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The Notion of Infinity in Plotinus and Cantor

1. Introduction

The notion of infinity is one that has a popular appeal that transcends disciplinary borders. Infinity is also a topic that weaves its way through the theories and systems of many philosophers and mathematical schools. It is the cradle of paradoxes, the space where ‚things happen that don't', the *topos* where the logic of part and whole collapses. As A. W. Moore clarifies:

> Two clusters of concepts [of the infinite] dominate, and much of the dialectic in the history of the topic has taken the form of oscillation between them. Within the first cluster we find: boundlessness; endlessness; unlimitedness; immeasurability; eternity; that which is such that, given any determinate part of it, there is always more to come; that which is greater than any assignable quantity. Within the second cluster we find: completeness; wholeness; unity; universality; absoluteness; perfection; self-sufficiency; autonomy. The concepts of the first cluster are more negative and convey a sense of potentiality ... The concepts in the second cluster are more positive and convey a sense of actuality.[1]

Plotinus' notion of infinity in the *Enneads* echoes the above oscillation between actual and potential infinity as well as seems to be one of the earliest attempts at contemplating the actual or completed infinity lately observed in modern mathematics and particularly in Cantor's set-theory of infinity. For Plotinus, infinity is integral to the ontological structure of the Three Hypostases: it is related to the inexhaustible and endless productive power of the One, the internal partlessness of Intellect's intelligibles and the infiniteness of the forming principles in the Soul. It is also attributed to the indefiniteness of Matter and the unending expansion of the material universe from the simplicity and unity of the Soul to the plurality and complexity of the perceptible world.

On this basis, the aim of this paper is to juxtapose the notion of infinity in Cantor's mathematics to the concept of infinity in Plotinus' ontology and metaphysics. This comparison elicits considerable discussion and results in a better understanding of both

1 See Moore, (1991), *The Infinite*, Routledge, NY, pp. 1-2.

realms.[2] We are struck by Plotinus' concept of infinity appearing throughout the definition of the Three Hypostases and Cantor's set-theoretical definition of infinity. In particular, interesting similarities and conceptual isomorphisms could be observed by comparing Cantor's mathematical notions of Absolute Infinite, Actual Infinite and the Generative Principles to Plotinus' conceptions of infinity in the One, the Intellect and the Soul. The parallels are obvious, but also subtle, and lead us in questions like these: Can you get to infinity by counting or does it require something else? How can the finite generate that which is infinite? Can you understand infinity rationally or does it require something else? Is the absolute infinity conceivable? Are all infinities the same, or are some more infinite than others? Is the actual infinity logical prior to the potential infinity? How does the infinite generate that which is finite? Can we recast the one/many problem in set theoretical terms?

2. The Clouds of Infinity

Aristotle considered two types of infinity: the ‚actual' and the ‚potential' infinite. To put it simply, his idea was that we can never conceive of the natural numbers as a whole. However, they are potentially infinite in the sense that given any finite collection we can always find a larger finite collection. The infinite can only exist potentially and never actually.[3] Also, infinity can never be traversed, that is, gone right through:[4]

> So that if someone asks whether you can traverse an infinity either in time or in length, we must say that in a way you can, and in a way you cannot. For you cannot traverse an infinity of actually existing [divisions], but you can of potentially existing ones. (*Physics* 8.8, 263b3-6)

According to Aristotle the infinite exists in the same way that the Olympic Games exist. The whole Olympic Games (lasting a couple of weeks) cannot exist at one particular moment, although obviously the Games as a whole can and do take place. When the final event takes place that completes their actualization, the first events that happened two weeks earlier no longer exist. And yet the Olympic Games as whole do exist in some meaningful sense. As Aristotle puts it, while some of the parts may cease to exist, the process as a whole does not fail.

Aristotle claimed that denying that the actual infinite exists and allowing only the potential infinite would be no hardship to mathematicians:[5]

2 It has been acknowledged Prof. Byron Kaldis' contribution in this paper with invaluable comments, suggestions and philosophical insights.

3 *Physics* 3.6, 206a14-23.

4 *Physics* 3.5, 204b9, 6.2, 233a22.

5 *Physics*, 3.4-8.

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untransversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.

According to Aristotle, the physical world is not infinite (for that would be an actual infinity). There is nothing existing right now that is infinite in extension. However, we can conceive of infinite processes – such as processes of division or processes of addition. These will be infinite in the sense of being unlimited or unending rather than in the sense of being inconceivably large.

Aristotle's notion of a potential infinity might be better described not as a form of infinity at all, but rather as an *ever-extending finitude*. The restrictions that Aristotle put on the infinite (never actual and never traversed) created a problem for his theory that there could not be a beginning for time. If time has no beginning and extends back unendingly, then the past will be infinite, in the sense that you will never reach a beginning. However, as the past evidently ‚has existed', then this means that there *has been* an actual infinite. If an infinite time has already elapsed, is this not an example of an infinity that exists actually? And has not the infinite set of past years been traversed?

On this basis, in the sixth century A.D., John Philoponus, in his treatise *On the Eternity of the World against Aristotle*, argued that the conception of a temporally infinite universe, understood as a successive causal chain, is impossible. The celestial spheres of Aristotelian theory have different periods of revolution, and in any given number of years they undergo different numbers of revolutions, some larger than others. The assumption of their motion having gone on for all eternity leads to the conclusion that infinity can be increased, even multiplied, which, according to Philoponus, is absurd:[6]

> Suppose the spheres do not revolve at equal speeds, but one takes thirty years, another twelve, and another in turn less, so that the sphere of the moon takes a month, and that of the fixed stars a day and a night. Suppose too that the movement of the heavens has no beginning, then the sphere of Saturn must have revolved an infinity of circuits, that of Jupiter nearly three times as many, while the circuits of the sun will be thirty times those of Saturn, those of the moon three hundred and sixty times as many, and those of the fixed stars more than ten thousand times. Is this not beyond all absurdity, if the infinite cannot be traversed even once, to entertain ten thousand times infinity, or rather infinity times infinity.

6 6th of 6 books against Aristotle, as recorded by Simplicius in *Physics* 1179, 15-21. Quoted in Sorabji (1983), p. 215.

In view of this ‚absurdness', Philoponus concluded that time does in fact have a beginning, and the world is not eternal, it must be finitely old. Philoponus' discussion of infinity and the eternity of world trace us back to Plotinus' metaphysics.

3. Plotinus' Notion of Infinity

Infinity *(apeiron)* is a central concept in Plotinus' philosophy; it portrays different aspects of the Three Hypostases – the One, the Intellect and the Soul – and reveals their distinctive ontological properties. Infiniteness *(apeiria)* highlights fundamental attributes of each Hypostasis and describes their original structure.

Initially the concept of infinity is related to the inexhaustible and endless productive power of the One. The One, as the originative source of all succeeding realities, is the supreme unified and unifying principle of all beings that coheres the whole intelligible universe before any complexity or multiplicity. It is the fundamental metaphysical principle that encompasses perfection, superabundance and plenitude;[7] it is the ‚seminal power of all things'.[8] The power of the One is ‚infinite'[9] not in size, number or measure but in its never-ending productive force.[10] The One produces endlessly in plenitude and spreads out its infinite power, like an inexhaustible spring that gives life to all derivatives.[11] Life is the by-product of the infinite activity of the One and the first and most perfect form of this life is Intellect: the Second Hypostasis of Being.

Plotinus' Intellect is a philosophical synthesis of Plato's theory of the Forms,[12] Aristotle's theory of the divine self-thinking Mind,[13] the Middle-Platonic idea of the Forms as thoughts of God,[14] as well as, influenced by Parmenides' theory of the Eleatic Being (frs 3 and 8). It is a unified multiple organism of intelligibles; a revealing unity-in-plurality of active intelligibles self-produced through the eternal contemplation of the One. Intellect exercises contemplation in order to conceive the infinite perfection of the

7 *Enneads* V.1.6.38 ff.; V.2.1.

8 *Enneads* III.8.10.1; V.1.7.9-10; V.3.15.33; V.4.1.36.

9 *Enneads* VI.9.6.10-12; V.5.6.15 ff.

10 Cf. Wallis (1997), p. 60, n. 2.

11 *Enneads* V.5.5.1-2; V.2.1.7-10; III 8.10.5-14.

12 Plotinus accepts the Platonic Forms as true and eternal living intelligences with a substantial content of their own. He identifies Intellect with the ‚all-complete ever-living being' of the *Timaeus* (29d7-47e2), with the ‚true being endowed with life and intelligence' of the *Sophist* (248e-249a), and with the second hypothesis in the second part of the *Parmenides* (137c-142a, 144e5; 155e5).

13 Plotinus accepts Aristotle's God as a divine living being, eternal and most good. More significantly, Plotinus inherits the Aristotelian and the Middle-Platonic position that at the divine level the object of intelligence is identified with the intellectualizing subject and this self-intellectualization is the essence of God. This divine identity is the purest form of intelligence, appearing only at the level of the immaterial intelligibles that constitute the divine Mind; cf. Aristotle's' *Metaphysics* 1074b 26-34 and *De Anima* 430-431.

14 This idea is mainly observed in the philosophy of Numenius and Albinus, cf. Wallis (1995), p. 54.

One.[15] But since Intellect could not apprehend a full contemplation of the One and it was unable to hold the One's received infinite and ultimate power, Intellect fragmented this power and transformed the infinite power of the One to the active self-thinking intelligence of the intelligibles.[16] Intellect's self-thinking activity is the reason for its intelligible perfection; the perfect identification between intelligible subject (Intellect) and intelligible object (Being). Intellect is united with its Being in the intelligibles and the intelligibles with individual intellects in Intellect.[17]

At the level of Intellect, infiniteness manifests the internal partlessness of the intelligible realm.[18] Intellect is ‚infinite' due to the internal infiniteness and the universal plurality of the intelligibles. Intellect is not just a synthesis of parts, but an organism of intelligibles; and an organic unity of intelligibles that mirror the Intellect's totality and universality. Intellect is infinite and it is exactly that partless infiniteness of the intelligibles that explains the Intellect's internal indestructibility and perfection (III.8.8.46-50). However, as Plotinus himself asks in *Ennead* VI.2.21.1, ‚how does the Intellect itself, while it remains unified in its essential structure, produce partial beings?' Plotinus answers this important problem by justifying the partless infiniteness of Intellect (lines 1-11). Intellect is complete intelligence without intervals or divisions;[19] it is an intelligible whole without parts where all things are lively present in perfect actuality and pure power. Intellect's greatness manifests the infiniteness of the intelligibles.

Since the intelligibles are the by-products of Intellect's self-thinking activity (VI.2.22.26-7), Intellect is not absolute unity, but simultaneous unity in plurality (IV.8.3.10). It is at the same time ‚one-many': ‚many' due to the plurality of the intelligibles, ‚one' because of the identity between Intellect and the intelligibles in a unified and undivided intelligible whole (V.1.8.26; V.3.15.22; VI.2.9.11-12, 15.14-16, 17.25; VI.6.13.52-3; VI.7.14.11-12).[20]

> For the intelligibles are many and they are one, and, being one, they are many by their infinite nature, and many in one and one over many and all together, and they are active towards the whole with the whole, and active towards the part again with the whole. [VI.5.6.1-5; trans. Armstrong modified]

Each intelligible is perfect and contemplates all intelligibles in every other intelligible:

> Intellect's intelligibles contemplate all things, not those that participate to coming to be, but those which belong to the essence of Being, and contemplate themselves in the others; for all the intelligibles there are transparent, and there is nothing dark or opaque; everything and all things are clear to the inmost part to

15 Cf. *Ennead* V.4.2.4-10.
16 Cf. *Enneads* VI.7.15.10-24; V.3.11; V.7.15;VI.2.22.26-7.
17 See *Enneads* V.1.4.26-9; V.9.8.3-7.
18 Cf. *Enneads* III.8.8.46, V.7.3.22, VI.2.21.10, VI.5.6.2, VI.7.13.7
19 Cf. also VI.7.14.18: ‚the division which is in Intellect is not of things confused'.
20 Cf. Gerson (1994), p. 44-45.

> everything; for light is transparent to light. Each intelligible in the intelligible world has everything in itself and contemplates all things in every other, so that all are everywhere and each and every one is all and the glory is infinite; for each of them is great, because even the small is great; the sun there is all the stars, and each star is the sun and all the others. [V.8.4.1-11; trans. Armstrong modified]

The above passages describe vividly the active infiniteness of Intellect. Intellect's infinite intellection is analogous to the transparency of light: as the light is transparent to light, each intelligible is transparent to every other intelligible within Intellect. The infinite transparency of Intellect's intelligible light expresses the ‚infinite glory' of Nous;[21] every intelligible is both actually itself and potentially all the others like the Euclidean theorems which each contains not only its own mathematical truth but also by implication the truth of geometry.[22] All the intelligibles are everywhere in the intelligible realm and thus each intelligible embraces all in infinite organic relation.

As Plotinus puts it in *Ennead* V.7.3.20-23, we should not hesitate to apply ‚infiniteness' at the level of Intellect. Intellect's infiniteness, as the infiniteness of the forming principles *(logoi)* in the Soul,[23] derives from the relationship between Intellect and the intelligibles.

> Now we have no need to be afraid of the infinity in seeds and forming principles, since Soul contains all the forming principles. Yes, in Intellect, as in Soul, there is again the infinity of these principles which are come out ready to be activated in the Soul. [V.7.3.20-23; trans. Armstrong modified]

In this passage Plotinus stresses two important things: firstly, the analogy between the infiniteness of Intellect's intelligibles and the infiniteness of the Soul's forming principles (*logoi*)[24] compared to the infiniteness of the seeds; secondly, that such an infiniteness is not an absurd consideration.

On the other hand, it is noteworthy that Plotinus uses, in some cases, the term ‚infiniteness' in a depreciatory sense. The most striking incidents are those where the term is attributed to the indefiniteness of Matter and the unending expansion of the material universe from the simplicity and unity of the Soul to the plurality and complexity of sensible world.[25] The clearest example of this discussion appears in *Ennead*

21 The ‚infinite glory' of Intellect in V.8.4.1-11 could be clearly compared to the ‚infinite greatness' of the intelligible world in VI.2.21.1-11. In the latter passage Plotinus again uses the metaphor of light compared to the glory of the intelligible world.

22 Cf. *Enneads* III.9.2; cf. V.9.8.4 ff.; cf. also Wallis (1995), p. 55.

23 For *logoi* as principles of meditation cf. Rangos (1999) *passim*.

24 Soul is an intelligible entity that includes within itself the intelligible seed of the formative principle. This formative principle is the expression and the activity of the Intellect in the Soul; the spermatic generator of Nature; cf. *Enneads* V.1.6.45; II.4.3.5-11; III.6.19.26-29.

25 For instance cf. *Enneads* I.6.9.20; I.8.6.42; II.4.15.4

II.4.15. For Plotinus, the indefiniteness of material world is contrasted with the infiniteness of the intelligible realm:

> For indefiniteness is present in a higher degree in that which is less defined ... That which exists in the intelligible world, which has a greater degree of existence, is indefinite only as an image. In contrast, that which is in the perceptible world has a less degree of existence ... and sunk down into the nature of the image, it is more truly indefinite. [II.4.15.24-28; trans. Armstrong modified]

The indefiniteness of Matter is the original version of indefiniteness and not *vice versa*. As such Matter can be identified only with evil itself: a kind of ‚unmeasuredness contrasted with measure', ‚formlessness contrasted with form', ‚insufficiency contrasted with self-sufficiency', indefiniteness contrasted with limit (I.8.3.12-15). Matter is ‚true indefiniteness' that *subsists originally* at the level of Matter and not at the level of Intellect (II.4.5.28). Therefore, the infiniteness of Matter is different from that of Intellect and so it has to explained and translated in different ways: ‚indefiniteness' in the case of Matter, ‚infiniteness' in the case of Intellect. Whereas the infiniteness of Matter is the reason of its sterility and poverty, the infiniteness of Intellect is the reason of its active partlessness, unity, perfection, all-togetherness and indestructibility. It is exactly these facets on the Plotinian theory of infinity that leads us to enquiry of infinity in modern mathematics from Galileo to Cantor.

4. The Mathematical Infinity: From Galileo to Cantor

Galileo, in his *Discorsi e Dimostrationi Matematiche, Intorno a Due Nuove Scienze* (1638), discussed several paradoxes of the infinite (e.g.: the circle paradox, the number of points in two unequal line segments, and the number of the perfect square-numbers in comparison to the number of natural numbers). He noticed that, on the one hand, ‚there are as many squares as there are numbers', and on the other hand ‚there are many more numbers than squares, since the larger portion of them are not squares'. It seems that the squares of natural numbers are as numerous as the natural numbers. Why? Because they can be put into one-to-one correspondence. Every distinct natural number has a distinct perfect square; and every distinct square number has a distinct natural number as its square root. Hence every member of one sequence has a unique counterpart on the other, and vice versa:[26]

26 Galileo Galilei, *The Discourses and Mathematical Demonstrations Relating to Two New Sciences*, translated by Henry Crew and Alfonso de Salvio; originally published in 1914 by The Macmillan Company; reprinted by Dover Publications Inc., New York, pp. 31–33.

> *Sagredo*: What then must one conclude under these circumstances?
> *Salviati*: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes ‚equal,' ‚greater,' and ‚less,' are not applicable to infinite, but only to finite, quantities.

Galileo's paradox violates our intuition that the whole is greater than the part (codified in Euclid's common notion 5). On the other hand, it is supported by another intuition that infinity is one, its size is undifferentiated: all infinite sets have the same number of elements. For Galileo it does not make sense to compare infinite quantities.

At this point we can introduce some mathematical terminology. A *set* is a collection of elements.[27] The term *cardinality* refers to the number of *members* in a set. Two sets which can be put into *one-to-one correspondence* are equal in cardinality (equivalent, equinumerous). We can regard it as the definition of equal magnitude for finite and infinite sets. One set is a *subset* of a second set if all its members belong to the second set. It is a *proper subset* if all its members belong to that second set and there are members of that second set that are not members of the first set. The *power set* of a given set is the set of all its subsets. The power set of any set, finite or infinite, possesses a greater cardinality than the original set.

Peter Suber[28] reserved the name of *self-nesting* for the property that a set can be put into one-to-one correspondence with at least one of its proper subsets. Infinite sets have the property of self-nesting; finite sets do not.

Galileo's paradox is paradoxical in the sense of its being counter-intuitive, but it is not contradictory. In fact, the collapse of the familiar logic of part and whole yields a new definition of infinity. Paradoxes of infinity become definitions of infinity: *mathematics defines the infinite set-theoretically as a set which is equinumerous to (equivalent: it can be put into a one-to-one correspondence with) a proper subset of itself (the part is ‚equal' to the whole, the same size as the whole).*[29]

The set-theoretic solution to Galileo's paradox is that the set of perfect squares and the set of natural numbers have the same cardinality even though one of these sets is a

27 We note that there is considerable mathematical and philosophical disagreement about how best to define the intuitive notion of a set since it has led to various paradoxes. For an overview of these discussions see A. A. Fraenkel et al., (1984), *Foundations of Set Theory*, 2nd revised edition, North Holland, Elsevier Science Publishers.

28 Peter Suber (1998), *Infinite Reflections*, St. John's Review, XLIV, 2, pp. 1-59.

29 Bernard Bolzano may have been the first to suggest this idea in his *Paradoxien des Unendlichen*, Section 20, published posthumously in 1851, where he stated: „... two unequal lengths [may be said to] contain the same number of points". It was used as a definition of ‚infinite' by Richard Dedekind in his *Essays on the Theory of Numbers* (1887): „A system *S* is said to be *infinite* when it is similar to a proper part of itself; in the contrary case *S* is said to be a *finite* system". Charles S. Peirce in 1881, in his *On the Logic of Number* (AJM, 1881, vol. 4, n.1/4, pp. 85-95), proposed (independently of Dedekind) an equivalent definition of the infinite.

proper subset of the other; while the one set (perfect squares) is ‚smaller' than the other set (natural numbers) since it is a proper subset of this set, on the other hand it is not ‚smaller' at all but the same (relative) size, since both sets have the same cardinality.

In view of these considerations we can interpret Philoponus' objections to the possibility of an infinite past as a question pertaining to the possibility of self-nesting as well as to the possibility of different ‚sizes' of the infinite. He discussed actual infinities almost as if they possessed cardinal numbers in the set-theoretical sense and was wondering about their addition and multiplication. Philoponus believed that the existence of (and the operations on) self-nesting sets is impossible and contradictory. Galileo believed that they surpass human understanding. Cantor used them as a basis for his set theory, he developed an ‚arithmetic' of infinite numbers, and, by considering their power sets, he built a never-ending tower of infinities.

In 1883 Cantor published his *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* where he explained that the role of transfinite numbers was to provide a description not of potential infinities, but of actual infinities. He wanted to provide a method of defining higher powers, i.e., he wanted to describe sets of higher power than the natural numbers.

Cantor made a distinction between *ordinal numbers* or *ordinals* and *cardinal numbers* or *cardinals*. A natural number can be used either to describe the size of a set, or to describe the position of an element in an ordered sequence. For finite sets and sequences these two notions coincide: 3, for instance, describes the position (third) of the element ‚c' in the set {a,b,c} and, also, gives the number of elements in the set. However, when dealing with infinite sets the two notions are different: if we write the set of natural numbers in ascending order {1,2,3,4...} and in descending order {...4,3,2,1} we have two sets which are of the same size ($\aleph_0$) but their elements are ordered in a different way since in the first case we have a set with no last element while in the second case we have a set with no first element. The ordinal number of the first set is ω and that of the second is ω^*. It is clear that $\omega \neq \omega^*$. The ordinal ω is the first number following the sequence of natural numbers. In a sense, this number represents a limit that the natural numbers approach but never quite reach.

In his *Grundlagen*, Cantor introduced two principles for generating ordinal numbers. The *first principle of generation* may be simply defined as a successive addition of units:

> (1). If α is an ordinal number (whether finite or transfinite) then there is a new ordinal number $\alpha + 1$ which is the immediate successor of α.

We use this principle to define the natural numbers when we say that for any arbitrary large finite number n there exists another finite natural number $(n+1)$. This principle alone allows us only to define finite numbers, and does not offer any method of transcending the finite to reach the infinite. Thus, Cantor introduced the *second principle of generation*:

> (2). Given any unending sequence of increasing ordinal numbers α there is a new ordinal number greater than all the α's, called their ‚limit'.

Thus, according to the second principle, given a limitless set of numbers, another number may be introduced as the first number larger than the entire set. Hence, given the infinite set of natural numbers (produced by the first principle), the second principle legitimates the existence of ω as the first number larger than the set of natural numbers. Using the two principles of generation, Cantor provided a distinction between number classes, naming the *first (I) number class* as the natural numbers, and the *second (II) number class* as the collection of all numbers which can be formed by means of the two principles of generation.

The second principle of generation is a *principle of transcendence*. While it is not possible to exhaust an infinite series by successively naming terms until the last one is reached, it may be exhausted through the application of the second principle of generation. By regarding infinite sets as completed unities, Cantor used his second principle to name the first number that is greater than that of the infinite set. These two principles guarantee the existence of a never-ending hierarchy of infinities.

The two principles of generation lead to a sequence of ordinals that is absolutely infinite, in the sense that the ordinals cannot be collected into a set. The generation of ordinals (or the notion of Absolute infinity) is not described in proper mathematical terms: the second principle is obscure and depends on an account of when a multiplicity is a set. If every multiplicity were taken to be a set, then applying the second principle to the set of all ordinals would lead to a contradiction. Cantor proved that Ω, the ‚system' of all ordinals is an inconsistent, absolutely infinite multiplicity: If Ω were consistent, that is, if it were a set, then „there would correspond to it a number δ greater than all numbers of the system Ω; but the number δ also occurs in the system Ω, because this system contains all numbers; δ would thus be greater than δ, which is a contradiction".[30] According to Cantor:

> There are, then, definite multiplicities which are not at the same time unities, that is, those multiplicities for which it is impossible that all their elements be really together. These I call ‚inconsistent systems'; the rest I call sets.[31]

Cantor refers to inconsistent multiplicities as *absolutely infinite*:

> Those totalities that cannot be conceived by us as ‚sets' [...] I have called for many years ‚absolutely infinite' totalities and I have sharply distinguished them from the transfinite sets.[32]

Cantor was forced to deny the actuality of the absolute. Inconsistent multiplicities exist only potentially. No possible collection can encompass all sets. This can only be because

30 Cantor's letter to Dedekind of 3 August 1899, in J. van Heijenoort (1967), From Frege to Godel. Asource book in Mathematical Logic, 1879–1931, Harvard University Press, Cambridge, Massachusetts, pp. 114-115.

31 Letter to Dedekind of 3 August 1899.

32 Letter to Hilbert of 15 November 1899, in Georg Cantor (1991), p. 389.

the set-theoretical universe is a ‚temporary' universe, it can never be completed, i. e., because it is irreducibly potential.[33]

Cantor's Absolute cannot be known by rational means. Set theory's formulation of the Reflection Principle specifies that the Absolute is inconceivable, unattainable, inexpressible and unthinkable, hence confirming a link between the infinite and the ineffable. According to Rucker, the Reflection Principle specifies that given any description D of something V, there will always be a partial something Vμ that satisfies D as well. Any described universe of set theory turns out to be only one of the Vμ sets, and not the whole universe. Every conceivable property of the Absolute is shared by some lesser entity[34] or, as Rucker puts it: ‚the mind does not attain to God, but to what is beneath Him'.[35]

On this rationale, Cantor's theory of Absolute Infinity is influenced by Plato and Platonism. It is on this influence that we are going to turn our attention to.

6. Cantor's Platonism

Georg Cantor, while developing his theory of sets, also concerned himself with questions pertaining to the meaning of unity, multiplicity, potentiality, actuality, transcendence, and the Absolute.

Cantor criticized Aristotle's distinction between the *potential* (indeterminate and increasable variable *finite* quantity) and the *actual* infinite and he purported the view that the latter should be broken down further into the *transfinite* (the relative and increasable actual infinite; a mathematical notion) and the *absolute* actual infinite (the unincreasable; a metaphysical and theological notion, absolutely unreachable by pure mathematical knowledge). Cantor believed that the mathematically determinable infinite exists between the finite and the mathematically indeterminable, ‚absolutely' infinite. He also believed that the use of the potential infinity in mathematics means variability over completed infinite domains which cannot themselves be variable. The actual infinity holds primacy over every potential infinity. In Cantor's words:

33 Ignacio Jane (1995), *The Role of the Absolute Infinite in Cantor's Conception of Set*, Erkenntnis 42, pp. 375-402.

34 Rucker (1982) writes: „The motivation behind the Reflection Principle is that the Absolute should be totally inconceivable. Now, if there is some conceivable property *P* such that the Absolute is the only thing having property *P*, then I can conceive of the Absolute as ‚the only thing with property *P*'. The Reflection Principle prevents this from happening by asserting that whenever I conceive of some very powerful property *P*, then the first thing I come up with that satisfies *P* will *not* be the Absolute, but will instead be some smallish rational thought that just happens to reflect the facet of the Absolute that is expressed by saying it has property *P*", p. 50.

35 Rucker (1982), p. 203.

> ... each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite.[36]

and:

> ... the potential infinite is only an auxiliary or relative (or relational) concept, and always indicates an underlying transfinite without which it can neither be nor be thought.[37]

The actual infinite is something completed:

> By an Actual-Infinite is to be understood a quantum which on the one hand is not variable, but rather is fixed and determined in all its parts – a genuine constant – but which at the same time surpasses in magnitude every finite quantity of the same kind.[38]

The Absolute Infinite, on the other hand, is the ‚true', ‚veritable infinity', which is never known and whose magnitude:

> ... cannot in any way be added to or diminished, and it is therefore to be looked upon quantitatively as an absolute maximum. In a certain sense it transcends the human power of comprehension, and in particular is beyond mathematical determination.[39]

Cantor's attitude towards the actual infinite shaped his definition of the concept of set. For him „A set is a Many that allows itself to be thought of as One".[40] If he did not consider sets like the set of integers N: 1,2,3... as a whole (One), i. e., as a completed set, he couldn't define his transfinite numbers. The unitary, completed, holistic character of an infinite set as a *Ding an sich* (thing in itself) served as a sound basis for the introduction of transfinite numbers. An infinite set is actually, and not potentially,

36 Georg Cantor (1886), *Über die verschiedenen Standpunkte in bezug auf das aktuelle Unendliche*, Zeitschrift für Philosophie und philosophische Kritik 88, pp 224-33. Quoted in Michael Hallett (1984), *Cantorian set theory and limitation of size*, Clarendon Press, Oxford, p. 25.

37 Georg Cantor (1887-8), *Mitteilungen zur Lehre vom Transfiniten I, II, Zeitschrift für Philosophie und philosophische* Kritik 91, 81-125, 252-70; 92, 250-265. Quoted in Hallett (1984), p. 25.

38 Georg Cantor (1887-8), p. 40. Quoted in Hallett (1984), p. 12.

39 Georg Cantor (1887-8), p. 405. Quoted in Hallett (1984), p. 13.

40 Georg Cantor, *Gesammelte Abhandlungen*, eds. A. Fraenkel and E. Zermelo, Berlin: Springer-Verlag 1932, p. 204. Quoted in Rudy Rucker (1982), *Infinity and the Mind*, The Harvester Press, p. 40. In the English translation of Cantor's *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (1883) we read: „By an ‚aggregate' or ‚set' I mean generally any multitude which can be thought of as a whole, i. e., any collection of definite elements which can be united by a law into a whole" (p. 204).

infinite. Cantor thus believed he was defining something related to the Platonic ‚*eidos*' or ‚*idea*' and to what Plato called a ‚*mikton*' (mixture) in the *Philebus*,[41] an ordered mixture of the ‚*peras*' (limit) and the ‚*apeiron*' (unlimited), what Cantor himself called the improper or potential infinite:

> [Plato] opposes this to the *apeiron*, i.e., to the unlimited, to the indeterminate, which I call the improper infinite, as well as to the *peras*, i. e., to the limit, and explains it as an ordered mixture of both.[42]

The set arises by bounding, by limiting, by closing the potential infinite, by making a whole of it. The limit is imposed to the unlimited by means of a law.[43] What does it mean for a mathematical object to exist? Cantor was an idealist and a realist. As he wrote to Paul Tannery:

> My idealism is related to the Aristotelian-Platonic kind, which as you know is at the same time a form of realism. I am just as much a realist as an idealist.[44]

In Cantor's view, mathematical objects exist in a Platonic realm as permanently and independently existing real abstract ideas. He distinguished three levels of existence:

a. *in Intellectu Divino* (in the mind of God)
b. *in abstracto* (in the mind of man)
c. *in concreto* (in the physical universe)

According to Cantor the Absolute Infinite exists only in the mind of God:

> The actual infinite arises in three contexts: *first* when it is realized in the most complete form, in a fully independent other-worldly being, in *Deo*, where I call it the Absolute Infinite or simply Absolute; *second* when it occurs in the contingent, created world; *third* when the mind grasps it in *abstracto* as a mathematical

41 *Philebus* 23c-31a deals with four kinds of constituents making up „all that exists in the present universe" (23c4). In order of discussion, there are the Unlimited, Limit, Mixture of the first two, and Cause of mixture: „[Soc.] Then the first I will call the infinite or unlimited, and the second the finite or limited; then follows the third, an essence compound and generated; and I do not think that I shall be far wrong in speaking of the cause of mixture and generation as the fourth" (27b7-27c1). Earlier, Socrates (16c10) sited the Pythagorean dictum that things always said to exist are composed of one and many, having „Limit and Unlimited within themselves connaturally".

42 Georg Cantor (1883), p. 204, n. 1.

43 It is interesting to juxtapose Cantor's definition of a set to Proclus' ideas (in *Theol. Plat.* III) about multiplicity as being itself unity: „and being is nothing else than a monad of many powers, a completed reality (hyparxis), and on this account being is a many in one (en polla)".

44 Georg Cantor (1991), *Georg Cantor Briefe*, H. Meschkowski and W. Nilson (eds.), Springer, New York, p. 323.

> magnitude, number, or order type. I wish to make a sharp contrast between the Absolute and what I call the Transfinite, that is, the actual infinities of the last two sorts, which are clearly limited, subject to further increase, and thus related to the finite.[45]

The existence of numbers *in abstracto* he called their *intra-subjective* or *immanent* reality:

> ... ‚intersubjective' or ‚immanent' reality of concepts or ideas could be said to correspond with the designation ‚adequate' in the sense that this word is used by Spinoza when he says, Ethics, part II, def. IV: ‚By an adequate idea, I mean an idea which, in so far as it is considered in itself, without relation to the object, has all the properties or intrinsic marks of a true idea'.[46]

and their existence *in concreto* their *trans-subjective* or *transient* reality.

Immanent and transient realities are intimately connected. The actual basis of the connection between these two kinds of reality, Cantor tells us, lies „in the unity of the universe, to which we ourselves belong".[47] Unity was a major theme of Cantor's philosophy. Though he maintained that to be considered existent and real in mathematics numbers must be distinctly differentiated from one another so as to correspond to *adequate ideas*, he stressed, over and over again, that these independent numbers in and for themselves organically coalesce into a unified whole in special ways.

On the basis of these considerations, Cantor maintained that knowledge could:

> only be obtained through concepts and ideas which, at most stimulated by external experience, are on the whole formed through inner induction and deduction as something which in a way already lay within us and was only awakened and brought to consciousness ...

Only conceptual knowledge is said (according to Plato) to afford true knowledge. The nearer, however, our presentations come to the truth... the nearer their objects must come to being real and vice versa. What is knowable is; what is not knowable is not, and to the same extent something is, it is also knowable.[48]

45 Georg Cantor, *Gesammelte Abhandlungen*, p. 378. Quoted in Rudy Rucker (1982), p. 9.

46 Georg Cantor (1883), p. 206, n. 5. Quoted in Hallett (1984), p. 17-8.

47 Georg Cantor (1883), p. 182. Quoted in Claire Ortiz Hill, *Abstraction and Idealization in Edmund Husserl and Georg Cantor prior to 1895*, in *Husserl or Frege, Meaning, Objectivity and Mathematics*, by Claire Ortiz Hill and Guillermo Rosado Haddock, Chicago: Open Court, 2000 and *Idealization IV. Historical Studies on Abstraction and Idealization Poznan studies in the philosophy of the sciences and the humanities* vol. 82, F. Coniglione, R. Poli, R. Rollinger (eds.), Rodopi, Amsterdam, 2004, pp. 217-43.

48 Georg Cantor (1883), p. 206-07 note 6. Quoted in Claire Ortiz Hill, *op.cit.*

Hence, we can ‚discover' the transfinite numbers and study them. Mathematics is ‚free in its development' and the only limitations are the consistency of its concepts in themselves and in relation to the already accepted concepts.

7. Infinity in Plotinus and Cantor

Bearing in mind the aforementioned considerations we can observe the parallels between Cantor's mathematics and Plotinus' metaphysics on the notion of infinity. The comparison is particularly apparent in juxtaposing Cantor's Absolute Infinite with Plotinus' infinity of the One; Cantor's Actual Infinite with Plotinus' infinity of Intellect and Cantor's Generative Principles with Plotinus' *logoi* – Soul's forming principles. More precisely, Cantor's Absolute Infinite and Plotinus' One are ‚inconceivable', ‚unattainable', ‚inexpressive', ‚unthinkable' and ‚ineffable'. Both constitute transcendent natures beyond measure, numbering and comprehension. Moreover, Cantor's Actual Infinite echoes the completed and perfect structure of Plotinus' intelligible world. As Actual Infinite is a whole determined in all its parts, a set of plurality in unity, Plotinus' Intellect is a partless unity-in-multiplicity consisted of a plurality of intelligibles in one unified organism. Each intelligible embraces the whole Intellect in an infinite organic relation where Intellect reflects the intelligibles and every intelligible reflect the Intellect. The intelligibles are not parts of the whole but mirror the totality and completeness of Nous in infinite perfection. Finally, Cantor's Generative Principles could be associated to Plotinus' Forming Principles in the Soul. As at the level of Intellect, Soul's *logoi* manifest the infiniteness of the intelligible world. Every soul self-includes the whole intelligible world within the forming principles. The *logoi*, as in the case of Cantor's Generative Principles, are actual and immanent infinites which include the transient reality of the intelligible world as adequate ideas within the Soul. On this rationale, Cantor's existence *in abstracto* could be contrasted with Plotinus' higher part of the soul (the intelligible part existing in human mind) and the existence *in concreto* (the perceptible part existing in the physical universe). Whereas the former is directed to the higher intelligible realm, the latter is related to the formulation and direction of the perceptible world. The Soul reflects the perfection and completeness of the intelligible realm upon the mirror of Matter; the original indefiniteness and sterility of the universe.

8. Conclusion

In conclusion, reading Plotinus from a mathematical perspective it is inescapable to observe one of the earliest attempts at contemplating the notions of absolute and actual infinity later found in Cantor's mathematics. First, Cantor's Absolute Infinite echoes Plotinus' infinite power of the One; both notions describe a transcendent nature which is inconceivable, ineffable and unthinkable. Second, Cantor's Actual Infinite traces

back to Plotinus' partless infinity of the intelligible realm; Cantor and Plotinus speak[s] of an actual infinite nature which is completed and perfect, an absolute unity-in-multiplicity determined in all its parts. The latter conception actually leads Cantor to the development of set theory. Finally, Cantor's definition of the Generative Principles as actual infinities, immanent and transient realities encapsulated as adequate ideas present some similarities with Plotinus' forming principles in the Soul. Particularly, Soul's higher intelligible part in Plotinus corresponds to Cantor's intelligible part existing in human mind as *in abstracto* and Soul's lower perceptible part corresponds to Cantor's perceptible part existing in the physical universe as *in concreto*. Nevertheless, the difference between Plotinus and Cantor is their theoretical context; whereas Plotinus' starting-point is completely philosophical contemplating different forms of the *apeiron* within a metaphysical context and dialectic framework, Cantor's thought is mathematical justifying different aspects of the infinity within deductive speculation.

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