# Introduction

The scientific realist claims that the physical sciences provide, or aim to provide, a true description of the underlying reality behind the manifest world of experience.[1] But much of physical theory is expressed in the language of mathematics, and if any form of scientific realism is to be grounded in a mathematical science, it is essential that the realist provide an account of how mathematics is applied to the physical world. Philosophers of applied mathematics often explicate the use of mathematics in the physical sciences in terms of the concept of a representation. In the popular "mapping account" of applied mathematics, it is argued that we use mathematics to represent certain physical structures (Brown, 1999 and 2012; Pincock, 2007 and 2012). The basic idea is that we identify a physical structure in the world and then map it onto the appropriate mathematical structure within our scientific theories (Brown, 2012; 6-7, and Pincock, 2012; 27-29). Formally, a representation occurs when a morphism can be specified between a relational system in the physical world and a mathematical structure. Based on the popularity of the mapping account of applied mathematics, it worth taking the time to see if this approach can provide a viable foundation for scientific realism. At first glance, scientific realism and the mapping account appear to be a match made in heaven. Following the mapping account, the realist would be able to suggest that mathematics is successfully applied when the relations that hold within a physical system are correlated with the appropriate mathematical structure.[2] However, the mapping account of applied mathematics has met with wide-ranging criticism (e.g. van Fraassen, 2008; Batterman, 2008; Bueno and Colyvan, 2011; and Berkovitz, 2015). For the realist, the most pressing concerns with the mapping account pertain to how a physical structure is identified and represented as a mathematical structure.

The mapping account is appealing to the scientific realist specifically because it is a variant of the copy theory of representation. In the copy theory of representation, we rep-

[1] Alternatively, this sentence could be made compatible with the usual concessions to approximate truth.

[2] Pincock notes that this is the condition for the successful application of mathematics within the mapping account (Pincock, 2012; 28).

resent an object (or physical relation) by copying it, or an aspect of it, onto the intended representation. Such an account would allow the scientific realist to refer to the mathematical structure of a scientific theory as a copy of the physical structure in the world. However, any copy theory of representation is subject to Goodman's (1976) criticism. Goodman notes that the copy theory of representation is "stopped at the start" by an inability to identify exactly what is being copied by the representation relation (Goodman, 1976; 9). In Goodman's view, we do not copy the object or relation itself, but rather how the object or relation is conceived. When we conceive of an object or relation, we construe or interpret it and "[i]n representing an object [or relation], we do not copy such a construal or interpretation–we achieve it" (Goodman, 1976: 9).[3] The problem is that the world does not come 'carved at its joints'.[4] Rather, the joints are constructs of our conceptual systems, i.e. theories.[5] Although Goodman's general philosophical position is controversial, his point is clear in the case of theoretical physics, where the "physical structure" that is being represented is not readily apparent. In fact, the "physical structure" itself has to be constructed out of a mathematical theory of the world. The constitutive role that mathematics plays in the physical sciences presents a serious problem for the mapping account of applied mathematics. If mathematics is applied in the construction of our physical conception of the world, then it has certainly overstepped the boundaries of the copy theory of representation. Rather, representation becomes essential to the very construction of the physical structure that is at the foundation of our scientific theories.

The same issue can be viewed from another perspective. At the heart of the mapping account lies a relation that maps a physical structure onto a mathematical structure. This relation is defined as a morphism, which is a mathematical relation.[6] The problem is that a

[3]The insertion of the phrase "or relation" is supported by Goodman's footnote on page 5.

[4]This issue has been recently addressed in the context of the mapping account by Beuno and Colyvan (2011).

[5]See, for instance, Cassirer 1923, Duhem 1954, Goodman 1976 and 1978, Putnam 1987, and van Fraassen 2008.

[6]Alternatively, we could take the representation relation to be a primitive and leave it unanalyzed. There are important non-reductionist accounts of mathematical representation, for instance Suarez 2010, but a discussion of these cases would take us too far afield.

morphism is defined as a structure preserving map, or function, from one domain of mathematical structure to another, and van Fraassen correctly notes that "to define a function we need to have the domain and range identified first–and the question at issue [is] precisely how that can be done without presupposing that we already have a physical-mathematical relation on hand" (van Fraassen, 2008: 120).[7] If the definition of a morphism requires that a mathematical structure be defined on a physical relation, so that it can be representable in the mapping account, than we are faced with a dilemma: either the mapping account fails to account for applied mathematics, or the initial mathematization of the world is somehow already present. Berkovitz (2015) argues that the mapping account implicitly assumes that physical structure is mathematical, in a neo-Kantian or Pythagorean sense. If we ignore the problematic Pythagorean option,[8] we are once again led into a consideration of how mathematics is initially applied in the construction of our physical conception of the world.

If the scientific realist wants to base a theory of applied mathematics on the popular mapping account, then they need to clarify how mathematical concepts are brought to bear on the construction of our physical conception of the world.[9] As with any problem of conception, the realist needs to pay attention to where the points of convention lie. When we formulate certain scientific theories, especially in theoretical physics, mathematics plays an integral role in both the *definition* and *relation* of scientific concepts. The *definitional* role of mathematics delimits the domain of study by imposing a mathematical structure on the world. However, the *relational* role of mathematics provides the governing structure on this domain. The relationship between the definitional and relational roles of mathematics in complicated by the fact that mathematical concepts do not come free of charge. Implicit in

[7]The word "was" was substituted for "is" to reflect the tense of the discussion.

[8]If the Pythagorean view is associated with a naturalistic view of mathematics then it is subject to Brown's (2012) criticism and any rationalistic Pythagorean view seems to either collapse into the neo-Kantian view, or rely on an unaccounted for insight that borders on the mystical.

[9]This concern becomes more pressing when we consider whether or not a mapping-like account of applied mathematics is essential to any form of scientific realism. In the widely influential semantic account, scientific theories are thought to present structures or models that can be used to represent physical systems (Ladyman, 1998; 416). Any such account must clarify how mathematical models represent physical structures and it is difficult to see how the realist can account for the relation between a model and the world without either assuming that the representation relation is primitive, or presenting a mapping-like account of the representation.

their definition is a set of constraints that limit the types of physical structures to which they can be applied. These constraints are a direct result of certain assumptions that concern the underlying relations between the basic elements of a mathematical theory. Applying a particular mathematical concept to the physical world then entails that the basic assumptions in the underlying mathematical structure, such that the concepts may be well-defined, are satisfied by the world. The structural constraints implicit in the mathematical concepts dictate the type of physical phenomena that the theory can accommodate (Morrison, 2000; 109).[10]

If we are to untangle the web of issues related to mathematical representation, it is best to look to scientific practice and consider how a given mathematical theory comes to be applied. This paper will shed light on the essential conceptual pre-structuring of the world inherent in the application of mathematics by presenting an analysis of the use of the differential calculus in physical theory.[11] Specifically, this paper will treat the supposedly simple application of the differential calculus in the modern definition of Newton's second law. The application of the differential calculus requires that the world be pre-structured mathematically. This pre-structuring constrains the form of the world as understood within Newtonian theory. The constraints are a direct result of the formulation of the mathematical structure of the differential calculus. Our focus on mathematical constraint will highlight the dual role that mathematics plays in the definition and relation of physical concepts. The constraints imposed by the use of the differential calculus fall squarely within the purview of the definitional role of mathematics and, as such, delimit the applicability of the mapping account. This focus on the definitional and relational characteristics of applied mathematics will also showcase the role of convention and draw attention to the viability of any form of scientific realism that is based on the mapping account.

The body of this paper is comprised of three sections. The first section will develop the conceptual foundation of the differential calculus and identify the pre-structuring of the

[10] But here "type" should indicate form rather than kind.

[11] Note that the use of the term 'pre-structuring' should not be taken as a temporal relation but rather a necessary conceptual pre-structuring in the logical sense.

world inherent in its application. The second section will present a basic definition of Newton's second law and a discussion of the constraints that the differential calculus imposes on the structure of the world as conceived within Newtonian physics. This section will conclude with a discussion of what we take to be the limits of scientific realism, as conceived under the umbrella of the mapping account of applied mathematics. Finally, the third section will present a case study of a famous thought experiment by John Norton, simply called 'the dome'. The pre-structuring of the world required by the differential calculus offers a firm foundation for the mapping account of applied mathematics, but it also precludes certain physical structures from being understood within the confines of any theory based on the differential calculus. The modern formulation of Newton's second law is such a theory. The dome thought experiment provides a nice example of a hypothetical physical structure that fails to meet the necessary conditions for the differential calculus to be well-defined. Therefore, this structure is excluded by the pre-structuring of the world inherent in the application of the differential calculus. On the basis of this argument, we suggest that Norton incorrectly claims that the dome demonstrates the indeterministic nature of Newton's second law. Newton's second law actually cannot be applied in the thought experiment. This case study was chosen because it demonstrates the inherent danger in assuming that mathematics can be applied in a world of arbitrary structure.

## The Conceptual Foundation of the Differential Calculus

The differential calculus plays an integral role in almost every theory of modern physics. In the modern formulation of Newtonian physics, it is constitutive of the very definition of motion. Formally, the differential calculus is applied to characterize the behaviour of a function in the infinitesimal neighbourhood of a point by providing a linear approximation to a function in that neighbourhood. But the differential calculus poses an interesting problem for any form of scientific realism based on mapping account of applied mathematics. The calculus cannot be applied to an arbitrary function, but only to functions of a specific form. Therefore, the calculus can only be applied within a physical conception of the world in which the world is structured in a particular way. This pre-structuring of the world is not accounted for in the mapping account of applied mathematics and is a clear example of the mathematical construction implicit the in application of mathematics in the physical sciences. Our treatment of the differential calculus will begin with the definition of the concept of a function, and trace its development through the concepts of approximation, continuity, and the infinitesimal, culminating in a discussion of the differential and its role in the differential calculus.[12]

The conceptual foundation of the differential calculus begins with the notion of a function. A function is a relation, or map, from one domain of mathematical elements or structure to another. Functions are applied in the physical sciences to represent, among other things, entities (e.g. electrons and planets), constraint surfaces (e.g. the top of a table or a space-time), and dynamical variables (e.g. force, position, and velocity) in the physical world. In each of these cases, a function serves to define the quantitative structure of the world by providing a map from some physical property, or structure, to a element, or structure, defined in $\mathbb{R}^n$, the n-dimensional space of real numbers. But how is a function is applicable to the world? Is this not the same question that lies at the heart of our discussion of the mapping account?

Within the conceptual system of a physical theory that is based on the differential calculus, the application of the concept of a function serves to define the initial mathematization of the world. This application of mathematics is itself a form of representation, but it is a representation akin to Goodman's characterization, in which we apply a representation to construe, classify, and interpret the world. In this sense, the application of the concept of a function serves to delimit the domain of study. But it is important to note that this initial

[12]In this section, We follow the development of the differential calculus provided by Loomis and Sternberg (1980). If the reader is familiar with the detailed formal development of the differential calculus, they may want to pass quickly through the mathematical parts of this section.

mathematization is both selective and productive. It is selective in that only those aspects of the world that are amenable to functional representation will enter into our physical conception of the world, e.g. extension and spatial-temporal location. When we conceive of the world as representable by functions, we limit our conception to only those aspects of the world that consistently allow such an interpretation. It is productive in that we fit the physical world for a "garb of ideas" to obtain an objective mathematical science (Husserl 1970; 54). The world as conceived through functions, is a quantitative mathematical world.[13]

We have barely gotten our feet wet, but the scientific realist might already feel slightly uneasy. If this initial mathematization of the world is a representation, in the sense of an interpretation, then there need not be any physical correlate to the mathematical structure. Rather, the mathematics is playing a definitional role that is constitutive of our physical conception. This issue is complicated by the fact that the definitional role of mathematics does not conclude with the application of the concept of a function, but rather only begins.

When we discuss functions in the differential calculus, we are usually interested in the behaviour of a function in the neighbourhood of a given point, but as we have already noted the differential calculus can only be applied to certain types of functions. In order to begin a discussion of the differential calculus, the functions we consider must satisfy four conditions:

> ***Condition 1:*** The function must be defined on at least one open neighbourhood of the point under consideration; except, maybe, the point itself.[14]
>
> ***Condition 2:*** The space of the function and the space of its domain must possess a norm (a definition of distance).[15]

[13]The scientific realist might protest that what is needed is not an exact quantitative world but only an approximation of the worlds inherent structure, however, any attempt to make the notion of approximation precise will have to provide a quantitative measure for the relation and, as such, would require an account of how this quantitative structure is defined and applied. This issue will be addressed in the next section.

[14]We allow for the possible exclusion of the point itself because, looking forward, the difference ratio of the calculus is not defined at the point under consideration.

[15]The concept of a norm allows us to provide a rigorous definition of distance and this provides us with a means to characterize an approximation and a coordinate system. In one dimension, it is customary to employ the absolute value of the difference between the elements, e.g. $|x - a|$, as the norm, but in multiple dimensions there are a few norms that work equally well.

***Condition 3:*** The function must possess a limit in the neighbourhood of the point under consideration.[16]

***Condition 4:*** The functions must be continuous.[17]

These conditions determine the form of the allowable physical structures that a theory based on the differential calculus can accommodate. These constraints are necessary for the concept of the infinitesimal to be well-defined and consistently applied. They represent the bare minimum that must be in place for our discussion of the differential calculus to begin.[18]

In the modern reformulation of the infinitesimal calculus, based on a rigorous foundation, infinitesimals are defined as functions that not only satisfy the previous four conditions, but also tend to zero as the element of their domain tends to zero, e.g. $\phi(t) \to 0$ as $t \to 0$. The difference ratio of the derivative is defined in terms of infinitesimals, $f'(x)$ is defined as $(f(x+h) - f(x))/t$ and this is simply the ratio of two infinitesimals (Loomis and Sternberg, 136). We usually say that the derivative $f'(x)$ exists and has a value $a$ if $(f(x+h)-f(x))/t-a$ approaches 0 as $t \to 0$, or equivalently if $((f(x+h)-f(x))-at)/t$ approaches 0 as $t \to 0$ (Loomis and Sternberg, 136). In this case, $\phi(t) = (f(x+h)-f(x))-at$ "is an infinitesimal that approaches 0 *faster than* $t$ (i.e., $\phi(t)/t \to 0$ as $t \to 0$)" (Loomis and Sternberg, 136). The fact that "$\phi t$ converges to 0 faster than $t$ as $t \to 0$ is exactly equivalent to the fact that the difference quotient of $f$ converges to $a$" (Loomis and Sternberg, 137).[19] Therefore, the study of the derivative is equivalent to the study of the behaviour of

[16] In the $\epsilon$,$\delta$-definition of a limit, we say that a function $f(x)$ tends to a limit $l$ as the element $x$ approaches $a$ if for every positive $\epsilon$ there exists a positive $\delta$ such that $0 < |x-a| < \delta \to |f(x) - l| < \epsilon$. It is important to note here that only functions possess a limit. Later on, when we discuss the differential calculus, keep in mind that the relation $(f(x+h) - f(x))/t$ expresses the ratio of two functions; we consider $t$ to be a function not a variable.

[17] A function is 'continuous at a given point' if the limit, as defined above, exists at that point and the limiting value of the function, taken from the left and the right, is the same as the value of the function at that point. A function is 'continuous' if it is continuous at all points of its domain. more intuitive way to talk about continuity is through Hausdorff continuity. We say a set of elements is Hausdorff continuous if every pair of elements can be separated by an open neighbourhood.

[18] One could easily reformulate the following discussion in terms of continuity conditions, but we will base our treatment of the calculus on a discussion of infinitesimals, due to their intuitive appeal.

[19] Note: two commas were removed from the quote to fit the quote into the sentence structure.

infinitesimals.

Following Loomis and Sternberg, we may identify two special classes of infinitesimals: "big oh", $O$, and "little oh", $o$ (Loomis and Sternberg, 136). A function falls under the class of "big oh", $f \in O$, if $f$ is Lipschitz continuous at $0$.[20] A function falls under the class "little oh", $f \in o$, if $f(x)/x \to 0$ as $x \to 0$.[21] Clearly, the numerator of the difference ratio of the derivative, written in the form $\phi(t) = (f(x+h) - f(x)) - at$, must be an infinitesimal of class "little oh". If this condition does not hold, the derivative cannot be well-defined.

The notion of an infinitesimal function defines an additional structure within our physical conception. We require that the functions we define on the world have a certain behaviour "in the small". But what type of constraint does this condition impose on the form of the functions we consider in the differential calculus? To answer this question, we will have to introduce the mathematical concept of a differential.

In our formal development of the concept of the differential, we will continue to follow Loomis and Sternberg (1980), and formally base the notion of a differential in terms of a general coordinate translation.[22] The coordinates of a function and its element are usually represented by an ordered pair containing the point under consideration and the value of the function at that point, $(a, f(a))$. We can always move the pair of elements to the origin by a coordinate translation of the form $s = f(x) - f(a)$ and $t = x - a$. In what follows, we will consider an ordered pair $(a, f(a))$ located near a point at which we would like to study the behaviour of a function. We can represent a general coordinate translation by the following diagram.

[20]a function is Lipschitz continuous if for all $x$ sufficiently close to $a$, $|f(x) - l| \leq c|x - a|$, where $l$ is the limit of the function and $c$ is a constant.

[21]From this definition we can show that "big oh" is a subset of "little oh", $o \subset O$.

[22]A coordinate translation can represent a passive shift in the coordinate system, or an active translation that describes the motion of an object.

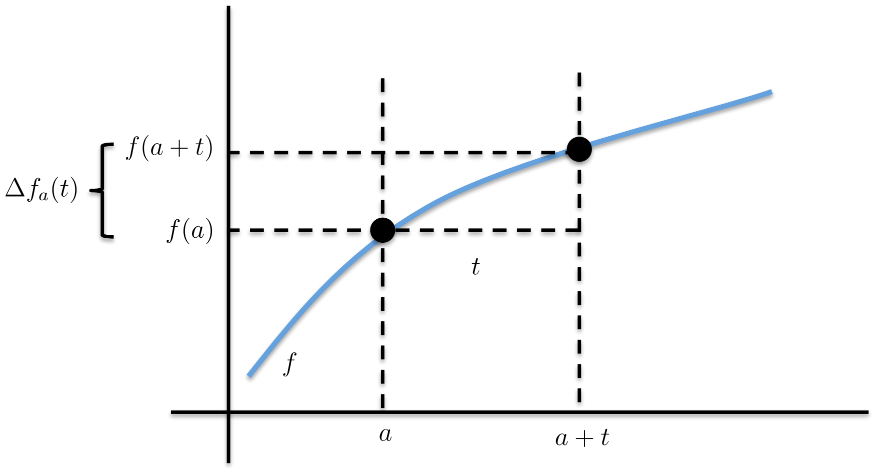


Figure 1: Diagram of a Coordinate Translation (Loomis and Sternberg, 141)

In the diagram it is clear that the image of $f$ under the translation is given by the relation $\Delta f_a(t) = f(a+t) - f(a)$. $\Delta f_a(t)$ is simply the change in $f$ brought about by the coordinate translation. The original curve, in the new coordinates, is the graph of $\Delta f_a(t)$.

We can now define the differential. In the new coordinate system, the equation for the tangent is given by the functional map $l(t) : t \to f'(a)t$; where $l(t)$ is the map from $t$ onto the tangent (Loomis and Sternberg, 141). From this definition of $l(t)$, it is clear that the existence of the derivative $f'(a) = \Delta f_a(t)/t$ as $t \to 0$ is exactly equivalent to saying that $\Delta f_a(t) - l(t)/t \to 0$ as $t \to 0$ (Loomis and Sternberg, 141). Therefore, for the derivative to be well-defined in the infinitesimal neighbourhood of the point under consideration, the difference between the map $\Delta f_a(t)$ and the tangent $l(t)$ in that neighbourhood, given by $\Delta f_a(t) - l(t)$, must be an infinitesimal of the class "little oh", $\Delta f_a(t) - l(t) = o$ (Loomis and Sternberg, 141). To put the same point another way, we can say that the difference between $\Delta f_a(t)$ and $l(t)$ must tend to zero faster that $t$. It can also be shown that the expression $\Delta f_a(t) - l(t) = o$ is unique (Loomis and Sternberg, 141). The differential is defined as the "unique linear approximation $l(t)$ ... of $f$ at $a$ and is designated $df_a$" (Loomis and Sternberg, 141). From this definition, it is clear that without a well-defined differential, a derivative cannot be defined in the infinitesimal neighbourhood of the point under consideration.

The concept of a differential provides a valuable tool for analyzing the behaviour of a function near a given point. In the limiting neighbourhood of the origin in the new coordinate system, the difference between an infinitesimal change in the function, $\Delta f_a(t)$, and the differential $df_a$ is an infinitesimal of order $o$. The existence of a derivative at a given point requires that the infinitesimal behaviour of the function, $\Delta f_a(t)$, can be uniquely approximated by a differential map, up to an infinitesimal of order $o$. This means that the existence of the differential in the neighbourhood of the origin entails that the behaviour of the function in the neighbourhood can be approximated by a unique tangent. This can be seen clearly in the following diagram:

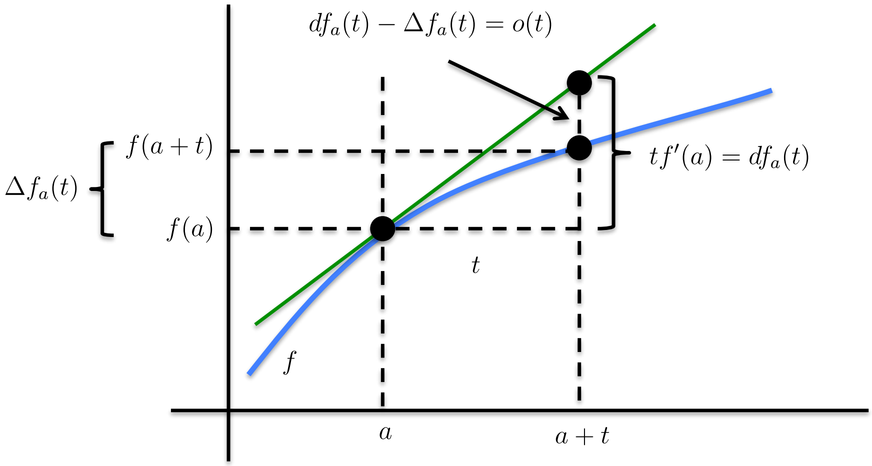


Figure 2: Diagram of a Coordinate Translation and the Differential (Loomis and Sternberg, 141)

The converse is also true. If the derivative does not exist, then the differential does not provide a unique approximation (up to class "little oh") of the function in the limiting neighbourhood of the point (Loomis and Sternberg, 146 -147). Therefore, the existence of a derivative entails the existence of a unique tangent that approximates the behaviour of the function up to class "little oh". Without this internal structure, the derivative cannot be defined. If we want to apply the differential calculus, then the functions we consider must possess this internal structure. This imposes a constraint on the form of any physical structure on which the differential calculus is applied.[23]

[23]But, here 'applied' should be read in the sense of a scientific realist's application of mathematics. Of course, one could apply the differential calculus to discrete systems by smoothing out the discontinuity through

We are now in a position to characterize the interpretation of the world that must be in place in order for the differential calculus to be applied. We can see that the functions that we apply within our scientific conception must be defined on at least one open neighbourhood of the point under consideration, possess a norm, possess a limit in the neighbourhood of the point under consideration, and be continuous, they must also possess a specific form such that the concepts and infinitesimal and differential can be well-defined and consistently applied. The scientific conception of the world, within any theory that applies the differential calculus, is defined and interpreted to possess this structure. When we interpret the world to possess a certain mathematical structure we at the same time construe and classify it. When we apply mathematical concepts to define the structure of the world, we project a mathematical structure onto the world in order to make it representable within the mapping account. And the choice among projectable mathematical concepts imposes a classification, which is simply a result of the governing mathematical conception that prevails within the larger theoretical structure.

The selective and productive interpretation of the world outlined in this section is based on the representation of the world given by the differential calculus, and is not based on any independent physical consideration. This presents a serious problem for the scientific realist, as the definitional role of mathematics does not necessarily possess a physical correlate. Rather, this mathematical pre-structuring of the world is a result of our intended representation the world given by the differential calculus. But this definitional role of mathematics is only half the story, and we now need to address the interrelation of mathematical concepts that takes place within a given physical theory.

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idealization. But in this case the realist could no longer suggest that the mathematical structure of the differential calculus in any sense represents the structure of the world.

## Newtonian Physics and the Differential Calculus

Newton's second law is simple enough to be familiar to almost every high school student and has remained a common discussion point in the philosophy of science. It expresses a relation between an impressed force on an object and the resulting change in the objects momentum. The modern definition of the law asserts that the force on an object is equal to the rate of change of the objects momentum, expressed as derivative of the momentum with respect to time. We write this symbolically as $\boldsymbol{F}(t) = d\boldsymbol{p}(t)/dt$. Within the mapping account, the scientific realist would want to claim that the differential relation maps a physical relation that holds between the physical force and the physical momentum of an object, which are represented by two time dependent vector functions $\boldsymbol{F}(t)$ and $\boldsymbol{p}(t)$, into $\mathbb{R}^3$, that is, if Newtonian theory were still accepted as a valid representation of the world.

But this simple narrative is untenable. We have noted that the application of the concept of a function is itself a representation, but one that serves to define the initial mathematization of the world. This mathematical construal, classification, and interpretation provides a quantitative structure to the world in order to provide a foundation for objective mathematical science. In this sense the interpretation of the world, as representable in $\mathbb{R}^3$, is both a selective and productive interpretation of the world that constitutes the basis of our physical conception. The use of functions delimits the conceptual system to only those aspects of the world that are amenable to functional representation. But what is more important in this case is the productive aspect of representation that fits the physical world for a "garb of ideas" to obtain an objective mathematical science (Husserl 1970; 54). For, the application of the concept of a function serves not only to define a quantitative structure on the world, but also to define which aspects of the world are to be represented as fundamental variables. Newton's second law produces an objective physical conception by setting a definition of inertial motion. Physical objects are thought to possess momentum, which remains constant unless a force acts on the object. Forces are construed to be a non-local relation that all objects enter into as a result of their possession of certain properties, e.g. mass.

When we apply these two functions, $\boldsymbol{F}(t)$ and $\boldsymbol{p}(t)$, we do not represent force or momentum in the sense of a copy theory of representation, but produce a specific mathematical/physical conception of the world.[24] Newton's second law expresses a relation within this conception of the world, and if it meaningful at all, then it is a "law" of the world as representable within $\mathbb{R}^3$. But this law cannot be applied to arbitrary momentum and force functions, due to the constraints implicit in the definition of the differential calculus. Rather, we need to pre-structure our conception of the world such that the concepts of the differential calculus can be consistently applied and well-defined.

In order to apply the differential calculus, the function that represents the objects momentum must satisfy four conditions: namely, it must be defined on at least one open neighbourhood of the point under consideration, possess a norm, possess a limit in the neighbourhood of the point under consideration, and be continuous. The first condition is satisfied by stipulating that momentum, as construed within the Newtonian conception, is specified by a function that is defined on the neighbourhood of any point on its trajectory. The second condition is satisfied by imposing a Euclidean metric on $\mathbb{R}^3$.[25] The third and fourth conditions require that we represent the world in such a way that only continuous functions define the momentum of any object. Motion, as construed within the Newtonian conception of the world, is continuous, that is if we wish to apply Newton's second law.

The application of the differential calculus also requires that the function that represents the momentum of an object possess a certain internal structure. This structure is necessary so that the concept of an infinitesimal and differential can be consistently applied and well-defined. Specifically, what we require is that the functions possess a certain structure "in the small". The concept of a differential provides a valuable tool for characterizing the

[24]This claim is also supported by the existence of equivalent energy-based formalizations of classical mechanics.

[25]The Euclidean metric is defined as: $\|x\| = (\sum_{i=1}^{3} x_i^2)^{\frac{1}{2}}$. The absolute time of Newtonian physics is a one-dimensional space, and the absolute value function, $|x|$, provides a sufficient definition of distance in that space.

behaviour of a function near a given point. The existence of a derivative requires that the infinitesimal behaviour of the function can be uniquely approximated by a unique tangent. This in turn requires that the function can be uniquely approximated by a differential map in the infinitesimal neighbourhood of a given point. Motion, as construed within any conception of the world based on the differential calculus, is defined to have this internal structure "in the small".

However, the pre-structuring does not end here. In the modern formulation of Newtonian theory, the momentum function, $\boldsymbol{p}(t)$, is defined in terms of two other functions; one mass function, $m(t)$, and one velocity function, $\boldsymbol{v}(t)$. Formally, the momentum is defined as the mass times the velocity, $\boldsymbol{p}(t) = m(t)\boldsymbol{v}(t)$.[26] In the case, the differentiability of $\boldsymbol{p}(t)$ requires that both $m(t)$ and $\boldsymbol{v}(t)$ be differentiable. Therefore, the functions $m(t)$ and $\boldsymbol{v}(t)$ must also possess the necessary internal structure "in the small". And finally, the velocity function is defined as the rate of change of position in time, expressed as derivative of the position, $\boldsymbol{x}(t)$, with respect to time, $\boldsymbol{v}(t) = d\boldsymbol{x}(t)/dt$, and the position functions as well must possess the necessary internal structure "in the small" such that the concepts of the differential calculus can be consistently applied and well-defined. All of this pre-structuring must be in place in order to form the Newtonian conception of the world.

So where does this leave the scientific realist? On the one hand, we have a theory that is supposed to represent a physical relation that holds in the world. On the other, we have a set of mathematical definitions that construe, classify, and interpret the world in order to apply the theory. This initial representation of the world imparts it with a mathematical structure, and this pre-structuring undermines any form of scientific realism based on a copy theory of representation, such as the mapping account.

The fact is that any mathematical scientific theory that is taken to represent certain physical features of the world must address the implicit mathematization of the world. Husserl is right to note that "[m]athematics and mathematical science, as a garb of ideas, or garb

[26] Against usual convention, the mass functions is defined to be time dependent in order to highlight the fact that the continuity conditions apply equally to the mass and velocity functions.

of symbols of the systematic mathematical theories, encompassing everything which, for scientists and the educated generally, represent the life world, dresses it up as "objectively actual and true" nature" (Husserl, 1970; 54). Newton's second law only expresses a relation in the objective pre-structured world represented in $\mathbb{R}^3$. This mathematical law cannot copy, or map, a physical relation because there is no conception free physical relation that it can represent, as understood within a copy theory of representation.

Within the mapping account, It appears as though only a contingent form of scientific realism can be supported. Given a certain mathematical conception of the world, certain law-like relations hold, but these relations cannot be said to represent any innate structure in the world. At this point one might wonder if there any viable alternative open to the scientific realist. The answer will depend on whether or not the scientific realist can make do without a copy theory of mathematical representation. There is no question of whether or not Newton's second law expresses a functional relation, or map, from one domain of mathematical elements, or structure, to another. The mapping account provides an accurate description of the structure of the law itself, but this is not really the issue. The real issue relates to how a given mathematical structure, as a whole, represents a supposedly physical structure.

The real problem is that any symbolism, mathematical or not, harbours the curse of mediacy (Cassirer, 1946; 7). What is symbolized or represented is not a copy of what exists. The scientific realist might respond by abandoning the mapping account and noting that was is needed is not some exact copy of the world, which may indeed be impossible, but rather, a rough approximation to its structure. It may be the case that all this supposed pre-structuring is simply a form of abstraction or idealization that is typical of science in general, and in this case the real problem is that of abstraction and idealization, not of copying. The focus on approximation may change the nature of the question, but not its substance. The idea that mathematics might approximate, rather than copy, a physical structure still requires a clarification of how mathematics is brought to bear on the world. The concept of approximation is equivocal, to say that a certain structure approximates another might indicate a closeness with respect to a given measure, an indication of similarity, the presence of common properties, or a number of other possible relations. The problem for the realist is to make the notion of approximation sufficiently precise without falling back onto the notion of approximately, or partially, copied structure. However, if the supposed 'closeness', 'similarity', or 'common property' is explicated in mathematical terms, then we end up right back where we started. The scientific realist needs to find a non-mathematical notion of approximation that is strong enough to support a viable realism but yet also weak enough to avoid the concerns associated with a copy theory of representation. Whether or not such an account can be found, the supposed marriage between the scientific realist and the mapping account is an unhappy one. Mathematics is not applied to map relations that hold within a physical system into an appropriate mathematical structure.

Although this paper has largely been a critical discussion of scientific realism and the mapping account of applied mathematics, it is not without practical importance. The pre-structuring inherent in the application of the differential calculus precludes certain structures from being well-defined within the confines of a Newtonian conception of the world. This pre-structuring limits the types of supposedly physical structures that the theory can accommodate. The form of the Newtonian conception of the world dictates the structure of the physical phenomena that the theory can accommodate, and in the final section, it is worthwhile to take a closer look at the constraints implicit in the application of Newton's second law.

# The Newtonian Conception of Motion and a Case Study of Norton's Dome

What is the Newtonian conception of motion? The first thing we should note is that since Newton's second law can only be applied in the infinitesimal neighbourhood of a given point, motion can only be defined up to an infinitesimal neighbourhood. If the realist wants to argue that Newton's second law is a true description of the phenomena, then it is a fuzzy description, and this inherent fuzziness is an unavoidable conclusion of the Newtonian conception of the world as governed by the differential calculus.

The description of motion, as defined by a Newtonian conception of the world, is complicated by the possibility of both constrained and unconstrained motion. In the case where the objects' motion is unconstrained, the objects' trajectory is defined solely with reference to the background space and time. If we consider a point-like object located at a particular point in space and time, we can discuss its motion with respect to a specified well-behaved force. At the initial time when the force is specified, the object is accelerated at a rate given by Newton's second law. The time evolution of the system can be specified by a unique trajectory in space and time. Since Newton's second law is an ordinary differential equation, and there are no additional constraints on the form of the position function, we can refer to the existence and uniqueness theorems of ordinary differential calculus in order to demonstrate that the trajectory of the object is indeed defined and unique (Kaplan,1973; 494-497). The structure of Newton's second law uniquely specifies the trajectory of an object with respect to the background space and time, and all of the conditions required for the differential calculus to be well-defined over every neighbourhood of each point along the trajectory are automatically satisfied.

The case of constrained motion is more complicated. A constraint imposes certain conditions on the form of an object's motion. For instance, we might consider the motion of an object constrained to the top of a billiards table. In this case, the motion of the object is constrained to the surface of the table and this imposes conditions on the form of the time evolution of the system given by Newton's second law. The problem that arises in the case of constrained motion is that the form of the constraint may impose undesirable conditions on the form of the function that defines the object's position. Since the differential calculus requires that this function possess a certain internal structure, only certain types of constraints will allow for the differential calculus to be well-defined in the neighbourhood of every point. The good news is that many of the constraints that we consider within Newtonian theory are constructed to satisfy these conditions. The bad news is that there are many constraints that impose conditions on the form of a function such that it will possess neighbourhoods on which Newton's equation of motion simply cannot be applied. We will now discuss such a case.

Norton (2003) presents a now famous thought experiment simply called 'the dome', which attempts to demonstrate that Newton's second law allows for indeterministic solutions. Norton asks us to consider a point-like ball, of unit mass, located at the top of a frictionless, perfectly rigid, dome. The shape of the dome is given by $h = (2/3g)r^{\frac{3}{2}}$, where $h$ is the height of the dome and $r$ is the radial arc length measured along the surface of the dome (Norton, 2008; 787).[27] . The ball is subject only to the force of gravity.[28] At the start of this thought experiment, the ball is located at the apex of the dome.

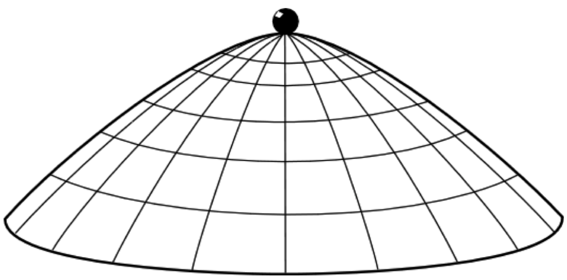

Figure 3: Norton's dome with static ball, taken from http://www.pitt.edu/ jdnorton/Goodies/Dome

Norton claims that the gravitational force, $F$, acting on the ball is given by: $F = (dh/dr) = r^{\frac{1}{2}}$ (Norton, 2008; 787). Since the ball has unit mass, Newton's second law states that the acceleration, $\frac{d^2r}{dt^2}$, is equal to this force, the result gives: $d^2r/dt^2 = r^{\frac{1}{2}}$. This is the equation of motion for a ball anywhere on the surface of the dome (Norton, 2008; 787).

We now come to the crux of Norton's argument. The equation of motion for a ball located at the apex is given by: $d^2r/dt^2 = 0$. One solution to this equation is: $r(t) = 0$

[27]From this point onwards the gravitational constant, $g$, will be set to 1

[28]Imagine that the dome is located within ahomogeneous gravitational field pointing downwards in the following diagram.

(Norton, 2008; 788). This is the solution that we naturally accept; that is, the radial coordinate of the ball remains constant. According to this criterion, the ball should not move. However, Norton claims that there is an alternative solution given by (Norton, 2008; 788):

$$r(t) = \begin{cases} \frac{1}{144}(t-T)^4 & \text{for t} \geq \text{T} \\ 0 & \text{for t} \leq \text{T.} \end{cases} \tag{1}$$

This solution states that the ball remains at the apex, for some arbitrary time, where it is subject to the equation of motion at the apex: $d^2r/dt^2 = 0$. However, spontaneously, the ball may begin to roll and is subsequently subject to the equation of motion for the surface of the dome: $d^2r/dt^2 = r^{\frac{1}{2}}$.

This solution states that the ball will remain at rest for some period of time, when $t \leq T$, and at $t = T$ the ball will spontaneously begin to roll down the dome. Notice the independence of these equations of motion on the radial direction the dome. If the ball is to move, there is no way of predicting the direction that it will go. Norton's conclusion is a result of the fact that the structure of the dome violates the Lipschitz condition and the associated existence and uniqueness theorem of ordinary differential calculus. It turns out that there is no way to predict at what time $T$ the ball will begin to roll. If we add this fact to the independence of the equations of motion on the radial direction of descent, we observe a true indeterminacy in both the time and direction of descent. This is Norton's demonstration of indeterminacy at work in Newtonian physics. The whole situation is summed up nicely in the following diagram.

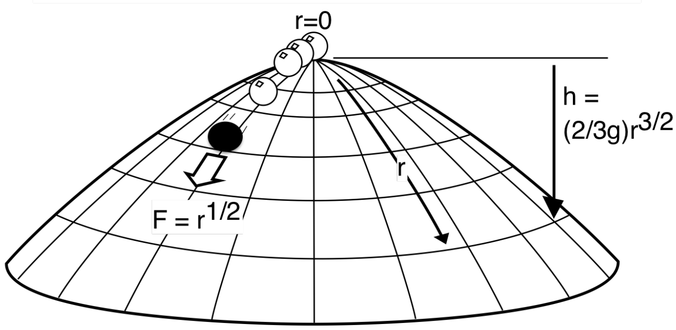


*Figure 1a.* Mass sliding on a dome

Figure 4: Norton's dome with falling ball, taken from http://www.pitt.edu/ jdnorton/Goodies/Dome

We intend to show that the problem with Norton's argument is that the differential calculus actually cannot be applied in the infinitesimal neighbourhood of the apex of the dome. This can be clearly seen if we consider the case of the ball rolling up the side of the dome towards the apex. We will show that as the ball approaches the infinitesimal neighbourhood of the apex, the differential structure breaks down. If the differential calculus cannot be defined in the infinitesimal neighbourhood surrounding the apex of the dome, then Norton cannot apply Newton's law in the infinitesimal neighbourhood of the apex, and the argument fails.

To begin, we can define a function, $\boldsymbol{x}(s)$ for the objects' position on the surface of the dome in terms of the arch length, $s$, measured from the apex. We can express the arc length, $s$, in terms of the time parameter, $t$, to define the objects position as a function of time, $\boldsymbol{x}(s(t))$. We will express the function $\boldsymbol{x}(s(t))$ in terms of a coordinate system fixed in the background Euclidean space. The origin of our coordinates will be centred on the apex of the dome with the apex itself occupying the point $(0, 1)$.

To simplify the problem, we can consider the case of a ball rolling up the right hand side of the dome.[29] In our coordinate system, the $x$ coordinate of the dome is given by $x_1(s) = -\frac{2}{3}(1-s)^{\frac{3}{2}} + \frac{2}{3}$, and the $y$ coordinate is given by $x_2(s) = 1 - \frac{2}{3}s^{\frac{3}{2}}$. The right half of the dome is then given by the equation $\boldsymbol{x}(s) = (x_1(s), x_2(s))$. The position of the ball along the dome as a function of time is then $\boldsymbol{x}(s(t)) = (x_1(s(t)), x_2(s(t)))$. Newton's second law states that

[29] We can define all of the other solutions from the radial symmetry.

$F = m\boldsymbol{x}''(s(t))$, where each prime indicates a derivative with respect to time, $t$. Expanding out the derivative by the chain rule, we find that $\boldsymbol{x}''(s(t)) = \ddot{\boldsymbol{x}}(s(t))s'(t) + \dot{\boldsymbol{x}}(s(t))s''(t)$, where each dot indicates a derivative with respect to arc length, $s$. We can immediately note that $\dot{\boldsymbol{x}}(s(t))$ is the tangent to the dome and $\ddot{\boldsymbol{x}}(s(t))$ is the normal to the dome. The behaviour of the derivative $\boldsymbol{x}''(s(t))$ in the infinitesimal neighbourhood of the apex is a function of the tangent and the normal to the dome. Therefore, we can get a good feel for how the ball will behave in the infinitesimal neighbourhood of the apex by studying the behaviour of the tangent and normal in that neighbourhood.

In our coordinate system, the tangent and the normal to Norton's dome are given by: $\dot{\boldsymbol{x}}(s(t)) = (\sqrt{(1-s)}, -\sqrt{s})$ and $\ddot{\boldsymbol{x}}(s(t)) = (-\frac{1}{2\sqrt{1-s}}, -\frac{1}{2\sqrt{s}})$, respectively. Immediately, we see that we are going to run into a problem. As the ball rolls towards the apex of the dome we see that the normal to the curve will blow up in the infinitesimal neighbourhood of the apex. This is simply a result of the fact that the curvature of the dome $\kappa(s) = \sqrt{(\ddot{\boldsymbol{x}}(s) \cdot \ddot{\boldsymbol{x}}(s))}$ blows up in the infinitesimal neighbourhood of the apex. Therefore, the derivative that represents the objects acceleration blows up as the object heads to the apex; Malament has come to the same conclusion (Malament, 2008). He suggests that the fact curvature blows up at the apex, shows that the apex of the dome has a zero fly-off speed, and might be considered to be a more of launching pad than a constraint surface (Malament, 2008; 13). Norton responded by noting that we could consider the ball, or in this case a bead, to be constrained to the surface of the dome by a perfectly rigid wire (Norton, 2008; 790). Norton claims that the wire would then provide the necessary constraint force to keep the ball on the surface of the dome, and Malament's concerns are easily alleviated. This apparent "solution" in no way alleviates Malament's concerns. The real problem is that the differential calculus simply cannot be applied in Norton's thought experiment.

Drawing from our discussion of the differential calculus, we can see what is going on. We know that the fact that the normal to the curve blows up in the infinitesimal neighbourhood of the apex indicates that the infinitesimal $\Delta\dot{\boldsymbol{x}}(s(t)) - d\dot{\boldsymbol{x}}(s(t)) = \dot{\boldsymbol{x}}(s(t+h)) -$

$\dot{\boldsymbol{x}}(s(t)) - l(s(t))$, where $l(s(t))$ is the tangent to the surface, is not of class "little oh". The issue is that $\Delta\dot{\boldsymbol{x}}(s(t))$ does not tend to its limit as fast as $h \to 0$. Therefore, when we take the derivative, we find that it blows up because the change in the function, $\Delta\dot{\boldsymbol{x}}(s(t))$, remains finite as $h \to 0$. If we cannot define an infinitesimal $\Delta\dot{\boldsymbol{x}}(s(t)) - d\dot{\boldsymbol{x}}(s(t))$ of class "little oh" then we cannot define a differential to the curve in the infinitesimal neighbourhood of the apex. If we cannot define a differential, then we cannot define a unique tangent, $l(s(t))$, to the curve, $\dot{\boldsymbol{x}}(s(t))$, that approximates the curve up to a class of "little oh". The problem is that we simply cannot determine the behaviour of the function $\dot{\boldsymbol{x}}(s(t))$ in the infinitesimal neighbourhood of the apex, because we cannot employ the concept of a differential to approximate the behaviour of the curve in that neighbourhood and if you cannot provide a unique linear approximation to the function $\dot{\boldsymbol{x}}(s(t))$ in the infinitesimal neighbourhood of the apex, then you cannot apply the differential calculus.

To get a feel for how pathological Norton's dome truly is, we can consider the normal force acting on the ball in the infinitesimal neighbourhood of the apex. The normal force on the ball over the surface of the dome is given by: $\boldsymbol{F}_{\perp}(s) = \sqrt{(1-s)}(-\sqrt{s}, -\sqrt{(1-s)})$; and its derivative is given by: $\dot{\boldsymbol{F}}_{\perp}(s) = (-\frac{1}{2}(1-2s)/(\sqrt{s}\sqrt{1-s}), 1)$. Right away, we see that the derivative blows up in the infinitesimal neighbourhood of the apex. Just as in our previous discussion, this indicates that we cannot define a differential to the force function that approximates the behaviour of the function in the infinitesimal neighbourhood of the apex. Therefore, we simply cannot define a well-behaved force acting on the ball in the infinitesimal neighbourhood of the apex. Geometrically, this is a result of the fact that the force swings though a finite angle in an infinitesimal neighbourhood.

The fundamental problem with Norton's thought experiment is that both of the functions employed in Newton's second law behave pathologically in the infinitesimal neighbourhood of the apex. All of this pathological behaviour is a simple result of applying Newtonian physics on a surface that is precluded by the pre-structuring of the world inherent in the Newtonian conception. Motion, as defined within the Newtonian conception of the world, takes place within a mathematically pre-structured world that possesses a specific structure "in the small". This pre-structuring limits the form of the phenomena that the theory can describe.

## Conclusion

Duhem was right to note that "[t]he role of the scientist is not limited to creating a clear and precise language in which to express concrete facts; rather, it is the case that the creation of this language presupposes the creation of a physical theory" (Duhem, 1954; 151). In the case of mathematics, the application of this language presupposes that our physical conception of the world has already been pre-structured mathematically. This initial mathematical pre-structuring of the world is a representation akin to Goodman's characterization, in which we apply a representation to construe, classify, and interpret the world. The choice among projectable mathematical concepts imposes a classification, which is simply a result of the governing mathematical conception that prevails within the larger theoretical structure. We saw that this initial mathematization is both selective and productive. It is selective in the sense that only those aspects of the world that are amenable to functional representation will enter into our physical conception of the world. And it is productive in the sense that we fit the physical world for a "garb of ideas" to obtain an objective mathematical science (Husserl 1970; 54). The world as conceived through mathematics, is a quantitative world. The real problem is that any symbolism, mathematical or not, harbours the curse of mediacy (Cassirer, 1946; 7). What is symbolized or represented is not a copy of what exists.

The mapping account of applied mathematics can only serve as a viable foundation for a contingent form of scientific realism. Given a certain mathematical conception of the world, certain law-like relations hold, but these relations cannot be said to represent any innate structure in the world. If a true scientific realism is to be grounded in a mathematical theory of the world, we must find an alternative to the mapping account of applied mathematics.

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