Frege and Hilbert on the Foundations of Geometry

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Acknowledgments & Note on the Text

This is the text of a talk given on October 14, 1994, at the University of Pittsburgh in the Department of Philosophy's Graduate Student Colloquium. The paper was written for a seminar given Fall 1987 by Wilfried Sieg (CMU Philosophy) and Ken Manders (Pittsburgh Philosophy), and was completed August 1988. Some minor revisions were made in Fall 1994 to prepare it as a talk.

The reader is warned that the paper has not been brought up-to-date to 2002 as far as other philosophers's work. Though my intention in presenting it and circulating it for comments in 1994 was to develop it for publication, a number of other philosophers have since written major papers on the topic of making sense of Frege's view in the Frege-Hilbert correspondence. As it seemed to me the moment of timeliness for making the paper's main point had passed, I did not attempt to publish it. I am making it available via e-print now (2002) in its embryonic form nevertheless, as it may be useful to some as a short and accessible introduction to the topic. Thanks to Aldo Antonelli, who encouraged me to publish it way back in 1994, and to Jamie Tappenden for citing it in spite of its being unpublished, in his forthcoming "Frege on Axioms, Indirect Proof and Independence Arguments in Geometry: Did Frege Reject Independence Arguments?".

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by Susan G. Sterrett

In his Foundations of Geometry, Hilbert formulated a modern approach to geometry that had been developing in the wake of the discovery of non-Euclidean geometries. Although his presentation was not as abstract as some others'[1], in that his axioms used the familiar language of Euclid, his formulation was explicit that the things called points, lines, and planes in the axioms, could be called anything one liked. And, whereas in Euclid's formulation, the axioms were meant to be statements whose truth was evident, the modern approach prided itself on not relying on any appeals to intuitions about space, or, even, to the meanings of words. Hilbert is reported to have remarked "One must be able to say at all times --- instead of points, straight lines, and planes--- tables, chairs, and beer mugs."[2].

His approach to the truth of geometrical axioms is recognized as a revolutionary step towards formalism in mathematics. One feature of the approach is that statements could be interpreted in different ways, so that, in general, whether a geometrical statement is true or not can depend on how the terms are defined. This is not quite historically accurate, though: to be more precise, Hilbert's approach was to give an axiomatization of geometry in groups of axioms which were independent of each other. Each group of axioms was associated with, or, rather, defined, a concept, such as "betweenness," or "incidence." Depending on whether certain statements or groups of statements were included among the axioms or not, different notions of point, line, and plane arose, being implicitly defined by the axioms one held to be true. Then a certain geometrical statement might be true or false, depending on the (implicit) definitions geometrical terms such as point, line and plane, received.

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1 e.g., Pasch.

2 Hilbert, Constance Reid, Springer Verlag (New York, 1970), p. 57.

This way of treating geometrical statements is generally regarded as a major step forward in the development of logic towards model theory, the study of the relationships between formal languages and their interpretations. However, Frege, who is generally regarded as having developed the modern logic that was the precursor to model theory, thought Hilbert seriously misguided on these points, and wrote to tell him so. A correspondence followed, and, as Frege began to appreciate the revolutionary character of Hilbert's work, he expressed alarm. Among other things, Frege responded to Hilbert's advance in axiomatizing geometry by saying that a statement or proposition cannot admit of different interpretations.

A typical approach to making sense of the Frege-Hilbert correspondence is to give an explanation of Frege's blindness in not recognizing the next step in the logical revolution he had instigated--to say what his blindspot consisted in, and why he had it. Other historical studies have focused on how the correspondence exhibits Frege and Hilbert's differing views on existence and consistency.

My interest here is different: I want to clarify their differences in order to see what sense could be made, even now, of Frege's view that statements do not admit of different interpretations.

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Reference to the Frege-Hilbert correspondence is slightly complicated by the involvement of other writers and by the use of the same title for several of the essays. The controversy developed as follows: In 1895, Frege initiated a correspondence with David Hilbert concerning formalism in mathematics, following up on a conversation they had had at a conference, which had been interrupted. Hilbert responded agreeably, and so, when Hilbert's The Foundations of Geometry was published in 1899, Frege sent Hilbert his comments and requests for clarification. He also commented on Hilbert's work to Heinrich Leibmann in 1900, sending along a copy of his correspondence with Hilbert to Leibmann as well. Later, in 1903, he reformulated his ideas in an essay entitled "On the Foundations of Geometry", to which Alwin Korselt responded in an essay of the same title. This was followed by yet another essay, of the same title, by Frege. The essays by Frege were more or less criticisms of Hilbert's approach to axiomatizing geometry, and Korselt proposed and defended a rather formalist version of Hilbert's work.

I include Frege's response to Korselt, and comments regarding geometry found in the Foundations of Arithmetic, because they are valuable in understanding Frege's views, particularly in showing his way of handling what (anachronistically) amounts to different interpretations of a theory, while still refusing to work with uninterpreted statements. I focus here on clarifying differences in their accounts of how concepts in a geometrical theory get their meaning, and, derivatively, in how knowledge of the truth of the theorems is obtained. Depending upon how different one thinks the processes of interpretation and elucidation are, and how crucial a role one thinks they play in the two accounts, this is a real philosophical difference.

Hilbert, Frege, and Korselt each comment on axiomatic formulations, and their relation to the subject they axiomatize. We can see differences in these views of what axioms are. For Frege, the comments are specific to axioms of geometry, since, at the time of writing these letters and essays, he considered geometry to be fundamentally different from arithmetic----- spatial intuition is required to establish the axioms of geometry, but not those of arithmetic. In his reply to Hilbert's second letter, he says this:

> It seems to me that you want to detach geometry entirely from spatial intuition and turn it into a purely logical science like arithmetic. The axioms which are usually taken to be guaranteed by spatial intuition and placed at the base of the whole structure are, if I understand you correctly, to be carried along in every theorem as its conditions---not of course in a fully articulated form, but included in the words 'point', 'line', etc. [5].

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[5] Frege, Gottlob, Philosophical and Mathematical Correspondence, University of Chicago Press, 1980, p. 43.

In his discussion of the Second Original Problem in his 1900 address, Hilbert describes what's involved in setting up a system of axioms:

> When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the definitions of those elementary ideas. . . [6]

Korselt's comments express a somewhat more formal view of axiom systems: He says that if an "axiom" contains an undefined word, then the axiom is a rule for how the word should be used---"a proposition determining the use of this sign", and:

> The signs of a formal theory have no "reference" at all; its laws merely give rules to which the intended interpretations actually are subject and to which those not intended are supposed to be subject. The rules and propositions of any game, e.g., chess, are the best example for this. [7]

The pictorial metaphors these passages suggest for an axiomatic system of geometry are roughly, then, respectively: For Frege, the picture is of a structure with a foundation of axioms whose truth is assured by spatial intuition. The whole structure of resulting theorems is then not only supposed to be tethered (via such axioms) to some ground of intuition that is to provide the source of meaning and truth in those theorems, but it's important that the foundation consisting of these axioms is so grounded <u>before</u> the rest of the structure is erected; the activity of erecting the structure requires axioms so grounded. [8] For Hilbert, the metaphor would be that of something being mirrored or represented, as in, say, a sketch: what's represented is the structure of the (scientific) theory. The relationships between the elements in the sketch (the mathematical theory) describe the relationships between the ideas of the science. And, there is the notion of

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[6] Ref. [8] , p. 9

[7] Korselt, "On the Foundations of Geometry", in ref. [2].

[8] This is consistent with what he says in "17 Key sentences of Logic," in items 15 and 16: Whereas, he says about logic, in item 15, that "The task of logic is to set up laws according to which a judgement is justified by others, irrespective of whether these are themselves true," his objection to treating geometry this way is consistent with his statement in item 16 that "Following the laws of logic can guarantee the truth of a judgement only insofar as our original grounds for making it, reside in judgements that are true."

implicit definition here: the element is what it is in virtue of its relationships to all the other things pictured (all the other elements and relationships in the theory). So, the sketch, suitably re-interpreted, can be used to represent something other than what was originally meant to be represented. (". . .it is surely obvious that every theory is only a scaffolding (schema) of concepts together with their necessary connections, and that the basic elements can be thought of in any way one likes.[9]") And, then, whatever is true about the things in the sketch, under the re-interpretation, become true statements about the things represented under the re-interpretation of the picture or sketch. But there is still some connection to the science one is axiomatizing here; the mathematical structure is supposed to represent the logical structure of relationships in that science. [10]

One difference between these two metaphors is that, in the image/representation metaphor that I associate with Hilbert, the representation (the sketch) can stand by itself. It may be created by looking at something and representing selected aspects of it, but, once created, it does stand of its own accord, and can be used to represent something else. So, one can examine it independently of examining, or of re-establishing connections to, the things that inspired the sketch. The connection acts as an umbilical cord that can be cut once the theory has been formulated. In the metaphor I've associated with Frege, as regards geometrical axioms, there is nothing that can be totally disconnected from intuition and examined independently---if the axioms aren't fully sensed propositions guaranteed by intuition, there aren't even any resulting theorems. For, on his view, you need to have axioms so grounded to be able to derive any consequences. (Below, I discuss what at first seems to be an exception to not being able to treat of anything disconnected from intuition---his comments on projective geometry, which, on his view, are not really in the discipline called geometry.)

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9 Frege, Gottlob, Philosophical and Mathematical Correspondence, p. 42.

10 In later editions of The Foundations of Geometry, the project is described as a logical analysis of our perception of space.

For Korselt, I take it, the metaphor for the idea of a formal axiomatic system would be that of someone sketching a picture of relationships out of his imagination, without having to look at anything. The mathematical structure can be conceived without benefit of anything else: it doesn't purport to be a representation of anything. That he compares axioms to the rules of a game shows that he can envision the laws of a formal theory being formulated independently of---or, even, as creating---what they describe. The connection to something they describe is incidental, and need not be made until after the mathematical structure is complete, if at all. This differs from both Hilbert's and Frege's views as expressed above, in that they see axioms as descriptive of some already existing discipline.

Korselt's description of axioms is more abstract than Hilbert's, although Hilbert manages to have a view that overlaps with both Korselt and Frege.[11] For both Hilbert and Korselt, the concepts of the theory are implicitly defined by the axioms. But, Hilbert agrees with Frege, in that the axioms express "facts of our intuition".[12] In the first exchange of letters between Hilbert and Frege, Frege raised a concern over "first creating symbolism, and then looking for an application for it", and they seem to be in agreement over this. Frege compared the process of formalizing relationships in science to a musician playing music: "a series of processes which were originally conscious must have become unconscious and mechanical so that the artist, unburdened of these things, can put his heart into the playing", and says:

> "The natural way in which one arrives at a symbolism seems to me to be this: in conducting an investigation in words, one feels the broad, inperspicuous and imprecise character of word language to be an obstacle, and to remedy this, one creates a sign language in which the investigation can be conducted in a more perspicuous way and with more precision. Thus the need comes first and then the satisfaction."[13]

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[11] This refers to Hilbert's views as expressed in this correspondence. Since Hilbert later objected to publishing the correspondence on the grounds that he had since changed his views, this discussion should not be taken as giving Hilbert's ultimate views. I am interested in the views Frege expressed in response to what Hilbert said at this time, and make no pretense of doing Hilbert scholarship here.

[12] Frege quotes Hilbert's phrase from the Foundations of Geometry (p. 3) with approval, on p. 25 of ref. [2].

[13] Frege, Gottlob, Philosophical and Mathematical Correspondence, p. 33

To which Hilbert replied: "I believe that your view of the nature and purpose of symbolism in mathematics is exactly right. I agree especially that the symbolism must come later and in response to a need, from which it follows, of course, that whoever wants to create or develop a symbolism must first study those needs."[14] So, at this point in time, at least, there is that much agreement between Hilbert and Frege on the role of symbolism.

However, this agreement either did not endure, or did not carry over into other areas: In later correspondence with Hilbert, Frege objected to Hilbert's work on several counts; basically, though, they were objections to different aspects of the procedure of implicit definition. Specifically, he objected to using axioms to define concepts, on the principle that axioms and definitions are just two very different kinds of statements, neither of which can perform the function of the other. Even apart from the objection that axioms could have no role in definition, he objected to axioms that contained words whose reference was not already established. Only definitions, which have the function of stipulating, may contain words whose reference is not established, and this only because their function is to give a reference to a word whose reference is not yet established, by way of using other words whose reference is already established.

Obviously, this method of definition would require some tethering if anything is to have a reference; that's the role of what he calls "explicatory propositions" (p.8), which aren't strictly part of mathematics, but of its propaedeutic. These are supposed to establish the reference by counting on "cooperative understanding, even guessing." One has to take the "language of ordinary life" as a starting point. Then, one tries to make a word in this language of ordinary life more precise by "ruling out by means of hints the unwanted references present in ordinary usage, and [by] indicating those that are intended." For Frege, it's important that this be done before setting down a single axiom, since axioms are assertions that express facts of (spatial) intuition.

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14 Frege, Gottlob, Philosophical and Mathematical Correspondence, p. 34

From his remarks, it seems that he wants to maintain that there is an important difference between the activity of interpreting a word in a theorem formulated in an uninterpreted language, in which one goes from linguistic items in an already formulated sentence to things that the words are stipulated to mean, and starting with words in ordinary language (which, of course, already have a sense) that must be made more precise by explication. (Such explication must precede the use of intuition to establish true axioms, and the inference of consequences from the axioms.)

It is probably the stringent criticism of using what we'd call statements in an uninterpreted language, or, in his words, of statements that contain words whose reference has not been fixed, that seems most unjustifiable. It is difficult at first to sort out why Frege, while maintaining that there is no such thing as an interpretation, seems in several places to recognize what amounts to the same thing. [15] And, he offers clarifications that, it seems, could take care of his own objections. But, I will try to make a case that there are some differences that could make a difference in one's account of epistemology of geometry, depending upon how crucial one finds the difference between "interpretations" (in the usual sense) and/or Frege's notion of "explication".

In the second essay entitled "On the Foundations. . .", in which Frege was responding to Korselt, he rejects the notion of interpretation: "The word interpretation is objectionable, for when properly expressed, a thought leaves no room for different interpretations" [16] Specifically, he argues against the view that a proposition can be subject to interpretation (that is, that it might need interpretation to give it a determinate sense): "What I call. . .a real proposition is a group of signs that expresses a thought; however, whatever has only the grammatical form of a proposition

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15 E.g., quote from Roberto Torretti [9] "to demand like Frege that the meaning of these concepts be intuitively elucidated shows a lack of understanding of the nature of logical consequence that is indeed astonishing in the founder of modern logic."

16 Frege, Gottlob, On the Foundations of Geometry, in On the Foundations of Geometry and Formal Theories of Arithmetic, p. 79.

I call a pseudo-proposition"[17]. To express a thought all the words occurring in the proposition must have a determinate reference. Axioms are special sorts of propositions. They are special in that they are either logical laws, or "a thought whose truth is certain without being provable by a chain of logical inferences." He isn't very explicit in this essay as to what <u>does</u> justify axioms, except to say that, for geometrical axioms, it is intuition. He considered it an error that the word "point" in Hilbert's axiomatic formulation could be "interpreted" in different ways. He was concerned that there was no guarantee that a unique reference would be determined by the method of implicit definition wherein the system of axioms is to implicitly define the concept "point".

He found this objectionable, since he felt that one could not use "point" unless its reference was determinate. And, since "point" had different references in different geometries, this at first appears an irreconcilable issue. However, Frege follows these comments with a description of a way one could give a determinate sense to "point" when used in this general sense. That the word "point" has different references in different geometries is no excuse for an ambiguous use of the word "point", he says. Rather, one should distinguish between the first-level and second-level concepts of "point" involved. Euclidean point-concept is a first-level concept; it applies to Euclidean points. One could also define a second-level concept, "within which, aside from the Euclidean point-concept, still other concepts fall. . .this second level concept must also be a completely determinate one; but it behaves towards the first-level concepts falling within it in a way similar to that in which a first-level concept behaves toward the objects falling under it".[18] So, allowing "point" to be something different in different geometries isn't ruled out by his rejection of interpretations, if one properly distinguishes the first- and second-level concepts involved. However, one cannot establish the truth of axioms containing second-level concepts by intuition, (since the only sort of spatial intuition we've got is that of Euclidean space).

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17 Frege, Gottlob, "On the Foundations of Geometry", in <u>On the Foundations of Geometry and Formal Theories of Arithmetic</u>, p. 69.
18 <u>ibid</u>., p. 68

There is a passage in The Foundations of Arithmetic, written years before, which seems close to recognizing alternate interpretations of a theory :

> . . .let us suppose two rational beings such that projective properties and relations are all they can intuit---the lying of three points on a line, of four points on a plane, and so on; and let what the one intuits as a plane appear to the other as a point, and vice versa, so that what for the one is the line joining two points for the other is the line of intersection of two planes, and so on with the one intuition always dual to the other. In these circumstances they could understand one another quite well and would never realize the difference between their intuitions. . .Over all geometrical theorems they would be in complete agreement, only interpreting the words differently in terms of their respective intuitions. With the word "point", for example, one would connect one intuition and the other another. We can therefore still say that this word has for them an objective meaning, provided only that by this meaning we do not understand any of the peculiarities of their respective intuitions. [19]

In this passage, unlike in the response to Korselt, there is a notion of (something at least analogous to) an uninterpreted word having an objective meaning, a meaning at a different level than the interpretations. It seems to me that what he calls the "objective meaning" of "point" here, which he wants to contrast with what's intuited, corresponds to what he calls a "second-level" concept. As explained above, a first-level concept is one under which objects fall, and a second-level concept is one within which first-level concepts fall. It was not his purpose in this passage to make the first-level/second-level distinction; he was using this example to make a different distinction: of "objective" (expressible in words) versus what is merely intuitable. But, it seems to me that this is an example of how a second-level concept of point is defined by theorems---in this case, the theorems on which these two rational beings agree. Each of the rational beings would have his own first-level concept of point, but the "objective meaning" (which would be second-level) of "point" would be constrained only by what the theorems of projective geometry legislate.

But, it's worth paying closer attention to how the meanings of both the "objective" second-level concept, and the "intuited" first-level concept are established---from the bottom up. That is, the

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[19] Frege, Gottlob, Foundations of Arithmetic , transl. by J. L. Austin, 1950, p. 35-36 (§26).

process is grounded in the intuition of these two rational beings, and the objective, second-level concept arises as a result of their differing intuitions. (This wouldn't happen in "real" geometry, though, because, for humans, Euclidean space is the only kind we can intuit.)[20]

And, the really fundamental observation is this: Frege meant this example to illustrate "what is independent of our sensation, intuition, and imagination,. . .but not what is independent of reason."[21] By this he meant the realm of logic, and, he conjectured, also of arithmetic, but not of geometry. Geometry is not as basic as logic and arithmetic; logic and arithmetic are to hold in the realm of thought in general, but geometry only in the realm of the "spatially intuitable". So, the projective geometry being discussed here is not part of the discipline he considers geometry, as it doesn't specify constraints on what's spatially intuitable:

> Conceptual thought alone can after a fashion shake off this yoke [of being intuitable, and hence subject to the axioms of geometry], when it assumes, say, a space of four dimensions or positive curvature. To study such conceptions is not useless by any means; but it is to leave the ground of intuition entirely behind. If we do make use of intuition even here, as an aid, it is still the same old intuition of Euclidean space, the only one whose structures we can intuit. Only then the intuition is not taken at its face value, but as symbolic of something else. . .[22]

He complains, in the second letter to Hilbert, that "a logical danger lies in the fact that you say, for example, 'the axiom of parallels', as if it were the same in every other geometry. Only the wording is the same; the thought-content is different in each particular geometry"[23], and adds that the error is one of confounding first- and second-level concepts. "The characteristics which you state in your axioms undoubtedly are all of a higher level than the first", he says, because they don't say what properties an object must have to fall under that concept. Rather, they describe second-level relations, such as the relations between the first-level concepts of "point" and "straight line".

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20 Frege, Gottlob, Foundations of Arithmetic, trans. by J.L. Austin, 1950, p. 20 (§14).
21 Frege, Gottlob, Foundations of Arithmetic, trans. by J.L. Austin, 1950, p. 36 (§26).
22 Ibid., p. 20 (§14)
23 Ref. [2],p. 19

He repeats this in the first essay entitled "On the Foundations. . ." and concludes: "Therefore, if any concept is defined by means of them, it can only be a second-level concept"[24]

This criticism seems aimed at showing how to rehabilitate the enterprise in a way that would avoid having an axiom containing terms that had no determinate reference, which did not meet his standards of expressing a thought. [25] And, in this same essay, in which he rejects the notion of an interpretation, Frege also gives accounts of ways one could understand Hilbert's project, as characterized by Korselt, that are not susceptible to his objections to propositions that contain undefined signs, to making inferences from them, or to interpretations in general. There, Frege gives an account of how to understand statements that contain undefined signs, while still rejecting the notion of interpretation (that is, rejecting the idea that a proposition could have multiple interpretations). He considers, as an example, "a > 1". Now, he says it is a mistake to consider "2 > 1" an interpretation of the proposition "a > 1". First of all, "a > 1" isn't even a proposition, since it contains an undefined sign, namely "a". But, it is a pseudo-proposition, and such a pseudo-proposition does have a role to play.

The pseudo-proposition "a > 1" is a part of the real proposition: "If a > 1, then a > 0." This proposition does express a thought, even though it contains an undefined sign.[26] Frege

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[24] Ref [2], p. 35

[25] This point was accepted by Hilbert's assistant, Paul Bernays, who later revised and expanded Hilbert's Foundations of Geometry. In Bernays' review of Frege's letter to Leibmann, he writes that Frege's point regarding the confusion of first- and second-level concepts is well-taken, but that the objection is overcome by taking proper care. Just as in abstract algebra, the axioms of group theory define what a group is ("or, more explicitly, on what conditions a domain of individuals and a binary function applying to them constitute the elements and the composition of a group"), so the Hilbert axioms of geometry define the concept of a three-dimensional Euclidean space. So, strictly speaking, the axioms don't define concepts such as "point", "straight line", and "incidence", but the second order concept of a three-dimensional Euclidean space. This judgement agrees with Frege's evaluation mentioned above, in that any concept so defined would be a second-level one (although Bernays adds that the definition of the second-level concept is of the whole structure, and is really an explicit one). I thank Wilfried Sieg for making me aware of this piece of Bernays".

[26] This remark is similar to the passage in On Sense and Reference in Geach and Black, p. 72, where he says; " In conditional clauses, also, there most often recognizably occurs an indefinite indicator, with a correlative indicator in the dependent clause. . . .In so far as each indicator relates to the other, both clauses together form a connected whole, which as a rule expresses only a single thought." I mention this to make the point that Frege is expressing a view he had already arrived at in thinking about language, and so is not an ad hoc way of dealing with Hilbert, nor special to mathematical statements.

explains why it is acceptable for a proposition to contain an undefined sign in this case: the proposition doesn't contain any undefined concepts:

> Here the letter 'a' only indicates, as did the words "something" and "it" above ["If something is greater than 1, then it is a positive number". "Something" and "it" refer to one another. If we break this connection by separating the propositions, each of them become senseless. [27]] The generality extends to the content of the whole propositional complex, not to the antecedent proposition by itself nor the consequent proposition by itself. [28]

So here is a way to allow (what Hilbert calls) axioms to contain undefined words. Frege's complaint was that nothing containing undefined words could be a proposition, and so neither could it be an axiom. But, on this account, what Hilbert calls axioms aren't propositions. He says that what Hilbert calls axioms are really pseudo-propositions. But that doesn't make the work worthless: "Mr. Hilbert. . .understands by an axiom. . .a pseudo-proposition from which. . .several real propositions can emerge which then do express thoughts." [29]. So it appears that by the latter part of the second essay, this objection is really to Hilbert's own account of what he is doing in his axiomatic treatment of geometry. He allows: "Hilbert's axioms are parts of a general theorem that has a sense, although the parts themselves do not." The explanation for this is:

> Neither the axiom nor the propositions that follow have a sense of their own; rather, the axiom is an antecedent pseudo-proposition and these propositions that follow are consequent pseudo-propositions which [together] form one or several real propositions whose parts they are. [30]

In Hilbert's axioms, then (which are pseudo-propositions on this view), the words "point", "plane", etc. play the role that "something" and "it" did in the earlier example: the role of "lending generality of content to the theorem, as do the letters in algebra." [31] A theory is a system of such general theorems that all have the same antecedent pseudo-propositions.

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[27] Ref. [2], p. 69
[28] Ref. [2], p. 70.
[29] Ref. [2], p. 76.
[30] Ref. [2], p. 77.
[31] Ref. [2], p. 80.

Theorems derived in such theories are then of the form "If [conjunction of all the pseudo-axioms], then [statement of theorem (which is a pseudo-proposition)]". The resulting proposition, as a whole, does express a thought. Then, such a proposition, which does contain undefined words but still expresses a thought, rather than playing the role of a statement that needs to be interpreted, plays the role of a general (real) proposition from which particular statements may be inferred. For example, the statement "If a is a square root of 1, then a is a fourth root of 1" does count as a proposition. And, one can infer from it: "If 1 is a square root of 1, then 1 is a fourth root of 1". But, the "a" is not an uninterpreted sign, of which "1" is an interpretation. It is a case of going from a general statement that expresses a thought to a particular statement expressing a thought. A part of the proposition---such as "a is a square root of 1" ---is not considered, except in its role as part of a genuine proposition.

The strategy of working only with "real" propositions may provide a way to make sense of inferring consequences of formal axioms while denying that these same axioms are propositions. But, since it denies that the formal axioms express thoughts, the strategy doesn't help in making sense of Hilbert's projects of showing axioms consistent with or independent of other ones. The axioms, being merely pseudo-propositions, don't express thoughts. And, on his view, it is only thoughts that can be said to be independent of or consistent with one another.

But Frege <u>does</u> sketch a way "real" axioms might be shown independent of each other. This is possible since real axioms express thoughts, and he allows that one might be able to make sense of a thought being independent of a group of thoughts.

Briefly, here's how it works: one maps ("set[s] up a correspondence between") words of a language (in which, of course, the reference of every word is fully determinate) onto other words of the same language, subject to some restrictions. These restrictions include mapping proper names to proper names, concept-words to concept-words of the same level, and so on. The signs whose references belong to logic (e.g., negation, identity, subsumption, and subordination of concepts) are not mapped to different signs. Then, one can show that a thought G is independent of a group of thoughts ß, if one can obtain from ß and G, respectively, a map to a group of true thoughts ß' and a false thought G'. This actually seems to be the sort of approach Hilbert did take:

> The axiom of parallels is independent of the other axioms. This is most simply shown in a well-known way as follows: Let the points, lines and planes of the ordinary (Cartesian) geometry. . .which lie in a fixed sphere be chosen as the elements of a space geometry and let the congruences of this geometry be replaced by linear transformations of ordinary geometry that map the fixed sphere into itself. By suitable interpretations it can be seen that in this *non-Euclidean" geometry* all axioms except [the parallel axiom] are valid. . [32]

So it seems that Frege has provided what he considers a solution to the problem of first- and second-level confusion, and a way of understanding "interpretation" of axioms such as Hilbert's that he is satisfied with. The question that arises is: If these objections of Frege's are met by properly distinguishing first-and second-level concepts, and by understanding Hilbert's work in terms of his proposal of handling "interpretations" by way of "pseudo-propositions", and a method of showing independence of axioms that seems quite in keeping with what Hilbert actually did[33] in his independence proofs, is there any real difference between Hilbert's and Frege's views of the axiomatization of geometry?

I think that depends. It depends on how important one thinks their accounts of how words get their meaning are to their respective views. Certainly Hilbert and Frege thought they disagreed on this. Frege seems never to have rescinded his view that axioms are special in that their truth is grounded in intuition, and this is strongly related to his stance on meaning. In an essay written in

32 Hilbert, David, Foundations of Geometry, Open Court,(1971), p. 32

33 Although Frege was here responding to Korselt's more formalist view.

1914 and published posthumously, the reasons he gives for why one---or, rather, why he, at least---cannot entertain the thought of the axiom of parallels being false has to do with how words get their meaning:

> I can only say: so long as I understand the words "straight line", "parallel" and "intersect" as I do, I cannot but accept the parallels axiom. If someone else does not accept it, I can only assume that he understands these words differently. Their sense is indissolubly bound up with the axiom of parallels.[34]

This indicates how central the meaning of words (or, perhaps, the understanding of words) is to his account of whether things are true: the truth of the axiom of parallels is supposed to just follow (not logically, but in a way that involves (Euclidean) spatial intuition) from understanding the words in the axiom, and if one doesn't accept the axiom, he thinks the only possible reason is that that person doesn't understand the words in the same way he does.

So, the question of how words get to mean what they do is quite central to his account of mathematical truth; meanings are all built up from primitive notions that can only be explicated, and this explication must take place <u>before</u> one gets started formulating sentences (real propositions) with them. So the source of meaning can be traced back to these primitive notions. The primitive notions are explicated, as he says repeatedly, via a "meeting of minds".[35] That one must do this before using the words (or signs) really is substantially different from Hilbert's notion of words as being implicitly defined, and Hilbert likewise identified it as a source of difference in their views: in a short note he wrote to Frege in 1900, he states as his opinion that "a concept can be fixed only by

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34 in "Logic in Mathematics", <u>Posthumous Writings</u>, p. 247

35 In an essay published posthumously, "Sources of Knowledge of Mathematics and Natural Sciences", he writes, in discussing the word "function", which is logically unanalyzable; "How does a child learn to understand grown-ups? Not as if it were already endowed with an understanding of a few words and grammatical constructions, so that all you would need to do would be to explain what it did not understand by means of the linguistic knowledge it already had. In reality of course children are only endowed with a capacity to learn to speak. We must be able to count on a meeting of minds with them just as in the case of animals with whom men can arrive at a mutual understanding. Neither is it possible, without a meeting of minds, to make designations of a logically unanalysable content intelligible to others." <u>Posthumous Writings</u>,p. 271.

its relations to other concepts", that he calls the statements of these relations axioms, and that he has arrived at the view:

> . . .that axioms (perhaps together with propositions assigning names to concepts) are the definitions of the concepts. I did not think up this view because I had nothing better to do, but I found myself forced into it by the requirements of strictness in logical inference and in the logical construction of a theory. I have become convinced that the more subtle parts of mathematics and the natural sciences can be treated with certainty only in this way; otherwise one is only going around in a circle.[36]

In a letter three years later (1903), Hilbert identifies the sharpening and refining of "the theory of concept formation" as crucial: "As I see it, the most important gap in the traditional structure of logic is the assumption made by all logicians and mathematicians up to now that a concept is already there if one can state of any object whether or not it falls under it. This does not seem adequate to me." He found implicit definition crucial to his work: "What is decisive is the recognition that the axioms that define the concept are free from contradiction."[37]

Going back to the metaphors at the beginning of this paper, one can see that they suggest why, on Hilbert's view of the mathematical theory (the system of axioms), the only alternative to implicit definition is "going around in a circle". If the theory, in my metaphor, is compared to a sketch representing the relations in some science, and the sketch, <u>once made</u>, can stand by itself, then this sketch, standing alone, isn't connected to the thing it represents anymore, except in the sense that it is supposed to represent it. So there aren't any connections to anything else. It makes sense, then, that an item in the sketch is defined by its relations to all the other things in the sketch. And we can see why someone with this picture of things would ask, exasperatedly, as Hilbert does, "what other respectable alternatives could there be"? To demand of every item in the sketch that it be reducible to other, known, items in the sketch <u>would</u> be circular. And, to demand that we can say of each thing in the world whether it is an instance of the particular thing in

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[36] Frege, Gottlob, <u>Philosophical and Mathematical Correspondence</u>, p. 51

[37] Frege, Gottlob, <u>Philosophical and Mathematical Correspondence</u>, p. 52.

the sketch doesn't make sense, if the sketch is to stand on its own as a representation of the relations of a discipline.

Likewise, this explains why Hilbert was able to ask some important questions about the axioms that don't have counterparts in Frege's account. In the metaphor I've associated with Hilbert, it makes sense to ask of one's axiomatic system, "Is this set of axioms consistent?" , since the object of examination is the sketch that can stand alone. In fact, that seems to be the innovation of his view. To Frege, this question doesn't make sense, as the axioms are axioms in virtue of being recognized as true, using spatial intuition. To look back on the axioms and ask if they're consistent just wasn't appropriate. Hence, the exchange between Hilbert and Frege that resulted in Hilbert's comment:

> I was very much interested in your sentence: "From the truth of the axioms it follows that they do not contradict one another" because for as long as I have been thinking, writing, lecturing about these things, I have been saying the exact reverse: If the arbitrarily given axioms do not contradict one another, then they are true, and the things defined by the axioms exist. This for me is the criterion of truth and existence. [38]

So, I think this much can be said: Hilbert and Frege differ, in ways they both considered crucial[39], as to how concepts get to mean what they do, and as to the criterion of truth. This is not to go so far as to say that they differ as to ultimate sources of truth and meaning. For, Hilbert's motivation in reducing questions in geometry to ones of arithmetic, and questions in proof theory to ones involving only finite procedures were presumably to put the issues on sounder epistemological bases, but not to totally reject intuition of any sort. But, the accounts of "how" do differ.

I find it surprising that Frege could make as much sense as he did of (Korselt's description of) Hilbert's work, given his views on meaning. And, I find it interesting that one can give the account

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38 Frege, Gottlob, Philosophical and Mathematical Correspondence, p. 42.

39 Hilbert had made revolutionary use of this criterion of existence early in his career: from C. Reid's biography of Hilbert: "To prove the finiteness of the basis of the invariant system, one did not actually have to construct it, as Gordan and all the others had been trying to do. One did not even have to show how it could be constructed. All one had to do was to prove that a finite basis, of logical necessity, must exist, because any other conclusion would result in a contradiction---and this was what Hilbert had done." (p. 33)

of "interpretation" that he did, while never working with uninterpreted statements. It means that someone who wants to work with only fully interpreted statements can do more than one might have thought possible. This was important to Frege, since his account of how words get to mean what they do relies on using only fully sensed propositions. If we think, as he did not, that the notion of "interpretation" is no more problematic than "explication", then the difference does not seem so great, and this shows only, as a point of interest, that one can often do with "sense" and fully sensed propositions, what is sometimes held could only be effected by "interpretation" of statements. But, if one considers interpretation problematic, or at least more problematic than "explication", the difference is a philosophical one.

And, the notion of an interpretation in philosophy of language has come to be regarded as problematic; most notably in the passages on rule-following in Wittgenstein's Philosophical Investigations, and the literature that has arisen discussing them. McDowell calls attention to the statement in #201 that "there is a way of grasping a rule which is not an interpretation." [40] And this in fact was Frege's point: he was not making a skeptical point about the ambiguity of geometrical axioms, but trying to point out that the modern formulation of them does not properly appreciate what is involved in understanding them.

For Frege, "explication" starts with taking words that are already in use, in some ordinary language shared by a group of people who are able to carry on communication with it; I think it not incidental that he mentions the new generations to whom the language is passed on; this is somewhat different than "agreement" by members of a group, which is all too easily assimilated to stipulation. Explication makes words already in use more precise, sometimes relying on non-linguistic procedures (a "meeting of minds") to make clear what is meant. To be a word in an ordinary language already in use among humans is to carry some objectivity along with it, since it is already "the common property of many people, and so not a part or a mode of the individual mind. For one

40 "Wittgenstein on Following a Rule," Synthese 58, 1984.

can hardly deny that mankind has a common store of thoughts which is transmitted from one generation to another."[41] Even if one considers this process every bit as problematic as effecting an interpretation, it is just different from implicit definition of concepts, and is associated with a very different account of what it is that makes a concept mean what it does. To be sure, one does something similar in interpretation, say, of "point", in connecting the word "point" with the one in use in ordinary language, but here we see a difference: we can axiomatize geometry, let the axioms define the concept "point", and then raise the question of whether, so defined, the concept contains all and only what's in our concept of "point" as the word is used in our ordinary language. There's the chance for mismatch, since there's the implicitly defined concept of point in addition to the one already in use, and so such questions make sense. We can see how different notions of point can arise. Similarly, we can axiomatize a type of space, and see whether our notion of space matches it. But, if we start with the word "point" in ordinary language, and the notion of space in use among humans, assuming that there is one, then (once ambiguities in the ordinary use have been clarified) there isn't any possibility of mismatch arising. And Frege did think there was a notion of space in use among humans, grounded in human behavior, if not in the way people think of it:

> . . .we cannot even know whether [space] appears the same to one man as to another; for we cannot, in order to compare them, lay one man's intuition of space beside another's. Yet there is something objective in it all the same; everyone recognizes the same geometrical axioms, even if only by his behaviour, and must do so if he is to find his way about the world.[42]

The appeal of implicit definition is that it can be made much more precise than this activity of "explication", and that's what Hilbert thought important. But Frege saw only the disadvantage of potential mismatch, and so of confusion. The consequence of their different accounts of meaning and criteria of truth explain why what seems a strength in Hilbert's work---that one can consider different types of geometries by setting up axiom systems for them, and ask of an axiomatization of

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41 In "On Sense and Meaning", Ref. 3, p. 59

42 Frege, Gottlob, The Foundations of Arithmetic, p. 35 (§26).

geometry whether it is consistent--- seemed, to Frege, an admission that one had cut oneself off from the source needed to guarantee the truth of the resulting theorems.

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The most striking thing about the Frege-Hilbert correspondence is the one usually neglected: Frege carefully explaining how the work in axiomatizing different geometries can be understood, without requiring that one regard axioms of geometry as capable of different interpretations. That is, it is possible to do the things usually regarded as possible only by using formal languages and interpretations of them---such as showing independence of statements, and admitting non-Euclidean geometries--- without ever going through the route of uninterpreted statements. Thus, for philosophers who take the point that "there is a way of following a rule that is not an interpretation," Frege's view that statements do not admit of interpretation can be mined for its positive views on what is involved in understanding a statement, rather than being diagnosed as some sort of blindspot that prevented him from foreseeing the possibility of model theory. Then, this view, rather than being seen as pre-analytic naiveté, becomes reborn as a post-analytic rejection of the tradition he is credited with inspiring.

But rejection is not quite accurate. Both these are characterizations of Frege's views with respect to modern logic. But, he did not develop them in ignorance of, nor in reaction to, the development of modern axiomatics and formal languages. His major work had been in developing a calculus of pure thought. He did not like to see it neglected in favor of, or taken for, a calculus of signs. However, he did not simply criticize Hilbert's formulation of a modern approach to geometry for failing to meet his standards that a thought, properly expressed, leaves no room for interpretation. He suggested how the project could be reworked to meet those standards.[43]

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[43] Although Frege does become a bit rude at some points in the correspondence, in his first letter about Hilbert's book he says: "I would not regard yourwork as a valuable one if I did not believe I could see roughly how such objections could be rendered harmless; but this will not be possible without considerable reshaping." (27 December 1899)

So, to revise my earlier statement: Frege's view, rather than looking like pre-analytic naiveté, appears instead as a post-analytic rehabilitation of the tradition he is credited with inspiring, calling it back from its excesses, or, rather, its anorexia, in priding itself on its refusal to partake of what he considered vital nourishment.

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