

# How Many Thoughts Can Fit in the Form of a Proposition?

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## Abstract

I argue here that Frege's eventual view on the relation between sentences and the thoughts they express is that, ideally, a sentence expresses exactly one thought, and a thought is expressed by exactly one (canonical) sentence. This may clash with some mainstream views of Frege, for it has the consequence of de-emphasizing the philosophical significance of the question of how it is possible for someone to regard one sentence as true yet regard another sentence that expresses the same thought as false. This account of Frege was developed by taking a long-range look at his writings over the course of his life.

## 1. Introduction and Overview

Let us agree on this much: people use sentences to communicate. On the view that sometimes it is thoughts that are communicated, then, sentences can be used to communicate thoughts. This was Frege's view. However, sentences are used not only to communicate thoughts, but to do other things as well. And, sometimes, in conversation and in writing in natural language, people rely on more than the sentence itself to communicate a thought. This, too, was Frege's view. The study of language is not the study of logic.

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Still, developing a logic can start with the study of language, and progress by clarifying how logic is different from language. Unlike the study of language, logic studies sentences only inasmuch as they are used to communicate thoughts, and logic is about using nothing more than sentences to do so. This approach is sometimes part of an anti-psychologistic program in logic, for these two differences between language and logic involve separating the communication of thoughts from psychological aspects of communication.

It can be a bit misleading to call this approach anti-psychologistic without qualification, though, for it is not against the use of psychology in places other than logic: in fact, it draws attention to the fact that psychology is involved in many cases of human communication. An anti-psychologistic view of logic based upon a conviction that psychology has no place in logic was not the only motivation a mathematician might have for distinguishing logic from natural language. For, by Frege's time, it was becoming clear to many mathematicians that natural language, no matter how well-suited it might be for conversation, prose and poetry, was not always up to the task of providing a language in which to prove theorems and show how the truth of one thought depends upon the truth of another. Relying on natural language, different mathematicians produced proofs whose conclusions were in conflict. What was needed was a means of referring to thoughts that allowed one to determine how thoughts were related to each other.

The anti-psychologistic aspect of Frege's approach is the conviction that relations between thoughts are not a matter of human psychology of any sort whatsoever, general or individual, and hence that any formal calculus of sentences meant to reflect the relationship between thoughts between should not involve psychology. In Frege's writings, there was never any wavering, never the slightest hint of compromise, on this point.

Another, less crucial goal, was that the means of expressing a thought using a sentence in this formal calculus be at same time a means of communicating thoughts that did not depend upon the contingencies of an individual human's psychology. This latter goal was something aimed at, but even Frege came to realize that there was no guarantee that it could be achieved in every case using the system of logical notation he had developed.

Sentences are used to enable people to grasp thoughts, but there can be cases where a sentence enables some people, but not others, to grasp the thought it expresses. Ultimately, communication of thoughts relies on some common understanding, including 'a store of concepts' held in common; Frege wrote that a common store of concepts is handed down from one generation to the next (Frege 1980, p. 59) , and that learning a language necessarily involves nonverbal communication. Once the common understanding of relevant concepts is achieved, though, it becomes possible for communication of thoughts to proceed such that the sentence alone communicates the thought, without any reference to particulars such as the context in which it was uttered or particulars of the psychological state of the speaker or hearer. When, however, this common understanding is lacking, as Frege fretted might be the case with certain concepts such as the mathematical/logical notion of a 'course of values' of a function used in his formulation of Basic Law V, Frege did not count on everyone grasping the thought expressed by the sentence. Hence different people might disagree on the truth of a sentence, yet their disagreement not be a matter of one person holding a grasped thought true and the other person holding the same thought false. The disagreement arises because they are not grasping the same thought.

That different people hearing or reading a sentence might not grasp the same thought by it was an unhappy situation that, as time went on, Frege learned to accept

as inescapable in practice --- in the general case. For the more specific and very urgent issue of giving a foundation for arithmetic, however, he at least began the project with the hope that the necessary common understanding might be achieved among mathematicians, if only the right formalism for expressing the thoughts used in arithmetic might be developed. That was the goal of the Begriffsschrift (Concept Script): to develop a formalism for statements that expressed the thoughts needed to prove theorems of arithmetic. It was not until he got down to working out the project in detail in the Grundgesetze (Basic Laws of Arithmetic) that he hit the little snags he regarded as temporary imperfections that might eventually be perfected.<sup>2</sup>

The formalism Frege developed in the Begriffsschrift is generally regarded as containing the fundamental features of modern symbolic logic --- and so as an historically significant breakthrough from previous logics, including subject-predicate logic and Boolean logic. Thus reverential appellations such as ‘the father of modern logic’ are heaped upon Frege. In light of such an accomplishment, many have wondered at Frege’s seeming antipathy to formalism in arithmetic, and to his criticisms of Hilbert’s formalization of geometry, especially his criticisms of implicit definitions of concepts such as the concept of point. Some sympathetic examinations of Frege’s views have already been offered (Resnik 1974, Sterrett 1994, Blanchette 1996, Antonelli and May 2000, Tappenden 2001). One point on which Frege did strenuously diverge from Hilbert was on the use of (what we would now call) uninterpreted statements as premises in proofs.

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<sup>2</sup> Here I am referring to Frege’s remark in the Grundgesetze ‘A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V), which logicians perhaps have not yet expressly enunciated, and yet is what people have in mind, for example, where they speak of the extensions of concepts.’ (Frege 1964, p. 3 -4) Here he does express confidence that the concept of course of values might be enunciated more clearly, and that, when it is, disputes about Basic Law V will be settled. His attitude towards the very different kind of problem later pointed out by Bertrand Russell was not one of confidence in overcoming it, and I am not referring to Russell’s paradox when speaking of ‘little snags’. Frege addressed Russell’s paradox in Appendix II to volume II of the Grundgesetze .

The thesis I will put forth in this paper --- that, ultimately, Frege came to the view that 'ideally, one sentence, one thought; one thought, one sentence' --- bears on questions about how to explain and understand Frege's criticisms of formalism in arithmetic and Hilbert's use of uninterpreted statements and implicit definition. These criticisms Frege made of his contemporaries are not ad hoc reactions or piece meal bits of philosophy, but reflect a more unified view about the relation between sentences and thoughts that was slowly being clarified as he tried to attain his original goals of putting mathematics on a solid foundation. The view about the relation between sentences and thoughts that developed over time was as much a result of frustrations encountered as of initial convictions vindicated, and one that he came to accept as practically necessary over time, rather than one chosen as a starting point.

## **2. Language and Logic**

According to Frege, it is the thought that matters -- at least to the logician. Sentences *express* thoughts, he said (Frege 1918, 328). But he also found it frustrating to have to use sentences to communicate thoughts. He lamented that he could not put a thought in the hands of his readers as a mineralogist might put a rock-crystal in the hands of audience members. 'One fights against language,' he wrote in a footnote to his essay 'Thought', 'and I am compelled to occupy myself with language although it is not my proper concern here.' (Frege 1918, 329-330) According to him, the logician is concerned only with the thought expressed by the sentence. The thought, however, cannot be handled on its own, it can only be dealt with as wrapped in a linguistic form.

If sentences express thoughts, then what is the problem? Twofold: sometimes --- often, in fact --- (i) the content of a sentence goes beyond the thought it expresses. Sometimes 'the opposite' happens, instead : (ii) the thought expressed goes beyond

the content of a sentence; the 'mere wording. . . does not suffice for the expression of the thought". (Frege 1918; 331)

The examples Frege gave of sentences in natural language in which the content of the sentence goes beyond the thought expressed by it ( i.e., of case (i) above) consisted of pairs of sentences that expressed the same thought, although one sentence in the pair was a transformed version of the other. Although the sentences expressed the same thought, one of the sentences had a content that the other did not. An example is the pair: 'Alfred has not come.' and 'Alfred has not yet come.' -- the latter sentence differs from the former by the addition of the word 'yet' and it creates expectations in the hearer that the first does not. Some other examples of transforming one sentence into another that expresses the same thought but has a different content are: (a) replacing 'but' for 'and' in a sentence, (b) adding 'still' or 'already' to emphasize part of a sentence, and (c) changing the verb from active to passive and the accusative into the subject.

In all these examples of case (i), the difference between the original and transformed sentences may not be trivial from the standpoint of what the hearer comes to understand or expect upon hearing it, so the *sentences* may well be said to differ. But, according to Frege's own remarks on such examples, the original and transformed sentence do express the *same thought*. This is because these transformations 'do not touch the thought, they do not touch what is true or false.'(Frege 1918; 331) Logical relationships are relationships between thoughts; the relationships between thoughts in which Frege was interested were relationships of one thought's dependence on another for justification. So, in the context of an endeavor in which we are concerned only with relationships between thoughts, rather than with expectations created in the hearer, we are not concerned with variations on a given sentence that do not affect its truth value. This point is made by saying that

the logician is not concerned with the difference between two such sentences.

'Alfred has still not come.', says Frege, is *not false* even if Alfred's arrival is not expected. 'Alfred has still not come.' is a different *sentence* than 'Alfred has not come.', but the two sentences express the same thought. (Frege 1918; 331) That is because a thought, for Frege, 'is something for which the question of truth can arise at all.' (Beany 1997, p. 328) That different sentences of our natural language can express the same thought is no problem for the logician: the logician just doesn't distinguish between them. In the Begriffsschrift, Frege had explained the purpose of his logical symbolism by comparing it to a specialized mechanical aid for seeing: a microscope. The comparison was meant to emphasize that logical symbolism is designed for a special purpose, as is a microscope. Logic is as poor a tool for capturing all the distinctions important to understanding conversation and poetry as is a microscope for viewing a landscape. The point here is that one must be modest about the aims of a particular logical symbolism. Or, rather, one must be clear about its purpose. For purposes of capturing what is relevant to the kind of content of a sentence of interest in doing logic, sentences that express the same thought should not be distinguished.<sup>3</sup> Let's call this activity --- the activity of, for the purposes of logic, identifying sentences that express the same thought --- *pruning* sentences.

What about the opposite situation, i.e., case (ii), when the thought expressed goes *beyond* the content of a sentence? To Frege, the task is clear: since logic is concerned only with thoughts, we need to augment such a sentence, so that the thought determined by the sentence is unique. The kind of sentence that logic is

<sup>3</sup> In a 1906 letter to Husserl, Frege wrote that, while it is not possible to say exactly when two propositions are merely equipollent and when they are congruent, this is not an obstacle in principle: 'All that would be needed would be a single standard proposition for each system of equipollent propositions, and any thought could be communicated by such a standard proposition. For given a standard proposition everyone would have the whole system of equipollent propositions, and he could make the transition to any one of them whose illumination was particularly to his taste.'(p. 303 in Beany (1997))

concerned with is the kind that expresses a thought. Case (i) was the case in which different sentences may express the same thought --- this the logician tolerates by not distinguishing between the various sentences, and perhaps selecting one as canonical and not using the others --- but case (ii) is not so easily accommodated, for it is intolerable that the same sentence should express different thoughts.

Hence, when a sentence contains indexicals (e.g., 'I', 'this', 'that', 'yesterday') or proper names (e.g., Dr. Lauben, Venus) the logician is in trouble if stuck with only the sentence to go on. Thus, Frege says that 'The words 'This tree is covered with green leaves' are not sufficient by themselves to constitute the expression of a thought, for the time of utterance is involved as well.' He continues: 'Only a sentence with the time-specification filled out, a sentence complete in every respect, expresses a thought.' (Frege 1918; 343) Let's call this activity *extending* a sentence, on analogy with a gardener's activity in doing the opposite of pruning a small offshoot -- here, each of a set of multiple offshoots is encouraged to grow and extend itself to become a distinguishable branch in its own right.

What, then, he asks, should a logician make of a sentence such as 'Dr Lauben was wounded'? This sentence expresses different thoughts to different people, depending upon the meaning each associates with 'Dr Lauben', which in turn may depend upon whether they are acquainted with him and what they know about him. Frege gives a long example in which various people associate different definite descriptions with the proper name 'Dr. Lauben', and are in different states of ignorance or knowledge about whether people with whom they are acquainted fit those descriptions. A fellow named Leo and a fellow named Rudolph both hear Dr. Lauben say aloud 'I was wounded.' Later, Rudolph hears Leo report aloud: 'Dr. Lauben was wounded.' Whether or not the statement made by Dr. Lauben and the statement made by Leo express the same thought to Rudolph is going to depend upon whether or not Rudolph knew that the

man he heard saying 'I was wounded' was Dr. Lauben. The point of these examples is that 'Dr. Lauben' is a proper name, but there may be different modes of determining the man to whom it refers ('the way that the object so designated is presented'). The logician does need to take these differences into account, Frege says, for different modes of determination for 'Dr Lauben' will result in different thoughts being expressed by the sentence 'Dr. Lauben was wounded'. (Frege 1918: 333)

Here, the problem is not a matter of difference in truth value of the different thoughts, for, says Frege, either the thoughts expressed by the sentence are all true or the thoughts expressed by it are all false. The problem is that knowledge of the truth of these thoughts can differ due to different hearers' mode of determination of the person to whom a proper name refers, and this indicates that the thoughts *are* different. In 'Thought', Frege addresses this kind of case --- i.e., the kind of case wherein the same sentence can be used to express different thoughts --- by adding a restriction on sentences that will be permitted in a logical treatment of any topic involving proper names. The restriction is this: *restrict the meaning (sense) of proper names so that no sentence expresses more than one thought*. Let's call this kind of activity *extending* a sentence, too, for, as in the other examples of case (ii), it is analogous to dealing with multiple offshoots by encouraging each offshoot to take its own shape, and so distinguishing each offshoot from each other. However, we do not ever use more than one proper name for an individual -- we may have multiple modes of determination that happen to determine the same object, but no proper name has as its meaning more than one mode of determination. In Frege's words: 'So we must really stipulate that for every proper name there shall be just one associated manner of presentation of the object so designated. It is often unimportant that this stipulation should be fulfilled, but not always.'(Frege 1918; 333)

### 3. Natural Language and the Formal Garden of Propositions

Thus, Frege requires that the sentences of one's natural language that are the concern of logic be in some cases extended (distinguished from each other) and in some cases pruned (identified with each other), so that the relationships that hold between the resulting sentences -- sentences the logician can, so to speak, hold in his hand and show to his audience --- express the relationships that hold between the thoughts they express. As described in the previous section, sentences that express the same thought are not distinguished from each other (metaphorically, the several branches are pruned down to a single branch). Sentences that do not determine exactly one thought are extended (so that they determine only one thought) or disambiguated such that several sentences, each of which determines exactly one thought, are obtained. The result is that, for the pile of sentences with which the logician deigns to work, each such sentence expresses exactly one thought, each thought is expressed by exactly one such sentence, and the relation of consequence between such sentences expresses the relation of consequence between the thoughts they express. Of course this is not true for all the sentences of one's *natural* language -- the point is that it is true of all the sentences the logician is working with after extending and pruning them per the prescriptions just described. Frege eventually came to see such prescriptions as necessary.

We can call the items that result from this process propositions, once they meet such prescriptions; it is irrelevant whether or not the resulting items also happen to be sentences of a natural language. In the Begriffsschrift, in explaining the value of the notation he introduced as a replacement for subject-predicate form, Frege said the symbolism he was presenting was a useful tool, if the task of philosophy was to 'break the power of words over the human mind' and to free thought 'from the taint of

ordinary linguistic means of expression.' (in Beany 1977; 51)

Some readers may take issue with the point just made above that for the pile of sentences with which the logician deigns to work, each thought is expressed by exactly one sentence, each sentence expresses exactly one thought, and the relation of consequence between sentences expresses the relation of consequence between the thoughts they express. I am well aware that not everyone who has encountered Frege's writings has the impression that Frege avoids the situation wherein a thought is expressed by more than one sentence. Nevertheless this is what Frege says in 'Thought'. He wrote 'Thought' over 25 years after writing the much-emphasized and more widely studied 'On Sense and Reference' and almost 40 years<sup>4</sup> after the publication of Begriffsschrift, the work in which he introduced the formalism suitable for doing arithmetic in a 'calculus of pure thought.' In the Begriffsschrift (which predated a distinction he later drew between sense and reference), he did begin to lay out a view that was later revised. As I see it, the vision and ideal he had are not rejected, but rather are better realized, in the view he later laid out in 'Thought.' In 'Thought' he explains more fully, and with examples, the process that I have referred to as the extending and pruning of sentences in the natural language required to obtain the kind of propositions that are fitting for the study of logic.

There's a similar progression in Frege's work concerning his attitude towards the relation between sentences and the thoughts they express. Frege's break with the traditional subject-predicate form of his predecessors, which he discusses in the Begriffsschrift, is accompanied by the statement in that early work that this break with tradition is warranted, 'that logic hitherto has always followed ordinary language and grammar too closely.'(in Beany (1977); 51) In the much later 'Thought', Frege writes

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<sup>4</sup> The Begriffsschrift (Concept-Script) was published in 1879; Grundlagen der Arithmetik (Foundations of Arithmetic) in 1884, 'On Sense and Reference' in 1892, and 'Thought' in 1918. 'On the Foundations of Geometry' and associated correspondence with Hilbert and others was written around 1900.

that, although he is not in the 'happy position' of the mineralogist who can exhibit the gem he is talking about, he is resolved to a kind of resentful contentment: 'Something in itself not perceptible by sense, the thought, is presented to the reader --- and I must be content with that --- wrapped up in a perceptible linguistic form.' It is not, however, a totally peaceful contentment: 'The pictorial aspect of language presents difficulties. The sensible always breaks in and makes expressions pictorial and so improper.' (Frege 1918; 334) The contentment he has achieved is the serenity of accepting what he cannot change.

Frege's explanation of this point --- that there are differences between the *linguistic forms* one needs in natural language (where sentences have additional functions not relevant to logic, such as the function of generating expectations in a hearer that enable conversations to be carried on effectively, and the function of generating ideational associations), and the *logical forms* one needs to establish the truths of arithmetic --- also illuminates his critique of Hilbert. For, once one sees the view he expresses in 'Thought' about the relationship between sentences and the thoughts they express as the view he was in the process of working towards when he responded to Hilbert's *Foundations of Geometry*, Frege's response to Hilbert's formalization of geometrical axioms seems quite natural.

Hilbert's axioms of geometry were (what we would call) uninterpreted: they were neither true nor false, until they received an interpretation. Frege's complaint was that the notion of an interpretation of a proposition was fundamentally incompatible with the notion of proposition required to do logic. It's easy to see why he thought so: logical relations hold between thoughts. A proposition --- the kind of extended and pruned sentence logicians deal with --- expresses a thought, and only one thought. On this view, the notion of interpretation has no place in logic.

In his correspondence with Hilbert, Frege wrote that 'one feels the broad,

imperspicuous and imprecise character of word language to be an obstacle, and to remedy this, one creates a sign language in which the investigation can be conducted in a more perspicuous way and with more precision.' He used a slightly different horticultural metaphor, the process of lignification, to illustrate a point about symbolism: Instituting a new symbolism is like the tree's new growth hardening --- *after* it has had a chance to take on the shape appropriate to performing its function. Then, additional new growth depends upon those hardened sections to support the delivery of nutrients to the newly forming branchtips. The point is that trees do not grow into a predetermined suit of armour made of bark. The rigidity provided by the bark comes only after new branch tips have had a chance to grow in a natural formation. The sign language of a science is not set independently of inquiring as to what signs are best suited to it; if developed appropriately, these signs can be used to hook imperceptible thoughts and wrap them in a perceptible form so there is something that can be held in one's hand, so to speak, and worked with. Signs always involve a compromise compared to what one wishes to communicate, for, after all, signs are perceptible and the thoughts they express are not. It is fundamental to Frege's view that having the right formalism available is important to being able to capture the kinds of imperceptible thoughts in which one is interested. Frege's remark to Hilbert that the need for symbolism comes first, and only later the satisfaction of that need, reflects this conviction. In correspondence, Hilbert expressed agreement with this last statement. (Hilbert 4.10.1895 in Frege (1980); 34)

Hilbert used axioms as implicit definitions of the concepts contained in them, though, and Frege didn't like that anymore than he liked the fact that Hilbert's axioms required interpretation in order to express a thought. However, as critical as Frege might have seemed of Hilbert, he did evaluate Hilbert's formalization of geometry with the idea of showing how one might achieve what Hilbert was after in a proper manner.

In fact, he outlined a way to make sense of Hilbert's method of showing axioms independent of each other. Frege's reconstruction of Hilbert's independence proofs, however, only work for (what Frege called) real propositions, which Hilbert's axioms were not. (Sterrett 1994)

Frege's method works as follows: one maps ('set(s) up a correspondence between') words of a language (in which, of course, the reference of every word is fully determinate) onto other words of the same language, subject to some restrictions. These restrictions include mapping proper names to proper names, concept-words to concept-words of the same level, and so on. The signs whose references belong to logic (e.g., negation, identity, subsumption, and subordination of concepts) are not mapped to different signs. Then, one can show that a thought  $G$  is independent of a group of thoughts  $\beta$ , if one can obtain from  $\beta$  and  $G$ , respectively, a map to a group of true thoughts  $\beta'$  and a false thought  $G'$ . In Sterrett 1994 I argued that this was in fact somewhat like the approach Hilbert actually took, and so it was striking that Frege distinguishes his method from methods that employ interpretations of statements. The significance of the difference between Hilbert and Frege, I concluded there, had to do with differences in their accounts of how words come to mean what they do. I will not repeat that discussion here, as it is readily available elsewhere.<sup>5</sup>

It should be clear by now that Frege is not drawing a distinction between referring to a thought and referring to the perceptible linguistic form in which the thought is wrapped. Frege's point was that the only way he's got to show anyone what thought

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<sup>5</sup> Giving a brief description of the contrast between Frege's account of elucidation and Hilbert's account of implicit definition risks mischaracterizing Hilbert as more formalist than he was, so I refer the reader to my discussion in Sterrett (1994). In Sterrett (1994) I distinguish the positions of Hilbert, Korselt (who responded to Frege on Hilbert's behalf), and Frege. The paper is available free online at the Philosophy of Science Archives server, at <http://philsci-archive.pitt.edu/>

he is referring to is by wrapping it in a perceptible linguistic form. Hence in talking of the thought G' to which G is mapped (via the mapping of words of a language as outlined above), Frege can hardly be talking about making substitutions of words in, and obtaining transformations of, anything other than sentences. Not just any old sentences of a natural language, however. These sentences or propositions are the result of extending and pruning sentences of the natural language so that each proposition expresses one and only one thought, and so that propositions that express the same thought are not distinguished from each other. I use the term 'proposition' here because Frege isn't including all sentences of natural language. He doesn't talk about the forms he has to wrap thoughts in other than as the forms in which the thoughts are wrapped; these forms are not self-subsistent. He was certainly against the idea of developing symbolic forms first and then looking for thoughts that might fit into them. And I don't think he ever meant to talk about these symbolic forms other than as used to express thoughts.

Thus, for Frege, the notion of logical consequence arises for relationships between the imperceptible thoughts that are wrapped up in perceptible forms, not to the forms of the wrappers themselves. One cannot communicate thoughts except by capturing them in such a perceptible wrapping, so proofs and derivations proceed by way of rules that apply to propositions or statements. However, these propositions always express a thought: they are never empty wrappers. They are not in need of interpretation.

#### **4. Departed Thoughts**

Hence, Frege says that if by sentence is meant the 'external, audible, or visible that is

supposed to express a thought”, then it does not make any sense to say that one sentence is independent of another. The context in which Frege wrote this was in arguing that Hilbert had erred in the specific way he had gone about trying to establish the independence of the parallel postulate from the other axioms of geometry. It was in this context that Frege said that Hilbert makes a mistake in calling anything 'the axiom of parallels”, for, as Frege put it in the passage quoted above, it is not the same in every geometry: 'Only the wording is the same; the thought-content is different in each particular geometry.’<sup>6</sup>

Frege means here to warn against mistaking the 'external, audible, or visible that is supposed to express a thought' for the thought . Logic is concerned with thoughts and how they are related to each other. So there is a realm of thought: it cannot be perceived by the senses but it is like perceptible things in that it does not need an owner, as ideas do. (Frege 1918, 337) Frege’s favorite example of a thought in his essay entitled 'Thought' is the Pythagorean theorem. Different people can grasp the thought, and it can be communicated by wrapping it in a perceptible linguistic form. But I don’t think Frege intends to alert the reader to the existence of a logical calculus of the 'the external, audible, or visible that is supposed to express a thought.' This would be a study of the relationships of linguistic forms, something Frege thought of interest for many purposes -- understanding conversations and writing poetry, for instance --- but decidedly not the subject matter of logic. Logic is about thoughts, it is about the laws of thought, the laws of the laws of science. It is about deriving proofs so that we can see how one thought depends upon another. It involves the linguistic forms in which these thoughts must be wrapped in order to be communicated, but only

<sup>6</sup> Frege’s analysis is that the fault lies in confounding first- and second- level concepts, such as the concept of point. There may be different first-level concepts of point, under which points fall: the Euclidean point-concept is one such first-level concept. If one likes, one may also define a second-level concept, within which the Euclidean point-concept and other first-level concepts, fall. A fuller discussion of Frege’s point is given in Sterrett 1994, page 9.

in the context of investigating which thoughts depend upon which other thoughts. It is not about relationships of dependence between perceptible linguistic forms. If there are such things as forms that exist as shed snakeskins left behind from departed thoughts, they are not the concern of logic; they are not the items of a calculus of pure thought.

For Frege, there is no such thing as a realm of linguistic forms within which no thoughts are wrapped but which are related to each other in virtue of their form by logical laws. The logical relations are not logical relations between linguistic forms.

## **5. The unity of thought and expression**

Frege did discuss examples of different sentences that expressed the same thought, even in 'Thought', arguing that 'the content of a sentence often goes beyond the thought expressed in it.' But, his response to this observation was not to posit a new kind of logical law or a new kind of logical relation to account for how such sentences were related. In 'Thought', he did not regard such situations as puzzles; he did not then consider them relevant to logic. Rather, his response to this observation about natural language was that the logician does not distinguish between such sentences.

Was Frege this blasé about different sentences that express the same thought because, on a view sometimes attributed to Frege, he thought that there are really two distinct things, sentences and thoughts, and thus that the distinction between sentences is a distinction that can be made only in the realm of what is derivable, and not in the realm of what is provable? I don't think that this is how Frege's views on sentences that express the same thought ought to be viewed.

Recall that what Frege said about pairs of distinct sentences that express the same thought was only that some such transformations between sentences must be

recognized as admissible. But this wasn't a matter of recognizing relationships that obtain in a realm of equipollent propositions. In his 1906 letter to Husserl, in fact, Frege suggested that equipollent propositions could all be communicated by a single standard proposition. (in Beany 1997, 302) In closing the letter, he remarks that the question of whether equipollent propositions are congruent 'could well be debated for a hundred years or more.' But he isn't concerned about the answer; he writes 'I do not see what criterion would allow us to decide this question objectively. . . . But I do find that if there is no objective criterion for answering a question, then the question has no place at all in science.' (in Beany 1997; 305) Placing significance on the difference in the relations that hold between sentences and the relations that hold between thoughts is attributing significance to exactly what, I think, he actually said ought to be de-emphasized.

We have seen that what Frege said about sentence transformations that do not affect the thought expressed was that sentences with differences that don't affect the thought expressed don't need to be distinguished when doing logic. All that the existence of transformations that yield two or more sentences expressing the same thought means to the logician is that, if propositions or statements admit of such transformations, one must recognize as admissible those transformations that do not affect the thought expressed. Once we see this point of Frege's, the apparition of the notion of derivability according to which things are not always as they seem disappears: i.e., the notion of derivability on which a thought when wrapped in a different wrapper might have different derivability relations disappears. There is a realm of thoughts (thoughts are not the property of individuals as ideas are, but they are not perceptible either), and it is distinct from the realm of perceptible things. (Frege 1918; 337) Logical laws are used in showing the relationships that exist between thoughts, via a proof. Thus, the realm in which logical rules apply involves both of

these realms, since it includes both thoughts and signs; and this in turn is due to the unavoidable situation that communication of thoughts requires that thoughts be wrapped in perceptible forms.

What about Frege's statement that a thought can be 'carved up' in different ways? Doesn't the fact that the same thought could be carved up in different ways mean that the same thought could be expressed by different sentences? Yes and No -- the difference being a matter of which language you are talking about. In natural language: Yes, the same thought can be expressed by different sentences that analyze the thought into subject and predicate differently; typically this will happen whenever the same sentence is transformed from the active to the passive voice. But in the formal language of the Begriffsschrift, the answer is No: the carving really captures the structure of the thought relevant to the kind of inferences one wants to be able to draw. That is, the whole point of the Begriffsschrift was that subject-predicate logic did not get at the structure of thought relevant to making inferences! In contrast, the formalism of the Begriffsschrift was created to ensure that all of the structure relevant to making inferences that were a matter of pure logic could be expressed. The point that the situation of having only subject-predicate logic available is restrictive in spite of allowing many options might be explained using the metaphor of a plant, as follows: that situation is like having only a certain kind of analysis of the plant available to you, for instance, having only the option of describing a plant in terms of dividing it up into the edible food it bears and the part of the plant that produces the edible food. What's limiting about this is not a matter of how many ways there are to carve up the plant, for in fact the edible-food and plant-that-produces-food way of carving up a plant permits many different ways of carving up the plant. Depending upon what part of the plant a creature is interested in consuming, one could analyze the plant into an edible product and the remainder of the plant that produces it in

different ways, just as the subject-predicate form allows one to express a thought in different ways depending upon what one chooses as the subject of the sentence. Rather (using the plant metaphor) the limitation is this: the available ways of analyzing the plant does not necessarily allow us to analyze the plant structure in the way required for investigations in natural science. Analyses of a plant based on edible parts of the plant does exhibit something about the structure of the plant, of course, but it also obscures some of the structure of the plant. What we want is a general method of carving up the plant in a way that allows the flexibility and precision to exhibit various kinds of structure in the plant, a way that permits the many different kinds of carving ups of the plant needed for making inferences we want to draw to conduct research about a variety of questions that interest us.

On this analogy, what's wrong with subject-predicate logic is that the kinds of 'carving up' of a thought it permits -- and it may permit a number of alternatives -- might not include the structure of the thought that is relevant to making the kinds of inferences in which one is interested. In contrast, the formalism of the Begriffsschrift , in which concepts are modelled on functions, is meant to introduce a formal language in which one can carve a thought in any way needed for making scientific inferences. The formalism provided in that work is supposed to be enough to permit making any inferences that are a matter of pure logic. This is not to say that the kind of structure sought for even when using the formalism of the Begriffsschrift may not be relative to the kinds of inferences one is interested in making (hence the formalism needed for chemistry and physics is left open in the Begriffsschrift; in my biological metaphor, the added formalism needed to carve the plant into its relevant parts might be the gene concept). It is to say that the formal language does not, as subject-predicate logic does, limit one to carving the plant into two parts according to a criterion that may never permit one to delineate the structure of the plant relevant to the inference in

which one is interested.

The advance Frege offered was not was a way of dissecting of a thought into formalism and unformed thought, but, rather, consisted in a formalism that permitted carving thoughts in more useful ways than previous formalisms allowed. The separation of thought from sentence underlying the distinction between provability and derivability is not something we find in Frege. To describe such a disconnect as part of Frege's view misdescribes Frege's notion of a proposition in the same way that Aristotle's notion of form would be misdescribed by using Plato's notion of form. That is, in Plato's philosophy, forms exist in a realm separate from the things of which they are forms. Aristotle, too, used a metaphor from biology to break from Plato: that there are male and female animals, he said, does not imply that male and female exists as something separable from male and female animals.

To use another metaphor: in a certain science fiction television series, there is a creature that can transform itself into various shapes, called a shape-shifter. These shape-shifters can separate from their shapes and meld together somehow in a realm in which they are shapeless. But this is, after all, fiction. To make the metaphor of shape-shifters who take on various shapes fit Frege's account of sentences as the forms within which thoughts are wrapped, let us leave the details of this particular science fiction story behind and stipulate that shape-shifters take on human forms, that a given shape-shifter cannot take on every form and, in fact, that the forms a particular shape-shifter takes on are not taken on by any other creature. (This corresponds to Frege's requirements that, in his formalism, thoughts are expressed by sentences, that more than one thought is expressible, and that no sentence expresses more than one thought.) Clearly, once we've figured out the shapes between which a particular shape-shifter can transform itself, we no longer need distinguish between those shapes.

The analogy to thoughts and the linguistic forms they take on is this: just as, in the science fiction story, a creature is apprehended via the senses by its sensible form, so a thought is expressed via a sentence. That, in effect, is Frege's unperturbed response in his essay 'Thought' to the examples in which there are several sentences that express the same thought, such as two sentences that differ only in the manner used to designate an object. That is, in contrast to the view that Frege is saying that there are two *different* calculi, one for thoughts and one for the linguistic forms in which they can be wrapped, Frege shows that he intends to avoid such commitments by stipulating that, when proper names are used, only one manner of presentation (e.g., for Venus, either 'the morning star' or 'the evening star', but not both) be permitted. Thus I do not think that, as is often supposed, that Frege developed a calculus of sentences associated with something called derivability in addition to the calculus of thoughts associated with provability. His remark to Husserl (quoted earlier) that he does not think there is room in science for the question of whether equipollent propositions are congruent bears this out.

Looking back from the present, some people attribute to Frege's Begriffsschrift the achievement of having developed a calculus of sentences related by derivability, which are accurately described in modern parlance as syntactic relations. This is not so, and Frege is explicit enough about what he was doing to make that clear. Frege's Begriffsschrift was to be a calculus of thoughts. There were reasons that the calculus had to involve symbolic formalism -- to clarify thoughts, and to express them -- but the calculus was not a 'topic-neutral' calculus of symbolic or syntactic forms. That may be what a modern logician sees in looking at the Begriffsschrift, but it doesn't sound much like Frege's description of the Begriffsschrift. What it does sound like, however, is Frege's description of Leibniz's vision, which, he said, 'was too grandiose for the attempt to realize it to go further than the bare preliminaries.'(in Beany, p. 50) Frege

thought Leibniz's vision of a universal calculus an excellent guiding vision, but what he said about his own achievement in the Begriffsschrift with respect to Leibniz's visionary aim was that 'even if this great aim cannot be achieved at the first attempt, one need not despair of a slow, step by step approach.' The project, Frege said, 'has to be limited provisionally' at first. And he identified the Begriffsschrift as one of the 'realizations of the Leibnizian conception in particular fields.' (50) He spoke of additions that would have to be made to extend it to geometry and then to the pure theory of motion, then mechanics, and then physics. These latter fields involve natural necessity as well as conceptual necessity.

In his correspondence with Hilbert, Frege writes that he thinks Hilbert is (mistakenly) treating geometry as if it were like arithmetic. Frege thought it an error to regard geometrical knowledge as having the same kind of basis as arithmetical knowledge. This is important, for it meant that Frege didn't think sentences or propositions of geometry were related to each other in the same way that statements of arithmetic were. The Begriffsschrift was to help in showing that arithmetical truths were truths of logic, but even this does not mean that the rules in the Begriffsschrift applied to topic-neutral sentences, for Frege did not take a formalist approach to arithmetic either. What I mean by this is that he did not allow (what we would now call) uninterpreted statements of arithmetic anymore than he did statements of geometry. In the Basic Laws of Arithmetic, he reiterates his requirement on axioms, i.e., that all the terms in them must be defined. That he is not always able to meet the requirement should not be cited as evidence that some of the concepts are implicitly defined or are uninterpreted and to be interpreted at a later date. Rather, Frege's explanation of such undefined concepts is found in a statement he makes in preliminary remarks in the Basic Laws: 'It will not always be possible to give a regular definition of everything, precisely because our endeavor must be to trace our way back to what is logically

simple, which as such is not properly definable. I must then be satisfied with indicating what I intend by means of hints.' ( Frege 1964; 32) The principles of the Begriffsschrift may apply to every science, but according to Frege they do not include all the principles nor, even, all the formalism needed to do geometry, kinematics, physics, or chemistry. These await future development, he said.

Thus, we must avoid the anachronism of splitting asunder a propositional form from a thought. For Frege, a proposition is a thought wrapped in a perceptible linguistic form, i.e., a propositional form. The perceptible linguistic form it is possible to wrap a thought in may not be uniquely determined for a given thought, but the thought must be wrapped in some perceptible linguistic form or other. Hence the proposition cannot survive such a dissection. Even in developing a calculus in which the ideal is 'one proposition, one thought; one thought, one proposition', a thought and its expression are not split apart. Throughout his correspondence with Hilbert, Frege seems concerned to speak of the proposition as a whole, i.e., a 'real' proposition expressing a thought. The kind of axioms Hilbert proposed, which were neither true nor false, and so which, on Frege's view, did not express thoughts, were not , on his view, proper subjects of logic.

On Frege's view, a thought is necessarily wrapped in linguistic form if it is to be communicated, studied, or used in reasoning. Thoughts are individuals for Frege, somewhat as trees and humans were individuals for Aristotle. Aristotle was concerned (at least in some of his works ) to hold out for the identity of an individual in spite of the different things that could be predicated of it, but in a way that didn't call for dissecting that individual into a self-subsistent form and something else. Similarly, what Frege thought was called for with respect to thoughts was a method of expressing an individual thought that exhibited the structure of the thought in such a way that we could see its relation to other thoughts, but in a way that didn't call for dissecting it into

a self-subsistent linguistic part and something else. Frege also seemed to recognize different *kinds* of relations between thoughts, that the relations that were crucial might be different for different investigations and different disciplines. The Begriffsschrift was meant to provide a calculus in which to express thoughts that met the needs of the discipline of logic, i.e., a calculus in which the logical relations between thoughts would be exhibited. Frege continually warned against the tendencies of some of his contemporaries to take the approach of attempting to separate the propositional form of a proposition from the thought it expresses and treat it as self-subsistent. The admonition to refrain from attempting such fatal dissections, though, is quite general. It is as old as Aristotle and as new as post-analytic philosophy.

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## REFERENCES

Antonelli, Aldo and Robert May ( 2000) Frege's new science. Notre Dame Journal of Formal Logic, 41, No. 3.

Beany, Michael, ed. (1997) The Frege Reader. Oxford: Blackwell Publishers.

Blanchette, Patricia. 1996: Frege and Hilbert on consistency. Journal of Philosophy 93, 317-336

Frege (1900a) Frege to Hilbert 6.1.1900. In Frege (1980), pgs. 43-48.

Frege, Gottlob 1918: Thought. In Beany (1997), pgs. 325-345.

Frege, Gottlob. 1964: The Basic Laws of Arithmetic. Exposition of the System. Translated and edited, with an introduction, by Montgomery Furth. Berkeley: University of California Press.

Frege, Gottlob. 1980: Philosophical and Mathematical Correspondence. Edited by Brian McGuinness and Translated by Hans Kaal. Chicago: University of Chicago Press.

Resnik, Michael. 1994: The Frege-Hilbert controversy', Philosophy and Phenomenological Research, 34, 386 - 403.

Ricketts, Thomas 1997: Frege's 1906 foray into metalogic. Philosophical Topics, 25, 169-188.

Sterrett, S. G. 1994: Frege and Hilbert on the foundations of geometry. Available on the Philosophy of Science e-print archives ( <http://philsci-archive.pitt.edu/> )

Tappenden, Jamie. 2001: Frege on axioms, indirect proof, and independence arguments in geometry: Did Frege reject independence arguments? Notre Dame Journal of Formal Logic , 41: 271- 315.