Regularity Relationalism and the Constructivist Project

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Abstract

It has recently been argued by Stevens (2015) that Harvey Brown and Oliver Pooley's 'dynamical approach' to special relativity should be understood as what might be called an ontologically and ideologically relationalist approach to Minkowski geometry, according to which Minkowski geometrical structure supervenes upon the symmetries of the best-systems dynamical laws for a material world with primitive topological or differentiable structure. Fleshing out the details of some such primitive structure, and a conception of laws according to which Minkowski geometry could so supervene, has been referred to by some as the 'constructivist project'. Here, it is explained that Nick Huggett's work on 'regularity relationalism' provides a framework for such an approach, and a relativistic version of Huggett's regularity relationalism is outlined for that purpose. Finally, by way of examples, it is shown that this approach fails in the simplest cases. Still, reasons are given for which this should not necessarily discourage an advocate of this interpretation of the dynamical approach.

1 Introduction

In *Physical Relativity* (2005), Harvey Brown promotes an unorthodox interpretation of special relativity, which has since been referred to as the 'dynamical approach'. In his recent review, Stevens (2015) explains that the dynamical approach argues for the priority of dynamics over geometry in the context of explanation, even claiming that Minkowski geometrical structure itself is explained by the fact that the dynamical laws governing the behavior of material objects are Lorentz-covariant. He also argues that the dynamical approach must therefore be what might be called an ontologically *and* ideologically relationalist approach to Minkowski geometry.[1] Like a traditional relationalist approach, it must provide a material ontology that excludes Minkowski spacetime. But unlike other traditional relationalist approaches, it must also provide a spatiotemporally scant ideology, the spatiotemporal components of which do not constitute Minkowski geometry. Then, with respect to some such metaphysics, it must be shown that Minkowski geometry is derivative of facts about the symmetries of the dynamical laws.

This, then, is the 'constructivist' project of the dynamical approach: to specify a primitive spatiotemporal structure which, while not constituting Minkowski geometry itself, could somehow support the use of Lorentz-covariant laws in systematizing material objects' degrees of freedom. This would seem to require that the dynamical laws be written in terms of more geometrical structure than is ascribed primitively to the material world. Stevens has reminded us of proposals that this might be

[1]Following Stevens (2015), Quine's terminology will be used, according to which the ontology lists the entities to whose existence the theory is committed, and the ideology lists those entities' primitive properties or relations (Quine 1951).

accomplished with a ‘liberalized’ humean conception of laws. Below, it will be explained how Nick Huggett’s work on ‘regularity relationalism’ sets the stage for such a project, and a relativistic version of Huggett’s regularity relationalism will be outlined for that purpose. Finally, it will be shown that this project fails in the simplest contexts, but reasons are given for which the advocate of this interpretation of the dynamical approach might trust in more complex supportive examples.

2 Huggett's Regularity Relationalism

Nick Huggett has promoted 'regularity relationalism' as a way to overcome the problems faced by a Leibnizian relationalist in a Newtonian world (2006). Leibnizian relationalism assumes a material ontology, and the spatiotemporal components of its ideology are only those relations encoded by Leibnizian spacetime: a 'simultaneity' equivalence relation; a temporal metrical relation on the space of simultaneity slices; and a spatial metrical relation on pairs of points in each simultaneity slice.[2] The trouble for Leibnizian relationalism is that this amounts to only part of the apparent geometrical structure of a world in which Newton's laws hold. Instead, the full geometrical structural presupposed by Newton's laws is captured in Galilean (or Neo-Newtonian) spacetime.[3] As illustrated in Figure 1, the latter is more structured—i.e. less symmetric—than Leibnizian spacetime.

For present purposes, the salient difference is Galilean spacetime's affine, or inertial, structure. By not underwriting any affine structure, Leibnizian relations do not account for the full geometrical structure assumed by Newton's laws.
A more formal way of expressing this problem is to say that Galilean spacetime is symmetric under translations, boosts, and static rotations, whereas Leibnizian spacetime is symmetric under those same transformations as well as accelerations

[2]This presentation of Leibnizian and Galilean spacetime symmetries draws on Earman's (1989, §§2.2–2.4).

[3]Or, if gravity is taken into account, the apparent geometrical structure is captured by Newton–Cartan spacetime. See recent work by Knox (2013) or Pooley [2013, §6.1.1].

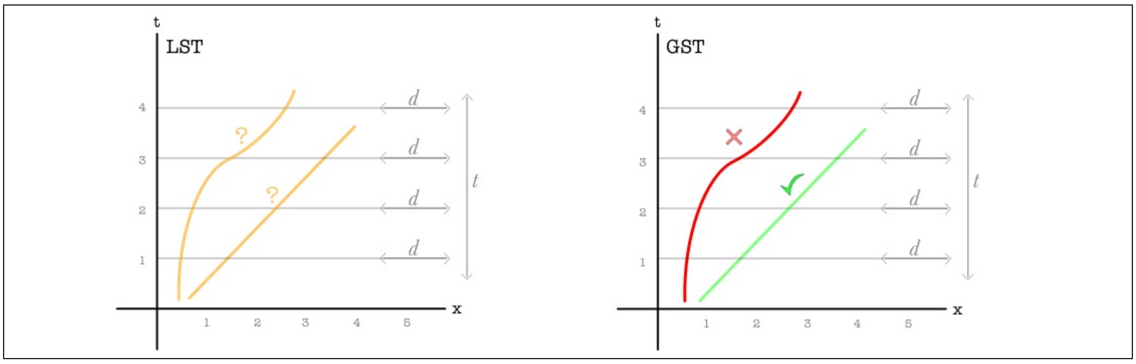

Figure 1: Following Earman's terminology (1989, ch. 2), Galilean spacetime has the same spatial (d) and temporal (t) metrical structure as Leibnizian spacetime. But in addition, it also has an affine structure, with respect to which lines are found to be or not to be straight.

and time-dependent rotations.

$$\mathcal{S}_{\text{G}} : \bar{r}\,' = \mathbf{R}\bar{r} + t\bar{v} + \bar{c} \quad ; \quad t' = t + d$$

$$\mathcal{S}_{\text{L}} : \bar{r}\,' = \mathbf{R}(t)\bar{r} + \bar{a}(t) \quad ; \quad t' = t + d$$

Now, in order to maintain that some world in which Newton's laws hold has less spatiotemporal structure than that of Galilean spacetime, the Leibnizian relationalist must give an account for the fact that $\mathcal{S}_{\text{L}}$ is a *supergroup* of $\mathcal{S}_{\text{G}}$. It might be argued that the relationalist's only forthright and honest response would be: to posit a more robust ideology, which would result in fewer spatiotemporal symmetries; or else to propose some new, empirically adequate dynamical laws with a larger symmetry group, which would be written in terms of less spacetime structure.[4] But rather than take either of these approaches, Huggett proposes a move that he calls 'regularity relationalism'.

[4]Moves like this are evaluated by Maudlin (1993) and Pooley (2013).

Regularity relationalism is effectively a way to have one's cake and eat it too.[5] The idea is simply that the Leibnizian relationalist can refuse to budge on taking only Leibnizian relations as primitive, by somehow explaining away the structural deficiency introduced above. To do so requires that one adopt a form of Humean supervenience, along with a 'liberalized' best-system conception of dynamical laws. As for the former, consider the following summary by David Lewis.

> Humean supervenience [...] is the doctrine that all there is to the world is a vast mosaic of local matters of particular fact, just one little thing and then another. [...] We have geometry: a system of external relations of spatiotemporal distances between points. Maybe points of spacetime itself, maybe point-sized bits of matter or aether or fields, maybe both. And at those points we have local qualities: perfectly natural intrinsic properties which need nothing bigger than a point at which to be instantiated. For short: we have an arrangement of qualities. And that is all. [...] All else supervenes on that.
>
> (1986, pp.ix–x)

As Lewis makes clear, the fundamental ontology of Humean supervenience amounts to the point-like parts of spacetime and/or physical fields, and the spatiotemporal component of its fundamental ideology includes some geometrical structure. As a form of relationalism, Huggett's regularity relationalist would naturally strike "points of spacetime itself" from the ontology, leaving the Humean mosaic to

[5]Huggett also hints at this kind of approach in an earlier publication (1999, pp. 22–3). The approach is referred to as 'have-it-all' relationalism by Pooley [2013, §6.3].

comprise only “point-sized bits of matter”. But the more interesting and salient characteristic of regularity relationalism comes up in its fundamental ideology. For according to Huggett’s form of Humean supervenience, the full geometrical structure required to make sense of the world’s dynamical laws is not taken as a primitive feature of the Humean mosaic. That is, in Lewis’s quotation above, the regularity relationalist would take “geometry”, “distances”, and “arrangement” to mean only those constituted by a history of Leibnizian relations, which do not underwrite the affine structure of Galilean spacetime.

Then, the ‘liberalized’ best-systems conception of dynamical laws is a version of the Mill-Ramsey-Lewis approach, according to which dynamical laws are theorems of the ‘best’ axiomatizations of the Humean mosaic (Lewis 1973, pp. 72–7). The basic view can be liberalized by taking into consideration those laws that are written in terms of more geometrical structure than appears in the Humean mosaic. If the equations of some such systematization have a better balance of simplicity and strength than those written in terms of the mosaic’s primitive geometry, then they are awarded the status of dynamical laws. Whatever extra geometrical structure plays a role in the expression of those dynamical laws is simply claimed to be part of the “all else” that supervenes upon the mosaic’s arrangement of qualities.[6]

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[6]Similar ‘liberalizations’ of Humeanism have been promoted in the context of quantum mechanics, in which Humean interpretations benefit from moving certain elements out of a theory’s fundamental ontology and into a part of the ontology’s best systematization. (Note, however, that in the present case, it is an element of the fundamental *ideology* that is being so moved.) Elizabeth Miller, for instance, promotes a Humean interpretation of Bohmian mechanics, according to which the

However controversial it may be, regularity relationalism is as helpful for an advocate of the dynamical approach to special relativity as it is for the Leibnizian relationalist in a Newtonian world. For note that the Leibnizian relationalist's troubling position is precisely that of the dynamical approach—namely, the position of having admitted less primitive geometrical structure than that in terms of which the dynamics is written, so that there is some work to be done in accounting for the difference.

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pilot wave itself becomes a part of the best systematization (2014). Similarly, Craig Callendar (2015, esp. p. 3161) does so with classical and quantum wavefunctions more generally. See also recent work by Michael Esfeld (2014), and by Harjit Bhogal and Zee Perry (2015). (Personal acknowledgement to be added after blind review.)

3 The Dynamical Approach as a Form of Regularity Relationalism

Huggett's regularity relationalism may provide the conception of natural laws necessary to understand the dynamical approach along the lines proposed here, but whereas Huggett assumes an ontology of point particles, and an ideology the spatiotemporal component of which comprises Leibnizian relations, Stevens has argued (2015) that Brown seems to assume an ontology of material events and an ideology having a spatiotemporal component that amounts to some sub-metrical structure—arguably a topological or differentiable structure. Interestingly, Stevens also points out John Norton's acknowledgement that one way around his own critiques of the dynamical approach might be to take a topologically structured spacetime as primitive (Norton 2008, p. 833), although no comments are made as to whether that topological structure could instead be assigned to the material objects themselves. A similar suggestion has also been made by Huggett in his review of Brown's book (Huggett 2009, p. 418). And more recently, Oliver Pooley, who coauthored with Brown two earlier presentations of the dynamical approach (1999; 2006), also argues that the dynamical approach could be construed as a relativistic version of Huggett's regularity relationalism (2013, pp. 571–2). He proposes taking a topological structure as primitive, and provides a simple illustration of how such a project would go.

Pooley describes a single material entity with point-like parts, the degrees of freedom of which can be modeled by a function into the reals—i.e. a single scalar field. He assumes the entity to be extended in four dimensions, and to have a structure that can be respected by some global mapping into $\mathbb{R}^4$—i.e. to have

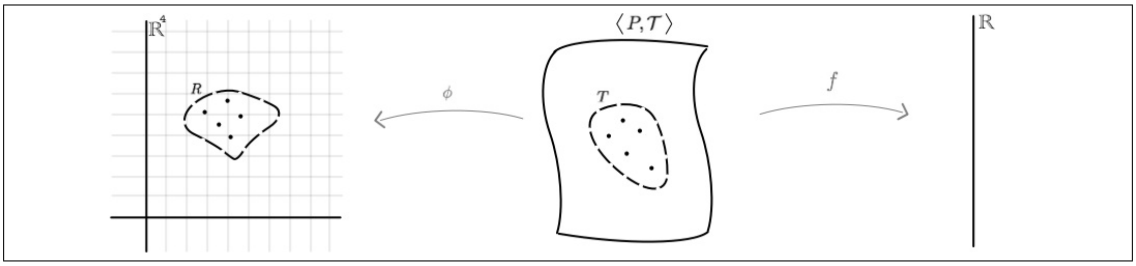

Figure 2: Pooley's material entity P, with a primitive Euclidean topological structure $\mathcal{T}$ (with open sets T). Its point-like parts have real-valued degrees of freedom, represented by f. A homeomorphism ϕ is shown mapping T onto R, the latter being an open subset of $\mathbb{R}^4$ with respect to the standard topology $\mathcal{R}^4$.

Euclidean topological structure.[7] As in Figure 2, his example can be illustrated by a topological structure $\mathcal{T}$ on a set P of field points, which together constitute a topological space $\langle P, \mathcal{T} \rangle$. The topological structure and the field points' degrees of freedom can be illustrated by a homeomorphism $\phi : \langle P, \mathcal{T} \rangle \to \mathbb{R}^4$ and a real-valued function $f : \langle P, \mathcal{T} \rangle \to \mathbb{R}$, respectively.

With that Humean mosaic in mind, Pooley's liberalized best-systems approach to dynamical laws is similar to Huggett's. It can be outlined in further detail as follows. First, note that mappings like ϕ provide coordinatizations according to which the field points' degrees of freedom might be given a mathematical description. For instance, the values could be described by the function $f_\phi = f \circ \phi^{-1}$, so that different coordinatizations ϕ bring about different descriptions f_ϕ. Now, suppose that $\langle P, \mathcal{T} \rangle$ is such that given all homeomorphisms ϕ, there exists a collection $\mathbf{S}$ of coordinatizations with respect to which each corresponding function f_ϕ solves some kind of equation. Then, suppose further that for some collection $\mathbf{B} \subset \mathbf{S}$, the f_ϕ all solve some particular equation L, which has a better combination of simplicity and strength than the equations solved by the f_ϕ corresponding to the

[7] Here, $\mathbb{R}^4$ is assumed to have the standard Euclidean topology, $\mathcal{R}^4$.

other coordinatizations in $\mathbf{S}$. If L is Lorentz-covariant, meaning that it preserves its form under elements of the Poincaré group, then Minkowski geometry could be said to supervene upon L's being the 'best' systematization of the field points' values and topological structure. Put differently, if the coordinatizations $\phi \in \mathbf{B}$, according to which the f_ϕ solve L, are related to one another by elements of the Poincaré group, then Minkowski geometry would be "grounded in the existence of a proper subset of the coordinate systems adapted to the ~~Leibnizian relations~~ [topological structure] with respect to which the description of the entire ~~relational history~~ [topological space] is the solution of particularly simple equations" (Pooley 2013, pp. 571–2, my strikeouts).

Pooley is right that this construal of the dynamical approach would support the explanatory claims mentioned in §1, which are discussed in further detail by Stevens (2015). It would also meet the other criteria outlined above: neither Minkowski spacetime nor its point-like parts feature in the ontology; the spatiotemporal components of the ideology do not constitute Minkowski geometry; and Minkowski geometry codifies the behavior of material objects as described by the (liberalized best-system) Lorentz-covariant dynamical laws. However, there is no doubt that this proposal is somewhat bolder than Huggett's. For whereas Huggett proposes an account of Euclidean affine structure by taking as primitive material objects with mass and charge and all the metrical relations encoded by Leibnizian spacetime, Pooley proposes an account of the full metrical structure of Minkowski spacetime by taking as primitive only a single set of scalar values endowed with mere Euclidean topological structure, which is far more symmetric than Leibnizian spacetime. For this reason, we might expect (like Pooley) that the constructivist project would fail in the simplest of settings.

4 The Simplest Setting

It has been argued that Huggett's work on 'regularity relationalism' provides the framework in which the constructivist project of the dynamical approach might be realized. Starting with set of field points endowed with some topological structure, the exercise of determining the symmetry group of its 'liberalized' best systematization has also been outlined. Thus the stage is set for case studies.

Now, a natural place to begin looking for a topological space $\langle P, \mathcal{T} \rangle$ of scalar field values, for which there exists a coordinatization ϕ with respect to which the corresponding f_ϕ solves a Lorentz-covariant equation, would obviously be in the space of solutions to familiar Lorentz-covariant scalar field equations. For example, a solution to the Klein–Gordon equation, purged of all geometrical structure but its induced Euclidean topology, is some such $\langle P, \mathcal{T} \rangle$. The question at hand, then, is whether or not solutions to the Klein–Gordon equation, when considered up to homeomorphism as $\langle P, \mathcal{T} \rangle$, admit any permissible *re*coordinatizations ϕ', for which the corresponding $f_{\phi'}$ solve any equations of a *different* covariance group.

In preparation for that exercise, this section gives a brief introduction to the simplest solutions of the Klein–Gordon equation, and also to those of its non-relativistic cousin, the Schrödinger equation. For as will be shown below, those solutions are simple enough (and similar enough) that the constructivist project fails in that context.

Recall, then, the time-dependent Schrödinger equation for a single, free, nonrelativistic particle.[8]

$$i\partial_t \psi + \frac{1}{2m}\nabla^2 \psi = 0 \tag{1}$$

[8] $\nabla = \partial_x + \partial_y + \partial_z$; $\nabla^2 = \nabla \cdot \nabla$. Natural units are chosen, so that $c = \hbar = 1$.

The Schrödinger equation is not Lorentz-covariant.[9] But consider now the Klein–Gordon equation for a single, free, relativistic particle.

$$-\partial_t^2\psi + \nabla^2\psi = m_0^2\psi \tag{2}$$

The Klein–Gordon equation is indeed Lorentz-covariant; unlike the Schrödinger equation, it retains its form under any coordinate transformation that is an element of the Poincaré group.

Now, despite their different covariance groups, these two equations both admit solutions of the following form,

$$\psi = Ae^{i(\bar{\kappa}\cdot\bar{x}-\omega t)} \tag{3}$$

where A is some scalar value.[10] Solutions of this form describe a wave-like distribution of both real and imaginary scalar values, as illustrated in Figure 3.

Unless otherwise noted, ψ is used in the following sections to represent a solution of this form—a complex-valued 'planewave' solution. Planewaves have constant

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[9]The Schrödinger equation is sometimes called 'Schrödinger-covariant', where the Schrödinger group is an adaption of the Galilean group. See discussions by Lévy-Leblond [1963, §III; 1967, §II], and more recently by Brown and Holland (1999).

[10]As above, natural units are being used. Recall that $\omega := 2\pi\nu$ is the angular frequency of a wave, and that $\bar{\kappa}$ is the wave-vector, which points in the direction of the 'phase velocity' $\bar{v}_p$, the velocity of propagation of a single wave crest. The angular wave*number* is $\kappa = |\bar{\kappa}| := \omega/v_p$.

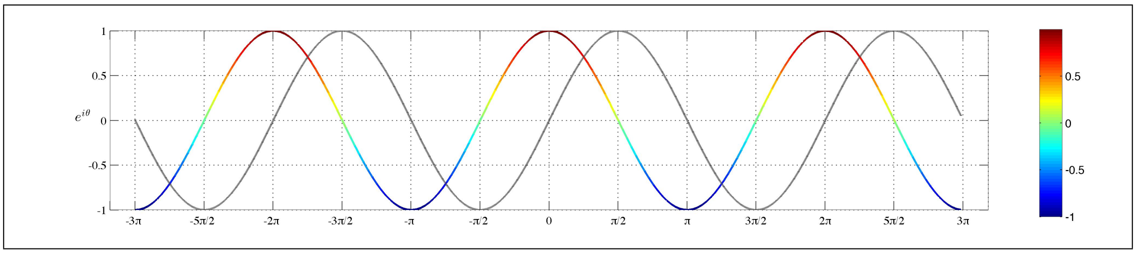

Figure 3: Real (in color) and imaginary (in grey) components of $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. As in all illustrations to follow, unless otherwise noted, the vertical axis represents the magnitude of the scalar function. For real-valued scalar functions, this magnitude corresponds to the color of the curve.

amplitude A, constant angular frequency ω, constant wave-vector $\bar{\kappa}$, and thus constant phase velocity $\bar{v}_p$. They also have a constant magnitude in all spatial directions orthogonal to the wavevector.

Now, despite their similarities in mathematical form, planewave solutions to the Schrödinger and Klein–Gordon equations will differ precisely insofar as do the equations' so-called 'dispersion relations' between ω and κ. For solutions to the Schrödinger equation, that dispersion relation is

$$\omega = \frac{\kappa^2}{2m} \tag{4}$$

whereas another dispersion relation holds for solutions to the Klein–Gordon equation.

$$\omega = \sqrt{\kappa^2 + m_0^2} \tag{5}$$

The differences in wave behaviour brought about by differences in these dispersion relations will become clear below.

In closing, it should be noted that while imaginary numbers may appear in ψ, they do not appear explicitly in the Klein–Gordon equation itself (Eqn.2), whereas they do in the Schrödinger equation (Eqn.1). Thus while the Schrödinger equation admits planewave solutions only in the form of Equation 3, the Klein–Gordon equation also admits planewave solutions that are entirely real-valued. Such solutions are still planewaves, in that they share the characteristics listed above. Indeed, the real-valued planewave solutions to the Klein–Gordon equation are nothing more than the real-valued components of the complex-valued planewave solutions ψ. Thus they can be written as such.

$$\psi_{\Re} = \Re\left(Ae^{i(\bar{\kappa}\cdot\bar{x}-\omega t)}\right) = A\cos(\bar{\kappa}\cdot\bar{x}-\omega t) \tag{6}$$

5 A First Attempt

As mentioned before delving into the details of planewaves, the Klein–Gordon equation is of interest as a Lorentz-covariant equation that is solved by a real-valued scalar function. Might its solutions serve as a supportive example for the constructivist project?

Complex-Valued Planewave Solutions

Let $P_{\mathbb{C}}$ be the set of field points that constitute the physical field represented by a complex-valued planewave solution ψ to the Klein–Gordon equation, as in Equation 3. When those field points are stripped of all geometrical structure but their induced Euclidean topology $\mathcal{T}$, there exists a homeomorphism $\phi : \langle P_{\mathbb{C}}, \mathcal{T}\rangle \to \mathbb{R}^4$, according to which f_ϕ serves as the real-valued component of a planewave solution to the Klein–Gordon equation.[11] But is the Klein–Gordon equation the only way to systematize $\langle P_{\mathbb{C}}, \mathcal{T}\rangle$? No: in fact, there are rival systematizations at play; the Klein–Gordon equation is not the only equation solved by the $f_{\phi'}$ corresponding to permissible coordinatizations ϕ' of $\langle P_{\mathbb{C}}, \mathcal{T}\rangle$. For instance, there is a certain set of permissible coordinatizations for which the corresponding $f_{\phi'}$ solve the Schrödinger equation.

To see this, recall from above that both the Klein–Gordon and Schrödinger

[11]Note that Figure 2 falls short in this context, since a complex-valued solution to the Klein–Gordon equation would have to be modelled by either *two* mappings, $\psi_{\Re} : \mathbb{R}^4 \to \mathbb{R}$ and $\psi_{\Im} : \mathbb{R}^4 \to \mathbb{R}$, or else by one mapping, $\psi_{\mathbb{C}} : \mathbb{R}^4 \to \mathbb{C}$. For simplicity, and for reasons outlined below, this discussion focuses primarily upon the real-valued components.

equations admit planewave solutions ψ of the form of Equation 3, and that for any given choice of amplitude A, mass m, and wavenumber κ, the planewave solutions to these two equations will differ only insofar as do their dispersion relations. And what kind of change in wave behaviour is effected by a difference in dispersion relation? Obviously, wave equations with different dispersion relations will have different angular frequencies ω for some given wavenumber κ. But another difference follows: this change in angular frequency ω will not be accompanied by a corresponding change in wavelength λ, because $\kappa := 2\pi/\lambda$, and no change in wavenumber κ is being proposed. Rather, the difference in dispersion relation affects a change in angular frequency that is accompanied by a change of phase velocity $\bar{v}_p$, since $v_p = \omega/\kappa$.[12] Both of these differences are illustrated in Figure 4, which shows the real-valued components of planewave solutions to the Klein–Gordon and Schrödinger equations with equal-valued wavenumbers κ. The difference in angular frequency ω can be seen by the relative difference in the number of crests that fit within some time, and the difference in phase velocity $\bar{v}_p$ can be seen by the relative difference in the 'slope' of the wavecrests.

The important point is that differences of this sort can be accommodated by homeomorphisms of the field points' Euclidean topology. Indeed, both can be accounted for by one and the same uniform 'stretching' of the time coordinate. In particular, if the time coordinate of Figure 4-ii were stretched by a small amount,

[12]As in footnote 10, $v_p = \nu\lambda = 2\pi\nu/\kappa = \omega/\kappa$. When the dispersion relation is linear, a change in κ does not result in a change in v_p, but merely in λ. Having non-linear dispersion relations, the Klein–Gordon and Schrödinger equations are examples of so-called 'dispersive' wave equations.

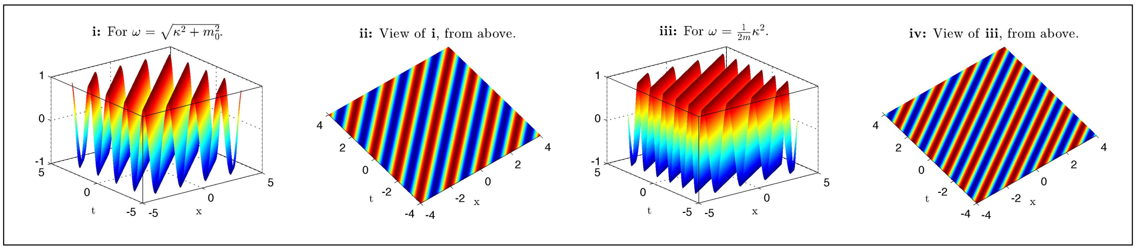

Figure 4: $\Re(\psi)$, the real-valued component of a planewave ψ. For simplicity, $A = m = \hbar = 1$ and $\kappa = 3$. Plots i and ii show $\Re(\psi)$ for the dispersion relation of the Klein–Gordon equation, while iii and iv show $\Re(\psi)$ for that of the Schrödinger equation. Plots ii and iv reveal a difference in $\bar{v}_p$, but not in λ.

the result would be the plot of Figure 4-iv.[13] The upshot of all this is that for any such $\langle P_{\mathbb{C}}, \mathcal{T}\rangle$, which by definition admits a coordinatization ϕ with respect to which f_ϕ serves as the real-valued component of a planewave solution to the Klein–Gordon equation, there is a permissible alternative coordinatization ϕ' with respect to which $f_{\phi'}$ serves as the real-valued component of a solution to the Schrödinger equation. What's more, the same can be said about the imaginary components $\Im(\psi)$ of these complex-valued solutions. As illustrated in Figure 3, the magnitude of the imaginary component of a complex-valued planewave is out of phase with, but otherwise equal to, that of the real-valued component. Therefore the imaginary-valued component of complex-valued solutions to these equations needn't be given separate attention when considering uniform coordinate transformations.[14] But all of this is simply to say that the $\langle P_{\mathbb{C}}, \mathcal{T}\rangle$ under consideration here is just as easily systematized by the Schrödinger equation.

[13]Specifically, the time dilation is: $t \to t' = \frac{\sqrt{\kappa^2+m_0^2}}{\kappa^2/2m}t$.

[14]Note that the phase difference would also be preserved in such a transformation. As illustrated in Figure 3, the real and imaginary components are out of phase by one quarter of a wavelength, and as in the discussion surrounding footnote 12 on page 16, no change in wavelength (wavenumber) is being proposed.

Following the programme laid out in §3, it now remains to be asked what the covariance group is for the Schrödinger equation. For present purposes, suffice it to say that it is not the Poincaré group; the Schrödinger equation is not Lorentz-covariant.[15] And because the Klein–Gordon equation does not systematize $\langle P_{\mathbb{C}}, \mathcal{T} \rangle$ in a way that is obviously simpler or in any way stronger than the Schrödinger equation does, this does not bode well for finding a single planewave solution to the Klein–Gordon equation that would serve as a supportive example of the constructivist project. In fact, plenty of other equations might be given as rival systematizations when dealing with such a simple example of $\langle P_{\mathbb{C}}, \mathcal{T} \rangle$. But of course the Klein–Gordon equation admits not only complex-valued, but also real-valued planewave solutions.

Real-Valued Planewave Solutions

Consider now some real-valued planewave solution $\psi_{\Re}$ to the Klein–Gordon equation, as in Equation 6. As before, let P be the set of field points that constitute the physical field, and purge those points of all but their induced Euclidean topological structure $\mathcal{T}$. Then once again, in order to determine whether the best systematization of $\langle P, \mathcal{T} \rangle$ is a Lorentz-covariant equation, it must first be determined whether there are any permissible recoordinatizations ϕ' according to which $f_{\phi'}$ solves another equation.

Because the Schrödinger equation does not admit real-valued planewave solutions, it is not in the running for the best-systems dynamical law of $\langle P, \mathcal{T} \rangle$. Still, this doesn't ensure the title for the Klein–Gordon equation. In fact there are other

[15]See footnote 9 on page 12.

equations that admit real-valued solutions of this same form. And more importantly, $\langle P, \mathcal{T} \rangle$ does in fact admit permissible coordinatizations ϕ' with respect to which the $f_{\phi'}$ solve these other equations.

Consider for example what I will call the 'Squaringer' equation.[16]

$$\partial_t^2 \psi + \frac{1}{4m^2} \nabla^4 \psi = 0 \tag{7}$$

Like the Klein–Gordon equation, this admits both complex- and real-valued planewave solutions. For both, the dispersion relations match those of the Schrödinger equation: $\omega = \kappa^2/2m$. And above, it was shown that complex-valued planewaves with this dispersion relation are homeomorphic to solutions to the Klein–Gordon equation. The same is therefore true of real-valued planewaves with this dispersion relation. That is, when a single, real-valued planewave solution to the Klein–Gordon equation is considered up to homeomorphism as $\langle P, \mathcal{T} \rangle$, there are permissible alternative coordinatizations ϕ' with respect to which $f_{\phi'}$ will solve the Squaringer equation. And because the Squaringer equation is not Lorentz-covariant, it serves as a threat to the claim that $\langle P, \mathcal{T} \rangle$ might be systematized best by a Lorentz-covariant equation.[17]

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[16]The name comes from its derivation. Following a common derivation of the Schrödinger equation (see Wachter's, for instance (2011, §1.1)), begin instead by *squaring* the nonrelativistic energy–momentum relation: $E = \frac{p^2}{2m} \longrightarrow E^2 = \frac{p^4}{4m^2}$.)

[17]The free-wave question would be another such threat: $-\partial_t^2 \psi + \nabla^2 \psi = 0$. The Squaringer equation has been chosen for its solutions' similarity to the already-familiar Schrödinger equation.

6 More General Solutions

It has been shown above that, as expected, both complex- and real-valued planewave solutions to the Klein–Gordon equation, when considered 'up to homeomorphism' as $\langle P, \mathcal{T} \rangle$, admit permissible coordinatizations ϕ' according to which the $f_{\phi'}$ solve other, equally simple and strong equations with different covariance groups. But of course a single planewave solution constitutes an extremely simple and symmetric space of field points, and one ought not to be surprised if such a $\langle P, \mathcal{T} \rangle$ cannot support a best-systems account of Minkowski geometrical structure.[18]

So, given that planewaves of the form of Equations 3 and 6 are not the most general solutions to Klein–Gordon equation, might its solutions of a more general form have characteristics such that the corresponding $\langle P, \mathcal{T} \rangle$ admit no permissible recoordinatizations such that the $f_{\phi'}$ solve even the few rival equations introduced above? That is, might some of the competition be eliminated by choosing a $\langle P, \mathcal{T} \rangle$ that comprises the point-like parts of a more complicated solution? This question could be addressed in two ways: first, by considering more general solutions to the Klein–Gordon equation as introduced above; second, by considering the solutions to the more general form of the Klein–Gordon equation itself.

In the space provided, this section will address the question in only the first way.

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[18]Indeed, for a single, real-valued planewave solution $\psi_{\Re}$, certain permissible coordinatizations of $\langle P, \mathcal{T} \rangle$ will yield no change in $f_{\phi'}$ along particular coordinates, allowing for systematizations of fewer variables. For example, under a recoordinatization ϕ' according to which $\partial_t f_{\phi'} = 0$, it turns out that $f_{\phi'}$ solves the Helmholtz equation: $\nabla^2 \psi = -m_0^2 \psi$. However, this will no longer be the case for more general solutions to the Klein–Gordon equation. (*cf.* fn. 19, p. 23.)

In doing so, it will be shown that by considering just slightly more complicated solutions to the Klein–Gordon equation, it becomes much more difficult to find permissible coordinatizations with respect to which the $f_{\phi'}$ solve any other equations at all. Thus the onus falls on the critic of this interpretation of the dynamical approach to show that some other non-Lorentz-covariant equation deserves the title of 'dynamical law' for the systems under consideration. Granted, these examples are still extremely simple in comparison with anything that might represent a more realistic, specially relativistic world, but of course that would only bolster the plausibility of the constructivist's best-systems claim.

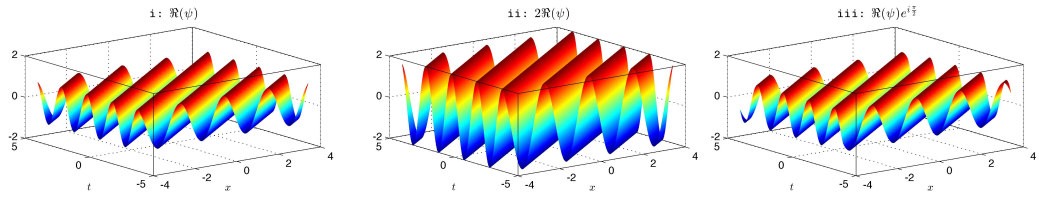
Linear Combinations of Planewave Solutions

It was reported in §4 that the Schrödinger and Klein–Gordon equations both admit complex-valued planewave solutions ψ of the form of Equation 3. Being linear equations, they also admit solutions of the form of any linear combination of its planewave solutions. That is, any solution of the form

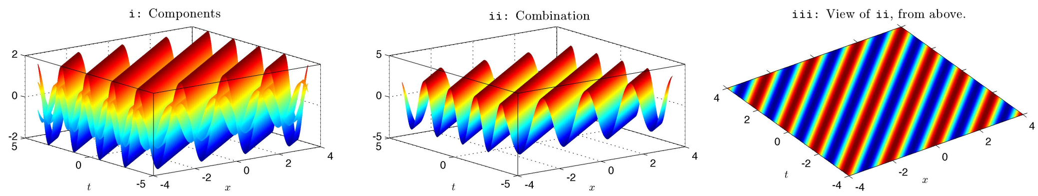
$$\psi = \sum_{\kappa} A_{\kappa} e^{i(\bar{\kappa}\cdot\bar{x}-\omega_{\kappa}t)}$$

will solve the Schrödinger equation (Eqn. 1), so long as the dispersion relation $\omega_{\kappa} = \kappa^2/2m$ holds for each of the summands. And of course, the same could be said for the Klein–Gordon equation, but with two changes: first, the dispersion relation that would have to hold for each of the summands is $\omega_{\kappa} = \sqrt{\kappa^2 + m_0^2}$; second, the linear combination claim would also hold for real-valued planewave solutions $\psi_{\Re}$.

What are the salient characteristics of linear combinations of real-valued planewaves? As shown below, linear combinations of planewaves sometimes constitute planewaves themselves, and at other times result in more complicated wave



(a) Three planewaves of equal $\bar{\kappa}$, ω, and thus $\bar{v}_p$.



(b) Sum of the three planewaves above.

Figure 5: The linear combination of three real-valued, planewave solutions to the Klein–Gordon equation ($\omega = \sqrt{\kappa^2 + m_0^2}$), all with equal wavevectors $\bar{\kappa}$, angular frequencies ω, and thus phase velocities $\bar{v}_p$. The summands are shown in 5(a): plot i has an amplitude of $A = 1$, and therefore matches Figure 4-i; plot ii differs from i by having a doubled amplitude, $A = 2$; and plot iii differs from plot i by a phase shift of $\frac{\pi}{2}$. Their sum is shown in 5(b): plot i shows the summands superimposed; plot ii shows their sum; plot iii shows their sum from above. As in earlier figures, $m_0 = \hbar = 1$ and $\kappa = 3$.

configurations. Consider first an example of the former: a linear combination of different planewaves, all of which have the same wavenumber κ, angular frequency ω, and therefore phase velocity $\bar{v}_p$, the sum of which is another planewave. Figure 5(a) illustrates three such planewave summands, all of which solve the Klein–Gordon equation. From the discussion above, it should be clear by inspection that each of the summands (i–iii) is homeomorphic to a solution to the Squaringer equation, introduced above. The three planewaves' sum is illustrated in Figure 5(b). There too, it should be clear by inspection that the resulting planewave is also homeomorphic to a solution to the Squaringer equation. From this it follows that a linear combination of planewave solutions to the Klein–Gordon equation, all having equal wave vectors and angular frequencies, results in another solution to the Klein–Gordon equation that, when taken up to homeomorphism as $\langle P, \mathcal{T} \rangle$,

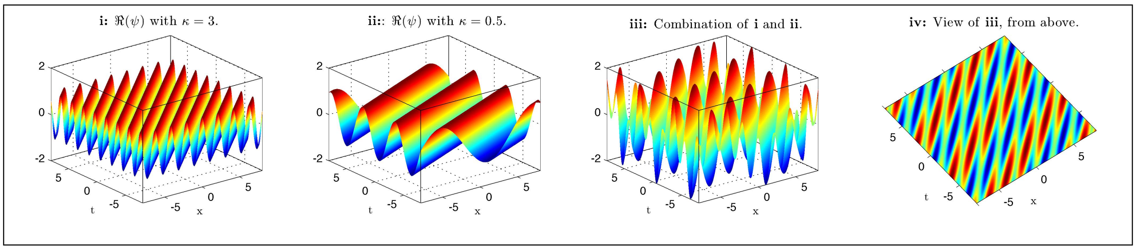


Figure 6: The sum of two real-valued planewave solutions to the Klein–Gordon equation, for which $\omega = \sqrt{\kappa^2 + m_0^2}$. As above, $A = m = \hbar = 1$, so that plot i matches Figure 4-i again. The difference in the summands' wavenumbers κ ensures that their sum is not a planewave.

admits alternative coordinate-dependent descriptions $f_{\phi'}$ from among which can be found solutions to rival equations like the Squaringer.

But now consider a linear combination of planewave solutions to the Klein–Gordon equation, the wavenumbers (and thus phase velocities) of which differ from one another, and the sum of which is therefore not a planewave. Figure 6 illustrates two such planewaves.[19] As before, it is clear by inspection that both of the summands (i, ii) are homeomorphic to a solution to any of the rival equations introduced above. However, it is less clear whether their sum (iii, iv), which is another real-valued solution to the Klein–Gordon equation, is also homeomorphic to solutions of those rival equations.

In fact it is. To see this for the Squaringer equation, consider what would result from the following process: first, consider two planewave solutions to the Squaringer equation with wave-vectors equal to those of the planewave summands illustrated in Figure 6; then, consider the linear combination of those planewaves. This linear combination is illustrated in Figure 7. By the fact that the summands (i, ii) solve the

[19]Note that for such solutions, the systematization mentioned in footnote 18 (p. 20) is no longer possible, as there are no dimensions along which the field values are constant.

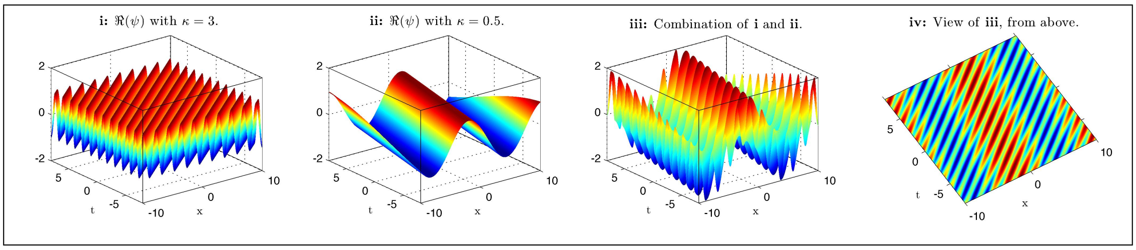


Figure 7: The sum of two real-valued planewave solutions to the Squaringer equation, for which $\omega = \kappa^2/2m$. As above, $A = m_0 = \hbar = 1$. Note that plots i and ii have the same value for κ as the corresponding plots in Figure 6.

Squaringer equation, it follows that the sum (iii, iv) does as well. Then, comparing the sum to Figure 6-iv reveals that the two are indeed homeomorphic—i.e. their differences can be accommodated by some permissible autohomeomorphism. To see this, review the illustrations and the caption to Figure 10.

From this it follows that a linear combination of two real-valued planewave solutions to the Klein–Gordon equation is no more viable as a supportive example of the constructivist's account of Minkowski geometry than a single planewave solution was. However, things change quite quickly when the number of summands increases. Consider the situation illustrated in Figure 8, which shows the linear combination of three planewave solutions to the Klein–Gordon equation, all having distinct wavenumbers. Is the sum homeomorphic to a solution to any of the three rival equations introduced above? If it is, those solutions can not be found by the process above, for the sum in Figure 8-iv is not obviously homeomorphic to what results from the preceding operation of taking planewave solutions to, say, the Squaringer equation, all having corresponding values for κ, and then taking their linear combination. The result of that process is illustrated in Figure 9. To see what the trouble is, consider the steps that would be involved in repeating the process mentioned above, and outlined in Figure 10. After using a time dilation to transform the first planewave summand into a solution to the Squaringer equation, it would

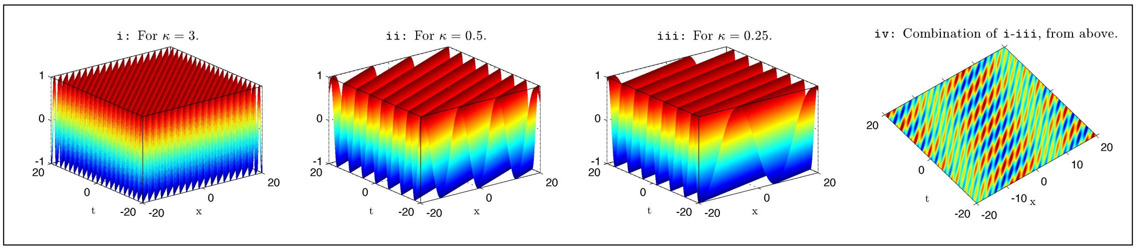


Figure 8: The linear combination of three planewave solutions to the Klein–Gordon equation, for which $\omega = \sqrt{\kappa^2 + m_0^2}$. As above, $A = m = \hbar = 1$. Plots i and ii match those of Figure 6 in a different scale.

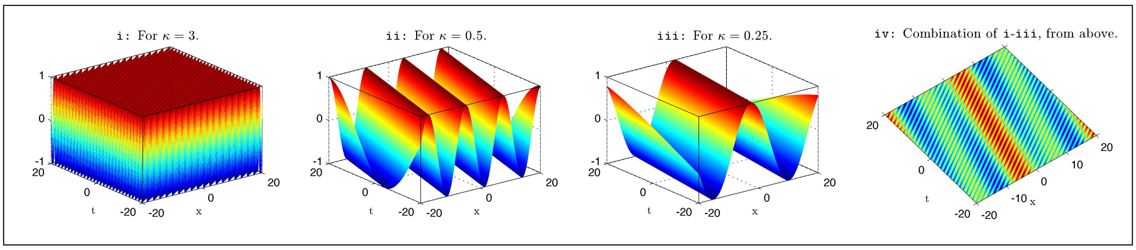


Figure 9: The linear combination of three planewave solutions to the Squaringer equation, for which $\omega = \kappa^2/2m$. As above, $A = m = \hbar = 1$. Plots i and ii match those of Figure 7 in a different scale.

once again be necessary to boost into the 'rest frame' of that planewave before using another time dilation to transform the second summand into a solution to the Squaringer equation. But when this process is attempted for the third summand, trouble arises. There is no mutual 'rest frame' for the first and second summands, from which one could carry out a time dilation on the third summand without affecting what was accomplished in the previous two operations. The problem is simply that there are now more planewaves than there are dimensions. Any time dilation that transforms the third planewave summand into a solution of the Squaringer equation will interfere with the dispersion relation of at least one of the first two planewave summands.[20]

[20]Note that this claim is made in regard to two-dimensional examples. In taking a four-dimensional topological structure as primitive, this would only be true in

Of course, it should be granted that this only shows that by taking up to homeomorphism a linear combination of a sufficient number of real-valued planewave solutions to the Klein–Gordon equation, all differing in wavenumber, there is no combination of boosts (shearing transformations) and dilations (stretching transformations) under which $\langle P, \mathcal{T} \rangle$ would be recoordinatized so that $f_{\phi'}$ solves one of the rival systematizations outlined above. Whether some other autohomeomorphism of $\mathcal{R}^4$ might do the trick, or whether there may be alternative coordinatizations ϕ' according to which the $f_{\phi'}$ solve some other equations altogether, is an open question. But for the time being, the lesson is simply that by taking a linear combination of planewave solutions to the Klein–Gordon equation with different wavenumbers, the obvious competition for the best-systems dynamical law of $\langle P, \mathcal{T} \rangle$ at least diminishes. Thus the onus is on the critic of the dynamical approach to find an autohomeomorphism of $\mathcal{R}^4$ such that the corresponding $f_{\phi'}$ would solve some otherwise-covariant equation, the simplicity and strength of which could rival those of the Klein–Gordon equation.

7 Summary and Reflections

One of the goals of this paper has been to use solutions to the Klein–Gordon equation as a case study in the constructivist's best-systems account of Minkowski spacetime structure. First, it was shown that the simplest, planewave solutions to the Klein–Gordon equation can indeed be recoordinatized—without any violation to

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the case of *five* or more planewave solutions to the Klein–Gordon equation, all differing in wavenumber. A similar comment applies to the claim made in footnote 19 (p. 23).

their topological structure—in such a way as to instantiate a solution to equations of other covariance groups. However, it has also been shown that one needn't venture far into the more complicated solutions of the Klein–Gordon equation in order to find candidate examples of $\langle P, \mathcal{T} \rangle$ or $\langle P_{\mathbb{C}}, \mathcal{T} \rangle$ for which rival systematizations are not so easily found. Indeed, this was shown to be the case for linear combinations of a small number of real-valued planewave solutions.

All of this would suggest that the constructivist's best-systems claim is more plausible than it might have seemed before considering some specific examples. What's more, it has been shown to be plausible in the relatively disadvantageous context of taking Euclidean topological structure as primitive. The enormous symmetry group of $\mathcal{R}^4$ allows a great number of alternative coordinatizations ϕ' to be considered, making it infeasible to argue that all possible rival systematizations have been considered. If a richer geometrical structure were taken as primitive, and if its symmetry group could be outlined in terms of specific coordinate transformations, then it would be much easier for the constructivist to ensure that all rival systematizations had in fact been considered. Alternatively, instead of enriching the topological structure being taken as primitive, the constructivist project might also be accomplished by considering a topologically structured *set* of scalar fields, or a topologically structured set of vector fields or tensor fields. Such examples, however, would require more room for discussion than allotted here.

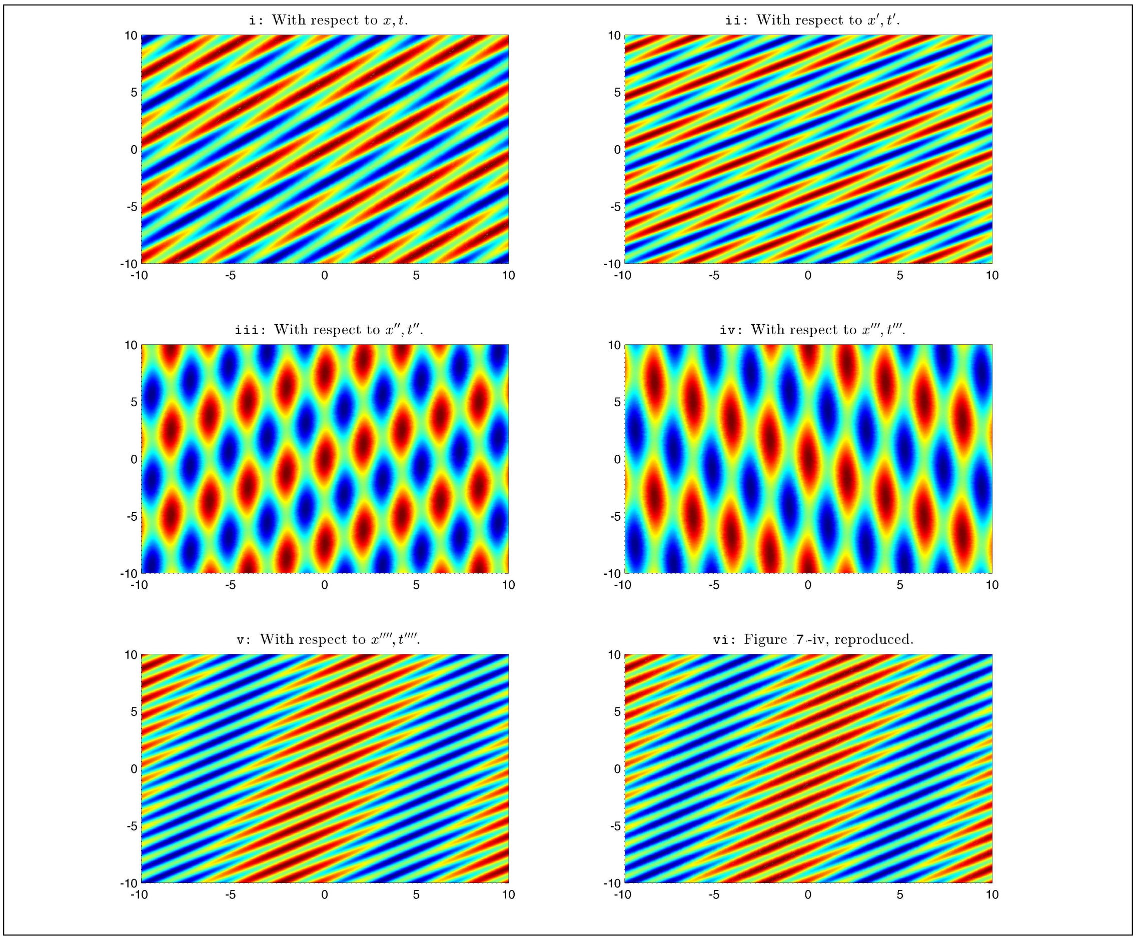

Figure 10: The relationship between plots 6-iv and 7-iv. Plot i shows the former, and plots ii–v show a series of transformations under which plot i comes to solve the Squaringer equation. That is, plots ii–iv show a series of transformations under which Figure 6-iv becomes Figure 7-iv. In reviewing these steps, it will be helpful to think of plot i in terms of the two planewave summands illustrated in Figure 6. To begin, plot ii shows the result of the time dilation, mentioned earlier (p. 17), under which the first planewave summand shown in Figure 6-i would come to satisfy the Squaringer equation. For this step, $x \to x' = x$ and $t \to t' = t\sqrt{\kappa^2 + m_0^2}/(\kappa^2/2m)$, with $\kappa = 3$. The following three steps must do the same for the second summand of Figure 6, but without ruining the work accomplished so far. Toward that end, plot iii shows the result of boosting into the 'rest frame' of the now-dilated first planewave summand. For this step, $t' \to t'' = t'$, and $x' \to x'' = x' - v_d t'$, where $v_d = 1.5$ is the phase velocity of the first planewave summand after the aforementioned time dilation. Another time dilation can now safely be carried out to transform the second planewave summand into a solution to the Squaringer equation. For this step, $x'' \to x''' = x''$, and $t'' \to t''' = -1.3454t''$. (The decimal value can be solved for by requiring that the final result of these steps be a solution to the Squaringer equation.) Finally, plot v shows the result of boosting back, so as to undo the changes brought about in moving from plot ii to plot iii. For this step, $t''' \to t'''' = t'''$, and $x''' \to x'''' = x''' + v_d t'''$. Plugging in values for κ, ω, and v_d, these steps combine to the overall transformation $x \to x'''' = x + 2.6149t''''$ and $t \to t'''' = -0.9454t$, under which the exponents of both planewave summands of Figure 6 transform to satisfying the dispersion relation of the Squaringer equation (p. 19).

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