \in \mathrm{dom}(\beta)$ set $B_n = \mathrm{int}_\rho(\{u \in Y \mid \beta_n \leq_\rho u\,\})$ and define

$$m \prec_B n \Leftrightarrow \beta_n \ll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which $\{\, B_n \mid n \in \mathrm{dom}(\beta) \,\}$ is a strong basis. The A-space $\mathcal{Y}$ with basic subspace Y_0 is *constructive*, if the numbering β is total, the restriction of $\ll$ to Y_0 is completely enumerable, and the neighbourhood filter of each point has an enumerable strong base of basic open sets. As a consequence, Y has an acceptable numbering x such that $\mathcal{Y}$ is effective [36]. Moreover, it is recursively separable with dense basis Y_0.

Since the topology ρ of a constructive A-space is not required to be the Scott topology (with respect to $\leq_\rho$), constructive d-completeness is too weak a completeness notion in this case.

Definition 3.21 A constructive A-space $\mathcal{Y}$ is *effectively complete*, if every enumerable directed subset S of Y with the property that for every $z \in S$ there is some $z' \in S$ with $z \ll z'$, has an upper bound $y \in Y$ which is also a limit point of S.

Obviously, given such a set S we can enumerate a subset S' such that any two elements of S' are comparable with respect to $\ll$ and for every $z \in S$ there is some $z' \in S'$ with $z \ll z'$. This gives us the following result.

Proposition 3.22 ([37]) *A constructive A-space $\mathcal{Y}$ is constructively complete if and only if it is effectively complete.*

Let $\mathcal{Y} = (Y, \rho)$ be an arbitrary topological T_0-space again. An open set V is an *f-set*, if there is an element $z_V \in V$ such that $V = \{\, y \in Y \mid z_V \leq_\rho y \,\}$. The uniquely determined element z_V is called an *f-element*. $\mathcal{Y}$ is an *f-space*, if the following two conditions hold:

1. If U and V are f-sets with nonempty intersection, then $U \cap V$ is also an f-set.

2. The collection of all f-sets is a basis of topology ρ.

An f-space is *constructive*, if the set of all f-elements has a total numbering α such that the restriction of the specialisation order to this set as well as the boundedness of two f-elements are completely recursive and there is a function su $\in R^{(2)}$ such that in the case that α_n and α_m are bounded, $\alpha_{\mathrm{su}(n,m)}$ is their least upper bound, and if the neighbourhood filter of each point has an enumerable base of f-sets.

Every f-space is an A-space with basic subspace the set of all f-elements. Moreover, for y, $z \in Y$ with y or z being an f-element, $y \ll z$ if and only if $y \leq_\rho z$. It follows that also every constructive f-space is a constructive A-space.

An essential property of continuous partial orders, just as of A- and f-spaces, is that their canonical topology has a basis with every basic open set B_n being an upper set generated by a point which is not necessarily included in B_n but in $\mathrm{hl}(B_n)$.

Definition 3.23 Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with a countable strong basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. We say that $\mathcal{T}$ is *effectively pointed*, if there is a function pd $\in P^{(1)}$ such that for all $n \in \mathrm{dom}(B)$ with $B_n \neq \emptyset$, $\mathrm{pd}(n)\downarrow \in \mathrm{dom}(x)$, $x_{\mathrm{pd}(n)} \in \mathrm{hl}(B_n)$ and $x_{\mathrm{pd}(n)} \leq_\tau z$, for all $z \in B_n$.

Obviously,

$$B_n \subseteq \{\, z \in T \mid x_{\mathrm{pd}(n)} \leq_\tau z \,\} \subseteq \mathrm{hl}(B_n).$$

Note that if $\mathcal{T}$ is effectively pointed, it is recursively separable with dense base $\{\, x_a \mid a \in \mathrm{range(pd)} \,\}$ [36].

Since $(T, \leq_\tau)$ is a partial order, we can equip T with the Scott topology σ.

Lemma 3.24 *Let $\mathcal{T}$ be effectively pointed. Moreover, let B be total and x be computable. Then the Scott topology on T is coarser than topology τ.*

Proof: Let $O \in \sigma$ and $y \in O$. As is shown in [35, Lemma 2.22] the set $\{\, x_{\mathrm{pd}(a)} \mid y \in B_a \,\}$ is directed with least upper bound y. Moreover, it is enumerable. By definition of the Scott topology there is therefore some $a \in \omega$ such that $y \in B_a$ and $x_{\mathrm{pd}(a)} \in O$. As O is upwards closed with respect to $\leq_\tau$, it follows that

$$y \in B_a \subseteq \{\, z \in T \mid x_{\mathrm{pd}(a)} \leq_\tau z \,\} \subseteq O.$$

In domain theory it is common to use sequences of points for approximation instead of sequences of basic open sets. Let $(y_a)_{a\in\omega}$ be a sequence of points of T. It is *recursive*, if there is some function $f \in R^{(1)}$ with $\mathrm{range}(f) \subseteq \mathrm{dom}(x)$ such that $y_a = x_{f(a)}$, for all $a \in \omega$. Any Gödel number of f is called an *index* of (y_a). We say that x *allows the computation of least upper bounds*, if there is a function $sp \in P^{(1)}$ such that, if n is an index of a recursive sequence (y_a) which is increasing with respect to the specialisation order and has a least upper bound in T, then $sp(n)\downarrow \in \mathrm{dom}(x)$ and $x_{sp(n)}$ is the least upper bound of (y_a). As we shall see now, the requirement that x allows the computation of least upper bounds not only implies that x allows effective limit passing, but has also strong impacts on the topology. Topology τ is *constructively order-consistent* (cf. [25]), if every recursive sequence of points of T which is increasing with respect to the specialisation order and has a least upper bound in T is eventually in any basic open set that contains its least upper bound.

Proposition 3.25 ([36]) *Let $\mathcal{T}$ be effective and effectively pointed. Moreover, let x be computable. Then x allows the computation of least upper bounds if, and only if, τ is constructively order-consistent and x allows effective limit passing.*

As we have seen above, if topology τ has a realiser for noninclusion with respect to some Mal'cev topology η, then η is effectively coarser than τ. For the results we are aiming for it will be important to know with respect to which Mal'cev topologies such a realiser exists. In the case of effectively pointed spaces and hence in the case of constructive domains as well as constructive A- and f-spaces there is an easy answer.

Proposition 3.26 ([36]) *Let $\mathcal{T}$ be effective and effectively pointed. Moreover, let x be acceptable. Then τ has a realiser for noninclusion with respect to every Mal'cev topology on T.*

If $\mathcal{T}$ is effectively pointed we always have for $n \prec_B m$ that $x_{\mathrm{pd}(n)} \in B_m$. In certain cases we need the converse implication to hold as well (cf. [35, Corr.]).

Definition 3.27 $\mathcal{T}$ is *strongly pointed*, if it is effectively pointed and the function pd is such that for $m, n \in \mathrm{dom}(B)$ with $x_{\mathrm{pd}(n)} \in B_m$ one has that $n \prec_B m$.

For strongly pointed and constructively order-consistent spaces also the converse of Proposition 3.10 holds.

Proposition 3.28 *Let $\mathcal{T}$ be strongly pointed and τ be constructively order-consistent. Moreover, let $(T, \leq_\tau)$ be constructively d-complete. Then $\mathcal{T}$ is constructively complete.*

Proof: Let $(B_{f(i)})_{i\in\omega}$ be a normed recursive enumeration of nonempty basic open sets. Then $(x_{\mathrm{pd}(f(i))})_{i\in\omega}$ is a recursive sequence of points of T that is increasing with respect to the specialisation order. Since T is constructively d-complete, it has a least upper bound, say y. We need to show that $\{ B_{f(i)} \mid i \in \omega \}$ is a strong base of the neighbourhood filter of y. Let $y \in B_m$. Then it suffices to show that for some $i \in \omega$, $f(i) \prec_B m$. Because τ is constructively order-consistent, there is an index i such that $x_{\mathrm{pd}(f(i))} \in B_m$ from which we obtain that $f(i) \prec_B m$ by strong pointedness.

As we shall see next, strong pointedness is a rather powerful notion.

Definition 3.29 A transitive relation $\lll$ on a set S has the *interpolation property*, if for all $m_1, m_2, n \in S$ so that $n \lll m_1, m_2$ there is some $a \in S$ with $n \lll a \lll m_1, m_2$.

Lemma 3.30 *Let $\mathcal{T}$ be strongly pointed. Then the strong inclusion relation $\prec_B$ has the interpolation property.*

Proof: Let $n, m_1, m_2 \in \mathrm{dom}(B)$ with $n \prec_B m_1, m_2$. Then $x_{\mathrm{pd}(n)} \in B_{m_1} \cap B_{m_2}$. Since $\mathcal{B}$ is a strong basis, there is some $a \prec_B m_1, m_2$ with $x_{\mathrm{pd}(n)} \in B_a$. It follows that $n \prec_B a$, as $\mathcal{T}$ is strongly pointed.

Proposition 3.31 *Let $\mathcal{T}$ be constructively complete, B be total, and x allow effective limit passing. Moreover, let $\prec_B$ be r.e. and have the interpolation property. Then $\mathcal{T}$ is strongly pointed.*

Proof: Let $s, r \in R^{(1)}$ such that $\varphi_{s(a)}$ is a total enumeration of W_a, if this set is not empty, and $W_{r(n)} = \{ m \in \omega \mid n \prec_B m \}$, for $n \in \omega$. Define $h \in R^{(1)}$ by

$$\varphi_{h(n)}(0) = \varphi_{s(r(n))}(0),$$

$$\varphi_{h(n)}(a+1) = \begin{cases} \text{first } m \text{ enumerated with } m \in W_{r(n)}, & \\ m \prec_B \varphi_{h(n)}(a) \text{ and } m \prec_B \varphi_{s(r(n))}(a+1) & \text{if such an } m \text{ exists,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since $\prec_B$ has the interpolation property, the function $\varphi_{h(n)}$ is total, for all $n \in \omega$. Thus, for each $n \in \omega$ such that B_n is not empty, $\big(B(\varphi_{h(n)}(a))\big)_{a\in\omega}$ is a normed enumeration of nonempty basic open sets, which converges to a point $z \in T$, as the space is constructively complete. Let the function $\mathrm{pt} \in P^{(1)}$ witness that x allows effective limit passing. Then $z = x_{\mathrm{pt}(h(n))}$.

We first show that $z \in \mathrm{hl}(B_n)$. Let to this end $m \in W_{r(n)}$. Then there is some $c \in \omega$ with $\varphi_{h(n)}(c) \prec_B m$. Since the sets $B(\varphi_{h(n)}(a))$ form a strong base of the neighbourhood filter of z, we have that $z \in B_m$.

Next, we show that $z \leq_\tau y$, for all $y \in B_n$. Let $B_m \in \mathcal{N}(z)$. Then there is again some $c \in \omega$ with $\varphi_{h(n)}(c) \prec_B m$. Because $\varphi_{h(n)}(c) \in W_{r(n)}$ we obtain that $n \prec_B m$. Consequently, $B_n \subseteq B_m$ and hence $y \in B_m$.

Set $\mathrm{pd} = \mathrm{pt} \circ h$. It follows that $\mathcal{T}$ is effectively pointed. As above we obtain that it is also strongly pointed.

Summing up what we have just shown as well as in Proposition 3.10, we obtain the following result.

Theorem 3.32 *Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with a countable strong basis $\mathcal{B}$ such that τ is constructively order-consistent. Moreover, let B be a total indexing of $\mathcal{B}$, x be an acceptable numbering of T, and $\prec_B$ be r.e. Then $\mathcal{T}$ is strongly pointed and constructively d-complete if, and only if, $\mathcal{T}$ is constructively complete and $\prec_B$ has the interpolation property.*

Proposition 3.33 *Let $\mathcal{T}$ be constructively complete such that all basic open sets are nonempty, and x be computable. Moreover, let $\prec_B$ be r.e. and have the interpolation property. Then $(T, \leq_\tau)$ is a constructive pre-domain and τ is coarser than the Scott topology on T.*

Proof: Let $\mathfrak{F}$ be the collection of all r.e. filters on $\mathrm{dom}(B)$ with respect to $\prec_B$, ordered under set inclusion. Then $(\mathfrak{F}, \subseteq)$ is a constructive pre-domain with basis $\{ \uparrow n \mid n \in \mathrm{dom}(B) \}$, where $\uparrow n = \{ m \in \mathrm{dom}(B) \mid n \prec_B m \}$ (cf. [1, Proposition 2.2.22]). As we have seen above, if $A \in \mathfrak{F}$, then we can construct a function $f_A \in R^{(1)}$ with $\mathrm{range}(f_A) \subseteq A$ that is decreasing with respect to $\prec_B$ and is such that for every $a \in A$ there is some $c \in \omega$ with $f_A(c) \prec_B a$. It follows that $\{ B_{f_A(n)} \mid n \in \omega \}$ is a strong base of the filter generated by $\{ B_m \mid m \in A \}$. Moreover, $(B_{f_A(n)})_{n\in\omega}$ is a normed recursive enumeration of nonempty basic open sets that converges to some point $y_A \in T$.

Next, we show that

$$A = \{ n \in \mathrm{dom}(B) \mid y_A \in B_n \}. \tag{1}$$

If $a \in A$, there is some $c \in \omega$ with $f_A(c) \prec_B a$. Thus, $y_A \in B_{f_A(c)} \subseteq B_a$. Conversely, if $n \in \mathrm{dom}(B)$ with $y_A \in B_n$, then there is some $m \in \omega$ with $f_A(m) \prec_B n$, as $\{ B_{f_A(n)} \mid n \in \omega \}$ is a strong base of $\mathcal{N}(y_A)$. Since $f_A(m) \in A$ and A is a filter with respect to $\prec_B$, we obtain that $n \in A$ as well.

As is easily verified, the assignment $F\colon A \mapsto y_A$ is an order-preserving as well as order-reflecting bijection between $(\mathfrak{F}, \subseteq)$ and $(T, \leq_\tau)$. We will now show that it commutes with the operation of taking least upper bounds of enumerable directed sets.

Apparently, the union over an enumerable directed subset of $\mathfrak{F}$ is in $\mathfrak{F}$ again. Let $\mathfrak{S}$ be such a subset. Then $F(\bigcup \mathfrak{S})$ is the least upper bound of $\{ F(A) \mid A \in \mathfrak{S} \}$. In order to prove this note that $F(\bigcup \mathfrak{S})$ is an upper bound of this set, as F is monotone. Let $z \in T$ be another upper bound. We need to show that $F(\bigcup \mathfrak{S}) \leq_\tau z$. Let to this end $n \in \mathrm{dom}(B)$ with $F(\bigcup \mathfrak{S}) \in B_n$. Then $n \in \bigcup \mathfrak{S}$ by (1). Thus, $n \in A$, for some $A \in \mathfrak{S}$, which implies that $F(A) \in B_n$. Hence $z \in B_n$ as well.

It remains to derive that $\tau \subseteq \sigma$. Let to this end $B_n \in \tau$. By definition of the specialisation order B_n is upwards closed with respect to $\leq_\tau$. Let $E \subseteq \mathrm{dom}(B)$ be r.e. such that $\{ x_a \mid a \in E \}$ is directed with least upper bound y. Moreover, assume that $y \in B_n$. Then $n \in F^{-1}(y)$. Since $F^{-1}(y) = \bigcup \{ F^{-1}(x_a) \mid a \in E \}$, it follows that there is some $a \in E$ with $x_a \in B_n$. Thus, $B_n \in \sigma$.

Combining this result with Proposition 3.31 and Lemma 3.24 we obtain the following consequence.

Theorem 3.34 *Let $\mathcal{T}$ be constructively complete such that all basic open sets are nonempty. Moreover, let B be total, x be acceptable, and $\prec_B$ be r.e. as well as have the interpolation property. Then $(T, \leq_\tau)$ is a constructive pre-domain and τ coincides with the Scott topology on T.*

4 Subspace indexings

In the previous sections we have investigated effectiveness properties in T_0 spaces. To this end we encoded the essential objects, the points and the basic open sets. In this section we will study ways of assigning indices to subspaces. For cardinality reasons, in general not all subspaces can be given an index.

We have already encountered two classes of subsets with indexing systems: the completely enumerable and the enumerable sets. For total numberings, each completely enumerable set is obviously also enumerable and any of its c.e. indices is an enumeration index of it. In the general, partial, case, however, this need not be true, as a complete enumeration will not only enumerate all indices of elements, but perhaps also numbers that are not used by the indexing.

By listing (indices of) all elements of a set we have effective access to all of them. However, this limits the kind of sets we can deal with in a computable way. Sometimes it suffices to enumerate a generating part of the set or certain properties of its points. Note that in general the set will not be uniquely determined by these properties and as a result different sets may have the same index. We think of the set as being given by other means. The index codes only a procedure generating certain useful properties. Several of the naming systems considered in this paper have also been studied in [10, 9] for Euclidean space and, more general, effectively given metric spaces.

4.1 Density indices

Definition 4.1 Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space and x be an indexing of T. A subset X of T is *effectively separable* if it is empty or has an enumerable subset Z that is dense in X with respect to the subspace topology. Every enumeration index of Z is called a *density index* of X.

Obviously, if X is enumerable, then every enumeration index of X is a density index of X. As follows from the next example, the converse is not true, in general.

Example 4.2 Consider the set $P^{(1)}$ of all unary partial recursive functions with some Gödel numbering. By applying a diagonal argument it is readily shown that the subset $R^{(1)}$ of all unary total recursive functions is not enumerable. We mentioned already that $P^{(1)}$ is a constructive domain with respect to the extension ordering. Moreover, $R^{(1)}$ with the Baire metric is a computable metric space, of which the functions that are 0 almost everywhere form an enumerable dense subspace. Note that the metric topology on $R^{(1)}$ is the subspace topology with respect to the Scott topology on $P^{(1)}$. Both spaces are constructively complete. In the case of Baire space this is a consequence of Theorem 3.18.

The situation is different, however, in the case of effectively separable closed subsets of effectively pointed spaces.

Proposition 4.3 *Let $\mathcal{T}$ be effective, effectively pointed and constructively complete. In addition, let x be acceptable and $\prec_B$ r.e. Then any nonempty effectively separable closed subset X of T is enumerable. Moreover, an enumeration index of every such set X can be computed from any of its density indices in a uniform way.*

Proof: Let $L \subseteq \omega$ and $\mathrm{pt} \in P^{(1)}$ witness that x is acceptable, let $s \in R^{(1)}$ such that $\varphi_{s(a)}$ is a total enumeration of W_a, if this set is not empty, and let $w \in R^{(1)}$ as well as $v \in R^{(2)}$ so that

$$W_{w(i)} = \{\, a \in \omega \mid (\exists n \in W_a)(\exists j \in W_i)\langle j, n\rangle \in L \,\}$$

and

$$W_{v(i,a)} = \{\, n \in W_{\varphi_{s(w(i))}(a)} \mid (\exists j \in W_i)\langle j, n\rangle \in L \,\}.$$

Define $g, f, k \in R^{(2)}$ by

$$\varphi_{g(i,a)}(0) = \varphi_{s(v(i,a))}(0),$$

$$\varphi_{g(i,a)}(c+1) = \begin{cases} \text{first } m \text{ enumerated with } m \in W_{v(i,a)}, & \\ m \prec_B \varphi_{g(i,a)}(c), \text{ and } m \prec_B \varphi_{s(v(i,a))}(c+1) & \text{if such an } m \text{ exists,} \\ \text{undefined} & \text{otherwise,} \end{cases}$$

$$\varphi_{f(i,a)}(c) = \max\{\, e \le c \mid (\forall 1 \le e' \le e)\varphi_{g(i,a)}(e')\downarrow_c \,\},$$

$$\varphi_{k(i,a)}(c) = 1 + \max\{\, m \le c \mid \varphi_{f(i,a)}(m) \ne \varphi_{f(i,a)}(m+1) \,\}.$$

Let the functions $q \in R^{(1)}$ and $p \in R^{(2)}$ be as in Lemma 3.5 and let $\mathrm{sb} \in P^{(3)}$ and $\mathrm{pd} \in P^{(1)}$, respectively, witness that $\mathcal{T}$ is effective and effectively pointed. Moreover, let $h \in R^{(2)}$ such that

$$\varphi_{h(i,a)}(0) = \varphi_{q(\mathrm{pd}(\varphi_{g(i,a)}(0)))}(0),$$

$$\varphi_{h(i,a)}(c+1) = \begin{cases} \varphi_{q(\mathrm{pd}(\varphi_{g(i,a)}(0)))}(c+1) & \text{if } \varphi_{f(i,a)}(c+1) = \varphi_{f(i,a)}(c) = 0, \\ \varphi_{p(\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c))), \varphi_{h(i,a)}(\varphi_{k(i,a)}(c)))}(c+1) & \\ & \text{if } 0 < \varphi_{f(i,a)}(c) = \varphi_{f(i,a)}(c+1), \\ \mathrm{sb}(\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c+1))), \varphi_{g(i,a)}(\varphi_{f(i,a)}(c)), \varphi_{h(i,a)}(c)) & \\ & \text{if } \varphi_{f(i,a)}(c) \ne \varphi_{f(i,a)}(c+1). \end{cases}$$

Now, assume that X is a nonempty closed subset of T and i is a density index of X. Note that $W_{w(i)}$ and $W_{v(i,a)}$ are not empty in this case, as X is not empty.

Claim 1 $(\forall c \in \omega)\varphi_{h(i,a)}(c+1) \prec_B \varphi_{h(i,a)}(c)$.

Proof of Claim: It suffices to consider the case that $\varphi_{f(i,a)}(c) \neq \varphi_{f(i,a)}(c+1)$. Moreover, because of the properties of the function sb, we only have to show that

$$x_{\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c+1)))} \in B_{\varphi_{g(i,a)}(\varphi_{f(i,a)}(c))} \cap B_{\varphi_{h(i,a)}(c)}.$$

By construction, $\varphi_{g(i,a)}(\varphi_{f(i,a)}(c+1)) \prec_B \varphi_{g(i,a)}(\varphi_{f(i,a)}(c))$. Hence,

$$x_{\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c+1)))} \in B_{\varphi_{g(i,a)}(\varphi_{f(i,a)}(c))}.$$

It follows that

$$x_{\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c)))} \leq_\tau x_{\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c+1)))}.$$

Since $x(\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c)))) \in B(\varphi_{h(i,a)}(c))$, by definition of $\varphi_{h(i,a)}$, we obtain that also $x(\mathrm{pd}(\varphi_{g(i,a)}(\varphi_{f(i,a)}(c+1)))) \in B(\varphi_{h(i,a)}(c))$.

As a consequence we have that $\left(B(\varphi_{h(i,a)}(c))\right)_{c\in\omega}$ is a normed enumeration of basic open sets which starts to converge to $x(\mathrm{pd}(\varphi_{g(i,a)}(0)))$, and that until in some first step $\overline{c} > 1$ the values $\varphi_{g(i,a)}(1), \ldots, \varphi_{g(i,a)}(e)$ are computed, for some $1 \leq e \leq \overline{c}$. As we have just seen, $x(\mathrm{pd}(\varphi_{g(i,a)}(e))) \in B(\varphi_{g(i,a)}(0)) \cap B(\varphi_{h(i,a)}(\overline{c}-1))$. Therefore, an index $m \prec_B \varphi_{g(i,a)}(0), \varphi_{h(i,a)}(\overline{c}-1)$ can be computed so that $x(\mathrm{pd}(\varphi_{g(i,a)}(e))) \in B_m$. Next the sequence will try to converge to $x(\mathrm{pd}(\varphi_{g(i,a)}(e)))$ in such a way that the basic open sets that are generated are strongly contained in B_m. This will be done until in some further step $\overline{c'} > \overline{c}$ the values $\varphi_{g(i,a)}(1), \ldots, \varphi_{g(i,a)}(e')$ with $e' > e$ are computed. Again we have that $x(\mathrm{pd}(\varphi_{g(i,a)}(e'))) \in B(\varphi_{g(i,a)}(e)) \cap B(\varphi_{h(i,a)}(\overline{c'}-1))$. Now, the sequence starts to converge to $x(\mathrm{pd}(\varphi_{g(i,a)}(e')))$, and so on.

Since $\mathcal{T}$ is constructively complete the enumeration converges to some point $z \in T$. Then $z = x_{\mathrm{pt}(h(i,a))}$.

Claim 2 $z \in X$.

Proof of Claim: As follows from the definition, the domain of the function $\varphi_{g(i,a)}$ is an initial segment of ω. We first consider the case that $\mathrm{dom}(\varphi_{g(i,a)})$ is finite. Then there is a maximal number e such that for all $e' \leq e$, $\varphi_{g(i,a)}(e')\downarrow$. It follows that the enumeration $\left(B(\varphi_{h(i,a)}(c))\right)_{c\in\omega}$ converges to $x(\mathrm{pd}(\varphi_{g(i,a)}(e)))$, i.e. $z = x(\mathrm{pd}(\varphi_{g(i,a)}(e)))$. By construction, $B(\varphi_{g(i,a)}(e))$ contains an element of the dense subset $\{\, x_j \mid j \in W_i \,\}$, say $x_{\overline{\jmath}}$. Then $x(\mathrm{pd}(\varphi_{g(i,a)}(e))) \leq_\tau x_{\overline{\jmath}}$. Since all basic open set $B(\varphi_{h(i,a)}(c))$, for $c \in \omega$, contain $x(\mathrm{pd}(\varphi_{g(i,a)}(e)))$, they also contain $x_{\overline{\jmath}}$. Note that the collection of these sets forms a strong base of the neighbourhood filter of $x(\mathrm{pd}(\varphi_{g(i,a)}(e)))$. Because X is closed, it follows that $x(\mathrm{pd}(\varphi_{g(i,a)}(e))) \in X$.

Let us next consider the case that $\mathrm{dom}(\varphi_{g(i,a)})$ is infinite. Then the function $\varphi_{g(i,a)}$ is total. It follows that there are strictly increasing sequences $(c_\nu)_{\nu\geq 1}$ and $(e_\nu)_{\nu\geq 0}$ such that $e_0 = 0$, $c_1 > 0$, and for all $\nu > 0$,

$$\begin{aligned}
&e_\nu \leq c_\nu, \\
&e_\nu = \varphi_{f(i,a)}(c_\nu) \neq \varphi_{f(i,a)}(c_\nu - 1), \text{ and} \\
&x_{\mathrm{pd}(\varphi_{g(i,a)}(e_\nu))} \in B_{\varphi_{g(i,a)}(e_{\nu-1})} \cap B_{\varphi_{h(i,a)}(c_\nu - 1)}.
\end{aligned}$$

Each $B(\varphi_{g(i,a)}(e_\nu))$ contains some x_{j_ν} with $j_\nu \in W_i$. It follows that $x(\mathrm{pd}(\varphi_{g(i,a)}(e_\nu))) \leq_\tau x_{j_\nu}$. Hence $x_{j_\nu} \in B(\varphi_{h(i,a)}(c_\nu - 1))$. Since $\varphi_{h(i,a)}(c_\nu - 1) \prec_B \varphi_{h(i,a)}(m)$, for $m < c_\nu - 1$, and because the c_ν with $\nu \geq 1$ are strictly increasing, we have that all sets $B(\varphi_{h(i,a)}(c))$ hit the dense subset $\{\, x_j \mid j \in W_i \,\}$. As above it follows that $z \in X$.

Claim 3 $(\forall y \in X)(\exists a \in \omega) y = x_{\mathrm{pt}(h(i,a))}$.

Proof of Claim: Let $b \in \omega$ with $W_b = \{ n \in \omega \mid y \in B_n \}$. Because $\{ x_j \mid j \in W_i \}$ is dense in X and $y \in X$, each open sets B_n with $n \in W_b$ hits the dense subset. It follows that $b \in W_{w(i)}$. Let $a \in \omega$ so that $b = \varphi_{s(w(i))}(a)$. Then $W_{v(i,a)} = W_b$. Because the topological basis of space $\mathcal{T}$ is strong, we have that W_b is filtered with respect to $\prec_B$. Hence $\varphi_{g(i,a)}$ is a total function and the collection of all sets $B(\varphi_{g(i,a)}(c))$ is a strong base of the neighbourhood filter of y. Now, let the sequences $(c_\nu)_{\nu \geq 1}$ and $(e_\nu)_{\nu \geq 0}$ be as above in the second case of Claim (2). Then $\varphi_{h(i,a)}(c_\nu) \prec_B \varphi_{g(i,a)}(e_\nu)$, for all $\nu > 0$. It follows that the filter generated by the sets $B(\varphi_{g(i,a)}(c))$, i.e., the neighbourhood filter of y, is contained in the filter generated by the sets $B(\varphi_{h(i,a)}(c))$, which is the neighbourhood filter of point $z \in T$. Thus $y \leq_\tau z$.

To prove that also $z \leq_\tau y$, it suffices to show that $y \in B(\varphi_{h(i,a)}(c_\nu))$, for all $\nu > 0$. By [36, Lemma 4.2] the enumeration $\big(B(\varphi_{g(i,a)}(c))\big)_{c \in \omega}$ converges to the least upper bound of the set of all points $x(\mathrm{pd}(\varphi_{g(i,a)}(c)))$. Thus, $x(\varphi_{g(i,a)}(e_\nu)) \leq_\tau y$, for all $\nu > 0$. Above we have seen that always $x(\mathrm{pd}(\varphi_{g(i,a)}(e_{\nu+1}))) \in B(\varphi_{k(i,a)}(c_{\nu+1} - 1))$. As $B(\varphi_{k(i,a)}(c_{\nu+1} - 1)) \subseteq B(\varphi_{k(i,a)}(c_\nu))$, we obtain that $y \in B(\varphi_{h(i,a)}(c_\nu))$, for all $\nu > 0$.

Let $r \in R^{(1)}$ with $W_{r(j)} = \{ \mathrm{pt}(h(j,a)) \mid a \in \omega \}$. Then $X = \{ x_m \mid m \in W_{r(i)} \}$. Thus X is enumerable with enumeration index $r(i)$.

As a consequence of Lemma 2.3 we have that a density index of the union of two effectively separable subsets of T can be computed from their density indices. Moreover, the collection of these sets has an intensional selection function.

Lemma 4.4 *A subset X of T has density index i if, and only if,*

$$\{ x_a \mid A \in W_i \} \subseteq X \subseteq \mathrm{cl}_\tau(\{ x_a \mid a \in W_i \}).$$

Corollary 4.5 *If a subset X of T has density index i, then* $\mathrm{cl}_\tau(X) = \mathrm{cl}_\tau(\{ x_a \mid a \in W_i \})$.

This result characterises the subspaces of $\mathcal{T}$ that may have the same density index. In particular it follows that exactly the closed sets are uniquely determined by their density indices.

4.2 Hit indices

Up to now we have represented subspaces by enumerating some particular or all of their elements. We will now use the open sets that hit the space. Note that we write $m \preccurlyeq_B n$ to mean that $m \prec_B n$ or $m = n$.

Definition 4.6 Let $\mathcal{T} = (T, \tau)$ be a T_0 space with countable strong basis $\mathcal{B}$ and B be a numbering of $\mathcal{B}$. A subset X of T is said to be *effectively covered* if there is some r.e. set $A \subseteq \mathrm{dom}(B)$ such that the following two conditions hold:

1. $(\forall m \in A) B_m \cap X \neq \emptyset$.
2. $(\forall n \in \mathrm{dom}(B))[B_n \cap X \neq \emptyset \Rightarrow (\exists m \in A) m \preccurlyeq_B n]$.

Every r.e. index of A is called *hit index* of X.

Lemma 4.7 *Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with countable strong basis $\mathcal{B}$. In addition, let B be a total numbering of $\mathcal{B}$ and x be a computable indexing of T. Then every effectively separable subset X of T is effectively covered. Moreover, a hit index of every such set X can be computed from any of its density indices in a uniform way.*

Proof: Let $L \subseteq \omega$ witness that x is computable and $f \in R^{(1)}$ with

$$W_{f(i)} = \{ n \in \omega \mid (\exists a \in W_i)\langle a, n\rangle \in L \}.$$

Then, if i is a density index of X, we have that

$$B_n \cap X \neq \emptyset \Leftrightarrow (\exists a \in W_i) B_n \cap \{ x_a \mid a \in W_i \} \neq \emptyset.$$

It follows that X is effectively covered with hit index $f(i)$.

Note that the converse is not true in general. In [8] a closed subset of an incomplete constructive metric space is exhibited, which is effectively covered but not effectively separable. As we will see next, it is true, however, in the case of constructively complete spaces.

Let X be a subset of T and τ^X the induced topology on X. Define $\mathcal{B}^X$ to be the collection of all sets $O \cap X$ with $O \in \mathcal{B}$, set $B_n^X = B_n \cap X$, and let

$$m \prec_{B^X} n \Leftrightarrow m \prec_B n.$$

Then $\mathcal{T}^X = (X, \tau^X)$ is also a countable T_0 space with countable basis $\mathcal{B}^X$, which is strong if $\mathcal{B}$ is strong. Moreover, $\mathcal{T}^X$ is effective if $\mathcal{T}$ is. Set $x_i^X = x_i$, if $i \in \mathrm{dom}(x)$ with $x_i \in X$, and let x^X be undefined in any other case. Then x^X is a numbering of X with $x^X \leq x$. Moreover, x^X is computable, if x is, and allows effective limit passing, if x does (cf. [37]). As is readily verified, the r.e. set $L \subseteq \omega$ and the function $\mathrm{pt} \in P^{(1)}$ witnessing that x is computable and allows effective limit passing, respectively, do the same for x^X.

Note that every index c of a converging recursive enumeration $\left(B_{g(a)} \cap X\right)_{a\in\omega}$ of relatively open sets that is normed with respect to $\prec_{B^X}$ is also an index of the enumeration $\left(B_{g(a)}\right)_{a\in\omega}$ normed with respect to $\prec_B$ that is converging to the same point.

Proposition 4.8 *Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with countable strong basis $\mathcal{B}$. In addition, let B be a numbering of $\mathcal{B}$ and x be an indexing of T that allows effective limit passing. Then every nonempty effectively covered subset X of T the induced subspace $\mathcal{T}^X$ of which is constructively complete, is effectively separable. Moreover, a density index of every such set X can be computed from any of its hit indices in a uniform way.*

Proof: Let i be a hit index for X and $n \in W_i$. Then B_n hits X. Let $y \in B_n \cap X$. Because $\mathcal{B}$ is a strong topological basis we have that there is some $m \prec_B n$ with $y \in B_m \cap X$. It follows that there is some $n' \in W_i$ with $n' \preccurlyeq_B m$. So, we have that $n' \prec_B n$ and $B_{n'}$ hits X. Now let $k \in R^{(2)}$ with

$$\begin{aligned}
&\varphi_{k(j,n)}(0) = n,\\
&\varphi_{k(j,n)}(a+1) = \text{ first } m \in W_i \text{ with } m \prec_B \varphi_{k(j,n)}(a).
\end{aligned}$$

As we have just seen, for $j = i$ we can always find such an index m. Thus $\varphi_{k(i,a)}$ is a total function. Since the subspace $\mathcal{T}^X$ is constructively complete, the enumeration $\big(B(\varphi_{k(i,n)}(a)) \cap X\big)_{a\in\omega}$ converges to some point $z \in X$. It follows that the enumeration $\big(B(\varphi_{k(i,n)}(a))\big)_{a\in\omega}$ does the same. So, $z = x(\mathrm{pt}(k(i,n)))$, where the function $\mathrm{pt} \in P^{(1)}$ witnesses that x allows effective limit passing. Let $h \in R^{(1)}$ with $W_{h(j)} = \{ \mathrm{pt}(k(j,n)) \mid n \in W_j \}$.

It remains to show that $\{ x_a \mid a \in W_{h(i)} \}$ is dense in X with respect to the subspace topology. By construction, $\{ x_a \mid a \in W_{h(i)} \} \subseteq X$. Let B_m meet X. Then there is some $n \in W_i$ with $n \preccurlyeq_B m$. Thus $x_{\mathrm{pt}(k(i,n))} \in B_n \cap X \subseteq B_m \cap X$.

This proposition generalises [9, Theorem 3.8(2)]. As follows from the next result, constructive completeness is transmitted to the subspaces induced by closed subsets.

Lemma 4.9 *Let $\mathcal{T}$ be constructively complete and X a closed subset of T. Then the induced subspace $\mathcal{T}^X$ is constructively complete as well.*

Proof: Let m be the index of a recursive enumeration of nonempty relatively basic open sets normed with respect to $\prec_{B^X}$. Then m is also an index of a recursive enumeration of nonempty basic open sets normed with respect to $\prec_B$. Since $\mathcal{T}$ is constructively complete, the latter enumeration converges to some point $z \in T$. It remains to show that $z \in X$.

Assume to the contrary that $z \notin X$. Note that $\overline{X}$ is open. Since the collection of sets $B(\varphi_m(a))$ with $a \in \omega$ is a base of the neighbourhood filter of z, there is some $c \in \omega$ such that $B(\varphi_m(c)) \subseteq \overline{X}$. By assumption m is an index of a normed recursive enumeration of nonempty relatively open sets, i.e. every set $B(\varphi_m(a))$ hits X, a contradiction.

The converse is also true.

Lemma 4.10 *Let $\mathcal{T}$ be effective, x computable and X be a subset of T such that the induced subspace $\mathcal{T}^X$ is constructively complete. Then X is a closed subset of T.*

Proof: Let $z \in \overline{X}$ and assume that every basic open set in the neighbourhood filter of z hits X. By Lemma 3.5 there is a normed recursive enumeration $\left(B_{f(a)}\right)_{a\in\omega}$ of basic open sets converging to z. Because of our the assumption, $\left(B_{f(a)} \cap X\right)_{a\in\omega}$ is a normed recursive enumeration of nonempty relatively basic open sets in the induced subspace $\mathcal{T}^X$. Since this subspace is constructively complete, the enumeration converges to some point $y \in X$. Then the enumeration $\left(B_{f(a)}\right)_{a\in\omega}$ converges to y as well, which means that $y = z$, a contradiction.

It follows that there is some basic open set containing z which is included in $\overline{X}$. Thus, $\overline{X}$ is open and X is closed.

Summing up we obtain the following result.

Theorem 4.11 *Let $\mathcal{T} = (T, \tau)$ be a constructively complete countable T_0 space with countable strong basis $\mathcal{B}$. In addition, let B be a numbering of $\mathcal{B}$ and x be an indexing of T that allows effective limit passing. Then every nonempty closed effectively covered subset X of T is effectively separable. Moreover, a density index of every such set X can be computed from any of its hit indices in a uniform way.*

As we have already seen, the collection of effectively separable subsets of T has an intensional selection function. With the theorem it follows that this holds in particular for the closed effectively covered subsets of T.

Corollary 4.12 *Let $\mathcal{T} = (T, \tau)$ be a constructively complete countable T_0 space with countable strong basis $\mathcal{B}$. Moreover, let B be a numbering of $\mathcal{B}$ and x be an indexing of T that allows effective limit passing. Then the collection of effectively covered closed subsets of T has an intensional selection function.*

From the above results we see that closed subsets are of importance in the study of effectively covered subsets of space T. As we shall see now, just as in the case of effective separability they are the ones that are uniquely determined by their hit indices.

Lemma 4.13 *Let X and X' be subsets of T such that X is effectively covered and $i \in \omega$ is a corresponding hit index. Then X' is effectively covered as well with hit index i if, and only if, $\mathrm{cl}_\tau(X) = \mathrm{cl}_\tau(X')$.*

Proof: Assume that i is a hit index of both X and X' and that $y \in \mathrm{cl}_\tau(X)$. Let B_n contain y. Then B_n intersects X. Thus, there is some $m \in W_i$ with $m \preccurlyeq_B n$. It follows that B_m hits X', as i is also a hit index of X'. Consequently, B_n intersects X' as well. This shows that $y \in \mathrm{cl}_\tau(X')$. The converse inclusion is obtained analogously.

Now, suppose that $\mathrm{cl}_\tau(X) = \mathrm{cl}_\tau(X')$. We show that X' is effectively covered as well with hit index of i. Let $m \in W_i$. Then B_m intersects X. Since $B_m \cap \mathrm{cl}_\tau(X') = B_m \cap \mathrm{cl}_\tau(X) \supseteq B_m \cap X \neq \emptyset$, we have that B_m hits also X'. For the verification of the second condition in Definition 4.6 let B_n intersect X'. Because $B_n \cap \mathrm{cl}_\tau(X) = B_n \cap \mathrm{cl}_\tau(X') \supseteq B_n \cap X' \neq \emptyset$, we obtain that B_n hits X. Thus, there is some $m \in W_i$ with $m \preccurlyeq_B n$.

4.3 Covering indices

Definition 4.14 Let $\mathcal{T} = (T, \tau)$ be a T_0 space with countable strong basis $\mathcal{B}$ and B be a numbering of $\mathcal{B}$. A subset X of T has *computable finite covers* if there is some r.e. set $A \subseteq \omega$ such that the following two conditions hold for all $i \in A$ and all $n \in \omega$ with $D_n \subseteq \mathrm{dom}(B)$:

1. $D_i \subseteq \mathrm{dom}(B)$ and $X \subseteq \bigcup \{ B_a \mid a \in D_i \}$, i.e., $\{ B_a \mid a \in D_i \}$ is a finite cover of X.
2. If $\{ B_a \mid a \in D_n \}$ is a finite cover of X, then there is some $i \in A$ so that for all $a \in D_i$ there exists some $b \in D_n$ with $a \preccurlyeq_B b$.

Each r.e. index of A is called *covering index* of X.

Note that a covering index does not code just one finite cover of X, but a family of finite covers of X from which any other such cover can be derived by taking supersets.

Let $\uparrow_{\leq_\tau} X = \{ z \in T \mid (\exists y \in X) y \leq_\tau z \}$ be the upper set generated by X with respect to the specialisation order. Note that each finite cover of X is also a finite cover of $\uparrow_{\leq_\tau} X$, and vice versa, as all open sets are upwards closed with respect to the specialisation order.

Lemma 4.15 *Let X be a compact subset of T with computable finite covers and let $i \in \omega$ be a covering index of X. Then*

$$\bigcap \{ \bigcup \{ B_a \mid a \in D_j \} \mid j \in W_i \} = \uparrow_{\leq_\tau} X.$$

Proof: By what has just been noted it suffices to show that the left hand side is included in the right hand side. Let to this end $z \in T$ with $z \notin \uparrow_{\leq_\tau} X$. It follows that for each $y \in X$ there is some index $n_y \in \mathrm{dom}(B)$ such that $y \in B_{n_y}$ but $z \notin B_{n_y}$. Then X is covered by the sets B_{n_y} with $y \in X$. Since X is compact, there is a finite subcover, i.e., there are points $y_1, \ldots, y_c$ such that $X \subseteq \bigcup_{\nu=1}^{c} B_{n_{y_\nu}}$, but $z \notin \bigcup_{\nu=1}^{m} B_{n_{y_\nu}}$. Let $D_m = \{n_{y_1}, \ldots, n_{y_c}\}$. Then there is some $j \in W_i$ with $X \subseteq \bigcup \{ B_a \mid a \in D_j \} \subseteq \bigcup_{\nu=1}^{c} B_{n_{y_\nu}}$. Thus, $z \notin \bigcup \{ B_a \mid a \in D_j \} \supseteq \bigcap \{ \bigcup \{ B_a \mid a \in D_j \} \mid j \in W_i \}$.

As follows from the proof, for every (not necessarily compact) subset X of T, $\uparrow_{\leq_\tau} X$ is the intersection of all its open neighbourhoods. Such sets are called *saturated*. The next result shows that the compact saturated subsets of T are uniquely determined by their covering indices.

Corollary 4.16 *Let X and X' be compact subsets of T. Moreover, let X have computable finite covers and let $i \in \omega$ be a corresponding covering index. Then X' has computable finite covers as well with covering index i if, and only if, $\uparrow_{\leq_\tau} X = \uparrow_{\leq_\tau} X'$.*

Proof: The implication from left to right is a consequence of the preceding lemma. The converse implication follows from the fact that, as a consequence of the right hand side, X and X' have the same strict finite covers.

A finite cover of a set X gives few information about the set X itself, but negative information about which points of the space can definitely not be in X, namely all those outside the cover. In order to have also some kind of positive information about X, one has to combine covering indices i of with hit indices j. Any such pair $\langle i, j\rangle$ will be called a *complete covering index* of X. Another possibility of obtaining also positive information is to require that every set in a finite cover meets X.

Definition 4.17 Let $\mathcal{T} = (T, \tau)$ be a T_0 space with countable strong basis $\mathcal{B}$, B be a numbering of $\mathcal{B}$, and X be a subset of T.

1. A finite subset $\mathcal{E}$ of $\mathcal{B}$ is a *strict finite cover* of X, if $X \subseteq \bigcup \mathcal{E}$ and each $B_a \in \mathcal{E}$ intersects X.

2. X has *computable strict finite covers*, if there is some r.e. set $A \subseteq \omega$ such that $n \in A$ if and only if $\{ B_a \mid a \in D_n \}$ is a strict finite cover of X, for all $n \in \omega$ with $D_n \subseteq \mathrm{dom}(B)$. Each r.e. index of A is called *strict covering index* of X.

Lemma 4.18 *Let $\mathcal{T} = (T, \tau)$ be a T_0 space with countable strong basis $\mathcal{B}$ and B be a total numbering of $\mathcal{B}$. Moreover, let $\prec_B$ be r.e. Then the following two statements hold:*

1. *A strict covering index of every subset X of T can be computed from any of its complete covering indices in a uniform way.*

2. *A complete covering index of every compact subset X of T can be computed from any of its strict covering indices in a uniform way.*

Proof: (1) Let $r \in R^{(1)}$ so that

$$W_{r(\langle i,j\rangle)} = \{ m \in \omega \mid (\exists n \in W_i)(\forall a \in D_n)(\exists b \in D_m) a \preccurlyeq_B b \wedge (\forall b \in D_m)(\exists c \in W_j) c \preccurlyeq_B b \}$$

and assume that $X \subseteq T$ has complete covering index $\langle i, j\rangle$. If $m \in W_{r(\langle i,j\rangle)}$, then the first condition guaranties that $\{ B_b \mid b \in D_m \}$ is a cover of X and the second that all open sets B_b with $b \in D_m$ meet X. Thus, $\{ B_b \mid b \in D_m \}$ is a strict cover of X. On the other hand, in case $\{ B_c \mid c \in D_e \}$ is a strict cover of X, it is in particular a finite cover of X. Hence, there is some $n \in W_i$ such that there is some $b \in D_e$ with $a \preccurlyeq_B b$, for all $a \in D_n$. In addition, each B_c with $c \in D_e$ hits X. It follows that $e \in W_{r(\langle i,j\rangle)}$. This shows that $r(\langle i, j\rangle)$ is a strict covering index of X.

(2) Let i be a strict covering index of some compact subset X of T. Then it is also a covering index of X. To see this, note that the first condition in Definition 4.14 holds trivially and the second as each finite cover has a strict subcover. It remains to show how a hit index of X can be obtained from i. Define to this end $h \in R^{(1)}$ by

$$W_{h(i)} = \{ m \in \omega \mid (\exists n \in W_i)(\exists a \in D_n) a \preccurlyeq_B m \}.$$

Obviously, B_m meets X, for each $m \in W_{h(i)}$. Next assume that B_c intersects X. Since X is compact, it has a strict finite cover of basic open sets, say $\{ B_a \mid a \in D_e \}$. Then $\{ B_a \mid a \in D_e \} \cup \{B_c\}$ is a strict finite cover of X as well. Let $D_n = D_e \cup \{c\}$. Then $n \in W_i$ and hence $c \in W_{h(i)}$. It follows that $h(i)$ is a hit index of X, whence $\langle i, h(i)\rangle$ is a complete covering index of X.

4.4 Indexing the complement

We will now consider a further way to encode information about the complement of a set.

Let $\mathcal{M} = (M, \delta)$ be an effectively given metric space as introduced in Example 3.17. Then the canonical topology Δ has the collection of sets $B_{\langle i,m\rangle} = \{ y \in M \mid \delta(\beta_i, y) < 2^{-m} \}$ with $i, m \in \omega$ as strong basis. Here, β is a numbering of the dense subset M_0. Let $B^c_{\langle i,m\rangle} = \{ y \in M \mid \delta(\beta_i, y) \leq 2^{-m} \}$. Note that each set B^c_n is closed, as inverse image of a closed real interval under the continuous distance function.

Definition 4.19 Let $\mathcal{M}$ be an effectively given metric space. A subset X of M admits *effective complement exhaustion* if the set $\{ n \in \omega \mid B^c_n \cap X = \emptyset \}$ is r.e. Any r.e. index of this set is called *complement exhaustion index* of X.

Lemma 4.20 *Let $\mathcal{M} = (M, \delta)$ be an effectively given metric space and X be a compact subset of M that has computable finite covers. Then X admits effective complement exhaustion. Moreover, a complement exhaustion index of every such set X can be computed from any of its covering indices in a uniform way.*

Proof: The lemma is a consequence of [9, Theorem 4.10(2)]. We give a proof in the framework of the present paper. Let i be a covering index of X. We show that

$$B^c_n \cap X = \emptyset \Leftrightarrow (\exists m \in W_i)(\forall b \in D_m)\delta(\beta_{\pi_1(b)}, \beta_{\pi_1(n)}) > 2^{-\pi_2(b)} + 2^{-\pi_2(n)}.$$

The right hand side implies that $X \subseteq \bigcup \{ B_b \mid b \in D_m \}$ and that B^c_n is disjoint from all B^c_b with $b \in D_m$, from which it follows that B^c_n is also disjoint from X.

Now, assume that B^c_n is disjoint from X. Since X is compact, there is some point $y \in X$ so that $\delta(\beta_{\pi_1(n)}, y) = \min \{ \delta(\beta_{\pi_1(n)}, z) \mid z \in X \}$. Then $\delta(\beta_{\pi_1(n)}, y) > 2^{-\pi_2(n)}$. Let $a \in \omega$ with $2^{-a} < \delta(\beta_{\pi_1(n)}, y) - 2^{-\pi_2(n)}$.

Recall that compact subspaces are totally bounded. Thus, there are finitely many points in X such that X is covered by the open spheres with radius $2^{-(a+1)}$ around these points. It follows that there exist indices $j_1, \ldots, j_e$ so that X is also covered by the spheres $B_{\langle j_\nu, a\rangle}$ with $\nu = 1, \ldots, e$. Since i is a covering index of X, we obtain that there is some $m \in W_i$ such that there exist some $\nu_b \in \{1, \ldots, e\}$ with $b \preccurlyeq_B \langle j_{\nu_b}, a\rangle$, for each $b \in D_m$. Thus,

$$\begin{aligned} 2^{-a} + 2^{-\pi_2(n)} &< \delta(\beta_{\pi_1(n)}, y) \\ &\leq \delta(\beta_{\pi_1(n)}, \beta_{\pi(j_{\nu_b})}) \\ &\leq \delta(\beta_{\pi_1(n)}, \beta_{\pi_1(b)}) + \delta(\beta_{\pi_1(b)}, \beta_{\pi(j_{\nu_b})}) \\ &\leq \delta(\beta_{\pi_1(n)}, \beta_{\pi_1(b)}) + 2^{-a} - 2^{-\pi_2(b)}, \end{aligned}$$

from which it follows that $\delta(\beta_{\pi_1(n)}, \beta_{\pi_1(b)}) > 2^{-\pi_2(n)} + 2^{-\pi_2(b)}$, for all $b \in D_m$.

Note that the set $E = \{ \langle c, e\rangle \mid \delta(\beta_{\pi_1(c)}, \beta_{\pi_1(e)}) > 2^{-\pi_2(c)} + 2^{-\pi_2(e)} \}$ is r.e. Let $t \in R^{(1)}$ with $W_{t(i)} = \{ n \in \omega \mid (\exists m \in W_i)(\forall b \in D_m)\langle b, n\rangle \in E \}$. Then $t(i)$ is a complement exhaustion index of X.

Observe that for a closed set X the existence of an effective complement exhaustion has a rather strong implication on the complement of X.

Lemma 4.21 *Let $\mathcal{M} = (M, \delta)$ be a constructive metric space and X be a closed subset that admits effective complement exhaustion. Then the complement $\overline{X}$ of X is completely enumerable. Moreover, a c.e. index of the complement of every such set X can be computed from any of its complement exhaustion indices in a uniform way.*

Proof: Note that by the regularity of metric spaces,

$$x_i \in \overline{X} \Leftrightarrow (\exists n \in \omega) x_i \in B_n \wedge B_n^c \cap X = \emptyset.$$

So, if $A, L \subseteq \omega$, respectively, witness that X admits effective complement exhaustion and x is computable, and if $f \in R^{(1)}$ such that $W_{f(i)} = \{\, i \in \omega \mid (\exists n \in \omega)\langle i, n\rangle \in L \wedge n \in A \,\}$, then $f(i)$ is a c.e. index of $\overline{X}$, in case that i is a complement exhaustion index of X.

5 Multifunctions

Multifunctions generalise the concept of function to the case that several values may be assigned to an argument.

Definition 5.1 Let S and S' be sets. A multifunction $F\colon S \rightrightarrows S'$ is given by a relation R_F between S and S'. For $y \in S$,

$$F(y) = \{\, z \in S' \mid (y, z) \in R_F \,\}$$

is the *image* or *value* of F at y.

The *domain* and the *range* of F are taken to be the sets

$$\operatorname{dom}(F) = \{\, y \in S \mid F(y) \neq \emptyset \,\}, \quad \operatorname{range}(F) = \{\, z \in S' \mid (\exists y \in S) z \in F(y) \,\},$$

which are the images of R_F under the projections to the first and second component. The *image* of a subset Y of S under F is $F(Y) = \bigcup \{\, F(y) \mid y \in Y \,\}$. For a subset Z of S', the *lower inverse* and the *upper inverse*, respectively, of F are

$$F^-(Z) = \{\, y \in S \mid F(y) \cap Z \neq \emptyset \,\}, \quad F^+(Z) = \{\, y \in S \mid F(y) \subseteq Z \,\}.$$

Lemma 5.2 *Let Z, Z_i be subsets of S' ($i \in I$). Then the following statements hold:*

1. $\overline{F^+(Z)} = F^-(\overline{Z})$, $\overline{F^-(Z)} = F^+(\overline{Z})$.
2. $\bigcup_{i \in I} F^+(Z_i) \subseteq F^+(\bigcup_{i \in I} Z_i)$.
3. $\bigcup_{i \in I} F^-(Z_i) = F^-(\bigcup_{i \in I} Z_i)$.

In the preceding section we introduced indexings for certain collections of subspaces. As we have seen, in general the indexed sets are not uniquely determined by their index.

Definition 5.3 Let S be a nonempty countable set. A *multinumbering* Θ of S is a multifunction $\Theta\colon \omega \rightrightarrows S$ that has S as its range.

Definition 5.4 Let S, S' be sets, $\mathfrak{S}'$ be a collection of subsets of S', θ be a (partial) numbering of S, and Θ' be a multinumbering of $\mathfrak{S}'$. Moreover, let $F\colon S \rightrightarrows S'$ such that $F(y) \in \mathfrak{S}'$, for all $y \in S$. Then we call F *effective with respect to* Θ' if there is some function $f \in P^{(1)}$ so that for all $i \in \mathrm{dom}(\theta)$, $f(i){\downarrow} \in \mathrm{dom}(\Theta')$ and

$$F(\theta_i) \in \Theta'_{f(i)}.$$

Let $f \in R^{(1)}$ witness that $F\colon S \rightrightarrows S'$ is effective with respect to c.e. enumeration indices. Then $g \in R^{(1)}$ with $W_{g(j)} = \{\, i \in \omega \mid j \in W_{f(i)} \,\}$ witnesses that $F^-\colon S' \rightrightarrows S$, the lower inverse of F, is effective with respect to c.e. enumeration indices as well.

If $F\colon S \rightrightarrows S'$ and $G\colon S' \rightrightarrows S''$ are effective with respect to enumeration indices, then so is $G \circ F$. Moreover, if $H\colon S \rightrightarrows S$ is effective with respect to enumeration indices, the same is true for the transitive closure H^* of H defined by $H^*(y) = \bigcup \{\, H^n(y) \mid n \in \omega \,\}$.

Now, let $\mathcal{T} = (T, \tau)$ and $\mathcal{T}' = (T', \tau')$, respectively, be T_0 spaces with countable strong bases $\mathcal{B}$ and $\mathcal{B}'$ and indexings B as well as B' of $\mathcal{B}$ and $\mathcal{B}'$. We always assume that T is countable and x is a numbering of T. The question we are interested in for the rest of this paper is whether effective multifunctions $F\colon T \rightrightarrows T'$ are continuous. Without restriction we assume that the multifunctions considered in what follows have only nonempty values.

Several continuity notions for multifunctions have been considered in the literature.

Definition 5.5 Let $F\colon T \rightrightarrows T'$ and $y \in T$.

1. F is *lower semi-continuous* at y if for each open set $O \in \tau'$ meeting $F(y)$ there is a neighbourhood V of y such that $V \subseteq F^-(O)$.

2. F is *upper semi-continuous* at y if for each open set $O \in \tau'$ containing $F(y)$ there is a neighbourhood V of y such that $V \subseteq F^+(O)$.

3. F is *continuous* at y if it is both lower and upper semi-continuous at y.

4. F is *lower semi-continuous*, *upper semi-continuous* and *continuous* in T, respectively, if it is lower semi-continuous, upper semi-continuous and continuous at each point of T.

Obviously, F is lower semi-continuous in T, exactly if $F^-(O) \in \tau$, for each $O \in \tau'$, and similarly upper semi-continuous in T, exactly if $F^+(O) \in \tau$, for each $O \in \tau'$.

Because of Lemma 5.2(3) it is no restriction if in the definition of lower semi-continuity we quantify only over all basic open sets meeting $F(y)$. This leads us to the following definition of when a multifunction is effectively lower semi-continuous.

Definition 5.6 A multifunction $F\colon T \rightrightarrows T'$ is said to be

1. *effectively pointwise lower semi-continuous*, if there is a function $d \in P^{(2)}$ such that for all $i \in \mathrm{dom}(x)$ and $n \in \mathrm{dom}(B')$ for which $F(x_i)$ intersects B'_n, $d(i,n){\downarrow} \in \mathrm{dom}(B)$, $x_i \in B_{d(i,n)}$, and $B_{d(i,n)} \subseteq F^-(B'_n)$.

2. *effectively lower semi-continuous*, if there is a function $g \in R^{(1)}$ so that for all $n \in \mathrm{dom}(B')$, $g(n) \in \mathrm{dom}(L^\tau)$ and $F^-(B'_n) = L^\tau_{g(n)}$.

Next, we want to find a definition of when a multifunction could be called effectively upper semi-continuous. Unfortunately, we cannot proceed as in the case of lower semi-continuity. If we assume, however, that the multifunction F is compact-valued, i.e. that each image $F(y)$ for $y \in T$ is compact, we can, without loss of generality, restrict the quantification in the definition of upper semi-continuity to all finite unions of basic open sets containing $F(y)$. Note that Berge [4] defines upper semi-continuity only for compact-valued multifunctions. For $n \in \omega$ with $D_n \subseteq \mathrm{dom}(B)$ let

$$U_n = \bigcup \{ B_a \mid a \in D_n \}.$$

In any other case, let U_n be undefined.

Definition 5.7 A compact-valued multifunction $F\colon T \rightrightarrows T'$ is said to be

1. *effectively pointwise upper semi-continuous*, if there is a function $t \in P^{(2)}$ such that for all $i \in \mathrm{dom}(x)$ and $n \in \mathrm{dom}(U')$ with $F(x_i) \subseteq U'_n$, $t(i,n)\downarrow \in \mathrm{dom}(B)$, $x_i \in B_{t(i,n)}$, and $B_{t(i,n)} \subseteq F^+(U'_n)$.
2. *effectively upper semi-continuous*, if there is a function $h \in R^{(1)}$ so that for all $n \in \mathrm{dom}(U')$, $h(n) \in \mathrm{dom}(L^\tau)$ and $F^+(U'_n) = L^\tau_{h(n)}$.
3. *effectively pointwise continuous* if it is both effectively pointwise lower semi-continuous and effectively pointwise upper semi-continuous.
4. *effectively continuous* if it is effectively lower semi-continuous and effectively upper semi-continuous.

Proposition 5.8 *Let x be computable and B' be total. Then following two statements hold:*

1. *Every effectively lower semi-continuous multifunction $F\colon T \rightrightarrows T'$ is effective with respect to hit indices.*
2. *Every compact-valued effectively upper semi-continuous multifunction $F\colon T \rightrightarrows T'$ is effective with respect to covering indices.*

Proof: We only derive the first statement. The second follows similarly. Let $L \subseteq \omega$ and $g \in R^{(1)}$, respectively, witness that x is computable and F is effectively lower semi-continuous. Moreover, let $f \in R^{(1)}$ such that

$$W_{f(i)} = \{ n \in \omega \mid (\exists m \in W_{g(n)})\langle i, m\rangle \in L \}.$$

Now, note that for $i \in \mathrm{dom}(x)$,

$$F(x_i) \cap B'_n \neq \emptyset \Leftrightarrow x_i \in F^-(B'_n) \Leftrightarrow x_i \in L^\tau_{g(n)} \Leftrightarrow (\exists m \in W_{g(n)}) x_i \in B_m.$$

It follows that $f(i)$ is a hit index of $F(x_i)$,

As has already been said, we are interested in whether and when the converse statements hold. Let to this end for a (compact-valued) multifunction $F\colon T \rightrightarrows T'$, $F^-(\tau')$ and $F^+(\tau')$, respectively, be the topologies on T generated by the subbases $\{ F^-(B'_n) \mid n \in \mathrm{dom}(B') \}$ and $\{ F^+(U'_n) \mid n \in \mathrm{dom}(U') \}$, called the *lower inverse image* under F of τ' and the *upper inverse image* under F of τ'. For $n \in \mathrm{dom}(B')$ and $n \in \mathrm{dom}(U')$, respectively, set $F^-_n = F^-(B'_n)$ and $F^+_n = F^+(U'_n)$. Then F^- and F^+ are indexings of the subbases $\{ F^-(B'_n) \mid n \in \mathrm{dom}(B') \}$ and $\{ F^+(U'_n) \mid n \in \mathrm{dom}(U') \}$, respectively.

Lemma 5.9 *Let $\prec_{B'}$ be r.e. and $F\colon T \rightrightarrows T'$. Then the following two statements hold:*

1. *If F is effective with respect to hit indices then $F^- \leq M$. In particular, $F^-(\tau')$ is a Mal'cev topology on T.*

2. *If F is effective with respect to covering indices then $F^+ \leq M$. In particular, $F^+(\tau')$ is a Mal'cev topology on T.*

Proof: (1) Let $f \in R^{(1)}$ witness that F is effective with respect to hit indices and $g \in R^{(1)}$ so that $W_{g(n)} = \{ i \in \omega \mid (\exists m \in W_{f(i)}) m \preccurlyeq_{B'} n \}$. Then we have for $i \in \mathrm{dom}(x)$ and $n \in \mathrm{dom}(B')$ that

$$x_i \in F^-(B'_n) \Leftrightarrow F(x_i) \cap B'_n \neq \emptyset \Leftrightarrow (\exists m \in W_{f(i)}) m \preccurlyeq_{B'} n \Leftrightarrow i \in W_{g(i)}.$$

Thus, $F^-_n = M_{g(n)}$, for $n \in \mathrm{dom}(B')(= \mathrm{dom}(F^-))$.

(2) Now, let $f \in R^{(1)}$ witness that F is effective with respect to covering indices and $g \in R^{(1)}$ so that $W_{g(n)} = \{ i \in \omega \mid (\exists m \in W_{f(i)})(\forall a \in D_j)(\exists b \in D_n) a \preccurlyeq_{B'} b \}$. In this case we have for $i \in \mathrm{dom}(x)$ and $n \in \mathrm{dom}(B')$ that

$$\begin{aligned} x_i \in F^+(U'_n) &\Leftrightarrow F(x_i) \subseteq U'_n \\ &\Leftrightarrow F(x_i) \subseteq \bigcup \{ B_a \mid a \in D_n \} \\ &\Leftrightarrow (\exists m \in W_{f(i)})(\forall a \in D_j)(\exists b \in D_n) a \preccurlyeq_{B'} b \\ &\Leftrightarrow i \in W_{g(n)}. \end{aligned}$$

Hence, $F^+_n = M_{g(n)}$, for $n \in \mathrm{dom}(U')(= \mathrm{dom}(F^+))$.

As is readily verified, the converse implications hold as well.

Lemma 5.10 *Let B' be total and $F\colon T \rightrightarrows T'$. Then the following two statements hold:*

1. *If $F^- \leq M$ then F is effective with respect to hit indices.*

2. *If $F^+ \leq M$ then F is effective with respect to covering indices.*

Proof: We only show the first statement. The other one follows in a similar way. Let $g \in R^{(1)}$ witness that $F^- \leq M$ and $f \in R^{(1)}$ with $W_{f(i)} = \{ n \in \omega \mid i \in W_{g(n)} \}$. Then we obtain for $i \in \mathrm{dom}(x)$ and $n \in \omega$ that

$$n \in W_{f(i)} \Leftrightarrow x_i \in M_{g(n)} \Leftrightarrow x_i \in F^-(B'_n) \Leftrightarrow F(x_i) \cap B'_n \neq \emptyset.$$

It follows that $f(i)$ is a hit index of $F(x_i)$.

Note that F is effectively pointwise lower semi-continuous exactly if $F^-(\tau') \subseteq_p \tau$, and effectively lower semi-continuous just if $F^-(\tau') \subseteq_e \tau$. Similarly, F is effectively pointwise upper semi-continuous if, and only if, $F^+(\tau') \subseteq_p \tau$, and effectively upper semi-continuous if, and only if, $F^+(\tau') \subseteq_e \tau$.

As a consequence of Theorem 3.14 we now obtain the continuity result we are looking for. All we need is that topology τ has a realiser for noninclusion with respect to $F^-(\tau')$ and/or $F^+(\tau')$. We say in this case that F has a *lower*, respectively, *upper witness for noninlcusion.*

Theorem 5.11 *Let $\mathcal{T}$ be effective, x be acceptable, and $\prec_{B'}$ be r.e. Then the following two statements hold:*

1. *Every multifunction $F\colon T \rightrightarrows T'$ that is effective with respect to hit indices and has a lower witness for noninclusion must be effectively pointwise lower semi-continuous. If $\mathcal{T}$ is also recursively separable, then it is even effectively lower semi-continuous.*

2. *Every compact-valued multifunction $F\colon T \rightrightarrows T'$ that is effective with respect to covering indices and has an upper witness for noninclusion must be effectively pointwise upper semi-continuous. If $\mathcal{T}$ is also recursively separable, then it is even effectively upper semi-continuous.*

As follows from [36], the witness for noninclusion requirement is indispensable for this result. Under somewhat stronger, but still very natural assumptions we obtain that the sufficient conditions used in the theorem are also necessary. Apply to this end Proposition 5.8, and Proposition 3.15 together with Lemma 2.8.

Corollary 5.12 *Let $\mathcal{T}$ be effective and recursively separable, let τ have a realiser for noninclusion with respect to itself, x be acceptable, B' be total, and $\prec_{B'}$ be r.e. Then the following two statements hold:*

1. *Every multifunction $F\colon T \rightrightarrows T'$ is effectively lower semi-continuous if, and only if, it is effective with respect to hit indices and has a lower witness for noninclusion.*

2. *Every compact-valued multifunction $F\colon T \rightrightarrows T'$ is effectively upper semi-continuous if, and only if, it is effective with respect to covering indices and has an upper witness for noninclusion.*

As we have seen in Proposition 3.26, for effectively pointed effective spaces τ has a realiser for noninclusion with respect to any Mal'cev topology.

Theorem 5.13 *Let $\mathcal{T}$ be effective and effectively pointed, x be acceptable, B' be total, and $\prec_{B'}$ be r.e. Then the following two statements hold:*

1. *Every multifunction $F\colon T \rightrightarrows T'$ is effective with respect to hit indices if, and only if, it is effectively lower semi-continuous.*

2. *Every compact-valued multifunction $F\colon T \rightrightarrows T'$ is effective with respect to covering indices if, and only if, it is effectively upper semi-continuous.*

In the remainder of this section we will consider the case that $\mathcal{T}'$ is an effectively given metric space and investigate which effective multifunctions have a lower and/or upper witness for noninclusion. As we shall see, we need not only know the basic open sets that meet a set X in this case, but also those that miss it. We call a pair $\langle i, j\rangle$ a *hit-and-miss index* of X, if i is a hit and j a complement exhaustion index of X.

Proposition 5.14 *Let $\mathcal{T}$ be effective and recursively separable and $\mathcal{T}'$ be an effectively given metric space. Moreover, let x be acceptable. Then every multifunction $F\colon T \rightrightarrows T'$ that is effective with respect to hit-and-miss indices has a lower witness for noninclusion.*

Proof: Let $i \in \mathrm{dom}(x)$ and $m \in \omega$ with $x_i \in F^-_m$. Then $F(x_i)$ intersects B'_m. Let $z \in F(x_i) \cap B'_m$. Moreover, let $m = \langle m_1, m_2\rangle$. It follows that $\delta(\beta_{m_1}, z) < 2^{-m_2}$. Let $a \in \omega$

with $2^{-a} < 2^{-m_2} - \delta(\beta_{m_1}, z)$. Then there is some point β_c in the dense subset of T' so that $\delta(\beta_c, z) < 2^{-a-1}$. Thus, we have

$$\begin{aligned}\delta(\beta_{m_1}, \beta_c) + 2^{-a-1} &\leq \delta(\beta_{m_1}, z) + \beta(\beta_c, z) + 2^{-a-1} \\ &< \delta(\beta_{m_1}, z) + 2^{-a-1} + 2^{-a-1} \\ &= \delta(\beta_{m_1}, z) + 2^{-a} \\ &< 2^{-m_2}.\end{aligned}$$

Set $m' = \langle c, a+1 \rangle$. Then $m' \prec_{B'} m$ and $z \in B'_{m'}$. In addition, $\mathrm{cl}_{\tau'}(B'_{m'}) \subseteq (B'_{m'})^c \subseteq B'_m$.

Thus, there is some $m' \prec_{B'} m$ with $(B'_{m'})^c \subseteq B'_m$ so that $F(x_i)$ is met by $B'_{m'}$. Let $f \in R^{(1)}$ witness that F is effective with respect to hit-and-miss indices and for $i, m \in \omega$, $\bar{s}(i, m)$ be the first element with respect to some fixed enumeration of $\{ \langle m', \bar{\imath}, \overline{m} \rangle \mid m' \prec_{B'} \overline{m} \wedge (\exists e \in W_{\pi_1(f(\bar{\imath}))}) e \preccurlyeq_{B'} m' \}$ with $\bar{\imath} = i$ and $\overline{m} = m$. Set $\hat{s}(i, m) = \pi_1(\bar{s}(i, m))$.

Now, assume in addition that $B_n \not\subseteq F_m^-$. Let $A \subseteq \omega$ witness that $\mathcal{T}$ is recursively separable. Then it follows with Lemma 3.11 that also $\{ x_\nu \in B_n \mid \nu \in A \} \not\subseteq F_m^-$. Hence, there is some $a \in A$ such that $x_a \in B_n$, and $F(x_a)$ and B'_m are disjoint. It follows that $F(x_a)$ and $(B'_{\hat{s}(i,m)})^c$ are disjoint as well. Let $L \subseteq \omega$ witness that x is computable. Moreover, let $h(i, n, m)$ be the first element in some fixed enumeration of $\{ \langle \bar{a}, \bar{\imath}, \bar{n}, \overline{m} \rangle \mid \bar{a} \in A \wedge \langle \bar{a}, \bar{n} \rangle \in L \wedge \hat{s}(\bar{\imath}, \overline{m}) \in W_{\pi_2(f(\bar{a}))} \}$ with $\bar{\imath} = i$, $\bar{n} = n$, and $\overline{m} = m$. Set $r(i, n, m) = \pi_1(h(i, n, m))$. It follows that $r(i, n, m) \in A \subseteq \mathrm{dom}(x)$ and $x_{r(i,n,m)} \in B_n \setminus F^-_{\hat{s}(i,m)}$. By Lemma 5.9 there is some $g \in R^{(1)}$ with $F_m^- = M_{g(m)}$. Set $s(i, m) = g(\hat{s}(i, m))$. Then (s, r) is a realiser for noninclusion of τ with respect to $F^-(\tau')$.

The following result is now a consequence of Theorem 5.11.

Theorem 5.15 *Let $\mathcal{T}$ be effective and recursively separable and $\mathcal{T}'$ be an effectively given metric space. Moreover, let x be acceptable. Then every multifunction $F \colon T \rightrightarrows T'$ that is effective with respect to hit-and-miss indices must be effectively lower semi-continuous.*

Let us see next when F has an upper witness for noninclusion.

Proposition 5.16 *Let $\mathcal{T}$ be effective and recursively separable and $\mathcal{T}'$ be an effectively given metric space. Moreover, let x be acceptable. Then every compact-valued multifunction $F \colon T \rightrightarrows T'$ that is effective with respect to complete covering indices has an upper witness for noninclusion.*

Proof: Let $i \in \mathrm{dom}(x)$ and $m \in \omega$ with $x_i \in F_m^+$. Then $F(x_i) \subseteq U'_m$. Let $z \in F(x_i)$. It follows that there is some $a_z \in D_m$ with $z \in B'_{a_z}$. As we have seen in the proof of Proposition 5.14, there is moreover some $a'_z \prec_{B'} a_z$ so that $z \in B'_{a'_z} \subseteq \mathrm{cl}_{\tau'}(B'_{a'_z}) \subseteq (B'_{a'_z})^c \subseteq B'_{a_z}$. Thus $F(x_i) \subseteq \bigcup \{ B'_{a'} \mid (\exists a \in D_m) a' \prec_{B'} a \} \subseteq U'_m$. Since $F(x_i)$ is compact, there is some $m' \in \omega$ such that for each $a' \in D_{m'}$ there exists some $a \in D_m$ with $a' \prec_{B'} a$. Moreover $F(x_i) \subseteq U'_{m'}$. Let $f \in R^{(1)}$ witness that F is effective with respect to complete covering indices. Then there is some $m'' \in W_{\pi_1(f(i))}$ so that for every $a'' \in D_{m''}$ there is some $a' \in D_{m'}$ with $a'' \preccurlyeq_{B'} a'$. It follows that there is some $m'' \in W_{\pi_1(f(i))}$ such that for each $a'' \in D_{m''}$ there exists some $a \in D_m$ with $a'' \prec_{B'} a$. Let $\bar{s}(i, m)$ be the first element in some fixed enumeration of $\{ \langle m'', \bar{\imath}, \overline{m} \rangle \mid m'' \in W_{\pi_1(f(i))} \wedge (\forall a'' \in D_{m''})(\exists a \in D_m) a'' \prec_{B'} a \}$ with $\bar{\imath} = i$ and $\overline{m} = m$. Set $\hat{s}(i, m) = \pi_1(\bar{s}(i, m))$. Then we have that $F(x_i) \subseteq U'_{\hat{s}(i,m)} \subseteq \mathrm{cl}_{\tau'}(U'_{\hat{s}(i,m)}) \subseteq \bigcup \{ \mathrm{cl}_{\tau'}(B'_{\bar{a}}) \mid \bar{a} \in D_{\hat{s}(i,m)} \} \subseteq \bigcup \{ (B'_{\bar{a}})^c \mid \bar{a} \in D_{\hat{s}(i,m)} \} \subseteq U'_m$.

Assume in addition that $B_n \not\subseteq F_m^+$. Moreover, let $A \subseteq \omega$ witness that $\mathcal{T}$ is recursively separable. Then we obtain with Lemma 3.11 that also $\{ x_\nu \in B_n \mid \nu \in A \} \not\subseteq F_m^+$. Hence, there is some $b \in A$ with $x_b \in B_n$ and $F(x_b) \not\subseteq U'_m$. Consequently, $F(x_b) \not\subseteq \bigcup\{ (B'_{\bar{a}})^c \mid \bar{a} \in D_{\hat{s}(i,m)} \}$ as well. Thus, there exists some $z \in F(x_b)$ so that $z \notin \bigcup\{ (B'_{\bar{a}})^c \mid \bar{a} \in D_{\hat{s}(i,m)} \}$. It follows that $\delta(z, \beta_{\pi_1(\bar{a})}) > 2^{-\pi_2(\bar{a})}$, for all $\bar{a} \in D_{\hat{s}(i,m)}$. Let $c \in \omega$ with $2^{-c} < \delta(z, \beta_{\pi_1(\bar{a})}) - 2^{-\pi_2(\bar{a})}$, for all these $\bar{a}$. Moreover, let $e \in \omega$ so that $\delta(z, \beta_e) < 2^{-(c+1)}$. Then $z \in B'_{\langle e, c+1\rangle} \cap F(x_b)$. Thus there is some $c' \in W_{\pi_1(f(b))}$ with $c' \preccurlyeq_{B'} \langle e, c+1\rangle$. So, we have

$$\begin{aligned}2^{-c} &< \delta(z, \beta_{\pi_1(\bar{a})}) - 2^{-\pi_2(\bar{a})}\\ &\leq \delta(z, \beta_e) + \delta(\beta_e, \beta_{\pi_1(c')}) + \delta(\beta_{\pi_1(c')}, \beta_{\pi_1(\bar{a})}) - 2^{-\pi_2(\bar{a})}\\ &< 2^{-(c+1)} + 2^{-(c+1)} - 2^{-\pi_2(c')} - 2^{-\pi_2(\bar{a})} + \delta(\beta_{\pi_1(c')}, \beta_{\pi_1(\bar{a})}),\end{aligned}$$

from which it follows that $\delta(\beta_{\pi_1(c')}, \beta_{\pi_1(\bar{a})}) > 2^{-\pi_2(c')} + 2^{-\pi_2(\bar{a})}$, for all $\bar{a} \in D_{\hat{s}(i,m)}$.

Now, let $L \subseteq \omega$ witness that x is computable. Furthermore, let

$$E = \{ \langle e, e'\rangle \mid \delta(\beta_{\pi_1(e)}, \beta_{\pi_1(e')}) > 2^{-\pi_2(e)} + 2^{-\pi_2(e')} \}.$$

Then E is r.e. Let $\bar{r}(i,n,m)$ be the first element in some fixed enumeration of $\{ \langle b, \bar{\imath}, \bar{n}, \overline{m}\rangle \mid b \in A \wedge \langle b, \bar{n}\rangle \in L \wedge (\exists c' \in W_{\pi_2(f(b))})(\forall \bar{a} \in D_{\hat{s}(\bar{\imath},\overline{m})})\langle c', \bar{a}\rangle \in E \}$ with $\bar{\imath} = i$, $\bar{n} = n$ and $\overline{m} = m$, and set $r = \pi_1 \circ \bar{r}$. It follows that $r(i,n,m) \in A \subseteq \mathrm{dom}(x)$, $x_{r(i,n,m)} \in B_n$ and $F(x_{r(i,n,m)}) \not\subseteq U'_{\hat{s}(i,m)}$. By Lemma 5.9 there is some $g \in R^{(1)}$ with $F_m^+ = M_{g(m)}$. Set $s(i,m) = g(\hat{s}(i,m))$. Then (s,r) is a realiser for noninclusion of τ with respect to $F^+(\tau')$.

By applying Theorem 5.11 again we obtain an analogue to Theorem 5.15 for the case of upper semi-continuity. Now recall that by Lemma 4.20 we can compute a complement exhaustion index from any covering index. Moreover, as we have seen in Lemma 4.18, complete and strict covering indices can be computed from each other. Therefore, every compact-valued multifunction that is effective with respect to strict covering indices and has values in an effectively given metric space is both effectively lower and upper semi-continuous.

Theorem 5.17 *Let $\mathcal{T}$ be effective and recursively separable and $\mathcal{T}'$ be an effectively given metric space. Moreover, let x be acceptable. Then every compact-valued multifunction $F\colon T \rightrightarrows T'$ that is effective with respect to strict covering indices must be effectively continuous.*

6 Outer semi-continuity

Rockafellar and Wetts [30] consider a further pair of semi-continuities for multifunctions: outer and inner semi-continuity, where inner semi-continuity coincides with lower semi-continuity. Let $\mathcal{T} = (T, \tau)$ and $\mathcal{T}' = (T', \tau')$, respectively, again be T_0 spaces with countable strong bases $\mathcal{B}$ and $\mathcal{B}'$ as well as indexings B and B' hereof. The outer semi-continuity of a multifunction $F\colon T \rightrightarrows T'$ is defined in terms of outer limits [30]:

$$\begin{aligned}&\limsup_{y \to \overline{y}} F(y)\\ &\quad = \{ \overline{z} \in T' \mid (\exists (y_\nu)_{\nu\in\omega} \subseteq T)(\exists (z_\nu)_{\nu\in\omega} \subseteq T') y_\nu \to \overline{y} \wedge z_\nu \to \overline{z} \wedge (\forall \nu \in \omega) z_\nu \in F(y_\nu) \}.\end{aligned}$$

Definition 6.1 A multifunction $F\colon T \rightrightarrows T'$ is *outer semi-continuous* at $\overline{y} \in T$ if

$$\limsup_{y \to \overline{y}} F(y) \subseteq F(\overline{y}).$$

Lemma 6.2 ([30]) *For any multifunction $F\colon T \rightrightarrows T$*

1. *F is outer semi-continuous at $y \in T$ if, and only if, for every $z \notin F(y)$ there are open neighbourhoods U of z and V of y such that V is disjoint from $F^{-}(U)$.*
2. *F is outer semi-continuous (everywhere) if, and only if, the relation*

$$R_F = \{\, (y, z) \in T \times T' \mid z \in F(y) \,\}$$

is closed in $T \times T'$.

The above definition is not well suited for effectivisation. We use the first characterisation in the preceding lemma instead. Obviuously, the open sets U and V depend on the points y and z. To be able to express this in our framework we need assume that both T and T' are countable and have numberings x and x', respectively.

Definition 6.3 A multifunction $F\colon T \rightrightarrows T'$ is *effectively (pointwise) outer semi-continuous* if there are functions $h, k \in P^{(1)}$ such that for all $i \in \mathrm{dom}(x)$ and $j \in \mathrm{dom}(x')$ with $x'_j \notin F(x_i)$, $h(i,j)\!\downarrow \in \mathrm{dom}(B)$, $k(i,j)\!\downarrow \in \mathrm{dom}(B')$, $x_i \in B_{h(i,j)}$, $x'_j \in B'_{k(i,j)}$, and $B_{h(i,j)} \cap F^{-}(B'_{k(i,j)}) = \emptyset$.

It should be noted that we do not have a "pointfree" effective version of global outer semi-continuity as we had in case of the other semi-continuity notions. Therefore, we need a weaker notion of effective closedness for a computable version of Lemma 6.2(2) than that of being the complement of a Lacombe set.

Definition 6.4 A subset O of T is *effectively (pointwise) open*, if there is some function $p \in P^{(1)}$ so that for all $i \in \mathrm{dom}(x)$ with $x_i \in O$, $p(i)\!\downarrow \in \mathrm{dom}(B)$ and $x_i \in B_{p(i)} \subseteq O$. The complements of effectively open sets are called *effectively closed* and the Gödel numbers of the respective functions p are said to be their *closedness indices*.

As an immediate consequence of the definition we have that any effectively outer semi-continuous multifunction is effective with respect to closedness indices.

Note that the product $\mathcal{T} \times \mathcal{T}' = (T \times T', \tau^{\times})$ with the product topology $\tau^{\times}$ has the basis $\mathcal{B} \times \mathcal{B}'$ with is indexed by $B^{\times}_{\langle m,n \rangle} = B_m \times B'_n$ and strong with respect to $\prec_{B^{\times}}$ defined by $\langle m, n \rangle \prec_{B^{\times}} \langle m', n' \rangle$ if $m \prec_B n$ and $m' \prec_{B'} n'$. The space $T \times T'$ is indexed by $x^{\times}$ defined similarly. Then $\mathcal{T} \times \mathcal{T}'$ is effective if $\mathcal{T}$ and $\mathcal{T}'$ are. In the same way, $x^{\times}$ is computable, if x and x' are, and allows effective limit passing if x and x' do so.

Lemma 6.5 *A multifunction $F\colon T \rightrightarrows T'$ is effectively outer semi-continuous if, and only if, the relation R_F is an effectively closed subset of $T \times T'$.*

As we did in case the other continuity notions in the preceding section, let us now study the question whether effective multifunctions are effectively outer semi-continuous. Again we will need both positive and negative information about the images under the multifunction.

Theorem 6.6 *Let $\mathcal{T}$ and $\mathcal{T}'$ be effective and x as well as x' be acceptable. Then every multifunction $F\colon T \rightrightarrows T'$ that is jointly effective with respect to density and closedness indices must be effectively outer semi-continuous.*

Proof: We show that the relation R_F is effectively closed. Let to this end $k, r \in R^{(1)}$ witness that F is effective with respect to density and closedness indices, respectively. Moreover, let $L' \subseteq \omega$ witness that x' is computable and let $s \in R^{(2)}$ such that for $\varphi_{s(i,m)}$ enumerates the set $\{\, a \in W_{k(i)} \mid \langle a, m\rangle \in L' \,\}$, if this set is nonempty. Define $f(i,m) = \varphi_{s(i,m)}(0)$ and $\bar{f}(i,m) = \langle i, f(i,m)\rangle$.

Let $\mathrm{pt} \in P^{(1)}$ witness that $x^{\times}$ allows effective limit passing and set $\hat{g}(n) = \mu c : \varphi_{r(\pi_1(\mathrm{pt}(n)))}(\pi_2(\mathrm{pt}(n)))\downarrow_c$. In addition, let $q \in R^{(1)}$ and $p \in R^{(2)}$ be as in Lemma 3.5 applied to $\mathcal{T} \times \mathcal{T}'$. Finally, define $h \in R^{(3)}$ by

$$\varphi_{h(n,i,j)}(a) = \begin{cases} \varphi_{q(\langle i,j\rangle)}(a) & \text{if } \hat{g}(n)\uparrow_a, \\ \varphi_{p(\bar{f}(i,\pi_2(\varphi_{q(\langle i,j\rangle)}(\hat{g}(n)-1))),\varphi_{q(\langle i,j\rangle)}(\hat{g}(n)-1))}(a - \hat{g}(n)) & \text{otherwise.} \end{cases}$$

By the recursion theorem there is then a function $d \in R^{(2)}$ with

$$\varphi_{h(d(i,j),i,j)} = \varphi_{d(i,j)}.$$

Let $g(i,j) = \hat{g}(d(i,j))$, and suppose that $g(i,j)\uparrow$, for some $i \in \mathrm{dom}(x)$ and $j \in \mathrm{dom}(x')$ with $(x_i, x'_j) \notin R_F$. Then $d(i,j)$ is an index of a normed recursive enumeration of basic open sets converging to (x_i, x'_j). Hence, $x^{\times}_{\mathrm{pt}(d(i,j))} = (x_i, x'_j)$. Note that $\varphi_{r(i)}(j)\downarrow$ as $(x_i, x'_j) \notin R_F$. Thus, $g(i,j)\downarrow$ as well, in contradiction to our assumption. It follows that $g(i,j)\downarrow$, for all $i \in \mathrm{dom}(x)$ and $j \in \mathrm{dom}(x')$ such that $(x_i, x'_j) \notin R_F$.

Assume next that that $B^{\times}(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1))$ meets R_F, for some $i \in \mathrm{dom}(x)$ and $j \in \mathrm{dom}(x')$ with $(x_i, x'_j) \notin R_F$. By Lemma 3.5 we have that $(x_i, x'_j) \in B^{\times}(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1))$. Hence, $x_i \in B(\pi_1(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))$ and $B'(\pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))$ meets $F(x_i)$. Consequently, $f(i, \pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))\downarrow$ and

$$x'_{f(i,\pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))} \in B'_{\pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1))}.$$

It follows that $d(i,j)$ is an index of a normed recursive enumeration of basic open sets converging to $(x(i), x'(f(i,\pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))))$, i.e.,

$$x^{\times}_{\mathrm{pt}(d(i,j))} = (x_i, x'_{f(i,\pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))}).$$

By definition, $x'(f(i,\pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))) \in F(x_i)$. Therefore,

$$(x_i, x'_{f(i,\pi_2(\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)))} \in R_F,$$

which implies that $g(i,j)\uparrow$ in contradiction to our assumption. It follows that

$$B^{\times}_{\varphi_{q(\langle i,j\rangle)}(g(i,j)-1)} \cap R_F = \emptyset,$$

for all $i \in \mathrm{dom}(x)$ and $j \in \mathrm{dom}(x')$ so that $(x_i, x'_j) \notin R_F$, showing that R_F is effectively closed.

Note that no witness for noninclusion condition was needed in this case.

Corollary 6.7 *Let $\mathcal{T}$ and $\mathcal{T}'$ be effective and x as well as x' be acceptable. Moreover, let $F \colon T \rightrightarrows T'$ be a multifunction that is effective with respect to density indices. Then F is effectively outer semi-continuous if, and only if, F is also effective with respect to closedness indices.*

By Lemma 4.7 a hit index of an effectively separable subspace can be computed from any of its density indices. As we have moreover seen in Theorem 5.13(1), every multifunction on an effectively pointed space is effective with respect to hit indices, exactly if it is effectively lower semi-continuous. If, in addition, the domain space of the multifunction is constructively complete, as in the case of constructive domains, then density indices can also be computed from hit indices, by Theorem 4.11. This allows to characterise the multifunctions that are jointly effective with respect to density and closedness indices.

Theorem 6.8 *Let $\mathcal{T}$ be effective and effectively pointed. Moreover, let $\mathcal{T}'$ be effective and x as well as x' be acceptable. Then the following two statements hold for any multifunction $F \colon T \rightrightarrows T'$:*

1. *If F is jointly effective with respect to density and closedness indices, then it must be both effectively lower semi-continuous and effectively outer semi-coninuous.*
2. *If, in addition, $\mathcal{T}$ is constructively complete then, if F is both effectively lower semi-continuous and effectively outer semi-coninuous, it is jointly effective with respect to density and closedness indices.*

Let us now consider the case that the value space $\mathcal{T}'$ is a constructive metric space. As we have seen in Theorem 5.15, multifunctions that are effective with respect to hit-and-miss indices are effectively lower semi-continuous in this case, if, in addition, $\mathcal{T}$ is recursively separable. Obviously, every closed subset of T' that admits effective complement exhaustion is also effectively closed and a closedness index of it can uniformly be computed from any of its complement exhaustion indices. This gives us the following result.

Theorem 6.9 *Let $\mathcal{T}$ be effective as well as recursively separable, and let x be acceptable. Moreover, let $\mathcal{T}'$ be a constructive metric space. Then every closed-valued multifunction $F \colon T \rightrightarrows T'$ that is jointly effective with respect to density and complement exhaustion indices must be both effectively lower semi-continuous and effectively outer semi-continuous.*

7 A Counterexample

In Theorem 5.15 we have seen that multifunctions between an effective, recursively separable space and an effectively given metric space are effectively lower semi-continuous, if they are jointly effective with respect to hit and to complement exhaustion indices. That is, in addition to the positive information on the values of the multifunction given by the hit indices, negative information, i.e. information about the complements of these values, was needed in the proof. As we will see next, the result is not true without this additional negative information.

Proposition 7.1 *There are constructive metric spaces $\mathcal{M}$ and $\mathcal{M}'$, and a multifunction $F \colon M \rightrightarrows M'$ which is effective with respect to enumeration indices, but not lower semi-continuous, and which is hence not effective with respect to complement exhaustion indices.*

Proof: The construction is based on an idea of Friedberg [20]. Let $\mathcal{M}$ be the Baire space and $\mathcal{M}'$ be the space of the natural numbers with the discrete metric. Moreover, define $f \in R^{(1)}$ by

$$\varphi_{f(i)}(n) = \begin{cases} 0 & \text{if } [(\forall a \leq i)\varphi_i(a) = 0] \vee (\exists c)[\varphi_i(c) \neq 0 \\ & \quad \wedge (\forall a < c)\varphi_i(a) = 0 \wedge (\exists j < c)(\forall b \leq c)\varphi_i(b) = \varphi_j(b)], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

As is ready verified, for all $\varphi_i, \varphi_j \in R^{(1)}$ with $\varphi_i = \varphi_j$ one has that $\varphi_{f(i)} = \varphi_{f(j)}$. Note that φ is an acceptable numbering of Baire space. Now, let x be the acceptable numbering of M' constructed in Example 3.17 and let $t \in R^{(1)}$ such that $\varphi_{t(n)}$ is the constantly n function. Then $x_{t(n)} = n$, for all $n \in \omega$. Let $k \in R^{(1)}$ with $W_{k(i)} = t(W_{f(i)} \cup \{1\})$ and define $F\colon R^{(1)} \rightrightarrows \omega$ by $F(\varphi_i) = \{\, x_a \mid a \in W_{k(i)} \,\}$. Then F is effective with respect to enumeration indices, and $F(g) = \omega$, if the first condition in the definition of function f holds, and $F(g) = \{1\}$, otherwise.

Assume that F is lower semi-continuous. We have that $F(\lambda n.0) = \omega$. Moreover, $\{0\}$ is open in the discrete topology on ω. Thus, $F(\lambda n.0)$ meets $\{0\}$, i.e., $\lambda n.0 \in F^{-}(\{0\})$. By the lower semi-continuity of F and the definition of the Baire metric there is thus some $m > 0$ such that $g \in F^{-}(\{0\})$, for all $g \in R^{(1)}$ with $g(n) = 0$, for $n < m$. Let $c = \max\{\, \varphi_i(m) + 1 \mid i < m \wedge \varphi_i \in R^{(1)} \,\}$ and define

$$\hat{g}(n) = \begin{cases} 0 & \text{if } n \neq m, \\ c & \text{otherwise.} \end{cases}$$

Then $\hat{g} \in R^{(1)}$. Moreover, $\hat{g} \in F^{-}(\{0\})$, since $\hat{g}(n) = 0$, for $n < m$. Let j be a Gödel number of $\hat{g}$. Then $j \geq m$, by construction of $\hat{g}$. In addition, $\hat{g}(m) \neq 0$. It follows that $F(\hat{g}) = \{1\}$, i.e., $\hat{g} \notin F^{-}(\{0\})$. Thus, F cannot be lower semi-continuous.

A research programme that has received much attention over the last decades suggests to embed important spaces of analytical mathematics in domains and to extend operations on the spaces to domain operations (cf. e.g. [6, 14, 12]). Domains have been used with great success in defining a mathematically sound semantics for programming language constructs. One of the aims of the programme is to provide such a semantics for languages with a real number data type. This implies as well the development of a new approach to real number computations. The intermediate results appearing in such computations are contained in the domains involved and can thus be perfectly controlled. There is also a well understood computability theory for domains which can be applied here.

An important result in this respect is Berger's Extension Theorem [5] saying that every effective function defined on a effectively dense subset of a constructive algebraic domain with only total values may be extended to an effective map on the domain in such a way that for every argument in the effectively dense subspace the value of the extended map covers the corresponding value of the given map with respect to the domain order. As we will see now, the result is no longer true in the case of multifunctions.

Definition 7.2 Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with countable basis $\mathcal{B}$, and x and B be numberings of T and $\mathcal{B}$, respectively.

1. A subset X of T is *effectively dense* if there is a function $d \in P^{(1)}$ with $d(n){\downarrow} \in \mathrm{dom}(x)$ and $x_{d(n)} \in B_n \cap X$, for every $n \in \mathrm{dom}(B)$.

2. A point $y \in T$ is *total* if the set $\uparrow_{\leq_\tau}\{y\}$ has a greatest element.

3. For subsets X, Y of T and a relation R on T, $X R_{EM} Y$ if for all $y \in X$ there is some $z \in Y$ with $y \, R \, z$ and for all $z \in Y$ there is some $y \in X$ with $y \, R \, z$.

Corollary 7.3 *There are constructive algebraic domains $\mathcal{Q}$ and $\mathcal{Q}'$, an effectively dense subspace M of $\mathcal{Q}$, a set M' of total elements of $\mathcal{Q}'$, and a multifunction $F \colon M \rightrightarrows M'$ that is effective with respect to enumeration indices, but is such that for no lower semi-continuous multifunction $G \colon Q \rightrightarrows Q'$ one has $F(y) \sqsubseteq_{EM} G(y)$, for all $y \in M$.*

Proof: Let $\mathcal{Q}$ be the set $P^{(1)}$ of all partial recursive functions with the extension ordering. As has already been said, the finite functions form an algebraic basis. Note that φ is an admissible numbering of this domain. Moreover, let M be the subset $R^{(1)}$ of all total recursive functions. Since $R^{(1)}$ contains the functions that are 0 almost everywhere, it is effectively dense in $P^{(1)}$.

Choose $\mathcal{Q}'$ to be the flat domain $\omega_\perp$ of the natural numbers. Then the natural numbers are the total elements of this domain. Let x' be an admissible indexing of $\mathcal{Q}'$. It follows that there is a function $d \in R^{(1)}$ such for all $i \in \omega$ for which $\beta(W_i)$ is directed, $x_{d(i)}$ is the least upper bound of $\beta(W_i)$. Let $p \in R^{(1)}$ with $W_{p(c)} = \{c+1\}$. Then $x_{d(p(c))} = \beta_{c+1} = c$.

Now, let $f \in R^{(1)}$ be defined as in the proof of the preceding proposition and $h \in R^{(1)}$ so that $W_{h(i)} = \{\, d(p(c)) \mid c \in W_{f(i)} \cup \{1\} \,\}$. Define $F \colon R^{(1)} \rightrightarrows \omega$ by $F(\varphi_i) = \{\, x_a \mid a \in W_{h(i)} \,\}$. Then we have again that F is effective with respect to enumeration indices, and $F(g) = \omega$, if the first condition in the definition of f holds, and $F(g) = \{1\}$, otherwise. As we have seen in the preceding proposition, F is not lower semi-continuous.

Assume that there is some lower semi-continuous multifunction $G \colon P^{(1)} \rightrightarrows \omega_\perp$ such that for all $g \in R^{(1)}$, $F(g) \sqsubseteq_{EM} G(g)$. Note that since all elements of $F(g)$ are maximal in the domain order on $\omega_\perp$, $F(g) \sqsubseteq_{EM} G(g)$ implies that $F(g) = G(g)$. Observe further that the metric topologies on $R^{(1)}$ and ω, respectively, are the subspace topologies of the Scott topologies on $P^{(1)}$ and $\omega_\perp$. Therefore, F is lower semi-continuous as well in this case, which is not true, as just stated. Thus, such a multifunction G cannot exist.

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