Australasian Journal of Logic

Contraction and revision

Shawn Standefer
The University of Melbourne

Abstract

An important question for proponents of non-contractive approaches to paradox is why contraction fails. One non-contractive theorist, Elia Zardini, offers the answer that paradoxical sentences exhibit a kind of instability. I elaborate this idea using revision theory, and I argue that while instability does motivate failures of contraction, it equally motivates failure of many principles that non-contractive theorists want to maintain.

1 Introduction

Non-contractive logics are receiving increased attention as ways of responding to the semantic and set-theoretic paradoxes.[1] These logics are often presented in a proof-theoretic form, characterized by their rejection of the structural rules of contraction, $(W \vdash)$ and $(\vdash W)$.

$$\frac{X, A, A, Z \vdash Y}{X, A, Z \vdash Y}\ (W \vdash) \qquad\qquad \frac{X \vdash Y, A, A, Z}{X \vdash Y, A, Z}\ (\vdash W)$$

There is a philosophical question of why contraction fails. One prominent non-contractive theorist, Elia Zardini, justifies the failure of contraction by an appeal to unstable states of affairs, using some ideas from revision theory.[2]

[1] See Petersen [2000], Zardini [2011], Shapiro [2011], Beall and Murzi [2013], Mares and Paoli [2014], and Murzi and Shapiro [2015], among others. See French and Ripley [2015] for a good overview of the area. There are other approaches that reject forms of contraction for the *conditional* but maintain the *structural rule* of contraction, e.g. Brady [2006], Field [2008, 2014, 2016], Beall [2009], and Bacon [2013]. My focus will be on the approaches that reject the structural rule.

[2] See Zardini [2011].

Revision theories of truth were originally developed as an alternative to fixed-point approaches to truth that could maintain classical logic.[3] In contrast with the non-contractive approaches to truth, revision theories of truth are usually defined model-theoretically. This disparity makes comparison between the two difficult.

In this paper, I will present Zardini's considerations of unstable states of affairs that motivate looking into revision theory (§2). I will then present a sequent calculus for a revision theory of truth (§3) and compare it to two non-contractive theories of truth (§4). Revision theory supports the original motivation, namely failure of contraction, but it equally supports rejecting many principles that non-contractive theorists wish to maintain. I conclude that, absent reason to think that instability undermines only structural rules, justifying the failure of contraction should be done in another way (§5).

2 Motivations

Rejecting contraction offers one way for a theory of truth to contain a naive truth predicate, i.e. a truth predicate for which A and $T\langle A\rangle$ can be substituted without change of semantic value.[4] One of the features of non-contractive theories is that they offer natural responses to many paradoxes beyond the familiar semantic paradoxes, such as paradoxes of validity.[5] One of the pressing philosophical challenges facing non-contractive approaches, however, is that of justifying the failures of contraction in a non-*ad hoc* way, which is to say providing a rationale for the failures of contraction that does not simply come down to the fact that permitting those instances of contraction would result in triviality.

Elia Zardini has developed a non-contractive theory of truth and proposed a metaphysical picture to justify the failures of contraction for his theory.[6] According to this picture, sentences represent states of affairs. Some states

[3]See Gupta [1982], Herzberger [1982], and Belnap Jr. [1982] for the original presentations of revision theory. See Gupta and Belnap [1993] for a comprehensive treatment of revision theory.

[4]$\langle\cdot\rangle$ is a name-forming operator for sentences. There are some caveats on the contexts in which the truth predicate can be removed or added freely, but we need not worry about those here.

[5]See Beall and Murzi [2013], Shapiro [2013], Priest [2015], and Weber [2014].

[6]See Zardini [2011, 2013, 2014]. The metaphysical picture is primarily developed in Zardini [2011].

of affairs are stable and obtain with all of their consequences. Some states of affairs are unstable, and these lead to failures of contraction. Zardini says:

> I think that the key to understanding what it is about the state-of-affairs expressed by a sentence that explains its failure to contract is given by thinking of that state-of-affairs as distinctively *unstable*. I conceive of instability as the *metaphysical* property attaching to states-of-affairs exemplification of which *causes* the exemplification of the *logical* property of failing to contract attaching to the corresponding sentences.[7]

The liar sentence, Tl, where $l = \langle {\sim}Tl \rangle$, provides the paradigm example.[8] If the state of affairs represented by Tl obtained, then so would that of ${\sim}Tl$, but these two states are incompatible. Zardini continues:

> Thus, the behavior of an unstable state-of-affairs as the one expressed by ${\sim}Tl$ resembles the behavior of physical states: both kinds of states, if they obtained, would lead to other states with which they would not co-obtain—although it is precisely the obtaining of the former states that would lead to the obtaining of the latter states.[9]

As Zardini remarks, in the case of paradoxical sentences, it is less clear what states of affairs they represent than sentences about, say, snow. Zardini remarks:

> The temporal dimension is of course absent in the case of an unstable state-of-affairs as the one expressed by ${\sim}Tl$, but there too we can discern a broadly analogous structure of '*stages of truth evaluation*,' with the transition from one stage to the other being governed by T-introduction (for evaluations of truth) and (the contrapositive of) T-elimination (for evaluations of untruth).[10]

Zardini says, in a subsequent footnote, that he is "appropriating and pushing into a certain direction some themes of the *revision-theoretic* tradition."[11]

[7] Zardini [2011, 504], emphasis in the original.

[8] I will use equations, such as $l = \langle {\sim}Tl \rangle$, to indicate the interpretation of certain names of sentences. In this case, l denotes the sentence ${\sim}Tl$.

[9] Zardini [2011, 504]

[10] Zardini [2011, 505]

[11] Zardini [2011, 505, fn. 13]

Shortly, I will formalize the picture that Zardini presents, which will suggest how to connect revision theory, properly presented, and non-contractive theories of truth. Before doing so, I will make a couple of remarks on Zardini's metaphysical picture.

The first is that the picture is relatively straightforward to understand for the examples given, namely the liar and a sentence about the color of snow. It is appealing to say that the state of affairs expressed by a sentence, A, cannot co-obtain with the state of affairs expressed by its negation, $\sim A$. It is less clear how to understand the incompatibility in the case of a Curry sentence such as Tc, where $c = \langle Tc \to p \rangle$, for some merely contingent truth p. While it is not a problem for p to be true, one surely does not want for p to be true as a matter of the logic of truth. A natural candidate for the state of affairs expressed by Tc to be incompatible with is the state of affairs expressed by $\sim\Box p$. This depends on some plausible assumptions about modality together with states of affairs including modal facts, e.g. there being a state of affairs of being a necessary truth. The situation with Tc contrasts with that of the Curry sentence, Tk, where $k = \langle Tk \to \bot \rangle$, with $\bot$ implying everything. The state of affairs expressed by Tk would be incompatible with other states of affairs, since it implies both p and $\sim p$. Tk fits more naturally with the instability idea than Tc.[12]

The second issue is that the appeal to stability, or rather, instability, is at odds with the underlying conception of truth. Naive truth, of the sort that Zardini favors, is often, and perhaps best, understood in terms fixed-point models, models that assign A and $T\langle A \rangle$ the same semantic values. Zardini does not endorse fixed-point models as the proper model theory for his theory of truth, but it will be useful to consider them. A standard way to define a fixed-point model is to start with a temporary, partial interpretation of the truth predicate, say the function f that assigns no truth value to any sentence.[13] One then constructs a transfinite sequence of interpretations, $f_0, f_1, \ldots, f_\omega, \ldots$ that monotonically increase in the sense that for $\alpha < \beta$, if $f_\alpha(A) = 1$, then $f_\beta(A) = 1$ and if $f_\alpha(A) = 0$, then $f_\beta(A) = 0$. There is bound to be a stage γ such that $f_\gamma = f_{\gamma+1}$. For the least such stage, γ, let $f = f_\gamma$. Then, f is a fixed-point interpretation of truth. There will, in many cases, be multiple fixed-points that can interpret the truth predicate.

[12] I will return to this example in §4, once the details of revision theory are on the table.

[13] A related formulation was originally presented by Kripke [1975]. An alternative method of obtaining fixed-points was demonstrated by Martin and Woodruff [1975]. Similar techniques were used in the context of set theory by Brady [1971] and Gilmore [1974].

In logics based on fixed-point models, only fixed-point interpretations of a particular predicate, such as a truth predicate, are considered.[14] In such models, there is no instability.[15] Once a sentence is in the extension or anti-extension of the truth predicate, it stays there. Some sentences, such as the liar sentence in Zardini's example, cannot be assigned a classical truth value by any fixed-points. As the example shows, doing so would violate monotonicity.

There is, then, some tension between the usual underlying model-theoretic, semantic view of naive truth and the appeal to instability in the metaphysical picture. Perhaps Zardini rejects the described view of naive truth, in which case the tension would be lessened.[16] Nonetheless, there would still be a question concerning the issue of instability. It is, then, worth delving further into the instability being invoked.

Note, also, that the structure of stages of truth evaluation that Zardini describes is highly suggestive of the structure of revision in revision theory. The T-Introduction rule advances one stage in the revision process while the T-Elimination rule goes back one stage. These transitions are, however, somewhat different than in naive or fixed-point approaches to truth, as will be made clearer below.

I will now turn to the details of revision theory, in particular presenting a sequent system for it to aid the comparison with Zardini's view.

3 Sequents for revision theory

I will begin by presenting some of the basics of the revision theory of truth.[17]

[14]See, for example, Kremer [1988], Gupta and Belnap [1993], or Kremer [2009].

[15]In this connection, it is worth briefly mentioning Field [2008]. Field uses a revision-theoretic construction to interpret a conditional, which does not obey the conditional form of contraction, and at each stage of revision, he constructs a fixed-point to interpret the truth predicate, holding fixed the interpretation of the conditional. There is instability in Field's construction, which leads to the failures of contraction for the conditional, but there is not instability in the construction of a new fixed-point interpretation for the truth predicate at each stage. Field's consequence relation, however, obeys the structural rule of contraction. The instability in Field's construction does not seem to fit the mould of Zardini's metaphysical picture.

[16]See Zardini [2015] for discussion.

[17]For details, see Gupta and Belnap [1993].

3.1 Semantics

In revision theory, one begins with a first-order language $\mathscr{L}$ interpreted via a ground model $M(=\langle D, I\rangle)$. For this paper, I will take M to be classical. The language is expanded to the language $\mathscr{L}^+$ by adding a new predicate, T, along with a canonical name, $\langle A\rangle$ for each sentence A of the expanded language. The sentences of the expanded language are also added to the domain of the model. Thus, the expanded language has an expanded model, $M^+(=\langle D^+, I^+\rangle)$. I^+ interprets names of sentences as follows: $I^+(\langle A\rangle) = A$. I will, henceforth, drop the superscripts on the model, domain, and interpretation.

The syntactic theory used here, namely quotation names, is relatively weak.[18] One can strengthen it by specifying additional sentences to add to the language and to the domain, e.g. a name l that denotes $\sim Tl$, in short, $l = \langle\sim Tl\rangle$. I will assume, for this paper, that there are liar sentences, such as $l = \langle\sim Tl\rangle$, and Curry sentences, such as $c = \langle Tc \to p\rangle$, in the object language.

The syntax of $\mathscr{L}$ is that of standard classical first-order logic. I will not need quantifiers for this paper, so I will be concerned only with sentences, rather than formulas. I will say that a formula that does not contain an occurrence of the truth predicate is T-free.

Atomic sentences that use the truth predicate are interpreted via *hypotheses*, h, which are functions from the domain to $\{1, 0\}$, the classical truth values. Evaluation of the truth predicate is determined by the following equation.

$$Val_{M+h}(Tt) = h(I(t)).$$

The hypotheses will obey the additional constraint that if $I(s)$ is a non-sentence, then $h(I(s)) = 0$.

The set of T-sentences for $\mathscr{L}^+$,

$$T\langle A\rangle =_{Df} A,$$

yields a *revision operator*, δ_M, for each model M. The revision operator is an operation from hypotheses to hypotheses that obeys the following identity.

$$\delta_M(h)(A) = Val_{M+h}(A)$$

[18]See Gupta [1982] Kremer [1988], or Ripley [2012] for more on quotation names, although the notation here differs from theirs.

For legibility, rather than writing Val_{M+h}, I will sometimes write $M + h$. The revision operator can be iterated to construct revision sequences, which are special sequences of hypotheses.[19]

We can define the validity of A in M, $\models_M A$, as:

$$\exists n \forall h \; M + \delta_M^n(h)(A) = 1.$$

A is valid, $\models A$, just in case A is valid in M, for all classical ground models M. I will, shortly, generalize this definition to be applicable to the sequent system.[20]

3.2 Sequents

Validity, as defined in §3.1, has a sound and complete proof system, C_0, which is a Fitch-style natural deduction system that uses indexed formulas, A^i, where i can be any integer.[21] Indices can be thought of as relative position in a revision sequence. The connective rules of C_0 are the usual classical connective rules for Fitch-style natural deduction, with the restriction that all premises of a rule must have the same index, e.g. from A^i and B^i, infer $A \,\&\, B^i$. In addition to those rules, there are three rules that change indices. For T-free formulas, B, there is an index shift rule: from B^i, infer B^j. There are also two truth rules.

- (TIn) from A^k, infer $T\langle A\rangle^{k+1}$.
- (TElim) from $T\langle A\rangle^{k+1}$, infer A^k.

If A^i can be derived with no open assumptions, then A^k can be derived, for any index k. Thus, for theorems one can ignore indices.

As is usual with Fitch-style natural deduction systems, the structural rule of contraction is somewhat hidden.[22] It will facilitate comparison with non-contractive logics to formulate a sequent system for revision theory. Sequents

[19]For ω-long sequences, revision sequences are just those obtained by iterating the revision operator on the initial hypothesis. For transfinite sequences, additional constraints are required at limit stages. For details, se Gupta and Belnap [1993, 167-168].

[20]It should be noted that this is a weaker sense of validity than the one in which Gupta and Belnap are primarily interested. It is, however, simpler and enough for the purposes of this paper.

[21]See Gupta and Belnap [1993, 156-166]. The completeness theorem is for general circular definitions.

[22]See Rogerson [2007] for discussion of contraction in Fitch-style natural deduction, as well as other forms of natural deduction. See Negri and von Plato [2001, 172-185] for

will be pairs of finite multisets of indexed formulas:[23]

$$A_1^{j_1}, \ldots, A_n^{j_n} \vdash B_1^{k_1}, \ldots, B_m^{k_m}.$$

I will, additionally, require that all indices be non-negative.[24]

The rules of the sequent system are divided into two groups. First the logical and structural rules.

$$A^i \vdash A^i$$

$$\frac{X, A^i \vdash Y}{X \vdash {\sim}A^i, Y}\ (\vdash {\sim})$$

$$\frac{X \vdash A^i, Y}{X, {\sim}A^i \vdash Y}\ ({\sim} \vdash)$$

$$\frac{X, A^i, B^i \vdash Y}{X, A \otimes B^i \vdash Y}\ (\otimes \vdash)$$

$$\frac{X \vdash A^i, Y \qquad U \vdash B^i, V}{X, U \vdash A \otimes B^i, Y, V}\ (\vdash \otimes)$$

$$\frac{X \vdash A^i, B^i, Y}{X \vdash A \oplus B^i, Y}\ (\vdash \oplus)$$

$$\frac{X, A^i \vdash Y \qquad U, B^i \vdash V}{X, U, A \oplus B^i \vdash Y, V}\ (\oplus \vdash)$$

$$\frac{X A^i \vdash B^i, Y}{X \vdash A \to B^i, Y}\ (\vdash \to)$$

$$\frac{X \vdash Y, A^i \qquad U, B^i \vdash V}{X, U, A \to B^i \vdash Y, V}\ (\to \vdash)$$

$$\vdash t = t^i$$

$$\frac{X, A(t)^i \vdash Y}{X, s = t^i, A(s)^i \vdash Y}\ (= \vdash)$$

$$\frac{X \vdash Y, A(t)^i}{X, s = t^i \vdash Y, A(s)^i}\ (\vdash =)$$

$$\frac{X, A^i, A^i \vdash Y}{X, A^i \vdash Y}\ (W \vdash)$$

$$\frac{X \vdash Y, A^i, A^i}{X \vdash Y, A^i}\ (\vdash W)$$

$$\frac{X \vdash Y}{X, A^i \vdash Y}\ (K \vdash)$$

$$\frac{X \vdash Y}{X \vdash A^i, Y}\ (\vdash K)$$

details on structural rules in tree-style natural deduction. See Restall [2014] for further comparison.

[23] The multisets are finite in the sense that only a finite number of formulas have a non-zero number of occurrences in the multiset. See Blizard [1988].

[24] This requirement is to simplify the presentation. It can be dropped, and doing so is advantageous for technical reasons, such as completeness, that will not concern me here.

Next, there are the rules that concern the truth predicate and base language sentences.[25]

$$\frac{X, A^i \vdash Y}{X, T\langle A\rangle^{i+1} \vdash Y}\ (T\vdash) \qquad \frac{X \vdash B^i, Y}{X \vdash B^k, Y}\ (\vdash IS)$$

$$\frac{X \vdash A^i, Y}{X \vdash T\langle A\rangle^{i+1}, Y}\ (\vdash T) \qquad \frac{X, B^i \vdash Y}{X, B^k \vdash Y}\ (IS\vdash)$$

In the rules $(IS \vdash)$ and $(\vdash IS)$, B must be T-free. Finally, the cut rule is included.

$$\frac{X \vdash Y, A^i \qquad A^i, U \vdash V}{X, U \vdash Y, V}\ (\text{cut})$$

I will call the system resulting from these rules LC_0T.[26]

Definition 1. *A sequent of indexed sentences $X \vdash Y$ is satisfied on $\mathscr{D}$ in $M + h$, $X \models^{\mathscr{D}}_{M+h} Y$, iff*

- *if for each $A_i^{j_i} \in X$, $M + \delta_M^{j_i}(h)(A_i) = 1$,*
- *then for some $B_m^{l_m} \in Y$, $M + \delta_M^{l_m}(h)(B_m) = 1$.*

A sequent of indexed sentences $X \vdash Y$ is valid in M, $X \models_M Y$, iff there is a natural number p such that for all hypotheses h, $X \vdash Y$ is satisfied in $M + \delta_M^p(h)$.

A sequent of indexed sentences $X \vdash Y$ is valid, $X \models Y$, iff it is valid in M, for all classical grounds models M.

The definition of validity is a bit cumbersome, so a bit of explanation is in order.[27] A sequent is satisfied in $M + h$ just in case revising h to reflect the index of each sentence results in either some premiss being false or some conclusion being true. The number of revisions may vary from sentence to

[25]The quantifier rules have been omitted, because they are not needed for the purposes of this paper. They can be added without difficulty.

[26]Bruni [2013] presents a cut-free sequent system for circular definitions, rather than truth. Bruni's system differs slightly from LC_0T, although the differences are not relevant here.

[27]It is less cumbersome than it would have been, had the indices been allowed to be negative.

sentence, as the indices vary. A sequent is valid in M just in case there is some number of revisions after which the sequent is bound to be satisfied, no matter the initial hypothesis. LC_0T is sound for this definition of validity.

Proposition 1. *For finite multisets of sentences X, Y, if the sequent $X \vdash Y$ is derivable in LC_0T, then $X \models Y$.*

Soundness is proved by the usual induction on the construction of proofs, so I omit the proof.

The derivations in LC_0T reflect the structure of revision in a straightforward way. All the instances of underivable sequents that will appear later have straightforward counterexamples.

Let us now turn to the comparison between revision theory and non-contractive systems.

4 Comparison

To start, I will first focus on one particular non-contractive system, Multiplicative Linear Logic with weakening, which is the propositional part of Zardini's favored logic. Towards the end of this section, I will briefly look at Additive Linear Logic with weakening. The logical and structural rules of the former are the following.

$$A \vdash A$$

$$\frac{X, A \vdash Y}{X \vdash {\sim}A, Y}\ (\vdash \sim)$$

$$\frac{X \vdash A, Y}{X, {\sim}A \vdash Y}\ (\sim \vdash)$$

$$\frac{X, A \vdash B, Y}{X \vdash A \rightarrow B, Y}\ (\vdash \rightarrow)$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{X, A \rightarrow B \vdash Y}\ (\rightarrow \vdash)$$

$$\frac{X, A, B \vdash Y}{X, A \otimes B \vdash Y}\ (\otimes \vdash)$$

$$\frac{X \vdash A, Y \qquad U \vdash B, V}{X, U \vdash A \otimes B, Y, V}\ (\vdash \otimes)$$

$$\frac{X \vdash A, B, Y}{X \vdash A \oplus B, Y}\ (\vdash \oplus)$$

$$\frac{X, A \vdash Y \qquad U, B \vdash V}{X, U, A \oplus B \vdash Y, V}\ (\oplus \vdash)$$

$$\vdash t = t$$

$$\frac{X, B(t) \vdash Y}{X, s = t, B(s) \vdash Y}\ (= \vdash)$$

$$\frac{X \vdash Y, B(t)}{X, s = t \vdash Y, B(s)}\ (\vdash =)$$

$$\frac{X \vdash Y}{X, A \vdash Y}\ (K\vdash) \qquad \frac{X \vdash Y, A \qquad A, U \vdash V}{X, U \vdash Y, V}\ (cut)$$

$$\frac{X \vdash Y}{X \vdash A, Y}\ (\vdash K)$$

The truth rules are the following.

$$\frac{X, A \vdash Y}{X, T\langle A\rangle \vdash Y}\ (T\vdash) \qquad \frac{X \vdash A, Y}{X \vdash T\langle A\rangle, Y}\ (\vdash T)$$

The system with the above rules will be called $MLLT$.

The appeal to revision theory is supposed to support the failure of contraction for Zardini. It does, as we can see with LC_0T. Contraction is lost when formulas are not T-free and have distinct indices.[28] For example, consider the start of a trivializing derivation for a liar sentence, $l = \langle{\sim}Tl\rangle$.[29]

$$\dfrac{\dfrac{\dfrac{Tl^i \vdash Tl^i}{Tl^i, {\sim}Tl^i \vdash}\ ({\sim}\vdash)}{Tl^i, T\langle{\sim}Tl\rangle^{i+1} \vdash}\ (T\vdash)}{l = \langle{\sim}Tl\rangle^{i+1}, Tl^i, Tl^{i+1} \vdash}\ (=\vdash)$$

The two Tls in the final sequent cannot be contracted, since their indices differ. That inference is invalid.

Let us look at a Curry derivation. Recall that $k = \langle Tk \rightarrow \bot\rangle$ and $c = \langle Tc \rightarrow p\rangle$, for some contingent p. The inconsistency Curry, Tk, is unstable in all revision sequences. Each revision will flip it back and forth between 1 and 0. The contingent Curry, Tc, will be stable in all revision sequences over ground models in which the value of p is 1, and unstable in all revision sequences over ground models in which the value of p is 0. This will be enough to ensure that the standard Curry-style derivations will be blocked for both.

[28] This way of rejecting contraction bears some similarity to the second proposal of Schroeder-Heister [2012]. That proposal says, roughly, that contraction is only permissible when both occurrences of the contracted formula come from the same sort of rule, i.e. both from an identity axiom or both from an introduction rule. The indices of LC_0T may provide a way to track Schroeder-Heister's distinction, but I will leave this for future work. I thank Rohan French for suggesting this connection.

[29] I will suppress the identity steps in later proofs.

$$\dfrac{\dfrac{Tc^i \vdash Tc^i \qquad p^i \vdash p^i}{Tc^i, Tc \rightarrow p^i \vdash p^i}\,(\rightarrow \vdash)}{Tc^i, Tc^{i+1} \vdash p^i}\,(T \vdash)$$

The usual derivation applies $(W \vdash)$ to the two copies of Tc at the final line, but, as they have different indices, that is unavailable. While Tc and Tk exhibit different *sorts* of instability, they each exhibit *enough* instability for the invalidity of the relevant contraction in revision theory.

The appeal to revision-theoretic ideas provides support for the rejection of contraction, as illustrated by the derivations above. It also points towards some special cases in which contraction could be be added, as the following variants of contraction are admissible. In these, B is T-free.

$$\dfrac{B^i, B^j, X \vdash Y}{B^i, X \vdash Y} \qquad\qquad \dfrac{X \vdash Y, B^i, B^j}{X \vdash Y, B^i}$$

One can, then, contract on T-free sentences, regardless of index.[30] Further, one can contract copies of a formula at a given index.

The revision-theoretic ideas, however, motivate rejection of other rules, namely the truth rules and the connective rules. For example, the sequent

$$T\langle A\rangle \vdash A$$

is not valid in revision theory.[31] One gets instead

$$T\langle A\rangle^{i+1} \vdash A^i,$$

but that is not enough to support the intersubstitutivity of $T\langle A\rangle$ and A, in general. The repercussions of this extend to the connective rules. The conclusion of the following $MLLT$ derivation cannot be obtained in LC_0T, for any indices.

$$\dfrac{\dfrac{Tl \vdash Tl \qquad \dfrac{Tl \vdash Tl}{Tl \vdash T\langle Tl\rangle}\,(\vdash T)}{Tl, Tl \vdash Tl \otimes T\langle Tl\rangle}\,(\vdash \otimes)}{Tl \otimes Tl \vdash Tl \otimes T\langle Tl\rangle}\,(\otimes \vdash)$$

[30]These can be extended to wider classes of sentences, such as the hereditarily T-free sentences, although I will not do so here.

[31]When talking about sequents without indices in revision theory, I will follow the convention that all the sentences have the same index.

The applications of the truth rules interfere with the later applications of the $\otimes$ rules. Consider the liar derivation in LC_0T again.

$$\dfrac{\dfrac{Tl^i \vdash Tl^i}{Tl^i, {\sim}Tl^i \vdash}\,(\sim\vdash)}{Tl^i, Tl^{i+1} \vdash}\,(T\vdash)$$

Erasing the indices, one obtains, in Multiplicative Linear Logic, a derivation of $Tl, Tl \vdash$, which yields

$$Tl \otimes Tl \vdash$$

by $(\otimes \vdash)$. A corresponding sequent is not derivable in LC_0T, since the differing indices of the two occurrences of Tl prevent application of $(\otimes \vdash)$. These issues extend readily to the other binary connectives. The failures of contraction that motivated looking at LC_0T will correspond to a barrier to the use of connective rules in LC_0T. There is, then, as much support for rejecting the connective rules as there is for rejecting contraction.

At this point, one might wonder if there is a way of adding a connective, $\bullet$, akin to multiplicative conjunction, obeying the following rules.

$$\dfrac{X, A^j, B^k \vdash Y}{X, A \bullet B^j \vdash Y}\,(\bullet\vdash) \qquad \dfrac{X \vdash A^j, Y \qquad U \vdash B^k, V}{X, U \vdash A \bullet B^j, Y, V}\,(\vdash\bullet)$$

$A \bullet B^j$ says, roughly, A holds here at index j and B holds at some point. It is an open question whether such a connective can be given a revision-theoretic semantics or otherwise added to LC_0T consistently with cut. If one could add $\bullet$ consistently, then it would introduce further divergences from Zardini's system: $\otimes$ is commutative, while, in general, $\bullet$ is not commutative, since it matters which "conjunct" contributes its index.[32] A natural response is to permit the index of the $\bullet$-formula to be that of either of its parts, but this can lead to triviality, given cut. To see this, consider a conjunctive Curry sentence, $o = \langle {\sim}(To \bullet {\sim}\bot)\rangle$.

[32] One could adopt an alternative version of $\bullet$ that takes its index from the right-hand "conjunct", but the issue concerning commutativity would remain.

$$
\begin{prooftree}
\AxiomC{$To^k \vdash To^k$}
\AxiomC{$\sim\perp^k \vdash \sim\perp^k$}
\RightLabel{$(\vdash \bullet)$}
\BinaryInfC{$To^k, \sim\perp^k \vdash To \bullet \sim\perp^k$}
\RightLabel{$(\sim \vdash)$}
\UnaryInfC{$\sim(To \bullet \sim\perp)^k, To^k, \sim\perp^k \vdash$}
\RightLabel{$(T \vdash)$}
\UnaryInfC{$To^{k+1}, To^k, \sim\perp^k \vdash$}
\RightLabel{$(\bullet \vdash)^*$}
\UnaryInfC{$To \bullet \sim\perp^k, To^k \vdash$}
\RightLabel{$(K \vdash)$}
\UnaryInfC{$To \bullet \sim\perp^k, To^k, \sim\perp^k \vdash$}
\RightLabel{$(\bullet \vdash)$}
\UnaryInfC{$To \bullet \sim\perp^k, To \bullet \sim\perp^k \vdash$}
\RightLabel{$(W \vdash)$}
\UnaryInfC{$To \bullet \sim\perp^k \vdash$}
\RightLabel{$(\vdash \sim)$}
\UnaryInfC{$\vdash \sim(To \bullet \sim\perp)^k$}
\RightLabel{$(\vdash T)$}
\UnaryInfC{$\vdash To^{k+1}$}
\AxiomC{$\vdash To^k$}
\AxiomC{$\vdash \sim\perp^k$}
\RightLabel{$(\vdash \bullet)$}
\BinaryInfC{$\vdash To \bullet \sim\perp^k$}
\RightLabel{$(\sim \vdash)$}
\UnaryInfC{$\sim(To \bullet \sim\perp)^k \vdash$}
\RightLabel{$(T \vdash)$}
\UnaryInfC{$To^{k+1} \vdash$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

This omits the derivation leading to the sequent $\vdash To^k$ in the right branch of the proof, but that simply mirrors the left branch with an adjustment of indices.[33] The step labelled $(\bullet \vdash)^*$ uses the relaxed $\bullet$-rule that permits the index of either part to be used.

So far, I have focused on the ways in which the revision-theoretic ideas, cashed out in LC_0T, do not deliver certain principles. I will provide two examples of ways in which they outstrip $MLLT$. First, one will always have distribution in LC_0T, which is not derivable in $MLLT$.

$$(A \oplus B) \otimes (A \oplus C)^i \vdash A \otimes (B \oplus C)^i$$

Second, one can derive

$$\vdash A \rightarrow (A \otimes A)^i,$$

which is not generally derivable in $MLLT$. As Zardini notes, this is the sequent to add to $MLLT$ to recover contraction for the relevant sentence A. The latter poses a problem for Zardini's view. The revision-theoretic ideas appear to support that sequent as much as they support rejecting contraction. The question, then, is what motivates the rejection of the sequent, since appeal to the reasons for failure of contraction are not available here, at least not directly.

The issues which I have been raising are not peculiar to $MLLT$: They arise in roughly the same form for Additive Linear Logic with weakening as

[33] This derivation uses $(W \vdash)$. Rejecting that rule, while blocking this derivation, would render circular the appeal to revision theory as motivating the failure of contraction.

well. For the remainder of this section, I will turn to Additive Linear Logic with weakening. The system $ALLT$ replaces the multiplicative rules for the binary connectives of $MLLT$ with additive rules. I will present only the & rules here.

$$\frac{X \vdash A, Y \qquad X \vdash B, Y}{X \vdash A \,\&\, B, Y}\,(\vdash \&) \qquad\qquad \frac{X, B \vdash Y}{X, A \,\&\, B \vdash Y}\,(\& \vdash)$$

$$\frac{X, A \vdash Y}{X, A \,\&\, B \vdash Y}\,(\& \vdash)$$

The corresponding rules for LC_0T require that indices on the displayed sentences match.[34]

As with $MLLT$, not all derivations of $ALLT$ can be reproduced in additive LC_0T. The problems arise for the rules with two displayed sentences in their premises, $(\vdash \&), (\vee \vdash)$, and $(\rightarrow \vdash)$. Here is an illustrative example, modelled on an earlier one.

$$\frac{Tl \vdash Tl \qquad \dfrac{{\sim}Tl \vdash {\sim}Tl}{Tl \vdash {\sim}Tl}\,(T \vdash)}{Tl \vdash Tl \,\&\, {\sim}Tl}\,(\vdash \&)$$

The endsequent is invalid in revision theory. The problem is that if one tries to put indices on the sentences, either the conjuncts have distinct indices, or the antecedents have different indices, in which case the contexts are distinct. Either way, the final rule cannot be applied. Examples along these lines extend readily to the other binary connectives and multiply. Nothing is gained on this front by adopting $ALLT$ over $MLLT$.

[34] Since LC_0T contains both contraction and weakening rules on the left and the right, the additive versions of its connective rules are derivable. See French and Ripley [2015] for the derivations of additive rules from multiplicative ones. For illustration, here are the additive versions of the rules for &.

$$\frac{X \vdash A^i, Y \qquad X \vdash B^i, Y}{X \vdash A \,\&\, B^i, Y}\,(\vdash \&) \qquad\qquad \frac{X, B^i \vdash Y}{X, A \,\&\, B^i \vdash Y}\,(\& \vdash)$$

$$\frac{X, A^i \vdash Y}{X, A \,\&\, B^i \vdash Y}\,(\& \vdash)$$

5 Conclusion

Zardini appeals to ideas from revision theory as the basis for a metaphysical picture of stable and unstable states of affairs to underwrite the rejection of contraction. I presented a sequent system, LC_0T, as a codification of the revision-theoretic ideas that enabled straightforward comparison with $MLLT$. Failures of contraction are witnessed in LC_0T by differing indices, vindicating Zardini's intuitive picture. The problem is that the revision-theoretic ideas equally support rejecting the naive truth rules and the connective rules.

One might think that the failure of the naive truth rules is due to some feature specific to *classical* revision theory, which was used for LC_0T, but this would be mistaken. As long as negation is interpreted so that

$$Val_{M+h}(\sim\sim A) = Val_{M+h}(A),$$

and there are some semantic values for which $Val_{M+h}(A) \neq Val_{M+h}(\sim A)$, then the naive truth rules will have counterexamples. A liar sentence works as an example. Suppose that

$$Val_{M+h}(Tl) \neq Val_{M+h}(\sim Tl).$$

It follows that

$$Val_{M+\delta_M(h)}(Tl) \neq Val_{M+\delta_M(h)}(\sim Tl),$$

and this will continue through a revision sequence, even into the transfinite as long as the inequality is maintained at limit stages, which it always can be.

There may be a different definition of consequence than the one studied here, perhaps defined in terms of preservation of eventually, stably having a designated value, that can preserve

$$T\langle A\rangle \leftrightarrow A \models C(T\langle A\rangle) \leftrightarrow C(A).$$

That sort of definition is likely to yield a complex logic with other failures of desired logical laws and rules.[35]

[35] For comparison, see the discussion of S^* by Gupta and Belnap [1993] and Kremer [1993], as well as the work on axiomatization by Horsten et al. [2012]. The definition of validity that is more important to Gupta and Belnap [1993], $S^\#$ will not help, as it maintains the features indicated above.

Invalidity in the revision-theoretic setting would tell equally against naive truth rules as against contraction. This presents a challenge for the proponent of a non-contractive approach to paradox who wants to appeal to the picture that Zardini presents, namely, to elaborate another framework for instability that motivates rejecting only contraction. Failing that, a different basis for rejecting contraction is needed.

Acknowledgments

I would like to thank Rohan French, Anil Gupta, Greg Restall, the audience at the Melbourne Logic Seminar, and anonymous referees for providing helpful feedback on this material.

This research was supported by the Australian Research Council, Discovery Grant DP150103801.

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