Physically Similar Systems: A History of the Concept

(Submitted)

by S G Sterrett

Department of Philosophy,

Wichita State University

susangsterrett@gmail.com

Chapter 2
Physically Similar Systems: A History of the Concept

S. G. Sterrett

Summary

The concept of *similar systems* arose in physics, and appears to have originated with Newton in the seventeenth century. This chapter provides a critical history of the concept of *physically similar systems*, the twentieth century concept into which it developed. The concept was used in the nineteenth century in various fields of engineering (Froude, Bertrand, Reech), theoretical physics (van der Waals, Onnes, Lorentz, Maxwell, Boltzmann) and theoretical and experimental hydrodynamics (Stokes, Helmholtz, Reynolds, Prandtl, Rayleigh). In 1914, it was articulated in terms of ideas developed in the eighteenth century and used in nineteenth century mathematics and mechanics: equations, functions and dimensional analysis. The terminology *physically similar systems* was proposed for this new characterization of similar systems by the physicist Edgar Buckingham. Related work by Vaschy, Bertrand, and Riabouchinsky had appeared by then. The concept is very powerful in studying physical phenomena both theoretically and experimentally. As it is not currently part of the core curricula of STEM disciplines or philosophy of science, it is not as well known as it ought to be.

Contents

2.1 Introduction

The concept of *similar systems* is one of the most powerful concepts in the natural sciences, yet one of the most neglected concepts in philosophy of science today. The concept of similar systems was developed specifically for physics, and its use in biology has generally been in terms of plant and animal physiology; hence the term *physically similar systems* is often used. It remains an open research question whether, and how, the concept of similar systems might be applied to sciences other than physics, such as ecology, economics, and anthropology.

This chapter is devoted to providing a history of the concept of physically similar systems. It also aims, in doing so, to increase understanding and appreciation of the concept of *similar systems* in philosophy. For, in addition to being neglected in philosophy of science, the concept of *similar systems* is also often not fully understood even when it is mentioned.

The concept of similar systems has been useful in developing methods for drawing inferences about the values of specific quantities in one system from observations on another system. Some know of the concept only in this derivative way, via applications to specific questions in physics, biology, or engineering.

That it has such useful applications has sometimes led to an underappreciation of the fundamental nature, immense power and broad scope of the concept. Yet its utility in practical matters of determining or predicting the value of a particular otherwise unobservable quantity is an important feature of the concept. For, it is due at least in part to the utility of methods involving the concept of similar systems in providing answers to some otherwise intractable problems that natural philosophers in the Renaissance such as Galileo Galilei and Isaac Newton reasoned using some version of the concept, and, later, in the late nineteenth and early twentieth century, that scientists further developed it. Thus, understanding of the concept developed over centuries. I will use the twentieth century understanding of *similar systems* to characterize the concept first, then go back to some early precursors from which it was developed and follow the path up to the twentieth century characterization of it. This history of the concept, though admittedly not exhaustively complete, should help clarify its role in reasoning and drawing inferences.

2.2 Similar Systems, the 20th century concept

The landmark year in clarifying and articulating the concept of physically similar systems was 1914. There were two papers with "Physically Similar Systems" in the title that year by Edgar Buckingham; one in July ("Physically Similar Systems") in the *Journal of the Washington Academy of Sciences* [2.1], and one in October 1914 ("On Physically Similar Systems: Illustrations of the Use of Dimensional Equations") in *Physical Review* [2.2]. Though the latter one is well known and highly cited, and the former one little known, I think that it is the former, i.e., the much shorter July 1914 piece, that represents a crucial link or advance, conceptually speaking. The October 1914 Buckingham paper is often credited for the theorem it contains, which is ironic: as Buckingham emphasized numerous times in later papers, a version of the theorem itself had been proven years before. His articulation and discussion of the notion of *physically similar systems*, however, was unusually reasoned and more general than any others accompanying the proof of the theorem.

For now, I just wish to characterize the concept as it is currently understood; for that, we look to the well-known October 1914 *Physical Review* paper [2.2]. The paper opens with a section "The Most General Form of Physical Equations," which is about describing a relation that holds among physical quantities of different kinds, by an equation. This is followed by a section introducing and making use of the principle of dimensional homogeneity, entitled "Introduction of Dimensional Conditions." After exhibiting those points in an example, comes "The General Form to Which Any Physical Equation is Reducible" which states as "a general conclusion from the principle of dimensional homogeneity" that

> "If a relation subsists among any number of physical quantities of n different kinds, and if the symbols Q_1, Q_2, . . . Q_n represent one quantity of each kind, while the remaining quantities of each kind are specified by their ratios r', r'', . . . , etc. to the particular quantity of that kind selected, then: any equation which describes this relation completely is reducible to the form
>
> $$\Psi(\Pi_1, \Pi_2, \ldots \Pi_i, r', r'' \ldots) = 0$$ "
>
> ([2.2], p. 350)

As this form of the equation will be key in defining the notion of similar systems, let us give it a proper name; I'll call it the *Reduced Relation Equation of 1914*. The number of Π's in this equation is the difference between "the number of fundamental units required in an absolute system for measuring the n kinds of quantity, and n, the kinds of quantity [involved in the relation]." The function Ψ is not defined in this form of the equation, but that is perfectly fine; we still consider it an equation --- it's just an equation in which the form of the function is not specified. The equation states, basically, that such a function relating the Π's and r's does exist, and the conclusion is that this equation, the *Reduced Relation Equation of 1914*, is another form of the original physical equation, i.e., that any physical equation can be reduced to this form. Next follows a short section illustrating how this conclusion can be applied to the same example given earlier in the paper to determine the relationships between some specific quantities in an elegant and particularly useful way. All this is done prior to, and independently of, defining the notion of physically similar systems.

It is in the section entitled "Physically Similar Systems", the sixth section of the paper, that the notion of similar systems is first presented. Referring to the equation in his paper shown above, which I have called the *Reduced Relation Equation of 1914* , Buckingham writes that "we may develop from it the notion of similar systems"; he develops it as follows:

> "Let S be a physical system, and let a relation subsist among a number of quantities Q which pertain to S. Let us imagine S to be transformed into another system S' so that S' "corresponds" to S as regards the essential quantities. There is no point of the transformation at which we can suppose that the quantities cease to be dependent on one another: hence we must suppose that some relation will subsist among the quantities Q' in S' which correspond to the quantities Q in S. If this relation in S' is of the same form as the relation in S and is describable by the same equation, the two systems are 'physically similar' as regards this relation. " ([2.2], p. 353)

This is the notion of *physically similar systems* still currently in use today. It was first articulated in 1914 by the physicist Edgar Buckingham. But it didn't arise from Buckingham's cogitations out of the blue. For its precursors, we have to go back to the Renaissance.

2.3 Newton and Galileo

2.3.1 Newton on Similar Systems

Newton seems to have been the first to use the term *similar systems*. He uses it more than once, but the text usually associated with the concept of similar systems is in Book 2, Proposition 32, where he writes:

> " Suppose two similar systems of bodies consisting of an equal number of particles, and let the correspondent particles be similar and proportional, each in one system to each in the other, and have a like situation among themselves, and the same given ratio of density to each other; and let them begin to move among themselves in proportional times, and with like motions (that is, those in one system among one another, and those in the other among one another.) And if the particles that are in the same system do not touch one another, except in the moments of reflection; nor attract, nor repel each other, except with accelerative forces that are inversely as the diameters of the correspondent particles, and directly as the squares of the velocities: I say, that the particles of those systems will continue to move among themselves with like motions and in proportional times." ([2.3], p. 327)

In his *Science of Mechanics*, Mach refers to Newton's concept of similar systems in the context of his own discussion of oscillatory motion. ([2.4], p. 203) Mach's critical-historical work on mechanics was written to be accessible to the nonspecialist; his critique is informative of the understanding of similarity and similar systems at that time. After generalizing one of his own conclusions, Mach remarks: "The considerations last presented may be put in a very much abbreviated and very obvious form by a method of conception first employed by Newton." He does not quite accept Newton's use of the term similar system there, though:

> "Newton calls those material systems *similar* that have geometrically similar configurations and whose homologous masses bear to one another the same ratio. He says further that systems of this kind execute similar movements when the homologous points describe similar paths in proportional times." ([2.4], p. 203)

Mach admires Newton's methodology here, but he points out an issue with Newton's use of the term *similar* : "Conformably to the geometrical terminology of the present day we should not be permitted to call mechanical structures of this kind (of five dimensions) *similar* unless their homologous linear dimensions as well as the times and the masses bore to one another the *same* ratio."

I gather that what Mach is saying is that the notion of similar in use at the time he is writing is the notion of geometrical similarity, in which there is a kind of shrinking or enlarging of *every* linear quantity of *each* dimension by the *same* ratio (for geometrical similarity there would usually not be more than three dimensions). That is, I believe he means that, if we are talking about a three-dimensional machine, similarity amounts to shrinking or enlarging quantities of each linear dimension *by the same ratio* while keeping the machine and all its parts exactly the same shape, i.e., while preserving every ratio of linear quantities within the same machine. Now of course areas and volumes will bear a different ratio to their homologues than quantities of the linear dimensions do (e.g., if the ratio is 1:3 for the linear dimension, it will be 1:9 for an area and 1:27 for a volume), but the similarity can be defined in terms of the linear dimensions alone. That is how geometrical similarity works. Mach is saying, I think, that a strict application of the notion of geometric similarity would require that the ratio between a quantity and its homologous quantity be the same for all five of the dimensions that Newton mentions for his case, and that the situation imagined in Newton's proposition does not satisfy that constraint.

However -- and what is significant and interesting -- Mach does not say that Newton is wrong here; rather, what he says is that what Newton was doing is better understood in Mach's day in terms of affine transformations:

> "The structures might more appropriately be termed *affined* to one another.
>
> We shall retain, however, the name phoronomically [kinematically] *similar* structures, and in the consideration that is to follow leave the masses entirely out of account." ([2.4] , p. 204)

It is clear that Newton was interested in more than this, that he wanted to employ the notion of similar systems to reason about forces, too; in fact he does so in the remarks that follow the quote above. ([2.3], p. 327 - 328; [2.5] p. 766 - 768) However, in leaving the masses out of the account, Mach is picking out from Newton's work what he wishes to endorse, and showing how the points he endorses ought to be understood in the terminology of the nineteenth century. Mach shows how to understand phoronomically [kinematically] similar structures for the topic of oscillation he has been discussing:

> "In two such similar motions, then, let
>
> the homologous paths be s and α s,
>
> the homologous times be t and β t ;
>
> whence the homologous velocities are $v = s/t$ and $\alpha v = \alpha/\beta \;\; s/t$,
>
> the homologous accelerations $\phi = 2s/t^2$ and $\varepsilon\,\phi = \alpha/\beta^2 \;\; 2s/t^2$
>
> Now all oscillations which a body performs under the conditions above set forth with any two different amplitudes 1 and α, will be readily recognized as *similar* motions." [2.4]

Thus, in spite of noting that *similar* generally means *geometrically similar* at the time he was writing, Mach indulges Newton in the use of the adjective "similar" to indicate phoronomically [kinematically] similar structures, which are, properly speaking (in the terminology of Mach's day), not related by *similarity* but by *affinity* [i.e., by affine transformations]. After showing how elegantly theorems about centripetal motion can be obtained by such means, he remarks:

It is a pity that investigations of this kind respecting mechanical and phoronomical *affinity* are not more extensively cultivated, since they promise the most beautiful and most elucidative extensions of insight imaginable. " ([2.4], p. 205)

Thus Mach sees the great power of the notion of similar systems. In terms of clarification of the notion itself, though, which is the topic of this article, Mach's attention in his critique of Newton is on the "similar" in *similar systems*; he does not here discuss criteria for something counting as a *system*.

Newton is recognized for the concept today, as he has been throughout all of the nineteenth and twentieth centuries. In their "Similarity of Motion in Relation to the Surface Friction of Fluids" paper in early 1914, Stanton and Pannell credit George Greenhill with pointing out that the idea that relations "applicable to all fluids and conditions of flow" existed was "foreshadowed by Newton in Proposition 32, Book II of the 'Principia.' " ([2.6], p. 199). A. F. Zahm's 1929 report "Theories of Flow Similitude" [2.7] , also credits Newton for a method of "dynamically similar systems", citing Newton's Propositions 32 and 33. Also in many more recent works, including ([2.8], p. 86ff), ([2.9], p. 39 - 41), and ([2.5], p. 766)

2.3.2 Galileo

Although Newton seems to have been the first to use the term *similar systems*, Galileo's reasoning certainly used a notion of similar systems akin to, if not prescient of, Newton's in discussing not only the motions of the bob of a pendulum, but the more complicated behavior of machines and structures with mass; this is especially clear in his *Dialogues Concerning Two New Sciences*. Galileo's dialogue begins with Salviati (usually taken to be the voice of Galileo), recounting numerous examples of a large structure that has the same proportions and ratios as a smaller structure but that is not proportionately strong. In these opening pages of the dialogue, Salviati explains to a puzzled Sagredo that "if a scantling can bear the weight of ten scantlings, a [geometrically] similar beam will by no means be able to bear the weight of ten like beams." ([2.10], m.p. 52 - 53). The phenomenon of the effect of size on the function of machines of similar design holds among natural as well as artificial forms, Salviati explains: "just as smaller animals are proportionately stronger or more robust than larger ones, so smaller plants will sustain themselves better. ([2.10] , m.p. 52 - 53)

Perhaps the most well-known of Salviati's illustrations is about giants:

> " I think you both know that if an oak were two hundred feet high, it could not support branches spread out similarly to those of an oak of average size. Only by a miracle could nature form a horse the size of twenty horses, or a giant ten times the height of a man --- unless she greatly altered the proportions of the members, especially those of the skeleton, thickening the bones far beyond their ordinary symmetry." ([2.10] , m.p. 52 - 53)

Although Galileo's work opens with the wise participant in the dialogue reminding the others of the reasons for the lack of giant versions of naturally occurring life-forms, it soon proceeds to a case of a *valid* use of a small (artificial) machine to infer the behavior of a large (artificial) machine. But *the basis for the similarity is not merely geometric similarity*. Later in this same work of Galileo's, Sagredo makes use of Salviati's statement that the 'times of oscillation' of bodies suspended by threads of different lengths "are as the square roots of the string lengths; or we should say that the lengths are as the doubled ratios, or squares, of the times." From this, Sagredo uses one physical pendulum to infer the length of another physical pendulum:

> "Then, if I understood correctly, I can easily know the length of a string hanging from any great height, even though the upper attachment is out of my sight, and I see only the lower end. For if I attach a heavy weight to the string down here, and set it in oscillation back and forth; and if a companion counts a number of its vibrations made by another moveable hung to a thread exactly one braccio in length, I can find the length of the string from the numbers of vibrations of these two pendulums during the same period of time. " ([2.10], m.p. 140)

The reasoning that Sagredo uses to infer the length of one pendulum (the larger) from another (the smaller) is based upon the constancy of the value of a certain ratio involving the length and the frequency of a pendulum's oscillations. What Sagredo derives from the constancy of that ratio for all pendulums is a *law of correspondence* telling him how to find the corresponding length in the large pendulum from the length of the small (or vice versa) and the number of oscillations of the two pendulums observed during the same time period. (The time period itself during which the oscillations are observed is not needed; what is needed is only the (square of the) ratio of the number of oscillations of the two pendulums.) He works out an example:

> ". . . let us assume that in the time my friend has counted twenty vibrations of the long string, I have counted two hundred forty of my thread, which is one braccio long. Then after squaring the numbers 20 and 240, giving 400 and 57,600, I shall say that the long string contains 57,600 of those units [misure] of which my thread contains 400; and since my thread is a

> single braccio, I divide 57,600 by 400 and get 144, so 144 braccia is the length of the string." ([2.10] , m.p. 140)

Salviati (the voice of Galileo) responds approvingly to Sagredo's claim that this method will yield the length of the string: "Nor will you be in error by a span, especially if you take a large number of vibrations."   This is reasoning much like Newton's use of similar systems, in that one pendulum is regarded as being similar to another pendulum, so that the period of oscillation and length of one of the pendulums is homologous to the period of oscillation and length of the other.

Of course Galileo's reasoning here is not presented as a general method, as it is specific to pendulums, whereas Newton's notion of similar systems is. Nor do we find in Galileo's discussion here any explicit criteria for something being a machine that could serve to delineate the sorts of things on which this kind of reasoning could be used. However, Galileo's discussion does make clear that the two quantities that are considered homologous -- the "time of vibration" and the length of the pendulum string -- are fixed features of a pendulum, in contrast to other quantities such as the amplitude of the oscillations, or the weight of the bob: "Take in hand any string you like, to which a weight is attached, and try the best you can to increase or diminish the frequency of its vibrations; this will be a mere waste of effort. On the other hand, we confer motion on any pendulum, by merely blowing on it [. . .] This motion may be made quite large . . . yet it will take place only in accord with the time appropriate to its oscillations."   ([2.10] , m.p. 141)

Thus each of the two quantities --- length of the string, time of vibration --- of a given pendulum determines the other. The point germane to the topic of the history of similar systems, though, is this: *every pendulum is related to every other pendulum* by a law of correspondence. The law of correspondence relates each of these two quantities in one pendulum to its homologue in another pendulum. I think we can see this as akin to how Newton conceived similar systems to be related: by a law of correspondence between quantities in one system and their homologous quantities in the similar system. Only the length of the string and the time of vibration show up as homologous properties in the comparison of the two pendulums. Thus Galileo makes a point of distinguishing between quantities that characterize a given pendulum (length of string; time of oscillation) and quantities that do not (amplitude of oscillation; weight of bob), in addition to making the point about how *some* behaviors of *all* pendulums are related to each other by a *law of correspondence*.

Because the point is so often missed, it may be helpful to state it a slightly different way: Clearly Galileo sees that in a pendulum's behavior, the quantities that characterize a pendulum's behavior are related to each other in a fixed (though nonlinear) relation, as evidenced by his remarks about the time of oscillation of a pendulum being determined by the length of its string. Yet, rather than illustrating that one can use this relation to figure out the value of one quantity associated with a certain pendulum by measuring another quantity associated with *that same pendulum*, what Galileo is doing here is using a completely different method of inference: establishing a *law of correspondence* between two different pendulums. Then, from an observation of one quantity obtained experimentally on another pendulum chosen or constructed for the purpose, the law of correspondence he has established is invoked to *infer* the value of *the homologous quantity* in the pendulum. (In the passage from Galileo quoted above, the method was used to infer the length of one pendulum from the length of another pendulum.) It is the articulation of this method that justifies including Galileo along with Newton in a history of the concept of physically similar systems. [2.11]

2.4 Late Nineteenth & Early Twentieth Century

2.4.1 Introduction

By the late nineteenth century, mechanics and the mathematics used in it had changed dramatically from Newton's -- at least in terms of many of the mathematical methods used. The concept of mechanical similarity survived these major changes, though, and quite easily accommodated itself to the more advanced mathematics developed for mechanics. In fact, the notion of mechanical similarity was developed further, and more rigorously, into different kinds of similarity in mechanics --- geometrical, kinematical, and dynamical -- and extended to other areas of physics that had become quantitative, such as heat and electricity. The concept of similar systems survived, too, sometimes explicitly, sometimes only implicitly and in practice. More problematically, during the nineteenth century, the term was sometimes used to refer to something other than the rigorous notions associated with the term that were being developed in physics.

The advances in mathematics and physics to which the concept of similar systems and, along with it, the concept of similarity, was rather easily incorporated were not merely superficial matters such as the use of a different notation for calculus. By the late nineteenth century, there was widespread use of the more advanced mathematical methods that had been developed: partial differential equations and associated analysis methods for continuum mechanics, hydrodynamics, gas theory, electricity, and magnetism. During the eighteenth century, there had been many advances in mathematics and mechanics that transformed the methods of inquiry used into ones we would be at home with even today. The question of what constitutes a system shifts from asking not only how to decide when a configuration of bodies constitutes a system (Newton and Galileo seem to have thought in terms of systems of that sort), to also being able to ask what features of a function (or equation) indicate that the relations between quantities that it expresses have also delineated a system. For, it is functions that the eighteenth century gave mechanics, and functions represented or expressed "relations among quantities in nature", as Hepburn puts it. ([2.12], p. 129) As noted in section 1 above, when Buckingham articulated the concept of *physically similar systems* in 1914 [2.2], he did so by providing the "most general form of an equation", and, as seen in the excerpt quoted above, he did so by describing that form in terms of an equation using an unknown function:

$$\Psi\ (\Pi 1, \Pi 2, \ldots \Pi i\ , r', r'' \ldots) = 0,$$

i.e., the equation I have called the *Reduced Relation Equation of 1914*.

Buckingham did his doctoral work at the very end of the nineteenth century. Where were people employing or talking about the notion of similar systems during the late nineteenth century? By then, some notion of similar systems was known in theoretical physics, where it was occasionally explicitly discussed using the term 'similar systems', as well as in many branches of engineering, where it was involved, albeit sometimes implicitly or obliquely, in experimental investigations. Then, too, there were activities and investigations that did not fit neatly into one or the other of these categories, or straddled them. How did various thinkers producing these works think about and express the concepts associated with mechanical similarity and similar systems?

2.4.2 Engineering and similarity 'laws'

Similar Structures

In engineering and science of the nineteenth century, the main notion invoked when reasoning with similar machines or systems was that of a 'similarity law' or a 'similarity principle.' James Thomson (1822 - 1892); (brother of William Thomson, Lord Kelvin (1824 - 1907)) gave an influential paper in 1875 entitled 'Comparison of similar structures as to elasticity, strength, and stability' [2.13] that tried to identify and lay out the methodology involved in the engineering design of structures such as bridges and buildings, but he used some other interesting examples such as obelisks and umbrellas, too. Thomson's examples are often about how to vary some quantity such that two structures of different sizes are similar in one of these respects I refer to as behavioral: i.e., elasticity, strength, or stability. Thomson's paper was built upon and expanded in 1899 (by Barr [2.14]) and again in 1913 (by Torrance [2.15]).

The principle James Thomson identified was meant to be general. Yet, there were still different *kinds* of comparisons. In his 1875 paper, which became more widely available when his collected works were published in 1912, Thomson distinguished between two kinds of comparisons of similar structures, which, he said, were "very distinct, and which stand remarkably in contrast each with the other." One kind of comparison of similar structures is "in respect to their elasticity and strength for resisting bending, or damage, or breakage by similarly applied systems of forces." The other, contrasting kind was "comparisons of similar structures as to their stability, when that is mainly or essentially due to their gravity [weight] or, as we may say, to the downward force which they receive from gravitation." ([2.16], p. 362)

Thomson offered a "comprehensive but simple and easily intelligible principle" for the first kind of comparison: "Similar structures, if strained similarly within limits of elasticity from their forms when free from applied forces, must have their systems of applied forces, similar in arrangement and of amounts, at homologous places, proportional to the squares of their homologous linear dimensions." His reasoning in establishing this principle is a deductive argument special to solid mechanics, the mechanics of deformable bodies. "To establish this we have only to build up, in imagination, both structures out of similar small elements or blocks, alike strained, with the same intensity and direction of stress in each new pair of homologous elements built into the pair of objects." ([2.16], p. 362 - 363). These small elements or blocks are imagined to be so small in relation to the overall body that the stresses in them can be considered homogenous throughout the element or block. This is how the principle is derived, but the point of emphasis for both scientific understanding and engineering practice was that "similar structures of different dimensions must not be similarly loaded . . . if they are to be stressed with equal severity." In saying that the structures must not be similarly loaded, he is drawing attention to the part of the principle that says that the loads in the two similar structures must vary by the squares of their linear dimensions, rather than by the simple multiplicative factor that the linear dimensions do.

This was commonly what was meant at the time by a "similarity principle" or, sometimes "similarity law" or "law of similarity." Each one covered a certain class of cases. The point of the "principle" was usually to state how one variable --- e.g., density, stiffness -- was to be varied as another, such as length, was varied. One form such reasoning took was to show how the ratio of variables of one type varied as a ratio of another type of variable did: for instance, "If the scale ratio for any two orifices, i.e., the ratio of any two corresponding linear dimensions, is S, the ratio of the areas of corresponding elements of the orifices will be S^2 , while if similarly situated with respect to the water surface, their depths are proportional to S." ([2.17], p. 136) However, sometimes the similarity law or principle for a certain kind of behavior was stated simply as a ratio, the implication being that that ratio was invariant for similar systems; setting the ratio equal to one and rearranging terms yielded the relations between quantities that must be maintained in order to achieve similarity of that type.

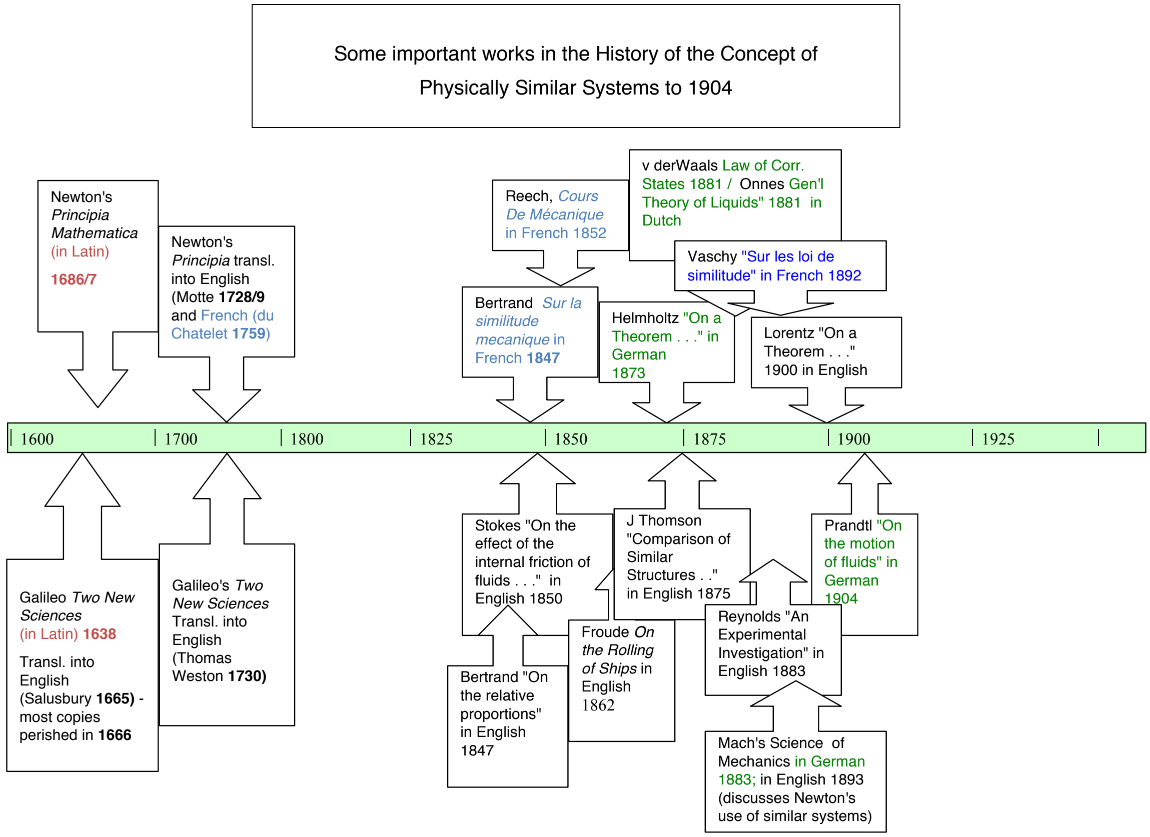

Figure 1. This timeline (not to scale) illustrates that the concept of similar systems is credited to Renaissance era thinkers Galileo and Newton, and was revived in the second half of the nineteenth century, when it was extended to chemistry, electromagnetic theory, heat, and thermodynamics.

Similar Interactions: A Law of Comparison for Model Ships

One of the most well-known engineering advances employing similarity and, implicitly, the notion of similar systems, was William Froude's (1810 - 1879) solution of significant, urgent and previously unsolved problems in ship design for the British Admiralty. ([2.18], p. 279; [2.5], [2.19], [2.20]) In the design of ships for stability and speed, not only does gravitational force enter into the consideration of a structure's behavior, but the ship's interaction with the water in which it is sitting or moving must also be considered.

Froude's reasoning about the stability of ships involved examining the motion of a pendulum in a resistive fluid ([2.21], p. 5ff, 15ff, 61): the same question Newton was addressing when he presented the proposition in which he introduced the idea of similar systems. Schaffer points out that, although the statement does not appear in the final version of the *Principia*, Newton had written that "if various shapes of ships were constructed as little models and compared with each other, one could test cheaply which was best for navigation" ([2.22], p. 90).

Unlike Newton, Froude does not seem to analyze the notion of similar systems in thinking about a pendulum in a resistive medium. However, the idea of relating quantities in one physical situation to those in another is predominant in Froude's work; it is, in fact, the topic of his main contributions to the problem of the efficient design of large ships driven by propellers. As Zwart has pointed out [2.5], the naval architect John Scott Russell had already constructed and tested many small models, but his experience had convinced him that the little models, though they had provided him with much pleasure, could provide no help in determining how large ships behaved. The exchange between Russell and Froude following Froude's reading of his 1874 paper was recorded in a transcript and so is available today ([2.23]), showing that the problem of how to extrapolate observations on the behavior of small models of ships when placed in water to the behavior of full size ships was considered unsolved when Froude took it on. ([2.23], [2.5] p. 15; [2.20] p. 128 - 130; [2.19]) Hagler also notes that Froude's confidence that the smaller model ships (some of which were over 20 feet long) could be used to infer the behavior of larger full scale ships was based in part on Rankine's investigations on streamlines. Froude explicitly discusses Rankine's work in his 1869 "The State of Existing Knowledge on the Stability, Propulsion and Seagoing Qualities of Ships." [2.20] He convinced the Admiralty to fund the construction of an experimental water tank to carry out the experiments he proposed. His methods for extrapolating from smaller, scale-models of ships in his water tank to the full size ship were vindicated when the Admiralty conducted full scale tests on the *HMS Greyhound* and Froude was able to compare the measurements taken on the full size *Greyhound* with those he had taken on his 1/16 model of the *HMS Greyhound* in his experimental tank. His "Law of Comparison" was soon adopted for all further ship design not only by the British Admiralty, but by the U. S. Navy, which constructed the Experimental Model Basin in Washington, D.C. in the 1890's. The Experimental Model Basin was constructed under the leadership of David Watson Taylor. Hagler [2.20] provides a good discussion of David Watson Taylor's writings on ship design; Taylor shows how the methodology used by the U.S. in almost all its naval design work in the first half of the twentieth century is ultimately traceable to this work Froude did in the nineteenth century.

Froude similarity was developed specifically for the purpose of using model experimentation for ship design. As with the similarity laws in mechanics, Froude similarity can be expressed in terms of a ratio, the Froude number, which is a dimensionless parameter. Though no notion of similar systems is defined, a nascent notion of similar systems was involved in practice, since similarity of situations is established when the Froude numbers for each of the two situations are equal. One formulation of the Froude number is $v / (gL)^{1/2}$ where v is a velocity, L is a length, and g the gravitational acceleration. The application of Froude similarity requires expertise; which velocity and characteristic length are relevant depends on the phenomenon being investigated. We can see from the form of the Froude dimensionless ratio, however, that quantities do not all scale linearly, much less by the same linear factor. Another point of note is that, as Froude similarity compares *homologous forces* as well as *homologous motions*, it is a kind of *dynamic* similarity, not merely a *kinematic* similarity.

Bertrand and Reech: The French Connection Between Newton and Froude

Many have pointed out that Froude took over results due to others, naming in particular French engineering professor Ferdinand Reech and French mathematician Joseph Bertrand, both of whom wrote on similarity methods in mechanics. ([2.24], p. 141ff; [2.25], p. 381; [2.26], p. 15; [2.18], p. 279) The extent to which this is true has been debated [2.24], but none deny that Froude holds a unique place as an experimentalist whose accomplishments advanced both the field of hydraulics and the industry of marine architecture. Ferdinand Reech (1805 - 1884), publishing in 1852 on topics he had lectured about much earlier, explicitly followed Newton's approach, discussing and deriving principles about how to relate observations of velocities and motions of one ship to other ships of different sizes. Like Newton, he considered bodies and forces on them, though he employed the term 'similar system' in his discussions when deriving laws of comparison. [2.28] It is Joseph Bertrand who seems to have taken a conceptual step beyond Newton, though he heaps quite a great deal of credit for his work upon Newton, as though he is doing little more than showing the consequences of Newton's theorems about similar systems.

Joseph Bertrand (1822 - 1900) produced many textbooks and treatises, including *Sur la similitude en mecanique*. He also published, in English in 1847, a sort of manifesto advocating that "persons occupied with the study of mechanics" attend to the theorem about similitude he derives using nineteenth century methods in mechanics, but for which he credits Newton. Of Newton's theorem about similar systems in the *Principia*, he writes:

> "This theorem constitutes a real theory of similitude in mechanics. It will be seen, that any system being given, there exists an infinite number of possible systems, which may be regarded

> as similar to it; and that, instead of a single kind of similitude, as in geometry, we may suppose four, viz., those of length, time, forces, and masses; each of these is, according to Newton's theorem, a consequence of the other three." ([2.27], p. 130)

Bertrand then went on in that same paper of 1847 to explain that he had "endeavoured to substitute . . . a proposition founded upon dynamic equations, and which does not differ mainly from the form employed by M. Cauchy to deduce from the equations of the movement of elastic bodies the laws of the vibrations of similar bodies, . . . but this theorem of M. Cauchy, although analogous to that of Newton, cannot be regarded as a corollary of the same"; using this instead, he deduces applications to laws of oscillation, centripetal force, speed of propagation of sound in various gases, and "a theorem relating to turbines." ([2.27], p. 130) Bertrand's concern seems to be twofold: (i) to get people in the field of mechanics to appreciate the power of the theory (or principle) of similitude in providing solutions to otherwise insoluble problems, and, (ii) to get people who use model experiments to understand the appropriate precautions that must be taken in designing experiments using small models to prevent errors that can be anticipated using the theory. He explains how the notion of similar systems, though it may look rather limited, is in fact sometimes indispensible, i.e., for problems not susceptible to a mathematical solution:

> "It is true that only proportional results can be deduced from [the principle]; and that, consequently, it will only serve to solve a question, when another of an analogous nature and of an equivalent analytical difficulty shall have been solved. It may, however, be of great utility to determine in certain cases the analogy which exists between the movements of the two systems, even supposing each of them not to be susceptible of strict theoretical determination." ([2.27], p. 131)

He gives an example of the usefulness of the principle: the performance of "experiments on a small scale" to ascertain "the value of a mechanical invention, which is too expensive to put in operation on a large scale." ([2.27], 131) What is interesting is that in this same paper where he is advocating use of the principle, he also discusses the kind of conundrums that arise in attempting to apply it to complicated cases such as a small-scale model of a locomotive; he cites an example of "an error which it is impossible to avoid, but which it is very essential to know." This 1847 paper published in England is thus a call to improving engineering practice by attending to theoretical derivations in mechanics, i.e., the theory of similitude. (Bertrand refers to it in the 1847 paper as the Cauchy theorem, which seems rather modest, for Cajori describes Bertrand as deriving "the principle of mechanical similitude" from "the principle of virtual velocities." ([2.25], p. 380) I mention Bertrand's 1847 paper here for its use of late eighteenth and nineteenth century mechanics.)

2.4.3 Similar Systems in Theoretical Physics: Lorentz, Boltzmann, Van der Waals and Onnes

Mechanical similarity held an important place among some researchers in theoretical physics in the late nineteenth century as well. The notion of similar systems was often employed in theories about the relationship of microscopic configurations to macroscopic phenomena, sometimes explicitly. Sometimes the term 'similar systems' was extended beyond the normal use it had had up to that time, too.

Lorentz

By the turn of the century, Henrik Lorentz (1853 - 1928) would note that "The consideration of similar systems has already proved of great value in molecular theory", as it had allowed Kamerlingh Onnes "to give a theoretical demonstration of Van der Waals's law of corresponding states." [2.29] The experimental confirmation of that law, Lorenz wrote, "has taught us that a large number of really existing bodies may, to a certain approximation be regarded as similar."

Lorentz had already developed a notion of corresponding states for use in electrodynamics by 1900. The context in which he made the observation above, though, was his paper "The Theory of Radiation and the Second Law of Thermodynamics", in which he was concerned with the question of the similarity in structure of different bodies that would be mandated by thermodynamics. ([2.29], p. 440) It would take us too far afield to explain everything that Lorentz was trying to do in this paper; here we restrict our discussion to what concept of 'similar systems' Lorentz employed or seems to have had in mind.

Lorentz' idea of 'similar systems' involves starting with one system and then constructing a second one from the first. Lorentz writes of 'comparing two systems'; what he says is that the systems he compares are : ". . . in a wide sense of the word, "similar", i.e., such that, for every kind of geometrical or physical quantity involved, there is a fixed ratio between its corresponding values in the two systems, . . ." [2.29] It is not clear on what basis he justifies being able to say that "We shall begin by supposing that, in passing from one system to another, the dimensions, masses and molecular forces may be arbitrarily modified", as this seems to require a certain kind of independence among the things being modified. He argues that "if the second system, as compared with the original one, is to satisfy Boltzmann's and Wien's laws", that "we shall find that the charges of the electrons must remain unaltered."

He first describes a certain system S which includes a "ponderable body" enclosed in a space. Some of the features of S are delineated (he ascribes "an irregular 'molecular' motion and the "power of acting on one another with certain 'molecular' forces" to the particles making up the body, for instance, and adds that some are electrically charged) but other features are not ("there may be other (molecular) forces of another kind, acting on the electron.") ([2.29], p. 443)   The description of the "really existing" system S is meant to pick out something that actually exists, in contrast to the system S', which "perhaps will be only an imaginary one." ([2.29], p. 444)   To complete the description of the state of S' , "we indicate, for each of the physical quantities involved, the number by which we must multiply its value in S, in order to obtain its value in S' at corresponding points and times." He then explores the constraints on these numbers; some are constrained by laws of motion, but others are not. This leaves him free to "imagine a large variety of systems S', similar to S, and which must be deemed possible as far as our equations of motion are concerned." ([2.29], p. 445)

Lorentz uses the notion of similar systems to explore the constraints on theory, as opposed to using theory to state how one can construct a system S' to be similar to a certain system S, in order to make inferences about one of the systems based upon observations about the other. This seems a different use of the notion than Galileo or Newton made of it; it also allows contemplation of unprecendented kinds of similarity. It may, Lorentz realizes, even give rise to systems of a different ontological status; he explains why that, too, may be useful:

> "It might be argued that two bodies existing in nature will hardly ever be similar in the sense we have given to the word, and that therefore, if S corresponds to a real system, this will not be the case with S'. But this seems to be no objection. Suppose, we have formed an image of a class of phenomena, with a view to certain laws that have been derived from observation or from general principles. If, then, we wish to know, which of the features of our picture are essential and which not, i.e., which of them are necessary for the agreement with the laws in question we have only to seek in how far these latter will still hold after different modifications of the image; it will not at all be necessary that every image which agrees in its essential characteristics with the one we have first formed corresponds to a natural object." ([2.29], p. 447-448)

Thus, Lorentz's exploratory use of similar systems in fields beyond mechanics was motivated by the example of van der Waals' and Onnes' highly successful results using mechanical similarity to derive new theoretical results.

Van der Waals and Onnes

In his 1881 "General Theory of Liquids", Onnes argued that van der Waals' 'Law of Corresponding States', which had just been published the previous year, could be derived from scaling arguments, in conjunction with assumptions about how molecules behaved. Van der Waals was impressed with the paper, and a long friendship between the two ensued. Van der Waals was awarded the Nobel Prize in Physics in 1910 for "The equation of state for gases and liquids' [2.30], and Onnes was awarded it in 1913 [2.31], for "Investigations into the properties of substances at low temperatures, which have led, amongst other things, to the preparation of liquid helium." In his lecture delivered for the occasion, Onnes highlighted the connection between his investigations into properties of substances at low temperatures and similarity principles:

> ". . . [F]rom the very beginning . . . I allowed myself to be led by Van der Waal's theories, particularly by the law of corresponding states which at that time had just been deduced by Van der Waals.
>
> This law had a particular attraction for me because I thought to find the basis for it in the stationary mechanical similarity of substances and from this point of view the study of deviations in substances of simple chemical structure with low critical temperatures seemed particularly important." ([2.32], p. 306)

What is special about the low temperatures Onnes needed to achieve in order to liquefy helium is that, according to the kinetic theory of gases on which van der Waals' equation of state was based, there would be much less molecular motion than in the usual kinds of cases considered. Onnes's approach in looking for the foundation of the law of corresponding states has a slightly different emphasis than the kinetic theory of gases. Boyle's Law (often called the ideal gas law) and van der Waals' equation were based on investigating the relationship between the microscale (the molecular level) and the macroscale (the properties of the substance, such as temperature and density.) But Onnes was instead looking at the foundation for the similarity of states. Like Van der Waals, he looked to mechanics and physics for governing principles, but Onnes pointed out that it was also useful to look at principles of similarity. At low enough temperatures, where motion of the molecules was not the predominant factor, the relevant principles of similarity would be principles of static mechanical similarity, as opposed to dynamical similarity.

The criterion for similarity Onnes developed arose out of investigations into the transition from one regime to another. This had been the case in work in hydrodynamics, too; In Osborne Reynolds work, discussed below, it was the critical point at which fluid flow underwent a transition from laminar to turbulent flow (or, in his terminology, from "lamellar" to "eddying" flow) that led to the identification of the dimensionless parameter that later became known as Reynolds Number. The Reynolds

Number is in a way a criterion of similarity, in that fluid systems with the same Reynolds Number will be in the same flow regime, regardless of the fluid. So it was with thermodynamics, Onnes showed: the critical point at which a substance undergoes a transition from the gaseous state to the liquid state led to the identification of a criterion of similarity of states that held for all substances.

Van der Waals was interested in the continuity of states and used the critical values of pressure, volume, and temperature in a brilliant way to normalize pressure, volume and temperature. He defined "reduced pressure", "reduced volume", and "reduced temperature" to yield an equation of state in which none of the parameters that are characteristic of a particular substance appear. As Levelt Sengers notes, "This is a truly remarkable result." The equation of state is "universal; all characteristics of individual fluids have disappeared from it or, rather, have been hidden in the reduction factors. The reduced pressures of two fluids are the same if the fluids are in *corresponding states*, that is, at the same reduced pressure and volume." ([2.33], p. 25) This is an important part of the history of similar systems in that the principle of corresponding states allowed the production of curves representative of all substances from experiments on a particular substance:

> "The principle of corresponding states . . . frees the scientist from the particular constraints of the van der Waals equation. The properties of a fluid can now be predicted if only its critical parameters are known, simply from correspondence with the properties of a well characterized reference fluid. Alternatively, unknown critical properties of a fluid can be predicted if its properties are known in a region not necessarily close to criticality, based on the behavior of the reference fluid." ([2.33], p. 26)

Onnes used this insight about corresponding states to set up an experimental apparatus to liquefy helium, which has an extremely low critical temperature. What is so exciting about his story is that he had to rely on the law of corresponding states to estimate the critical temperature so that he would know where to look --- that is, so that he would know what conditions to create in order for helium to liquefy. What is especially relevant to the history of the notion of *physically similar systems* is that he did more than just use van der Waals' law of corresponding states. He also gave a foundation for it that was independent of the exact form of van der Waals' equation and did not depend on results in statistical mechanics. Instead, he used *mechanical similarity*:

> "Kamerlingh Onnes's (1881) purpose is to demonstrate that the principle of corresponding states can be derived on the basis of what he calls the principle of similarity of motion, which he ascribes to Newton. He assumes, with Van der Waals, that the molecules are elastic bodies of constant size, which are subjected to attractive forces only when in the boundary

> layer near a wall, since the attractive forces in the interior of the volume are assumed to balance each other . . . He realizes this can be valid only if there is a large number of molecules within the range of attraction . . . [Onnes] considered a state in which N molecules occupy a volume v, and all have the same speed u (no Maxwellian distribution!). The problem is to express the external pressure p, required to keep the system of moving particles in balance, as a function of the five parameters. He solves this problem by deriving a set of scaling relations for M, A, v, u, and p, which pertain if the units of length, mass, and time are changed." ([2.33], p. 30)

Onnes provides a criterion for corresponding states based on these scaling relations, along with assumptions about what the molecular-sized objects are like. Sengers remarks:

> "Two fluids are in corresponding states if, by proper scaling of length, time and mass for each fluid, they can be brought into the same "state of motion." It is not clearly stated what he means by this, but he must have had in mind an exact mapping of the molecular motion in one system onto that of another system if the systems are in corresponding states." ([2.33], p. 30)

Sengers illustrates what being in the same "state of motion" means "in modern terms":

> ". . . suppose a movie is made of the molecular motions in one fluid. Then, after setting the initial positions and speed of the molecules, choosing the temperature and volume of a second fluid appropriately, and adjusting the film speed, a movie of the molecular motion in a second fluid can be made to be an exact replica of that in the first fluid." ([2.33], p. 30)

Appeal to such imagined visual images is very much in keeping with nineteenth century science, and one can see here an attempt to generalize Newton's use of similar systems in the *Principia* to thermodynamics. Onnes used the principle of corresponding states for more than visualizing, though, and, even, for more than theorizing; he used it to show how one could make a prediction about one fluid from knowledge about another. Wisniak explains:

> Kamerlingh Onnes proposed to use the law of corresponding states to examine the possibility of cooling hydrogen further by its own expansion. He then used this law to predict from the known experience with oxygen what was to be expected from the apparatus for the cooling of hydrogen: [quoting Onnes:] 'But let us return to the thermodynamically corresponding substances. If two such substances are brought in corresponding engines and if these engines are set in motion with corresponding velocities, then they will run correspondingly as

> long as there is given off a corresponding quantity of heat in the corresponding times by the walls of the machine.' ([2.34], p. 569)

Thus Onnes has, not just corresponding motions and times, as in mechanical similarity, but corresponding quantities of heat. Wisniak continues:

> He [Onnes] then introduced the notion of thermodynamically corresponding operations to argue that 'if then in a model, working with oxygen, after a given time a given volume of liquid oxygen is found, there will be obtained in the corresponding hydrogen apparatus after the corresponding time a corresponding volume of liquid hydrogen.' " ([2.34], p. 569)

By 'model' here, Onnes clearly means physical model, and the model includes the contained gases such as oxygen and hydrogen. The model is an actual physical model: a physical setup, an actual, physical machine. By the end of the nineteenth century, the physics of machines included the thermodynamics of machines. And, as in Newton and Galileo's day, one could talk both about imagined similar systems, and about actual similar machines.

Maxwell and Boltzmann

As several scholars have noted, Ludwig Boltzmann (1844 - 1906) mentioned "similar systems" in his investigations into the theory of gases, too. It's been noted that, in his 1884 and 1887 papers, Boltzmann "tried to deepen the foundation of the new theory [that was to become known as statistical mechanics] by introducing the concept of 'Ergoden' -- meaning a collection (ensemble) of similar systems (of gas molecules) having the same energy but different initial conditions" ([2.35], pp. 56-57). Stephen G Brush, also citing Boltzmann's 1884 and 1887 papers, remarks that

> "There has been considerable confusion about what Maxwell and Boltzmann really meant by ergodic systems. It appears that they did not have in mind completely-deterministic mechanical systems following a single trajectory unaffected by external conditions; [. . .]
>
> In fact, when Boltzmann first introduced the words Ergoden and ergodische, he used them not for single systems but for collections of similar systems with the same energy but different conditions. In these papers of 1884 and 1887, Boltzmann was continuing his earlier analysis of mechanical analogies for the Second Law of Thermodynamics, and also developing what is now (following J. Willard Gibbs) known as "ensemble" theory. Here again, Boltzmann was following a trail blazed by Maxwell, who had introduced the ensemble concept in his 1879

> paper. But while Maxwell never got past the restriction that all systems in the ensemble must have the same energy, Boltzmann suggested more general possibilities and Gibbs ultimately showed that it is most useful to consider ensembles in which not only the energy but also the number of particles can have any value, with a specified probability." ([2.36] pgs 75 - 76).

What these commentators on Boltzmann are referring to in mentioning the influence of Maxwell are Maxwell's remarks in his "On Boltzmann's Theorem on the average distribution of energy in a system of material points." [2.37] There, Maxwell wrote, speaking of the case "in which the system is supposed to be contained within a fixed vessel":

> I have found it convenient, instead of considering one system of material particles, **to consider a large number of systems similar to each other in all respects except in the initial circumstances of the motion**, which are supposed to vary from system to system, the total energy being the same in all. In the statistical investigation of the motion, we confine our attention to the number of these systems which at a given time are in a phase such that the variables which define it lie within given limits. (Emphasis in bold added.)
>
> " If the number of systems which are in a given phase (defined with respect to configuration and velocity) does not vary with the time, the distribution of the systems is said to be *steady*. " ([2.37], pgs. 715ff)

It is not clear how the use of the notion of similar systems here, i.e., in forming ensembles in thermodynamics in order to study their behavior statistically, might be related to either Newton's notion of similar systems or the notion involved in the principle of corresponding states. It is certainly a use of similar systems that is very different from using one system experimentally to infer the values of quantities in another. So, if, as Brush's comment implies, Boltzmann was thinking of more general kinds of similar systems, it seems he was no longer restricting the notion of similar systems to systems that are behaviorally similar to each other with respect to motions, and he was not restricting its use to the use of one system or machine to infer the behavior of another.

Yet Boltzmann's departure from Newton's use of the term similar systems was almost certainly not a matter of confusion on Boltzmann's part about the notion in the sense Newton had used it, for Boltzmann's encyclopedia entry on models [2.38] shows that he was well aware of, and respected the distinctive nature of, the use of experimental models of machines, in which one machine is specially constructed in order to infer the behavior of another. Boltzmann, in fact, associates the latter kind of model with Newton's insights.

On the approach in which physical models constructed with our own hands are actually a continuation and integration of our process of thought, Boltzmann says in that encyclopedia article ("Model"), "physical theory is merely a mental construction of mechanical models, the working of which we make plain to ourselves by the analogy of mechanisms we hold in our hands." [2.38] In contrast, Boltzmann explicitly described experimental models as of a different sort than the kind with which he was comparing mental models, and explained why they must be distinguished:

> "A distinction must be observed between the models which have been described and those experimental models which present on a small scale a machine that is subsequently to be completed on a larger, so as to afford a trial of its capabilities. Here it must be noted that a mere alteration in dimensions is often sufficient to cause a material alteration in the action, since the various capabilities depend in various ways on the linear dimensions. Thus the weight varies as the cube of the linear dimensions, the surface of any single part and the phenomena that depend on such surfaces are proportionate to the square, while other effects --- such as friction, expansion and condition of heat, etc., vary according to other laws. Hence a flying-machine, which when made on a small scale is able to support its own weight, loses its power when its dimensions are increased. The theory, initiated by Sir Isaac Newton, of the dependence of various effects on the linear dimensions, is treated in the article UNITS, DIMENSIONS OF. ([2.38])

The use of a flying-machine to illustrate the point was not incidental; in his "On Aeronautics", Boltzmann urged research into solving the problem of flight, and expressed his opinion that experimentation with kites was the appropriate approach. The complexities of airflow over an airplane wing, he said, were too difficult to study using hydrodynamics. ([2.39], p. 256) Yet, the basis for extrapolating from experiments on a kite or flying machine from one observed situation to another, unobserved, situation (even with a machine of the same size) owes something to hydrodynamics. The dimensionless parameters yielding the appropriate correspondences between homologous quantities for kites and flying-machines were provided by Helmholtz's innovative use of the equations of hydrodynamics.

2.4.4 Similar systems in theoretical physics

Stokes and Helmholtz

Hermann von Helmholtz (1821 - 1894) , like Ludwig Boltzmann and so many other physicists of the nineteenth century, contributed to the scientific literature on research into flight. Some of these contributions took the form of investigations concerning the earth's atmosphere. Six of the twenty papers in the important and selective 1891 anthology "The Mechanics of the Earth's Atmosphere: A collection of translations by Cleveland Abbe" [2.40] were by Helmholtz; one of these was his 1873 "On a Theorem Relative to Movements That Are Geometrically Similar in Fluid Bodies, Together with an Application to the Problem of Steering Balloons." [2.41], [2.42] It is the only one of Helmholtz's papers in that volume that explicitly addresses an application to the problem of flight. What is relevant to the history of the concept of similar systems is the kind of reasoning he uses in the paper.

Helmholtz's starting point is "the hydro-dynamic equations" which, he argues, can be considered "the exact expression of the laws controlling the motions of fluids." ([2.41], p. 67; [2.42]) What about the well-known contradictions between observations and the consequences of the equations? Those, he argues, are only apparent contradictions, which disappear once the phenomenon of "surfaces of separation" are no longer neglected; his "On Discontinuous Motions in Liquids" [2.43] [2.44], also included in the same collection of translations, aims to establish their existence.

The "Discontinuous Motions" paper [2.43] is an extraordinarily interesting contribution to the methods of reasoning by analogy between fluid currents, electrical currents, and heat currents. For, the paper begins by pointing out that "the partial differential equations for the interior of an incompressible fluid that is not subject to friction and whose particles have no motion of rotation" are precisely the same as the partial differential equations for "stationary currents of electricity or heat in conductors of uniform conductivity." ([2.43], p. 58) Yet, he notes, even for the same configurations and boundary conditions, the behavior of these different kinds of currents can differ. How can this be? It would be easy to assume that the difference is a matter of the equations being, in the case of hydrodynamics, an "imperfect approximation to reality", possibly due to friction or viscosity. Yet, Helmholtz argues, various observations indicate this is not plausible. Instead, he proposes, the difference in behavior between fluid currents on the one hand and electrical and heat currents on the other is due to "a surface of separation" that exists or arises in the case of the fluid. In some situations, "the liquid is torn asunder", whereas electricity and heat flows are not. Though the main point of the paper is to propose his detailed account of what happens in the liquid to cause this difference (the pressure becomes negative), it is interesting, especially in the context of nineteenth century, that Helmholtz is discussing a case in which physical entities described by the same partial differential equations do *not* behave in the same way. Yet, once the existence of discontinuous motions in fluids is recognized, Helmholtz says, the contradictions that "have been made to appear to exist between many apparent consequences of the hydro-dynamic equations on one hand and the observed reality on the other" will then "disappear." ([2.41], p. 67)

The problem with the hydrodynamic equations is not that are wrong, for they are not; they are "the exact expressions of the laws controlling the motions of fluids." The problem is that "it is only for a relatively few and specially simple experimental cases that we are able to deduce from these differential equations the corresponding integrals appropriate to the conditions of the given special cases." So, the hydrodynamic equations are impeccable; it's their solution that is the problem. Simplifying is not going to work, either, since in some cases "the nature of the problem is such that the internal friction [viscosity] and the formation of surfaces of discontinuity can not be neglected." These surfaces of discontinuity present a very fundamental problem to finding a neat solution, too, for "The discontinuous surfaces are extremely variable, since they possess a sort of unstable equilibrium, and with every disturbance in the whirl they strive to unroll themselves; this circumstance makes their theoretical treatment very difficult." Theory being of very little use in prediction here, "we are thrown almost entirely back upon experimental trials, . . . as to the result of new modifications of our hydraulic machines, aqueducts, or propelling apparatus."

That was how things stood but, Helmholtz says, there is another method, one that is neither a matter of prediction from theory nor an experimental trial of the machine whose behavior one wishes to predict. His description deserves to be read closely:

> In this state of affairs [the insolubility of the hydrodynamic equations for many cases of interest] I desire to call attention to an application of the hydro-dynamic equations that allows one to transfer the results of observations made upon any fluid and with an apparatus of given dimensions and velocity over to a geometrically similar mass of another fluid and to apparatus of other magnitudes and to other velocities of motion." ([2.41], p. 68)

The method Helmholtz is referring to, which he presented in this now-classic paper in 1873, thus differs from deducing predictions from theory in the same way that Newton's notion of similar systems and Galileo's use of one pendulum to inform him about another differ from deducing predictions from theory: theory is involved in the inference, but the *way* that theory is involved is to allow someone to "transfer the results of observations" made on one thing (system, machine, mass of fluid, apparatus) over to another thing (system, machine, mass of fluid, apparatus).

The way Helmholtz proceeds to establish this different "application of the hydro-dynamic equations" appeals to a formalism not available to either Galileo or Newton, though: "[t]he equations of motion in the Eulerian form introducing the frictional forces, as is done by Stokes." Although Helmholtz does not use the term 'similar system' here, Stokes did use it, in his "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums", presented in 1850. [2.45] In that paper, before attempting a solution of some flow equations, Stokes first examined "the general laws which follow merely from the dimensions of the several terms which appear in the equations." To do this, Stokes had employed 'similar systems':

> "Consider any number of similar systems, composed of similar solids, oscillating in a similar manner in different fluids or in the same fluid. Let a, a', a'' . . . be homologous lines in the different systems; T, T', T'' . . . corresponding times, such for example as the times of oscillation from rest to rest. Let x, y, z be measured from similarly situated origins, and in corresponding directions, and t from corresponding epochs, such for example as the commencements of oscillations when the systems are beginning to move from a given side of the mean position. " ([2.45], 1850)

Then, Stokes says, the form of the equations shows that the equations being satisfied for one system will be satisfied for all the systems, if certain relations between the quantities in those equations are met, which he lays out. He adds the condition needed in order for the systems to be dynamically similar; then, if we "compare similarly situated points", the motions in the systems will also be similar, and the "resultants [of pressure of the fluids on the solids] in two similar systems are to one another" in a certain ratio that he shows how to obtain. Stokes does not end there; the paper contains further discussion about establishing similarity between the two systems, having to do with how the fluids are confined. This much about Stokes should give a general idea of how he conceived of and used the notion of 'similar systems.'

Helmholtz' approach probably owes much to Stokes; David Cahan's study "Helmholtz and the British Scientific Elite: From Force Conservation to Energy Conservation" identifies Stokes as one of the British elite with whom Helmholtz built a relationship during the 1850s and 1860s [2.46] Helmholtz does refer to Stokes, to be sure, but there is also something creative in what he does in his own paper. Helmholtz turns the idea of how the eulerian equations for flow are related to similar systems around, so that he sees how one might, in principle at least, use the equations in conjunction with model experiments on ships to inform us about how to predict and direct the motions of balloons (dirigibles).

The discussion and derivation of the conclusions Helmholtz reaches for all the cases he considers in his 1873 paper [2.41] is too long to summarize here, but a few points can be mentioned:

(i) Helmholtz's strategy is to consider two given fluids and use the hydrodynamic equations to infer the way or ways in which their quantities must be related. For the first fluid, the direction of its coordinate axes are designated x, y, and z; the components of velocity associated with them are designated u, v, and w. The time t, fluid density ε, pressure p, and coefficient of friction k (viscosity) are also named, which allows him to construct the equations of motion of the first fluid in eulerian form. The second fluid is then given designations of U, V, W for the components of velocity (in coordinate axes X, Y, Z), the pressure P, the fluid density E, and the viscosity constant by K. Three additional constants q, r, and n are named, so that the quantities in the second fluid can then be related to the designated quantities in the first fluid such that the quantities in the second fluid will also satisfy the equations of motion that were constructed for the first fluid. For example, the densities of the two fluids are related by $E = r\varepsilon$; their coefficients of friction are related by $K = qk$; and the velocity components, by $U = nu$, $V = nv$, and $W = nw$. Then the pressures must be related by $P = n^2 r p +$ constant, and the times in the two fluids must be related by $T = q t / n^2$. Putting the terms for the quantities of the second fluid expressed in terms of the quantities of the first fluid into the equations of motion for the first fluid shows that they satisfy those equations.

(ii) The nature of the two fluids determines how their densities and coefficients of friction are related to each other, so two of the three constants, q and r, are determined. Helmholtz then considers various kinds of cases (e.g., compressible vs incompressible, cohesive vs non-cohesive (liquid vs gaseous fluids), certain boundary conditions, whether friction can be neglected), and what they permit to be inferred about the third undetermined constant n. The paper contains a variety of interesting remarks, some of great practical significance, about how other quantities of the two fluids (e.g., velocity of sound) must be related to each other.

(iii) When Helmholtz comes to addressing the practical problem mentioned in the title: "driving balloons forward relative to the surrounding air," he uses, not two masses of air in which two different air balloons are situated, but, rather: for the second fluid, a *mass of air in which an air balloon is situated*, and, for the first fluid, a *mass of water in which a ship is situated*. He writes: "our propositions allow us to compare this problem [driving balloons forward relative to the surrounding air] with the other one that is practically executed in many forms, namely, to drive a ship forwards in water by means of oar-like or screw-like means of motion. . . . we must . . . imagine to ourselves a ship driven along under the surface. Such a balloon which presents a surface above and below that is congruent with the submerged surface of an ordinary ship scarcely differs in its powers of motion from an ordinary ship." ([2.41], p. 73) Then, letting "the small letters of the two above given systems of hydro-dynamic equations refer to water and the large letters to the air" he examines the practical conditions under which he can "apply the transference from ship to balloon with complete consideration of the peculiarities of air and water."

Helmholtz's discussion contains many subtle points concerning what would need to be considered if actually building the kind of ship needed to model an air balloon. As he indicates, the practical considerations involved in applying the method are not trivial and can sometimes even be prohibitive; nevertheless the point is that the approach he outlines permits one to make a proper analysis of any such comparison, or "transference" using the hydro-dynamic equations, and can sometimes yield a solution when the hydro-dynamic equations are insoluble. Evidence of the influence and significance of this particular paper of Helmholtz's into the twentieth century appears in Zahm's "Theories of Flow Similitude" [2.7]. Zahm identifies three methods, one with Isaac Newton, one with Stokes and Helmholtz, and one with Rayleigh. The sole paper by Helmholtz cited there is this paper of 1873. [2.41]

The significance to the history of physically similar systems is that Helmholtz's account of his method involves a differential equation, that the equation is so central to the account, and that how it is involved is stated so clearly. What is not stated very clearly is whatever it is that plays the role of system; sometimes Helmholtz seems to be saying the transference is from one mass of fluid to another; other times, that it is between the objects situated within the fluid. If we denote whatever ought to play that role by the term system, though, we would say that, in Helmholtz's analysis, the hydro-dynamic equations are not only the core of the criterion for allowing "transference" of results observed in one situation to another, but they indirectly give a criterion for, and thus specify, what a system is, i.e., what the similarity in 'similar systems' is between. If we use the term system this way, then it is implicit in Helmholtz's account that a system is the mass and its configuration (including anything situated within the mass), with boundary conditions, to which the partial differential equation applies. We might also take note of the fact that what the equation applies to is in equilibrium (though not necessarily static equilibrium). The governing differential equations are important, too, in the specification of what quantities need to be considered in the analysis.

Yet, Helmholtz is careful not to overreach concerning what can be deduced from the *form* of an equation; as he points out in his "Discontinuous Motions" paper [2.43] when investigating the example of fluid being "torn asunder": *just because a certain situation is governed by an equation of the same form as another equation governing a different situation, does not in itself guarantee that the two situations will exhibit analogous behavior* --- even when the configuration and boundary conditions are also analogous. It is for the confluence of all these points that I consider Helmholtz' 1873 paper [2.41] such a major contribution to the history of the concept of similar systems.

Reynolds

Osborne Reynolds' (1842 - 1912) work and influence on similarity was immense, but it was by no means his only major achievement. [2.47] Unless one has invested the time required to read a significant part of his work, any evaluation of his achievements and influence will sound like hyperbole. I mention here only his most significant contribution relevant to the history of the concept of similar systems.

The decisive difference Reynolds made in the notion of similar systems was to show that it applied beyond well-behaved regimes. In fact, he showed, it applied during the transition between well-behaved regimes and chaotic ones. And, not only that, but that the critical point of transition between well-behaved (laminar flow) and chaotic (turbulent flow) regimes could be characterized, and characterized by a parameter that was independent of the fluid. Stokes put it well in the statement he made in his role as President of the Royal Society on the occasion of presenting a Royal Medal to Reynolds on November 30, 1888:

> "In an important paper published in the *Philosophical Transactions* for 1883, [Osborne Reynolds] has given an account of an investigation, both theoretical and experimental, of the circumstances which determine whether the motion of water shall be direct or sinuous, or, in other words, regular and stable, or else eddying and unstable. The dimensions of the terms in the equations of motion of a fluid when viscosity is taken into account involve, as had been pointed out, the conditions of dynamical similarity in geometrically similar systems in which the motion is regular; but when the motion becomes eddying it seemed no longer to be amenable to mathematical treatment. But Professor Reynolds has shown that the same conditions of similarity hold good, as to the average effect, even when the motion is of the eddying kind; and moreover that if in one system the motion is on the border between steady and eddying, in another system it will also be on the border, provided the system satisfies the above conditions of dynamical as well as geometrical similarity." ([2.45], p. 234)

Stokes does not here use the term 'similar systems', but that is what he means in using the grammatical construction: "if in one system . . ., in another system it will also . . ., provided the system satisfies the above conditions of dynamical as well as geometrical similarity." What this means is that there are some (experimentally determined) functions of a certain (dimensionless) parameter that describe the behavior of fluids, whatever the fluid. The parameter is not a single measured quantity such as distance, velocity, or viscosity; rather, it is a ratio involving a number of quantities (e.g., density, velocity, characteristic length, and viscosity). The ratio is without units, as it is dimensionless. Reynolds is often cited for coming up with the criterion of dynamical similarity, but obviously, the idea predated his work, as Stokes' statement recognizes. Rather, what Reynolds did that was so decisive for the future of hydrodynamics (and aerodynamics) was, as he explained in a letter to Stokes, that there was a *critical value (or values)* for 'what may be called the parameter of dynamical similarity [the dimensionless parameter mentioned earlier, which is now known as Reynolds number].' ([2.49], p. 233.)

In the excerpt from his statement quoted above, Stokes puts his finger on why what Reynolds did was so significant in terms of a fundamental understanding of fluid behavior, but Reynolds' 1883 paper also had practical significance for research in the field as well. Stokes continued:

> "This is a matter of great practical importance, because the resistance to the flow of water in channels and conduits usually depends mainly on the formation of eddies; and though we cannot determine mathematically the actual resistance, yet the application of the above proposition leads to a formula for the flow, in which there is a most material reduction in the number of constants for the determination of which we are obliged to have recourse to experiment." ([2.48], p. 234)

It is not surprising that interest in applying the methods of similar systems grew in the subsequent years.

Prandtl

Prandtl's work in experimental hydrodynamics and aerodynamics is singularly prominent in work done in the field in Germany in the twentieth century. Ludwig Prandtl (1873 - 1953) was an ex-engineer-turned-professor in the Polytechnic at Hanover conducting research on air flow when he presented a paper at the Third International Congress of Mathematicians in 1904: "Motion of fluids with very little viscosity" [2.50]. It didn't make much of a splash -- except with Felix Klein, then a prominent mathematician at the University of Gottingen. In his paper, Prandtl laid out a plan to treat flow around bodies. What he proposed was that the problem be analyzed into several distinct questions: (i) what happened at the boundary of the "skin" that formed against the body, and what happened on each side of it, i.e., (ii) what happened in the fluid on the side of the boundary that was within the "skin", and (iii) what happened in the fluid on the other side of the boundary, within the main fluid stream. [2.50] Prandtl showed that, in the mainstream, the mathematical solutions that were obtained by neglecting viscosity could be applied to even these real fluids. In the part of the flow under the "skin" formed around the body, however, viscosity did have to be taken into account. And, crucially, what happened in the mainstream -- the formation of vortices --- set conditions for what happened on the other side of the boundary, via setting boundary conditions at the interface between the two layers. Klein saw the potential of Prandtl's approach and brought him to a post in Gottingen right away. [2.11]

In Gottingen, Prandtl then made use of the knowledge that had been developed about hydrodynamical similarity, using a water tank for some of his most famous experiments. Rather than towing an object in the water, though, Prandtl used a water-wheel to move the fluid in the water tank, much like fans were being used to push air through wind tunnels (which by then were replacing the whirling arm or moving railcar apparatuses used earlier in aerodynamical research.) Prandtl's results for airfoils were based on hydrodynamical similarity and, hence, on the concept of dynamically similar systems. His approach went beyond that, too, including fundamental questions he addressed by combining mathematical solutions and experimental results in an uncommon kind of synthesis. William Lanchester in England also employed dynamic similarity and authored significant works about his theoretical and experimental research in aerodynamics; his visit to Prandtl in 1908 may have contributed somewhat to Prandtl developing these ideas, since Prandtl was in a position to understand Lanchester's work, and appreciate its significance. [2.11]

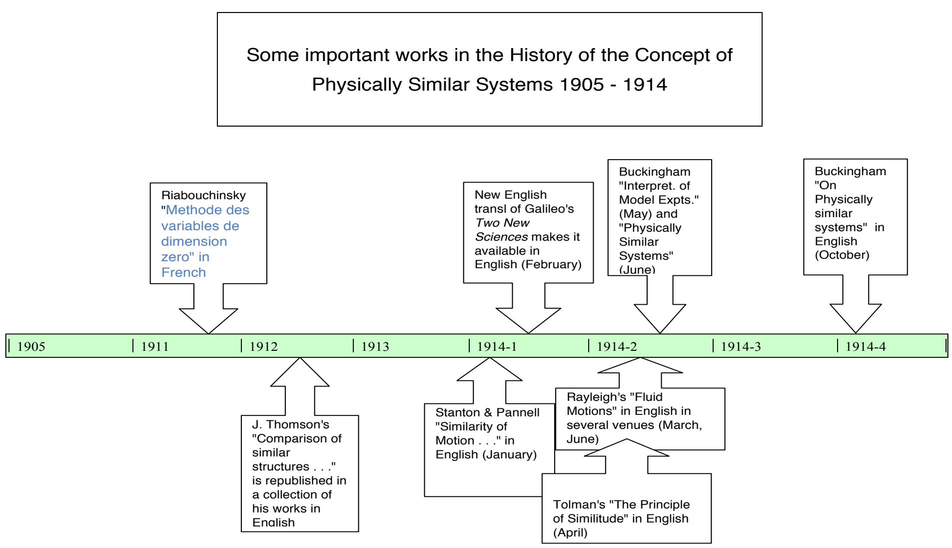

Figure 2. This timeline (not to scale) shows there was a lot of discussion about and interest in issues regarding similarity in 1914 and the years immediately preceding. In 1914 the term "physically similar systems" comes into use.

Rayleigh

Lord Rayleigh (John William Strutt) (1842 - 1919) became a proponent of dynamic similarity in Great Britain. The context of his advocacy of the method was part scientific, part political. The scientific part was an appreciation for the significance of dynamical similarity in effective research; the political part was a feeling that Britain ought not be left behind in aeronautical research. His political, social, and professional prominence put him in a position to be an effective advocate. He was the first president of the British Advisory Committee on Aeronautics, founded in 1909. Its first report includes his "Note as to the Application of the Principle of Dynamical Similarity [2.51]; he introduces the topic by first citing Lanchester for one application of the principle of dynamical similarity, then noting his own communications of "a somewhat more general statement which may be found to possess advantages." The next year, 1910-11, the committee's annual report included two papers on dynamical similarity, one of them by Rayleigh, under the "General Questions in Aerodynamics" section of the report. [2.52] In 1911-12, the annual report mentions plans for experiments on an airship to determine its resistance "by towing tests in the William Froude National Tank." [2.53] Under a section "The Law of Dynamical Similarity and the Use of Models in Aeronautics" the report notes its significance to all their research: "The theory relating to dynamical similarity explained by Lord Rayleigh and Mr. Lanchester in the first of the Annual Reports of the Committee is of fundamental importance in all applications of the method of models to the determination of the forces acting on bodies moving in air or in water," [2.52] The next year, the annual report noted that "Much evidence has now been accumulated in favour of the truth of the law of dynamical similarity to which attention was drawn by Lord Rayleigh and Mr. Lanchester in the first Report of this Committee" [2.54]

In June of 1914, the journal *Nature* featured a kind of survey paper, "Fluid Motions", based on "a discourse delivered at the Royal Institution on March 20" by Rayleigh. [2.55] Here, we see Rayleigh actively campaigning for wider appreciation and use of the principle, which he credits Stokes with having "laid down in all its completeness." We know that Stokes explicitly used the notion of similar systems in developing and explaining the use of the principle, so it is fair to say that Rayleigh means his discussion and use of it to be consistent with Stokes' notion of similar systems.

In this paper, Rayleigh pointed out that it appeared that viscosity was important in many cases where it was so small that it seemed improbable that it should matter. When viscosities were low, as in water, one would not expect that the actual value of viscosity would be a significant factor in water's qualitative behavior. As explained above, Osborne Reynolds' results on fluid flow in pipes had shown that it is; Reynolds began to suspect that viscosity was important even in water when he observed unexpected changes in fluid flow as the temperature was varied. Since viscosity varies with temperature, he investigated the effect of viscosity and found that it was indeed important for fluid flow through pipes, even for nonviscous fluids such as water. Rayleigh added that Reynolds also investigated cases where viscosity was the "leading consideration", as Rayleigh put it, in remarking that "It appears that in the extreme cases, when viscosity can be neglected and again when it is paramount, we are able to give a pretty good account of what passes. it is in the intermediate region, where both inertia and viscosity are of influence, that the difficulty is the greatest" [2.55] This is the lead-in to his advocacy for the law of dynamic similarity: "But even here we are not wholly without guidance." What is this guidance? He continues:

> There is a general law, called the law of dynamical similarity, which is often of great service. In the past this law has been unaccountably neglected, and not only in the present field. It allows us to infer what will happen upon one scale of operations from what has been observed at another." ([2.55], p. 364)

Rayleigh also notes: "But the principle is at least equally important in effecting a comparison between different fluids. If we know what happens on a certain scale and at a certain velocity in *water*, [emphasis in the original] we can infer what will happen in *air* on any other scale, provided the velocity is chosen suitably." This is, of course, the point Helmholtz had made in 1873. Rayleigh notes that the point applies only in the range where the velocities are small in comparison to the velocity of sound. [2.55]

Rayleigh gives an example of a use of the principle which permits one observation or experiment to be regarded as representative of a whole class of actual cases: i.e., the class of all the other cases to which it is similar, even though the cases may have very different values of measurable quantities such as velocity. The important fact about the situation is expressed by the formula for the dimensionless parameter, which picks out the cases to which it is similar: "It appears that similar motions may take place provided a certain condition be satisfied, viz. that the product of the linear dimension and the velocity, divided by the kinematic viscosity of the fluid, remain unchanged." ([2.55], p. 364) Put more specifically, the important feature of a particular situation is the value of this dimensionless parameter; what Rayleigh is saying is that, even in cases of a different fluid, so long as this dimensionless product is the same (and, of course, that one is in the applicable velocity range for which it was derived), the motions will be similar.

One might think that, by 1914, when the use of wind tunnels had become recognized as essential to practical aeronautical research, this principle would have become accepted and would no longer be in question, at least among aeronautical researchers. But if Rayleigh's estimation of the state of the profession is correct, apart from Lanchester's work, this wasn't so, even as late as March of 1914; he says that "although the principle of similarity is well established on the theoretical side and has met with some confirmation in experiment, there has been much hesitation in applying it, . . ." He especially mentions problems in its acceptance in aeronautics due to skepticism that viscosity, which is extremely small in air, should be considered an important parameter: "In order to remove these doubts it is very desirable to experiment with different viscosities, but this is not easy to do on a moderately large scale, as in the wind channels used for aeronautical purposes."

Rayleigh tries to persuade the reader of the significance of the effects of viscosity on the velocity of fluid flow by relating some experiments he performed with a cleverly designed apparatus in his laboratory. The apparatus consisted of two bottles containing fluid at different heights, connected by a tube with a constriction, through which fluid flowed due to the difference in "head," or height of fluid, in the two bottles. The tube with the constriction contained fittings that allow measurement of pressure head at the constriction, and on either side of it. To investigate the effects of viscosity, Rayleigh varied the temperature of the fluid, which changes the fluid viscosity, and he observed how the velocity of the fluid flowing between the two bottles was affected. The kind of relationship he establishes and uses is of the form Galileo employed in reasoning from one pendulum to another. In other words, he worked in terms of ratios (ratios of velocities, ratios of viscosities, ratios of heads), and he employed the fact that some ratios are the square root of others. [2.56] He took the experimental results he reported in this 1914 paper to conclusively settle the question of the relevance of viscosity to fluid motions. This is an example of the kind of exploratory work that can be involved in order to answer one of the questions needed in order to use the principle of similarity properly: what quantities are relevant to the behavior of interest (in the range of interest)? Although the researcher's experience and judgment are involved, sometimes new experiments should be, and are, conceived and carried out to help determine the question.

Rayleigh delivered this "discourse" in early 1914. [2.55] 1914 was a very special year for the concept of similar systems, and deserves a section all its own.

2.5. 1914: The Year of "Physically Similar Systems"

In terms of an advance in the understanding and formalization of physically similar systems, 1914 was a landmark year, just as 1850 (Stokes' paper [2.45]), 1873 (Helmholtz's paper [2.41]), and 1883 (Reynold's paper [2.56]) would still be nineteenth century landmarks in any history of the concept of dynamical similarity. Going back to earlier eras, many would also consider the dates 1638 (Galileo's *Two New Sciences* [2.10]) and 1673 (Newton's *Principia* [2.3]) significant to the concept of similar systems. My review above suggests additions to the above list of dates in the nineteenth century that should be recognized as important in the history of the concept of similar systems: the years around 1880 (van der Waals paper [2.57]) and 1881 (Onnes' paper [2.33]). The role of the notion of similar systems in both the development and the understanding of the principle of corresponding states in physical chemistry should enjoy far more recognition among philosophers of science than it has to date, and perhaps Lorentz ought to be included, too, for his recognition of the importance of the method of similar systems. A strong argument could also be made for including a date commemorating one of Froude's influential achievements in the nineteenth century list.

In contrast, however, dates for the papers by Maxwell and Boltzmann using the term 'similar systems' should not be included on this list, in my view. This exclusion is not a lack of generosity, but an effort at clarification. Their use of the term "similar system" in statistical mechanics, a term that already had a fairly well-defined meaning in the theories of mechanical similarity and dynamical similarity, may have caused, or at least contributed to, confusion about the concepts of 'similar system' and similarity as they are used in connection with mechanical and dynamical similarity. As we shall see, confusions about these concepts came to a head in 1914; perhaps it is no coincidence that at least one source of the confusion was a proposal by someone known for his work in statistical thermodynamics.

2.5.1 Overview of relevant events of the year 1914

In the part of 1914 leading up to Buckingham's landmark paper in October 1914 [2.2] that developed the notion of physically similar systems, hardly a month went by without some major work concerning similarity and similar systems appearing:

In January 1914, Stanton and Pannell publish a major compendium of work [2.6] done at Britain's National Physical Laboratory over the previous four years, *Investigation into Similarity of Motions*

In February 1914, a much-anticipated English translation of Galileo's *Two New Sciences* [2.10] is published.

In March 1914, Rayleigh delivers his lecture *Fluid Motions* [2.55] at the Royal Institute (March 20, 1914)

In April 1914, Richard Chace Tolman's "The Principle of Similitude" appears in *Physical Review* [2.58], and Rayleigh's Fluid Motions [2.55] is published in the periodical *Engineering*, **97** (April 8, 1914).

In May 1914, Buckingham gives a paper on *The Interpretation of Model Experiments* to the Washington Academy of Sciences. [2.59]

In June 1914, Rayleigh's review article "Fluid Motions" is published in *Nature*. [2.55]

In July 1914, Buckingham's "Physically Similar Systems" in *Journal of the Washington Academy of Science* [2.1]

In October 1914, Buckingham's "Physically Similar Systems: Illustrations of the Use of Dimensional Equations." [2.2]

And sometime during 1914, Philipp Forchheimer's *Hydraulik* was published, which contains a section on "The Law of Similarity." (Das Ahnlichkeitgesetz [umlaut on A]) *Hydraulik* becomes a highly regarded compendium and reference work on Hydraulics for many decades afterwards. In the concluding paragraph of the section on the law of similarity, Forchheimer writes that every hydraulic equation that fulfills the law of similarity can be expressed in the form of an equation consisting of an unidentified function F of three dimensionless ratios set equal to an unidentified constant. He indicates that the law of similarity is shown to be merely a special case of the general law according to which all the terms of any of the equations of importance in mechanics, need to be of equal dimension, inasmuch as the law of similarity treats one body as a prototype, and the others as copies of it.

2.5.2   Stanton and Pannell

In January of 1914, T. E. Stanton and J. R. Pannell read their paper "Similarity of Motion in Relation to the Surface Friction of Fluids" [2.61] to the Royal Society of London. Stanton was superintendent of Britain's National Physical Laboratory (NPL) Engineering Department. The paper was a compendium of the work done there on similarity, and had been submitted to the Society in December 1913. It begins with references to Helmholtz's and Stokes' work using equations for non-ideal fluid flow, refers to Newton's *Principia* on similar motions, and uses Rayleigh's equation for fluid resistance. It explains that Stanton and Pannell's work involves investigating "the conditions under which similar motions can be produced under practical conditions." The work had been carried out due in part to interest in the possibilities of using small scale models in wind tunnels for engineering research. With one exception, they began, experimental study of similar motions of fluids was very recent:

> Apart from the researches on similarity of motion of fluids, which have been in progress in the Aeronautical Department of the National Physical Laboratory during the last four years, the only previous experimental investigation on the subject, as far as the authors are aware, has been that of Osborne Reynolds . . . ([2.61], p. 200)

Stanton and Pannell cite several of Reynolds' major discoveries: (i) that there is a critical point at which fluid flow suddenly changed from "lamellar motion" to "eddying motion"; (ii) that the critical velocity is directly proportional to the kinematical viscosity of the water and inversely proportional to the diameter of the tube, and (iii) that for geometrically similar tubes, the dimensionless product: (critical velocity) x (diameter) / (kinematic viscosity of water) is constant.

Stanton and Pannell also noted a complication: surface roughness needed to be taken into account; this is a matter of geometry on a much smaller scale making a difference. However, the overall approach of the use of dimensionless parameters to establish similar situations was still seen to be valid, as indicated by their extensive experiments:

> From the foregoing it appears that similarity of motion in fluids at constant values of the variable vd/ν [velocity x diameter / kinematic viscosity of water] will exist, provided the surfaces relative to which the fluids move are geometrically similar, which similarity, as Lord RAYLEIGH pointed out, must extend to those irregularities in the surfaces which constitute roughness. In view of the practical value of the ability to apply this principle to the prediction of the resistance of aircraft from experiments on models, experimental investigation of the conditions under which similar motions can be produced under practical conditions becomes of considerable importance, . . .
> By the use of colouring matter to reveal the eddy systems at the back of similar inclined plates in streams of air and water, photographs of the systems existing in the two fluids when the value of vd/ν was the same for each, have been obtained, and their comparison has revealed a remarkable similarity in the motions. ([2.61], p. 201)

In referring to the dimensionless parameter vd/ν as a "variable", what Stanton and Pannell meant was that their equation for the resistance R includes a function of this dimensionless parameter, i.e., resistance R = (density) x (velocity)^2 x (some function of vd/ν). As they put it, $R = \rho v^2 F(vd/\nu)$, where $F(vd/\nu)$ indicates some unspecified function of vd/ν. Hence, vd/ν is a variable in the sense that the relation for resistance includes an unspecified function of vd/ν . It is also a variable in a more practical sense: it can be physically manipulated.

Stanton and Pannell presented this relation as a consequence of the Principle of Dynamical Similarity (in conjunction with assumptions about what "the resistance of bodies immersed in fluids moving relatively to them" depends on. Evidently, it was Rayleigh who suggested the generalization; they cite Rayleigh's contribution in the Report to the Advisory Committee for Aeronautics, 1909 - 1910 ([2.51], p. 38) Rayleigh had there spoken of the possibility of taking a more general approach than current researchers were taking in applying the "principle of dynamical similarity."

In presenting the results they obtained at the National Laboratory in the paper, it is noteworthy that the results are presented in graphs where one of the variables plotted is the term $R/\rho v^2$, which is just another expression for the unspecified function, and is dimensionless. What this implies is that the laboratory experiments are not conceived of in terms of the values of individual measurable quantities such as velocity but in terms of the value of a dimensionless parameter.

Rayleigh, too, presented a kind of survey paper in early 1914, as mentioned above . In that March 1914 paper [2.55], Rayleigh noted that the principle of dynamical similarity "allows us to infer what will happen upon one scale of operations from what has been observed at another." That is, one use of the principle is to use an observation or experiment as representative of a whole class of actual cases: all the other cases to which it is similar, even though the cases may have very different values of measureable individual quantities such as velocity. The important fact of the situation is the dimensionless parameter just mentioned: "It appears that similar motions may take place provided a certain condition be satisfied, viz. that the product of the linear dimension and the velocity, divided by the kinematic viscosity of the fluid, remain unchanged." [2.55]

A consequence of this fact is that, even in cases of a different fluid, so long as this dimensionless product is the same, the motions will be similar: no mention of the fluid! Not only is this striking claim correct, but it is responsible for a particularly useful application of Stanton and Pannell's work, of which they were well aware: tests done on water can be used to infer behavior about systems where the fluid is air. Not because air and water are similar -- the relevant fluid properties are very different, in fact --- but because the dimensionless parameter relating a number of the features of the fluid and of the situation is the same. Air and water are about as different as can be: "The fluids used in the majority of the experiments have been air and water. The physical properties of these are so widely different that observations on others are hardly necessary . . . " ([2.61], p. 202) Just as the theorem of corresponding states in physical chemistry allowed the construction of a function such that the values for many different kinds of fluids all fell on the same line, so here, too: that the function of the variable is the same for air, water, and oil is experimentally illustrated by figure xx from the paper.

2.5.3 Buckingham and Tolman

Buckingham's background in 1914

Edgar Buckingham (1867 - 1940) was a physicist who had been working at the National Bureau of Standards in Washington, D.C. since 1906. He had little previous experience or background in aeronautics when he began working on issues related to aeronautical research. His involvement arose as a consequence of efforts afoot to establish a government agency devoted to aeronautical research in the United States, modeled on the British Advisory Committee for Aeronautics; one spot was allocated for a physicist from the National Bureau of Standards. [2.62] How did it end up that it was Buckingham, then, who authored the paper that has become such a landmark in hydrodynamics and aerodynamics? In a letter to Rayleigh in 1915, Buckingham explained the origins of his 1914 paper "On physically similar systems: illustrations of the use of dimensional equations":

> "Some three or four years ago, having occasion to occupy myself with practical hydro- and aerodynamics, I at once found that I needed to know more about the method [of dimensions] in order to use it with confidence for my own purposes. Since you and the few others who have made much use of the method of dimensions have generally referred to it somewhat casually as to a subject with which everyone was familiar, I supposed that the hiatus in my education would be easily filled." [2.63]

But it was not:

> ". . . upon looking through your collected papers, the "Sound" [probably a reference to Rayleigh's *Theory of Sound*], Stokes's papers, and a few standard books such as Thompson and Tait [*Principles of Mechanics*] and Routh's *Rigid Dynamics* I was amazed at my failure to find any simple but comprehensive exposition of the method which could be used as a textbook. . . . Each one of your numerous applications of the method seemed perfectly clear, and yet their simplicity gave them the appearance of magic and made the general principle rather elusive." [2.63]

It is noteworthy that Buckingham mentions looking at the main *mechanics* textbooks used in Britain, rather than engineering texts. Approaching aerodynamics from the point of view of a physicist was consistent with the kind of community in which Buckingham worked and had been educated. He had earned an undergraduate degree in physics at Harvard University (graduating in 1887) and a doctorate in physics from Leipzig in 1894. Descriptions of him as "an engineer" or "physicist-engineer", as in Maila Walter's book [2.8] are somewhat misleading. After a few years as a physics professor, Buckingham worked as a physicist at US government agencies; first at the USDA Bureau of Soils (where he did very original theoretical work, applying energy methods), then at the National Bureau of Standards. [2.11] Involving physicists on aerodynamical research planning made sense, but it also helped cultivate a more prestigious image of a research institution concerned with aerodynamics in 1914. Buckingham seemed aware of this, as evidenced by his remark to Rayleigh about the latter's *Nature* article on the principle of dynamical similarity; he wrote Rayleigh that "a note, such as the one in *Nature* of March 18th, which has your authority behind it, has an effect far more important in the present state of affairs than any detailed exposition of the subject, however good, because physicists will be sure to read it." [2.63]

One of Buckingham's special areas of expertise within physics was thermodynamics. He didn't view thermodynamics as merely a subspecialty in physics, though, but rather as an enlightened view of science in which thermodynamics encompassed all of classical mechanics. In his 1900 book *Outline of a Theory of Thermodynamics*, Buckingham had written:

> "Thermodynamics . . . aims at the study of all the properties or qualities of material systems, and of all the forms of energy which they possess. It must, therefore, be held, in a general sense, to include pure dynamics, which is then to be looked upon as the thermodynamics of systems of which a number of non-mechanical properties are considered invariable. For 'thermodynamics', in this larger sense, the more appropriate name 'energetics' is often used, the word 'thermodynamics' being reserved to designate the treatment of problems which are directly concerned with temperature and heat." ([2.65], p. 16)

Buckingham's approach towards formalizing physics in his 1900 book on the foundations of thermodynamics had been to make the formalism he proposed as flexible as possible, and to build as few assumptions into it as possible. In generalizing the existing science of dynamics, he chose to regard as variable certain properties that are often considered invariable in dynamics. As Buckingham obtained his doctorate in Leipzig under Wilhelm Ostwald, a friend of Boltzmann who was often engaged with him in discussions and debates about foundational issues in science, Buckingham was familiar with debates in philosophy of science. [2.11] Buckingham developed (if he had not already had) a penchant for asking foundational questions, too; in his new role of advisor on research into aeronautics, he set for himself the task of discerning the foundations of the methods he saw being used in aeronautical research.

Buckingham's papers at the Washington Academy of Sciences in 1914

By the middle of 1914, Buckingham had figured out some things about the foundations of the methods used in aerodynamical research. As his note to Rayleigh indicates, he had been concentrating on understanding how "the method of dimensions", or dimensional analysis, was employed in aerodynamical and hydrodynamical research. On May 23, 1914, he presented a paper entitled "The interpretation of experiments on models" to the Washington Academy of Sciences in Washington, D.C., of which he was a member; 27 people were present, and four discussed the paper afterwards. [2.59] The account published in the academy's journal stated that "The speaker began by deducing a general theorem regarding the form which physical equations must have in order to satisfy the requirement of dimensional homogeneity." Dimensional homogeneity is an exceedingly general requirement of an equation; if the terms in an equation have any units (as equations in physics do), the equation is not really considered an equation if it does not meet the requirement of dimensional homogeneity. Thus this deduction is of something very fundamental in physics; it is about the logic of equations. The account continues:

> "The theorem may be stated as follows: If a relation subsists among a number of physical quantities, and if we form all the possible independent dimensionless products of powers of those quantities, any equation which describes the relation is reducible to the statement that some unknown function of these dimensionless products, taken as independent arguments, must vanish." [2.59]

The antecedent of the theorem is extremely general: "if a relation subsists among a number of physical quantities. . ." ; what is striking is that the antecedent of the theorem is not a requirement that the relation mentioned *be known*, only that it *exist*. The theorem was described as a "general summary of the requirement of dimensional homogeneity." The report on Buckingham's talk added that the method of determining the number and forms of the independent dimensionless products was explained. There is no mention of similar systems in the journal's account of this May 1914 talk, but it does add that the theorem "may be looked at from various standpoints and utilized for various purposes", and that "several illustrative examples" were given showing the "practical operation of the theorem." [2.59]

In July of 1914, the academy's journal featured a short, six page paper by Buckingham. The topic identified was more general than model experiments, and this time it did mention 'similar systems'; in fact, the paper is titled "Physically Similar Systems." That Buckingham meant the July paper to be seen as a generalization of the earlier paper on the interpretation of model experiments is indicated in the closing sentence of the paper: "A particular form of this theorem, known as the principle of 'dynamical similarity' is in familiar use for the *interpretation of experiments on mechanical models*; but the theorem is equally applicable to problems in heat and electromagnetism." (emphasis added) ([2.2], p. 353)

Like the May 1914 talk, the short July 1914 paper is notable for the generality of its approach. It did not imply that there were any set fundamental quantities, nor how many there were. It did not talk about physics, even. It spoke of quantities, relations between quantities, and equations. It is spare and elegant. It begins: "Let n physical quantities, Q, of n different kinds, be so related that the value of any one is fixed by the others. If no further quantity is involved in the phenomenon characterized by the relation, the relation is complete and may be described by an equation of the form $\Sigma M Q_1^{b1} Q_2^{b2} Q_3^{b3} \ldots Q_n^{bn} = 0$, in which the coefficients M are dimensionless or pure numbers." [2.1] He makes it clear that it is a matter of choice which units are to be regarded as fundamental ones. "Let k be the number of fundamental units needed in an absolute system for measuring the n kinds of quantity. Then among the n units required, there is always at least one set of k which are independent and not derivable from one another, and which might therefore be used as fundamental units, the remaining (n - k) being derived from them."

Together, these allow him to say how the quantities *other than* those that are taken to be among the k fundamental quantities *are related to* those fundamental quantities. Denoting the fundamental units by $[Q_1]$ through $[Q_k]$ -- in this July 1914 paper he sometimes uses the square brackets indicate the *units* of the enclosed quantity -- and the remaining (n - k) units that are derived from them by $[P_1]$, $[P_2]$, and so on up to $[P_{n-k}]$, we get (n - k) equations that relate the units of the (n - k) Ps to the units of the k Qs. Putting these requirements in terms of dimensions rather than units allows one to apply the requirement of dimensional homogeneity -- doing so for each of the fundamental units gives k equations; each of the k equations is a result of setting the exponents of one of the units to zero. It can then be shown that the number of independent dimensionless parameters Π_i s is (n - k). [2.1]

The generality of the treatment here marks this work on similar systems by Buckingham's off from the earlier work by Stokes in 1850 [2.45] and Helmholtz in 1873 [2.41]. Whereas Stokes spoke of "similar systems, composed of similar solids, oscillating in a similar manner" and of comparing "similarly situated points in inferring from the circumstance that [the relevant hydrodynamical equations] are satisfied for one system that they will be satisfied for all [the other similar] systems" [2.45] Buckingham spoke of an undetermined function whose arguments were dimensionless parameters. Buckingham spoke of varying the quantities (Qs and Ps above) in ways that "are not entirely arbitrary but subjected to the (n - k - 1) conditions that [certain] dimensionless Π_i's remain constant." [2.1]

Putting it in other terms, Buckingham characterized systems as similar in terms of a (non-unique) set of invariants. His emphasis is on the principle of dimensional homogeneity, which is really about the *logic* of the *equations* of physics. The concept of similar systems arises from reflecting on how the principle of dimensional homogeneity might actually be put to use, what it might allow one to infer. After the paper's opening pages, in which he laid out the observations about the nature of equations that express relations in nature (i.e., wherein the value of one quantity is fixed by the others) stated above, he writes: "The chief value of the principle of dimensional homogeneity is found in its application to problems in which it is possible to arrange matters so that the [dimensionless ratios] r's and the [dimensionless parameters] Π's of [the set of linear equations relating the P's to the Q's and the (unknown) function ϕ of the dimensionless r's and Π's] remain constant", so that the unknown function ϕ takes on a fixed value, thus giving a definite relation between the Ps and Qs in terms of the value of the unknown function ϕ . As he remarks, the point is not that dimensional analysis *provides* the function ϕ or even the *value* ϕ takes on once the values of the invariants are set. Rather, the principle allows one to express the relations between quantities in terms of ϕ , which has a fixed value if all its arguments (the dimensionless parameters) are fixed. Hence, doing an experiment on one case yields the relation for all the cases in which the dimensionless parameters that are the arguments of ϕ have the same value, even if the individual quantities from which those parameters are formed are all different.

Though Buckingham was, he said, only aiming to give a clear treatment of the same idea that Stokes and others had stated, a lot had happened in mathematics and physics (especially in physical chemistry and thermodynamics), in the intervening decades. In their works on similar systems, Stokes and Helmholtz worked with physical equations, the partial differential equations of fluids and fields; Buckingham, as a physicist, was certainly cognizant of and competent in working with them, too, but in the July 1914 paper on similar systems, he worked with (more abstract) dimensional equations. The goal here, in this lean paper that featured no examples or applications, was to get straight on things that (so far as he was aware) had not yet been articulated by others who had employed the method. He would later write to Rayleigh about these first papers on the method:

> "I had therefore . . . to write an elementary textbook on the subject for my own information. My object has been to reduce the method to a mere algebraic routine of general applicability, making it clear that Physics came in only at the start in deciding what variables should be considered, and that the rest was a necessary consequence of the physical knowledge used at

> the beginning; thus distinguishing sharply between what was assumed, either hypothetically or from observation, and what was mere logic and therefore certain.
>
> The resulting exposition is naturally, in its general form, very cumbersome in appearance, and a large number of problems can be handled vastly more simply without dragging in so much mathematical machinery." [2.63]

His exposition treats of a system S characterized very abstractly: "The quantities involved in a physical relation pertain to some particular physical system which may usually be treated as of very limited extent." ([2.1], p. 352) The system constructed to be similar to it, likewise, is described very formally: "Let S' be a second system into which S would be transformed if all quantities of each kind Q involved in [the equation expressing the physical relation pertaining to the system] were changed in some arbitrary ratio, so that the r's for all quantities of these kinds remained constant, while the particular quantities Q_1, Q_2, . . . Q_k changed in k independent ratios. " ([2.1], p. 352) After completing the specification of the constraints on how the quantities change in concert with each other so that S' also satisfies the relation: "Two systems S and S' which are related in the manner just described are similar as regards the physical relation in question."

The exposition may have been cumbersome, but the point is elegant and spare: the constraints that must be satisfied in constructing the system S' are just these: to keep the value of the dimensionless parameters that appear in the general form of the equation -- the arguments of the function ϕ -- the same in S' as in S. So, what is crucial is to identify a set of dimensionless parameters that can serve as the arguments of the undetermined function ϕ . For Buckingham, unlike for some predecessors writing about similar systems or dynamic similarity, the method underlying the construction of physically similar systems is not a method peculiar to mechanics; it applies to *any equation describing a complete relation that holds between quantities*.

Richard Chace Tolman's "Principle of Similitude"

Meanwhile, another physicist in the United States was publishing on similitude, too, though with considerably less rigor. Richard Chace Tolman (1881 - 1948) was an assistant professor of the relatively new field of physical chemistry at the University of California when Onnes won the Nobel Prize for his work in physical chemistry on the liquefication of helium; Onnes delivered his Nobel Prize Lecture in December 1913. [2.66],[2.31] As noted above, Onnes had aimed to "demonstrate that the principle of corresponding states can be derived on the basis of what he calls the principle of similarity of motion, which he ascribes to Newton." [2.32]

Tolman published "The Principle of Similitude" in the March 1914 *Physical Review*, in which he proposed the following:

> "The fundamental entities out of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed exactly similar in every respect to the present universe." ([2.58], p. 244)

Tolman then (he claimed) showed that he could derive a variety of laws, including the ideal gas law, from the principle of similitude he had proposed, proceeding in somewhat the same way as Onnes had proceeded in showing that the principle of corresponding states was a consequence of mechanical similarity. Tolman seemed to appeal to a criterion that the two universes should be observationally equivalent:

> . . . let us consider two observers, O and O', provided with instruments for making physical measurements. O is provided with ordinary meter sticks, clocks and other measuring apparatus of the kind and size which we now possess, and makes measurements in our present physical universe. O', however, is provided with a shorter meter stick, and corresponding altered clocks and other apparatus so that he could make measurements in the miniature universe of which we have spoken, and in accordance with our postulate obtain exactly the same numerical results in all his experiments as does O in the analogous measurements made in the real universe. ([2.58], p. 245)

He brings up some other considerations, some from physics (Coulomb's Law), some from the theory of dimensions, and then tries to show how various physical relations, such as the ideal gas law, can be deduced from simple physical assumptions and his proposed principle of similitude. For relations involving gravitation, however, a contradiction arises; his response is to use the contradiction as motivation to propose a new criterion for an acceptable theory of gravitation. He concludes that his proposed principle is a new relativity principle: the "principle of the relativity of size."

Tolman believes that, in his paper, he has lain out transformation equations that specify the changes that have to be made in lengths, masses, time intervals, energy quantities, etc., in order to construct a miniature world such that

> "If, now, throughout the universe a simultaneous change in all physical magnitudes of just the nature required by these transformation equations should suddenly occur, it is evident that to

> any observer the universe would appear entirely unchanged. The length of any physical object would still appear to him as before, since his meter sticks would all be changed in the same ratio as the dimensions of the object, and similar considerations would apply to intervals of time, etc. From this point of view we can see that it is meaningless to speak of the absolute length of an object, all we can talk about are the relative lengths of objects, the relative duration of lengths of time, etc., etc. The principle of similitude is thus identical with the principle of the relativity of size." ([2.58], p. 255)

Tolman's suggestion differs from the concept of similar systems mentioned so far, though the difference may not be obvious. Others working on similar systems where quantities or paths were homologous between similar systems noted that there were limits of applicability; they recognized the fact that there are ranges in which size matters (e.g., surface tension matters disproportionately at small scales (Froude [2.21]); the restriction in Helmholtz' 1873 paper that velocities must be small with respect to the velocity of sound [2.41], Reynolds' recognition of the role of "mean range" of molecules in transpiration [[2.11]). Helmholtz even explicitly discussed the practical difficulties of constructing models of a different size than the configuration modeled, raising the question of whether in some cases it may not be possible to do so. [2.41] Tolman not only does not recognize such limits; he suggests making the denial that they exist a principle of physics. It seems pretty clear that Tolman is here modeling his exposition on Einstein's 1905 paper on the special theory of relativity. Tolman proposes that the relativity of size be regarded along the lines of the relativity of motion: in his paper on special relativity, Einstein had considered it a principle that observers cannot tell one state of unaccelerated motion from another; Tolman proposes to do the same for the statement that observers not be able to distinguish an appropriately constructed model universe from the actual one [2.58], if inhabiting it as an appropriately transformed being and using appropriately constructed or transformed instruments. There is a confusion in Tolman's reasoning. While it is quite natural to say that a desirable principle of nature, and a desirable constraint on measuring systems, is that it should not matter to the project of pursuing truth that one observer in the actual world is using one system of measurement and another observer in the actual world is using another system of measurement, Tolman seems here to be confusing that requirement with a requirement that miniature universes constructed from the materials of the actual universe be indistinguishable from the actual, full size, universe by the miniature observers inhabiting those miniature universes.

Buckingham's *Physical Review* paper & Reply to Richard Chace Tolman

It's rather obvious that the notion of similar systems --- one system being transformed into another system S' in such a way that it "corresponds" to S ("as regards the essential quantities") -- is relevant to evaluating the claim Tolman made in his 1914 "Principle of Similitude" paper [2.58] that the universe could be transformed overnight into an observationally indistinguishable miniature universe. The notion of similar systems is also relevant to Stanton and Pannell's "similarity of motion" paper [2.61], in that it is a more general treatment of the methodology of model testing ("the principle of dynamical similarity") given there. In the next paper Buckingham wrote on the topic [2.2], in addition to presenting the generalized treatment found in the July 1914 version of "Physically Similar Systems," he addressed both these related topics on which major papers had appeared in the earlier part of the year: experimental models and Tolman's claims about the possibility of an observationally indistinguishable miniature universe. The October 1914 *Physical Review* featured Buckingham's "On Physically Similar Systems: Illustrations of the Use of Dimensional Equations"; his manuscript is dated June 18th of that year. [2.2]

In his 1914 *Physical Review* paper [2.2], Buckingham says that his purpose in presenting how the notion of physically similar systems can be developed from the principle of dimensional homogeneity in that paper was to provide background against which to respond to Tolman's proposed "principle of similitude." He makes several points relevant to addressing Tolman's proposal for a new principle in physics in developing "the notion of physical similarity" and "the notion of physically similar systems":

> (i) It is only "the phenomenon characterized by the relation [expressed by the equation whose existence was assumed at the start]" that "occurs in a similar manner" in both systems: "we say that the bodies or systems are *similar with respect to this phenomenon*. (emphasis added)" Buckingham specifically points out that systems that are "said to be 'dynamically similar' " might not be similar "as regards some other dynamical relation"; two dynamically similar systems might not "behave similarly in some different sort of experiment."
>
> (ii) There is a more general conception of similarity than dynamical similarity, and it too "follows directly from the dimensional reasoning, based on the principle of homogeneity."
>
> (iii) Tolman's proposed "Principle of Similitude" is not clearly stated, but inasmuch as Buckingham understands it, it seems to him "merely a particular case" of the theorem Buckingham presents in the paper. Buckingham reasons as follows: The way Tolman proceeds is to select four specific independent kinds of quantity (length, speed, quantity of electricity, electrostatic force), subjects these four kinds of quantity to four arbitrary conditions, then finds the conditions that some other kinds of quantities are subject to "in passing from the actual universe to a miniature universe that is physically similar to it." ([2.2], p. 356) I take Buckingham's point to be that, inasmuch as what Tolman is concluding is correct, it can be

> concluded using the principle of dimensional homogeneity without the aid of the "new" principle that Tolman proposed in his March 1914 *Physical Review* paper.

Having already remarked that the notion of similar systems used in constructing and using a model propeller is generalizable beyond mechanics, he then goes on to show how the principle involved in doing so --- the "method of dimensions" -- applies in problems ranging from electrodynamics (energy density of a field, the relation between mass and radius of an electron, radiation from an accelerated electron) to thermal transmission, and, finally, at a higher level, to the kind of "bird's-eye view" question to which his interest tended to migrate: "the relation of the law of gravitation to our ordinary system of mechanical units."

The question he asks about the role of the law of gravitation in determining units of measure is a bit different. It is about the number of "fundamental units," and the question Buckingham asks can be put in terms of similar systems: if it is in fact true that in mechanics three fundamental units suffice to describe mechanical phenomena (more if thermal and electromagnetic phenomena are to be described), then it would be correct to conclude that:

> "a purely mechanical system may be kept similar to itself when any three independent kinds of mechanical quantity pertaining to it are varied in arbitrary ratios, by simultaneously changing the remaining kinds of quantity in ratios specified by [the constraint of dimensional homogeneity] . . . For instance, we derive a unit of force from independent units of mass, length, and time, by using these units in a certain way which is fixed by definition, and we thereby determine a definite force which is reproducible and may be used as a unit. Now by Newton's law of gravitation it is, in principle, possible to derive one of the three fundamental units of mechanics from the other two." ([2.2], p. 372-373)

Buckingham then describes a laboratory experiment from which a unit of time can be derived from units of mass and length -- *if* one assumes Newton's law of gravitation to hold. To be clear: Buckingham is granting that people have sometimes reduced the number of fundamental units to two, such as when a unit of time is derived from units for mass and length, when working on specific problems. What he is concerned to show is that, in order to do so, they have had to use assumptions about the law of gravitation. He is not unaware that the current state of physics indicates Newton's law of gravitation is not the final word, and is pointing out the role that a law of gravitation plays in such reductions of the number of fundamental units to two. Put in terms of similar systems, the question is: how many degrees of freedom do we have in constructing a system S' that is similar to

S? How many quantities can be varied in an arbitrary ratio when we transform S into S', a system that is physically similar to it?

Buckingham points out that, even in the domain of mechanics, it depends. It depends on what phenomenon the relation between quantities characterizes. As he emphasized, the notion of physical similarity and physically similar systems involve only similarity *with respect to* a specified relation. (Recall that the analysis started with the quantities involved in a given equation, where that equation describes a relation that relates a certain number of kinds of quantities such that any one was determined by all the others, and the relation characterized a phenomenon of interest.) In developing a general methodology, Buckingham had considered *any* such relation; that is, all possible relations that could exist among the given kinds of quantities. In practice, this means that, if, on the contrary, we consider only some such relations ("all our ordinary physical phenomena [which] occur subject to the attraction of an earth of constant mass and under such circumstances that the variation of gravity with height is of no sensible importance"), we can take advantage of some features of specific relations. However, for precise geodesy and astronomy, one needs to be explicit about the law of gravitation.

Buckingham's answer to the question Tolman's paper raises about the possibility of constructing observationally indistinguishable miniature universes thus bifurcates into two cases, depending on whether or not the phenomenon that we are interested in observing in the miniature universe is influenced by the law of gravitation. If not, then it might not be impossible to construct a miniature universe, as Tolman suggests, that will be similar to the universe (as regards that phenomenon.) On the other hand, if the phenomenon is influenced by the law of gravitation, more things must be taken into account: "the gravitational forces in the miniature universe must bear to the corresponding gravitational forces in the actual universe a ratio fixed by the law of gravitation." He points out that the effect of the law of gravitation on the phenomena of interest shows up in the process of constructing similar systems. If we erroneously try to independently choose three units rather than letting the third be determined by the first two fundamental units chosen, we run into trouble because the measured values for corresponding speeds and forces won't correspond to the values in the actual universe -- unless, that is, the third unit is allowed to be fixed by the law of gravitation in terms of the first two.

The points about physically similar systems, systems of units, and the law of gravitation seem to be questions in the logic of physics. Yet, the main claim of Buckingham's papers on physically similar systems can actually be stated in terms of a theorem about the *symbolism of relations* between physical quantities.

This is seen in the "convenient summary" with which he concludes the paper:

> A convenient summary of the general consequence of the principle of dimensional homogeneity consists in the statement that any equation which describes completely a relation subsisting among a number of physical quantities of an equal or smaller number of different kinds, is reducible to the form $\Psi (\Pi_1, \Pi_2, \ldots \Pi_i, \text{etc.}) = 0$ in which the Π 's are all the independent dimensionless products of the form $Q_1^{\,x} Q_2^{\,y}$ . . ., etc. that can be made by using the symbols of all the quantities Q. ([2.2], p. 376)

The equation $\Psi (\Pi_1, \Pi_2, \ldots \Pi_i, \text{etc.}) = 0$ in the quote from Buckingham above is what I called *The Reduced Relation Equation of 1914* in Section 1 of this article.

2.5.4 Precursors of the "pi-theorem" in Buckingham's 1914 papers

This article is devoted to the history of the notion of physically similar systems. Buckingham's 1914 papers are considered a landmark in the development of our current notion of physically similar systems, due to the articulation of what a physically similar system is and how it is related to the symbolism used to express relations in physics. First, Buckingham showed that *The Reduced Relation Equation of 1914* followed from the principle of the homogeneity of a physical equation. Then, he showed how the notion of 'physically similar systems' could be developed from it.

However, since Buckingham's name has since become attached to the so-called 'pi-theorem', and the full contents of his 1914 papers are often ignored, being inaccurately viewed as doing little more than presenting the pi-theorem, I want to emphasize that what has become known as the pi-theorem itself is not actually due to Buckingham. There were, in fact, many precursors who proved the same result, with varying levels of generality.

Vaschy and Bertrand

The 'pi-theorem" is referred to in France as the Vaschy-Buckingham Pi Theorem. In 1892, Vaschy (1857 - 1899) published "Sur les lois de similitude en physique" ([2.67], [2.68]), in which he stated the result about the number of parameters required to state a given relationship that is often attributed to Buckingham. However, unlike Buckingham, Vaschy did not mention dimensions or dimensional equations. He spoke of quantities and units, and did so as though they were the same sort of thing, though he did speak of some units as fundamental and others as derived. More precisely, Vaschy's theorem is:

"Let a_1 , $a_{2,}$, $a_{3,}$, . . a_n be physical quantities, of which the first p are distinct fundamental units and the last (n - p) are derived from the p fundamental units (for example, a_1 could be a length, a_2 a mass, a_3 a time, and the (n-3) other quantities would be forces, velocities, etc.; then p = 3). If between these n quantities there exists a relation F (a_1 , $a_{2,}$, $a_{3,}$, . . a_n) = 0, which remains the same whatever the arbitrary magnitudes of the fundamental units, this relationship can be transformed in another relationship between at most (n - p) parameters, that is f (x_1 , $x_{2,}$, $x_{3,}$, . . x_{n-p}) = 0 , the parameters x_1 , $x_{2,}$, $x_{3,}$, . . x_{n-p} being monomial functions of a_1 , $a_{2,}$, $a_{3,}$, . . a_n. " [2.68]

The parameters x_1 , $x_{2,}$, $x_{3,}$, . . x_{n-p} play the same role as the dimensionless Π 's in Buckingham's theorem. Vaschy then shows how to obtain reduced relations for the pendulum and for a telegraph cable. What is notable is that he produces a pair of ratios, not just one ratio, in each case, and he expresses the result as an unknown function of these parameters (x_i's) set equal to zero. He does not use the terminology of systems, but he is interested in laws of similitude (in the sense of the similarity 'laws' of section 2.4.2) that can be derived from them, citing one by W Thomson (Lord Kelvin) in the case of the telegraph line. The conditions of Vaschy's theorem are not exactly the same as in Buckingham's theorem, but Vaschy does emphasize that his reasoning does not assume any particular system of units, and he does derive the key move to the *Reduced Relation Equation of 1914 .* The case is strong for crediting Vaschy's paper with containing the "pi-theorem."

Some have also argued that Joseph Bertrand provided an even earlier, though less general, proof of the pi-theorem in 1878, in "Sur l'homogeneite dans les formules de physique." ([2.67], p. 209) This is the same Joseph Bertrand (1822 - 1900) cited above for the much earlier 1847 work drawing attention to the principle of similitude, in which he mentioned "an infinite number of possible systems, which may be regarded as similar to" a given system, and provided a new basis for Newton's theorem of similarity using a result by Cauchy involving the principle of virtual velocities.

These two works by Bertrand thirty years apart reflect an important late nineteenth century development that permitted using a logical principle about t*he equations of physics*, i.e., the homogeneity of equations of physics, rather than a principle of physics itself. This late nineteenth-century development was the idea of coherence as a constraint on a system of units; the idea, that is, of a coherent system of units. Coherence of a system of units, and its importance in connecting dimensional analysis and similarity, is discussed in Sterrett [2.70].

Riabouchinsky

Sometime after 1914, Buckingham became aware that Dimitri Riabouchinsky (1882 - 1962) had also proved a mathematical theorem about the number of dimensionless parameters needed to express a given physical relation, using the methods of dimensional analysis, in 1911. [2.71] Riabouchinsky (spelled Riabouchinski in Buckingham's papers), was a scientist who had provided the private funding for the Aerodynamic Institute of Koutchino associated with the University of Moscow, which had a wind tunnel; hence Riabouchinsky was, like Buckingham, faced with the problem of understanding how to interpret model experiments. After becoming aware of Riabouchinsky's proof, Buckingham credited him prominently for the proof in his writings. In a paper in 1921, discussing the desire that had arisen for a more systematic procedure for obtaining the results that Rayleigh and others had obtained using dimensional methods, he wrote: "Such a routine procedure is provided by formulating the requirement of dimensional homogeneity as a general algebraic theorem, which was first published by Riabouchinski (sic), and which will be referred to as the Π theorem." ([2.72], p. 696) Buckingham speculated that he might have seen a notice of Riabouchinski's result in one of the Annual Reports of the British Advisory Committee on Aeronautics [2.73], and that "Guided . . . by the hint contained in this abstract, the present writer came upon substantially the same theorem, . . . The theorem does not differ materially from Riabouchinski's, except in that he confined his attention to mechanical quantities." ([2.72], p. 696n.)

2.6 Physically Similar Systems: the path in retrospect

We are now in a position to survey the path from Newton's theorem about similar systems of bodies in the seventeenth century to Buckingham's development of the notion of similar systems from what I have called the *Reduced Relation Equation of 1914*, in the early twentieth century. Painting what we can see in retrospect in broad brush strokes, the picture of this path is that there are several key ideas that made the twentieth century notion of physically similar systems possible. The first of these is the notion of a function developed in the eighteenth century, and the second is the notion of a coherent system of units developed in the late nineteenth century.

Brian Hepburn identifies Leonhard Euler as a key eighteenth century figure linking Newton's age and ours, and has argued that the concept of a function was crucial to the development of what we now know as Newtonian mechanics. Whereas Newton's mechanics "dictated how motions are generated in time by forces" and "would treat of the actual process of moving bodies," Hepburn says, for Euler, in contrast, "the central object of investigation in mechanics is the [mathematical] function." [2.12] He points out that equilibrium relations are the most important among relations, and hence that "sets of quantities" characterized "states" -- I would amend this to "states of a system." The notion of a function allowed the concept of a system to be expressed in terms of the interrelatedness of some quantities -- if one quantity changed, any of the others in the system might be affected, too. The notation of a function set to 0, i.e., $f(x_1, x_{2,} \ldots x_n) = 0$ can be used to express this interrelatedness. The notion of equilibrium and an equation of state, which are expressible by the functional notation, are important in this newer notion of a system; what this new notion of system eventually replaced was the notion of a system as a configuration of particles and/or bodies. The notion of a similarity law likewise progressed from simply a single ratio to express an invariant relation, to a function with multiple arguments, each of which was a dimensionless ratio.

When Bertrand invoked the principle of virtual velocities in 1847 ([2.25], p. 380) to derive the principle of mechanical similitude, he was using the notion of a function, but he was still using considerations and principles of mechanics. By 1878, he could take a much more general approach, using a principle that was a constraint on *the equation* expressing relations between the physical quantities, rather than the system of bodies and particles itself. Independently, many others could do so, too: Vaschy in France and Riabouchinsky in Russia, and they were not the only ones. In physical chemistry, van der Waals and Onnes, thinking of collections of molecules as systems, could apply these more formal notions of similar systems to come up with a way to predict the behavior of one substance based on only its critical points, along with observations about how another substance behaved. The amazing success of this approach in physical chemistry seems to have encouraged extending the approach of similar systems to electromagnetic theory and the kinetic theory of gases.

That the time was right in 1914 for deriving the pi-theorem and the Reduced Relation Equation of 1914 is clear from the fact that so many had already done it by then. That Buckingham was the one to write what has become the landmark paper articulating the notion of physically similar systems, which he developed from the *Reduced Relation Equation of 1914* in the Π–theorem , then, appears to be a matter of timing, at least in part: when he was suddenly asked to devote time to the question of the value of model experiments using wind tunnels, it was the early twentieth century, when the notion of a system was readily expressible by the notation for a function, when coherent systems of units in every part of physics was something that could be assumed, and someone with a doctorate in physics would have a facility with formal methods applied to equations.

Around the same time, or shortly thereafter, D'Arcy Wentworth Thompson wrote his classic work, *On Growth and Form* [2.74], on the mathematicization of biology. In that work, he carried the use of similitude in physics over into biology and he, too, explicitly cites Newton (for his use of similitude), as well as Galileo (for his discussion of scaling and similitude), Boltzmann, Helmholtz and numerous publications on aerial flight. A detailed discussion of D'Arcy Thompson on similitude may be found in Chapter 6 ("The Physics of Miniature Worlds") of *Wittgenstein Flies A Kite* ([2.11], pgs 117 - 130)

How do things stand today, in the early twenty-first century? Certainly there are pockets in many disciplines -- physics, hydrodynamics, aerodynamics, the geological and other sciences, hydrology, mechanics, biology, and more -- where researchers recognize the value of thinking in terms of physically similar systems. However, it is not really a staple of the basic curriculum. Few philosophers of science understand the concept or why it is significant. This article is offered to help improve at least the latter situation.

Acknowledgements

Work on this paper was supported in part by a Visiting Fellowship at the Center for Philosophy of Science at the University of Pittsburgh in 2010, during which my project was the history of the concept of physically similar systems. This paper also incorporates some earlier work published in Chapters 6 ("The Physics of Miniature Worlds") and 7 ("Models of Wings and Models of the World") of *Wittgenstein Flies A Kite: A Story of Models of Wings and Models of the World*. Thanks also to Brian Hepburn and George Smith for conversations about Newton's use of similar systems, and to Jasmin Ozel for translating parts of Forchheimer's *Hydraulik*.

References

2.1 E. Buckingham: Physically similar systems, J. Wash. Academy of Sciences, **93**, 347 - 353 (1914)

2.2 E. Buckingham: On physically similar systems: illustrations of the use of dimensional equations, Physical Review, **4**, 345–376 (1914)

2.3 Isaac Newton, Andrew Motte, Florian Cajori, and R. T. Crawford: *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World* (University of California Press, Berkeley, Calif 1946)

2.4 Ernst Mach, *The Science of Mechanics: a Critical and Historical Account of Its Development* Transl by Thomas J. McCormack. New Introduction by Karl Menger. Sixth Edition, with Revisions Through the Ninth German Edition. (Open Court Pub. Co, La Salle, Ill 1960)

2.5 Sjoerd D. Zwart: Scale modelling in engineering: Froude's case, in A. W. M. Meijers (Ed.) *Philosophy of technology and engineering sciences*, Vol. 9, p.759 - 798 (North Holland/Elsevier, Amsterdam, The Netherlands 2009)

2.6 T. E. Stanton and J. R. Pannell: Similarity of motion in relation to the surface friction of fluids, Philosophical Transactions of the Royal Society of London, Series A, **214**, 199 - 224 (1914)

2.7 A. F. Zahm: Theories of Flow Similitude, Report No. 287 (National Advisory Committee for Aeronautics, Washington DC 1928)

2.8 Maila L. Walter: *Science and Cultural Crisis: an intellectual biography of Percy Williams Bridgman* (1882 - 1961) (Stanford University Press, Stanford, Calif 1990)

2.9 Edwin T. Layton: Escape from the Jail of Shape: Dimensionality and Engineering Science, in Peter Kroes and Martijn Bakker (Eds): *Technological development and science in the Industrial Age: new perspectives on the science-technology relationship*. (Kluwer Academic Publishers, Dordrecht; Boston 1992)

2.10 Galilei Galileo, Stillman Drake (Trans.): *Two new sciences: including centers of gravity & Force of percussion. (2nd edition)* (Wall & Emerson, Toronto 2000)

2.11 Susan G. Sterrett: Wittgenstein Flies A Kite: a story of models of wings and models of the world. (Pi Press/Penguin, New York 2006)

2.12 Brian S. Hepburn: *Equilibrium and Explanation in 18th Century Mechanics* (Doctoral Dissertation, Department of History and Philosophy of Science, University of Pittsburgh, Pittsburgh 2007)

2.13 James Thomson: Comparison of similar structures as to elasticity, strength, and stability, Transactions of the Institute of Engineers and Shipbuilders of Scotland, **54**, 361 (1875)

2.14 Archibald Barr: Comparisons of Similar Structures & Machines, Inst. of Engineers and Shipbuilders in Scotland, **42**, 322 - 360 (1899)

2.15 R. P. Torrance: Use of models in engineering design, Engineering News, December 18 (1913)

2.16 James Thomson: Joseph Larmor and James C. Thomson (Eds) *Collected papers in physics and engineering* (Cambridge University Press, Cambridge 1912)

2.17 A. H. Gibson: *Hydraulics and its applications* (D. Van Nostrand Company, New York (1908)

2.18 Olivier Darrigol: *Worlds of Flow: a history of hydrodynamics from the Bernoullis to Prandtl* (Oxford University Press, Oxford, New York 2005)

2.19 R. H. M. Robinson: Experimental Model Basin -- I, Scientific American Supplement, **1698**, 37 - 38 (1908)

2.20 Gina Hagler: *Modeling ships and space craft: the science and art of mastering the oceans and sky* (Springer, New York 2013)

2.21 William Froude: *On the Rolling of Ships* (Parker, Son, and Bourn, London 1862)

2.22 Simon Schaffer: Fish and Ships: models in the age of reason in Soraya de Chadarevian and Nick Hopwood, *Models: the third dimension of science* (Stanford University Press, Stanford Calif 2004)

2.23 William Froude: On Experiments with H. M. S 'Greyhound', in *Transactions of the Royal Institution of Naval Architects*, **15**, 36 - 73 (1874)

2.24 William Denny: Mr Mansel's and the late Mr Froude's Methods of Analysing the Results of Progressive Speed Trials in *Transactions of the Institution of Engineers and Shipbuilders in Scotland*, **28**, (1885)

2.25 Florian Cajori: *A History of Mathematics* (Macmillan, New York 1894)

2.26 R. Ettema: *Hydraulic modeling concepts and practice*. (American Society of Civil Engineers, Reston, Va 2000)

2.27 M. J. Bertrand: *On the Relative Proportions of Machinery, Considered with regard to their powers of working Newton's London Journal of Arts and Sciences*, 31 (1847)

2.28 Ferdinand Reech: Cours de Mécanique d'après la nature généralement flexible et élastique des corps (Carilian-Gœury et V^{or} Dalmont, Paris 1852)

2.29 H. A. Lorentz: The theory of radiation and the Second Law of Thermodynamics, *KNAW (*Koninklijke Nerlandse Akademie van Wetenschappen) *Proceedings*, **3 , 436 - 450** (Huygens Institute, Royal Netherlands Academy of Arts and Sciences, Amsterdam 1901)

2.30 The Nobel Prize in Physics 1910". *Nobelprize.org* <http://www.nobelprize.org/nobel_prizes/physics/laureates/1910/>

2.31 "The Nobel Prize in Physics 1913". *Nobelprize.org* <http://www.nobelprize.org/nobel_prizes/physics/laureates/1913/>

2.32 Heike Kamerlingh Onnes: Investigations into the Properties of Substances at Low Temperatures, which Have Led, amongst Other Things, to the Preparation of Liquid Helium, *Nobel Lectures, Physics 1901 - 1921* (Elsevier Publishing Company, Amsterdam 1967)

2.33 J. M. H. Sengers (Levelt Sengers): *How Fluids Unmix: discoveries by the school of Van der Waals and Kamerlingh Onnes* (Koninklijke Nerlandse Akademie van Wetenschappen, Amsterdam 2002)

2.34 Jaime Wisniak: Heike Kamerlingh -- The Virial Equation of State, *Indian Journal of Chemical Technology*, **10**, 564 - 572 (2003)

2.35 Jagdish Mehra, Helmut Rechenberg: *The Historical Development of Quantum Theory*, Volume 5 (Springer, New York 1987)

2.36 Stephen G. Brush: Ludwig Boltzman and the Foundations of Natural Science, in Ilse Maria Fasol-Boltzmann, Gerhard Ludwig Fasol (Eds.) *Ludwig Boltzmann (1844 - 1906): Zum Hundertsten Todestag* (Springer, Austria 2006)

2.37 James Clerk Maxwell: On Boltzmann's Theorem on the average distribution of energy in a system of material points, W. D. Niven (Ed.) The Scientific Papers of James Clerk Maxwell, Vol. II (Cambridge University Press, Cambridge 1890)

2.38 Ludwig Boltzmann: Model, *Encyclopedia Britannica 10th Edition*, **30**, 788 - 791 ("The Times" Printing House, London 1902)

2.39 Ludwig Boltzmann: On Aeronautics (Ingo Pollman and Marco Mertens, Transl.) in Susan G Sterrett, *Wittgenstein Flies A Kite: a story of models of wings and models of the world* (Pi Press/Penguin, New York 2005/6)

2.40 Cleveland Abbe: *Mechanics of the Earth's Atmosphere: A Collection of Translations*, Smithsonian Miscellaneous Collections No. 843 (The Smithsonian Institution, Washington DC 1891)

2.41 Hermann von Helmholtz: On a Theorem Relative to Movements that are Geometrically Similar in Fluid Bodies, Together with an Application to the Problem of Steering Balloons, *Mechanics of the Earth's Atmosphere: A Collection of Translations* by Cleveland Abbe. Smithsonian Miscellaneous Collections No. **843**, 67 - 77 (The Smithsonian Institution, Washington DC 1891) Original in German [2.42.].

2.42 Hermann von Helmholtz: Monatsberichte of the Royal Academy of Berlin, June 26, 1873, 501 - 514 (1891)

2.43 Hermann von Helmholtz: On Discontinuous Motions in Liquids, *Mechanics of the Earth's Atmosphere: A Collection of Translations* by Cleveland Abbe. Smithsonian Miscellaneous Collections No. **843**, 58 - 66 (The Smithsonian Institution, Washington DC 1891) Original in German [2.44].

2.44 Herman von Helmholtz: Monatsberichte of the Royal Academy of Berlin, April 23, 1868, 215 - 228 (1868)

2.45 George Gabriel Stokes: On the effect of the internal friction of fluids on the motion of pendulums, Transactions of teh Cambridge Philosophical Society, **9,** 8. (1850)

2.46 David Cahan: Helmholtz and the British Scientific Elite: From Force Conservation to Energy Conservation, *Notes and Records of the Royal Society*, 66, 55 - 68 (2011)

2.47 D. M. McDowell and J. D. Jackson: *Osborne Reynolds and engineering science today* (Manchester University Press, Manchester 1970)

2.48 George Gabriel Stokes, Joseph Larmor (Ed.) *Memoir and Scientific Correspondence of the Late George Gabriel Stokes, Bart., Sc.D., LL. D., D.c.l., Past Pres R. S.*, Volume 1. (1907)

2.49 Osborne Reynolds: Letter to George Stokes, April 25, 1883, in [2.48], p. 233.

2.50 Ludwig Prandtl: Motion of fluids with very little viscosity, English trans. in J. A. K. Ackroyd, B. P. Axcell, A. I. Ruban, (Eds.), *Early Developments of Modern Aerodynamics* (Butterworth-Heinemann, Oxford, UK 2001), p. 77. Original in German, L. Prandtl, in *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg 1904*, A. Krazer, ed., Teubner, Leipzig, Germany (1905), p. 484.

2.51 Rayleigh, Sir (John William Strutt): Note as to the application of the principle of dynamical similarity, *Report of the Advisory Committee for Aeronautics* 1909 - 1910 (British Advisory Committee for Aeronautics, London 1910)

2.52 *Report of the Advisory Committee for Aeronautics* 1910 - 1911 (British Advisory Committee for Aeronautics, London 1911)

2.53 *Report of the Advisory Committee for Aeronautics* 1911 - 1912 (British Advisory Committee for Aeronautics, London 1912)

2.54 *Report of the Advisory Committee for Aeronautics* 1912 - 1913 (British Advisory Committee f or Aeronautics, London 1913)

2.55 Rayleigh, Sir (John William Strutt): Fluid Motions, Lecture delivered at the Royal Institute, March 20, 1914, printed in Engineering, **97**, 442 - 443; (April 8, 1914) and in Nature, **93**, 364 - 365 (June 4, 1914) (1914)

2.56 Osborne Reynolds: An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and the law of resistance in parallel channels, Phil. Trans. R. Soc. **174**, 935–982 (1883)

2.57 J. D. van der Waals: Coefficients of expansion and compression in corresponding states, Amst. Ak. Vh., **20**, 32 - 43 (1880)

2.58 R. C. Tolman: The Principle of Similitude, Physical Review, **3**, 244 (1914)

2.59 Edgar Buckingham: The interpretation of experiments on models, J of the Wash Academy of Sciences, **93**, 336. (1914)

2.60 Philipp Forchheimer: *Hydraulik* (Teubner, Leipzig, Berlin 1914)

2.61 T. E. Stanton and J. R. Pannell: Similarity of motion in relation to the surface friction of fluids, Philosophical Transactions of the Royal Society of London, Series A, **214**, 199 - 224 (1914)

2.62 Joseph R. Chambers: *Cave of the Winds: The remarkable history of the Langley Full-Scale Wind Tunnel* (NASA, Washington, DC 2014)

2.63 Edgar Buckingham: Letter to Lord Rayleigh (John William Strutt) dated November 13, 1915, handwritten on official National Bureau of Standards stationery.

2.64 Maila L. Walter: *Science and cultural crisis: an intellectual biography of Percy Williams Bridgman (1882 - 1961)* (Stanford University Press, Stanford, Calif 1990) (same as 2.8)

2.65 Edgar Buckingham: *An outline of a theory of thermodynamics* (Macmillan, New York 1900)

2.66 John G. Kirkwood and Oliver R. Wulfepstein: Richard Chace Tolman: biographical memoir (National Academy of Sciences, Washington DC 1952)

2.67 John J. Roche: The mathematics of measurement: a critical history (Anthione Press, London; Springer-Verlag, New York 1998)

2.68 A. Vaschy: Sur les lois de similitude en physique, Annales Telegraphiques, **19**, 25 - 28 (1892) Translation by Andrew C Palmer in appendix to: Dimensional Analysis and Intelligent Experimentation (World Scientific Publishing Company, Hackensack, NJ and London 2008)

2.69 J. Bertrand: Sur l'homogénéité dans les formules de physique, Comptes rendus, **86**, 916 - 920.

2.70 S. G. Sterrett: Dimensional Analysis and Similarity, in A. W. M. Meijers (Ed.) *Philosophy of technology and engineering sciences*, Vol. 9, p.799 - 823 (North Holland/Elsevier, Amsterdam, The Netherlands 2009)

2.71 D. P. Riabouchinsky: Methode des variables de dimension zero, et son application en aerodynamique, L'Aerophile, Septembre 1, 407-408 (1911)

2.72 E. Buckingham: Notes on the theory of dimensions, Philosophical Magazine, **42**, 696 - 719 (1921)

2.73 British Advisory Committee on Aeronautics: Abstract 134, Annual Report of the British Committee on Aeronautics for 1911-1912, p. 260 (1912)

2.74 D'Arcy Wentworth Thompson: *On Growth and Form* (Cambridge University Press: Cambridge [England] 1917)