Continuous Versions of Haack's Puzzles: Equilibria, Eigen-States and Ontologies

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Abstract

This article discusses some continuous limit cases of Susan Haack's *crossword puzzle metaphor* for the coherent development and foundation of science. The main objective of this discussion is to build a bridge between Haack's *foundherentism* and the epistemological framework of *objective cognitive constructivism*, including its key metaphor of *objects as tokens for eigen-solutions*. The historical development of chemical affinity tables is used to illustrate our arguments.

Keywords: Bayesian statistics; Cognitive constructivism; Eigen-solutions; Foundherentism; Invariant entities; Münchhausen trilemma; Objectivity criteria; Ontology; Statistical significance.

Daí, o senhor veja: tanto trabalho, ainda, por causa de uns metros de água mansinha, só por falta duma ponte.

There you see, my lord: So much work, still, just in order to overcome some harmless waters, for the lack of a good bridge.

João Guimarães Rosa (1956); Grande Sertão: Veredas.

Here the methods of thermodynamics have brought quantitative precision in palce of the old, vague ideas of chemical affinity, and thus chemistry has made a greatest advance towards the status of an exact science since the early chemists, Lavoisier, Richter, and Dalton laid the foundations of stoichiometry.

Lewis and Randal [43, p.2]; The Scope of Thermodynamics.

1 Introduction

For a long time, the philosopher Susan Haack has been working with the metaphor of *science as a crossword puzzle*, in which she compares the scientific activity, that is, the activity of formulating scientific theories and solving empirical models, with the activity of solving a crossword game. Susan Haack makes use of the crossword metaphor to develop the epistemological framework of *foundherentism*, see Haack [29–32]. My goal in this article is to build a bridge between Haack's crossword metaphor and a celebrated aphorism stating that *objects are tokens for eigen-solutions*. The eigen-solution aphorism, due to Heinz von Foerster, is of vital importance in the epistemological framework of *objective cognitive constructivism*, a framework that we have explored in several published articles, see Stern [74–81].

Figure 1. Crosswords and Equilibrium Games.

In order to build such a bridge between foundherentism and objective cognitive constructivism, I generalize the standard discrete crossword metaphor to continuous crossword puzzles. While the latter preserve the basic analogy of foundherentism, they also exhibit some essential properties studied in objective cognitive construtivism. Solving such continuous puzzles involves finding special states called equilibrium-states, steady-states or, more generally, eigen-states or *eigen-solutions* of a continuous system. I will also describe and analyze *four essential properties* of systemic eigen-solutions, namely, the properties of being *precise*, *stable*, *separable* and *composable*. Finally, I will relate systemic eigen-solutions to basic entities of a scientific theory, that is, to entities that belong to that theory's ontology. At this point, I take the precaution of quoting Susan Haack on the limits of analogical thinking:

> Of course, the role of an analogy is only to suggest ideas, which then have to stand on their own feet; of course, the usefulness of one analogy by no means precludes the possibility that others will be fruitful too; and of course, an analogy is only an analogy. Haack [30, p.255].

Other articles already published by the Bayesian research group of the University of São Paulo complement the informal analogical approach taken in this article presenting arguments in the context of formal logic and mathematical statistics, as explained in following sections.

In this article I make frequent use of pictures and images, offered as auxiliary resources aiming to stimulate visual insights, as intuitive complements to starker and dryer text arguments and explanations. This section has four illustrative figures: Figure 1a celebrates 100 years of crossword puzzles. Figures 1b to 1d display scales and balances based on achieving (or used to achieve) simple, complex and abstract equilibrium states. The balances depicted at Figure 1b and 5c can be modeled by means of a simple linear system, as discussed in Section 3. Some of the simplest and easiest to use but, at the same time most effective scientific models, are formulated in the continuum as linear systems. For this reason, we will use several linear models to exemplify the ideas presented in the following sections.

Figures 1a to 1d display postage stamps. A postage stamp is a token that can be redeemed or exchanged for well established goods or services. Other figures used to illustrate this article display coin and paper currencies, exemplifying the tokens used as the basic exchange medium in modern economies. Efficient markets are characterized by well defined *equilibrium prices*, in which these currencies are nominated (or the other way around), see Bayoumi et al. [5], Bordo [7] and Portes and Atal [58].

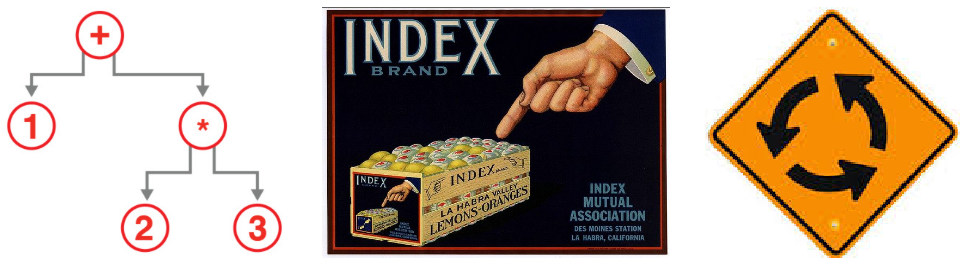

Figure 2. Finite Regress, Infinite Regress and Circular argumentation.

These prices can be characterized as economic eigen-values, exhibiting the very same aforementioned four essential properties of systemic eigen-solutions, see Cerezetti [11], Cerny [12] and Ingrao and Israel [34].

2 Epistemology, Foundherentism and Crosswords

Epistemology, or knowledge theory, aims to know how it is possible to know, that is, its (main) goal is to understand human understanding (see also the final comments at section 5.3). This article's focus is empirical science, including some examples in economics that may suggest the possibility of extending the frameworks under discussion to broader horizons. Within this scope, a basic question addressed by epistemology is this: How can we prove that a given hypothesis is true? In 1968, the philosopher Hans Albert gave a very general answer to this question, formulated as – the *Münchhausen trilemma*, a contemporary version of earlier forms of the trilemma attributed to Agrippa the Skeptic or Sextus Empiricus, see Albert [1, p.18] and Stern [81]. The trilemma states that there are only three alternative options for proving a hypothesis, each one of them plagued by particular epistemological problems. This trilemma is named after the Baron of Münchhausen, a literary character known by his elaborate lies, conveying the idea that, in empirical science, no proof should be taken as definitive and absolute. Avoiding minutiae and technicalities, we can state the three alternatives (or horns) of the Münchhausen trilemma as follows: Finite Regress, Infinite Regress, and Circular argumentation.

Finite regress requires a finite deduction procedure starting from *foundational statements*, namely, axioms, unquestionable ideals, empirical "facts", "real" observations, etc. Such a finite deduction (also called *dogmatic*) procedure is analogous to a computational procedure starting from known numerical values that are combined according to specified arithmetic operators to find at the end the desired result. For example, the operation tree depicted at Figure 2a indicates how to compute the expression $(1 + (2 * 3)) = 7$, proceeding from the numerical values at the leaves of the tree, to obtain the final result at its root. Notice that in Logic and Computer Science a tree grows upside down, having its root at the top, and its leaves at the bottom. The problem with this horn of the trilemma is that it just transfers the *onus probandi* or shifts the burden of proof down the line (or down the branches of the deduction tree) towards the initial axioms or "facts". But the fundamental problem remains: How can one be sure that the fundational statements at the bottom of the deduction tree are "really" true?

Infinite regress avoids landing the burden of proof on someone's lap by creating an infinite line, for example: I can claim to know something because my father knows it, and that he knows it because his father (my grandfather) knew it, and so on, ad infinitum, see Figure 2b.

Meanwhile, *circular argumentation* simply shifts the burden of proof back and forth among a finite number of players, see Figure 2c. For example: I can make a statement that is guaranteed to be true because my friend attests that I never lie, while my friend's testimony is guaranteed because I vouch for him. The problems of using infinite regress or circular argumentation for proving a statement are quite obvious. Nevertheless, good stories in the literary genre of detective fiction (or the work of detectives in real life) often show how important it is to follow circular arguments in search of inconsistencies. On the one hand, a good detective can use inconsistencies to detect false statements; on the other hand, crosschecking a wide web of mutually supporting statements found free of inconsistencies is a method often used by good detectives to corroborate the whole story. In this sense, the coherence of circular arguments may be taken as a provisional or tentative indication of truth.

The philosopher Susan Haack denounces the mutual exclusion of the alternatives in the Münchhausen trilemma. She claims that both, finite regress and (coherent) circular argumentation, are important in corroborating scientific hypotheses, an epistemological position she calls – *foundherentism*. Furthermore, in order to advance the case for foundherentism, Susan Haack uses the *crossword puzzle metaphor*. This metaphor is concisely explained in Haack [29, p.198-199]:

> How reasonable a crossword entry is depends on how well it is supported by its clue and any already-completed intersecting entries, how reasonable those other entries are, independent of the entry in question, and how much of the crossword has been completed. How warranted an empirical claim is depends, analogously, on how well it is supported by experience and background beliefs, how warranted those background beliefs are, independent of the claim in question, and how much of the relevant evidence the evidence includes.
>
> [T]he natural sciences, at least, have come up with deep, broad and explanatory theories which are well anchored in experience and interlock surprisingly with each other, and, as plausibly filling in long, much-intersected entries in a crossword puzzle greatly improves one's prospects of completing more of the puzzle, these successes have enabled further successes.

Observing the development of natural sciences, and the work of actual scientists, foundherentism seems to be a compelling general approach to the problem of justification of scientific hypotheses. Indeed, long after she began working with the crossword metaphor, Susan Haack found that a similar analogy had been used by Albert Einstein:

> The liberty of choice [of scientific concepts and theories] is of a special kind; it is not in any way similar to the liberty of a writer of fiction. Rather, it is similar to that of a man engaged in solving a well-designed word puzzle. He may, it is true, propose any word as the solution; but, there is only one word which really solves the puzzle in all its parts. It is a matter of faith that nature – as she is perceptible to our five senses – takes the character of such a well-formulated puzzle. The successes reaped up to now by science... give a certain

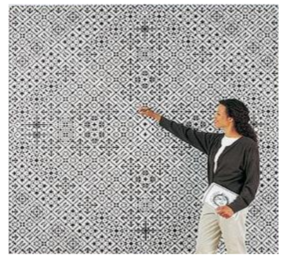

Figure 3. The theory of everything as a huge crossword

encouragement to this faith.
Albert Einstein [19], as quoted in Haack [30, p.1].

Nevertheless, some topics deserve further investigation: I would like to clarify what is the positive role of either (a) direct verification of fundamental hypotheses, or (b) coherent circular argumentation (beyond the trivial requirement of coherence as the absence of contradictions), in construing, proving or corroborating a scientific theory. I also want to investigate the best, strongest or most appropriate forms of presenting a scientific hypothesis H in the scope of a statistical model or logical formalism. Finally, I want to relate these investigations to desirable properties of measures of evidence in support of scientific hypotheses.

These lines of investigation address some demands made by critics of Haack's work, like Brian Lightbody, see next quotation, even though our "objective virtuous criteria" are quite different from the ones proposed by that author.

> It would seem foundherentism is in need of some additional, "objective", virtuous criteria to explicate precisely what Haack means by the evaluation of [C] evidence for [a statement] p. Lightbody [44, p.19].

The next section presents some ideas concerning limit cases of the crossword metaphor, ideas that will be used as the starting point for our investigations. Similar pathways have already been explored by Jeanne Peijnenburg and David Atkinson, see next quotation, even though the specific limit cases to be explored in this article are quite different from the ones proposed by those authors.

> A common objection to coherentism is that it cannot account for truth... By stretching Susan Haack's crossword metaphor to its limits, we show that there are circumstances under which this objection is untenable. Atkinson and Peijnenburg [3, abstract].

2.1 Large and Amazing Discrete Crosswords

If a scientific theory is similar to a crossword puzzle, then a theory of everything, a theory that explains every single aspect of the universe, is analogous to a huge crossword, see Figures 3a and 3b. Moreover, Atkinson and Peijnenburg argue as follows:

> As the complexity of the crossword increases, the ambiguity in general decreases: it becomes more and more difficult to come up with different solutions.

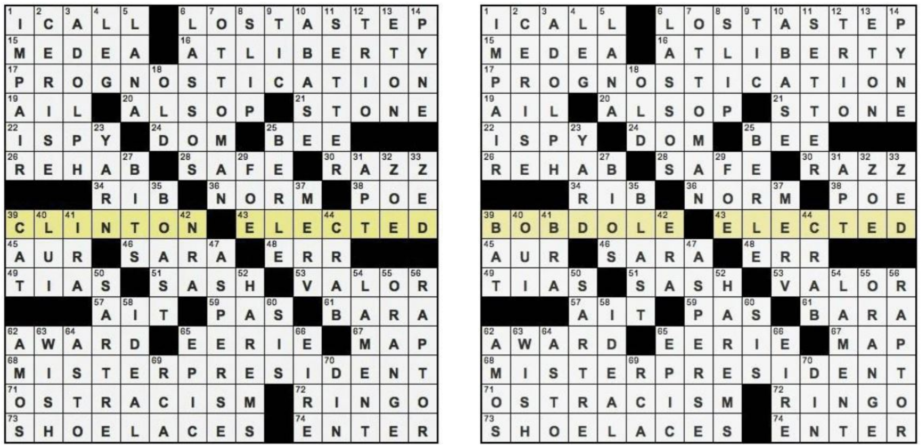

Figure 4. Most amazing crossword ever seen. Clues: ACROSS: (17) forcast, (68) title for 39th next year, (39+43) lead story in tomorrow newspaper; DOWN: (23) sewing shop purchase, (27) short writings, (35) trumpet, (39) black halloween animal, (40) french-101 word, (41) provider of support - short, (42) much-debated political inits.

> If the size of the crossword is finite, then the number of words that can be filled in on the basis of the given clues will also be finite. ... The number of coherent ways of filling in a finite crossword, with a finite alphabet, irrespective of lexical constraint, is finite. In the end, if the crossword puzzle is sufficiently complicated, there might be only one solution. Atkinson and Peijnenburg [3, p.353-354].

This seems to be a very powerful argument explaining how coherence arguments can be used to access, at least in a limit case, the truthfulness of a very complex hypothesis or of a very large scientific theory. Nevertheless, I would like to have similar arguments that can be used to access the truthfulness of more modest theories or even small empirical models, like those found in the ordinary practice of science. I would like to have an operational notion of truth that can be used in the context of theories of limited scope, but capable of attaining very high precision like, for example, Newtonian physics, Lavoisier's chemistry or Ohm and Kirchhoff's circuit theory.

With this goal in mind, I will now analyze some remarkable properties exhibited by good crossword puzzles of standard size. As a concrete example, I will study the puzzle nominated by the crossword community as the best of its kind – the most amazing crossword ever seen, namely, the New York Times puzzle from November 05, 1996, one day before the USA presidential elections, see Figure 4a and the following partial list of clues for this game.

This crossword game has some standard features, commonly found in other well designed puzzles, as well as some extraordinary features that are rarely seen.

– First, we should notice that this puzzle is of standard size, that is, it uses a 15 by 15 grid of white spaces, for alphabet letters, and black spaces, for separators.

– Second, we notice that this puzzle complies with the standard rules of crossword games, namely: (A) Each *separate* word has a correct spelling and significance defined by a standard dictionary; (B) Words must comply to a *composition rule* based on letter coincidence wherever they join at grid intersections.

– Third, we notice that the solution of this crossword has three *special entries* of maximum length, running all the way across the grid – at the 3rd (Prognostication), 8th (Clinton-elected) and 13th (MisterPresident) horizontal lines, see Figure 4a. Such special entries of long or maximum length are a common feature of well designed puzzles, fulfilling a very special purpose to be analyzed in the sequel.

Finally, this most amazing crossword game exhibits the extraordinary feature of allowing two alternative solutions: At the 8th horizontal line, either the entry Clinton-elected or the entry BobDole-elected generate a correct solution for the game, see Figures 4a and 4b. Furthermore, the expression *correct solution* can be interpreted in two distinct ways: First, each solution is correct according to the standard rules of the game, that is, there are corresponding admissible down (vertical) entries with (dictionary) meanings that match the clues, that also have matching letters at grid intersections. Second, both solutions are correct according to English grammar and historical criteria that transcend the standard rules of the game: The set of correct special entries of this game, either – "Prognostication Bill Clinton-elected MisterPresident" or "Prognostication Bob Dole-elected MisterPresident" – span all possible outcomes of the USA presidential election held the day after this crossword publication.

Special entries of maximum length play very important roles in well designed games, roles that will be of great importance in our epistemological investigations. First, it is important to notice that such a special entry makes a very *precise* or tight fit, meaning that the probability of a random miss-fit is very low.[1]

Moreover, these special entries of maximum length can be thought as long bolts, that hold the entire puzzle together, giving great *stability* to the game. For instance, once a player gets one of these special entries correctly, many other pieces of the puzzle seem to naturally fall into their proper place. Using the crossword community jargon, we may then feel confident to "ink in" such a cluster of inter-related entries.

3 Equilibrium States in Linear Systems

In this section we examine some continuous physical or mathematical systems exhibiting special solutions, called *equilibrium states*, having essential properties analogous to the properties of the special entries examined in the last section, namely, equilibrium states are *precise*, *stable*, *separable* and *composable*. In the following the sections we will explain the meaning of these properties and examine why they are the keys to build an epistemological framework known as *objective cognitive constructivism.*

Equilibria are characterized as invariant states of a system. Let $x(t)$ be a real vector giving the coordinates of a system with n degrees of freedom at time t, $x(t) = [x_1(t), x_2(t), \dots x_n(t)]$. For example, two coordinates $x(t) = [x_1(t), x_2(t)]$ can be used to describe the position of a point in a plane, and three coordinates

[1] Let $f = 0.13$ be the approximate frequency of the most common letter in the English language (the letter e), and $s = 15$ be the size of the grid. Under these conditions, we can argue that the probability of a random miss-fit should be less than f^s, approximately 5.1×10^{-14}. This value is approximately of the same order of magnitude of the relative uncertainty of the best known physical constants. For example, according to CODATA – The Committee on Data for Science and Technology – the relative uncertainty of the best estimate of Rydberg's constant currently available is approximately 5.9×10^{-12}.

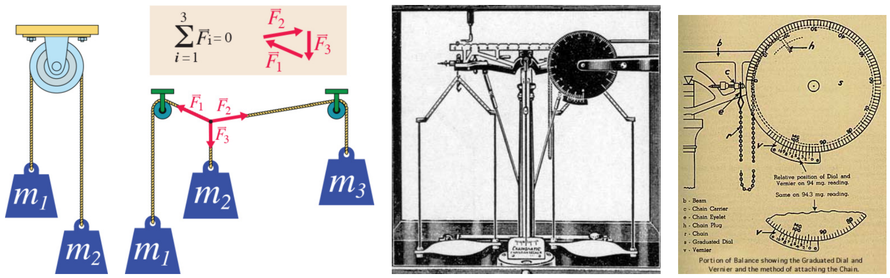

Figure 5. *Pulley Systems* and Analytical Balance.

$x(t) = [x_1(t), x_2(t), x_3(t)]$ can be used to describe the position of a point in space. The time index, t, can be an integer number, $t = 0, 1, 2, 3, \ldots$, so that $x(t)$ describes the position of the system at the ticks of a discrete clock (seconds, days, etc.). Alternatively, the time index, t, can be a real number, so that $x(t)$ describes the position of the system in continuous time.

Given the current coordinates of the system, $x(t)$, an evolution function, $f\left(x(t)\right) = x(t+1)$, gives the coordinates of the system at the next tick of the clock. For continuous time systems, the discrete evolution equation must be replaced by a differential equation. An equilibrium state is a state that remains strictly invariant in time, that is, x is an equilibrium state if $x = f(x)$.

The notions of eigen-solution, eigen-vector, eigen-state, eigen-form, eigen-behavior, etc., to be examined in the next sections, generalize the notion of equilibrium state as follows: The basic form of the system must still remain invariant, but lesser characteristics, like relative size or position, may vary through time. For example, in many important systems, an eigen-solution x evolves by a simple scaling operation: $f(x) = \alpha x = [\alpha x_1, \alpha x_2, \ldots \alpha x_n]$. In this case, if we can manage to place the system at coordinates x at time t, in the next time interval the basic form x will only be expanded or contracted by the scale parameter α. The following sub-sections present equilibrium states in some simple systems, giving concrete examples intended to clarify these notions. In the following sections, these simple systems will also be used to explain important epistemological concepts.

3.1 Pulley Systems and Balances

Figure 5a displays the simplest pulley system, made by one pulley and two masses with fixed weights. This system is unable to attain a stable equilibrium: The heaviest of the two masses will fall all the way down. In contrast, let us consider the far more interesting pulley system, displayed in Figure 5b. This system has three masses with fixed weights $[\mu_1, \mu_2, \mu_3]$ suspended by strings that, using a system of two pulleys, transmit their weight forces to a free moving pivot P. This system can attain a stable equilibrium by 'adjusting' its geometry in order to balance vertical and horizontal projections of the weight forces exerted at pivot P. The system's geometry is described by the angles, $[a_1, a_2]$, formed by the left and right strings attached to pivot P and the horizontal axis. The third string is kept vertical, hence, $a_3 = 270°$. The equilibrium conditions for this pulley system can be summarized in the following system of equations:

$$S\,\mu = \mathbf{0}\ ,\quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}\ ,\quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ ,\quad S = \begin{bmatrix} \sin(a_1) & \sin(a_2) & -1 \\ \cos(a_1) & \cos(a_2) & 0 \end{bmatrix}\ .$$

This system of equations has an external structure, based on linear algebra, but it also has a non-linear internal structure, for some of the matrix coefficients in the linear system are trigonometric functions. Given the weights, $[\mu_1, \mu_2, \mu_3]$, we can find the systemic solution, $[a_1, a_2]$, using an appropriate numerical method, see for example Luenberger [46]. It is interesting to observe that, if two of the masses are known, one can find out the value of the third (unknown) mass by measuring the angles $[a_1, a_2]$; this is the working principle of many mechanical balances, as the one displayed in Figure 5c and 5d. Finally, it is worthwhile to pay attention to a peculiar scale invariance of this system: Any solution, $[a_1, a_2]$, for given masses $[\mu_1, \mu_2, \mu_3]$, is also a solution for masses $\alpha[\mu_1, \mu_2, \mu_3]$. This result (known in physics as a gauge invariance) can be interpreted as a liberty to choose the unit in which we want mass to be measured.

3.2 Four Essential Properties of Equilibrium States

Based on practical experiments with pulleys and other similar mechanical systems, physicists and engineers make some truly amazing statements concerning *four essential properties* exhibited by such systems and their (equilibrium or invariant) solutions, namely:

Stability property: The static equilibrium under study is a stable state: If slightly perturbed, the system will regenerate the state of equilibrium. The other way around, the concept of stability means that, with skill and effort, it is possible to build practical pulley systems (or other similar systems) that actually behave in the physical world according to the pertinent equations of the idealized theoretical model.

Precision property: The equilibrium state under study is precisely defined by the equations describing it. The very symbol of equality (=) used in the linear system conveys this idea, indicating an exact solution. The other way around, the idea of precision means that, with skill and effort, it is possible to build practical systems so that they obey the pertinent equations of the idealized theoretical model with very high precision. For example, Figures 5c and 5d display analytical balances, mechanical instruments similar to the pulley system under scrutiny. Purely mechanical analytical balances, commonly found at a chemistry laboratory, can be accurate to five parts in a million, or 5×10^{-6}, and electromechanical instruments of even higher precision are commercially available.

Separability and Compositionality properties: These properties relate to how several similar systems (or parts thereof) can work separately or be put to work together. At a concrete level, we could say, for example, that we can use a good commercial balance to weigh several masses, and then experiment with the already weighed masses on a different pulley system, obtaining the expected results. As another example, we could generalize the pulley system by adding several interacting pulleys, strings and weights, as one can find in Physics 101 textbooks, see Allen [2], Cross et al. [15] and Halliday and Resnick [33], see also Dugas [17] and Duhem [18] for historical perspectives.

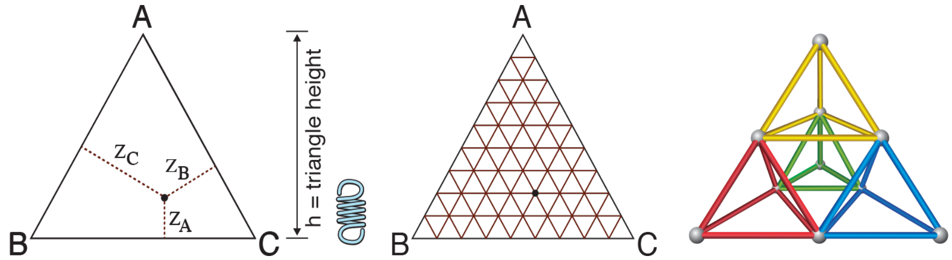

Figure 6. The *Spring System* in Equilibrium

From a more abstract perspective, the solutions to such linear systems can be handled by means of standard coupling and decoupling principles for linear operators, or by applying standard composition and decomposition techniques of Linear Algebra, or by employing standard matrix factorization algorithms, see Golub and Loan [24], Recski [61], Sadun [66] and Stern [76]. For example, when formulating the equations for the pulley system, we have implicitly used the concept of basis for a linear space, decomposing, by orthogonal projections, all the involved forces in their vertical and horizontal components.

3.3 Spring Systems and Compositional Graphs

Let us now consider the *spring system*, displayed in Figure 6, a system that is very similar to the pulley system already studied. Nevertheless, in contrast to the former, this system is characterized by variable forces and a fixed geometry (fixed angles).

The spring system represents a mechanical apparatus where three springs, σ_1, σ_2 and σ_3, are extended to lengths $[z_1, z_2, z_3]$. The springs have one end joined together at a free moving pivot P, and have the other end free to move along a side of an equilateral triangle of height h. Hence, at equilibrium, pivot P will remain inside the triangle. Also, each of the springs, represented by the line segment of length z_i between its extremities, will stay perpendicular to the triangle's side along which it can slide. The equilibrium conditions for the spring system can be summarized in the following system of linear equations:

$$S\,\mu(z) = \mathbf{0} \;\text{ and }\; \mathbf{1}\,z = h \ , \;\text{ where }\; \mathbf{1} = \left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right] ,$$

$$z = \left[\begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array}\right] \ , \quad \mu(z) = \left[\begin{array}{c} \kappa_1(z_1 - r_1) \\ \kappa_2(z_2 - r_2) \\ \kappa_3(z_3 - r_3) \end{array}\right] \ , \quad \mathbf{0} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] ,$$

$$S = \left[\begin{array}{ccc} \sin(270^\circ) & \sin(30^\circ) & \sin(150^\circ) \\ \cos(270^\circ) & \cos(30^\circ) & \cos(150^\circ) \end{array}\right] = \left[\begin{array}{ccc} -1 & 1/2 & 1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{array}\right] .$$

Hooke's law was enunciated in 1678 by means of the anagram – *ut tensio, sic vis* – as the extension, so the force. In order to take a spring σ, with rest length r_σ and elasticity constant κ_σ, and extend this spring to length z_σ, one has to exert a *driving force* $\mu_\sigma = \kappa_\sigma(z_\sigma - r_\sigma)$. In order to obtain the projection of the forces exerted by the three springs, $[\mu_1, \mu_3, \mu_3]$, into the vertical and horizontal axes, we use the

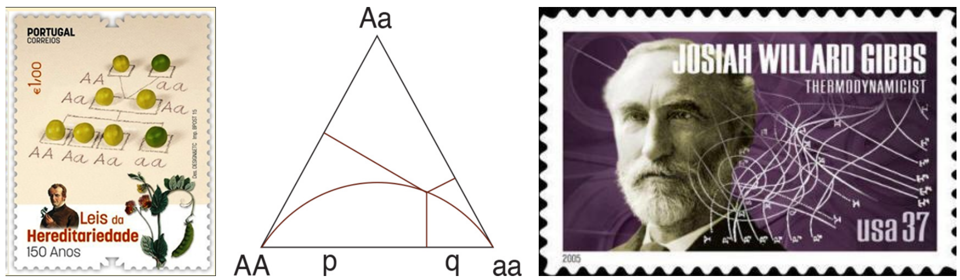

Figure 7. Mendel Rules, Hardy-Weinberg and Gibbs Equilibrium Laws.

(fixed) *geometric coefficients* in matrix S. At equilibrium, the resultant horizontal and vertical forces must cancel, hence the zero vector right hand side for the first two linear equations.

The third equation in the linear system is a normalization condition: It states Viviani's theorem, that is, the sum of the three components of vector z add up to h, the height of the triangle. If we take $h = 1$, vector $[z_1, z_2, z_3]$ can be interpreted as the fractional composition of a system with three constituent parts. Hence, graphical representations like Figures 6a and 6b are known as *compositional graphs* (also known as *de Finetti diagrams*). The spring system (or compositional graphs) can be generalized for handling n interacting springs and, in this case, can be represented geometrically in the n-dimensional simplex, see Figure 6c. Further interpretations for this system are given in the next sections.

3.4 Statistical Equilibria and Sharp Hypotheses

Let us recall the simplest case of Mendel's genetic inheritance rules, for a single locus with two alternative alleles, a and A, and write genotype frequencies as the vector $z = [z_1, z_2, z_3] = [\Pr(aa), \Pr(aA), \Pr(AA)]$. The normalization condition for probabilities, $z_1+z_2+z_3 = 1$, implies that vector z can be represented in a bi-dimensional space (the simplex), as seen in the last sub-section. Mendel rules, see Figures 7a, relate single allele frequencies, $[p, q] = [\Pr(a), \Pr(A)]$, to genotype frequencies: $z = [p^2, 2pq, q^2]$. Again, the normalization condition for probabilities requires that $p + q = 1$. Hence, genotype frequencies can be written as $z = [p^2, 2p(1-p), (1-p)^2]$, further constraining the vector z of genotype frequencies into a one-dimensional curve, see Figure 7b.

The Hardy-Weinberg equilibrium hypothesis, formalized in an appropriate statistical model, states that, under appropriate regularity conditions (like panmixia), genotype frequencies in a biological population must fall within the precise one-dimensional curve depicted in Figure 7b, a curve that lies inside a much larger bi-dimensional space of possibilities, see Pereira and Stern [54, 55].

Using similar ideas, Josiah Willard Gibbs, Figure 7c, advanced the modeling of equilibrium states in chemical reaction networks in the framework of statistical physics; simple versions of such models and their historical and conceptual development are going to be analyzed in the following sections. In these models, the molecules of chemical substances play a role that is analogous to that of genotypes in the models of population genetics, while atoms of chemical elements play a role that is analogous to that of genetic alleles in a biological population. Moreover, it is possible to derive

Figure 8. Affinity Table by Étienne Geoffroy (1718).

linearized equilibrium conditions for these models that closely resemble the system of equations for the spring system seen in the last sub-section, see next Section, Stern and Nakano [82] and the bibliography therein.

In statistical science, a hypothesis H that requires an extremely tight fit (that is, a hypothesis that constrains the parameters of the statistical model into a lower dimensional locus of the parameter space) is called a *sharp* or *precise hypothesis*. On the one hand, testing the veracity, truthfulness, or *statistical significance* of a sharp hypothesis poses several technical challenges, requiring the development of statistical significance measures with characteristics that are adequate for this specific purpose. On the other hand, equilibrium states are represented by mathematical models specified by equation constraints that, in turn, can be subject to empirical testing using statistical models with sharp hypotheses.

The Bayesian research group of the University of São Paulo has developed a statistical significance measure specially designed to overcome the technical difficulties associated with handling sharp hypotheses. This measure, $\mathrm{ev}\,(H \mid X)$, is known as the *Bayesian epistemic value* of hypothesis H given the observed data X, or as the *Bayesian evidence value* of data X in support of hypothesis H, see Pereira and Stern [54], Pereira et al. [56]. For examples of statistical models developed to accommodate practical needs of empirical science in situations similar to the illustrative examples presented in this paper, see Lauretto et al. [41, 42] and Pereira and Stern [55]. We will return to this topic sections 5.3 and 6.1.

4 Affinity Tables as Analytical Crosswords

In this Section, we use the historical evolution of chemical affinity tables to illustrate the generalization steps from the discrete to the continuous crossword analogy in science. At the beginning of their history, affinity tables share many characteristics of discrete puzzles. Later on in the course of their evolution, affinity tables display data used to compute equilibrium states in chemical reaction networks characterized as continuous systems. This historical and conceptual transition from discrete to continuous system representations used in the theories and practices of an empirical science will be used to build the bridge we are seeking between Science as a crossword

Base/ Acid	Vitriolic	Nitric	Muriatic	Acetic	Mephitic
Barytes	65	62	36	28	14
Potash	62	58	32	26	9
Soda	58	50	31	25	8
Lime	54	44	20	19	12
Ammonia	46	38	14	20	4
Magnesia	50	40	16	17	6
Alumina	40	36	10	15	2

Figure 9. Numerical Affinities by Guyton de Morveau (1786).

$$\begin{array}{ccc} & \underline{\text{Muriate of Potash, KCl}} & \\ \begin{array}{c}\text{Muriate of}\\ \text{Barytes,}\\ BaCl_2\end{array} & \left\{\begin{array}{ccc}\text{Muriat.Ac.} & 32 & \text{Potash}\\ 36 & + & 9\ (=45)\\ \text{Barytes} & 14\ (=46) & \text{Meph.Ac.}\end{array}\right\} & \begin{array}{c}\text{Mephite}\\ \text{of Potash,}\\ K_2CO_3\end{array} \\ & \text{Mephite of Barytes, } BaCO_3 & \end{array}$$

$$BaCl_2 + K_2CO_3 \rightarrow 2KCl + BaCO_3 \ ,$$

$$\text{Quiescent affinities} = 36 + 9 = 45 < \text{Divellent affinities} = 32 + 14 = 46 \ .$$

Figure 10. Reaction Diagram by Guyton de Morveau (1786).

puzzle metaphor and Objects as tokens for eigen-solutions metaphor.

In the XVIII century, a chemical reaction was conceived as a one-directional chemical transformation, in which two or more chemical substances, the reactants, are decomposed, and their parts are rearranged to form new chemical substances, the products of the reaction. This situation is analogous to the physical system with one pulley and two masses studied in Section 3, where the system goes all the way in the direction of the strongest force. At this time, chemical affinities were thought as forces holding together the parts of a chemical substance. In order to predict in practice which chemical reactions actually occur, chemical substances had to be ranked by their chemical affinity.

Figure 8a shows the affinity table published in 1718 by Étienne François Geoffroy, the first table of this kind, praised at the time as a *system of axioms* for the science of chemistry, see Stern [80]. We can think of this table as a discrete puzzle, where we have to rearrange the symbols corresponding to chemical substances in the correct order, namely, according to the rank established by the strength of their chemical affinities. Conceived as a game, this exercise resembles the sliding puzzle at Figure 8b.

In 1786, Guyton de Morveau devised a far more sophisticated way to present an affinity table. Each chemical substance is conceived as made of a base and an acid part. Integer numbers, as those in Figure 9, convey the force holding the two parts together. The diagram in Figure 10 shows how to proceed to predict if a given reaction will or will not occur, see see Stern [80] and the references therein for further details. The affinity numbers are arbitrary integer numbers in a specified range, in Figure 9,

Figure 11. Magic Squares by Albrecht Dürer, Benjamin Franklin and Su Hui

between 1 and 70. The table is considered correct if the all inequalities obtained in the reaction diagrams point in the right direction, indicating the substitution reaction that actually occurs.

Magic squares are puzzles, like those displayed in Figure 11, where a matrix or table must be filled with integer numbers under restrictions imposed by equality and inequality constraints. The examples at Figure 11 require the sum over all rows, columns and some diagonals to be equal to a constant, in these examples, 34, 260 and 15. Typical inequality constraints demand that numbers in a table can not be repeated or are in a certain range (like single-digits or *sudoku*). Morveau affinity tables are similar to magic-square puzzles. In this case, the constraints imposed on the numerical table are all inequalities, namely, a finite range, plus linear inequalities derived from the observed direction of substitution in chemical reactions.

In this conceptual framework, reactions are always conceived as uni-directional and irreversible. However, people began to realize that when the inequality resulting from the reaction diagram was weak, the resulting reaction was slow or even incomplete, that is, both reactants and products could steadily coexist.

Almost one century latter, in 1879, Cato Maximilian Guldberg and Peter Waage presented a new conceptual framework for chemical reactions, in which reactions are not conceived as irreversible but as reversible, not going all the way in one direction, but reaching a point of equilibrium. There is a close analogy between such a reversible reaction systems of chemical reactions and the spring system studied in section 3:

For a reaction network, instead of a matrix S of fixed geometric coefficients, we have a matrix S of fixed *stoichiometric coefficients*, given the proportions in which chemical substances combine. Specifically, matrix element $S_{i,j}$ expresses the (fixed) *stoichiometric coefficient* of substance j in reaction i (following the standard convention of a negative sign for reactants and a positive sign for products). Meanwhile, vector element z_j expresses the molar fraction of chemical substance j present in the system, and the affinity vector element μ_j expresses the chemical driving force promoting the formation of substance j. These chemical forces are computable thermodynamic functions (or potentials) of the molar fractions, z, of systemic state variables, like temperature, pressure, etc., and of thermodynamic constants as those in Figure 12; see Kondepudi and Prigogine [38, p.477] for explanations about the physicochemical meaning of these constants. Nevertheless, having in mind the goals of this paper, we will only comment some general aspects of the form and structure of this table.

The thermodynamic constants in Figure 12 have the form of real numbers, in con-

The following properties are listed at T = 298.15K:

$\Delta_f H^0$	Standard enthalpy of formation	$\Delta_f G^0$	Standard Gibbs energy of formation
S^0_m	Standard molar entropy	C_{mp}	Molar heat capacity at constant pressure

The standard state pressure is 100 kPa (1 bar). An entry of 0.0 for $\Delta_f H^0$ for an element indicates the reference state of that element. Blanks indicate no data available.

Molecular formula	Name	State	$\Delta_f H^0$ kJ mol^{-1}	$\Delta_f G^0$ kJ mol^{-1}	S^0_m J mol^{-1} K^{-1}	C_{mp} J mol^{-1} K^{-1}
Compounds not containing carbon						
Ac	Actinium	gas	406.0	366.0	188.1	20.8
Ag	Silver	cry	0.0	0.0	42.6	25.4
AgBr	Silver bromide	cry	−100.4	−96.9	107.1	52.4
$AgBrO_3$	Silver bromate	cry	−10.5	71.3	151.9	
AgCl	Silver chloride	cry	−127.0	−109.8	96.3	50.8
$AgClO_3$	Silver chlorate	cry	−30.3	64.5	142.0	
Al	Aluminum	cry	0.0	0.0	28.3	24.4
		gas	330.0	289.4	164.6	21.4

Figure 12. Standard Thermodynamic Constants

trast to integer numbers at Morveau table. These constants are used to compute quantities analogous to the elastic constants in the spring system displayed in Figure 6 that, in turn, are are used to compute fractional compositions analogous to ones portrayed at Figures 6a and 7b; for further details see Knox [37, ch.4 and 11], Kondepudi and Prigogine [39, ch.2 and 9], Reichl [62, ch.3], and Shell [68, ch.20].

At this point, the historical metamorphosis of affinity tables is complete: From a purely discrete puzzle, to a fixed point or equilibrium state in a continuous system. The precise formulation of thermodynamic equilibria for chemical reaction networks was achieved in 1923 by Théophile Ernest de Donder; for more detailed historical accounts congruent with the perspective taken in this article, see Donder [6], Goupil [28] and Stern [80]. In Stern and Nakano [82], we explore in great detail the formal structure of linear systems and minimum divergence optimization problems describing chemical flows for complex reaction networks.

5 Dynamic Invariants and Eigen-Solutions

The concept of eigen-solution generalizes the idea of equilibrium to dynamic states: Eigen-solutions are conceived as invariant forms of movement − and this is not an oxymoron, an apparent contradiction. For the chemical reaction networks studied in the last section, the equilibrium state is static only from a macroscopic point of view: In more detail, from a microscopic perspective, reactants and products are constantly being decomposed and regenerated. Many times, in order to attain a state of equilibrium, one can not maintain a static position, for the desired equilibrium can only be achieved dynamically. As an informal example, in order to keep proper balance when riding a bicycle, one has to keep moving. Figure 13 presents a few systems with this distinctive characteristic, all involving circular motions; Figure 18 presents a system exhibiting similar characteristics, to be analyzed in the sequel.

Figures 14 and 15 display a system of two coupled oscillators: In Figure 14, a system of transverse oscillators, two beads on an elastic string; in Figure 15, a system of longitudinal oscillators, two pendula coupled by a spring. Figures 14a and 15a

Figure 13. Systems characterized by dynamic invariants or moving equilibria.

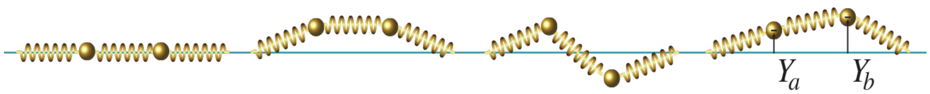

Figure 14. Eigen-Solutions for Two Coupled Transverse Oscillators

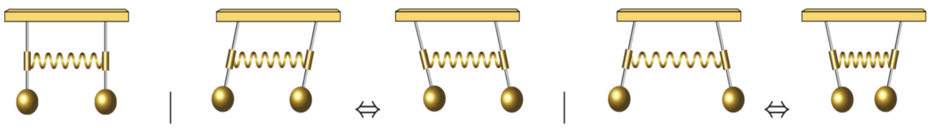

Figure 15. Eigen-Solutions for Two Coupled Longitudinal Oscillators

display the static invariant states for these systems. In contrast, Figures 14b,c and 15b,c display dynamic invariant states known as the normal modes of movement for these systems. There are two normal modes, also called eigen-solutions, for each one of these systems: In Figure 15b, the distance between the two masses remain the same, and the center of mass oscillates back and forth, while in Figure 15c, the center of mass remains fixed, while the distance between the two masses oscillates, contacting and expanding. Similarly, in Figure 14b, the transverse displacement of the two beads is the same, while in Figure 14c the two displacements are of the same magnitude but opposite in sign. In both cases the pertinent (transverse or longitudinal) displacements oscillate with a characteristic frequency, known as the eigen-value of the corresponding eigen-solution.

Hence, if the two particles in one of these systems are placed according to a pertinent eigen-solution, this specific form will be preserved (up to an oscillating scaling factor). In this way, the dynamic system constantly re-generates its own characteristic eigen-forms. This is why these forms are called eigen-solutions: *Eigen* is the German expression corresponding to *auto* in Latin or *self* in English.

Furthermore, each of the systems above exhibits a fascinating property, that can be stated as a remarkable *composition / decomposition rule*: Any free movement of the system is a (linear) composition of its two eigen-solutions, that is, the set of linear superpositions of its two eigen-forms spans all possible movements of the system. In a more technical jargon: The discrete set of eigen-solutions for the dynamical system forms a *basis* for a functional space that can be used to describe its general evolution.

Eigen-solutions are also *stable*, in the following sense: As the system evolves in time, the energy stored at each normal mode remains the same. Moreover, eigen solutions are *precisely*, defined: The symmetries of each system impose strict invariant forms, the eigen-solutions, and precise oscillating factor frequencies, the eigen-values.

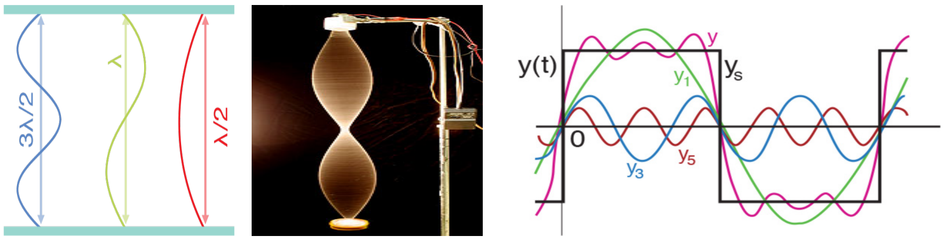

Figure 16. Eigen-Solutions and Orthogonal Basis for the Vibrating String.

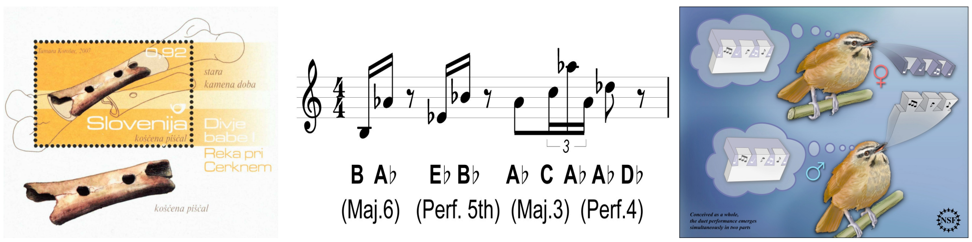

Figure 17. Cultural and Biological Eigen-values.

Similar systems with n (instead of two) identical coupled oscillators exhibit n distinct eigen-solutions. Finally, it is possible to take the limit of n going to infinity with the purpose of modeling a continuous system. Figure 16 displays a vibrating string, the continuous limit of the transverse coupled oscillators, see [89]. The eigen-solutions of the continuous (homogeneous) string are exact trigonometric functions, and the corresponding eigen-values, are multiples of the harmonic sequence $(1, 1/2, 1/3, 1/4, \ldots)$. The composition/ decomposition rules and other important properties of the continuous string and their eigen-solutions are natural generalizations of the properties of the discrete systems of transverse oscillators. For formal but intuitive accounts of the basic properties of discrete and continuous vibrating systems and its eigen-solutions, see Crawford [14], French [22], Sadun [66] and Stern [76].

5.1 Objects are Tokens for Eigen-Solutions

The abstract structure of the composition/ decomposition rules for the vibrating string seen in the last section and its eigen-solutions is known as a Hilbert space, see Butkov [9, ch.11], Byron and Fuller [10, ch.5], Luenberger [47] and Sadun [66, ch.6]. The vibrating strings of a harp or a guitar give very good approximations of such a system, for these musical instruments are designed and made with the specific purpose of "playing" according to these very same rules. Of course, men and women have been using intuitive versions of these abstract rules to play music, long before mankind knew anything about Hilbert spaces. Figure 17a displays a postal stamp showing a forty thousand year old Neanderthal bone flute found at the Divje Babe archeological site, in modern Slovenia.

The NSF poster at Figure 17c displays a couple of plain-tailed wrens singing a very complex love duet whose parts fit precisely together, like a jigsaw puzzle, see Mann [51]. Birds do what they do, without further elucubration. In contrast, men can and do talk about the things they (or the birds) do. However, this talking activity requires the things they do to be named. For example, Figure 17b shows the standard names given in western music to the notes and intervals used in the wren melody, see [90]. These names point at or stand for entities directly related to eigen-solutions. Musical notes are names for eigen-values (invariant vibration frequencies of the continuous string), and basic musical intervals (and consonant musical chords) are names for certain quotients of musical note frequencies (harmonic quotients).

In general, the entities that we can recognize and name for a given system, are related to operational invariants of that same system, hence the aphorism: *Objects are tokens for eigen-solutions.* Heinz von Foerster formulated this aphorism in a slightly different form: *Objects are tokens for eigen-behaviors*, stressing that the most important entities in our lives, or simply the objects that we are able to recognize as such, are those invariant entities that we constantly re-encounter in, or re-generate by, our own activity in life.

5.2 The Scientific Production Diagram

Objects as tokens for eigen-solutions: This is the central metaphor of our version of *objective cognitive constructivism.* Moreover, the *four essential properties* of eigen-solutions, namely, to be *precise*, *stable*, *separable* and *composable*, are the key characteristics that allow us to easily identify the most important entities related to a given system, namely, the system's core operational invariants, those being, after all, the objects we should care to name, the entities for which we should create distinctive labels, so that they can be re-presented in language.

In previous sections we have studied some simple systems described by well defined mathematical models. The goal of this section is to show how it is possible to extend the use of the – Objects as tokens for eigen-solutions – metaphor to much broader systems. Unfortunately, very detailed mathematical models covering all relevant aspects of the "life" or way of existence of such general or abstract systems are often unavailable. Nevertheless, we hope to demonstrate how the essential properties of eigen-solutions can still be clearly recognized and used to identify the most important objects for the system's modus vivendi, for its way of life or form of perpetual operation.

Scientific systems are the prime object of our attention, specifically, we focus on abstract system comprised by the means and methods of a scientific discipline, including pertinent theories, mathematical models, experimental supplies and instrumentation, computational equipment, data analysis methods, and all other pertinent resources needed by the scientists working in that field to keep doing whatever they do. Contemporary science relies on statistical methods for validation of its theories and hypotheses. Therefore, we pay special attention to the way that theoretical entities as well as observational data are represented in statistical models.

Figure 18 shows the Scientific production diagram, a schematic representation of how objects (or identifiable invariants) emerge in the scientific system, see Krohn and Küppers [40] and Stern [81]. At the right side of the diagram we have experimental means and methods. In a statistical model, experimental observations usually correspond to entities represented in the *sample space* and to quantities of interest assigned to Latin letter variables. At the bottom, we have operations related to the

Theoretical	|	Metaphysical	|	Experimental
Mathematical formalization	$\Rightarrow$	Causal explanation	$\Rightarrow$	Hypotheses formulation
$\Uparrow$				$\Downarrow$
Speculative interpretation		Eigen-solution verification*		Experiment (trial) design
$\Uparrow$				$\Downarrow$
Statistical modeling	$\Leftarrow$	Data acquisition	$\Leftarrow$	Technological implementation
Parameter space (Θ)	|	Operational	|	Sample space (X)

Figure 18. The Scientific production diagram.

technological implementation of experiments, data acquisition and statistical modeling. The theoretical side of the diagram, at the left, corresponds to the interpretation of mathematical and statistical models and the mathematical formalization of theories. In statistical models, theoretical entities are usually represented in the *parameter space* with quantities of interest assigned to Greek letter variables. At the top of the diagram, we have metaphysics, that is, the formulation of causal explanations for the phenomena under study. These causal explanations, often intertwined with the mathematical formalism of theoretical models, provide valuable ways of understanding the phenomena under study and also may, eventually, lead to the formulation of new hypotheses. The endless cycling around this diagram represents typical scientific activities and the modus operandi of science itself.

5.3 Verification of Eigen-Solutions

The four essential propreties of eigen-solutions, namely, separability, compositionality, stability and precision, make them magical objects – magical in the sense of wonderful or astonishing, as in the Magic Squares studied in section 4. Furthermore, the essential properties of eigen-solutions can withstand a process of empirical verification, or else succumb by empirical falsification. In the practice of science, such verification tasks are accomplished by representing these essential properties as hypotheses in statistical models, hypotheses that can, in turn, be tested in light of evidence provided by empirical data.

Verification, confirmation or corroboration are all terms used to convey a similar general idea, but each of them is also heavily overloaded with very specific meanings related to distinct epistemological or statistical frameworks. In the scope of the objective cognitive constructivism epistemological framework, we have chosen the term *verification* to express high truth-values of scientific hypotheses, values that, in turn, correspond to strong statistical support (or statistical significance) in light of observed empirical data. Section 6.1 further elaborates on the objective nature of these truth-values; for more detailed discussions, see Stern [76, 78, 79, 81].

As already mentioned in section 3.4, the Bayesian research group of the University of

São Paulo has developed a statistical significance measure tailor made for scrutinizing statistical hypotheses directly related to the essential properties of eigen-solutions. This measure, $\text{ev}(H \mid X)$, is the *Bayesian epistemic value* of hypothesis H given the observed data X, or as the *Bayesian evidence value* of data X in support of hypothesis H, see Pereira and Stern [54] and Pereira et al. [58]. The formal mathematical and logical properties of $\text{ev}(H \mid X)$ makes it the right tool for the job of empirical verification in the epistemological framework of objective cognitive constructivism, as discussed in Borges and Stern [8], Esteves et al. [20], Izbicki and Esteves [35], Madruga et al. [48, 49], Silva et al. [69], Stern [73, 78, 79] and Stern and Pereira [83].

Scientific hypothesis testing is the objective cognitive constructivism counterpart of the foundationalist aspect of foundherentism. Of course, it does not offer a path of finite regress from fundational statements concerning dogmatic axioms or unquestionable facts. Instead it offers a path for the straightforward statistical verification of fundamental scientific hypotheses, hypotheses that, in turn, are directly related to (the four essential properties of) eigen-solutions (obtained in the endless cycle of scientific production); for a more detailed discussion, see Stern [74–77, 80, 81].

The stability and precision properties of eigen-solutions provide a firm foundation for all that we came to know as "exact sciences" and their technological achievements. The separability and compositionality properties of the eigen-solutions of a good scientific theory provide "articulation points" of reality (as reality is seen through this theory), giving us an opportunity for "carving nature at its joints". The last expressions under quotation marks are traditionally used to describe desired characteristics of an ontology, the subject of section 6.

Before ending this section, let us make a brief comment about another topic of interest of mainstream epistemology, namely, the somber thema of skeptical worries. Radical forms of skepticism, solipsism or subjectivism are inexpugnable castles, as explained by I.J. Good [27, ch.8, p.93]. Nevertheless, witnessing and understanding the magical or wonderful nature of eigen-solutions, together with the verification of their essential properties by good statistical analysis based on honest and carefully obtained empirical data, constitute our best hope for breaking the melancholic spell of skepticism; for further comments, see Stern [78].

6 Scientific Ontologies

In information science, an *ontology* is a formal definition of terms (words) and their (semantic) interrelations, reflecting the way they are used in the domain of discourse of a given scientific discipline, Gómez-Pérez et al. [26, ch.1], Lim et al. [45, ch.1]. Our previous considerations take us to the inexorable conclusion that the key elements of a scientific ontology must be characterized as systemic eigen-solutions in necessarily circular production schemata. From the perspective of cognitive constructivism, this is also the ultimate available foundation for affirming the existence (or for asserting the good ontological status) of an object of knowledge: Any investigation concerning the existence of an object as a *Ding an sich*, is futile. Any attempt to access a *thing in itself* outside its pertinent production diagram is vain, for there is no way of reaching an object that is not mediated by the means and methods used by the relevant discipline in which it emerges, see Rasch [59, 60] and Stern [75].

At this point, some similarities and also some differences concerning the role played by the notions of circularity and fundamental justification in Haack's foundherentism and in objective cognitive constructivism should be clear. In the foundherentist epistemological framework, coherent circular argumentation makes an integral part of the justification of scientific hypotheses. In objective cognitive constructivism however, the notion of circularity plays a far more radical role: Not only the justification of hypotheses, but the very objects of knowledge these hypotheses refer to, owe their existence to intrinsically circular production schemata, see Stern [81]. Furthermore, in the foundherentist framework, some special or foundational statements must find some external justification by means to be specified by their particular context. In objective cognitive constructivism however, key elements in the ontology of a scientific discipline must be characterized as systemic eigen-solutions, and fundamental hypotheses concerning these key ontological elements must be verified (or not) by the degree of compliance with the four essential properties of eigen-solutions.

6.1 Objectivity Criteria

Coherence in a scientific production diagram is manifested by the emergence of eigen-solutions and other less formally characterized invariant entities. Good scientific objects, that is, *objective* entities in this process are – by definition – those that comply with the four essential properties of eigen-solutions, namely, as seen in previous sections, precision, stability and conformance to case specific composition and decomposition rules (rules that, in turn, in conjunction with other relevant systemic properties, allow the emergence of invariant features). Moreover, good objects deserve to be rewarded by receiving a name, so that they can be re-presented in language.

Furthermore, the strong essential properties characterizing an eigen-solution, in conjunction with statistical significance measures specially developed to evaluate an object's compliance with these properties, makes this combination (epistemological framework and statistical methods) specially appropriate to judge the *degree of objectivity* of tokens for corresponding eigen-solutions.

In real life, it is not the case that whatever we try to do works, no matter our strongest individual or collective wishful thinking; it is not true that anything goes, see Stern [75, 76]. Hence, one should expect the objective version of cognitive constructivism we have been developing, in conjunction with its accompanying statistical methods, to be able to separate the wheat from the chaff, to discern strong and weakly objective invariants. More specifically, science needs operational methods able to distinguish strong from weakly objective eigen-solutions, to discern plausible, trustworthy, reliable or, in the limit of full statistical support, true theories, from inaccurate, unsound, implausible or, in the limit of null statistical support, false hypotheses. As already explained in sections 3.4 and 5.3, the Bayesian epistemic value of hypothesis H given the observed data X, $\mathrm{ev}\,(H \mid X)$, is a statistical tool that was tailor-made for this task.

Figure 19a to 19d contrasts some strong and weak economic tokens. These figures depict, respectively: (18a) a Drachma coin showing the Owl of Athens emblem, from 450 BC (the typical workman's day pay at the time, equivalent to six Obolos, Charon's pay to carry the dead to afterlife); (18b) a Drachma of 1944, during the German occupation period; (18c) a Greek Drachma of 1973; and (18d) a Euro coin minted in 2002. While I was writing this article, alarming news of Greek politics called the

Figure 19. Tokens for Strong and Weak Invariants

Figure 20. Tokens for very Weak Invariants

world's attention, reminding us that currencies are not all made equal. As mentioned in Section 1, the quality of these tokens can be studied within the scope of efficient market theories, and judged by evaluating the epistemic value of the corresponding economic eigen-values, see Cerezetti [11], Cerny [12] and Ingrao and Israel [34].

Finally, Figure 20a to 20c display German Notgeld (emergency money) of the Interbellum period between the first and second world wars. Inefficient markets are unstable, imprecise and prone to arbitrage. The images used to illustrate these notes should serve to remind us how the deterioration of key invariants of the economic system can lead to production break-down or even catastrophic collapse, and also to remind us of our political and social responsibility to preserve the conditions that maintain economic stability and to perfect mechanisms that promote its efficiency.

6.2 Ontological Alignments

In the scientific production diagram, the emergence of a well characterized eigen-solution depends on a well adjusted integration of all components in its production cycle or, paraphrasing Susan Haack's quotation on section 2, it requires that components in its production cycle *interlock surprisingly well with each other.* Moreover, the practice of empirical science, as well as other less formalized human activities, require good interlocking properties across far more extensive domains. This is the topic under examination in this section.

Figures 19a to 19d display four version of Greek currency, issued in a time period ranging from 450 BC to the present. Most of these coins are called by the same name, the Drachma, and all of them display the same emblematic figure, the Owl of Athens. Moreover, they have all served a same basic function: Being a common exchange medium for buying and selling goods and services in the Greek economy. Hence, on the one hand, all these currencies could be considered as different versions of the "same thing". However, on the other hand, 450 BC Athens was quite different from modern Greece, with sovereignty over distinct although overlapping territories, speaking distinct although related languages, and even worshiping different gods,

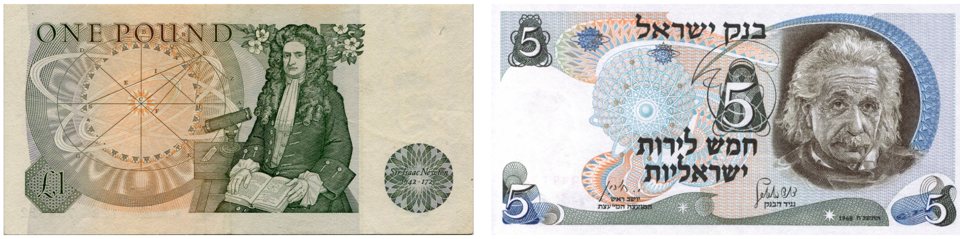

Figure 21. Synchronic and Diachronic ontological alignments.

hence placing these currencies in very distinct contexts. Facing this opposition, it is natural to pose the question: Can we "convert" prices nominated in these currencies, diachronically (between distinct time dates) within the time period they span?

Figures 21a and 21b display two banknotes, a British Pound and an Israeli Pound. These two banknotes circulated simultaneously from 1978 to 1983, although in distinct economies. Hence, it is natural to pose similar questions concerning the possibility of synchronic (at the same time) price conversions. Of course, it is unavoidable that a given price will be interpreted in different ways at distinct times, or at distinct places, or even by two distinct individuals. Nevertheless, such price conversions are a bare necessity for making commerce over extended periods of time or for trading across the seas. Hence, it should not come as a surprise that such price conversions are considered by merchants as a routine operation, for example: At 22/07/2015, the conversion rate from British Pound (GBP) to Israeli Shekel (ILS) was approximately 5.95. The specifics of how to make (and also how to make sense of) such price conversions are a topic of active research in statistics and computational finance; for technical details, see for example Balk [4] , Biggeri and Ferrari [6], Schultze and Mackie [65], Selvanathan and Prasada-Rao [67], Staines [71], Takada and Stern [85, 86], and Wang [88].

In the objective cognitive constructivism epistemological framework, these examples of price conversion are conceived as making an *ontological alignment*, that is, establishing (approximate) functional equivalences or compatibilities between (objects designated by) terms, or making (approximate) translations of words used in two different ontologies, where each of these ontologies concerns a distinct system (like a national economy or a scientific discipline), and each system is characterized by its own production diagram.

The banknotes displayed in Figures 21a and 21b depict, respectively, Sir Isaac Newton and Albert Einstein, creators of Newtonian Physics, Special Relativity, and General Relativity. Words like *space*, *time*, *simultaneity*, *synchronization*, *past*, *future*, *velocity*, *mass*, *momentum*, *energy*, etc., belong to the ontology of each one of these three theories, despite having in each one of them a rather distinct significance, see French [21, 23], Sachs and Wu [64], Stern [72] and Taylor and Wheeler [87]. Nevertheless, it is perfectly feasible to build effective ontological alignments between the three theories, so much so that these theories can share the same system of units of measurement. On an even more fundamental level, these theories can share "the same" *universal constants*; for further comments, see Cohen et al. [13], Goldberg [25], Planck [57, p.170-172], and Spiridonov [70].

Few concepts offer more dramatic examples of successive ontological alignments

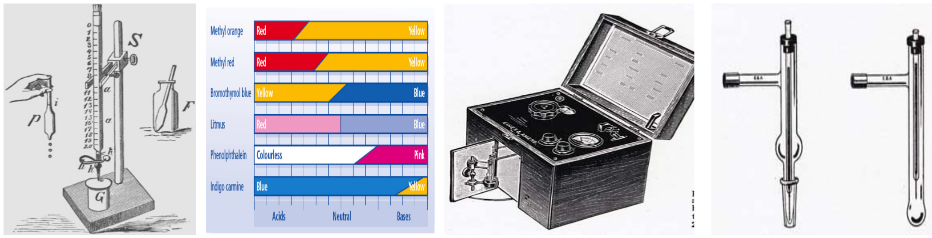

Figure 22. More synchronic and diachronic alignments.

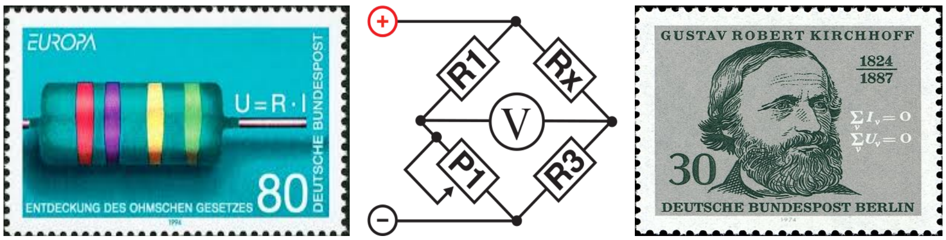

Figure 23. Ohm's formula, Kirchhoff's law and Wheatstone bridges.

than chemical affinity, in its radical evolution from middle age Alchemy to modern Physicochemistry, see Goupil [28] and Stern [80]. Changes in experimental methods require similar translation efforts. Figure 22a to 22d display scientific instrumentation used for determining the acidity of a solution, a measurement that, in turn, is often used to determine equilibrium constants for chemical reactions. Titration by volumetric analysis with the help of halochromic indicators, and potentiometric analysis with the help of glass electrodes, are very distinct procedures. In contemporary chemistry, both methods are in current use, a situation that requires the construction of good ontological alignments, including efficient calibration methods, effective standardization procedures, etc., see Johansson [36], Madsen [50], Martell and Motelaitis [52], Mirsky [53], Rossotti and Rossotti [63], and Szabadváry [84].

Figures 23a and 23c celebrate Ohm's law (1827) and Kirchhoff's laws (1845). Put together, these laws describe a (purely resistive) electrical circuit as a linear system, a theory that is formally very similar to the linear systems studied in Section 3, see Halliday and Resnick [33]. Figure 23b depicts a Wheatstone bridge, a prototypical electrical circuit used to measure an unknown electrical resistance or an unknown electrical potential. This circuit is, in turn, a key element for the construction of electronic pH-meters, as displayed in Figures 22c and 22d. A few expert users or designers of chemical instrumentation may need to be familiar with these concepts; however, a pH-meter is a very user-friendly equipment, in the sense that a professional chemist can use it without further knowledge of electronic circuits. Not surprisingly, concepts related to circuit theory are usually placed in formal ontologies for (and the practice of) disciplines distinct from Chemistry, like Electrical Engineering.

One of the most amazing features of the crossword game studied in section 2, was its power to (at specific points) harmonically conform to two different sets of rules (for crosswords games, and for colloquial English), to meaningfully cross the boundary between two different "realities". In the same way, the electronic pH-meter is a most

most amazing instrument, capable of establishing relevant conceptual bridges between two distinct ontologies, constituting a material link between the means and methods of the corresponding scientific disciplines.

Or course, scientific ontologies are not fixed in time, but very dynamic. Also the scope and boundaries of scientific disciplines are subject to changes and revisions. This process can be seen as an adaptive struggle, an evolutive effort, that seeks optimal structures for scientific disciplines and aims to improve their forms of communication.

7 Final Remarks

My goal in this article was to build a bridge between Haack's crossword metaphor and Heinz von Foerster's aphorism stating that Objects are tokens for eigen-solutions. This bridge should be able to foster and facilitate the exchange of insights and intuitive understandings provided by the epistemological frameworks of foundherentism and objective cognitive constructivism. Nevertheless, notwithstanding all mutual benefits offered by comparative studies, these two epistemological frameworks remain quite distinct, and any temptation of naive reductionism should be avoided. Furthermore, in this article, this conceptual bridge was mostly used in one direction: To cross from foundherentism to objective cognitive contructivism. Hence, in this comparative study effort, many topics remain unexplored, awaiting further research, for example:

(1) How could the idea of ontological alignment be supported in the foundherentist framework?

(2) How could ontological alignments be represented using the crossword metaphor?

(3) The universe of scientific measurement instrumentation seems to be populated by many amazing devices, capable of (or based on the possibility of) "translating" quantities of interest between the realms of different disciplines. Could the study of some concrete examples be helpful in investigating the aforementioned questions?

(4) Parallel to the constant restructuring of scientific ontologies, a similar adaptive process takes place in the organization of university curricula, departamental structures, and other social aspects of the scientific community organization. Could the study these topics benefit from the approach presented in this article?

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